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ON THE STRUCTURE OF SOME MODULI SPACES OF FINITE FLAT GROUP SCHEMES

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To P. Deligne for his 65th birthday

Abstract. We consider the moduli space, in the sense of Kisin, of finite flat models of a 2-dimensional representation with values in a finite field of the absolute Galois group of a totally ramified extension of $\mathbb{Q}_p$. We determine the connected components of this space and describe its irreducible components in the case of an irreducible Galois representation. These results prove a modified version of a conjecture of Kisin.

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1. Introduction and Notations

Let p be an odd prime and k a finite field of characteristic p. Let $W = W(k)$ be the ring of Witt vectors with coefficients in k and $K_0 = W[\frac{1}{p}]$ its fraction field. We consider a finite, totally ramified extension K/K_0 and denote by $e = [K : K_0]$ the degree of the extension. Let us fix a uniformizer $\pi \in \mathcal{O}_K$ and an algebraic closure $\bar{K}$ of K.

Let $\mathbb{F}$ be a finite field of characteristic p and $\rho\colon G_K \to \mathrm{GL}(V_{\mathbb{F}})$ a continuous representation of the absolute Galois group $G_K = \mathrm{Gal}(\bar{K}/K)$ of K in a finite dimensional $\mathbb{F}$-vector space $V_{\mathbb{F}}$ whose dimension will be denoted by d.

This datum is equivalent to a finite commutative group scheme $\widetilde{\mathcal{G}} \to \operatorname{Spec} K$ with an operation of $\mathbb{F}$: The $\bar{K}$-valued points become an $\mathbb{F}$-vector space with a natural action of G_K and we want $\widetilde{\mathcal{G}}(\bar{K})$ and $V_{\mathbb{F}}$ to be isomorphic as $\mathbb{F}[G_K]$-modules.

If $\mathbb{F}'$ is a finite extension of $\mathbb{F}$, the representation ρ induces a representation ρ' on $V_{\mathbb{F}'} = V_{\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{F}'$.

By the construction in Kisin's article [Ki], there is a projective $\mathbb{F}$-scheme $\mathcal{GR}_{V_{\mathbb{F}},0}$ whose $\mathbb{F}'$-valued points parametrize the isomorphism classes of finite flat models of $V_{\mathbb{F}'}$, i.e., finite flat group schemes $\mathcal{G} \to \operatorname{Spec} \mathcal{O}_K$ with an operation of $\mathbb{F}'$ such that the generic fiber of $\mathcal{G}$ is the G_K-representation on $V_{\mathbb{F}'}$ in the above sense.

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Our aim is to analyse the structure of (some stratification of) $\mathcal{GR}_{V_{\mathbb{F}},0}$ in the case $d = 2$ and $k = \mathbb{F}_p$.

First we recall some constructions from [Ki], see also [PR2]. We assume $k = \mathbb{F}_p$ to simplify the situation.

For each n let $\pi_n \in \bar{K}$ be a p^n-th root of the uniformizer π such that $\pi_n^p = \pi_{n-1}$ for all n. Define $K_\infty = \bigcup_{n\geqslant 1} K(\pi_n)$ and denote by $G_{K\infty} = \mathrm{Gal}(\bar{K}/K_\infty)$ the absolute Galois group of K_∞.

For each algebraic extension $\mathbb{F}'$ of $\mathbb{F}$ we denote by $\phi\colon \mathbb{F}'((u)) \to \mathbb{F}'((u))$ the homomorphism which takes u to its p-th power and which is the identity on the coefficients:

$$\phi\Big(\sum_i a_i u^i\Big) = \sum a_i u^{pi}.$$

Denote by $\mathrm{Mod}^{\phi}_{/\mathbb{F}'((u))}$ the category of finite dimensional $\mathbb{F}'((u))$-modules M together with a ϕ-linear map $\Phi\colon M \to M$ such that the linearization $\mathrm{id}\otimes\Phi\colon \phi^*M \to M$ is an isomorphism. The morphism are $\mathbb{F}'((u))$-linear maps commuting with Φ. By [Ki, 1.2.6, Lemma 1.2.7], there is an equivalence of abelian categories

$$\mathrm{Mod}^{\phi}_{/\mathbb{F}'((u))} \longleftrightarrow \left\{\begin{matrix}\text{continuous } G_{K\infty}\text{-representations}\\ \text{on finite dimensional } \mathbb{F}'\text{-vector spaces}\end{matrix}\right\}$$

which preserves the dimensions and is compatible with finite base change $\mathbb{F}''/\mathbb{F}'$. This is a version with coefficients of the equivalence of categories of Fontaine (cf. [Fo, A3]).

Denote by $(M_{\mathbb{F}}, \Phi)$ the d-dimensional $\mathbb{F}((u))$-vector space with semi-linear endomorphism Φ, associated to the restriction of the Tate-twist $V_{\mathbb{F}}(-1)$ to G_{K_∞} under the above equivalence. By the descriptions in [Ki], the finite flat models $\mathcal{G} \to \mathrm{Spec}\,\mathcal{O}_K$ of $V_{\mathbb{F}}$ correspond to $\mathbb{F}[\![u]\!]$-lattices $\mathfrak{M} \subset M_{\mathbb{F}}$ satisfying $u^e\mathfrak{M} \subset \langle\Phi(\mathfrak{M})\rangle \subset \mathfrak{M}$. Here $\langle\Phi(\mathfrak{M})\rangle = (\mathrm{id}\otimes\Phi)\phi^*\mathfrak{M}$ is the $\mathbb{F}[\![u]\!]$-lattice in $M_{\mathbb{F}}$ generated by $\Phi(\mathfrak{M})$.

Under this description the multiplicative group schemes correspond to the lattices $\mathfrak{M}$ such that $\langle\Phi(\mathfrak{M})\rangle = \mathfrak{M}$ and the étale group schemes correspond to the lattices with $u^e\mathfrak{M} = \langle\Phi(\mathfrak{M})\rangle$. These lattices will be called multiplicative resp. étale.

This construction is compatible with base change in the following sense. Suppose $\mathfrak{M}_{\mathbb{F}} \subset M_{\mathbb{F}}$ is a $\mathbb{F}[\![u]\!]$-lattice corresponding to a finite flat model $\mathcal{G}$ of $V_{\mathbb{F}}$. If $\mathbb{F}'$ is a finite extension of $\mathbb{F}$ with $n = [\mathbb{F}' : \mathbb{F}]$, then the $\mathbb{F}'[\![u]\!]$ lattice

$$\mathfrak{M}_{\mathbb{F}'} = \mathfrak{M}\,\widehat{\otimes}_{\mathbb{F}}\,\mathbb{F}' \subset M_{\mathbb{F}'} = M_{\mathbb{F}}\,\widehat{\otimes}_{\mathbb{F}}\,\mathbb{F}'$$

corresponds to the finite flat model $\mathcal{G}' = \mathcal{G}\boxtimes_{\mathbb{F}}\mathbb{F}'$ of $V_{\mathbb{F}'}$. Here the exterior tensor product $\mathcal{G}\boxtimes_{\mathbb{F}}\mathbb{F}'$ is the following group scheme: Choose an $\mathbb{F}$-basis $e_1, \ldots, e_n$ of $\mathbb{F}'$. Then $\mathcal{G}\boxtimes_{\mathbb{F}}\mathbb{F}' = \prod_{i=1}^n \mathcal{G}$ and $z \in \mathbb{F}'$ operates via the matrix $A \in \mathrm{GL}_n(\mathbb{F})$ describing the multiplication by z on $\mathbb{F}'$ in the fixed $\mathbb{F}$-basis.

The scheme $\mathcal{GR}_{V_{\mathbb{F}},0}$ is constructed as a closed subscheme of the affine Grassmannian Grass $M_{\mathbb{F}}$ for $\mathrm{GL}(M_{\mathbb{F}})$ and its closed points are given by

$$\mathcal{GR}_{V_{\mathbb{F}},0}(\mathbb{F}') = \{\mathbb{F}'[\![u]\!]\text{-lattices } \mathfrak{M} \subset M_{\mathbb{F}'}\colon u^e\mathfrak{M} \subset \langle\Phi(\mathfrak{M})\rangle \subset \mathfrak{M}\} \qquad (1.1)$$

for every finite extension $\mathbb{F}'$ of $\mathbb{F}$.

In the following we will forget about the Galois representation and finite flat group schemes and will consider lattices. We will drop the condition $p \neq 2$. All results hold for arbitrary p, except those using the interpretation of the closed points as finite flat group schemes. We will always assume that there exists a finite flat model for $V_{\mathbb{F}}$.

For each $\mathbb{Q}_p$-algebra embedding $\psi\colon K \to \bar{K}_0$ we fix an integer $v_\psi \in \{0, \ldots, d\}$. Denote by $\boldsymbol{v} = (v_\psi)_\psi$ the collection of the v_ψ and by $\boldsymbol{r} = \check{\boldsymbol{v}}$ the dual partition, i.e., $r_i = \#\{\psi\colon v_\psi \geqslant i\}$.

Kisin constructs closed reduced subschemes

$$\mathcal{GR}^{\boldsymbol{v},\mathrm{loc}}_{V_{\mathbb{F}},0} \subset \mathcal{GR}_{V_{\mathbb{F}},0}$$

whose $\mathbb{F}'$-valued points are given by

$$\mathcal{GR}^{\boldsymbol{v},\mathrm{loc}}_{V_{\mathbb{F}},0}(\mathbb{F}') = \{\mathfrak{M} \in \mathcal{GR}_{V_{\mathbb{F}},0}(\mathbb{F}')\colon J(u|_{\langle\Phi(\mathfrak{M})\rangle/u^e\,\mathfrak{M}}) \leqslant \boldsymbol{r}\} \tag{1.2}$$

for a finite extension $\mathbb{F}'$ of $\mathbb{F}$ (cf. [Ki, Prop. 2.4.6]). Here $J(u|_{\langle\Phi(\mathfrak{M})\rangle/u^e\,\mathfrak{M}})$ denotes the Jordan type of the nilpotent endomorphism on $\langle\Phi(\mathfrak{M})\rangle/u^e\,\mathfrak{M}$ induced by the multiplication with u. If $d = 2$ this means $J(u|_{\langle\Phi(\mathfrak{M})\rangle/u^e\,\mathfrak{M}}) = (a, b)$, where $a \geqslant b$ is such that $\langle\Phi(\mathfrak{M})\rangle/u^e\,\mathfrak{M}$ decomposes into two maximal u-stable proper subspaces of dimension a and b. Recall that for $d = 2$

$$(a_1, b_1) \leqslant (a_2, b_2) \;\Leftrightarrow\; \begin{cases} a_1 \leqslant a_2, \\ a_1 + b_1 = a_2 + b_2 \end{cases} \tag{1.3}$$

for pairs $(a_i, b_i) \in \mathbb{Z}^2$ with $a_i \geqslant b_i$. The local structure of $\mathcal{GR}^{\boldsymbol{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$ is linked to the structure of the local models studied in [PR1]. These schemes are named *closed Kisin varieties* in [PR2].

Kisin conjectures in [Ki, 2.4.16] that, if $\mathrm{End}_{\mathbb{F}[G_K]}(V_{\mathbb{F}}) = \mathbb{F}$, the connected components of $\mathcal{GR}^{\boldsymbol{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$ are given by the open and closed subschemes on which both the rank of the maximal multiplicative subobject and the rank of the maximal étale quotient are fixed. In [Ki, 2.5] he proves this conjecture in the case $d = 2$, $k = \mathbb{F}_p$ and $v_\psi = 1$ for all ψ. For $d = 2$ and $v_\psi = 1$ for all ψ this result is generalized by Imai to the case of arbitrary k (see [Im]). In this paper we want to analyse the situation in the case $k = \mathbb{F}_p$, $d = 2$ but arbitrary $\boldsymbol{v}$. It turns out that the conjecture is not true in general. Our main results are as follows.

