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SOME ENUMERATIVE GLOBAL PROPERTIES OF VARIATIONS OF HODGE STRUCTURES

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For Pierre Deligne, whose work has incomparably enriched our subject

Abstract. The global enumerative invariants of a variation of polarized Hodge structures over a smooth quasi-projective curve reflect the global twisting of the family and numerical measures of the complexity of the limiting mixed Hodge structures that arise when the family degenerates. We study several of these global enumerative invariants and give applications to questions such as: Give conditions under which a non-isotrivial family of Calabi–Yau threefolds must have singular fibres? Determine the correction terms arising from the limiting mixed Hodge structures that are required to turn the classical Arakelov inequalities into exact equalities.

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I. Introduction

Among the principal invariants measuring the complexity of a variation of Hodge structure (VHS) of weight n are the *Chern classes of the Hodge bundles.* Of particular interest is how the *singularities or degeneracies of the VHS affect these Chern classes.* These are the main themes of the present work.

We will be almost exclusively concerned with variations of Hodge structure parametrized by a smooth curve S and where the local monodromies around the points s_i of degeneracies are unipotent. In this case there are canonical extensions $\mathcal{F}_e^p$ to S of the Hodge filtration bundles, which are defined initially on $S^* = S\backslash\{s_1, \ldots, s_N\}$. The Chern classes then are the quantities

$$\begin{aligned}\delta_p &=: \deg\left(\mathcal{F}_e^{n-p}/\mathcal{F}_e^{n-p+1}\right),\\ \Delta_p &=: \delta_0 + \cdots + \delta_p \geqslant 0,\end{aligned}$$

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where the inequality follows from the curvature properties of the extended Hodge bundles. In particular, setting $E = s_0 + \cdots + s_N$ these properties imply that

$$\Delta_p =: \delta_0 + \cdots + \delta_p = 0 \iff \nabla \mathcal{F}_e^{n-p} \subseteq \mathcal{F}_e^{n-p} \otimes \Omega_S^1(\log E),$$

where ∇ is the Gauss–Manin connection.

In the case of weight $n = 1$ and Hodge number $h^{1,0} = 1$, the VHS is given by a family of elliptic curves. Assuming that the fibres are nodal curves without multiple components, we have

$$\delta_0 = \deg j_S,$$

where j_S is the composition of the extended period mapping $S \to \mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}^*$, where $\mathcal{H}^* = \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$, with the j-function. Thus, we may think of the Δ_p's as extensions to a general VHS of the degree of the classical j-function. It is known (cf. [Ko]) that in the classical case

$$12\delta_0 = \sum_i m_i, \tag{I.1}$$

where the singular fibre over s_i is of type I_{m_i} in Kodaira's notation; this is a prototype of one type of result we would like to extend to more general situations.

In this paper we shall distinguish between

(i) an abstract variation of Hodge structure $(\mathcal{H}_{\mathbb{Z},e}, \mathcal{F}_e^p, \nabla, S)$; and
(ii) the case when the VHS arises from a geometric family

$$f\colon X \to S \tag{I.2}$$

of varieties $X_s = f^{-1}(s)$.

We shall refer to (ii) as the *geometric case*, and we are particularly interested in understanding which results hold in the general case (i) or can only be established in the geometric case (ii).

For example, in the $n = 1$ and $h^{1,0} = 1$ case, a trivial consequence of the existence of the j-function is that a VHS without degeneracies is *isotrivial*.[1] In the case $n = 3$ and $h^{3,0} = 1$, of the type of VHS that arises from a family of Calabi–Yau threefolds, one may ask if the same result holds. This is not the case as there are complete curves in the moduli space $\mathcal{M}_3$ of smooth genus 3 curves (cf. [Di]), and decomposing the Hodge structure on the third cohomology of the Jacobians into its primitive and unprimitive parts and taking the corresponding primitive part of intermediate Jacobians gives such a family without degeneracies. We are, however, able to show that:

If there is a family (I.2) *of Calabi–Yau threefolds without singular fibres, then*

$$h^{2,1} > h^{1,1} + 12. \tag{I.3}$$

[1] A variation of Hodge structure is said to be *isotrivial* if it becomes trivial on a finite branched covering of S. Equivalently, the global monodromy group should be finite, or the $\mathcal{F}_e^p$ should all be stable under ∇. A family (I.2) is *isotrivial* if it becomes a product over a Zariski open set of a branched covering of S. In the case where the X_s are Calabi–Yau this is equivalent to saying that the Kodaira–Spencer maps $T_sS \to H^1(\Theta_{X_s})$ should be zero.

Interestingly, it is the quantity[2]

$$(24+\chi(X_\eta))\delta_0+12(\delta_0+\delta_1) \tag{I.4}$$

that in some sense plays the role of $\deg j_S$. In section V we shall give a result expressing this quantity in terms of topological data arising from the singularities of the singular fibres, which may be viewed as an extension of (I.1).

As a particularly interesting special case, a family (I.2) of Calabi–Yau threefolds is said to be of *mirror quintic type* if $h^{2,1}(X_\eta)=1$. In this case the moduli space $\Gamma\backslash D$ for polarized Hodge structures with $h^{3,0}=h^{2,1}=1$ has dimension 4, but any non-trivial VHS can depend on only 1 parameter, consistent with the well-known fact that the number of moduli of X_η is equal to 1. This situation is in many ways reminiscent of the $n=1$ and $h^{1,0}=1$ case, and from (I.3) we infer

> *Any non-isotrivial family of CY threefolds of mirror quintic type must have singular fibres.*

In this regard we pose the

Question. *Does a non-isotrivial VHS of weight* 3 *with* $h^{3,0}=h^{2,1}=1$ *necessarily have degeneracies?*

After the introductory material in Section II, we shall discuss 1-parameter Calabi–Yau n-fold fibrations in the cases $n=1$, 2, 3. Section III is a brief summary of the classical elliptic surface case, presented in a way to illustrate possible extensions.

In Section IV we discuss the $n=2$ case. Here, there are non-isotrivial families without singular fibres.[3] In this case an interesting quantity is

$$\Delta\chi=\chi(X)-\chi(S)\chi(X_\eta) \tag{I.5}$$

measuring the deviation from multiplicativity of topological Euler characteristics in the fibration (I.2). (In general, (I.5) is computed by the sum of Euler characteristics of the singular fibres.) In that section, and later in Section V in the $n=3$ case, we shall give formulas expressing $\Delta\chi$ in terms of topological and combinatorial data on the singular fibres, under a relative minimality assumption. These results generalize the Kodaira formula

$$\Delta\chi=\Big(\sum_i m_i\Big),$$

where $\Delta\chi=\chi(X)$ since $\chi(X_\eta)=0$.

In Section VI we shall build on some of the work ([Pe1], [Pe2], [JZ], [V] and [VZ1]–[VZ5]) on *Arakelov inequalities*, which give upper bounds on degrees of Hodge bundles. These results are purely Hodge theoretic, and in fact make no use of the rational structure on the local system. For the classical $n=1$ case the inequality is

$$\delta_0\leqslant\frac{1}{2}(h^{1,0}-h_0^{1,0})(2g-2+N), \tag{I.6}$$

[2]Here, η is a generic point of S and X_η is a generic fibre of (I.2).

[3]In general there seems to be an odd-even pattern for families (I.2), perhaps reflecting the odd-even behaviour of the classical Picard–Lefschetz transformations.

where $g = g(S)$ is the genus of the parameter curve, N is the number of singular fibres, and

$$h_0^{1,0} = \text{rank of the } \mathcal{O}_{S,\eta}\text{-module } \ker\big(\mathcal{H}_\eta^{1,0} \xrightarrow{\theta_{0,\eta}} \mathcal{H}_\eta^{0,1} \otimes \Omega^1_{S,\eta}\big),$$

where θ_0 is the mapping induced by ∇. In that section, for the cases of weights $n = 1, 2, 3$ we are able to refine the Arakelov inequalities to equalities. For example, in the $n = 1$ case when we have (see Section VI for explanation of notation)

$$\delta_0 = \frac{1}{2}\Big[(h^{1,0} - h_0^{1,0})(2g-2) + \sum_i \operatorname{rank} \overline{N}_i\Big] - \frac{1}{2}\Big[(-\delta_0') + \sum_{s \in S^*} v_s(\det \bar{\theta}) + \sum_i \nu_{s_i}(\det A')\Big], \qquad \text{(I.7)}$$

where $\operatorname{rank} \overline{N}_i$ is computed from the action of the logarithm of monodromy in the limiting mixed Hodge structure (LMHS) at a point s_i of degeneracy of the VHS, $\delta_0' \leqslant 0$ is the degree of $\mathcal{H}_{0,e}^{2,0}$, and the quantities in the sums are non-negative and zero for all but a finite number of points. We have

$$0 \leqslant \operatorname{rank} \overline{N}_i \leqslant h^{1,0} - h_0^{1,0}$$

and

$$\begin{cases} \operatorname{rank} \overline{N}_i = 0 \iff s_i \textit{ is not a singularity}, \\ \operatorname{rank} \overline{N}_i = h^{1,0} - h_0^{1,0} \iff \textit{the LMHS is of Hodge–Tate type.} \end{cases} \qquad \text{(I.8)}$$

In (I.7), $\bar{\theta}_{0,s}$ is the mapping $\mathcal{H}_s^{1,0}/\mathcal{H}_{0,s}^{1,0} \to (\mathcal{H}_{0,s}^{1,0})^\perp \otimes \Omega^1_{S,s}$ induced by $\theta_{0,s}$ using $(\mathcal{H}_{0,e}^{1,0})^\perp \subset \check{\mathcal{H}}_e^{1,0} \cong \mathcal{H}_e^{0,1}$.

There are similar, perhaps even more interesting, results in the $n = 2, 3$ cases. For $n = 3$ it turns out that the correct quantities to bound are not δ_0 and δ_1 but rather

$$\begin{cases} \Delta_0 = \delta_0, \\ \Delta_1 = \delta_0 + \delta_1 \end{cases}$$

which are non-negative and vanish if and only if the VHS is isotrivial.

For $n = 2$ we find the classical Arakelov inequality

$$\delta_0 \leqslant (h^{2,0} - h_0^{2,0})(2g-2) \qquad \text{(I.9)}$$

with equality holding if and only if

(i) the induced Kodaira–Spencer maps $\bar{\theta}_{0,s}$ in $S^* = S \backslash \{s_1, \ldots, s_N\}$ are all fibrewise injective;
(ii) the subbundle $\mathcal{H}_{0,e}^{2,0} \oplus \mathcal{H}_{0,e}^{1,1} \oplus \overline{\mathcal{H}}_{0,e}^{2,0}$ is a flat subbundle of $\mathcal{H}_e$, and therefore gives a sub-VHS; and
(iii) denoting by $\hat{H}_i^r = \bigoplus_{p+q=r} \hat{H}_i^{p,q}$ the Hodge structure on the r^{th} graded piece $\operatorname{Gr}_r(\mathrm{LMHS}_i)$ for the limiting mixed Hodge structure at s_i, we have

$$\hat{h}_i^{2,0} = h_{0,s_i}^{2,0}$$

for all i.

The third condition is satisfied if LMHS_i is of Hodge–Tate type, but not conversely.

The Arakelov inequality is given by correcting (I.9) by subtracting from the right-hand side a sum of three non-negative terms corresponding to (i), (ii), (iii).

For $n = 3$ we find inequalities

$$\begin{cases} \text{(a)} \qquad \delta_0 \leqslant \left(h^{3,0} - h_0^{3,0} + \frac{1}{2}(h^{2,1} - h_0^{2,1})\right)(2g - 2 + N), \\ \text{(b)} \qquad \delta_0 + \delta_1 \leqslant \left(h^{3,0} - h_0^{3,0} + h^{2,1} - h_0^{2,1}\right)(2g - 2 + N) \end{cases} \tag{I.10}$$

with equality holding if and only if conditions similar to (i), (ii), (iii) above are satisfied. For the most interesting condition (iii) we have

$$\text{(a)} \iff \hat{h}_i^{3,0} = h_{0,s_i}^{3,0} \text{ for each singular point,}$$

$$\text{(b)} \iff \hat{h}_i^{3,0} = h_{0,s_i}^{3,0} \text{ and } \hat{h}_i^{2,1} = h_{0,s_i}^{2,1} + \hat{h}_i^{3,2} \text{ for each singular point.}$$

Again, these conditions are satisfied if LMHS_i is of Hodge–Tate type, but not conversely.[4]

We note that for $n = 1$ we have two expressions for $\delta = \delta_0$, one arising from the Grothendieck–Riemann–Roch theorem and one arising from the Arakelov equality. Comparing these gives a formula for the sum of the correction terms in the Arakelov equality. We work this out for elliptic surfaces, where the only correction terms are where the Kodaira–Spencer maps fail to be fibrewise injective; in particular this includes when the fibre is smooth with automorphisms.

We also work this out when $n = 3$ and X_η is Calabi–Yau of mirror quintic type. In [GGK2] the possible unipotent degernerations have been classified and the contributions to δ_0 and to $\delta_0 + \delta_1$ of each type are analyzed. This leads to a generic global Torelli result for the physicists mirror quintic family, a result obtained earlier by S. Usui.

In section VII we explore a different type of enumerative question. Namely, when we have a VHS $(\mathcal{H}_\mathbb{Z}, \mathcal{F}^p, \nabla, S)$ of weight n and without degeneracies and where S has arbitrary dimension, a theorem of Deligne (cf. [Z]) gives that

$$H^{n+r}(S, \mathcal{H}_\mathbb{Z}) \textit{ has a Hodge structure of weight } n + r.$$

An interesting question is: *What data is needed to determine the Hodge numbers* $H^{p,q}(S, \mathcal{H}_\mathbb{C})$? What we find is that in the case of a general VHS, aside from simple cases there is not much that we are able to say. However, in the geometric case (I.2) of a family of threefolds (which need not be Calabi–Yau's) over a curve, one finds that

(i) from Deligne's theorem alone one may determine $H^{4,0}(S, \mathcal{H}_\mathbb{C})$,
(ii) to determine $H^{3,1}(S, \mathcal{H}_\mathbb{C})$ and $H^{2,2}(S, \mathcal{H}_\mathbb{C})$ one needs additional information coming from the VHS arising from the other $R_f^q\mathbb{Q}$ $(q \neq 3)$.

We find this to be instructive in that it helps to further illuminate what additional information is contained in variations of Hodge structures that arise from geometry.

[4] *Note added in proof*: We have just seen the interesting paper [MVZ] in which there are results related to, but seemingly different from, those given above. See also [VZ1]–[VZ5].

II. Notations and Terminology

II.A. *Hodge theoretic preliminaries*

We will be considering a *polarized variation of Hodge structure* (VHS) over a smooth quasi-projective algebraic curve S^*. The variation of Hodge structure will be denoted $(\mathcal{H}_{\mathbb{Z}}, \mathcal{F}^p, \nabla, S^*)$, where

- $\mathcal{H}_{\mathbb{Z}}$ is a local system on S^*;
- $\mathcal{F}^p$ is a filtration of $\mathcal{H} = \mathcal{H}_{\mathbb{Z}} \otimes \mathcal{O}_{S^*}$ that induces on each fibre H_s a polarized Hodge structure of weight n;[5]
- $\nabla\colon \mathcal{H} \to \mathcal{H} \otimes \Omega^1_{S^*}$ is the Gauss–Manin connection with kernel $\mathcal{H}_{\mathbb{C}}$ and satisfying
$$\nabla(\mathcal{F}^p) \subset \mathcal{F}^{p-1} \otimes \Omega^1_{S^*}.$$

There is a canonical completion of S^* to a smooth projective curve S; then

$$S^* = S \backslash E,$$

where $E = \{s_1, \ldots, s_N\}$ is the set of punctures. Around each s_i there is a local monodromy transformation T_i; unless stated otherwise we shall assume that

- T_i is unipotent.

There is then the canonical *Deligne extension* of the VHS to $(\mathcal{H}_{\mathbb{Z},e}, \mathcal{F}^p_e, \nabla, S)$ over S.[6] Then $\mathcal{H}_e$ is a vector bundle over S filtered by subbundles $\mathcal{F}^p_e$ and where the transversality condition on the Gauss–Manin connection becomes

$$\nabla\colon \mathcal{F}^p_e \to \mathcal{F}^{p-1}_e \otimes \Omega^1_S(\log E). \tag{II.A.1}$$

We set

$$\begin{cases} \mathcal{H}^{p,n-p} = \mathcal{F}^p/\mathcal{F}^{p+1}, \\ \mathcal{H}^{p,n-p}_e = \mathcal{F}^p_e/\mathcal{F}^{p+1}_e. \end{cases}$$

Of importance throughout this paper will be the maps induced by ∇, called the *Kodaira–Spencer maps*[7]

$$\mathcal{H}^{p,q} \xrightarrow{\theta_q} \mathcal{H}^{p-1,q+1} \otimes \Omega^1_S(\log E) \tag{II.A.2}$$

that capture the first order infinitesimal information in the VHS over S^*. At the punctures s_i, the map θ_{q,s_i} on the fibre is given by the residue of ∇.

Over S^* the polarized Hodge structures on the fibres induce *Hodge metrics* in each vector bundle $\mathcal{H}^{p,q}$. There is then an associated *metric connection* whose corresponding curvature is

$$\Theta_{\mathcal{H}^{p,q}} = {}^t\theta_q \wedge \bar{\theta}_q + \bar{\theta}_{q-1} \wedge {}^t\theta_{q-1}, \tag{II.A.3}$$

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[5]We shall not distinguish between a locally free sheaf $\mathcal{E}$ of $\mathcal{O}_{S^*}$-modules and the corresponding analytic vector bundle E with fibres E_s, $s \in S^*$. Although we will work in the analytic category, all of the analytic objects will have an algebraic structure.

[6]Throughout this work the subscript "e" will stand for extension.

[7]Here we shall understand that $q = n - p$.

where the superscript t denotes the Hermitian adjoint. In terms of local unitary frames

$$\begin{cases} e_\alpha \ \text{ for } \mathcal{H}^{p,q}, \\ e_\mu \ \text{ for } \mathcal{H}^{p-1,q+1}, \\ e_\rho \ \text{ for } \mathcal{H}^{p+1,q-1} \end{cases}$$

with the Kodaira–Spencer maps given by

$$\begin{cases} \theta_q(e_\alpha) = \sum_\mu A^\mu_\alpha e_\mu, \\ \theta_{q-1}(e_\rho) = \sum_\alpha A^\alpha_\rho e_\alpha \end{cases}$$

the curvature matrix for $\mathcal{H}^{p,n-p}$ is given by

$$(\Theta_{\mathcal{H}^{p,n-p}})_{\alpha\overline{\beta}} = \sum_\mu A^\mu_\alpha \wedge \overline{A}^\mu_\beta - \sum_\rho A^\alpha_\rho \wedge \overline{A}^\beta_\rho.$$

The Chern form is

$$\begin{aligned} c_1\left(\Theta_{\mathcal{H}^{p,q}}\right) &=: \frac{\sqrt{-1}}{2\pi}\operatorname{Tr}\left(\Theta_{\mathcal{H}^{p,q}}\right) \\ &= \frac{\sqrt{-1}}{2\pi}\left(\sum_{\alpha,\mu} A^\mu_\alpha \wedge \overline{A}^\mu_\alpha\right) - \frac{\sqrt{-1}}{2\pi}\left(\sum_{\alpha,\rho} A^\alpha_\rho \wedge \overline{A}^\alpha_\rho\right). \end{aligned} \tag{II.A.4}$$

This "alternation" of signs has always been an important aspect of the curvature properties of the Hodge bundles.

Near a puncture, it is a result of Schmid [Sch] that:

(i) the forms $c_1(\Theta_{\mathcal{H}^{p,q}})$ are integrable and define closed, (1, 1) currents on the completion S;
(ii) the de Rham cohomology class represented by $c_1(\Theta_{\mathcal{H}^{p,q}})$ gives the Chern class $c_1(\mathcal{H}^{p,q}_e)$ of the canonically extended Hodge bundle.

For our purposes, taking the case $p = n$, $q = 0$ we have

$$c_1\left(\Theta_{\mathcal{H}^{n,0}}\right) = \frac{\sqrt{-1}}{2\pi}\left(\sum_{\alpha,\mu} A^\mu_\alpha \wedge \overline{A}^\mu_\alpha\right)$$

so that setting[8]

$$\delta = \int_S c_1\left(\Theta_{\mathcal{H}^{n,0}}\right)$$

we have

$$\delta \geqslant 0 \text{ with equality if and only if } \theta_0 = 0. \tag{II.A.5}$$

[8]For almost all of this paper we shall only be concerned with $\mathcal{H}^{n,0}_e$ and $\mathcal{H}^{n-1,1}_e$, and we shall set $\delta = \delta_0$ and $\lambda = \delta_1$ in terms of the general notations given above.

Next, letting $\theta_0 \leftrightarrow \{A^\mu_\alpha\}$ and $\theta_1 \leftrightarrow \{A^\rho_\mu\}$ as above, we obtain, reflecting the alternation of signs, a cancellation that gives

$$c_1\left(\Theta_{\mathcal{H}^{n,0}}\right) + c_1\left(\Theta_{\mathcal{H}^{n-1,1}}\right) = \frac{\sqrt{-1}}{2\pi}\left(\sum_{\mu,\rho} A^\rho_\mu \wedge \overline{A}^\rho_\mu\right).$$

Setting

$$\lambda = \int_S c_1\left(\Theta_{\mathcal{H}^{n-1,1}}\right)$$

we obtain

$$\delta + \lambda \geqslant 0 \text{ with equality if and only if } \theta_1 = 0. \tag{II.A.6}$$

This pattern continues, but in this work we shall only need the first two cases.

Finally we recall that given a puncture and a local coordinate s such that $s = s_0$ corresponds to the puncture, there is defined on the fibre H_{e,s_0} a polarized limiting mixed Hodge structure (LMHS). The Hodge filtration F^p is defined by the F^p_{e,s_0}; the weight filtration W_m is defined using the logarithm N of the unipotent monodromy transformation T. The following are properties of the LMHS:

$$\begin{cases} \text{(i)} & N\colon W_m \to W_{m-2}, \\ \text{(ii)} & N^k : \mathrm{Gr}^W_{n+k} \xrightarrow{\sim} \mathrm{Gr}^W_{n-k} \text{ is an isomorphism,} \\ \text{(iii)} & N(F^p) \subseteq F^{p-1}. \end{cases} \tag{II.A.7}$$

The $\mathbb{Q}$-structure and weight filtration change under a scaling

$$s \to \lambda s$$

by $\exp(\lambda N)$, from which it follows that the HS on the graded pieces and adjacent extensions of Hodge structures are well-defined. The polarizations are defined using the polarizing form, evaluated at s_0, of the given VHS.

