CONTEMPORARY MATHEMATICS

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Horizons of Fractal Geometry and Complex Dimensions

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Fractal Geometry and Complex Dimensions,
In celebration of the 60th birthday of Michel Lapidus
June 21–29, 2016
California Polytechnic State University,
San Luis Obispo, California

Robert G. Niemeyer
Erin P. J. Pearse
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Editors

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Preface: Horizons of Fractal Geometry and Complex Dimensions

Here we showcase various recent results from the fertile and active nexus of research topics: dynamical systems, fractal geometry, number theory and quasicrystals. The contributions presented reflect different aspects of these topics and form new connections; they have been carefully selected, peer reviewed and are aimed at both the specialist and those that are new to the fields.

This volume complements and records the outcomes of the 2016 Summer School on Fractal Geometry and Complex Dimensions, an event which was funded by the National Science Foundation and hosted by the Mathematics Department of California Polytechnic State University, and which, in part, celebrated the 60th birthday of Prof. M. L. Lapidus. The event successfully brought together experts from 14 countries and was attended by over 100 academics. During this event new collaborative projects were formed — results of which are featured within. We hope to have captured the exciting atmosphere of the event in this issue through the original research, survey and expository articles.

We are indebted and grateful to a number of anonymous referees for their invaluable help and suggestions in preparing this issue. We express our gratitude to the National Science Foundation for sponsoring the summer school; without this support, this issue and the event could not have come to fruition.

This issue is divided into four parts, each centered around a different theme: dimension gaps and the mass transfer principle, fractal strings and complex dimensions, Laplacians on fractal domains and SDEs with fractal noise, and aperiodic order (Delone sets and tilings).

We begin with the subject of dimension gaps and the mass transfer principle. A goal often encountered in mathematics is to develop relationships between independently defined invariants of classes of mathematical objects. In conformal dynamics, there is a large body of work devoted to showing that the critical exponent of the Poincaré series associated to a conformal dynamical system, for instance Kleinian groups acting on hyperbolic space or rational maps acting on the Riemann sphere, coincides with the Hausdorff dimension of the corresponding limit set or Julia set, respectively. The article by K. Falk gives a survey of this topic, concentrating on an important notion in geometric group theory, namely amenability, as a tool to determine if the aforementioned invariants coincide or not. This work is complemented by a comprehensive survey article by D. Allen and S. Troscheit on the celebrated mass transfer principle of V. Beresnevich and S. Velani, and its applications to metric number theory. This result was originally motivated by a conjecture of Duffin and Schaeffer, and allows one to transfer a Lebesgue measure statement for a different $\limsup$ set of balls to a f-dimensional Hausdorff measure statement for a lim sup set of balls which are obtained by 'shrinking' the original balls in a certain manner according to the dimension function f. This is a remarkable result given that the Lebesgue measure can be considered to be 'coarser' than the Hausdorff measure.

Our second theme concerns the theory of fractal strings and complex dimensions. The theory of complex dimensions, as introduced by M. L. Lapidus and M. van Frankenhuijsen, is a $\mathbb{C}$-valued extension of non-integer notions of dimension such as the Hausdorff dimension and the Minkowski dimension. These non-integer notions of dimension assign a single number to a given set $U \subseteq \mathbb{R}^d$. By contrast, the set of complex dimensions provides a richly structured geometric invariant of U, typically an infinite but discrete subset of $\mathbb{C}$ of which the Minkowski dimension is a distinguished member. The complex dimensions describe not just the order of magnitude of the scaling properties of U, but also the oscillatory aspects. The research expository article by M. L. Lapidus gives a detailed overview bringing the reader to the frontiers of the subject. The author includes a historical account of the subject, an introduction to a higher dimensional theory as well as a presentation of two new concepts: quantized number theory and fractal cohomology. Moreover, many conjectures, open problems and future research directions are stated.

In our third theme we turn to the closely related topic of Laplacians on fractal domains and SDEs with fractal noise. We begin with the article of P. Pollack and C. Pomerance concerning the asymptotic behaviour of $\mathcal{N}$ – the eigenvlaue counting function of a Laplacian on a domain with a fractal boundary. The first asymptotic term of $\mathcal{N}$ is given by the dimension of the open domain and depends on its volume. The Weyl-Berry-Lapidus conjecture states that if the boundary of the open domain is Minkowski measurable in dimension d, then the second asymptotic term is of order d and that it depends on the Minkowski content in dimension d. This conjecture has been shown to hold in $\mathbb{R}$. The present article shows that in $\mathbb{R}^2$ it is false. Indeed, here the authors present two fractal sprays which possess the same volume and whose boundaries have the same Minkowski content, but with different spectral counting functions. Indeed, they show that the two zeta functions never coincide on the interval $(1, 2)$. This leads us to the article of T. Eichinger and S. Winter where they investigate the second asymptotic term of $\mathcal{N}$ for Laplacians on domains with fractal boundaries which are not Minkowski measurable. Returning to the one-dimensional setting, in M. van Frankenhuijsen's contribution to this issue, a description of how fractal strings can be used to understand the Riemann hypothesis through the relationship between geometric and spectral oscillations is given. This exposition illuminates a number of open questions, some of which are explicitly stated, while others are only suggested.

Complementing these work are the articles by M. Kesseböhmer, T. Samuel and H. Weyer, and M. Zähle and E. Schneider. In the first of these articles, measure geometric Laplacians for discrete distributions are investigated. In the article of M. Zähle and E. Schneider, uniform local contraction properties of the fractional integral operators are established and a higher-dimensional variant of the Doss-Sussman approach to path-wise global solutions is presented.

Our final theme concerns aspects of quasicrystals, specifically tilings and Delone sets. Here we have two contributions, one focusing on tilings associated with iterated function systems, and the other on Delone sets. In the latter of these two articles, a connection with the Duffin-Schaeffer conjecture is also made.

Iterated function systems were pioneered in the 1980s by M. F. Barnsley and J. Hutchinson as a way to generate fractals. This is still one of the most widely used method and has brought about a rich body of work. The article by M. F. Barnsley and A. Vince examines tilings associated with iterated function systems consisting of similitudes and gives conditions under which the resulting tiling is periodic, quasi-periodic and non-periodic.

Delone sets are ubiquitous in the study of aperiodic order (mathemtical quasicrystals) and have applications in coding theory as well as in algorithms that find approximate solutions to NP-hard optimisation problems. M. Baake and A. Haynes provide elegant and interesting generalisations of some of the most classical results in metric number theory, in particular Khinchin's theorem and the Duffin-Schaeffer conjecture, to the case of Diophantine approximations where the distance to the nearest integer is replaced by the distance to a Delone set.

We hope that you enjoy reading this volume as much as we have enjoyed compiling it.

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The theme of the contributions is fractals and dynamics and content is split into four parts, centered around the following themes: Dimension gaps and the mass transfer principle, fractal strings and complex dimensions, Laplacians on fractal domains and SDEs with fractal noise, and aperiodic order (Delone sets and tilings).

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