For $(a, b) \in \mathbb{Z}^2$ with $a \geqslant b$, we introduce a locally closed subscheme of the affine Grassmannian

$$\mathcal{G}_{V_{\mathbb{F}}}(a, b) \subset \mathrm{Grass} M_{\mathbb{F}},$$

with closed points the lattices $\mathfrak{M}$ such that the elementary divisors of $\langle\Phi(\mathfrak{M})\rangle$ with respect to $\mathfrak{M}$ are given by (a, b). Then

$$\mathcal{GR}^{\boldsymbol{v},\mathrm{loc}}_{V_{\mathbb{F}},0} = \bigcup_{(a,b)\leqslant \boldsymbol{r}} \mathcal{G}_{V_{\mathbb{F}}}(a, b),$$

where $\boldsymbol{r}$ again denotes the dual partition of $\boldsymbol{v}$.

Theorem 1.1. *Assume that $(M_{\mathbb{F}'}, \Phi) = (M_{\mathbb{F}} \,\widehat{\otimes}_{\mathbb{F}}\, \mathbb{F}', \Phi)$ is simple for all finite extensions $\mathbb{F}'$ of $\mathbb{F}$.*

(i) *If* $\mathcal{G}_{V_{\mathbb{F}}}(a,\,b) \neq \varnothing$*, there exists a finite extension* $\mathbb{F}'$ *of* $\mathbb{F}$ *such that*

$$\mathcal{G}_{V_{\mathbb{F}'}}(a,\,b) = \mathcal{G}_{V_{\mathbb{F}}}(a,\,b) \otimes_{\mathbb{F}} \mathbb{F}' \cong \mathbb{A}^n_{\mathbb{F}'}$$

for $n = \lfloor \frac{a-b}{p+1} \rfloor$.

(ii) *The scheme* $\mathcal{GR}^{\boldsymbol{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$ *is geometrically connected and irreducible. There exists a finite extension* $\mathbb{F}'$ *of* $\mathbb{F}$ *such that* $\mathcal{GR}^{\boldsymbol{v},\mathrm{loc}}_{V_{\mathbb{F}},0} \otimes_{\mathbb{F}} \mathbb{F}'$ *is isomorphic to a Schubert variety in the affine Grassmannian for* $\mathrm{GL}(M_{\mathbb{F}'})$*, that is the closure of some* $\mathrm{GL}(\mathfrak{M}_0)$*-orbit for a suitable lattice* $\mathfrak{M}_0 \subset M_{\mathbb{F}'}$.

The dimension of $\mathcal{GR}^{\boldsymbol{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$ *is either* $\lfloor \frac{r_1-r_2}{p+1} \rfloor$ *or* $\lfloor \frac{r_1-r_2}{p+1} \rfloor - 1$*. Here* $r_i = \#\{\psi\colon v_\psi \geqslant i\}$.

In the treatment of the reducible case we consider the set $\mathcal{S}(\boldsymbol{v})$ of isomorphism classes $[M']$ of one dimensional objects in $\mathrm{Mod}^{\phi}_{/\bar{\mathbb{F}}((u))}$ which admit an $\bar{\mathbb{F}}[\![u]\!]$-lattice $\mathfrak{M}_{[M']} \subset M'$ such that $\langle \Phi(\mathfrak{M}_{[M']}) \rangle = u^{e-r_1}\mathfrak{M}_{[M']}$. We will define subschemes

$$X^{\boldsymbol{v}}_{[M']} \subset \mathcal{GR}^{\boldsymbol{v},\mathrm{loc}}_{V_{\mathbb{F}},0} \otimes_{\mathbb{F}} \bar{\mathbb{F}}.$$

A lattice defines a closed point of $X^{\boldsymbol{v}}_{[M']}$ if it admits a Φ-stable subobject isomorphic to $\mathfrak{M}_{[M']}$. A lattice $\mathfrak{M}$ is called $\boldsymbol{v}$*-ordinary* iff it defines a closed point of $X^{\boldsymbol{v}}_{[M']}$ for some $[M'] \in \mathcal{S}(\boldsymbol{v})$. The subscheme of non-$\boldsymbol{v}$-ordinary points will be denoted by $X^{\boldsymbol{v}}_0$. We will prove the following theorem.

Theorem 1.2. *Assume that* $(M_{\mathbb{F}'},\,\Phi) = (M_{\mathbb{F}} \widehat{\otimes}_{\mathbb{F}} \mathbb{F}',\,\Phi)$ *is reducible for some finite extension* $\mathbb{F}'$ *of* $\mathbb{F}$.

(i) *The subschemes* $X^{\boldsymbol{v}}_0$ *and* $X^{\boldsymbol{v}}_{[M']}$ *are open and closed in* $\mathcal{GR}^{\boldsymbol{v},\mathrm{loc}}_{V_{\mathbb{F}},0} \otimes_{\mathbb{F}} \bar{\mathbb{F}}$ *for all isomorphism classes* $[M'] \in \mathcal{S}(\boldsymbol{v})$.
(ii) *The scheme* $X^{\boldsymbol{v}}_0$ *is connected.*
(iii) *For each* $[M'] \in \mathcal{S}(\boldsymbol{v})$ *the scheme* $X^{\boldsymbol{v}}_{[M']}$ *is connected. If it is non empty, it is either a single point, or isomorphic to* $\mathbb{P}^1_{\bar{\mathbb{F}}}$.
(iv) *There are at most two isomorphism classes* $[M'] \in \mathcal{S}(\boldsymbol{v})$ *such that* $X^{\boldsymbol{v}}_{[M']} \neq \varnothing$.

The structure of the subscheme $X^{\boldsymbol{v}}_0$ of non-$\boldsymbol{v}$-ordinary lattices is much more complicated than in the (absolutely) simple case. In general $X^{\boldsymbol{v}}_0$ has many irreducible components of varying dimensions. The proof of this theorem implies a modified version of Kisin's conjecture in the case $k = \mathbb{F}_p$ and $d = 2$, as follows.

For an integer s denote by

$$\mathcal{GR}^{\boldsymbol{v},\mathrm{loc},s}_{V_{\mathbb{F}},0} \subset \mathcal{GR}^{\boldsymbol{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$$

the open and closed subscheme where the rank of the maximal Φ-stable subobject $\mathfrak{M}_1$, satisfying $\langle \Phi(\mathfrak{M}_1) \rangle = u^{e-r_1}\mathfrak{M}_1$, is equal to s.

Corollary 1.3. *Assume* $p \neq 2$ *and let* $\rho\colon G_K \to V_{\mathbb{F}}$ *be any two dimensional continuous representation of* G_K*. Assume that* $\mathrm{End}_{\mathbb{F}'[G_K]}(V_{\mathbb{F}'})$ *is a simple algebra for all finite extensions* $\mathbb{F}'$ *of* $\mathbb{F}$*. Then* $\mathcal{GR}^{\boldsymbol{v},\mathrm{loc},s}_{V_{\mathbb{F}},0}$ *is geometrically connected for all* s.

Furthermore

(i) *If* $s = 1$ *and* $\mathrm{End}_{\mathbb{F}'[G_K]}(V_{\mathbb{F}'}) = \mathbb{F}'$ *for all finite extensions* $\mathbb{F}'$ *of* $\mathbb{F}$*, then* $\mathcal{GR}^{\boldsymbol{v},\mathrm{loc},s}_{V_{\mathbb{F}},0}$ *is either empty or a single point.*
If $s = 1$ *and* $\mathrm{End}_{\mathbb{F}'[G_K]}(V_{\mathbb{F}'}) = M_2(\mathbb{F}')$ *for some finite extension* $\mathbb{F}'$ *of* $\mathbb{F}$*, then* $\mathcal{GR}^{\boldsymbol{v},\mathrm{loc},s}_{V_{\mathbb{F}},0}$ *is either empty or becomes isomorphic to* $\mathbb{P}^1_{\mathbb{F}'}$ *after extending the scalars to* $\mathbb{F}'$*.*

(ii) *If* $s = 2$*, then* $\mathcal{GR}^{\boldsymbol{v},\mathrm{loc},s}_{V_{\mathbb{F}},0}$ *is either empty or a single point.*

This paper is a shortened version of the authors diploma thesis [He] written at the University of Bonn. In this version I will not give all computations in detail but sometimes refer to [He].

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2. Some Notations in the Building

The method of the paper is to determine all lattices in the building of $\mathrm{GL}_2(\bar{\mathbb{F}}((u)))$ that correspond to closed points of $\mathcal{GR}^{\boldsymbol{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$. As we know that the scheme we study is a closed reduced subscheme of the affine Grassmannian, we can get information on the structure of $\mathcal{GR}^{\boldsymbol{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$ by looking at its closed points.

For the rest of this paper, we fix the following notations: Let $(M_{\mathbb{F}}, \Phi)$ be the object in $\mathrm{Mod}^{\phi}_{/\mathbb{F}((u))}$ corresponding to the 2-dimensional Galois representation ρ on $V_{\mathbb{F}}$. For a finite extension $\mathbb{F}'$ of $\mathbb{F}$ we write $M_{\mathbb{F}'}$ for the base change $M_{\mathbb{F}} \widehat{\otimes}_{\mathbb{F}} \mathbb{F}'$ and we again write Φ for the ϕ-linear endomorphism $\Phi \otimes \mathrm{id}$ of $M_{\mathbb{F}'}$. Similarly we write $(M_{\bar{\mathbb{F}}}, \Phi)$ for the object obtained from $(M_{\mathbb{F}}, \Phi)$ by extending the scalars to $\bar{\mathbb{F}}$.

Let $\boldsymbol{v} = (v_\psi)_\psi$ be a collection of integers $v_\psi \in \{0, 1, 2\}$ for every $\psi\colon K \to \bar{K}_0$. Define

$$d' = \sum_{\psi} v_\psi. \tag{2.1}$$

Denote by $\boldsymbol{r} = \check{\boldsymbol{v}}$ the dual partition, i.e.,

$$r_1 = \#\{\psi\colon v_\psi \geqslant 1\}$$
$$r_2 = \#\{\psi\colon v_\psi \geqslant 2\}.$$

Denote by $\mathcal{B}$ the Bruhat–Tits building for $\mathrm{GL}_2(\mathbb{F}((u)))$. For any finite extension $\mathbb{F}'$ of $\mathbb{F}$ the building for $\mathrm{GL}_2(\mathbb{F}'((u)))$ will be denoted by $\mathcal{B}_{\mathbb{F}'}$. We write

$$\bar{\mathcal{B}} = \bigcup_{\mathbb{F}'/\mathbb{F}} \mathcal{B}_{\mathbb{F}'}$$

for the building for $\mathrm{GL}_2\left(\bar{\mathbb{F}}((u))\right)$.

We choose an $\mathbb{F}((u))$-basis e_1, e_2 of $M_{\mathbb{F}}$. Denote by $\mathfrak{M}_0 = \langle e_1, e_2\rangle$ the standard lattice in the standard apartment $\mathcal{A}_0$ determined by e_1, e_2. In this apartment we choose the following coordinates:

Let $(m, n)_0$ denote the lattice $\langle u^m e_1, u^n e_2\rangle$. Further, we consider another set of coordinates given by $[x, y]_0 = \left(\frac{x+y}{2}, \frac{y-x}{2}\right)_0$ for $x, y \in \mathbb{Z}$ with $x \equiv y \mod 2$; i.e., $(m, n)_0 = [m-n, m+n]_0$.