II.B. *Algebro-geometric preliminaries*

We will consider families of projective algebraic varieties given by the fibres $X_s = f^{-1}(s)$ of a connected surjective morphism

$$f\colon X \to S \tag{II.B.1}$$

between smooth projective varieties and where the base space S is a curve.

Define the *relative dualizing sheaf* for this general setting by

$$\omega_{X/S} := \omega_X \otimes f^*(TS),$$

where ω_X is the canonical bundle. Indeed, one has $(R^q_f\mathcal{E})^\vee \cong R^{n-q}_f(\mathcal{E}^\vee \otimes \omega_{X/S})$ if $\mathcal{E}$ is locally free and n is the relative dimension.

Outside the *discriminant locus* $E = \{s_1, \ldots, s_N\}$ the fibres are smooth. Unless mentioned otherwise we will assume that the singular fibres are *normal crossing divisors* (NCD's)

$$X_{s_i} = \sum_{i,\alpha} m^i_\alpha X^i_\alpha,$$

where the X^i_α are smooth and meet transversely. We will also assume that there are no multiple fibres; i.e., $\gcd_\alpha\{m^i_\alpha\} = 1$. Usually we will assume that the X_{s_i} are reduced normal crossing divisors; i.e., the non-zero m^i_α are equal to 1. Locally we may assume that f is given by

$$s = z_1^{\mu_1} \cdots z_{n+1}^{\mu_{n+1}},$$

where the μ_i reflect the m^i_α. In the reduced normal crossing case the non-zero μ_i are equal to 1. In this situation the local monodromy is unipotent, not just quasi-unipotent.

One exception to the above is in Section V, where we shall consider the situation where the singular fibre has ordinary double points (ODP's). Then the map f is given locally by

$$s = z_1^2 + \cdots + z_{n+1}^2.$$

Unless stated otherwise we shall assume that the smooth fibres are Calabi–Yau (CY) varieties of dimension n; in this paper we shall study the cases $n = 1, 2, 3$. Denoting by X_η the generic fibre, we thus have

$$h^{n,0}(X_\eta) = 1, \quad h^{1,0}(X_\eta) = \cdots = h^{n-1,1}(X_\eta) = 0.$$

We shall make the important assumption of *relative minimality*; i.e.,

$$f^*\left(f_*(\omega_X)\right) \cong \omega_X. \tag{II.B.2}$$

This has the implication

$$c_1(X)^2 = 0, \tag{II.B.3}$$

that will be frequently used. We refer to Section V.G for further discussion of this.

We now assume that we are in the reduced normal crossing case and set

$$D = f^{-1}(E) = \bigcup_{s_i \in \Delta} X_{s_i}.$$

An important result of Steenbrink [St] is that the canonically extended Hodge bundles are given by

$$\mathcal{H}_e^{p,q} \cong R_f^q \Omega^p_{X/S}(\log D). \tag{II.B.4}$$

Here, the relative log differentials are as usual defined by

$$\Omega^p_{X/S}(\log D) = \Omega^p_X \big/ f^*(\Omega^1_S(\log E)) \wedge \Omega^{p-1}_X(\log D).$$

These sheaves are locally free (loc. cit.) and

$$\begin{cases} \Omega^p_{X/S}(\log D) = \bigwedge^p \Omega^1_{X/S}(\log D), \\ \omega_{X/S} = \Omega^n_{X/S}(\log D), \end{cases}$$

where $\omega_{X/S}$ is the relative dualizing sheaf.

We shall systematically use the following sequences, where (b) and (c) define the sheaves $\mathcal{F}_p$ and $\mathcal{G}$:

$$\begin{cases} \text{(a)} \quad 0 \to f^*(\Omega^1_S) \to \Omega^1_X \to \Omega^1_{X/S} \to 0, \\ \text{(b)} \quad 0 \to \Omega^p_{X/S} \to \Omega^p_{X/S}(\log D) \to \mathcal{F}_p \to 0, \\ \text{(c)} \quad 0 \to f^*(\mathcal{H}_e^{n,0}) \to \omega_{X/S} \to \mathcal{G} \to 0 \end{cases} \tag{II.B.5}$$

and where in the third sequence

$$\mathcal{G} = 0 \iff \text{the relative minimality assumption (II.B.2) is satisfied.}$$

Since

$$\omega_X \cong f^*(\Omega^1_S) \otimes \omega_{X/S}$$

by definition, and in the RNCD case

$$\mathcal{H}^{n,o}_e \cong (\mathcal{H}^{0,n}_e)^\vee \cong (R^n_f(\mathcal{O}_X))^\vee \cong R^0_f(\omega_{X/S}),$$

we have

$$f_*(\omega_X) \cong \Omega^1_S \otimes \mathcal{H}^{n,0}_e.$$

Then using that we are in the situation of a CY fibration, the injective sheaf map

$$f^*(f_*(\omega_X)) \to \omega_X$$

is surjective if and only if the first map of (II.B.5)(c) is surjective, thus verifying the above assertion.

From the above we infer that, assuming minimality,

$$c_1(X) = f^*(c_1(S) - c_1(\mathcal{H}^{n,0}_e)).$$

The right-hand side may be rewritten to give

$$c_1(X) = (2 - 2g - \delta) f^*([\eta]), \tag{II.B.6}$$

where $[\eta] \in H^2(S)$ is a generator. This of course implies (II.B.3).

We shall use the notations

$$\begin{cases} \tilde{c}_i = c_i(\Omega^1_{X/S}), \\ \hat{c}_i = c_i(\Omega^1_{X/S}(\log D)). \end{cases}$$

Then from (II.B.5) and the vanishing of $c_1(\mathcal{F})$ as discussed below, we have

$$\tilde{c}_1 = \hat{c}_1 = \delta f^*[\eta], \tag{II.B.7}$$

which gives also that

$$\tilde{c}_1^2 = \hat{c}_1^2 = 0.$$

We set

$$\mathcal{F} = \mathcal{F}_1$$

and note that in general

$$\operatorname{supp} \mathcal{F}_p \subseteq D_{\text{sing}}.$$

This implies that the Chern classes

$$c_i(\mathcal{F}_p) = 0 \quad \text{for } i = 0,\, 1.$$

A pair of formulas that will be used extensively arises by putting the sequences (II.B.5) in a diagram and giving a resolution of the sheaf $\mathcal{F}$. For this it is notationally convenient to denote a singular fibre simply by $Y = \bigcup Y_\alpha$. Then we have an exact diagram

$$\left\{\begin{tikzcd} 0 \arrow[r] & f^*(\Omega^1_S) \arrow[r] \arrow[d] & \Omega^1_X \arrow[r] \arrow[d] & \Omega^1_{X/S} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & f^*(\Omega^1_S(\log E)) \arrow[r] \arrow[d] & \Omega^1_X(\log D) \arrow[r] \arrow[d] & \Omega^1_{X/S}(\log D) \arrow[r] \arrow[d] & 0 \\ & \mathcal{O}_Y \arrow[r] & \bigoplus_\alpha \mathcal{O}_{Y_\alpha} \arrow[r] & \mathcal{F} & \end{tikzcd}\right.$$

and a resolution (here for $n = 1, 2, 3$)

$$0 \to \mathcal{F} \xrightarrow{\rho} \bigoplus_{\alpha<\beta} \mathcal{O}_{Y_{\alpha\beta}} \xrightarrow{\sigma} \bigoplus_{\alpha<\beta<\gamma} \mathcal{O}_{Y_{\alpha\beta\gamma}} \xrightarrow{\tau} \bigoplus_{\alpha<\beta<\gamma<\lambda} \mathcal{O}_{Y_{\alpha\beta\gamma\lambda}} \to 0. \qquad \text{(II.B.8)}$$

Here, the first map ρ is constructed from the composition of the *restriction* mapping

$$\Omega^1_{X/S}(\log D) \to \bigoplus_{\alpha<\beta} \Omega^1_{Y_\alpha}(\log Y_{\alpha\beta}),$$

together with the *residue* mapping

$$\bigoplus_{\alpha<\beta} \Omega^1_{Y_\alpha}(\log Y_{\alpha\beta}) \to \bigoplus_{\alpha<\beta} \mathcal{O}_{Y_{\alpha\beta}},$$

with attention to signs. The next mapping is given by

$$\sigma\Bigg(\bigoplus_{\alpha<\beta} F_{\alpha\beta}\Bigg) = \sum_{\alpha<\beta<\gamma} F_{\alpha\beta}\big|_{Y_{\alpha\beta\gamma}},$$

where $F_{\alpha\beta} \in \mathcal{O}_{Y_{\alpha\beta}}$. The third mapping τ is similarly defined. The above maps depend upon a choice of ordering of the index set for the components of D.

We shall now discuss how one proves the exactness of (II.B.8). In order to not have the notation obscure the ideas, we shall restrict to the situation where

$$Y_{s_0} = Y_1 \cup Y_2 \cup Y_3,$$

and we assume there is one triple point and set

$$p = Y_{123} = Y_1 \cap Y_2 \cap Y_3.$$

Locally we may choose coordinates x, y, z on X and s on S so that f is given by

$$xyz = s,$$

and

$$Y_1 = \{x = 0\}, \quad Y_2 = \{y = 0\}, \quad Y_3 = \{z = 0\}.[9]$$

Then $\varphi \in \Omega^1_{X/S}(\log D)$ is given as

$$\varphi = f(x,\,y,\,z)\frac{dx}{x} + g(x,\,y,\,z)\frac{dy}{y} + h(x,\,y,\,z)\frac{dz}{z},$$

where

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0. \tag{II.B.9}$$

To define the part

$$\Omega^1_{X/S}(\log D) \xrightarrow{R_3} \Omega^1_{Y_3}(Y_{31}) \oplus \Omega^1_{Y_3}(Y_{32})$$

of the restriction map, we use (II.B.9) to write

$$\varphi = (f-h)\frac{dx}{x} + (g-h)\frac{dy}{y},$$

and then

$$R_3(\varphi) =: (f(x,\,y,\,0) - h(x,\,y,\,0))\,\frac{dx}{x} \oplus (g(x,\,y,\,0) - h(x,\,y,\,0))\,\frac{dy}{y}.$$

We then use Poincaré residues to map

$$R_3(\varphi) \to \mathrm{res}_{Y_{31}}(R_3(\varphi)) \oplus \mathrm{res}_{Y_{32}}(R_3(\varphi))$$

and then the composition $\rho_3(\varphi)$ is given by

$$\rho_3(\varphi) = (f(0,\,y,\,0) - h(0,\,y,\,0)) \oplus (g(x,\,0,\,0) - h(x,\,0,\,0))\,.$$

Thus

$$\rho_3(\varphi) \in \mathcal{O}_{Y_{31}} \oplus \mathcal{O}_{Y_{32}}.$$

Similarly,

$$R_2(\varphi) = (f(x,\,0,\,z) - g(x,\,0,\,z))\frac{dx}{x} \oplus (h(x,\,0,\,z) - g(x,\,0,\,z))\frac{dz}{z}$$

and

$$R_2(\varphi) \to \rho_2(\varphi) = \mathrm{res}_{Y_{21}}(R_2(\varphi)) \oplus \mathrm{res}_{Y_{32}}(R_2(\varphi)),$$

where

$$\rho_2(\varphi) = (f(0,\,0,\,z) - g(0,\,0,\,z)) \oplus (h(x,\,0,\,0) - g(x,\,0,\,0)) \in \mathcal{O}_{Y_{21}} \oplus \mathcal{O}_{Y_{32}}.$$

Note that

$$\rho_3(\varphi)\big|_{Y_{32}} = -\rho_2(\varphi)\big|_{Y_{32}}.$$

It is clear that $\sigma(\rho(\varphi)) = 0$ and that

$$\mathcal{F} \xrightarrow{\rho} \bigoplus_{\alpha<\beta} \mathcal{O}_{Y_{\alpha\beta}} \xrightarrow{\sigma} \mathcal{O}_p \to 0$$

is exact. Thus we have to show:

$$\rho(\varphi) = 0 \implies \varphi \in \Omega^1_{Y/S}.$$

[9]With apologies to the reader, we will use the ordering $3 < 2 < 1$ as it makes the end result appear more symmetric.

We have

$$\rho(\varphi)=0 \iff \begin{cases} f(x,\,y,\,0)-h(x,\,y,\,0)=xA_3(x,\,y),\\ g(x,\,y,\,0)-h(x,\,y,\,0)=yB_3(x,\,y),\\ f(x,\,0,\,z)-g(x,\,0,\,z)=xC_2(x,\,z),\\ h(0,\,y,\,z)-f(0,\,y,\,z)=-zA_1(y,\,z),\\ g(0,\,y,\,z)-f(0,\,y,\,z)=-y(C_1(y,\,z),\\ h(x,\,0,\,z)-g(x,\,0,\,z)=-zB_2(x,\,z). \end{cases}$$

Thus

$$\begin{aligned} h(x,\,y,\,z)&=f(x,\,y,\,z)-xA_3(x,\,y)+zD(x,\,y,\,z),\\ h(x,\,y,\,z)&=g(x,\,y,\,z)-yB_3(x,\,y)+zE(x,\,y,\,z), \end{aligned}$$

and modulo $\Omega^1_{Y/S}$

$$\begin{cases} \text{(i)} \quad \varphi=f\dfrac{dx}{x}+g\dfrac{dy}{y}+(f-xA_3)\dfrac{dz}{z},\\ \text{(ii)} \quad \varphi=f\dfrac{dx}{x}+g\dfrac{dy}{y}+(g-yB_3)\dfrac{dz}{z}. \end{cases}$$

The case where there are quadruple points is a direct extension of this argument. □

For $\mathcal{E}$ a coherent sheaf with Chern classes c_i, for frequent latter reference we record the formulas

$$\begin{cases} \mathrm{ch}_0(\mathcal{E})=\operatorname{rank}\mathcal{E}_\eta,\\ \mathrm{ch}_1(\mathcal{E})=c_1,\\ \mathrm{ch}_2(\mathcal{E})=\dfrac{c_1^2-2c_2}{2,}\\ \mathrm{ch}_3(\mathcal{E})=\dfrac{c_1^3-3c_1c_2+3c_3}{6},\\ \mathrm{ch}_4(\mathcal{E})=\dfrac{c_1^4-4c_1^2c_2+2c_2^2+4c_1c_3-4c_4}{24}, \end{cases} \tag{II.B.10}$$

where in the first equation $\operatorname{rank}\mathcal{E}_\eta$ is the rank of $\mathcal{E}$ at a generic point. For the Todd genus we have

$$\begin{cases} \mathrm{Td}_1(\mathcal{E})=c_1/2,\\ \mathrm{Td}_2(\mathcal{E})=\dfrac{c_1^2+c_2}{12},\\ \mathrm{Td}_3(\mathcal{E})=\dfrac{c_1c_2}{24},\\ \mathrm{Td}_4(\mathcal{E})=\dfrac{-c_1^4+3c_2^2+4c_1^2c_2+c_1c_3-c_4}{720} \end{cases} \tag{II.B.11}$$

Formal properties such as

$$\begin{cases} \mathrm{Td}(\mathcal{E}+\mathcal{E}')=\mathrm{Td}\;\mathcal{E}\cdot\mathrm{Td}\;\mathcal{E}',\\ \mathrm{ch}(\mathcal{E}+\mathcal{E}')=\mathrm{ch}(\mathcal{E})+\mathrm{ch}(\mathcal{E}') \end{cases}$$

will be used without further mention. We shall set

$$\widetilde{\mathrm{Td}}(\mathcal{E}) = \text{Todd polynomial for } \mathcal{E} \text{ with } c_i \text{ replaced by } (-1)^i c_i.$$

In case $\mathcal{E}$ is locally free with dual $\check{\mathcal{E}}$

$$\widetilde{\mathrm{Td}}(\mathcal{E}) = \mathrm{Td}(\check{\mathcal{E}}). \tag{II.B.12}$$

We also have

$$\widetilde{\mathrm{Td}}(\mathcal{E} + \mathcal{E}') = \widetilde{\mathrm{Td}}(\mathcal{E}) \cdot \widetilde{\mathrm{Td}}(\mathcal{E}'). \tag{II.B.13}$$

For $\mathcal{E}$ a coherent sheaf on X, the GRR is

$$\mathrm{ch}(f_!(\mathcal{E})) \cdot \mathrm{Td}\ S = f_*(\mathrm{ch}(\mathcal{E}) \cdot \mathrm{Td}\ X). \tag{II.B.14}$$

We may then rewrite (II.B.14) as

$$\begin{aligned} \mathrm{ch}(f_!(\mathcal{E}))\widetilde{\mathrm{Td}}(\Omega^1_S) &= f_*(\mathrm{ch}(\mathcal{E}) \cdot \big(f^* \Big(\widetilde{\mathrm{Td}}(\Omega^1_S)\Big) \cdot \widetilde{\mathrm{Td}}(\Omega^1_{X/S})\big) \\ &= f_*\big(\mathrm{ch}(\mathcal{E}) \cdot \widetilde{\mathrm{Td}}(\Omega^1_{X/S})\big) \cdot \widetilde{\mathrm{Td}}(\Omega^1_S), \end{aligned}$$

where the second step uses the push-pull formula. Cancelling the $\widetilde{\mathrm{Td}}(\Omega^1_S)$ from both sides leads to the version of GRR that we shall use in Section III:

$$\mathrm{ch}(f_!(\mathcal{E})) = f_*\big(\mathrm{ch}(\mathcal{E}) \cdot \widetilde{\mathrm{Td}}(\Omega^1_{X/S})\big). \tag{II.B.15}$$

In case all the fibres of (II.B.1) are smooth we denote by

$$\xi = TX/S$$

the tangent bundle along the fibres of. Then, in this case

$$\Omega^1_{X/S} \cong \check{\xi}$$

and (II.B.14) is

$$\mathrm{ch}(f_!(\mathcal{E})) = f_*(\mathrm{ch}(\mathcal{E}) \cdot \mathrm{Td}\ \xi). \tag{II.B.16}$$

Referring to the sequences (II.B.6), we shall set

$$\begin{cases} \tilde{c}_i = (-1)^i c_i\big(\Omega^1_{X/S}\big), \\ \hat{c}_i = (-1)^i c_i\big(\Omega^1_{X/S}(\log D)\big). \end{cases} \tag{II.B.17}$$

Then

$$\begin{cases} \widetilde{\mathrm{Td}}(\Omega^1_{X/S}) = \mathrm{Td}(\tilde{c}_1, \tilde{c}_2, \tilde{c}_3, \dots), \\ \widetilde{\mathrm{Td}}(\Omega^1_{X/S}(\log D)) = \mathrm{Td}(\hat{c}_1, \hat{c}_2, \hat{c}_3, \dots). \end{cases}$$

In case the fibres of (II.B.1) are all smooth

$$\mathrm{Td}\,\xi = \mathrm{Td}(\tilde{c}_1, \tilde{c}_2, \tilde{c}_3 \dots) = \mathrm{Td}(\hat{c}_1, \hat{c}_2, \hat{c}_3, \dots). \tag{II.B.18}$$

Finally we remark that

(II.B.19) *A family of Calabi–Yau varieties is isotrivial if and only if the equivalent conditions*

$$\begin{cases} \theta_0 = 0, \\ \delta = 0 \end{cases}$$

are satisfied.

This follows from (II.A.5), and the well known fact that for $s \in S^*$ the differential of the VHS is given by the usual Kodaira–Spencer map

$$T_s S \to \operatorname{Hom}\left(H^{n,0}(X_s),\, H^{n-1,1}(X_s)\right) \cong H^1(\Theta_{X_s}),$$

where the isomorphism on the right depends on a a choice of trivialization

$$\omega_{X_s} \cong \mathcal{O}_{X_s}.$$

III. Elliptic Surfaces

In part to set a context for what follows, and in part because it is such a beautiful story, we want to recall briefly some elements of the theory of minimal elliptic fibrations, due to Kodaira [Ko] and nicely recounted in [BHPV, Sect. V.7–13]. In this section we shall not assume NCD's (as we will treat all of Kodaira's fiber types). We begin with two examples.

Example. The Fermat pencil

$$s(x_0^3 + x_1^3 + x_2^3) - x_0 x_1 x_2 = 0$$

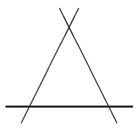
gives an elliptic surface

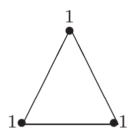
$$f\colon X \to \mathbb{P}^1;$$

using s as a coordinate on $\mathbb{P}^1$, the discriminant locus is

$$\Delta = \left\{0,\, \frac{1}{3},\, \frac{\zeta}{3},\, \frac{\zeta^2}{3}\right\}, \qquad \zeta = e^{2\pi\sqrt{-1}/3}.$$

Each singular fibre is of type I_3 in Kodaira's classification; the picture is

$X_{s_i} =$ 

dual graph 

(III.1)

Equivalently, letting as usual

$$\Gamma(N) = \ker\left(\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})\right),$$

the semi-direct product $\Gamma(3) \ltimes \mathbb{Z}^2$ acts on $\mathcal{H} \times \mathbb{C}$ by

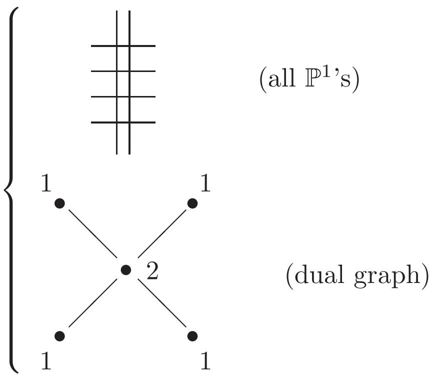
$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix},\, (m_1,\, m_2)\right)(\tau,\, z) = \left(\frac{a\tau + b}{c\tau + d},\, \frac{z + m_1 + m_2\tau}{cz + d}\right).$$

This gives a family of elliptic curves parametrized by $\mathcal{H}/\Gamma(3)$, and adding curves of type (III.1) over the four cusps gives an alternate description of the Fermat pencil.

Example. Fixing $\tau \in \mathcal{H}$ and setting $E_\tau = \mathbb{C}/\mathbb{Z}\{1, \tau\}$, we quotient $\mathbb{P}^1 \times E_\tau$ by the involution

$$(-1)(s, z) = (-s, -z).$$

This gives an isotrivial family with Kodaira type I_0^* singular fibres over $s = 0, \infty$, which are described by



(III.2)

To an elliptic curve E_τ with Weierstrass equation

$$y^2 = 4x^3 - g_2 x - g_3,$$

where g_2 and g_3 are the well known expressions in terms of τ, one associates the j-function

$$j(\tau) = \frac{1728 g_2^3}{g_2^3 - 27 g_2^2}.$$

Then $j(\tau)$ is invariant under $\mathrm{SL}_2(\mathbb{Z})$, and setting $\mathcal{H}^* = \mathcal{H} \cup \mathbb{Q}$ one has

$$j\colon \big(\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}^*\big) \xrightarrow{\sim} \mathbb{P}^1,$$

where $j^{-1}(\infty)$ corresponds to the cusp in the fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$. Their existence of j immediately implies that

(III.3) *Any non-isotrivial family of elliptic curves parametrized by a complete curve must have a singular fibre.*

Indeed, if the family is $f\colon X \to S$, then the composition j_S in

$$S \to (\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}^*) \xrightarrow{j} \mathbb{P}^1$$

gives a non-constant meromorphic function on S over whose poles must lie singular curves. $\square$

We set

$$\tilde{\delta} = \frac{1}{12}\deg j_S,$$

and solely for this section,

$$\delta := \deg(f_*(\omega_{X/S})).$$

In the case when the fibres of $f\colon X \to S$ are reduced normal crossings, i.e., of type I_m in Kodaira's classification, $\delta = \deg(\mathcal{H}_e^{1,0})$, and we shall see that

$$\delta = \tilde{\delta}. \tag{III.4}$$

Thus, although while one does not (yet?) have an extension of the j-function to other situations,[10] in the reduced normal crossing case we do have an extension of its degree. This simple observation will play a critical role in this work.

For later reference we shall reproduce Kodaira's table. Recall that the minimality of X implies that X contains no -1 curves in its fibres and that each irreducible component X_α^i of a properly singular fibre has self-intersection $(X_\alpha^i)^2 = -2$. The notations δ_0 and $\tilde{\delta}_0$ refer to the contribution of the singular fibre to δ and $\tilde{\delta}$ respectively, while κ is the order of monodromy. We have not labeled multiplicities of components in the dual graphs, though all individual-point intersections of components have multiplicity 1 (except type III).

We shall now apply the Grothendieck–Riemann–Roch formula for the sheaf $\mathcal{O}_X$, in the form (I.B.14). The formula is

$$\mathrm{ch}(f_!\,(\mathcal{O}_X)) = f_*(\tilde{\mathrm{T}}\mathrm{d}(\tilde{c}_1,\, \tilde{c}_2)).$$

For the left-hand side we have

$$\begin{aligned} f_!(\mathcal{O}_X) &= 1 - R_f^1 \mathcal{O}_X \\ &= 1 - (R_f^0 \omega_{X/S})^{\vee} \\ &= 1 + f_* \omega_{X/S}, \end{aligned}$$

by the property of the dualizing sheaf. Thus

$$\int_S \mathrm{ch}_1\,(f_!(\mathcal{O}_X)) = \delta.$$

For the right-hand side,

$$\begin{cases} c_1(X) = (2 - 2g - \delta) f^*[\eta], \\ \quad\ \tilde{c}_1 = \delta f^*[\eta] \end{cases}$$

from (II.B.6) and (II.B.7), which still hold in the present setting (with δ as just defined). Using (II.B.5)(a) it follows that

$$\begin{cases} \tilde{c}_1^2 = 0, \\ \tilde{c}_2 = c_2(X); \end{cases}$$

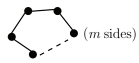
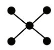
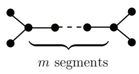
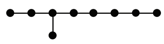
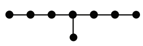
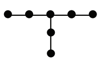
and so

$$\int_S f_*\big(\tilde{\mathrm{T}}\mathrm{d}_2(\tilde{c}_1,\, \tilde{c}_2)\big) = \int_X \frac{c_2(X)}{12} = \frac{\chi(X)}{12} \implies 12\delta = \chi(X) = \sum_i \chi(X_i),$$

where X_i are the singular fibres.

[10] One exception is the case of K3's of Picard rank 19 (and to some extent, rank 18); see [Dor] (resp. [CD].)

Table 1.

Fiber type	monodromy	κ	δ_0	$\tilde{\delta}_0$	dual graph
I_0	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	1	0	0	
I_1	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	∞	1	1	
I_m	$\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$	∞	m	m	
I_0^*	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	2	6	0	
I_m^*	$\begin{pmatrix} -1 & -m \\ 0 & -1 \end{pmatrix}$	∞	$m+6$	m	
II^*	$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$	6	10	0	
III^*	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	4	9	0	
IV^*	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$	3	8	0	
II	$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$	6	2	0	
III	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	4	3	0	
IV	$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$	3	4	0	

It is elementary to check that their Euler characteristics are as listed in the 4$^{\text{th}}$ column of the table: except for X_i of type I_0 (smooth) or IV,

$$\chi(X_i) = \sum_{\alpha} \chi(X^i_\alpha) - \sum_{\alpha<\beta} \chi(X^i_{\alpha\beta}) = \underbrace{2(\#\,\text{vertices}) - (\#\,\text{edges})}_{\text{in the dual graph}}. \tag{III.5}$$

No singular fibre contributes zero (in contrast to type I K3 degenerations in Section IV below). In particular, we note that

$$\tilde{\delta} \neq 0 \underset{\delta \geqslant \delta_0}{\Longrightarrow} \delta \neq 0 \implies \text{presence of singular fibres,}$$

which gives another proof of (III.3).

Next, we turn to the semistable case, which will form the basis of our higher-dimensional generalizations. From Kodaira's table we see that (III.4) holds, since all singular fibres are of type I_m. Thus, in the case where the singular fibres are reduced normal crossing divisors, *the degree of the Hodge bundle may be used in place of the j-function*, which suggests the importance (in general semistable families) of the former as a measure of the non-isotriviality of families of CY's.

If the singular fibre X_{s_i} is of type I_{m_i}, then the dual graph has m_i edges and m_i vertices, so that by (III.5)

$$X_{s_i} \text{ contributes } m_i \text{ to } \chi(X).$$

This leads to Kodaira's relation

$$12\delta = \sum_i m_i, \tag{III.6}$$

which can be interpreted in two ways:

(i) *The contribution to* $\deg \mathcal{H}_e^{1,0}$ *of the singular fibres is expressed in terms of the intersection numbers of the components of the singular fibres*;

(ii) *The contribution to* $\deg \mathcal{H}_e^{1,0}$ *of the singular fibres is expressed in terms of the monodromies.*

A moment's reflection show that (i) and (ii) remain true for a family of curves of any genus $g \geqslant 1$, assuming of course semi-stable reduction and minimality.

We shall see below that (i) remains true for the quantity $(24+\chi(X_\eta))\delta+12(\delta+\lambda)$ for $n = 3$.

Finally, we observe that the relations arising from the GRR place global constraints on the combinations of singular fibres that may occur, since the sum of the contributions of the singular fibres must be divisible by 12. Thus

$$\begin{cases} I_1,\ I_2,\ I_3,\ I_4^* & \text{does not occur,} \\ I_1,\ I_2,\ I_3,\ I_6 & \text{does occur.} \end{cases}$$

The first example at the beginning of this section is of type I_3, I_3, I_3, I_3.

IV. Semi-stable Families of K3 Surfaces

We first observe that the analogue of (III.3) does not hold.

Example. Let $|X_t|$ be a general pencil of quartic surfaces in $\mathbb{P}^3$. The singular X_{t_i} each have one ordinary double point p_i. The Picard–Lefschetz transformation T_i around t_i is finite of order 2, i.e., $T_i^2 = I$. Denoting by $\Gamma \subset \Gamma_\mathbb{Z}$ the global monodromy group the period mapping extends across the t_i to give a non-constant map

$$\tau\colon \mathbb{P}^1 \to \Gamma\backslash D,$$

where D is the relevant period domain. Thus:

> *The moduli space* $\Gamma_\mathbb{Z}\backslash D$ *for polarized Hodge structures with* $h^{2,0} = 1$, $h^{1,1} = 19$ *contains complete curves.*

Following Atiyah [At], one may construct a family of smooth K3's $X \to S$, where $S \to \mathbb{P}^1$ is the 2-sheeted covering branched at the t_i. The construction is local over a disc Δ around t_i. Let

$$Y \subset \Delta \times \mathbb{P}^3$$

be the smooth surface given by the corresponding family of K3's parametrized by Δ. Let $\Delta' \to \Delta$ be the standard 2-sheeted covering and

$$Y' = \Delta' \times_\Delta Y$$

the fibre product. Then Y' has an ordinary double point corresponding to $\{0\} \times p_i$ and the blowup $\hat{Y} = \mathrm{Bl}_{p_i} Y'$ is a smooth threefold. The fibre $\hat{Y}_0$ over the origin of $\hat{Y} \to \Delta'$ is a smooth surface with two -1 curves lying over p_i. Contracting one of those curves gives a new smooth threefold which fibres smoothly over Δ' with K3 surfaces as fibres.

Equivalently we may blow $\hat{Y}$ down along one of the rulings of the quadric $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ given by the proper transform of p_i. When this is done we obtain the same family.

Turning to use of Riemann–Roch theorems, especially GRR, there seems to be an even-odd phenomenon whereby they fail to yield direct information on the degrees of Hodge bundles for families of CY's of even dimension, but do so when the fibres are of odd dimension. For example, for a family $f\colon X \to S$ where all fibres are smooth K3's, the GRR using $\mathcal{O}_X$ and $\Omega^1_{X/S}$ only confirms that $\chi(X_s) = 24$ and $h^{1,1}(X_s) = 20$. When there are singular fibres one does not find that they give a contribution to $\delta = \deg \mathcal{H}_e^{2,0}$. This is not unexpected since, as noted above, there are non-isotrivial families without singular fibres.[11] What Riemann–Roch theorems do give is expressions for the contribution of singular fibre,[12] denoted for simplicity of notation by $Y = \bigcup Y_\alpha$, to the discrepancy

$$\Delta\chi = \chi(X) - \chi(S)\chi(X_\eta)$$

from multiplicativity of the Euler characteristic. In general this is given by

$$\sum_\alpha \chi(Y_\alpha) - \sum_{\alpha<\beta} \chi(Y_{\alpha\beta}) + \sum_{\alpha<\beta<\gamma} \chi(Y_{\alpha\beta\gamma}) - 24,$$

but under our assumption of relative minimality GRR yields a nontrivial simplification. We set

$$\begin{cases} f = \text{number of components } Y_\alpha \text{ of } \tilde{Y} = Y^{[0]}, \\ \varepsilon = \text{number of components } Y_{\alpha\beta} = Y_\alpha \cap Y_\beta \text{ of } Y^{[1]}, \\ v = \text{number of components } Y_{\alpha\beta\gamma} = Y_\alpha \cap Y_\beta \cap Y_\gamma \text{ of } Y^{[2]}. \end{cases}$$

We also write $\{C_\lambda\}_{\lambda=1}^{\varepsilon}$ for the $\{Y_{\alpha\beta}\}$. Thus,

> v, ε, f *are the number of respective faces, edges, vertices in the dual graph.*

[11]From the different perspective of "Arakelov equalities" in section VI, we will find expressions in terms of their LMHS of the contributions of singular fibres to the various $\deg \mathcal{H}_e^{p,q}$'s.

[12]We continue to assume Y is a reduced NCD.

Denoting by $g_\lambda = g(C_\lambda)$ the genus of the smooth irreducible curve C_λ, we shall show that

$$\Delta\chi = \varepsilon - v - \sum_\lambda g_\lambda.\text{[13]} \tag{IV.1}$$

Following the proof we will illustrate the formula in the two cases of semi-stable degenerate fibres that Kulikov found in his work [Ku] on minimal semi-stable degenerations of K3's (cf. also [P] and [PP]).

Proof of (IV.1). We have

$$\chi(X) = \int_X c_3(X).$$

Using the notations in (II.B.5) and the additivity of the Chern character

$$\mathrm{ch}(\Omega^1_X) = f^*(\mathrm{ch}(\Omega^1_S)) + \mathrm{ch}(\Omega^1_{X/S}(\log Y)) - \mathrm{ch}(\mathcal{F}).$$

The left-hand side expands, using our assumption of minimality and (II.B.3), to

$$3 + c_1(\Omega^1_X) - c_2(\Omega^1_X) + \left(-\frac{1}{2}c_1(\Omega^1_X)c_2(\Omega^1_X) + \frac{1}{2}c_3(\Omega^1_X)\right).$$

Expanding the right-hand side using (II.B.8) and comparing terms gives, after simplification, the equations

$$\begin{cases} c_1(\Omega^1_X) = (-\chi(S)+\delta)[X_\eta], \\ c_1(\Omega^2_X) = c_2(\Omega^1_{X/S}(\log Y)) - \displaystyle\sum_{\alpha<\beta}[Y_{\alpha\beta}], \\ c_3(\Omega^1_X) = -\chi(S)\chi(X_\eta)[p] - \mathrm{ch}_3(\mathcal{F}). \end{cases}$$

From this we conclude that

$$\Delta\chi = \int_X \mathrm{ch}_3(\mathcal{F}). \tag{IV.2}$$

Next we use the resolution (II.B.8), which in this case reduces to

$$0 \to \mathcal{F} \to \bigoplus_{\alpha<\beta} \mathcal{O}_{Y_{\alpha\beta}} \to \bigoplus_{\alpha<\beta<\gamma} \mathcal{O}_{Y_{\alpha\beta\gamma}} \to 0$$

to obtain

$$\mathrm{ch}(\mathcal{F}) = \sum_{\alpha<\beta} \mathrm{ch}(\mathcal{O}_{Y_{\alpha\beta}}) - \sum_{\alpha<\beta<\gamma} \mathrm{ch}(\mathcal{O}_{Y_{\alpha\beta\gamma}}).$$

The right-hand side is

$$\underbrace{\sum_{\alpha<\beta}[Y_{\alpha\beta}]}_{\mathrm{ch}_2(\mathcal{F})} + \underbrace{\frac{1}{2}\sum_{\alpha<\beta} c_3(\mathcal{O}_{Y_{\alpha\beta}}) - \sum_{\alpha<\beta<\gamma}[Y_{\alpha\beta\gamma}]}_{\mathrm{ch}_3(\mathcal{F})}.$$

Using the GRR applied to the inclusion maps in the situation

$$U \subset W \subset X$$

[13] The corresponding formula for an elliptic surface with a fibre of type I_m is simply (cf. (III.5)).

$$\Delta\chi = m.$$

of a smooth curve U in a smooth surface W in X, together with the adjunction formula, we may evaluate $\mathrm{ch}_3(\mathcal{F})$ to find

$$\int_X \mathrm{ch}_3(\mathcal{F}) = \varepsilon - \sum_\lambda g_\lambda - v.$$

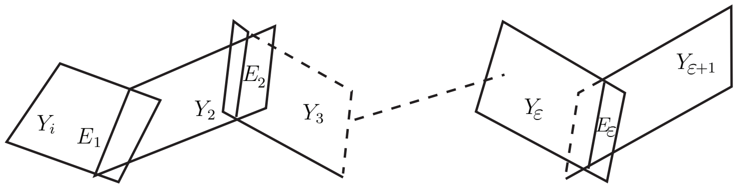
This gives

$$\Delta\chi = -\int_X c_3(\Omega^1_X) - \chi(S)\chi(X_\eta) = \varepsilon - v - \sum_\lambda g_\lambda$$

as claimed. □

Example. There are two types of Kulikov semi-stable degenerate fibres

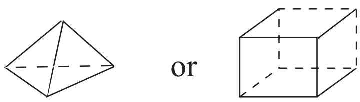
(I)



Here $E_1, \ldots, E_\varepsilon$ are smooth elliptic curves ($g_\lambda = 1$), $Y_1, \ldots, Y_{\varepsilon+1}$ are rational surfaces and $Y_2, \ldots, Y_\varepsilon$ are ruled elliptic surfaces, from which it follows that $v = 0$, $\varepsilon = \sum_{\lambda=1}^{\varepsilon} g_\lambda$, and

Type I degenerations make no contribution to $\Delta\chi$.

(II) Then $Y = \bigcup_\alpha Y_\alpha$, where all the Y_α are rational surfaces and all $Y_{\alpha\beta}$ are rational curves ($g_\lambda = 0$). The "schematic" polytope of the configuration, e.g.[14]



is dual to a triangulation of S^2. Thus $f - \varepsilon + v = \chi(S^2) = 2$ and so

Type II degenerations contribute $f - 2$ to $\Delta\chi$.

For a minimal family $f\colon X \to S$ of K3 surfaces, possibly having singular fibres, the degree δ of $\mathcal{H}^{2,0}_e = f_*(\omega_{X/S})$ enters algebro-geometrically through the ordinary (Hirzebruch) Riemann–Roch formula for an ample line bundle $L \to X$. Setting $\omega = c_1(L)$, using (II.B.5) the Hilbert polynomial for L has as coefficients

$$\chi(X, L^k) = \frac{\omega^3}{6}k^3 + \frac{\chi(S) - \delta}{2}k^2 + \text{lower order terms in } k.$$

Thus, for large k

The next to highest order term in $h^0(X, L^k)$ decreases in proportion to δ.

This is a purely algebro-geometric statement for which we do not know of an argument that does not use Hodge theory.

[14]There are literally thousands of possibilities.

V. Calabi–Yau Threefold Fibrations

V.A. *Statement of the main result*

We first establish/recall some notations. Let

$$f_0\colon X_0 \to S$$

be a relatively minimal (cf. subsection G to this section) family of CY's for which X_0 and S are smooth.[15] We will allow two types of singular fibres.

(i) semi-stable fibres $X_{s_i} = \bigcup_\alpha X^i_\alpha$;

(ii) fibres Y_j with ordinary double points (ODP's) $\{p^j_\beta\}$ and no other singularities.[16]

In case (ii) we denote by Δ_j the number of ODP's on Y_j. We also set

$$D_0 = \sum_{i,\alpha} X^i_\alpha.$$

We recall from Section II.B our notations

$$\begin{aligned}\delta &= \deg \mathcal{H}^{3,0}_e = -\deg R^3 f_* \mathcal{O}_{X_0},\\ \lambda &= \deg \mathcal{H}^{2,1}_e = -\deg R^2 f_* \Omega^1_{X_0/S}(\log D_0),\\ \Delta\chi_0 &= \chi(X_0) - \chi(X_\eta)\chi(S),\end{aligned}$$

and in this section shall use the additional notations

$$\chi^i_2 = \sum_{\alpha<\beta} \chi(X^i_{\alpha\beta}),$$

$$g^i_{\alpha\beta\gamma} = \text{genus}(X^i_{\alpha\beta\gamma}) \text{ for } \alpha<\beta<\gamma$$

(*topological data*; here the subscript "2" refers to codimension 2 strata);

$$I^i_2 = \sum_{\alpha<\beta} \deg[(X^i_{\alpha\beta})^2] - \deg\Big[\Big(\sum_{\alpha<\beta} X^i_{\alpha\beta}\Big)^2\Big]$$

(*intersection data*);

$$\begin{aligned}\varepsilon_i &= \text{number of edges } X^i_{\alpha\beta\gamma}\ (\alpha<\beta<\gamma);\\ v_i &= \text{number of vertices } X^i_{\alpha\beta\gamma\delta}\ (\alpha<\beta<\gamma<\delta).\end{aligned}$$

(V.A.1) **Theorem.** *With the above notations we have*

(a) $$\Delta\chi_0 = \sum_j \Delta_j + \sum_i \left\{\chi^i_2 + \frac{1}{2} I^i_2 + 6\left(\sum_{\alpha<\beta<\gamma} g^i_{\alpha\beta\gamma}\right) - 6\epsilon_i + 6v_i\right\}$$

(b) $$(24 + \chi(X_\eta)) + 12(\lambda+\delta)\delta = 2\sum_j \Delta_j + \sum_i \left\{\chi^i_2 - 3I^i_2\right\}.$$

---

[15] We will use the modification described below to replace X_0 by our usual notation X.

[16] We shall use the notation Y_i instead of $X_i = X_{s_i}$ to distinguish the ODP singular fibres from the NCD ones. We note that for the desingularization we shall use, we will have

$$\sum_j h^{3,0}(\tilde{Y}_j) = h^{3,0}(X_\eta).$$

Remark. (i) If there are only ODP's, (a) and (b) reduce to

$$\Delta\chi_0 = \sum_j \Delta_j,$$

$$(24 + \chi(X_\eta)) + 12(\lambda+\delta)\delta = 2\sum_j \Delta_j.$$

(ii) In (V.A.1)(a) the terms in braces can be replaced by

$$\sum_\alpha \chi(X^i_\alpha) - \sum_{\alpha<\beta} \chi(X^i_{\alpha\beta}) + \sum_{\alpha<\beta<\gamma} \chi(X^i_{\alpha\beta\gamma}) \\ - \sum_{\alpha<\beta<\gamma<\delta} \chi(X^i_{\alpha\beta\gamma\delta}) - \chi(X_\eta)\cdot\{\text{number of semistable singular fibres}\}.$$

This is a *weaker* result (which does not use the minimality) but may in some cases be easier to actually compute with.

Corollary. *If $f\colon X_0 \to S$ has no singular fibres and is not isotrivial, we have*

$$h^{2,1} > h^{1,1} + 12. \tag{V.A.2}$$

Proof. If there are no singular fibres, (b) above gives

$$\lambda + \delta = \frac{-\delta(\chi(X_\eta)+24)}{12}.$$

Non-isotriviality gives $\delta > 0$, and moreover from (II.A.5) and (II.A.6) both δ and $\lambda+\delta$ have the same sign. Thus

$$\chi(X_\eta) + 24 < 0.$$

Since

$$\chi(X_\eta) = 2(h^{1,1} - h^{2,1})$$

we obtain (V.A.2). □

A family of CY threefolds is said to be of *mirror quintic type* if $h^{2,1} = 1$ (cf. [GGK2]). Denote the right-hand side of (V.A.1)(b) by

$$\sigma(X_0) =: \text{measure of the singularities in the singular fibres.}$$

From (b) we have for a family of mirror quintic type

$$\sigma(X_0) = 2(h^{1,1}+11)\delta + 12(\lambda+\delta).$$

Thus for such families we have that *the measure of the singularities of the singular fibres increases both with δ and $\lambda+\delta$.* This may be viewed as an analogue of Kodaira's result III.6 for elliptic surfaces.