Let $q \in \mathbb{F}((u))^\times$ and set $k = v_u(q) \in \mathbb{Z}$, where v_u is the valuation on $\mathbb{F}((u))$ with $v_u(u) = 1$. The basis e_1, $qe_1 + e_2$ of $M_\mathbb{F}$ defines another apartment $\mathcal{A}_q$ which is branching off from the standard apartment at the line defined by $x = k$. Using the Iwasawa decomposition we find

$$\mathcal{B} = \bigcup_{q\in\mathbb{F}((u))} \mathcal{A}_q.$$

For arbitrary $q \in \mathbb{F}((u))$ we choose coordinates in the apartments $\mathcal{A}_q$, similar to the case of $\mathcal{A}_0$. Define

$$(m, n)_q = [m-n, m+n]_q := \langle u^m e_1, u^n(qe_1 + e_2)\rangle \in \mathcal{A}_q.$$

Remark 2.1. (i) The systems of coordinates in the various apartments are compatible in the following sense: For any $x, y \in \mathbb{Z}$ with $x \equiv y \mod 2$ and $q, q' \in \mathbb{F}((u))$ we have $[x, y]_q = [x, y]_{q'}$ if and only if $x \leqslant v_u(q - q')$, which implies

$$[x, y]_q = [x, y]_{q'} \;\Leftrightarrow\; [x, y]_q \in \mathcal{A}_q \cap \mathcal{A}_{q'} \;\Leftrightarrow\; [x, y]_{q'} \in \mathcal{A}_q \cap \mathcal{A}_{q'}.$$

(ii) We will make use of these coordinates for arbitrary points in the building (not only points corresponding to lattices). We see that $[x, y]_q$ defines a lattice if and only if $x, y \in \mathbb{Z}$ and $x + y \in 2\mathbb{Z}$.

(iii) We extend the above notations in the obvious way to the buildings $\bar{\mathcal{B}}$ and $\mathcal{B}_{\mathbb{F}'}$ for arbitrary finite extensions $\mathbb{F}'$ of $\mathbb{F}$.

(iv) Two points $[x, y]_q, [x', y']_q \in \mathcal{A}_q$ define the same point in the building for $\mathrm{PGL}_2(\mathbb{F}((u)))$ if and only if $x = x'$. Thus the projection from $\mathcal{B}$ onto the building for $\mathrm{PGL}_2(\mathbb{F}((u)))$ is given by the projection onto the x-coordinate for every apartment $\mathcal{A}_q \subset \mathcal{B}$.

Definition 2.2. Let $\mathfrak{M}$ and $\mathfrak{M}'$ be lattices in $M_\mathbb{F}$. Let a, b be the elementary divisors of $\mathfrak{M}'$ with respect to $\mathfrak{M}$, i.e., there exists a basis e'_1, e'_2 of $\mathfrak{M}$ such that $\mathfrak{M}' = \langle u^a e'_1, u^b e'_2\rangle$. Define

$$d_1(\mathfrak{M}, \mathfrak{M}') = |a - b|,$$
$$d_2(\mathfrak{M}, \mathfrak{M}') = a + b.$$

Remark 2.3. These quantities have the following meaning in the building:

If $\mathfrak{M} = [x, y]_q$ and $\mathfrak{M}' = [x', y']_{q'}$, then $d_2(\mathfrak{M}, \mathfrak{M}') = y - y'$. Note that $d_2(\mathfrak{M}, \mathfrak{M}')$ might be negative. Further $d_1(\mathfrak{M}, \mathfrak{M}')$ is the distance between $\mathfrak{M}$ and $\mathfrak{M}'$ in the building for $\mathrm{PGL}_2(\mathbb{F}((u)))$. Here, the distance between two lattices joined by an edge is equal to 1. We see that the distance $d_1(\mathfrak{M}, \mathfrak{M}')$ only depends on x, x' (and on $v_u(q - q')$), while $d_2(\mathfrak{M}, \mathfrak{M}')$ only depends on y and y'.

Using this remark, we can extend the distances d_1 and d_2 in an obvious way to the whole building $\mathcal{B}$ (and to $\bar{\mathcal{B}}$, $\mathcal{B}_{\mathbb{F}'}$). For example

$$d_1([x, y]_q, [0, 0]_0) = \begin{cases} x & \text{if } x \geqslant 0,\ v_u(q) \geqslant 0, \\ -x & \text{if } x < 0,\ x \leqslant v_u(q), \\ x - 2v_u(q) & \text{if } v_u(q) < x,\ v_u(q) < 0. \end{cases}$$

Lemma 2.4. *Define d' as in (2.1). The closed points $z \in \mathcal{GR}^{\boldsymbol{v},\mathrm{loc}}_{V_{\mathbb{F}},0}(\bar{\mathbb{F}})$ correspond to the lattices $\mathfrak{M} \subset M_{\bar{\mathbb{F}}}$ which satisfy*

$$d_1(\mathfrak{M},\, \langle \Phi(\mathfrak{M})\rangle) \leqslant r_1 - r_2,$$
$$d_2(\mathfrak{M},\, \langle \Phi(\mathfrak{M})\rangle) = 2e - d'.$$

Proof. If $\mathfrak{M} \subset M_{\bar{\mathbb{F}}}$ is any lattice and if $a \geqslant b$ are the elementary divisors of $\langle \Phi(\mathfrak{M})\rangle$ with respect to $\mathfrak{M}$, then the conditions are

$$a - b \leqslant r_1 - r_2,$$
$$a + b = 2e - d' = 2e - (r_1 + r_2)$$

and this implies $u^e\mathfrak{M} \subset \langle \Phi(\mathfrak{M})\rangle \subset \mathfrak{M}$. The Jordan type of u on $\langle \Phi(\mathfrak{M})\rangle / u^e\,\mathfrak{M}$ is given by

$$J(u|_{\langle \Phi(\mathfrak{M})\rangle / u^e\,\mathfrak{M}}) = (e - a,\, e - b).$$

Assuming $a \geqslant b$ we find:

$$J(u|_{\langle \Phi(\mathfrak{M})\rangle / u^e\,\mathfrak{M}}) \leqslant \boldsymbol{r} \;\Leftrightarrow\; \begin{cases} b \geqslant e - r_1, \\ a + b = 2e - d' = 2e - (r_1 + r_2). \end{cases}$$

The lemma follows easily from this. □

Definition 2.5. A lattice $\mathfrak{M}$ is called $\boldsymbol{v}$-*admissible* if it satisfies

$$d_1(\mathfrak{M},\, \langle \Phi(\mathfrak{M})\rangle) \leqslant r_1 - r_2 \quad \text{and} \quad d_2(\mathfrak{M},\, \langle \Phi(\mathfrak{M})\rangle) = 2e - d'.$$

Let $\mathfrak{M}$ be a lattice in $M_{\mathbb{F}}$ and $A \in \mathrm{GL}_2(\mathbb{F}((u)))$ be a matrix. We will use the notation $\mathfrak{M} \sim A$ if $\mathfrak{M}$ admits a $\mathbb{F}[\![u]\!]$-basis $b_1,\, b_2$ satisfying $\Phi(b_i) = Ab_i$. Similarly we will use the notation $M_{\mathbb{F}} \sim A$ (use a $\mathbb{F}((u))$-basis of $M_{\mathbb{F}}$).

Lemma 2.6. *For $i = 1,\, 2$, let $z_i \in \mathcal{GR}^{\boldsymbol{v},\mathrm{loc}}_{V_{\mathbb{F}},0}(\bar{\mathbb{F}})$ be closed points corresponding to lattices $\mathfrak{M}_i = [x_i,\, y_i]_{q_i} \in \bar{\mathcal{B}}$. Then $y_1 = y_2$.*

Proof. Choose $A,\, B \in \mathrm{GL}_2(\bar{\mathbb{F}}((u)))$ such that $\mathfrak{M}_2 = A\mathfrak{M}_1$ and $\mathfrak{M}_1 \sim B$. Then $\mathfrak{M}_2 \sim \phi(A)BA^{-1}$. Using the theory of elementary divisors it follows that

$$v_u(\det B) = d_2(\mathfrak{M}_i,\, \langle \Phi(\mathfrak{M}_i)\rangle) = (p-1)v_u(\det A) + v_u(\det B)$$

and hence $v_u(\det A) = 0$ which yields the claim. □

Definition 2.7. For each $m \in \mathbb{Z}$ define the following subset of $\bar{\mathcal{B}}$:

$$\bar{\mathcal{B}}(m) := \bigcup_{q \in \bar{\mathbb{F}}((u))} \{[x,\, y]_q \in \mathcal{A}_q \colon y = m\}.$$

Viewing $\mathcal{GR}^{\boldsymbol{v},\mathrm{loc}}_{V_{\mathbb{F}},0}(\bar{\mathbb{F}})$ as a subset of $\bar{\mathcal{B}}$, Lemma 2.6 implies:

$$\mathcal{GR}^{\boldsymbol{v},\mathrm{loc}}_{V_{\mathbb{F}},0}(\bar{\mathbb{F}}) \subset \bar{\mathcal{B}}(m)$$

for some $m = m(\boldsymbol{v}) \in \mathbb{Z}$. We will give explicit expressions for this integer below. The subset $\bar{\mathcal{B}}(m)$ is a tree which is (as a topological space) isomorphic to the building for $\mathrm{PGL}_2(\bar{\mathbb{F}}((u)))$.

Note that not every vertex in $\bar{\mathcal{B}}(m)$ represents a lattice: A vertex $[x,\, m]_q \in \bar{\mathcal{B}}(m)$ represents a lattice $\mathfrak{M} \subset M_{\bar{\mathbb{F}}}$ if and only if $x \equiv m \bmod 2$.

Remark 2.8. By construction we have

$$\mathcal{GR}_{V_{\mathbb{F}},0} \subset \operatorname{Grass} M_{\mathbb{F}},$$

where $\operatorname{Grass} M_{\mathbb{F}}$ denotes the affine Grassmannian for GL_2. Since the determinant condition in (1.2) fixes the dimension

$$\dim \langle \Phi(\mathfrak{M})\rangle / u^e\,\mathfrak{M} = \sum_{\psi} v_\psi = d',$$

the closed subscheme $\mathcal{GR}^{\boldsymbol{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$ lies in a connected component of this Grassmannian: If $\mathfrak{M} = A\mathfrak{M}_0$ defines a closed point (where $\mathfrak{M}_0$ is the standard lattice and A is a matrix), then the valuation of $\det A$ is determined by the dimension of $\langle \Phi(\mathfrak{M})\rangle / u^e\,\mathfrak{M}$. Recall that the connected components of $\operatorname{Grass} M_{\mathbb{F}}$ are exactly the subsets where the valuation of the determinant is fixed.

3. The Absolutely Simple Case

In this section we will analyse the structure of $\mathcal{GR}^{\boldsymbol{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$ in the case where $(M_{\mathbb{F}}, \Phi)$ is absolutely simple, i.e., for every (finite) extension $\mathbb{F}'/\mathbb{F}$ there is no proper Φ-stable subobject of $(M_{\mathbb{F}'}, \Phi)$.

Lemma 3.1. *If $(M_{\mathbb{F}}, \Phi)$ is absolutely simple, there exists a finite extension $\mathbb{F}'$ of $\mathbb{F}$, a basis e_1, e_2 of $M_{\mathbb{F}'}$ and $a \in \mathbb{F}'$, $s \in \mathbb{Z}$ satisfying*

$$0 \leqslant s < p^2 - 1 \quad \text{and} \quad s \not\equiv 0 \mod (p+1)$$

such that

$$M_{\mathbb{F}'} \sim \begin{pmatrix} 0 & au^s \\ 1 & 0 \end{pmatrix}.$$

Proof. This follows from [Ca, Cor. 8], except we need to check that $s \not\equiv 0 \mod (p+1)$. If $p+1 | s$, then there would be a proper Φ-stable subspace of $M_{\mathbb{F}''}$ for a quadratic extension $\mathbb{F}''$ of $\mathbb{F}'$, namely $\langle \sqrt{a} u^{s/p+1} e_1 + e_2 \rangle \subset M_{\mathbb{F}'} \widehat{\otimes}_{\mathbb{F}'} \mathbb{F}'[\sqrt{a}]$.