V.B. *Resolving ordinary double points*

To do this we blow up p^j_β in the usual way to obtain

$$f\colon X \to S,$$

which is *non* relatively minimal if $\sum \Delta_j > 0$. The fibre Y_j is replaced by

$$\tilde{Y}_j = Z_j + 2\left(\sum_{\epsilon=1}^{\Delta_i} W_j^\epsilon\right),$$

where Z_j is the proper transform of Y_i and each W_j^ϵ is a $\mathbb{P}^3$. Also, $\Delta\chi_0$ is replaced by

$$\Delta\chi = \chi(X) - \chi(X_\eta)\chi(S) = \Delta\chi_0 + 3\Big(\sum_j \Delta_j\Big).$$

Each

$$Q_j^\epsilon =: Z_j \cap W_j^\epsilon$$

is a quadric surface in $\mathbb{P}^3$. We let

$$Z = \sum_j Z_j, \quad W = \sum_{j,\epsilon} W_j^\epsilon$$

and

$$D = \sum_i X_{s_i} + Z + W.$$

There is then an exact diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & & \mathcal{O}_W(W) \arrow[d] & \\
0 \arrow[r] & f^{-1}(\Omega_S^1) \arrow[r] \arrow[d] & \Omega_X^1 \arrow[r] \arrow[d] & \Omega_{X/S}^1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & f^{-1}(\Omega_S^1(\log D_S)) \arrow[r] \arrow[d] & \Omega_X^1(\log D) \arrow[r] \arrow[d] & \Omega_{X/S}^1(\log D) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_D \arrow[r] \arrow[d] & \bigoplus_{i,\alpha} \mathcal{O}_{X_\alpha^i} \oplus (\mathcal{O}_Z \oplus \mathcal{O}_W) \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0, &
\end{tikzcd} \tag{V.B.1}$$

where we denote by $D_S \subset S$ the set of points over which the fibres are singular and where (cf. (II.B.8))

$$0 \to \mathcal{F} \to \left(\bigoplus_i \bigoplus_{\alpha<\beta} \mathcal{O}_{X_{\alpha\beta}^i}\right) \oplus \left(\bigoplus_{j,\epsilon} \mathcal{O}_{Q_j^\epsilon}\right) \to \bigoplus_{i,\alpha<\beta} \mathcal{O}_{X_{\alpha\beta}^i}$$
$$\to \bigoplus_{i,\alpha<\beta<\gamma} \mathcal{O}_{X_{\alpha\beta\gamma}^i} \to \bigoplus_{i,\alpha<\beta<\gamma<\delta} \mathcal{O}_{X_{\alpha\beta\gamma\delta}^i} \to 0 \tag{V.B.2}$$

is exact. Denoting by $\sigma\colon X \to X_0$ the blowup so that $f = f_0 \cdot \sigma$, we find that

$$R^3_f \mathcal{O}_X \cong R^3_{f_0} \mathcal{O}_{X_0} \cong \mathcal{H}_e^{0,3},$$
$$R^2_f \Omega^1_{X/S}(\log D) \cong R^2_{f_0} \Omega^1_{X_0/S}(\log D_0) \cong \mathcal{H}_e^{1,2}.$$[17]

Defining

$$\omega_{X/S} =: \omega_X \otimes f^{-1}(\Omega^1_S)$$

one finds

$$f_*(\omega_{X/S}) \cong (R^3_f \mathcal{O}_X)^\vee = \mathcal{H}_e^{3,0},$$

while

$$\begin{cases} \mathcal{H}_e^{3,0} \cong \mathcal{O}_S(\delta), \\ \mathcal{H}_e^{2,1} \cong \mathcal{O}_S(\lambda). \end{cases}$$

In case there are actually double points so that $f\colon X \to S$ is not relatively minimal one has

$$\omega_{X/S} \cong f^{-1}(\mathcal{H}_e^{3,0}) \otimes \mathcal{O}_X(3W) \cong \mathcal{O}_X(\delta \cdot X_\eta + 3W).$$

Thus

$$c_1(\omega_X) = (2g - 2 + \delta)X_\eta + 3W,$$
$$c_1(X) = (2 - 2g - \delta)X_\eta - 3W.$$

V.C. *Intersection data on X*

For simplicity of notation, we drop the subscripts and just write W for $\sum_{j,\epsilon} W_j^\epsilon$.[18] Let

- $H \subset W$ be a general hyperplane
- L the intersection of two general hyperplanes
- $Q = Z \cdot W \equiv 2H$
- $E = Z \cdot Q \equiv 4L$.

Recalling that $X_\eta = Z + 2W$ we obtain the following, listed here for reference in subsections D, E below.

(i) $W^2 = W \cdot (-\frac{1}{2}Z) = -\frac{1}{2}Q = -H$.

(ii) $W^3 = -\frac{1}{2}Z \cdot (-\frac{1}{2}Z \cdot W) = \frac{1}{4}Z \cdot Q = \frac{1}{4}E = L$.

(iii) $W^4 = -\frac{1}{2}Z \cdot (-\frac{1}{2}Z \cdot (-\frac{1}{2}Z) \cdot W) = -\frac{1}{8}Z \cdot E = -\Delta \cdot p$, where $\Delta \cdot p$ is 0-cycle of degree equal to the number Δ of double points.

(iv) $H \cdot W = -W^3 = -L$.

(v) $H \cdot W^2 = -W^4 = \Delta \cdot p$.

(vi) $H^2 = -H \cdot W^2 = -\Delta \cdot p$.

(vii) $L \cdot W = W^3 \cdot W = -\Delta \cdot p$.

(viii) $c_1(X)^2 = c_1(\Omega^1_X)^2 = (-3W)^2 = -9H$.

(ix) $c_1(X)^3 = -c_1(\Omega^1_X)^3 = -27W^3 = -27L$.

(x) $c_1(X)^4 = c_1(\Omega^1_X)^4 = 81W^4 = -81\Delta \cdot p$.

[17] A residue-theoretic argument for this may be found in [G1].

[18] If the reader wishes, just think of having only one double point.

Claim 1. *We have*

$$c_2(X)\cdot W = 2L,$$
$$c_2(X)\cdot H = 2\Delta\cdot p,$$
$$c_3(X)\cdot W = -2\Delta\cdot p.$$

Proof. Using adjunction, the computation of $c_1(\omega_X)$ above, and (V.C)(i), we have

$$\begin{aligned} N^*_{W/X}\otimes\omega_W \cong \omega_X|_W &\cong \mathcal{O}_W(3W\cdot W)\\ &\cong \bigoplus^{\Delta}\mathcal{O}_{\mathbb{P}^3}(-3); \end{aligned}$$

tensoring with $\omega_W^*\cong\bigoplus^{\Delta}\mathcal{O}_{\mathbb{P}^3}(4)$,

$$N^*_{W/X}\cong\bigoplus^{\Delta}\mathcal{O}_{\mathbb{P}^3}(1).$$

Next, the total Chern class

$$\begin{aligned} \bigoplus^{\Delta} c(\mathbb{P}^3) &= (1+H)^4 = 1+4H+6L+4\Delta\cdot p\\ \implies \bigoplus^{\Delta}\mathrm{ch}(\mathbb{P}^3) &= 3+4H+2L+\frac{2}{3}\Delta\cdot p. \end{aligned}$$

Dualizing the standard exact sequence

$$0\to TW\to TX\big|_W\to N_{W/X}\to 0$$

gives

$$\mathcal{O}_{\mathbb{P}^3}(1)\cong N^*_{W/X}\to i_W^*(\Omega^1_X)\to\Omega^1_W.$$

Writing $i\colon W\hookrightarrow X$, this yields

$$i_*(\mathrm{ch}(i^*\Omega^1_X)) = i_*(\mathrm{ch}\,\mathcal{O}_{\mathbb{P}^3}(1)) + i_*(\widetilde{\mathrm{ch}}\,\mathbb{P}^3),$$
$$\mathrm{ch}(\Omega^1_X)\cdot W = i_*\Big(1+H+\frac{1}{2}L+\frac{1}{6}\Delta\cdot p\Big) + i_*\Big(3-4H+2L-\frac{2}{3}\Delta\cdot p\Big).$$

The left-hand side of this equation expands to

$$\begin{aligned} 4+[(2g-2+\delta)X_\eta+3W] &+ \Big[\frac{9}{2}W^2-c_2(X)\Big]\\ &+\Big[\frac{27}{6}W^3-\frac{1}{2}\big((2g-2+\delta)X_\eta+3W\big)c_2(X)-\frac{1}{3}c_3(X)\Big]+\mathrm{ch}_4(\Omega^1_X). \end{aligned}$$

Thus we obtain

$$\begin{aligned} 4W-3H+\Big(\frac{9}{2}L-c_2(X)\cdot W\Big)+\Big(\frac{-9}{2}\Delta_p+\frac{3}{2}H\cdot c_2(X)-\frac{1}{3}c_3(X)\cdot W\Big)&\\ = 4W-3H+\frac{5}{2}L-\frac{1}{2}\Delta\cdot p.& \quad\square \end{aligned}$$

Claim 2. *We have*

$$\deg\bigl(c_2(X)\cdot X^i_{\alpha\beta}\bigr) = \chi(X^i_{\alpha\beta}) + \deg_X\bigl[(X^i_{\alpha\beta})^2\bigr] - \deg_{X^i_{\alpha\beta}}\bigl[(\omega_{X^i_{\alpha\beta}})^2\bigr],$$
$$\sum_{\alpha<\beta}\deg\bigl(c_2(X)\cdot X^i_{\alpha\beta}\bigr) = \chi^i_2 + \sum_{\alpha<\beta}\deg\bigl[(X^i_{\alpha\beta})^2\bigr] - 4I^i_2.$$

Proof. For the inclusion $\iota\colon X^i_{\alpha\beta}\hookrightarrow X$ we have

$$\begin{aligned}\deg(c_2(X)\cdot X^i_{\alpha\beta}) &= \deg(c_2(\iota^*TX))\\ &= \deg(c_2(\iota^*\Omega^1_X)) = \deg(\mathrm{ch}_2(\iota^*\Omega^1_X)).\end{aligned}$$

Next, by adjunction and the description of $c_1(\omega_X)$ above

$$\bigwedge\nolimits^2 N^*_{X^i_{\alpha\beta}/X}\otimes\omega_{X^i_{\alpha\beta}} \cong \omega_X\big|_{X^i_{\alpha\beta}}$$
$$\implies \bigwedge\nolimits^2 N^*_{X^i_{\alpha\beta}/X}\cong\omega^*_{X^i_{\alpha\beta}} \implies c_1\bigl(N^*_{X^i_{\alpha\beta/X}}\bigr) = -c_1\bigl(\omega_{X^i_{\alpha\beta}}\bigr).$$

Together with the exact sequence

$$0\to N^*_{X^i_{\alpha\beta}/X}\to\iota^*\Omega^1_X\to\Omega^1_{X^i_{\alpha\beta}}\to 0,$$

this yields

$$\begin{aligned}\mathrm{ch}(\iota^*\Omega^1_X) &= \mathrm{ch}(\Omega^1_{X^i_{\alpha\beta}}) + \mathrm{ch}(N^*_{X^i_{\alpha\beta}/X}),\\ \iota^*(4+(2g-2+\delta)X_\eta - c_2(\Omega^1_X)+\dots) &= \left(2+(\omega_{X^i_{\alpha\beta}}) + \frac{(\omega_{X^i_{\alpha\beta}})^2}{2} - \chi(X^i_{\alpha\beta})_p\right)\\ &\quad + \left(2-(\omega_{X^i_{\alpha\beta}}) + \frac{(\omega_{X^i_{\alpha\beta}})^2}{2} - c_2(N^*_{X^i/X})\right),\\ -\iota^*c_2(\Omega^1_X) &= (\omega_{X^i_{\alpha\beta}})^2 - \chi(X^i_{\alpha\beta}) - (X^i_{\alpha\beta})^2.\end{aligned}$$

Since $\omega_X\cong\omega_{X/S}$ is trivial in a neighborhood of X_i, and X_i is semistable, we see that

$$\omega_{X^i_{\alpha\beta}}\cong\mathcal{O}_{X^i_{\alpha\beta}}\Bigl(-\sum_{\gamma\neq\alpha,\beta}X^i_{\alpha\beta\gamma}\Bigr)$$

which leads to

$$\begin{aligned}\sum_{\alpha<\beta}(K_{X^i_{\alpha\beta}})^2 &= \sum_{\alpha<\beta}X^i_{\alpha\beta}\left(\sum_{\gamma\neq\alpha,\beta}X^i_\gamma\right)^2 = \sum_{\alpha<\beta}X^i_{\alpha\beta}(X^i_\alpha+X^i_\beta)^2\\ &= 2\sum_{\alpha<\beta}(X^i_{\alpha\beta})^2 + \sum_{\alpha\neq\beta}(X^i_\alpha)^3X^i_\beta\\ &= 2\sum_{\alpha<\beta}(X^i_{\alpha\beta})^2 + \sum_\alpha(X^i_\alpha)^3\sum_{\beta\neq\alpha}X^i_\beta\\ &= 2\sum_{\alpha<\beta}(X^i_{\alpha\beta})^2 - \sum_\alpha(X^i_\alpha)^4.\end{aligned}$$

Now

$$
\begin{aligned}
4\left(\sum_{\alpha<\beta} X^i_{\alpha\beta}\right)^2 &= \left(\sum_{\alpha\neq\beta} X^i_{\alpha\beta}\right)^2 \\
&= \left(\sum_{\alpha}\left\{X^i_\alpha \cdot \sum_{\beta\neq\alpha} X^i_\beta\right\}\right)^2 = \left(-\sum_\alpha (X^i_\alpha)^2\right)^2 \\
&= \sum (X^i_\alpha)^4 + 2\sum_{\alpha<\beta}(X^i_{\alpha\beta})^2,
\end{aligned}
$$

so

$$
\sum_{\alpha<\beta}(\omega_{X^i_{\alpha\beta}})^2 = 2\sum_{\alpha<\beta}(X^i_{\alpha\beta})^2 - 4\left(\sum_{\alpha<\beta} X^i_{\alpha\beta}\right)^2 + 2\sum_{\alpha<\beta}(X^i_\beta)^2 = 4I^i_2. \qquad \square
$$

V.D. *Push forward data on X*

Claim 3. *For the inclusion $i\colon W \subset X$ we have*

$$
\mathrm{ch}(i_*\mathcal{O}_W(W)) = W - \frac{1}{2}H + \frac{1}{6}L - \frac{\Delta}{24}\cdot p.
$$

Proof. From

$$
0 \to \mathcal{O}_X \to \mathcal{O}_X(W) \to i_*\mathcal{O}_W(W) \to 0
$$

we have

$$
\begin{aligned}
\mathrm{ch}(i_*\mathcal{O}_W(W)) &= \mathrm{ch}(\mathcal{O}_X(W)) - \mathrm{ch}(\mathcal{O}_X) \\
&= 1 + W + \frac{W^2}{2} + \frac{W^3}{6} + \frac{W^4}{24} - 1.
\end{aligned}
$$

The result follows from the intersection formulas in Section V.C above. $\square$

Claim 4. *For the inclusion $j\colon Q \subset X$ we have*

$$
\mathrm{ch}(j_*\mathcal{O}_Q) = 2H - L + \frac{2}{3}\Delta \cdot p.
$$

Proof. The Grothendieck–Riemann–Roch formula gives

$$
\mathrm{ch}(j_*\mathcal{O}_Q)\,\mathrm{Td}\,X = j_*\left(\mathrm{ch}(\mathcal{O}_Q)\,\mathrm{Td}\,Q\right).
$$

In this calculation, we will omit any terms involving the NCD fibres X_{s_i}, as they do not contain any double points. With this understood

$$
\begin{aligned}
\text{left-hand side} &= (Q + \mathrm{ch}_3(j_*\mathcal{O}_Q) + \mathrm{ch}_4(j_*\mathcal{O}_Q)) \\
&\quad \times \left(1 + \left\{\left(1 - g - \frac{\delta}{2}\right)X_\eta - \frac{3}{2}W\right\} + \left\{-\frac{3}{4}H + \frac{1}{12}c_2(X)\right\} + \cdots\right)
\end{aligned}
$$

("$\cdots$" are higher order terms)

$$
= Q + (\mathrm{ch}_3(\mathcal{O}_Q) + 3L) + \left(\mathrm{ch}_4(\mathcal{O}_Q) - \frac{3}{2}W\cdot\mathrm{ch}_3(\mathcal{O}_Q) + \frac{11}{6}\Delta\cdot p\right),
$$

where we have used
$$\begin{cases} Q \cdot W = 2H \cdot W = -2L, \\ Q \cdot H = -2\Delta \cdot p, \\ c_2(X) \cdot Q = 4\Delta \cdot p. \end{cases}$$
The right-hand side is
$$\begin{aligned} j_*(\operatorname{Td} Q) &= j_*\left(1 + (L_1 + L_2) + \Delta \cdot p\right) \\ &= Q + 2L + \Delta \cdot p, \end{aligned}$$
where L_1, L_2 are lines from two different rulings on Q.

Combining, we have for the degree 6 terms
$$2L = \operatorname{ch}_3(\mathcal{O}_Q) + 3L \implies \operatorname{ch}_3(i_*\mathcal{O}_Q) = -L.$$
Using this and $W \cdot L = -\Delta \cdot p$, we have for the degree 8 terms
$$\Delta \cdot p = \frac{11}{6}\Delta \cdot p - \frac{3}{2}\Delta \cdot p + \operatorname{ch}_4(i_*\mathcal{O}_Q) \implies \frac{2}{3}\Delta \cdot p = \operatorname{ch}_4(i_*\mathcal{O}_Q). \qquad \square$$

Claim 5.

(a) $$\operatorname{ch}\big(\iota_*\mathcal{O}_{X^i_{\alpha\beta}}\big) = X^i_{\alpha\beta} - \frac{1}{2}\iota_*\big(K_{X^i_{\alpha\beta}}\big) - \frac{1}{12}\big(X^i_{\alpha\beta}\big)^2 + \frac{1}{6}\deg_{X^i_{\alpha\beta}}\big(K_{X^i_{\alpha\beta}}\big)^2 \cdot p$$

(b) $$\operatorname{ch}\big(\iota_*\mathcal{O}_{X^i_{\alpha\beta\gamma}}\big) = X^i_{\alpha\beta\gamma} + \big(1 - g^i_{\alpha\beta\gamma}\big) \cdot p$$

(c) $$\operatorname{ch}\big(\iota_*\mathcal{O}_{X^i_{\alpha\beta\gamma\delta}}\big) = X^i_{\alpha\beta\gamma\delta} = p.$$

Proof. The first term of each formula is obvious. For (a),
$$\operatorname{ch}\underbrace{\left(\iota_!\mathcal{O}_{X^i_{\alpha\beta}}\right)}_{\iota_*\mathcal{O}_{X^i_{\alpha\beta}}}\operatorname{Td}(X) = \iota_*(\operatorname{Td}(X^i_{\alpha\beta})),$$ [19]
$$\begin{aligned} &\left(X^i_{\alpha\beta} + \operatorname{ch}_3\big(\iota_*\mathcal{O}_{X^i_{\alpha\beta}}\big) + \operatorname{ch}_4\big(\iota_*\mathcal{O}_{X^i_{\alpha\beta}}\big)\right)\left(1 + \left(\frac{\delta}{2} + g - 1\right)X_\eta + \frac{1}{12}c_2(X) + \cdots\right) \\ &\qquad = \iota_*\left(1 - \frac{1}{2}K_{X^i_{\alpha\beta}} + \frac{\big(K_{X^i_{\alpha\beta}}\big)^2 + \chi(X^i_{\alpha\beta})p}{12}\right) \end{aligned}$$
yielding in $H^6(X)$:
$$\operatorname{ch}_3 = -\frac{1}{2}\omega_{X^i_{\alpha\beta}}$$
and in $H^8(X)$:
$$\operatorname{ch}_4 + \frac{1}{12}\underbrace{\left(c_2(X) \cdot X^i_{\alpha\beta}\right)}_{\chi(X^i_{\alpha\beta})p + (X^i_{\alpha\beta})^2 - \deg((\omega_{X^i_{\alpha\beta}})^2)} = \frac{1}{12}\deg\left(\big(\omega_{Y^i_{\alpha\beta}}\big)^2\right)p + \frac{1}{12}\chi(X^i_{\alpha\beta})_p,$$
$$\operatorname{ch}_4 = \frac{1}{6}\deg\big(\omega_{X^i_{\alpha\beta}}\big)^2 p - \frac{1}{12}(X^i_{\alpha\beta})^2. \qquad \square$$
The proof of (b) is similar.

---

[19] Here, and in a few lines below, the term above the horizontal brace is equal to the one in the parenthesis.

Now by (V.B.2),

$$\begin{aligned}\operatorname{ch}(\mathcal{F}) = \sum_{i,\alpha<\beta} \operatorname{ch}\big(i_*\mathcal{O}_{X^i_{\alpha\beta}}\big) + \sum_{j,\epsilon} \operatorname{ch}\big(j_*\mathcal{O}_{Q^\epsilon_j}\big) \\ - \sum_{i,\alpha<\beta<\gamma} \operatorname{ch}\big(\mathcal{O}_{X^i_{\alpha\beta\gamma}}\big) + \sum_{i,\alpha<\beta<\gamma<\delta} \operatorname{ch}\big(\mathcal{O}_{X^i_{\alpha\beta\gamma\delta}}\big).\end{aligned}$$

To evaluate this we use Claims 4, 5 together with the implication

$$\sum_{\alpha<\beta} \deg_{X^i_{\alpha\beta}}\big((\omega_{X^i_{\alpha\beta}})^2\big) = 4I^i_2$$

from Claim 2 to have for the terms of resp. degrees 4, 6, 8 in the right-hand side:

$$\begin{aligned}\operatorname{ch}_2(\mathcal{F}) &= \sum_{i,\alpha<\beta} X^i_{\alpha\beta} + 2\sum_{j,\epsilon} H^\epsilon_j,\\ \operatorname{ch}_3(\mathcal{F}) &= \frac{-1}{2}\sum_{i,\alpha<\beta} L_*\big(\omega_{X^i_{\alpha\beta}}\big) - \sum_{i,\alpha<\beta<\gamma} X^i_{\alpha\beta\gamma} - \sum_{j,\epsilon} L^\epsilon_j,\\ \operatorname{ch}_4(\mathcal{F}) &= \frac{-1}{12}\sum_{i,\alpha<\beta} \deg\big((X^i_{\alpha\beta})^2\big)p + \frac{2}{3}\sum_i I^i_2\cdot p + \sum_{i,\alpha<\beta<\gamma} g^i_{\alpha\beta\gamma}\cdot p\\ &\qquad - \sum_i \varepsilon_i\cdot p + \sum_i v_i\cdot p + \frac{2}{3}\Delta\cdot p.\end{aligned}$$

In particular we have

$$\begin{aligned}&\frac{1}{2}\int_X (\operatorname{ch}_2(\mathcal{F}))^2 + 6\int_X \operatorname{ch}_4(\mathcal{F})\\ &\qquad = \frac{1}{2}\sum_i \deg\left(\left(\sum_{\alpha<\beta} X^i_{\alpha\beta}\right)^2\right) - 2\Delta - \frac{1}{2}\sum_{i,\alpha<\beta}\deg\big((X^i_{\alpha\beta})^2\big)\\ &\qquad\quad + 4\sum_i I^i_2 + 6\sum g^i_{\alpha\beta\gamma} - 6\sum \varepsilon_i + 6\sum v_i + 4\Delta\\ &\qquad = \frac{7}{2}\sum_i I^i_2 + 2\Delta + 6\sum_i\left(\sum_{\alpha<\beta<\gamma} g^i_{\alpha\beta\gamma} - \varepsilon_i + v_i\right),\end{aligned}$$

where above we have combined the 1st and 3rd terms to obtain $\frac{-1}{2}\sum_i I^i_2$.