The constructions in [Ca] give a basis after extending the scalars to the algebraic closure $\bar{\mathbb{F}}$ of $\mathbb{F}$, but of course this also gives a basis after a finite field extension, as there are only finitely many equations to solve. See also [Im, Lemma 1.2]. ☐

For the rest of this section we fix the basis e_1, e_2 of Lemma 3.1 as the standard basis of $M_{\bar{\mathbb{F}}}$ and use the coordinates introduced in Section 2. Furthermore we fix the point

$$P_{\mathrm{irred}} := \left[\frac{s}{p+1}, -\frac{s}{p-1}\right]_0 \in \mathcal{A}_0 \subset \mathcal{B}. \tag{3.1}$$

Proposition 3.2. (i) *The map Φ extends to a map $\bar{\mathcal{B}} \to \bar{\mathcal{B}}$ also denoted by Φ.*

(ii) *Let $[x, y]_0 \in \mathcal{A}_0$ be any point in the standard apartment. Then*

$$\Phi([x, y]_0) = [-px + s, py + s]_0.$$

(iii) *For any $q \in \bar{\mathbb{F}}((u))^\times$ with $k = v_u(q)$ and $[x, y] \in \mathcal{A}_q \backslash \mathcal{A}_0$, i.e., $x > k$, the map Φ is given by*

$$\Phi([x, y]_q) = [px - 2pk + s, py + s]_{q'} \in \mathcal{A}_{q'}$$

for some $q' \in \bar{\mathbb{F}}((u))^\times$ with $v_u(q') = -pk + s \neq k$.

(iv) *The point* P_{irred}*, as defined in* (3.1)*, satisfies* $\Phi(P_{\mathrm{irred}}) = P_{\mathrm{irred}}$.

(v) *If* $Q \in \mathcal{B}$ *is an arbitrary point, then*

$$\begin{aligned} d_1(Q,\, \Phi(Q)) &= (p+1)d_1(Q,\, P_{\mathrm{irred}}),\\ d_2(Q,\, \Phi(Q)) &= (p-1)d_2(Q,\, P_{\mathrm{irred}}). \end{aligned}$$

Proof. (i) We can use the expressions in (ii) and (iii) to extend Φ.

(ii) We have

$$\begin{aligned} \Phi(u^m e_1) &= u^{pm}\Phi(e_1) = u^{pm}e_2,\\ \Phi(u^n e_2) &= u^{pn}\Phi(e_2) = au^{pn+s}e_1 \end{aligned}$$

and hence $\Phi((m,\, n)_0) = (pn+s,\, pm)_0$. The statement follows.

(iii) We put $v_u(q) = k$ and $\phi(q) = \alpha u^{pk}$ for some $\alpha \in \bar{\mathbb{F}}[\![u]\!]^\times$. If $\mathfrak{M} = (m,\, n)_q$, then

$$\langle \Phi(\mathfrak{M})\rangle = \big\langle u^{pm}e_2,\, u^{pn}\phi(q)e_2 + au^{pn+s}e_1\big\rangle.$$

As $\mathfrak{M} = [m-n,\, m+n]_q \notin \mathcal{A}_0$ we have $m > n+k$. Hence

$$\begin{aligned} \langle \Phi(\mathfrak{M})\rangle &= \big\langle u^{pm}e_2 - \alpha^{-1}u^{p(m-n-k)}(u^{pn}\phi(q)e_2 + au^{pn+s}e_1),\, u^{pn}\phi(q)e_2 + au^{pn+s}e_1\big\rangle\\ &= \big\langle u^{p(m-k)+s}e_1,\, u^{p(n+k)}(q'e_1 + e_2)\big\rangle \end{aligned}$$

with $q' = \alpha^{-1}au^{-pk+s}$. And thus $\Phi((m,\, n)_q) = (p(m-k)+s,\, p(n+k))_{q'}$ with $v_u(q') = -pk+s \neq k$, as $k \not\equiv 0 \mod (p+1)$.

(iv) Obvious.

(v) If $\mathfrak{M} = [x,\, y]_q$, then the statement on d_2 follows immediately from (ii) and (iii):

$$d_2(\mathfrak{M},\, \langle\Phi(\mathfrak{M})\rangle) = (p-1)y + s = (p-1)\, d_2\Big([x,\, y]_q,\, \Big[\frac{s}{p+1},\, -\frac{s}{p-1}\Big]_0\Big).$$

For the statement on d_1 first assume that $\mathfrak{M} = [x,\, y]_0 \in \mathcal{A}_0$. Then (ii) implies

$$d_1(\mathfrak{M},\, \langle\Phi(\mathfrak{M})\rangle) = |(p+1)x - s| = (p+1)\, d_1\Big([x,\, y]_0,\, \Big[\frac{s}{p+1},\, -\frac{s}{p-1}\Big]_0\Big).$$

If $\mathfrak{M} = [x,\, y]_q \in \mathcal{A}_q\backslash\mathcal{A}_0$, then $x > k$ and $px - 2pk + s > -pk + s$ which implies $\langle\Phi(\mathfrak{M})\rangle \notin \mathcal{A}_0$. Now (iii) implies

$$\begin{aligned} d_1(\mathfrak{M},\, \langle\Phi(\mathfrak{M})\rangle) &= (x-k) + |-pk+s-k| + (px - 2pk + s - (-pk+s))\\ &= (p+1)(x-k) + (p+1)\Big|k - \frac{s}{p+1}\Big|\\ &= \begin{cases} (p+1)\Big(x - \dfrac{s}{p+1}\Big) & \text{if } k > \dfrac{s}{p+1},\\ (p+1)\Big(x - k + \dfrac{s}{p+1} - k\Big) & \text{if } k < \frac{s}{p+1}, \end{cases} \end{aligned}$$

using $k \neq -pk+s$. In both cases the claim follows. □

Remark 3.3. The Lemma shows that the absolutely simple case is exactly the case discussed in [PR2, 6.d, A1]. The fixed point in the building is the point P_{irred} and its projection onto the building for $\mathrm{PGL}_2(\bar{\mathbb{F}}((u)))$ lies on the edge between two vertices. The set of lattices $\mathfrak{M}$ with $d_1(\mathfrak{M},\, \langle\Phi(\mathfrak{M})\rangle) \leqslant r_1 - r_2$ is identified with a ball around this fixed point.

Let $\boldsymbol{v}$ be a collection of integers as in the introduction. By Lemma 2.4 and Proposition 3.2 (v) we find $\mathcal{GR}^{\boldsymbol{v},\mathrm{loc}}_{V_{\mathbb{F}},0}(\bar{\mathbb{F}}) \subset \bar{\mathcal{B}}(m(\boldsymbol{v}))$, where

$$m(\boldsymbol{v}) = (2e - d' - s)/(p-1). \tag{3.2}$$

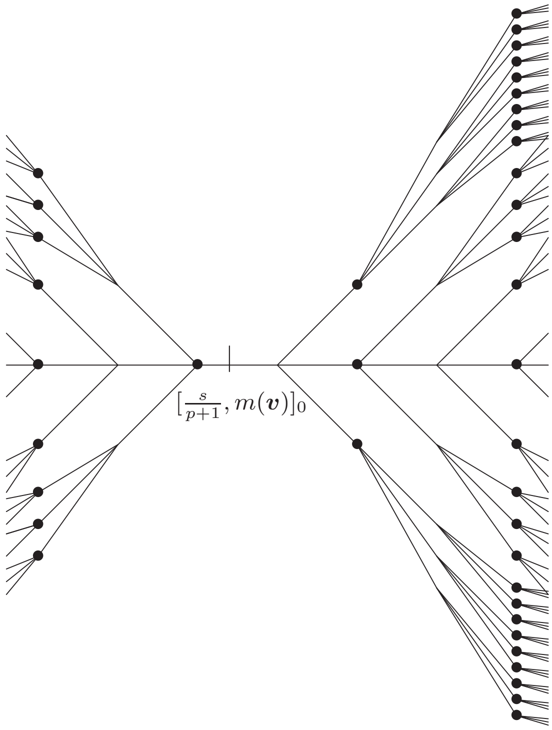


FIGURE 1. This picture illustrates the subset of $\boldsymbol{v}$-admissible lattices in the case $p = 3$ and $\mathbb{F} = \mathbb{F}_3$. This subset is given by all lattices $\mathfrak{M} \in \bar{\mathcal{B}}(m(\boldsymbol{v}))$ satisfying $d_1(\mathfrak{M}, P_{\mathrm{irred}}) \leqslant (r_1 - r_2)/(p+1)$. The fat points correspond to $\boldsymbol{v}$-admissible lattices.

Corollary 3.4. *The scheme $\mathcal{GR}^{\boldsymbol{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$ is empty if $2e - d' \not\equiv s \bmod (p-1)$.*

Proof. This follows from Lemma 2.4 and Proposition 3.2. □

Now we want to define locally closed subschemes of $\mathrm{Grass}\, M_{\mathbb{F}}$ on which the elementary divisors of $\langle \Phi(\mathfrak{M}) \rangle$ with respect to $\mathfrak{M}$ are fixed. Define a function

$$E\colon\ \mathrm{Grass}\, M_{\mathbb{F}} \to \mathbb{Z}^2.$$

For an extension field L of $\mathbb{F}$ and an L-valued point $z \in (\mathrm{Grass}\, M_{\mathbb{F}})(L)$ consider the $\mathbb{F}[\![u]\!] \,\widehat{\otimes}_{\mathbb{F}}\, L$-lattice $\mathfrak{M}_z$ in $M_{\mathbb{F}} \,\widehat{\otimes}_{\mathbb{F}}\, L$ corresponding to z. Then $E(z) = (j_1, j_2)$, where $j_1 \geqslant j_2$ are the elementary divisors of $\langle \Phi(\mathfrak{M}_z) \rangle$ with respect to $\mathfrak{M}_z$. Recall that there is a partial order on the pairs $(a, b) \in \mathbb{Z}^2$ given by (1.3).

Lemma 3.5. *The function E is lower semi-continuous with respect to the Zariski topology on* $\operatorname{Grass} M_{\mathbb{F}}$.

Proof. Let $\eta \rightsquigarrow z$ be a specialization and let $\mathfrak{M}_\eta$ and $\mathfrak{M}_z$ be the lattices corresponding to the points η and z. Denote by $E(\eta) = (a(\eta), b(\eta))$ resp. $E(z) = (a(z), b(z))$ the elementary divisors of $\langle \Phi(\mathfrak{M}_\eta) \rangle$ with respect to $\mathfrak{M}_\eta$ (resp. the elementary divisors of $\langle \Phi(\mathfrak{M}_z) \rangle$ with respect to $\mathfrak{M}_z$). We mark the specialization by a morphism $f\colon \operatorname{Spec} R \to \mathcal{GR}^{\boldsymbol{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$, where R is a discrete valuation ring with uniformizer t. The morphism f defines a $R[\![u]\!]$-lattice $\mathfrak{M}_R$ in $M_{\mathbb{F}} \widehat{\otimes}_{\mathbb{F}} R$. After choosing a basis we find a matrix $C = (c_{ij})_{ij} \in \mathrm{GL}_2(R((u))) \cap M_2(R[\![u]\!])$ such that $\mathfrak{M}_R \sim C$. Using theory of elementary divisors we find

$$b(\eta) = \min_{i,j} v_u(c_{ij}) \leqslant \min_{i,j} v_u(\bar{c}_{ij}) = b(z)$$

and hence $E(\eta) \geqslant E(z)$ which yields the claim. □

Definition 3.6. Let $(a, b) \in \mathbb{Z}^2$ such that $a \geqslant b$. The *Kisin variety* associated to (a, b) is

$$\mathcal{G}_{V_{\mathbb{F}}}(a, b) = E^{-1}(a, b) \subset \operatorname{Grass} M_{\mathbb{F}}.$$

The argument of [Ki, Proposition 2.1.7] shows that there exists an integer i and a fixed lattice $\mathfrak{M}_0$ such that

$$u^i \mathfrak{M}_0 \subset \mathfrak{M} \subset u^{-i} \mathfrak{M}_0$$

for all $\mathfrak{M} \in E^{-1}(a, b)$. Hence $E^{-1}(a, b)$ lies in a closed subscheme of $\operatorname{Grass} M_{\mathbb{F}}$. By Lemma 3.5, this is a locally closed subset and it will be considered as a subscheme with the reduced scheme structure (see also [PR2]).

Now we want to analyse the structure of $\mathcal{G}_{V_{\mathbb{F}}}(a, b)$ and $\mathcal{GR}^{\boldsymbol{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$. We will make use of the following fact.

Lemma 3.7. *Let b_1, b_2 be any basis of $M_{\bar{\mathbb{F}}}$. There exists a morphism*

$$\chi\colon \mathbb{A}^1_{\bar{\mathbb{F}}} \to \operatorname{Grass} M_{\bar{\mathbb{F}}}$$

such that $\chi(z) = \langle b_1, zu^{-1}b_1 + b_2 \rangle$ for every closed point $z \in \mathbb{A}^1_{\bar{\mathbb{F}}}$. The morphism χ extends in a unique way to a morphism

$$\bar{\chi}\colon \mathbb{P}^1_{\bar{\mathbb{F}}} \to \operatorname{Grass} M_{\bar{\mathbb{F}}}.$$

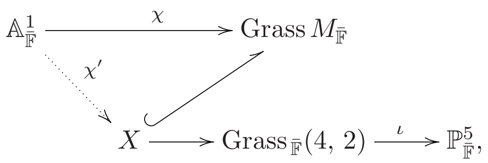
The image of the point at infinity is given by $\bar{\chi}(\infty) = \langle u^{-1}b_1, ub_2 \rangle$.

Proof. Consider the family

$$\langle b_1, Tu^{-1}b_1 + b_2 \rangle_{\bar{\mathbb{F}}[T][\![u]\!]} \subset M_{\bar{\mathbb{F}}} \widehat{\otimes}_{\bar{\mathbb{F}}} \bar{\mathbb{F}}[T]$$

of lattices on $\mathbb{A}^1 = \operatorname{Spec} \bar{\mathbb{F}}[T]$. This family defines the morphism χ.

Let X be the closed subscheme of $\operatorname{Grass} M_{\bar{\mathbb{F}}}$ consisting of all lattices $\mathfrak{M}$ that satisfy $u\langle b_1, b_2 \rangle \subset \mathfrak{M} \subset u^{-1}\langle b_1, b_2 \rangle$ and that lie in the same connected component of $\operatorname{Grass} M_{\bar{\mathbb{F}}}$ as $\langle b_1, b_2 \rangle$. The scheme X is identified with a closed subscheme of the (ordinary) Grassmann variety $\mathrm{Grass}_{\bar{\mathbb{F}}}(4, 2)$ of 2-dimensional subspaces in $\bar{\mathbb{F}}^4$. The morphism χ factors as follows:



where ι is the Plücker embedding. As X is projective, the valuative criterion shows that χ extends in a unique way to $\mathbb{P}^1$. We view $\mathrm{Grass}_{\bar{\mathbb{F}}}(4, 2)$ as the quotient $\mathrm{GL}_{2,\bar{\mathbb{F}}}\backslash V$, where V is the scheme of 2×4 matrices of rank 2 and $\mathrm{GL}_{2,\bar{\mathbb{F}}}$ acts on V by left multiplication. Now, the computations using Plücker coordinates gives

$$\chi'(z) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ z & 0 & 0 & 1 \end{pmatrix}, \quad \iota(\chi'(z)) = (-z : 0 : 0 : 0 : 1 : 0)$$

for all closed points $z \in \mathbb{A}^1(\bar{\mathbb{F}})$. Hence the extension to $\mathbb{P}^1$ is

$$(z_1 : z_2) \mapsto (-z_1 : 0 : 0 : 0 : z_2 : 0).$$

The image of the point at infinity is

$$\bar{\chi}((1:0)) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \mapsto (-1 : 0 : 0 : 0 : 0 : 0).$$

This is the lattice $\langle u^{-1}b_1,\, ub_2\rangle$. □

Remark 3.8. In the building, the $\mathbb{F}$-valued points of the image of the morphism $\bar{\chi}$ can be illustrated in the following way (if the morphism is defined over $\mathbb{F}$):

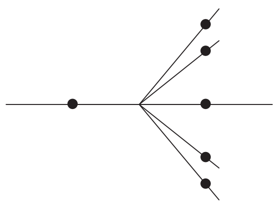


FIGURE 2. The morphism $\bar{\chi}$ in the building for $p = 5$ and $\mathbb{F} = \mathbb{F}_5$.

Similarly, we can define morphisms $\bar{\chi}_1, \bar{\chi}_2 : \mathbb{P}^1_{\bar{\mathbb{F}}} \to \mathrm{Grass}\, M_{\bar{\mathbb{F}}}$ such that

$$\mathrm{im}(\bar{\chi}_1) = \{\langle u^{n-1}b_1,\, u^{-(n-1)}(zu^{-1}b_1 + b_2)\rangle : z \in \bar{\mathbb{F}}\} \cup \{\langle u^{-n}b_1,\, u^n b_2\rangle\},$$
$$\mathrm{im}(\bar{\chi}_2) = \{\langle u^{n}b_1,\, u^{-n}(zb_1 + b_2)\rangle : z \in \bar{\mathbb{F}}\} \cup \{\langle u^{-n}b_1,\, u^n b_2\rangle\}.$$

Theorem 3.9. *Assume that* $(M_{\mathbb{F}}, \Phi)$ *is absolutely simple. Fix a finite extension* $\mathbb{F}'$ *of* $\mathbb{F}$ *such that the normal form for* Φ *of Lemma 3.1 is defined over* $\mathbb{F}'$.

(a) *For any* $(a, b) \in \mathbb{Z}^2$ *with* $a \geqslant b$:

$$\mathcal{G}_{V_{\mathbb{F}}}(a, b) \neq \varnothing \;\Leftrightarrow\; a + b \equiv s \mod (p-1), \quad \begin{cases} pa + b \equiv s \mod (p^2 - 1), \text{ or} \\ pa + b \equiv ps \mod (p^2 - 1). \end{cases}$$

This condition being satisfied, there exists an isomorphism

$$\mathcal{G}_{V_{\mathbb{F}}}(a,\, b) \otimes_{\mathbb{F}} \mathbb{F}' \cong \mathbb{A}^n_{\mathbb{F}'}$$

with $n = \lfloor \frac{a-b}{p+1} \rfloor$. *Further*

$$\overline{\mathcal{G}_{V_{\mathbb{F}}}(a,\, b)} = \bigcup_{(a',b') \leqslant (a,b)} \mathcal{G}_{V_{\mathbb{F}}}(a',\, b').$$

(b) *The scheme* $\mathcal{GR}^{\boldsymbol{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$ *is geometrically connected and irreducible. After extending the scalars to* $\mathbb{F}'$ *it becomes isomorphic to a Schubert variety in the affine Grassmannian for* $M_{\mathbb{F}'} = M_{\mathbb{F}} \,\widehat{\otimes}_{\mathbb{F}}\, \mathbb{F}'$ *with dimension given by*

$$\dim \mathcal{GR}^{\boldsymbol{v},\mathrm{loc}}_{V_{\mathbb{F}},0} = \left\lfloor \frac{r_1 - r_2}{p+1} - (-1)^{\epsilon} \frac{s}{p+1} \right\rfloor + \left\lfloor (-1)^{\epsilon} \frac{s}{p+1} \right\rfloor,$$

with $\epsilon = \lfloor \frac{r_1 - r_2}{p+1} \rfloor + \lfloor \frac{s}{p+1} \rfloor + \frac{2e - d' - s}{p-1}$.

[Here, as in the rest of the paper, $\lfloor x \rfloor$ denotes the integral part of a real number x.]

Proof. (a) Assume $\mathfrak{M} \in \mathcal{G}_{V_{\mathbb{F}}}(a,\, b) \neq \varnothing$. Without loss of generality, we may assume $\mathfrak{M} = [x,\, y]_0 \in \mathcal{A}_0$: if $\mathfrak{M}$ is an arbitrary lattice, then there exists a lattice $\mathfrak{M}' \in \mathcal{A}_0$ such that $d_i(\mathfrak{M},\, P_{\mathrm{irred}}) = d_i(\mathfrak{M}',\, P_{\mathrm{irred}})$ for $i = 1, 2$ (compare Fig. 1, for example).

By Lemma 3.2 (v) and Definition 2.2, the condition for $\mathfrak{M} = [x,\, y]_0 \in \mathcal{G}_{V_{\mathbb{F}}}(a,\, b)$ is

$$(p+1) d_1(\mathfrak{M},\, P_{\mathrm{irred}}) = a - b \qquad (p-1) d_2(\mathfrak{M},\, P_{\mathrm{irred}}) = a + b.$$

By an explicit computation of these distances, this is equivalent to

$$\left| x - \frac{s}{p+1} \right| = \frac{a-b}{p+1}, \quad y + \frac{s}{p-1} = \frac{a+b}{p-1}.$$

The second equation gives $s \equiv a + b \bmod (p-1)$ and the sum of both equations gives $s \equiv pa + b \bmod (p^2 - 1)$ if $(p+1)x > s$ and $ps \equiv pa + b \bmod (p^2 - 1)$ if $(p+1)x < s$ (using the fact that $x + y$ and $x - y$ are even).

Conversely, suppose $s \equiv a + b \bmod (p-1)$ and $s \equiv pa + b \bmod (p^2 - 1)$ and define

$$x = \frac{a - b + s}{p+1}, \quad y = \frac{a + b - s}{p-1}.$$

Then we have $y \in \mathbb{Z}$ and $x + y \in 2\mathbb{Z}$. Thus $[x,\, y]_0$ defines a lattice $\mathfrak{M} \in \mathcal{G}_{V_{\mathbb{F}}}(a,\, b)$. If $ps \equiv pa + b \bmod (p^2 - 1)$ we use

$$x = \frac{s - (a-b)}{p+1}, \quad y = \frac{a + b - s}{p-1}.$$

Now fix the sum $a + b$ and denote by y the integer solving the equation

$$(p-1)y + s = a + b.$$

Let us assume that $x_0 := \lfloor \frac{s}{p+1} \rfloor \equiv y \bmod 2$ (the case $x_0 \not\equiv y \bmod 2$ admits a similar treatment). In this case $[x_0,\, y]_0$ defines a lattice $\mathfrak{M}_0$ and we denote by X the connected component of $\operatorname{Grass} M_{\mathbb{F}'}$ containing $\mathfrak{M}_0$, i.e., $X(\bar{\mathbb{F}}) = \{\mathfrak{M} \in \bar{\mathcal{B}}(y)\}$.