A consequence of the above calculations is

$$\begin{cases}\operatorname{ch}_2(\mathcal{F})\cdot W = 2H\cdot W = -2L,\\ \operatorname{ch}_2(\mathcal{F})\cdot H = -\operatorname{ch}_2(\mathcal{F})\cdot W\cdot W = 2L\cdot W = -2\Delta\cdot p,\\ \operatorname{ch}_3(\mathcal{F})\cdot W = -L\cdot W = \Delta\cdot p.\end{cases} \tag{V.D.1}$$

Using Claim 2 we have

$$\deg(\operatorname{ch}_2(\mathcal{F})\cdot c_2(X)) = \sum_i \chi^i_2 - 4\sum_i I^i_2 + \sum_{i,\alpha<\beta}\deg\big((X^i_{\alpha\beta})^2\big) + 4\Delta.$$

Below we will see that

$$c_2\big(\Omega^1_{X/S}(\log D)\big) = c_2(X) - \mathrm{ch}_2(\mathcal{F}) + 2H,$$

which with a little work gives

$$\deg\big(\mathrm{ch}_2(\mathcal{F}) \cdot c_2(\Omega^1_{X/S}(\log D))\big) = \sum_i \chi_2^i - 3\sum_i I_2^i + 4\Delta. \tag{V.D.2}$$

V.E. *Proof of Theorem V.A.1*

We recall our notations

$$\begin{cases} c_i = c_i(X), \\ \hat{c}_i = c_i\big(\Omega^1_{X/S}(\log D)\big) \end{cases}$$

$$\begin{aligned} \Delta\chi &= \chi(X) - \chi(X_\eta)\chi(S) \\ &= \int_X c_4(\Omega^1_X) + 2(h^{2,1} - h^{1,1})(2-2g). \end{aligned}$$

From the diagram (V.B.1) we obtain

$$\mathrm{ch}(\Omega^1_X) = f^*\,\mathrm{ch}\big(\Omega^1_S\big) + \mathrm{ch}\big(\Omega^1_{X/S}(\log D)\big) + \mathrm{ch}(i_*\mathcal{O}_W(W)) - \mathrm{ch}(\mathcal{F}). \tag{V.E.1}$$

In $H^2(X)$ this gives

$$-c_1 = (2g-2)X_\eta + \hat{c}_1 + W,$$

while from the end of Section V.B

$$-c_1 = (2g - 2 + \delta)X_\eta + 3W;$$

hence

$$\hat{c}_1 = \delta X_\eta + 2W.$$

From the intersection relations in Section V.C above we obtain

$$\begin{cases} c_1^2 = -9H, \\ c_1^3 = -27L, \\ c_1^4 = -81\Delta\cdot p; \end{cases} \qquad \begin{cases} \hat{c}_1^2 = -4H, \\ \hat{c}_1^3 = -8L, \\ \hat{c}_1^4 = -16\Delta\cdot p. \end{cases}$$

In $H^4(X)$ the equation (V.E.1) gives

$$\begin{aligned} \frac{1}{2}c_1^2 - c_2 &= \frac{1}{2}\hat{c}_1^2 - \hat{c}_2 - \frac{1}{2}H - \mathrm{ch}_2(\mathcal{F}) \\ \implies \hat{c}_2 &= c_2 - \mathrm{ch}_2(\mathcal{F}) + 2H. \end{aligned}$$

Using (V.D.1), Claim 1, and Section V.C(vi), this implies

$$H\hat{c}_2 = 2\Delta\cdot p.$$

In $H^6(X)$ we have from (V.E.1)

$$-\frac{1}{6}c_1^3 + \frac{1}{2}c_1c_2 - \frac{1}{2}c_3 = \frac{1}{6}\hat{c}_1^3 - \frac{1}{2}\hat{c}_1\hat{c}_2 + \frac{1}{2}\hat{c}_3 + \frac{1}{6}L - \mathrm{ch}_3(\mathcal{F}),$$

which using the intersection relations (in V.C, Claim 1, (V.D.1)) gives

$$\begin{aligned}\hat{c}_3 &= -c_3 + 4L + (2-2g)X_\eta \cdot c_2 + 2\,\mathrm{ch}_3(\mathcal{F}) \\ &\implies W \cdot \hat{c}_3 = 0.\end{aligned}$$

Finally, in $H^8(X)$ we have (using $\hat{c}_4 = 0$ since $\Omega^1_{X/S}(\log D)$ is locally free of rank 3)

$$\frac{1}{4}c_1^4 - c_1^2 c_2 + \frac{1}{2}c_2^2 + c_1 c_3 - c_4 = \frac{1}{4}\hat{c}_1^4 - \hat{c}_1^2\hat{c}_2 + \frac{1}{2}\hat{c}_2^2 + \hat{c}_1\hat{c}_3 - \frac{1}{4}\Delta \cdot p - 6\,\mathrm{ch}_4(\mathcal{F}).$$

A somewhat lengthy computation using essentially all the intersection relations gives

$$\Delta\chi = \int_X 6\,\mathrm{ch}_4(\mathcal{F}) + \frac{1}{2}\mathrm{ch}_2(\mathcal{F})^2 + \hat{c}_2 \cdot \mathrm{ch}_2(\mathcal{F}) - 2\Delta \cdot p,$$

which using the computation after Claim 5 equals

$$\begin{aligned}\frac{7}{2}\sum_i I_2^i + 2\Delta + 6\sum_i\left(\sum_{\alpha<\beta<\gamma} g^i_{\alpha\beta\gamma} - \varepsilon_i + v_i\right) + \sum_i \chi_2^i - 3\sum_i I_2^i + 4\Delta - 2\Delta \\ = \frac{1}{2}\sum_i I_2^i + \sum_i \chi_2^i + 6\sum_i\left(\sum_{\alpha<\beta<\gamma} g^i_{\alpha\beta\gamma} - \varepsilon_i + v_i\right) + 4\Delta.\end{aligned}$$

Since $\Delta\chi_0 = \Delta\chi - 3\Delta$, this establishes (a) in the theorem.

Turning to part (b), we apply GRR to $f\colon X \to S$ and the element

$$\mathcal{A} := 36[\mathcal{O}_X] - 12\big[\Omega^1_{X/S}(\log D)\big] \in K_0(X).$$

This gives

$$\mathrm{ch}(f_!(\mathcal{A}))\,\mathrm{Td}\ S = f_*(\mathrm{ch}(\mathcal{A}) \cdot \mathrm{Td}\ X).$$

The left-hand side is (recall that η is a generic point of S)

$$(36\delta[\eta] + 6\chi(X_\eta) + 12\lambda[\eta])\left(1 + \frac{1}{2}\chi(S)[\eta]\right).$$

The right-hand side is

$$\begin{aligned}12 f_*\Bigg(\bigg\{&-\hat{c}_1 - \frac{1}{2}\hat{c}_1^2 + \hat{c}_2 - \frac{1}{6}\hat{c}_1^3 + \frac{1}{2}\hat{c}_1\hat{c}_2 - \frac{1}{2}\hat{c}_3 - \frac{1}{24}\hat{c}_1^4 \\ &+ \frac{1}{6}\hat{c}_1^2\hat{c}_2 - \frac{1}{12}\hat{c}_2^2 - \frac{1}{6}\hat{c}_1\hat{c}_3\bigg\}\bigg\{1 + \frac{c_1}{2} + \frac{c_1^2 + c_2}{12} + \frac{c_1 c_2}{24} + \mathrm{Td}_4(X)\bigg\}\Bigg).\end{aligned}$$

In $H^2(S)$ this gives

$$\begin{aligned}\big(3\chi(X_\eta)\chi(S) + 12\lambda + 36\delta\big)[p] = &-\frac{1}{2}\hat{c}_1^4 + 2\hat{c}_1^2\hat{c}_2 - \hat{c}_2^2 - 2\hat{c}_1\hat{c}_3 - \hat{c}_1^3 c_1 + 3c_1\hat{c}_1\hat{c}_2 \\ &- 3c_1\hat{c}_3 - \frac{1}{2}\hat{c}_1^2 c_1^2 + c_1^2\hat{c}_2 - \frac{1}{2}c_2\hat{c}_1^2 + c_2\hat{c}_2 - \frac{1}{2}\hat{c}_1 c_1 c_2\end{aligned}$$

the right-hand side of which simplifies to

$$-2\Delta \cdot p + \hat{c}_2\,\mathrm{ch}_2(\mathcal{F}) + 3\chi(S)\chi(X_\eta) - \delta\chi(X_\eta)[p].$$

Taking degrees of both sides and using (V.D.2) yields

$$12\lambda + 36\delta + \chi(X_\eta)\delta = 2\Delta + \sum_i \chi_2^i - 3\sum_i I_2^i. \qquad \square$$

V.F. *Example: The Fermat quintic pencil*[20]

We refer to [GGK1], section IV.A for the discussion of SSR applied to the Fermat/pentahedron pencil

$$(x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5) - s(x_0x_1x_2x_3x_4) = 0.$$

In that reference we obtain the family

$$f\colon X_0 \to \mathbb{P}^1,$$

where

$$X_0 = \mathbb{P}^4 \text{ blown up successively along 5 Fermat quintic surfaces } F.$$

Thus

$$\begin{aligned}\chi(X_0) &= \chi(\mathbb{P}^4) + 5\chi(F)\\ &= 5 + 5\cdot(55) = 280,\end{aligned}$$

using the total Chern class

$$c(F) = \frac{(1+H)^4}{(1+5H)} = 1 - H\cdot F + 11H^2\cdot F \quad \text{and} \quad H^2\cdot F = 5.$$

Now $\chi(X_\eta) = -200$ and so

$$\Delta\chi_0 = \chi(X_0) - \chi(\mathbb{P}^1)\chi(X_\eta) = 680.$$

For the fibre $X_{s_0} = \bigcup_{i=1}^5 X_\alpha$ ($s_0 = \infty$) of maximal unipotent monodromy we compute

$$\begin{cases} I_2 = \displaystyle\sum_{\alpha<\beta}\deg\big(X_{\alpha\beta}^2\big) - \deg\left[\left(\sum_{\alpha<\beta} X_{\alpha\beta}\right)^2\right] = 10,\\ \varepsilon = 10, \quad v = 5,\\ \chi_2 = \displaystyle\sum_{\alpha<\beta}\chi(X_{\alpha\beta}) = 30 + 50 = 80,\\ g_{\alpha\beta\gamma} = 0.\end{cases}$$

Here, $30 = 10\chi(\mathbb{P}^2)$ and 50 is the total number of points blown up.

All the other fibres have only ODP's. To calculate how many we use (a) in theorem (V.A.1) to have

$$\begin{aligned} 680 &= \Delta + 80 + \frac{1}{2}\cdot 10 + 6\cdot 0 - 6\cdot 10 + 6\cdot 5\\ &= \Delta + 55\\ \implies \Delta &= 625.\end{aligned}$$

In fact, there are 5 such singular fibres, each with 125 ODP's.

[20] For an explanation of why this is relatively minimal, see Section V.G below.

It will be shown (cf. Section V.G) that $\delta = 1$; thus (b) gives

$$12\lambda + 36 - 200 = 2 \cdot 625 + \underbrace{80 - 3 \cdot 10}_{},$$

where the term 50 over the brace is the contribuation of the maximally unipotent monodromy fibre. Solving we obtain

$$\lambda = 122.$$

V.G. *On relative minimality*

Given an irreducible variety X having as dualizing sheaf ω_X a line bundle K_X and with Kodaira dimension $\kappa(X) \geqslant 0$, we recall by definition X is *minimal* if K_X is nef. The *minimal model program* (cf. [KM], [Dr] and the references cited therein) says that a given X_0 should be birationally equivalent to a minimal X, which for $\dim X \geqslant 3$ will in general be singular. This suggests the

Definition. A family $f \colon X \to S$, where S is a smooth curve and the relative dualizing sheaf $\omega_{X/S}$ is assumed to be a line bundle and with $\kappa(X_\eta) \geqslant 0$, is *relative minimal* (RM) if $\omega_{X/S}$ is relatively nef.[21]

The latter condition is[22]

$$\omega_{X/S} \cdot C \geqslant 0$$

for any curve $C \subset f^{-1}(s)$ lying in a fibre.

Example. If $f \colon X \to S$ is an elliptic surface where X is smooth, then relative minimality is equivalent to there being no (-1) curves *in a fibre*. According to Kodaira [Ko], such exists and is unique. We note that X itself may not be minimal. For example, if X is a rational elliptic surface, the single-valued sections of $f \colon X \to S$ are $\mathbb{P}^1$'s which are (-1)-curves. Blowing these down would make f ill-defined as a regular map.[23]

Referring to the exact sequence (c) in (II.B.5) we have: Assuming that X is smooth, f semi-stable, and X_η Calabi–Yau,

$$f \colon X \to S \text{ is RM} \iff \mathcal{G} = 0. \tag{V.G.1}$$

Proof. If as usual the singular fibres are $X_{s_i} = \bigcup_\alpha X^i_\alpha$, then

$$\omega_{X/S} = \delta X_\eta + \sum_i m^i_\alpha X^i_\alpha,$$

where the $m^i_\alpha \geqslant 0$. Then, from the definition of $\mathcal{G}$

$$\mathcal{G} = 0 \iff \text{all } m^i_\alpha = 0.$$

Thus $\mathcal{G} = 0 \implies$ RM.

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[21] In this paper we are primarily concerned with the case where X_η is Calabi–Yau, so that $\kappa(X_\eta) = 0$.

[22] Equivalently, $\omega_X \cdot C \geqslant 0$, since

$$\omega_X \cong \omega_{X/S} \otimes f^{-1}\omega_S.$$

[23] For example, let X be the blowup of the base locus of a general pencil of cubic curves in $\mathbb{P}^2$.

For the converse, assume for simplicity of notation that we have one singular fibre $\bigcup X_\alpha$ (we drop the i index) with some $m_{\alpha_0} \neq 0$. By our assumption of semi-stability, for $\alpha \neq \alpha_0$

$$X_\alpha \cdot X_{\alpha_0} = X_\alpha \cap X_{\alpha_0} =: D_\alpha.$$

Around the singular fibres we have

$$\frac{1}{m_{\alpha_0}} K_X = \sum_{\alpha \neq \alpha_0} \frac{m_\alpha}{m_{\alpha_0}} X_\alpha + X_{\alpha_0}.$$

We may assume that $\alpha_0 \geqslant \alpha$ for all α and not all $\alpha = \alpha_0$. From $X_\eta \cdot X_{\alpha_0}$ we have

$$0 = \left(\sum_{\alpha \neq \alpha_0} X_\alpha + X_{\alpha_0} \right) \cdot X_{\alpha_0}$$

which gives

$$\begin{aligned} \frac{1}{m_{\alpha_0}} K_X \cdot X_{\alpha_0} &= \sum_{\alpha \neq \alpha_0} \left(\frac{m_\alpha}{m_{\alpha_0}} - 1 \right) X_\alpha \cdot X_{\alpha_0} \\ \implies K_X \cdot X_{\alpha_0} &= \sum_{\alpha \neq \alpha_0} (m_\alpha - m_{\alpha_0}) D_\alpha \in \mathrm{CH}^1(X_{\alpha_0}). \end{aligned}$$

This is *minus* an effective divisor on X_{α_0}, and so there exists a curve $C \subset X_{\alpha_0}$ with $K_X \cdot C < 0$. □

Examples. This will be a sketch of the constructions of families of relatively minimal semi-stable CY n-fold fibrations with $g = 0$, $\delta = 1$.

Let $\hat{X}$ be a smooth toric Fano $(n+1)$-fold, and consider a regular anticanonical pencil: more precisely, starting with a reflexive $(n+1)$-polytope Δ and assuming the toric variety $\mathbb{P}_\Delta$ obtained from a triangulation of the dual Δ° is smooth, we take a Δ-regular Laurent polynomial $\varphi \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_{n+1}^{\pm 1}]$; the family is then

$$X_t := \overline{\{t\varphi = 1\}} \subset \mathbb{P}_\Delta, \qquad t \in \mathbb{P}^1.$$

The base locus consists of smooth hypersurfaces in each component of $\mathbb{D}_\Delta := \mathbb{P}_\Delta \backslash (\mathbb{C}^*)^{n+1}$, and one blows $\mathbb{P}_\Delta$ up along successive proper transforms of these codimension 2 subvarieties to obtain $f\colon X \to \mathbb{P}^1$ with fibres X_t. Since $(K_{\mathbb{P}_\Delta}) = -\mathbb{D}_\Delta$, by a simple blow-up argument[24] we have $(K_X) = -\{\text{proper transform of } \mathbb{D}_\Delta\} \equiv -X_\eta$. So f is relatively minimal, the $\sum m_\alpha^i X_\alpha^i$ term disappears, and $K_X = (2g_s - 2 + \delta)X_\eta = (-2+\delta)X_\eta \implies \delta = 1$.

This set of examples includes:

- $n = 1$: many of the rational elliptic modular surfaces, plus some "non-modular" ones.
- $n = 2$: Fermat quartic K3 family and others with (and without) $\mathrm{rank}(\mathrm{Pic}(X_\eta)) = 19$;
- $n = 3$: Fermat quintic family (the construction above is the one in section IV.A of [GGK1]).

[24] It is enough to notice that $dx \wedge \frac{dy}{y}$ becomes $du \wedge dy$ upon substituting $x = yu$.

VI. Arakelov Equalities

In this section we shall refine the Arakelov inequalities (cf. [Pe1], [Pe2], [JZ], [V]) to give, in the case of VHS's of weights $n = 1, 2, 3$, an exact expression for the degrees of the Hodge bundles $\mathcal{H}_e^{n,0}$ and in the case $n = 3$ also for $\mathcal{H}_e^{2,1}$. For n odd the GRR gives an alternate expression for these degrees, and it is of interest to compare them, which will also be done below.

VI.A. *The weight one case*

We shall break the analysis into a sequence of steps, with the end result given by (VI.A.10) below.

Step one: *Some general considerations.*

(i) Suppose we are given a sequence of vector bundles and constant rank bundle maps

$$0 \to \mathcal{F}_0 \to \mathcal{F} \xrightarrow{\psi} \check{\mathcal{F}} \to \mathcal{B} \to 0. \tag{VI.A.1}$$

Setting

$$\begin{cases} \delta = \deg \mathcal{F}, \\ \delta_0 = \deg \mathcal{F}_0, \\ \beta = \deg \mathcal{B} \end{cases}$$

we have

$$2\delta = \delta_0 - \beta.$$

(ii) Suppose now that the mapping

$$\mathcal{F} \xrightarrow{\psi} \check{\mathcal{F}}$$

is symmetric. Then we have

$$\mathcal{B} \cong \check{\mathcal{F}}_0,$$

the sequence (VI.A.1) is self-dual, and we have

$$2\delta = 2\delta_0.$$

(iii) Suppose now that we have a line bundle $\mathcal{L}$ and (VI.A.1) is replaced by

$$0 \to \mathcal{F}_0 \to \mathcal{F} \xrightarrow{\psi} \mathcal{L} \otimes \check{\mathcal{F}} \to \mathcal{B} \to 0. \tag{VI.A.2}$$

Then setting

$$\begin{cases} r = \operatorname{rank} \mathcal{F}, \\ \lambda = \deg \mathcal{L} \end{cases}$$

we have

$$2\delta = r\lambda + \delta_0 - \beta.$$

(iv) If the map

$$\mathcal{F} \to \mathcal{L} \otimes \check{\mathcal{F}}$$

is symmetric, then setting $r_0 = \operatorname{rank} \mathcal{F}_0$ we have

$$2\delta = (r - r_0)\lambda + 2\delta_0. \tag{VI.A.3}$$

(v) Finally, we want to give the correction term to (VI.A.3) when at finitely many points of S, ψ drops rank from its rank at a generic point. We claim that

$$2\delta = (r - r_0)\lambda + 2\delta_0 - \sum_{s\in S}\big(\nu_s(\det\overline{\psi})\big), \tag{VI.A.4}$$

assuming that ψ is symmetric. Here, $\overline{\psi}_s\colon \mathcal{F}_s/\mathcal{F}_{0,s} \to \mathcal{L}_s \otimes (\mathcal{F}_s/\mathcal{F}_{0,s})\check{}$ is the induced mapping on stalks, and $\nu_s(\det\overline{\psi})$ is the order of vanishing of $\overline{\psi}_s$. Note that the sum on the right-hand side is over finitely many points of S.

To explain why such a formula should be true, we consider the simpler case of (VI.A.1) with $\mathcal{F}_0 = 0$. Thus, $\mathcal{L} \cong \mathcal{O}_S$ in (VI.A.2) and $\overline{\psi}_s = \psi_s$. We then have

$$0 \to \mathcal{F} \xrightarrow{\psi} \check{\mathcal{F}} \to \mathcal{B} \to 0,$$

where $\mathcal{B}$ is a skyscraper with stalk isomorphic as a vector space to $\mathbb{C}^{k_\alpha}$ at finitely many points s_α of S. Then

$$\begin{aligned} c_1(\check{\mathcal{F}}) &= c_1(\mathcal{F}) + c_1(\mathcal{B}) \\ &= c_1(\mathcal{F}) + \deg\mathcal{B}\cdot[\eta] \\ &= c_1(\mathcal{F}) + \Big(\sum_\alpha k_\alpha\Big)[\eta]. \end{aligned}$$

Next, the GRR gives

$$\begin{array}{ccc} \mathrm{ch}(f_!(\mathcal{B}))\,\mathrm{Td}(\mathrm{pt}) & = & f_*(\mathrm{ch}\,\mathcal{B}\cdot\mathrm{Td}\,S) \\ \| & & \| \\ \sum_\alpha k_\alpha & & f_*(c_1(\mathcal{B})) \\ & & \| \\ & & \deg\mathcal{B}, \end{array}$$

where we have used that $\mathrm{ch}\,\mathcal{B} = c_1(\mathcal{B})$ since rank $\mathcal{B}_\eta = 0$ at a generic point. Thus we have on the one hand

$$2\delta = \deg\mathcal{B} = \sum_\alpha k_\alpha,$$

while on the other hand we have

$$\begin{aligned} k_\alpha &= \dim\check{\mathcal{F}}_{s_\alpha}/\psi_{s_\alpha}(\mathcal{F}_{s_i}) \\ &= \dim(\mathrm{coker}\,\psi_{s_\alpha}). \end{aligned}$$

More generally, in (VI.A.2),

$$0 \to \check{\mathcal{F}}_0 \otimes L \to \mathcal{B} \to \mathcal{S} \to 0,$$

where $\mathcal{S}$ is a skyscraper sheaf with

$$c_1(\mathcal{S}) = \Big(\sum_\alpha k_\alpha\Big)[\eta].$$

Now

$$c_1(\mathcal{F}_0) - c_1(\mathcal{F}) + c_1(\check{\mathcal{L}}_1 \otimes L) - c_1(\mathcal{B}) = 0;$$

i.e.,

$$c_1(\mathcal{F}) - 2c_1(\mathcal{F}) + rc_1(\mathcal{L}) - c_1(\check{\mathcal{F}}_0 \otimes L) - c_1(\mathcal{S}) = 0$$

and thus

$$2c_1(\mathcal{F}_0) - 2c_1(\mathcal{S}) + (r - r_0)c_1(\mathcal{L}) + \sum_\alpha k_\alpha = 0$$

and (VI.A.4) follows by noting that $\dim(\operatorname{coker}\psi_{s_\alpha}) = \nu_{s_\alpha}(\det\overline{\psi})$.