For each $m \geqslant 0$, there is a morphism $f_m\colon \mathbb{A}^{2m+1}_{\mathbb{F}'} \to X$ given by the family of lattices

$$\left\langle u^{(x_0+y)/2}u^{m+1}e_1,\ u^{(y-x_0)/2}u^{-(m+1)}\left(\left(\sum_{i=1}^{2m+1} T_i u^{i+x_0}\right)e_1 + e_2\right)\right\rangle \subset M_{\mathbb{F}'}\ \widehat{\otimes}_{\mathbb{F}'}\ (\mathbb{F}'[T_1,\ \dots,\ T_{2m+1}])$$

on $\mathbb{A}^{2m+1}_{\mathbb{F}'} = \operatorname{Spec}\mathbb{F}'[T_1,\ \dots,\ T_{2m+1}]$. Let $V_m \cong \mathbb{A}^{2m+1}_{\mathbb{F}'}$ be its image. We have

$$\begin{aligned}\bar{\mathcal{B}}(y) &\supset V_m(\bar{\mathbb{F}}) \\ &= \left\{\left\langle u^{(x_0+y)/2}u^{m+1}e_1,\ u^{(y-x_0)/2}u^{-(m+1)}(qe_1+e_2)\right\rangle\colon q = \sum_{i=1}^{2m+1} a_i u^{i+x_0}\right\},\end{aligned} \tag{3.3}$$

with $a_1,\ \dots,\ a_{2m+1} \in \bar{\mathbb{F}}$. Similarly, define for $m \geqslant 0$ a morphism $g_m\colon \mathbb{A}^{2m}_{\mathbb{F}'} \to X$ given by the family of lattices x

$$\left\langle u^{(x_0+y)/2}u^{-m}\left(e_1 + \left(\sum_{i=0}^{2m-1} T_i u^{i-x_0}\right)e_2\right),\ u^{(y-x_0)/2}u^m e_2\right\rangle \subset M_{\mathbb{F}'}\ \widehat{\otimes}_{\mathbb{F}'}\ (\mathbb{F}'[T_0,\ \dots,\ T_{2m-1}])$$

and let $U_m \cong \mathbb{A}^{2m}_{\mathbb{F}'}$ be its image. We have

$$\begin{aligned}\bar{\mathcal{B}}(y) &\supset U_m(\bar{\mathbb{F}}) \\ &= \left\{\left\langle u^{(x_0+y)/2}u^{-m}(e_1+qe_2),\ u^{(y-x_0)/2}u^m e_2\right\rangle\colon q = \sum_{i=0}^{2m-1} a_i u^{i-x_0}\right\},\end{aligned} \tag{3.4}$$

with $a_0 \dots a_{2m-1} \in \bar{\mathbb{F}}$. It is easy to see that every lattice $\mathfrak{M} \in \bar{\mathcal{B}}(y)$ is either of the form (3.3) or of the form (3.4) for some $m \geqslant 0$. Thus

$$X = \Big(\bigcup_{m\geqslant 0} V_m\Big) \cup \Big(\bigcup_{m\geqslant 0} U_m\Big).$$

We claim

$$\begin{aligned}\mathfrak{M} \in V_m(\bar{\mathbb{F}}) &\Rightarrow d_1(\mathfrak{M},\ P_{\mathrm{irred}}) = 2m+2-\xi, \\ \mathfrak{M} \in U_m(\bar{\mathbb{F}}) &\Rightarrow d_1(\mathfrak{M},\ P_{\mathrm{irred}}) = 2m+\xi,\end{aligned} \tag{3.5}$$

where $\xi = \frac{s}{p+1} - x_0$ denotes the fractional part of $\frac{s}{p+1}$.

Indeed, if $\mathfrak{M} \in V_m(\bar{\mathbb{F}})$, then $\mathfrak{M} = [x_0+2m+2,\ y]_q$ for some $q \in \bar{\mathbb{F}}((u))$ with $v_u(q) > x_0$ and hence

$$d_1(\mathfrak{M},\ P_{\mathrm{irred}}) = x_0 + 2m + 2 - \frac{s}{p+1} = 2m+2-\xi.$$

The statement on U_m follows by a more complicated computation or by some symmetry argument: The choice of apartments $\mathcal{A}_q$ and coordinates $[-,\ -]_q$ depends on the order of e_1 and e_2. Interchanging e_1 and e_2 yields expressions for the lattices $\mathfrak{M} \in U_m$ similar to the above expressions for V_m (if $\mathfrak{M} \in U_m$ is a lattice, then $\mathfrak{M} = [-x_0+2m,\ y]_q$ for some q) while it maps the point P_{irred} to $\left[-\frac{s}{p+1},\ -\frac{s}{p-1}\right]_0$ and hence the claim follows by the same computation.

Now equation (3.5) together with Proposition 3.2 (v) implies

$$\begin{aligned} V_m(\bar{\mathbb{F}}) &\subset \mathcal{G}_{V_{\mathbb{F}}}(a_{\text{odd}}(m),\, b_{\text{odd}}(m))(\bar{\mathbb{F}}), \\ U_m(\bar{\mathbb{F}}) &\subset \mathcal{G}_{V_{\mathbb{F}}}(a_{\text{even}}(m),\, b_{\text{even}}(m))(\bar{\mathbb{F}}) \end{aligned} \tag{3.6}$$

for some $(a_{\text{odd}}(m),\, b_{\text{odd}}(m))$, $(a_{\text{even}}(m),\, b_{\text{even}}(m)) \in \mathbb{Z}^2$ with

$$\begin{aligned} a_{\text{odd}}(m) + b_{\text{odd}}(m) = a_{\text{even}}(m') + b_{\text{even}}(m') &= (p-1)y + s, \\ a_{\text{odd}}(m) - b_{\text{odd}}(m) &= (p+1)(2m+2-\xi), \\ a_{\text{even}}(m) - b_{\text{even}}(m) &= (p+1)(2m+\xi) \end{aligned} \tag{3.7}$$

and $0 < \xi < 1$ implies that all these pairs are pairwise distinct when m runs over all positive integers.

As U_m and V_m cover X, the inclusions in (3.6) are actually equalities. Furthermore $V_m = \mathcal{G}_{V_{\mathbb{F}}}(a_{\text{odd}}(m),\, b_{\text{odd}}(m))$ as schemes, as both are reduced locally closed subschemes of Grass $M_{\mathbb{F}'}$ with the same underlying point set. Finally (3.7) yields

$$\dim V_m = 2m + 1 = \lfloor 2m + 2 - \xi \rfloor = \left\lfloor \frac{a_{\text{odd}}(m) - b_{\text{odd}}(m)}{p+1} \right\rfloor.$$

The conclusion for U_m is similar.

To finish the proof of (a), it remains to show that $U_m \subset \overline{V_m}$ and $V_{m-1} \subset \overline{U_m}$. We will prove the first assertion: the second is proved in the same way.

Let $z_1 \in U_m$ be an arbitrary point corresponding to a lattice

$$\mathfrak{M}_1 = \big\langle u^{(x_0+y)/2} u^{-m}(e_1 + q e_2),\, u^{(y-x_0)/2} u^m e_2 \big\rangle$$

with $q = \sum_{i=0}^{2m-1} a_i u^{i-x_0}$ and let $z_2 \in V_m$ be the point corresponding to

$$\mathfrak{M}_2 = \big\langle u^{(x_0+y)/2} u^{m+1} e_1,\, u^{(y-x_0)/2} u^{-(m+1)} e_2 \big\rangle.$$

There exists a basis b_1 and b_2 of $M_{\bar{\mathbb{F}}}$ such that

$$\begin{aligned} \langle b_1,\, b_2 \rangle &= \mathfrak{M}_0 = [x_0,\, y]_0, \\ \langle u^{-m} b_1,\, u^m b_2 \rangle &= \mathfrak{M}_1, \\ \langle u^{m+1} b_1,\, u^{-(m+1)} b_2 \rangle &= \mathfrak{M}_2. \end{aligned}$$

Explicitly, we may choose

$$b_1 = u^{(x_0+y)/2}(e_1 + q e_2), \qquad b_2 = u^{(y-x_0)/2} e_2.$$

Applying Lemma 3.7 (resp. Remark 3.8) with the basis ub_1, $u^{-1}b_2$, we obtain a morphism $\chi\colon \mathbb{A}^1_{\bar{\mathbb{F}}} \to \operatorname{Grass} M_{\bar{\mathbb{F}}}$ that is given by $\chi(z) = \langle u^{m+1} b_1,\, u^{-(m+1)}(z u b_1 + b_2) \rangle$ on closed points and we easily find $\operatorname{im} \chi \subset V_m \otimes_{\mathbb{F}'} \bar{\mathbb{F}}$. As $\bar{V}_m \otimes_{\mathbb{F}'} \bar{\mathbb{F}}$ is projective, the morphism χ extends to a morphism from $\mathbb{P}^1$ to $\bar{V}_m \otimes_{\mathbb{F}'} \bar{\mathbb{F}}$ and the point at infinity is mapped to z_1 (Fig. 3 illustrates the image of the morphism $\bar{\chi}$ in the building. The fat points are the lattices in the image of $\bar{\chi}$). Hence $z_1 \in \bar{V}_m(\bar{\mathbb{F}})$ and the claim follows.

(b) For a given collection $\boldsymbol{v}$ we have

$$\mathcal{GR}^{\boldsymbol{v},\text{loc}}_{V_{\mathbb{F}},0} = \bigcup_{\substack{a+b=2e-d' \\ e-r_1 \leqslant b \leqslant a \leqslant e-r_2}} \mathcal{G}_{V_{\mathbb{F}}}(a,\, b), \tag{3.8}$$

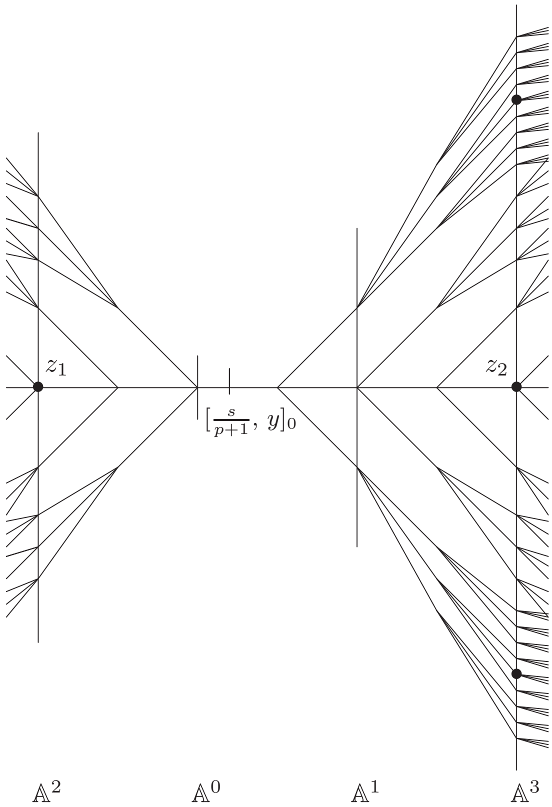


FIGURE 3. The stratification with affine spaces in the building. Fat points mark the image of an exemplary morphism $\bar{\chi}$.

where d' is the integer defined in (2.1). Hence the scheme is geometrically irreducible, because the restriction of the order "$\leqslant$" on the pairs

$$\{(a, b) \in \mathbb{Z}^2 : a + b = m(\boldsymbol{v})\},$$

where $m(\boldsymbol{v})$ is given by (3.2), is a total order. Of course this also implies connectedness.

The dimension of $\mathcal{GR}^{\boldsymbol{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$ is given by the dimension of the maximal affine space in (3.8). We assume that ϵ is even, i.e., $\lfloor \frac{r_1 - r_2}{p+1} \rfloor + x_0 \equiv m(\boldsymbol{v}) \bmod 2$. The computations in the other case are similar.

In this case the affine subspace of maximal dimension consists of all lattices $\mathfrak{M} \in \bar{\mathcal{B}}(m(\boldsymbol{v}))$ with

$$d_1(\mathfrak{M}, P_{\mathrm{irred}}) = d_1\Big(\Big[x_0 - \Big\lfloor \frac{r_1 - r_2}{p+1} \Big\rfloor, m(\boldsymbol{v})\Big]_0, P_{\mathrm{irred}}\Big)$$

(if the latter distance is $\leqslant \frac{r_1-r_2}{p+1}$) or of the lattices with

$$d_1(\mathfrak{M},\, P_{\mathrm{irred}}) = d_1\Big(\Big[x_0 + \Big\lfloor\frac{r_1-r_2}{p+1}\Big\rfloor,\, m(\boldsymbol{v})\Big]_0,\, P_{\mathrm{irred}}\Big)$$

(if $d_1([x_0 - \lfloor\frac{r_1-r_2}{p+1}\rfloor,\, m(\boldsymbol{v})]_0,\, P_{\mathrm{irred}}) > \frac{r_1-r_2}{p+1}$).

Hence its dimension is either $n := \lfloor\frac{r_1-r_2}{p+1}\rfloor$ (in the first case) or $n-1$ (in the second case). This yields the claim on the dimension:

$$\begin{aligned}\dim \mathcal{GR}^{\boldsymbol{v},\mathrm{loc}}_{V_{\mathbb{F}},0} &= \Big\lfloor\frac{r_1-r_2}{p+1} - \frac{s}{p+1}\Big\rfloor + \Big\lfloor\frac{s}{p+1}\Big\rfloor \\ &= \begin{cases} n & \text{if } \frac{s}{p+1} - \Big\lfloor\frac{s}{p+1}\Big\rfloor \leqslant \frac{r_1-r_2}{p+1} - \Big\lfloor\frac{r_1-r_2}{p+1}\Big\rfloor, \\ n-1 & \text{if } \frac{s}{p+1} - \Big\lfloor\frac{s}{p+1}\Big\rfloor > \frac{r_1-r_2}{p+1} - \Big\lfloor\frac{r_1-r_2}{p+1}\Big\rfloor. \end{cases}\end{aligned}$$

We further see that the set of $\boldsymbol{v}$-admissible lattices is exactly the set of lattices $\mathfrak{M}$ in $\bar{\mathcal{B}}(m(\boldsymbol{v}))$ with

$$\begin{aligned} &d_1(\mathfrak{M},\, [x_0,\, m(\boldsymbol{v})]_0) \leqslant n && \text{if } x_0 + \Big\lfloor\frac{r_1-r_2}{p+1}\Big\rfloor - \frac{s}{p+1} \leqslant \frac{r_1-r_2}{p+1}, \\ &d_1(\mathfrak{M},\, [x_0+1,\, m(\boldsymbol{v})]_0) \leqslant n-1 && \text{otherwise,}\end{aligned} \tag{3.9}$$

and hence this is the set of lattices whose elementary divisors $(a,\, b)$ with respect to a lattice $\mathfrak{N}$ satisfy $(a,\, b) \leqslant (a_{\max},\, b_{\max})$ for some given integers $a_{\max}$, $b_{\max}$. For $\mathfrak{N}$ we choose one of the lattices

$$[x_0,\, m(\boldsymbol{v})]_0,\;\; [x_0,\, m(\boldsymbol{v})-1]_0 \quad \text{or} \quad [x_0+1,\, m(\boldsymbol{v})]_0,\;\; [x_0+1,\, m(\boldsymbol{v})-1]_0$$

depending on the cases as listed in (3.9) and on $x_0 - m(\boldsymbol{v}) \bmod 2$. Since we know that $\mathcal{GR}^{\boldsymbol{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$ is reduced, we find that it is isomorphic to a Schubert variety in the affine Grassmannian after extending the scalars to $\mathbb{F}'$.

If ϵ is odd, then the maximal affine subspace consists of all lattices $\mathfrak{M} \in \bar{\mathcal{B}}(m(\boldsymbol{v}))$ with

$$d_1(\mathfrak{M},\, P_{\mathrm{irred}}) = d_1\Big(\Big[x_0 + 1 + \Big\lfloor\frac{r_1-r_2}{p+1}\Big\rfloor,\, m(\boldsymbol{v})\Big]_0,\, P_{\mathrm{irred}}\Big)$$

(if the latter distance is $\leqslant \frac{r_1-r_2}{p+1}$) or of the lattices with

$$d_1(\mathfrak{M},\, P_{\mathrm{irred}}) = d_1\Big(\Big[x_0 + 1 - \Big\lfloor\frac{r_1-r_2}{p+1}\Big\rfloor,\, m(\boldsymbol{v})\Big]_0,\, P_{\mathrm{irred}}\Big)$$

(if $d_1([x_0 + 1 + \lfloor\frac{r_1-r_2}{p+1}\rfloor,\, m(\boldsymbol{v})]_0,\, P_{\mathrm{irred}}) > \frac{r_1-r_2}{p+1}$).

We find

$$\begin{aligned}\dim \mathcal{GR}^{\boldsymbol{v},\mathrm{loc}}_{V_{\mathbb{F}},0} &= \Big\lfloor\frac{r_1-r_2}{p+1} + \frac{s}{p+1}\Big\rfloor + \Big\lfloor-\frac{s}{p+1}\Big\rfloor \\ &= \begin{cases} n & \text{if } 1 - \Big(\frac{s}{p+1} - \Big\lfloor\frac{s}{p+1}\Big\rfloor\Big) \leqslant \frac{r_1-r_2}{p+1} - \Big\lfloor\frac{r_1-r_2}{p+1}\Big\rfloor, \\ n-1 & \text{if } 1 - \Big(\frac{s}{p+1} - \Big\lfloor\frac{s}{p+1}\Big\rfloor\Big) > \frac{r_1-r_2}{p+1} - \Big\lfloor\frac{r_1-r_2}{p+1}\Big\rfloor, \end{cases}\end{aligned}$$

and the conclusion for the isomorphism with a Schubert variety is similar. □

As a consequence of the theorem, we may determine the cases when $\mathcal{GR}^{\boldsymbol{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$ is a single point.

Corollary 3.10. *Denote by* $\xi = \frac{s}{p+1} - \lfloor \frac{s}{p+1} \rfloor$ *the fractional part of* $\frac{s}{p+1}$.

$$\mathcal{GR}^{\boldsymbol{v},\mathrm{loc}}_{V_{\mathbb{F}},0} = \{*\} \Leftrightarrow \begin{cases} 0+\xi \leqslant \dfrac{r_1-r_2}{p+1} < 2-\xi & \text{if } \left\lfloor \dfrac{s}{p+1} \right\rfloor \equiv \dfrac{2e-d'-s}{p-1} \mod 2, \\ 1-\xi \leqslant \dfrac{r_1-r_2}{p+1} < 1+\xi & \text{if } \left\lfloor \dfrac{s}{p+1} \right\rfloor \not\equiv \dfrac{2e-d'-s}{p-1} \mod 2. \end{cases}$$

Proof. This is just the case where the dimension of $\mathcal{GR}^{\boldsymbol{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$ is zero. See [He, Cor. 3.10] for a more detailed discussion. □

4. The Reducible Case

In this section we want to analyse the case, where $(M_{\mathbb{F}}, \Phi)$ admits a proper Φ-stable subobject, at least after extending the scalars to some finite extension of $\mathbb{F}$. Before we start to determine the set of $\boldsymbol{v}$-admissible lattices in the building, we want to formulate the precise statement on the connected components of $\mathcal{GR}^{\boldsymbol{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$. We first define some open and closed subschemes of $\mathcal{GR}^{\boldsymbol{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$.

Definition 4.1. For $a \in \bar{\mathbb{F}}^{\times}$ and $j \in \mathbb{Z}_{\geqslant 0}$ define $(\mathfrak{M}^j(a), \Phi^j_a)$ by

$$\mathfrak{M}^j(a) = \bar{\mathbb{F}}[\![u]\!], \quad \Phi^j_a(1) = au^j.$$

Definition 4.2. A $\boldsymbol{v}$-admissible lattice $\mathfrak{M} \subset M_{\bar{\mathbb{F}}}$ is called $\boldsymbol{v}$-*ordinary* if there exists a short exact sequence

$$0 \to (\mathfrak{M}^{e-r_1}(a), \Phi^{e-r_1}_a) \to (\mathfrak{M}, \Phi) \to (\mathfrak{M}^{e-r_2}(b), \Phi^{e-r_2}_b) \to 0 \qquad (4.0.1)$$

for some $a, b \in \bar{\mathbb{F}}^{\times}$.

Remark 4.3. The determinant condition in (1.2) implies that

$$u^{e-r_1}\mathfrak{M} \subset \langle \Phi(\mathfrak{M}) \rangle \subset u^{e-r_2}\mathfrak{M} \qquad (4.0.2)$$

for all $\boldsymbol{v}$-admissible lattices $\mathfrak{M}$. Hence the $\boldsymbol{v}$-ordinary lattices are the lattices which admit a Φ-stable subobject with the minimal possible elementary divisors. If a $\boldsymbol{v}$-admissible lattice $(\mathfrak{M}, \Phi)$ admits a subobject isomorphic to $(\mathfrak{M}^{e-r_1}(a), \Phi^{e-r_1}_a)$ for some $a \in \bar{\mathbb{F}}^{\times}$, then the quotient has no u-torsion by (4.0.2) and is isomorphic to $(\mathfrak{M}^{e-r_2}(b), \Phi^{e-r_2}_b)$ for some $b \in \bar{\mathbb{F}}^{\times}$, because the sum of the elementary divisors is fixed by (1.2). Hence $(\mathfrak{M}, \Phi)$ is $\boldsymbol{v}$-ordinary in this case.

Denote by $\mathcal{S}(\boldsymbol{v})$ the set of isomorphism classes of one dimensional $\bar{\mathbb{F}}((u))$-modules M' with ϕ-linear map $\Phi' \neq 0$ such that M' admits a (unique) lattice $\mathfrak{M}_{[M']} \subset M'$ with $\langle \Phi(\mathfrak{M}_{[M']}) \rangle = u^{e-r_1}\mathfrak{M}_{[M']}$. The elements of $\mathcal{S}(\boldsymbol{v})$ are in bijection with the elements of $\bar{\mathbb{F}}^{\times}$: For each $a \in \bar{\mathbb{F}}^{\times}$ there is a unique isomorphism class represented by

$$(M_a, \Phi_a) = (\mathfrak{M}^{e-r_1}(a)[\tfrac{1}{u}], \Phi^{e-r_1}_a). \qquad (4.0.3)$$

Set $X = \mathcal{GR}^{\boldsymbol{v},\mathrm{loc}}_{V_{\mathbb{F}},0} \otimes_{\mathbb{F}} \bar{\mathbb{F}}$. On X there is a universal sheaf of $\bar{\mathbb{F}}[\![u]\!] \,\widehat{\otimes}_{\bar{\mathbb{F}}}\, \mathcal{O}_X = \mathcal{O}_X[\![u]\!]$-lattices $\mathcal{M} \subset M_{\mathbb{F}} \,\widehat{\otimes}_{\mathbb{F}}\, \mathcal{O}_X$ satisfying

$$u^e \mathcal{M} \subset (\mathrm{id} \otimes \Phi)\phi^* \mathcal{M} \subset \mathcal{M}.$$

For each $[M'] \in \mathcal{S}(\boldsymbol{v})$ define a sheaf of $\mathcal{O}_X$-modules

$$\mathcal{F}_{[M']} = \mathcal{H}om_{\mathcal{O}_X[\![u]\!],\Phi}(\mathfrak{M}_{[M']} \,\widehat{\otimes}_{\mathbb{F}}\, \mathcal{O}_X,\, \mathcal{M}) \tag{4.0.4}$$

where the subscript Φ indicates that the homomorphism have to commute with the semi-linear maps that are part of the data.