We conclude this step with the important

Observation. *The formula* (VI.A.4) *remains true if ψ is a rational* (= *meromorphic*) *map and $\nu_s(\det\overline{\psi})$ is the usual valuation.*

The proof follows by replacing $\mathcal{L}$ by $\mathcal{L} \otimes \mathcal{O}_S([D])$, where D is the divisor of $(\det\overline{\psi})$ and applying (IV.A.4) to this case.

Step two: *Application to VHS of weight one.*

We want to apply the above considerations to the case where

$$\begin{cases} \mathcal{F} = \mathcal{H}_e^{1,0}, \\ \mathcal{L} = \Omega_S^1(\log E), \\ \psi = \theta. \end{cases}$$

Here, E is a set of points $s_i \in S$ where the VHS has degenerations and θ is the mapping induced by the Gauss–Manin connection. The sheaf sequence (VI.A.2) is then

$$0 \to \mathcal{H}_{0,e}^{1,0} \to \mathcal{H}_e^{1,0} \xrightarrow{\theta} \check{\mathcal{H}}_e^{1,0} \otimes \Omega_S^1(\log E) \to \mathcal{B} \to 0. \qquad \text{(VI.A.5)}$$

Here we have used the identification $\mathcal{H}_e^{1,0} \cong \check{\mathcal{H}}_e^{1,0}$ and the regularity of the Gauss–Manin connection for the canonical extension to infer that θ maps $\mathcal{H}_e^{1,0}$ to $\check{\mathcal{H}}_e^{1,0} \otimes \Omega_S^1(\log E)$. Setting $S^* = S\backslash E$, there are two types of degenerate points. To describe these we observe that for all $s \in S$

$$\theta_s(\mathcal{H}_{e,s}^{1,0}) \subseteq \left(\mathcal{H}_{0,e,s}^{1,0}\right)^\perp \otimes \Omega_{S,s}^1(\log E)$$

with equality holding for almost all s. The first type are the $s_\alpha \in S^*$ such that equality does not hold, in which case the quotient

$$\Omega_{S,s_\alpha}^1 \otimes \left(\mathcal{H}_{0,s_\alpha}^{1,0}\right)^\perp / \theta_{s_\alpha}(\mathcal{H}_{s_\alpha}^{1,0})$$

is a finite dimensional vector space. We denote by

$$\dim(\operatorname{coker}\bar{\theta}_{s_\alpha}) = v_{s_\alpha}(\det\bar{\theta}_{s_\alpha})$$

the dimension of this vector space.

The other are the points $s_i \in E$ where the VHS degenerates. To give a first "coordinate" discussion of their contribution it is convenient to use the classical language of period matrices.[25] Denoting by s a local coordinate on S^*, we may give the VHS by a normalized period matrix

$$\Omega(s) = (I\ Z(s)),$$

[25]Subsequently we shall formulate the conclusion in terms of the LMHS, which will be the method used in the higher weight cases where period matrix calculations are less transparent.

where I is the $h^{1,0} \times h^{1,0}$ identity and $Z(s)$ is a holomorphic matrix of the same size and with

$$\begin{cases} Z(s) = {}^tZ(s), \\ \operatorname{Im} Z(s) > 0. \end{cases}$$

The row vectors in $\Omega(s)$ give a basis for $H_s^{1,0}$, and by subtracting linear combinations of vectors in $H_s^{1,0}$ we may identify $H_s^{0,1}$ with vectors

$$(\underbrace{0, \ldots, 0}_{h^{1,0}}, \underbrace{*, \ldots, *}_{h^{1,0}}) \cong \mathbb{C}^{h^{1,0}}.$$

When this is done, in terms of the local coordinate s the Kodaira–Spencer mapping is given by

$$\theta_s = Z'(s)$$

viewed as a symmetric mapping

$$\mathcal{O}_s^{h^{1,0}} \to \mathcal{O}_s^{h^{1,0}}.$$

We then have that

$$\dim(\ker \theta_s) \geqslant h_0^{1,0}$$

and the s_α are the points where strict inequality holds.

Around a singular point s_0 we have

$$Z(s) = \left(\frac{\log s}{2\pi\sqrt{-1}}\right) B + H(s),$$

where $H(s)$ is a holomorphic matrix, and where B is an integral matrix satisfying

$$\begin{cases} B = {}^tB, \\ B \geqslant 0. \end{cases}$$

The log of the monodromy matrix is given by

$$N = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}.$$

We claim that, using the identification

$$\Omega^1_{S,s_0}(\log s_0) \cong \mathcal{O}_{S,s_0}\left[\frac{ds}{s}\right] \tag{VI.A.6}$$

given by the choice of coordinate s, there is a commutative diagram

$$\begin{tikzcd} \mathcal{H}_e^{1,0} \otimes \mathcal{O}_{S,s_0} \arrow[r, "\theta_{s_0}"] \arrow[d, phantom, "\wr\|"] & \check{\mathcal{H}}_e^{1,0} \otimes \mathcal{O}_{S,s_0} \arrow[d, phantom, "\wr\|"] \\ \mathcal{O}_{S,s_0}^{h^{1,0}} \arrow[r, "B"] & \mathcal{O}_{S,s_0}^{h^{1,0}}. \end{tikzcd} \tag{VI.A.7}$$

This is just a reformulation of the facts that θ is induced by the Gauss–Manin connection ∇ and that

$$\operatorname{Res}_{s_0}(\nabla) = N.$$

We observe that

$$\theta_{s_0} = 0 \quad \text{on } \mathcal{H}^{1,0}_{0,e;s_0}$$

so that

$$H^{1,0}_{0,e;s_0} \subseteq \ker B.$$

To give the formula for the contribution of s_0 to δ, we consider the induced map

$$\mathcal{H}^{1,0}_e/\mathcal{H}^{1,0}_{0,e} \xrightarrow{\bar\theta} (\mathcal{H}^{1,0}_e/\mathcal{H}^{1,0}_{0,e})\check{} \otimes \Omega^1_S$$

as a symmetric, meromorphic map between equal rank vector bundles. Writing

$$B = \begin{pmatrix} B' & 0 \\ 0 & 0 \end{pmatrix},$$

where $B' = {}^tB^1 > 0$, we may non-canonically write

$$\mathcal{H}^{1,0}_e = \mathcal{F} \oplus \mathcal{G};$$

upon a choice of a local coordinate s and frames for $\mathcal{F}$ and $\mathcal{G}$, we have

$$\bar\theta = \begin{pmatrix} \frac{B'}{s} + \cdots & * \\ 0 & A(s) \end{pmatrix},$$

where $A(s)$ is a holomorphic matrix and the terms labelled "$\cdots$" and "$*$" are also holomorphic. We may assume that

$$\mathcal{H}^{1,0}_e \subseteq \mathcal{G}$$

and may further decompose

$$\mathcal{G} = \mathcal{G}' \oplus \mathcal{G}'',$$

where $\mathcal{G}'' = \mathcal{H}^{1,0}_{0,e}$ and

$$A(s) = \begin{pmatrix} A'(s) & * \\ 0 & 0 \end{pmatrix}$$

with $\det A'(s) \not\equiv 0$. Then rank $B' = \operatorname{rank} \overline{B}$ and

$$\nu_{s_0}(\det \bar\theta) = -\operatorname{rank} B' + \nu_{s_0}(\det A').$$

From this we conclude that:

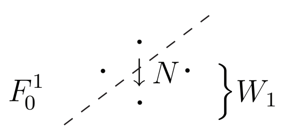
(VI.A.8) *The contribution to δ of the singular point s_0 is given by*

$$\operatorname{rank} \overline{B} - \nu_{s_0}(\det A').$$

Thus, δ goes up with the "size" of monodromy and goes down with the failure of the Kodaira–Spencer to be injective on the "finite part" of $\mathcal{H}^{1,0}_e$.

The way to think about the $-\nu_{s_0}(A')$ term is this: Imagine a VHS that is a direct sum of which one term has a singularity at s_0 but the other does not. The degree of $\mathcal{H}^{1,0}_e$ is additive, and the failure of the regular Kodaira–Spencer map on the second factor will contribute to decreasing the over all degree, just as it would if we were at a non-singular point for the whole VHS.

To put this in the form we shall use below for the cases $n = 2$, 3 we picture the Hodge diamond for the LMHS as



Here, $\hat{H}^1 =: \mathrm{Gr}_1$ (LMHS) is a pure Hodge structure of weight 1.[26] Now the fibre $H^{1,0}_{0,s_0}$ of $\mathcal{H}^{1,0}_0$ at s_0 is a subspace of $\ker N \cap F^1_{e,s_0}$. Thus

$$H^1_{0,s_0} \subset W_1(\mathrm{LMHS})$$

which implies that H^1_{0,s_0} projects isomorphically to a subspace, denoted by $H^{1,0}_0$, in $\hat{H}^{1,0}$. It follows that for the induced map

$$\overline{N}\colon F^1_0 \to H_0/F^1_0$$

we have $\operatorname{rank} \overline{N} = \operatorname{rank} \overline{B}$ and

$$\begin{cases} \quad \operatorname{rank}(\overline{N}) = \hat{h}^{1,1}, \\ \dim(\ker \overline{N}) = \hat{h}^{1,0} - h^{1,0}_0 = -\dim(\operatorname{coker} \overline{N}) = \nu_{s_0}(A'). \end{cases}$$

We note that the contribution of s_0 to δ is maximized when $\hat{h}^{1,0} = h^{1,0}_0$, which is equivalent to the LMHS being of Hodge–Tate type.

To put this in more intrinsic form, we use the limiting mixed Hodge structure (LMHS) as given in [Sch] and trivialization (VI.A.6) to have the equivalent form

$$\mathcal{F}^1_{e,s_0}/\mathcal{F}^1_{0,e,s_0} \xrightarrow{\overline{N}} (\mathcal{F}^1_{e,s_0}/\mathcal{F}^1_{0,e,s_0})\check{}. \tag{VI.A.9}$$

Then from (IV.A.8) and the above we conclude that *the contribution to* $\deg \mathcal{B}$ *of the singular point* s_0 *is* $(h^{1,0} - h^{1,0}_0) \times (\operatorname{rank} \overline{N} - \nu_{s_0}(\det A'))$.

Putting everything together we obtain the result

$$\delta = \frac{1}{2}(h^{1,0} - h^{1,0}_0)(2g-2) + \delta_0 - \frac{1}{2}\sum_{s\in S} \nu_s(\bar{\theta}), \tag{VI.A.10}$$

where $\nu_s(\bar{\theta})$ has the interpretation explained above with $\bar{\theta}$ is considered as a meromorphic map from $\mathcal{H}^{1,0}_e/\mathcal{H}^{1,0}_{0,e}$ to $(\mathcal{H}^{1,0}_e\check{}/\mathcal{H}^{1,0}_e) \otimes \Omega^1_S$.

An alternativee way of writing (VI.A.10), which perhaps makes more clear the contribution of the singular points to δ, is

$$\begin{aligned} \delta = \frac{1}{2}\left[(h^{1,0} - h^{1,0}_0)(2g-2) + \sum_i \dim(\operatorname{Im} \overline{N}_i)\right] \\ - \left[(-\delta_0) + \frac{1}{2}\left(\sum_{s\in S^*} \nu_s(\bar{\theta}) + \sum_i \nu_{s_i}(A'_i)\right)\right], \end{aligned} \tag{VI.A.11}$$

where A'_i is A' as above around s_i. In this formula all the terms (except $2g-2$ in case $g=0$) are non-negative.

We next claim that

$$\delta_0 \leqslant 0 \textit{ with equality if and only if } \mathcal{H}^{1,0}_{e,0} \textit{ is a flat subbundle of } \mathcal{H}_e.$$

This is because first of all

$$\Theta_{\mathcal{H}^{1,0}_e}\big|_{\mathcal{H}^{1,0}_{0,e}} = 0$$

[26]In general, for a LMHS we recall our notation $\hat{H}^r = \bigoplus_{p+q=r} \hat{H}^{p,q}$ for the r^{th} graded piece.

by (II.A.2) and (II.A.3). Next, we recall that the curvature of $\mathcal{H}^{1,0}_{0,e}$ is of the form

$$\Theta_{\mathcal{H}^{1,0}_{0,e}} = A \wedge {}^t\overline{A},$$

where A is a matrix of (1, 0) forms giving the 2$^{\text{nd}}$ fundamental form of $\mathcal{H}^{1,0}_0$ in $\mathcal{H}^{1,0}$, and that the Chern form is given by

$$c_1\bigl(\Theta_{\mathcal{H}^{1,0}_{0,e}}\bigr) = \frac{\sqrt{-1}}{2\pi}\operatorname{Tr}(A \wedge {}^t\overline{A}) \leqslant 0$$

with equality if and only if $A = 0$. Finally,

$$\delta_0 = \deg \mathcal{H}^{1,0}_{0,e} = \int_S c_1\bigl(\Theta_{\mathcal{H}^{1,0}_{0,e}}\bigr) \leqslant 0$$

with equality if and only if $c_1(\Theta_{\mathcal{H}^{1,0}_{0,e}}) = 0$. □

(VI.A.12) **Corollary** (Refined Arakelov inequality). *We have*

$$\delta \leqslant \bigl(h^{1,0} - h^{1,0}_0\bigr)(2g - 2 + N).$$

Moreover, we have equality if and only if the conditions

(i) $\mathcal{H}^{1,0}_{0,e}$ *is a flat subbundle of* $\mathcal{H}_e$;

(ii) *the induced Kodaira–Spencer maps* $\bar{\theta}_s$ *are injective for all* $s \in S^*$; *and*

(iii) *the induced monodromy logarithms* $\overline{N}_i$ *are all of maximal rank.*

This has the following geometric interpretation: Condition (i) is equivalent to the VHS having a fixed part $\mathcal{H}^{1,0}_{0,e} \oplus \overline{\mathcal{H}}^{1,0}_{0,e}$. Condition (ii) is self-explanatory. As noted above, condition (iii) means that for the variable part of the VHS the LMHS at each singular point is of Hodge–Tate type.

Example. In order to illustrate the general result and because it is a nice story in its own right, we shall work out (VI.A.10) in the elliptic curve case. Thus assume we have a relatively minimal, non-isotrivial elliptic surface

$$f\colon X \to S$$

with unipotent monodromies whose logarithms are conjugate to

$$N_i = \begin{pmatrix} 0 & m_i \\ 0 & 0 \end{pmatrix}, \qquad m_i \in \mathbb{Z}^+,$$

at points $s_1, \ldots, s_N \in S$, where X_{s_i} is singular of type I_{m_i}. We then have $\mathcal{H}^{1,0}_0 = 0$ and, recalling our notation $E = s_1 + \cdots + s_N$, there is an exact sequence

$$0 \to \mathcal{H}^{1,0}_e \xrightarrow{\theta} \check{\mathcal{H}}^{1,0}_e \otimes \Omega^1_S(\log E) \to \mathcal{B} \to 0, \tag{VI.A.13}$$

where θ is induced by the Gauss–Manin connection and $\mathcal{B}$ is a skyscraper sheaf.

Lemma. $\mathcal{B}$ *is supported at the points* $s_i \in S$.

Proof. Because the first two terms in (VI.A.13) are line bundles, it will suffice to show that the map on fibres at each s_i is injective. Choosing a local coordinate s centered at s_i so as to have an isomorphism

$$\Omega^1_S(\log E)_{s_i} \cong \mathcal{O}_{S,s_i} \otimes \left(\frac{ds}{s}\right)$$

the induced map on fibres in the map

$$F^1_{e,s_i} \xrightarrow{\overline{N}_i} H_{e,s_i}/F^1_{e,s_i}$$

induced by N_i in the LMHS. For s near to s_i, choosing a canonical basis δ, γ for $H_1(X_s, \mathbb{Z})$ with

$$\begin{aligned} T\delta &= \delta, \\ T\gamma &= \gamma + m_i\delta, \end{aligned}$$

we have a normalized generator $\omega(s)$ for $\mathcal{H}^{1,0}_{e,s_i}$ whose period vector is

$$\left[1,\, m_i \frac{\log s}{2\pi\sqrt{-1}} + h(s)\right],$$

where $h(s)$ is holomorphic. In the dual basis δ^*, γ^*, viewed as a multi-valued frame for $\mathcal{H}_e$ away from s_i, we have

$$\omega(s) = \delta^* + \left(m_i \frac{\log s}{2\pi\sqrt{-1}} + h(s)\right)\gamma^*.$$

Then

$$\nabla\omega(s) = \left(\frac{m_i}{2\pi\sqrt{-1}}\left(\frac{ds}{s}\right) + h'(s)ds\right)\gamma^*.$$

Since $N_i = \operatorname{Res}_{s_i}\nabla$ we find that

$$N_i(\omega(s_i)) = [0,\, m_i],$$

where

$$\begin{aligned} &H_{e,s_i} \cong \mathbb{C}^2 = \{[x,\, y]\} \\ &\cup \\ &F^1_{e,s_i} \cong \mathbb{C} \;= \{[x,\, 0]\}. \end{aligned}$$

Thus $H_{e,s_i}/F^1_{e,s_i} \cong \mathbb{C}$ with

$$\overline{N}_i(\omega(s_i)) = m_i. \qquad \square$$

We now have two formulas for the degree δ of $\mathcal{H}^{1,0}_e$, namely

$$\begin{cases} \text{(a)} \quad \delta = \dfrac{1}{12}\displaystyle\sum_{i=1}^{N} m_i, \\ \text{(b)} \quad \delta = g - 1 + \dfrac{1}{2}\Big(N - \displaystyle\sum_{s\in S^*} \nu_s(\theta)\Big). \end{cases} \tag{VI.A.14}$$

We also know that

$$N \leqslant \sum_i m_i$$

with equality if and only if all $m_i = 1$. It is of interest to compare these. For this we set

$$\overline{\mathcal{M}}_1 = \mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}^*,$$

where $\mathcal{H}$ is the upper-half-plane and $\mathcal{H}^* = \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$. Then we have maps

$$S \xrightarrow{\tau} \overline{\mathcal{M}}_1 \xrightarrow{j} \mathbb{P}^1,$$

where τ is the Torelli or period map and j is the usual j-function. As in section III we set

$$j_S = j \circ \tau.$$

We note that

$$j_S^{-1}(\infty) = \sum_i m_i s_i$$

and, as earlier,

$$\deg j_S = \sum_i m_i = 12\delta.$$

Let R be the ramification divisor of j_S.[27] Then we may write

$$R = R_\infty + R_0 + R_1 + R_{\text{other}},$$

where

$$R_z = \text{ramification divisor of } j_S \text{ over } z \in \mathbb{P}^1.$$

Thus

$$R_\infty = \sum_i (m_i - 1)s_i$$

while R_0 and R_1 give the ramification of j_S at points $s \in S^*$ where X_s has complex multiplication. Setting

$$\begin{cases} r = \deg R, \\ r_z = \deg R_z \end{cases}$$

the Riemann–Hurwitz formula for j_S

$$\begin{aligned} 2g - 2 &= -2\deg j_S + r \\ &= -2\sum_i m_i + \sum_i (m_i - 1) + r_0 + r_1 + r_{\text{other}} \\ &= -\sum_i m_i - N + r_0 + r_1 + r_{\text{other}}. \end{aligned}$$

Substituting in (VI.A.14)(b) and using (VI.A.14)(b) we obtain

$$\delta = \frac{1}{14}\left[r_0 + r_1 + r_{\text{other}} - \sum_{s \in S^*} \nu_s(\theta) \right]. \tag{VI.A.15}$$

Recalling that θ_s is the differential of the period map, and assuming for simplicity that the period map is not ramified on $j_S^{-1}(0)$ and $j_S^{-1}(1)$ we see that we have

$$r_{\text{other}} = \sum_{s \in S^*} \nu_s(\theta)$$

so that (VI.A.15) reduces to

$$r_0 + r_1 = 14\delta.$$

[27] For $z \to z^k$ we are using the terminology: the *ramification degree* is k while the contribution to the *ramification divisor* is $k - 1$.

If we are in a "modular" case, and there are N_1 order 4 and N_0 order 6 complex multiplication curves in a family, then the Riemann–Hurwitz becomes

$$2g-2=-\sum m_i - N_\infty + 2N_0 + N_1$$

since e.g. we know that r_0 is a sum of $(3-1)$'s. One can go even further: since $\sum m_i = \deg(j_S^{-1}(\infty)) = \deg(j_S)$, we have $12\delta = \deg(j_S) = \deg(j_S^{-1}(0)) = 3N_0 \implies N_0 = 4\delta$ and similarly $N_1 = 6\delta$. So in the modular case, one may first use the $\{m_i\}$ to compute δ, then we have simply

$$2g-2 = -12\delta - N_\infty + 2(4\delta) + (6\delta) = 2\delta - N_\infty$$

or

$$\delta = \frac{1}{2}\left((2g-2)+N\right),$$

so that the equality holds in the Arakelov inequality.

It is interesting to look at a few modular examples. Consider $X_1(5)$ and $X(3)$: these have singular fibres $I_5/I_1/I_1/I_5$ and $I_3/I_3/I_3/I_3$ respectively. In both cases $\delta = 1$ and $N_\infty = 4$, so $g = 0$. More generally, $X(n)$ has $N_\infty = \frac{n^2}{2}\prod_{p|n}(1-\frac{1}{p^2})$ (product is over primes p dividing n with $1 < p \leqslant n$) singular fibres, each of type I_n. This yields

$$\begin{aligned}\delta &= \frac{n^3}{24}\prod_{p|n}\left(1-\frac{1}{p^2}\right),\\ 2g-2 = 2\delta - N_\infty &= \frac{n^2(n-6)}{12}\prod_{p|n}\left(1-\frac{1}{p^2}\right),\\ g &= 1+\frac{n^2(n-6)}{24}\prod_{p|n}\left(1-\frac{1}{p^2}\right).\end{aligned}$$

If n is prime, i.e., for $X(p)$, this gives $g_S = 1+\frac{p^2(p-6)}{24}\left(\frac{p^2-1}{p^2}\right) = 1+\frac{1}{24}(p-6)(p^2-1)$. So we recover from this (and the more general formula for composite n) the well-known fact that $S = Y(3)$, $Y(4)$, $Y(5)$ are the only genus 0 $Y(n)$'s. One can also do the genus of $Y_1(p)$, $p \geqslant 3$, the problem with 3 or composite n being that the fibres are not all semistable. This is the quotient induced by adding the matrix

$$\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$$

to $\Gamma(p) := \ker\{\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/p)\}$, which is of order p. This is ramified (to order p) at exactly $\frac{p-1}{2}$ of the I_p's, and over no other points. So Riemann–Hurwitz says

$$\begin{aligned}\frac{(p-6)(p-1)(p+1)}{12} &= 2g_{Y(p)} - 2 = p(2g_{Y_1(p)}-2) + \frac{p-1}{2}(p-1),\\ \frac{p-1}{12}\{(p-6)(p+1)-6(p-1)\} &= 2pg_{Y_1(p)} - 2p,\\ g_{Y_1(p)} &= 1+\frac{(p-1)(p-11)}{24}.\end{aligned}$$

Again, this is good only for primes bigger than (not including) 3. What is true is that $g_{Y_1(n)}$ is zero for $n = 3, \ldots, 10$ and 12.

Example: Pencils of plane curves. An explicit example where the quantities appearing in the Arakelov equality can be explicitly computed is given by a general pencil $|X_s|_{s\in\mathbb{P}^1}$ of plane curves of degree d. Blowing up the base locus we obtain a minimal[28] fibration

$$f\colon X \to \mathbb{P}^1,$$

where $X = (\mathbb{P}^2$ blown up at d^2 points) is a smooth surface. There are N singular fibres X_{s_i} which are irreducible plane curves with a single node. We will show that

$$\begin{cases} \text{(i)} \quad \deg \mathcal{H}_e^{1,0} = h^{1,0} = \dfrac{(d-1)(d-2)}{2}, \\ \text{(ii)} \quad N = 3(d-1)^2, \\ \text{(iii)} \quad \text{for the Kodaira–Spencer maps } \theta_s\colon \mathcal{H}_s^{1,0} \to \mathcal{H}_s^{0,1}\otimes\Omega^1_{S,s} \text{ on} \\ \qquad\ \ \text{stalks away from the singular fibres } \det\theta_s \text{ has } d^2-1 \text{ zeroes.} \end{cases} \tag{VI.A.16}$$

It is reasonable to expect that the zeroes are simple; i.e.

$$\begin{gathered} \dim(\operatorname{coker}\theta_s) = 1 \ \text{ at } \ d^2-1 \\ \text{distinct points of } S^* = S\backslash\{s_1,\, \dots,\, s_N\} \end{gathered}$$

but we shall not attempt to show this.

We consider a projective plane $\mathbb{P}^2 = \mathbb{P}(\check{V})$, where $V \cong \mathbb{C}^3$. Denoting as above by $\mathbb{P}^1$ the parameter space of the pencil we claim that

$$\mathcal{H}_e^{1,0} \cong V^{(d-3)} \otimes \mathcal{O}_{\mathbb{P}^1}(1).$$

To see this, we let $F,\, G \in V^{(d)}$ span the pencil and

$$\Omega = \sum_{i=0}^{2} (-1)^i x^i dx^0 \wedge \cdots \wedge \hat{dx}^i \wedge \cdots \wedge dx^2,$$

where $x^0,\, x^1,\, x^2 \in V$ are coordinates. Then for $H \in V^{(d-3)}$ the Poincaré residue

$$\operatorname{Res}_{X_s}\left(\frac{H\Omega}{F+sG}\right) \in \mathcal{H}^{1,0}_{e,s}$$

establishes the isomorphism

$$V^{(d-3)} \otimes \mathcal{O}_{\mathbb{P}^1}(1)_s \cong \mathcal{H}^{1,0}_{e,s}. \tag{VI.A.17}$$

This gives (i) in (VI.A.16).

Next, if $[s_0,\, s_1]$ are homogeneous coordinates in $\mathbb{P}^1$, the singular points of the fibres are given by the solutions to

$$\sigma_i =: s_0 F_{x_i}(x) + s_1 G_{x_i}(x) = 0, \qquad i = 0,\, 1,\, 2,$$

in $\mathbb{P}^1 \times \mathbb{P}^2$.[29] Now the

$$\sigma_i \in H^0\big(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^1}(1) \otimes \mathcal{O}_{\mathbb{P}^2}(d-1))\big)$$

[28]Recall from Section V.G that minimality means that there are no -1 curves in the fibres.

[29]Any solution to these equations sastisfies $s_0F(x) + s_1G(x) = 0$ by the Euler relation.

so that

$$\deg\{\sigma_0 = \sigma_1 = \sigma_2 = 0\} = 3(d-1)^2.$$

This establishes (ii).

Using (VI.A.17) and setting $E = s_1 + \cdots + s_N$ we have for the Kodaira–Spencer maps θ induced by the Gauss–Manin connection a commutative diagram

$$\left\{\begin{tikzcd} \mathcal{H}_e^{1,0} \arrow[r, "\theta"] \arrow[d, phantom, "\wr\Vert"] & \check{\mathcal{H}}_e^{1,0} \otimes \Omega^1_{\mathbb{P}^1}(\log E) \arrow[d, phantom, "\wr\Vert"] \\ V^{(d-3)} \otimes \mathcal{O}_{\mathbb{P}^1}(1) \arrow[r, "\bar\theta"] & \check{V}^{(d-3)} \otimes \mathcal{O}_{\mathbb{P}^1}(N-2) \end{tikzcd}\right. \tag{VI.A.18}$$

which defines the section

$$\bar\theta \in \operatorname{Hom}\big(V^{(d-3)}, \check{V}^{(d-3)}\big) \otimes \mathcal{O}_{\mathbb{P}^1}(N-3).$$

In fact, it is easy to see that

$$\bar\theta \in \operatorname{Sym}^2\big(\check{V}^{(d-3)}\big) \otimes \mathcal{O}_{\mathbb{P}^1}(N-3)$$

and we suspect that

$$\det\theta \not\equiv 0.$$

If so, then

$$\deg(\det\theta) = \binom{d-1}{2}(N-3)+1. \tag{VI.A.19}$$

Now, and this is the key point, at each of the $N = 3(d-1)^2$ nodal curves, the induced map on the fibres of the vector bundles in (VI.A.18) has rank one. Hence, $\det\theta$ vanishes to order $\binom{d-1}{2} - 1$ at these N points. Thus

$$\begin{aligned} (\#\text{ finite zeroes of }\det\theta) &= \binom{d-1}{2}(N-3)+1-\left(\binom{d-1}{2}-1\right)N \\ &= d^2-1. \end{aligned}$$

VI.B. *General formulations in the absence of singularities*

We shall formulate the general setting for computing the degrees of the Hodge bundles in a VHS of weight n without degeneracies and where the Kodaira–Spencer maps have constant rank. Then in the next two sections we shall give the correction terms in the cases $n = 2$ and 3, following the method of step (iv) in the preceding section, at points in S^* where the Kodaira–Spencer map drops rank, and using the analysis of the LMHS to express the contributions of degeneracies to these degrees.

We begin by recalling/establishing some notations:

- $\mathcal{H}^{n-p,p} \xrightarrow{\theta_p} \mathcal{H}^{n-p-1,p+1} \otimes \Omega^1_S$,
- $\mathcal{H}_0^{n-p,p} = \ker\theta_p$, $\quad$ rank $= \mathcal{H}_0^{n-p,p} = h_0^{n-p,p}$,
- $\mathcal{H}_\#^{n-p-1,p+1} = \operatorname{coker}\theta_p$, $\quad$ rank $= \mathcal{H}_\#^{n-p-1,p+1} = h_\#^{n-p-1,p+1}$,
- $\Theta_{\mathcal{H}^{n-p,p}} = -\theta_{p-1} \wedge {}^t\bar\theta_{p-1} + {}^t\theta_p \wedge \bar\theta_p$,
- $\Theta_{\mathcal{H}^{n-p,p}}\big|_{\mathcal{H}_0^{n-p,p}} = -\theta_{p-1} \wedge {}^t\bar\theta_{p-1}$,

- $B_p \in A^{1,0}\big(\mathrm{Hom}(\mathcal{H}_0^{n-p,p}, \mathcal{H}_\#^{n-p,p})\big)$

is the matrix of (1, 0) forms giving the 2nd fundamental form of $\mathcal{H}_0^{n-p,p}$ in $\mathcal{H}^{n-p,p}$; here we have used the important consequence

$$\check{\mathcal{H}}_\#^{n-p,p} \otimes \Omega_S^1 \cong \mathcal{H}_0^{p,n-p} \tag{VI.B.1}$$

of the dualities resulting from the polarization and the properties of the Kodaira–Spencer mappings.

We also note the relations

- $$\begin{cases} h^{n-p,p} - h_0^{n-p,p} = h^{n-p-1,p+1} - h_\#^{n-p-1,p+1}, \\ h_\#^{n-p-1,p+1} = h^{n-p-1,p+1} + h_0^{n-p,p} - h^{n-p,p} \end{cases}$$

and the exact sequences

- $$0 \to \mathcal{H}_0^{n-p,p} \to \mathcal{H}^{n-p,p} \xrightarrow{\theta_p} \mathcal{H}^{n-p-1,p+1} \otimes \Omega_S^1 \to \mathcal{H}_\#^{n-p-1,p+1} \to 0; \tag{VI.B.2}$$

$$\Theta_{\mathcal{H}_0^{n-p,p}} = -\theta_{p-1} \wedge {}^t\bar{\theta}_{p-1} - B_p \wedge {}^t\overline{B}_p. \tag{VI.B.3}$$

We set

- $$\alpha_p = \frac{\sqrt{-1}}{2\pi}\,\mathrm{Tr}\big({}^t\theta_p \wedge \bar{\theta}_p\big|_{\mathcal{H}_0^{n-p,p}}\big) \geqslant 0,$$
- $$\beta_p = \frac{\sqrt{-1}}{2\pi}\,\mathrm{Tr}\big(B_p \wedge {}^t\overline{B}_p\big) \geqslant 0,$$

where we note that

$$\mathrm{Tr}(\theta_{p-1} \wedge {}^t\bar{\theta}_{p-1}) = \mathrm{Tr}({}^t\theta_{p-1} \wedge \bar{\theta}_{p-1}).$$

For Chern forms and degrees we set

- $$c_1(\Theta_{\mathcal{H}^{n-p,p}}) = -\alpha_{p-1} + \alpha_p, \quad \int_S \alpha_p = a_p, \tag{VI.B.4}$$
- $$c_1(\Theta_{\mathcal{H}_0^{n-p,p}}) = -\alpha_{p-1} - \beta_p, \quad \int_S \beta_p = b_p.$$

We note that

(VI.B.5) $\deg \mathcal{H}_0^{n-p,p} \leqslant 0$ *with equality if and only if* $a_{p-1} = b_p = 0$. *In this case,* $\nabla \mathcal{H}_0^{n-p,p} = 0$ *and* $\mathcal{H}_0^{n-p,p}$ *is a summand in a real sub-VHS.*

The reason is that from (VI.B.3) and its consequences

$$\deg \mathcal{H}_0^{n-p,p} = 0 \implies \alpha_{p-1} = \beta_p = 0$$

which implies that $B_p = 0$ and also

$$\begin{cases} \theta_p\big|_{\mathcal{H}_0^{n-p,p}} = 0, \\ {}^t\theta_{p-1}\big|_{\mathcal{H}_0^{n-p,p}} = 0 \end{cases}$$
$$\implies \nabla\big|_{\mathcal{H}_0^{n-p,p}} = 0.$$

Then $\mathcal{H}_0^{n-p,p} \oplus \overline{\mathcal{H}}_0^{n-p,p}$ is a flat subbundle of $\mathcal{H}$ that at each point gives a real sub-Hodge structure. We also set

- $$c_1(\Omega^1_S) =: \omega, \qquad \int_S \omega = 2g-2$$

From (VI.B.1) we have

$$c_1\big(\mathcal{H}^{n-p-1,p+1}_{\#}\big) = h^{n-p-1,p+1}_{\#}\omega - [\alpha_{n-p} + \beta_{n-p+1}].$$

Setting

$$\begin{cases} \delta_p = \deg \mathcal{H}^{n-p,p}, \\ \delta^0_p = \deg \mathcal{H}^{n-p,p}_0 \end{cases}$$

and using (VI.B.1) and (VI.B.4) gives linear relations involving δ_p, δ^0_p, q and the a_q's and b_q's. Using the non-negativity of the a_q's and b_q's leads to Arakelov inequalities in this case (cf. [Pe2]).

VI.C. *The weight two case*

We shall show that

> *For a weight two VHS without degeneracies and where the Kodaira–Spencer maps θ_0, θ_1 have constant rank, we have for $\delta = \delta_0 = \deg \mathcal{H}^{2,0}$*
> $$\delta = (h^{2,0} - h^{2,0}_0)(2g-2) - (a_0 + b_0 + b_1). \tag{VI.C.1}$$
> *Thus $\delta \leqslant (h^{2,0} - h^{2,0}_0)(2g-2)$, with equality holding if and only if*
> $$\mathcal{H}^{2,0}_0 + \overline{\mathcal{H}}^{2,0}_0 + \mathcal{H}^{1,1}_0 \tag{VI.C.2}$$
> *is a flat subbundle of $\mathcal{H}$.*

Proof. As a general observation, if $n = 2m$ is even, then from (II.A.4) and the results of Schmid discussed just below there, we have

$$\deg \mathcal{H}^{m,m}_e = 0.$$

The sequences (VI.B.2) are, using (VI.B.3)

$$\begin{aligned} &0 \to \mathcal{H}^{2,0}_0 \to \mathcal{H}^{2,0} \to \mathcal{H}^{1,1} \otimes \Omega^1_S \xrightarrow{\theta_1} \check{\mathcal{H}}^{1,1}_0 \otimes \Omega^1_S \to 0, \\ &0 \to \mathcal{H}^{1,1}_0 \to \mathcal{H}^{1,1} \to \mathcal{H}^{0,2} \otimes \Omega^1_S \longrightarrow \check{\mathcal{H}}^{2,0}_0 \otimes \Omega^1_S \to 0. \end{aligned} \tag{VI.C.3}$$

The first sequence gives

$$\deg \mathcal{H}^{2,0} = \deg \mathcal{H}^{2,0}_0 + h^{1,1}(2g-2) - h^{1,1}_0(2g-2) + \deg \mathcal{H}^{1,1}_0. \tag{VI.C.4}$$

From the second sequence

$$h^{1,1}_0 = h^{1,1} - h^{2,0} + h^{2,0}_0,$$

and from (VI.B.4)

$$\begin{cases} \deg \mathcal{H}^{1,1}_0 = -a_0 - b_1, \\ \deg \mathcal{H}^{2,0}_0 = -b_0. \end{cases}$$

Substituting these into (VI.C.4) gives the result. □

Remark. The result also follows from taking degrees in the the second sequence.

Next, as in step (iv) in Section VI.A we allow the rank of the mapping from (VI.C.3)

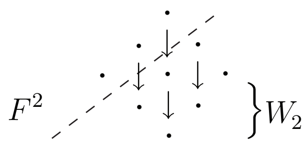
$$\bar{\theta}\colon \mathcal{H}^{2,0}/\mathcal{H}_0^{2,0} \to \mathcal{H}^{1,1} \otimes \Omega_S^1/\ker\theta_1$$

induced by θ_0 to drop at finitely many points. The result is

$$\delta = (h^{2,0} - h_0^{2,0})(2g-2) - \left[\sum_{s\in S} \dim(\operatorname{coker}\bar{\theta}_s) + (a_0 + b_0 + b_1)\right]. \tag{VI.C.5}$$

This gives the same inequality as before, with equality holding if and only if (VI.C.2) is a flat subbundle of $\mathcal{H}$ and all the induced maps θ_s are fiberwise injective.

When there is a degeneracy of the VHS at a point s_0, we draw the Hodge diamond of the LMHS as



and set $\hat{H}^q = \operatorname{Gr}_q$ (LMHS). As in the $n=1$ case we have for the fibre $H_{0,s_0}^{2,0}$ of $\mathcal{H}_0^{2,0}$ at s_0 that

$$H_{0,s_0}^{2,0} \subseteq W_2(\text{LMHS}).$$

From the above we see that for the mapping $\overline{N}$ induced by $\operatorname{Res}_{s_0}(\nabla)$

$$\mathcal{H}_{e,s_0}^{2,0}/\mathcal{H}_{0,s_0}^{2,0} \xrightarrow{\overline{N}} \mathcal{H}_{e,s_0}^{1,1}/\ker\bar{\theta}_1$$

we have

$$\begin{cases} \operatorname{rank}\overline{N} = \hat{h}^{2,2} + \hat{h}^{2,1}, \\ \dim(\ker\overline{N}) = \hat{h}^{2,0} - h_0^{2,0} = \dim(\operatorname{coker}\overline{N}). \end{cases}$$

Again the contribution of the singular point to δ is maximized when $\hat{h}^{2,0} = h_0^{2,0}$ in which case the contribution is $h^{2,0} - h_0^{2,0}$. The formula (VI.C.5) is

$$\boxed{\begin{aligned} \text{(VI.C.6)}\quad \delta = {} & (h^{2,0} - h_0^{2,0})(2g-2+N) \\ & - \left[\sum_{s\in S^*} \nu_s(\det\theta) + (a_0+b_0+b_1) + \sum_i (\hat{h}_i^{2,0} - h_i^{2,0})\right]. \end{aligned}}$$

From this we draw the conclusion:

The degree δ of $\mathcal{H}_e^{2,0}$ is maximized when

(i) *the induced Kodaira–Spencer map $\bar{\theta}_s$ is injective for all $s \in S^*$;*
(ii) *$\mathcal{H}_0^{2,0} \oplus \mathcal{H}_0^{1,1} \oplus \overline{\mathcal{H}}_0^{0,2}$ is a flat subbundle of $\mathcal{H}_e$; and*
(iii) *for each singular point s_i, we have*

$$\hat{h}_i^{2,0} = h_0^{2,0}.$$

If $\mathcal{H}_0^{2,0} = 0$, which might be thought of as the generic case, condition (iii) is equivalent to $\hat{h}_i^{2,0} = 0$. In particular, it is satisfied if the LMHS is of Hodge–Tate type (but not conversely).

VI.D. *The weight three case*

The objective here is to compute the two quantities

$$\begin{cases} \delta = \deg \mathcal{H}_e^{3,0} = \deg \mathcal{F}_e^3 \geqslant 0, \\ \delta + \lambda = \deg \mathcal{H}_e^{3,0} + \deg \mathcal{H}_e^{2,1} = \deg \mathcal{F}_e^2 \geqslant 0. \end{cases}$$

As before, we first do this assuming no degeneracies of the VHS and constancy of rank of the Kodaira–Spencer mappings. The sequences (VI.B.2) and duality (VI.B.1) give

(i) $0 \to \mathcal{H}_0^{3,0} \to \mathcal{H}^{3,0} \to \mathcal{H}^{2,1} \otimes \Omega_S^1 \to \check{\mathcal{H}}_0^{1,2} \otimes \Omega_S^1 \to 0;$

(ii) $0 \to \mathcal{H}_0^{2,1} \to \mathcal{H}^{2,1} \to \mathcal{H}^{1,2} \otimes \Omega_S^1 \to \check{\mathcal{H}}_0^{2,1} \otimes \Omega_S^1 \to 0;$

(iii) $0 \to \mathcal{H}_0^{1,2} \to \mathcal{H}^{1,2} \to \mathcal{H}^{0,3} \otimes \Omega_S^1 \to \check{\mathcal{H}}_0^{3,0} \otimes \Omega_S^1 \to 0.$

We note that (iii) is dual to (i) and that (ii) is essentially the same sequence as in the weight one case giving

$$\lambda = \deg \mathcal{H}^{2,1} = \frac{1}{2}(h^{2,1} - h_0^{2,1})(2g-2) - (a_0 + b_1). \tag{VI.D.1}$$

Using this together with (i) we obtain

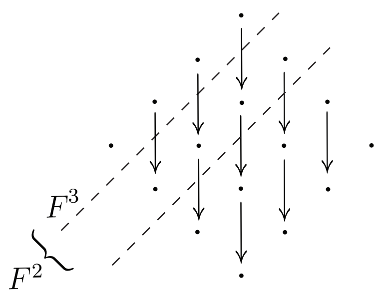
$$\delta = \left(h^{3,0} - h_0^{3,0} + \frac{1}{2}(h^{2,1} - h_0^{2,1}) \right)(2g-2) - (a_0 + b_1 + a_1 + b_2), \tag{VI.D.2}$$

$$\begin{aligned} \delta + \lambda &= \deg \mathcal{H}^{3,0} + \deg \mathcal{H}^{2,1} \\ &= \left(h^{3,0} - h_0^{3,0} + h^{2,1} - h_0^{2,1}\right)(2g-2) - \left(2(a_0+b_1) + a_1 + b_2\right). \end{aligned} \tag{VI.D.3}$$

If we do not assume the Kodaira–Spencer maps have constant rank, then

$$\begin{cases} \text{(VI.D.1) is corrected by subtracting } \sum_s \dim(\operatorname{coker} \bar{\theta}_{1,s}) \\ \text{(VI.D.2) is corrected by subtracting } \sum_s \dim(\operatorname{coker} \bar{\theta}_{0,s}). \end{cases} \tag{VI.D.4}$$

Finally, if we allow degeneracies so that the LMHS has a picture



then

$$\begin{cases} \text{(VI.D.2) is corrected by adding } \hat{h}^{3,3} + \hat{h}^{3,2} + \hat{h}^{3,1} =: \hat{f}^3, \\ \text{(VI.D.3) is corrected by adding } \hat{f}^3 + \hat{h}^{2,3} + \hat{h}^{2,2} + \hat{h}^{3,2} =: \hat{f}^2. \end{cases} \tag{VI.D.5}$$

We note that

$$\begin{cases} \text{(i)} \quad \hat{f}^3 \leqslant h^{3,0} - h_0^{3,0} \text{ with equality } \iff \hat{h}^{3,0} = h_0^{3,0}, \\ \text{(ii)} \quad \hat{f}^2 \leqslant h^{3,0} - h_0^{3,0} + h^{2,1} - h_0^{2,1} \\ \qquad\qquad\qquad \text{with equality } \iff \begin{cases} \hat{h}^{3,0} = h_0^{3,0}, \\ \hat{h}^{2,1} = h_0^{2,1} + \hat{h}^{3,2} \end{cases} \end{cases} \tag{VI.D.6}$$

In summary we have

$$\boxed{\begin{aligned} \delta &= \left(h^{3,0} - h_0^{3,0} + \frac{1}{2}(h^{2,1} - h_0^{2,1})\right)(2g-2+N) - (A_3+B_3+C_3), \qquad \text{(VI.D.7)} \\ \delta + \lambda &= \left(h^{3,0} - h_0^{3,0} + h^{2,1} - h_0^{2,1}\right)(2g-2+N) - (A_2+B_2+C_2), \end{aligned}}$$

where

$$\begin{cases} A_3 = 0 \iff \mathcal{H}_0^{3,0} \oplus \overline{\mathcal{H}}_0^{3,0} \textit{ is a flat subbundle of } \mathcal{H}, \\ B_3 = 0 \iff \textit{the induced Kodaira–Spencer maps } \bar{\theta}_{0,s} \textit{ are fiberwise injective}, \\ C_3 = 0 \iff \textit{the LMHS's all satisfy } \hat{h}^{3,0} = h_0^{3,0}. \end{cases}$$

$$\begin{cases} A_2 = 0 \iff (\mathcal{H}_0^{3,0} \oplus \mathcal{H}_0^{2,1}) \oplus \overline{(\mathcal{H}_0^{3,0} \oplus \mathcal{H}_0^{2,1})} \textit{ is a flab subbundle of } \mathcal{H}, \\ B_2 = 0 \iff \textit{the induced Kodaira–Spencer maps } \bar{\theta}_{0,s}, \bar{\theta}_{1,s} \textit{ are fiberwise injective}, \\ C_2 = 0 \iff \textit{the LMHS's all satisfy } \hat{h}^{3,0} = h_0^{3,0},\ \hat{h}^{2,1} = h_0^{2,1} + \hat{h}^{3,2}. \end{cases}$$

Example. In [GGK1] we have classified the possible unipotent monodromy degenerations of a family of Calabi–Yau threefolds of mirror quintic type. Assuming that the Yukawa coupling is not identically zero (the general case), we have

$$\textit{A singularity of type I contributes} \quad \begin{cases} 3/2 & \text{to } \delta, \\ 4 & \text{to } \delta+\lambda; \end{cases}$$

$$\textit{A singularity of type } II_1 \textit{ contributes} \quad \begin{cases} 0 & \text{to } \delta, \\ 2 & \text{to } \delta+\lambda; \end{cases}$$

$$\textit{A singularity of type } II_2 \textit{ contributes} \quad \begin{cases} 3/2 & \text{to } \delta, \\ 2 & \text{to } \delta+\lambda. \end{cases}$$

Now consider the "physicists' example", namely the VHS of mirror quintic type arising from the family

$$\{\psi(x_0^5 + \cdots + x_4^5) - 5x_0 \cdots x_4 = 0\}/(\mathbb{Z}/5\mathbb{Z})^5. \tag{VI.D.8}$$

Since this family has monodromy of order 5 at ∞, to obtain a family with a locally liftable period mapping we must pull it back under the mapping $t \to t^5$. When this is done the monodromies are unipotent and it is well known (cf. [GGK1] and the references cited there), that

$$\left\{\begin{matrix}\text{there is one singularity of type I} \\ \text{there are five singularities of type II}_1\end{matrix}\right\}.$$

Since $\delta > 0$ must be an integer and $\delta \leqslant 3/2$, we conclude

For the family (VI.D.8) *the period map*

$$\tau\colon \mathbb{P}^1 \to \Gamma\backslash D_\Sigma$$

has degree one mapping to its image.

Here, Γ is a subgroup of finite index in $G_\mathbb{Z}$ containing no elements of order five. This is a generic *global Torelli theorem* for the family (VI.D.8), a result obtained earlier by S. Usui (talk given on May 11, 2008 at the conference on "Hodge Theory, BSIR" and to appear in *Proceedings of the Japan Academy*).

VII. Analysis of the Hodge Structure Associated to a VHS

One theme of this work is to study implications on a global VHS that are present in the geometric case and that go beyond those following from the existing purely Hodge theoretic standard methods. In this section we shall illustrate this in the case of Deligne's theorem. Informally we may state one of the conclusions as follows:

In the geometric case there are global constraints on the Kodaira–Spencer maps that are not present for a general VHS.

VII.A. *Deligne's theorem and a consequence*

We consider a polarized VHS of weight n over a smooth complete variety S. This is given by a local system $\boldsymbol{H}$ on S such that on

$$\mathcal{H} = \boldsymbol{H} \otimes \mathcal{O}_S$$

there is the usual data $\mathcal{F}^p$, ∇, $\mathcal{H}^{p,q}$, where $p+q=n$, etc.[30] We shall omit reference to the polarization, which will be understood to be present. In this section, $\dim S$ is arbitrary; in the next section we shall take S to be a curve.

(VII.A.1) **Theorem (Deligne).** $H^r(S, \boldsymbol{H})$ *has a canonical polarized Hodge structure of weight* $r+n$.

A proof of this result and its extension to the case when S is an affine curve is given in Zucker [Z]. The generalization to a VHS over a quasi projective space of arbitrary dimension is given in [CKS]. In the general case one must use the intersection cohomology associated to the local system $H_\mathbb{Z}$.

In this and the next section we shall discuss the question

(VII.A.2) *What can one say about the Hodge numbers*

$$h^{P.Q}(S, \boldsymbol{H}) =: \dim H^{P,Q}(S, \boldsymbol{H}), \quad P+Q = r+n?$$

[30] In this section we shall use the notation $\mathbf{H}$ for $\mathcal{H}_\mathbb{C}$, as it seems to us more appropriate to the present discussion.

There are some general results, given by (I.A.7), (VII.A.9) and (VII.A.10) below. In the next section we shall see by illustration that these can be considerably refined in the geometric case.

We begin by recalling the idea behind the proof of (VII.A.1). There is a resolution

$$0 \to \boldsymbol{H} \to \mathcal{H} \xrightarrow{\nabla} \Omega^1_S \otimes \mathcal{H} \xrightarrow{\nabla} \Omega^2_S \otimes \mathcal{H} \to \cdots$$

so that

$$H^m(S, \boldsymbol{H}) \cong \mathbb{H}^m(S, \Omega^\bullet_S \otimes \mathcal{H}). \tag{VII.A.3}$$

This step holds for any local system. In the case of a VHS a Hodge filtration is defined on $\Omega^\bullet_S \otimes \mathcal{H}$ by

$$F^p\left(\Omega^r_S \otimes \mathcal{H}\right) = \Omega^r_S \otimes \mathcal{F}^{p-r}.$$

Because of

$$\nabla \mathcal{F}^p \subseteq \Omega^1_S \otimes \mathcal{F}^{p-1}$$

we see that

the $F^p\Omega^\bullet_S \otimes \mathcal{H}$ are subcomplexes of $\Omega^\bullet_S \otimes \mathcal{H}$.

The main step is to show the usual Kähler identities can be extended to a VHS with the Hodge metrics used along the fibres and an arbitrary Kähler metric on the base. This then has the consequence:

(VII.A.4) *The resulting spectral sequence*

$$E_1^{r,s} = \mathbb{H}^{r+s}(S, \operatorname{Gr}^r(\Omega^\bullet_S \otimes \mathcal{H})) \Rightarrow H^{r+s}(S, \boldsymbol{H})$$

degenerates at E_1.

We shall now discuss the implications of this.

For this we shall consider the VHS as giving rise to a family of Hodge bundles underlying a Higgs structure in the sense of Simpson [Sim]. We denote by

$$\Omega^{r-1}_S \otimes \mathcal{H}^{p+1,q-1} \xrightarrow{\theta} \Omega^r_S \otimes \mathcal{H}^{p,q} \xrightarrow{\theta} \Omega^{r+1}_S \otimes \mathcal{H}^{p-1,q+1} \tag{VII.A.5}$$

the Kodaira–Spencer maps induced by ∇.[31] Then

$$\nabla^2 = 0 \implies \theta^2 = 0,$$

and we set

$$\left(\Omega^r_S \otimes \mathcal{H}^{p,q}\right)_\nabla = \text{cohomology of (VII.A.5)}. \tag{VII.A.6}$$

The maps θ also induce

$$H^s\left(\Omega^{r-1}_S \otimes \mathcal{H}^{p+1,q-1}\right) \xrightarrow{\boldsymbol{\theta}} H^s\left(\Omega^r_S \otimes \mathcal{H}^{p,q}\right) \xrightarrow{\boldsymbol{\theta}} H^s\left(\Omega^{r+1}_S \otimes \mathcal{H}^{p-1,q+1}\right) \tag{VII.A.7}$$

and we set

$$\left(H^s\left(\Omega^r_S \otimes \mathcal{H}^{p,q}\right)\right)_\nabla = \text{cohomology of (VII.A.7)}.$$

[31] We shall omit the subscripts on the Kodaira–Spencer maps θ; the subscripts were used in the notations in Section I.A, where it was important to keep track of the p index.

Discussion. The groups (VII.A.6) have been extensively used in the literature to draw geometric consequences — cf. the lectures [GMV] for an exposition up until that time. One may think of them as Koszul type invariants at a generic point of a VHS.

From (VII.A.4) we have the isomorphism

$$\boxed{H^{P,Q}(S,\,\boldsymbol{H}_{\mathbb{C}}) \cong \bigoplus_{\substack{p+r=P\\ q+s=Q\\ p+q=w}} \left(H^s\left(\Omega^r_S \otimes \mathcal{H}^{p,q}\right)\right)_\nabla \quad \text{(VII.A.8)}}$$

expressing the $H^{P,Q}(S,\,\boldsymbol{H}_{\mathbb{C}})$ in terms of the global cohomology of complexes constructed from the Kodaira–Spencer maps. A corollary is

$$h^{P,Q}(S,\,\boldsymbol{H}_{\mathbb{C}}) = \sum_{\substack{p+r=P\\ q+s-Q\\ p+q=w}} \dim\big(H^s(\Omega^r_S \otimes \mathcal{H}^{p,q})\big)_\nabla.$$

We also have the relation of Euler characteristics

$$\sum_Q (-1)^Q h^{P,Q}(S,\,\boldsymbol{H}) = \sum_{\substack{r+s=P\\ p+s-w}} (-1)^{p+q} h^q(\Omega^r_S \otimes \mathcal{H}^{s,p}). \tag{VII.A.9}$$

To give an illustration of the consequence (VII.A.8) of the proof of Deligne's theorem we have for $r = 0,\,1$ that

$$H^{r+n,0}(S,\,\boldsymbol{H}) \cong \ker\big\{H^0(\Omega^r_S \otimes \mathcal{H}^{n,0}) \to H^0(\Omega^{r+1}_S \otimes \mathcal{H}^{n-1,1})\big\}. \tag{VII.A.10}$$

In words:

> *A global section $\varphi \in H^0(\Omega^r_S \otimes \mathcal{H}^{n,0})$ that satisfies $\boldsymbol{\theta}(\varphi) = 0$ then also satisfies*
> $$\nabla\varphi = 0.$$

If we write $\nabla = \nabla' + \theta$, then this is the implication

$$\boldsymbol{\theta}(\varphi) = 0 \implies \nabla'\varphi = 0.$$

Note that for $r = \dim S$ we have simply

$$H^{n+r,0}(S,\,\boldsymbol{H}) \cong H^0\left(\Omega^r_S \otimes \mathcal{H}^{n,0}\right). \tag{VII.A.11}$$

Example. The simplest example is a family

$$f\colon X \to S \tag{VII.A.12}$$

of smooth curves over a curve S. Taking $\boldsymbol{H} = R^1_f\mathbb{Z}$ we have that

$$H^r(S,\,\boldsymbol{H}) \textit{ is a sub-Hodge structure of } H^{r+1}(X). \tag{VII.A.13}$$

We recall that the Leray spectral sequence for (VII.A.12) degenerates at E_2 and we have additively

$$\begin{cases} H^2(X,\,\mathbb{Q}) \cong H^2(S,\,R^0_f\mathbb{Q}) \oplus \underbrace{H^1(S,\,R^1_f\mathbb{Q})} \oplus H^0(S,\,R^2_f\mathbb{Q}), \\ H^1(X,\,\mathbb{Q}) \cong H^1(S,\,R^0_f\mathbb{Q}) \oplus \underbrace{H^0(S,\,R^1_f\mathbb{Q})}. \end{cases}$$

The terms over the braces are the ones given by (VII.A.13) for $r = 0$, 1 respectively. The assertion (VII.A.11) gives

$$H^0(\Omega_X^2) \cong H^0\big(\Omega_S^1 \otimes \mathcal{H}^{1,0}\big).$$

Since

$$\mathcal{H}^{1,0} = R_f^0 \Omega_{X/S}^1$$

we have maps

$$H^0\big(\Omega_S^1 \otimes \mathcal{H}^{1,0}\big) \to H^0\big(X,\, f^{-1}(\Omega_S^1) \otimes \Omega_{X/S}^1\big) \to H^0(\Omega_X^2)$$

and, as may be seen directly, the composite is an isomorphism.

As for $H^0(\Omega_X^1)$, the part arising from $H^1(S,\, R_f^0\mathbb{C})$ is just $f^*(H^0(\Omega_S^1))$. The part over the brace is

(i) $$\ker\big\{H^0(S,\, \mathcal{H}^{1,0}) \xrightarrow{\boldsymbol{\theta}} H^0\big(\Omega_S^1 \otimes \mathcal{H}^{0,1}\big)\big\}.$$

Now $H^0(S,\, R_f^1\mathbb{C})$ is the *fixed part*

(ii) $$\ker\big\{H^0(S,\, \mathcal{H}) \xrightarrow{\nabla} H^0\big(\Omega_S^1 \otimes \mathcal{H}\big)\big\}$$

and (VII.A.10) gives

$$\text{left-hand side of (i)} \subset \text{left-hand side of (ii)}.$$

This is a non-trivial result.

VII.B. *Analysis of Deligne's theorem in a geometric example*

We consider a family of threefolds

$$f\colon X \to S \tag{VII.B.1}$$

with smooth fibres over a complete curve S. Having in mind the usual philosophy that the most interesting cohomology is in the middle dimension, we assume that

$$h^{1,0}(X_\eta) = h^{2,0}(X_\eta) = 0.$$

We set

$$H^2(X_\eta) = H^{1,1}(X_\eta) = V$$

thought of as a constant local system over S of rank v. We also assume that the local system $\boldsymbol{H}$ given by $R_f^3\mathbb{Z}$ (mod torsion) has no fixed part. We shall say that *the VHS given by the* $H^3(X_s)$ *is non-degenerate.*

Setting $h^{p,q} = h^{p,q}(X_\eta)$ we give the

Definition. We shall call

$$\delta,\, \lambda,\, g,\, v,\, h^{3,0},\, h^{2,1} \tag{VII.B.2}$$

the known quantities.

Here we recall our notation

$$\begin{cases} \delta = \deg \mathcal{H}^{3,0}, \\ \lambda = \deg \mathcal{H}^{2,1}. \end{cases}$$

Also, $g = g(S)$ is the genus of S. These quantities are all Hodge theoretic and are invariant under deformations of (VII.B.1). This is in contrast to quantities such as

$$\dim H^i(S, \mathcal{H}^{p,q}), \ \dim H^i(\Omega^1_S \otimes \mathcal{H}^{p,q}), \ \dim(\ker \boldsymbol{\theta}) \ \text{ and } \ \dim(\operatorname{coker} \boldsymbol{\theta}) \tag{VII.B.3}$$

which are not in general invariant under deformation of (VII.B.1).

The main points of this section are the following:

(VII.B.4) *From Deligne's theorem alone, one may determine $h^{4,0}(S, \boldsymbol{H}) = h^{4,0}(X)$ in terms of known quantities. However, to determine $h^{3,1}(X)$ and that part $\hat{h}^{2,2}(X)$ of $h^{2,2}(X)$ coming from the VHS of the $H^4(X_s)$, one needs the quantities* (VII.B.2) *and* (VII.B.3).

(VII.B.5) *In the geometric situation all of the $h^{p,q}(X)$ are expressible in terms of the known quantities* (VII.B.2).

In addition, we shall see that quantities such as

$$\dim(\operatorname{coker} \rho) + \dim(\ker \sigma)$$

in the sequence

$$H^0(\mathcal{H}^{1,3}) \xrightarrow{\rho} H^0(\Omega^1_S \otimes \mathcal{H}^{0,3}) \to H^{1,3}(S, \boldsymbol{H}) \to H^1(\mathcal{H}^{1,2}) \xrightarrow{\sigma} H^1(\Omega^1_S \otimes \mathcal{H}^{0,3}) \tag{VII.B.6}$$

are expressible in terms of the known quantities.

We begin showing that

$$h^{4,0}(X) = \delta - h^{3,0}(g-1). \tag{VII.B.7}$$

Proof. From (VII.A.11)

$$h^{4,0}(X) = h^0\big(\Omega^1_S \otimes \mathcal{H}^{3,0}\big) = \chi\big(\Omega^1_S \otimes \mathcal{H}^{3,0}\big)$$

by the RR theorem, and since by our assumed non-degeneracy of the VHS and the curvature property

$$(\Theta_{\mathcal{H}^{3,0}})_{\alpha\overline{\beta}} = \sum_\mu \theta^\alpha_\mu \wedge \bar{\theta}^\beta_\mu$$

of the Hodge bundle $\mathcal{H}^{3,0}$, we have the vanishing result

$$h^1\big(\Omega^1_S \otimes \mathcal{H}^{3,0}\big) = 0. \qquad \square$$

We next show that

$$h^{3,0}(X) = 0. \tag{VII.B.8}$$

Proof. From (VII.A.10) we have

$$\begin{aligned} H^{3,0}(X) &= \ker\big\{H^0(S, \mathcal{H}^{3,0}) \xrightarrow{\boldsymbol{\theta}} H^0\big(\Omega^1_S \otimes \mathcal{H}^{2,1}\big)\big\} \\ &= 0. \end{aligned}$$

Alternatively, the degeneration at E_2 of the Leray spectral sequence for (VII.B.1) gives additively

$$\begin{array}{ccccc} H^3(X) \cong H^0(S,\, R^3_f\mathbb{C}) & \oplus & H^1(S,\, R^2_f\mathbb{C}) & \oplus & H^2(S,\, R^1_f\mathbb{C}) \\ \wr\| & & \wr\| & & \\ H^0(S,\, \boldsymbol{H}) & & H^1(S)\otimes V. & & \end{array}$$

The first term is zero by the assumed non-degeneracy of the VHS, and the third term is out because $R^1_f\mathbb{C}=0$. This gives

$$\begin{cases} H^{3,0}(X)=0, \\ H^{2,1}(X)\cong H^{1,0}(S)\otimes V, \end{cases}$$

so that we have (VII.B.8) as well as

$$h^{2,1}(X)=gv. \tag{VII.B.9}$$

□

A similar argument gives

$$\begin{aligned} H^2(X) &\cong H^0(S,\, R^2_f\mathbb{C})\oplus H^2(S,\, R^0_f\mathbb{C}) \\ &\cong (H^0(S,\,\mathbb{C})\otimes V)\oplus H^2(S,\,\mathbb{C}) \end{aligned}$$

which implies that

$$h^2(X)=h^{1,1}(X)=gv+1. \tag{VII.B.10}$$

Next, the Leray spectral sequence gives additively

$$\begin{array}{ccccc} H^4(X) \cong H^0(S,\, R^4_f) & \oplus & \underbrace{H^1(S,\, R^3_f\mathbb{C})} & \oplus & H^2(S,\, R^2_f\mathbb{C}) \\ \wr\| & & & & \wr\| \\ H^0(S,\,\mathbb{C})\otimes V^* & & & & H^2(S,\,\mathbb{C})\otimes V \end{array} \tag{VII.B.11}$$

The term over the brace is the one of interest; it is

$$H^4(S,\,\boldsymbol{H}).$$

We have

$$\begin{cases} h^{3,1}(X)=h^{3,1}(S,\,\boldsymbol{H}), \\ h^{2,2}(X)=h^{2,2}(S,\,\boldsymbol{H})+2v. \end{cases}$$

From the multiplicativity of the Euler characteristic

$$\chi(X)=\chi(S)\chi(X_\eta) \tag{VII.B.12}$$

and the consequences of (VII.B.7)–(VII.B.12) we infer that

$$2h^{3,1}(X)+h^{2,2}(X)=:F(\delta,\,\lambda,\,v,\,h^{3,0},\,h^{2,1}) \tag{VII.B.13}$$

is a known quantity. Of course, it can be computed out but the explicit formula for F is neither important nor illuminating.

The last step in the proof of (VII.B.5) is to express $h^{3,1}(X)$ in terms of known quantities. For this we use the exact sheaf sequence

$$0\to f^{-1}(\Omega^1_S)\to\Omega^1_X\to\Omega^1_{X/S}\to 0.$$

The additivity of the sheaf cohomology Euler characteristics give

$$\chi(\Omega^1_X) = \chi(f^{-1}(\Omega^1_S)) + \chi(\Omega^1_{X/S}). \tag{VII.B.14}$$

The left-hand side is

$$-h^1(\Omega^1_X) + h^2(\Omega^1_X) - h^3(\Omega^1_X) = -(gv+1) + gv - h^{3,1}(X),$$

so that we have

$$-\,h^{3,1}(X) = \chi(f^{-1}(\Omega^1_S)) + \chi(\Omega^1_{X/S}) + 1. \tag{VII.B.15}$$

We next use that, for any sheaf $\mathcal{E}$ on X, the Leray spectral sequence gives

$$\chi(X,\, \mathcal{E}) = \sum_{p,q} (-1)^{p+q} h^p(S,\, R^q_f \mathcal{E}).$$

Thus

$$\begin{aligned}\chi(X,\, f^{-1}(\Omega^1_S)) &= \sum_{\substack{a=0,1\\ b=0,1,2,3}} (-1)^{a+b} H^a\left(S,\, \Omega^1_S \otimes \mathcal{H}^{0,b}\right)\\ &= \sum_b (-1)^b \chi(S,\, \Omega^1_S \otimes \mathcal{H}^{0,b}),\end{aligned}$$

and similarly

$$\begin{aligned}\chi(X,\, \Omega^1_X) &= \sum_{\substack{a=0,1\\ b=0,1,2,3}} (-1)^{a+b} H^a(S,\, R^b_f \Omega^1_{X/S})\\ &= \sum_b (-1)^b \chi(S,\, \mathcal{H}^{1,b}).\end{aligned}$$

Using the R-R for vector bundles on S, all of the $\chi(S,\, \Omega^1_S \otimes \mathcal{H}^{0,b})$ and $\chi(S,\, \mathcal{H}^{1,b})$ are expressible in terms of known quantities, as therefore is the right-hand side of (VII.B.4). □

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