Proposition 4.4. (i) *For each $[M'] \in \mathcal{S}(\boldsymbol{v})$ the sheaf $\mathcal{F}_{[M']}$ is a coherent $\mathcal{O}_X$-module.*

(ii) *A closed point $x \in \mathcal{GR}^{\boldsymbol{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$ corresponds to a non-$\boldsymbol{v}$-ordinary lattice if and only if $\mathcal{F}_{[M']} \otimes \kappa(x) = 0$ for all $[M'] \in \mathcal{S}(\boldsymbol{v})$.*

Proof. (i) For the isomorphism class $[M']$ we choose a representative of the form M_a defined in (4.0.3). Let $U = \operatorname{Spec} A \subset X$ an affine open. We claim

(a) $\mathrm{Hom}_{\mathrm{A}[\![\mathrm{u}]\!],\Phi}(\mathfrak{M}_{[\mathrm{M}']} \,\widehat{\otimes}_{\bar{\mathbb{F}}}\, \mathrm{A},\, \mathcal{M}(\mathrm{U}))$ *is a finitely generated A-module.*

(b) *If $V = \operatorname{Spec} \mathrm{B} \subset \mathrm{U}$ is an affine open, we have*

$$\mathrm{Hom}_{B[\![u]\!],\Phi}(\mathfrak{M}_{[M']} \,\widehat{\otimes}_{\bar{\mathbb{F}}}\, B,\, \mathcal{M}(V)) \cong \mathrm{Hom}_{A[\![u]\!],\Phi}(\mathfrak{M}_{[M']} \,\widehat{\otimes}_{\bar{\mathbb{F}}}\, A,\, \mathcal{M}(U)) \otimes_A B. \tag{4.0.5}$$

This implies the first part of the Proposition.

Proof of (a). Because $\mathfrak{M}_{[M']} \,\widehat{\otimes}_{\bar{\mathbb{F}}}\, A$ is a free $A[\![u]\!]$-module of rank one, a morphism is given by the image of 1 and hence

$$\mathrm{Hom}_{A[\![u]\!],\Phi}(\mathfrak{M}_{[M']} \,\widehat{\otimes}_{\bar{\mathbb{F}}}\, A,\, \mathcal{M}(U)) \cong N_A \subset \mathcal{M}(U),$$

where N_A is the A-submodule of all $v \in \mathcal{M}(U)$ satisfying $\Phi(v) = au^{e-r_1}v$. We claim that the reduction modulo u^{e+1} induces an injective homomorphism

$$N_A \hookrightarrow \mathcal{M}(U)/u^{e+1}\mathcal{M}(U),$$

and hence N_A is finitely generated as an A-module, because the scheme X is noetherian. Now, if $0 \neq v = u^n w \in N_A$ with $n \geqslant 0$ and $w \in \mathcal{M}(U)\backslash u\mathcal{M}(U)$, then

$$u^{pn}\Phi(w) = \Phi(u^n w) = au^{e-r_1+n}w$$

and hence $0 \leqslant e - r_1 - (p-1)n \leqslant e$ which implies $n \leqslant e$.

Proof of (b). We have the following commutative diagram

$$\begin{array}{ccccc}
\mathrm{Hom}_{A[\![u]\!],\Phi}(\mathfrak{M}_{[M']} \,\widehat{\otimes}_{\bar{\mathbb{F}}}\, A,\, \mathcal{M}(U)) & \xrightarrow{\cong} & N_A & \hookrightarrow & \mathcal{M}(U) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Hom}_{B[\![u]\!],\Phi}(\mathfrak{M}_{[M']} \,\widehat{\otimes}_{\bar{\mathbb{F}}}\, B,\, \mathcal{M}(V)) & \xrightarrow{\cong} & N_B & \hookrightarrow & \mathcal{M}(V) \cong \mathcal{M}(U) \,\widehat{\otimes}_A\, B.
\end{array}$$

As N_A is a finitely generated A-module, we do not need to complete the tensor product to obtain N_B from N_A (there are only finitely many denominators). Hence (4.0.5) is an isomorphism.

(ii) Let $[M'] \in \mathcal{S}(\boldsymbol{v})$ be an isomorphism class and suppose that $x \in \mathcal{GR}^{\boldsymbol{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$ is a closed point corresponding to a lattice $\mathfrak{M}$ such that $\mathcal{F}_{[M']} \otimes \kappa(x) \neq 0$, i.e., there exists a non trivial morphism

$$f\colon \mathfrak{M}_{[M']} \to \mathfrak{M}.$$

As both sides are free $\bar{\mathbb{F}}[\![u]\!]$-modules and the morphism is non trivial, it is injective. We have to convince ourselves that $\operatorname{coker} f$ has no u-torsion: in this case $\mathfrak{M}$ is the extension of free $\bar{\mathbb{F}}[\![u]\!]$-modules of rank 1 (an extension of $\operatorname{coker} f$ by $\operatorname{im} f$), and hence $\mathfrak{M}$ is $\boldsymbol{v}$-ordinary.

We write $f(1) = u^n v$ for some $n \in \mathbb{Z}$ and some $v \in \mathfrak{M}\backslash u\mathfrak{M}$. We have to show that $n = 0$. Because of $\Phi(f(1)) = f(\Phi(1))$, we find $\Phi(v) = au^{e-r_1-(p-1)n} \in \Phi(\mathfrak{M}) \subset u^{e-r_1}\mathfrak{M}$ for some $a \in \bar{\mathbb{F}}^{\times}$ and hence $n = 0$, as desired.

Conversely, if $\mathfrak{M}$ is $\boldsymbol{v}$-ordinary, then, by definition, there is a subobject isomorphic to $(\mathfrak{M}^{e-r_1}(a), \Phi_a^{e-r_1})$. This is isomorphic to $\mathfrak{M}_{[M']}$ for some $[M'] \in \mathcal{S}(\boldsymbol{v})$ and hence defines the desired non-trivial arrow $\mathfrak{M}_{[M']} \to \mathfrak{M}$. □

Definition 4.5. For each isomorphism class $[M'] \in \mathcal{S}(\boldsymbol{v})$ define

$$X^{\boldsymbol{v}}_{[M']} = \{x \in \mathcal{GR}^{\boldsymbol{v},\mathrm{loc}}_{V_{\mathbb{F}},0} \otimes_{\mathbb{F}} \bar{\mathbb{F}} \colon \mathcal{F}_{[M']} \otimes \kappa(x) \neq 0\}.$$

Further define

$$X^{\boldsymbol{v}}_0 = \mathcal{GR}^{\boldsymbol{v},\mathrm{loc}}_{V_{\mathbb{F}},0} \otimes_{\mathbb{F}} \bar{\mathbb{F}} \setminus \bigcup_{[M'] \in \mathcal{S}(\boldsymbol{v})} X^{\boldsymbol{v}}_{[M']}.$$

By the proposition below these subsets are open and closed and hence they come along with a canonical scheme structure.

Proposition 4.6. (i) *The subset $X^{\boldsymbol{v}}_{[M']}$ is open and closed for each $[M'] \in \mathcal{S}(\boldsymbol{v})$.*

(ii) *The subset $X^{\boldsymbol{v}}_0$ is open and closed.*

Proof. (i) It is clear that $X^{\boldsymbol{v}}_{[M']}$ is closed, as $\mathcal{F}_{[M']}$ is coherent. We show that it is closed under cospecialization.

Let $\eta \rightsquigarrow x$ be a specialization with $x \in X^{\boldsymbol{v}}_{[M']}$ and assume that x is a closed point. We mark this specialization by $\operatorname{Spec} R \to X$, where R is a discrete valuation ring with uniformizer t and residue field $\bar{\mathbb{F}}$. Denote by $\mathfrak{M}_R$ the $R[\![u]\!]$-lattice in $M_{\bar{\mathbb{F}}} \mathbin{\widehat{\otimes}}_{\bar{\mathbb{F}}} R$ defined by this morphism. Because of $\mathcal{F}_{[M']} \otimes \kappa(x) \neq 0$, there is a non trivial morphism $\mathfrak{M}_{[M']} \to \mathfrak{M}_x$ and hence there is a basis vector $b_1 \in M_{\bar{\mathbb{F}}}$ such that $\Phi(b_1) = au^{e-r_1}b_1$ for some $a \in \bar{\mathbb{F}}^{\times}$. As $\mathfrak{M}_R$ is a free $R[\![u]\!]$-module, there is a basis of $\mathfrak{M}_R$ such that

$$\mathfrak{M}_R \sim \begin{pmatrix} \alpha & \gamma \\ 0 & \delta \end{pmatrix},$$

for $\alpha, \gamma, \delta \in R[\![u]\!]$, with $\alpha \equiv au^{e-r_1} \mod t$. But the determinant condition in (1.2) implies $v_u(\alpha) \geqslant e - r_1$. Hence $v_u(\alpha) = e - r_1$ and $\eta \in X^{\boldsymbol{v}}_{[M'']}$ for some $[M''] \in \mathcal{S}(\boldsymbol{v})$. If $[M'] = [M'']$ we are done.

Assume $[M'] \neq [M'']$. As $X^{\boldsymbol{v}}_{[M'']}$ is closed, we have $x \in X^{\boldsymbol{v}}_{[M']} \cap X^{\boldsymbol{v}}_{[M'']}$. In this case $\mathfrak{M}_x$ admits two linear independent subspaces:

$$\mathfrak{M}_x \sim \begin{pmatrix} au^{e-r_1} & 0 \\ 0 & bu^{e-r_1} \end{pmatrix}$$

and hence $e - r_1 = e - r_2$. Now we easily deduce $\mathcal{GR}^{\boldsymbol{v},\mathrm{loc}}_{V_{\mathbb{F}},0} = \{\mathfrak{M}_x\}$ and the claim follows.

(ii) This follows from the first part of the proposition together with the fact that the one-dimensional Φ-invariant subspaces of $M_{\bar{\mathbb{F}}}$ which admit an integral model $\mathfrak{M}$ with $\langle \Phi(\mathfrak{M}) \rangle = u^{e-r_1}\mathfrak{M}$ run over a finite set of isomorphism classes of one-dimensional objects:

Assume that there are two different one-dimensional Φ-stable subspaces $\langle b_1 \rangle$ and $\langle b_2 \rangle$ of $M_{\bar{\mathbb{F}}}$ such that $\Phi(b_i) = a_i u^{e-r_1} b_i$, for $i = 1, 2$. Then b_1 and b_2 are linear independent.

If $a_1 \neq a_2$, then $\langle b_1 + q b_2 \rangle$ is not Φ-stable for all $q \in \bar{\mathbb{F}}((u))^\times$ and hence there are only two isomorphism classes.

If $a_1 = a_2$, then there is a unique such isomorphism class given by $[M_a]$. □

As we will see, the open and closed subschemes $X^{\boldsymbol{v}}_{[M']}$ and $X^{\boldsymbol{v}}_0$ of $\mathcal{GR}^{\boldsymbol{v},\mathrm{loc}}_{V_{\mathbb{F}},0} \otimes_{\mathbb{F}} \bar{\mathbb{F}}$ are connected and hence turn out to be the connected components of $\mathcal{GR}^{\boldsymbol{v},\mathrm{loc}}_{V_{\mathbb{F}},0} \otimes_{\mathbb{F}} \bar{\mathbb{F}}$.

Now we want to determine the subset of $\boldsymbol{v}$-admissible lattices in the building. As we are assuming that $(M_{\mathbb{F}}, \Phi)$ is reducible, at least after extending scalars, there exists a finite extension $\mathbb{F}'$ of $\mathbb{F}$ and a basis e_1, e_2 of $M_{\mathbb{F}'} = M_{\mathbb{F}} \widehat{\otimes}_{\mathbb{F}} \mathbb{F}'$ such that

$$M_{\mathbb{F}'} \sim \begin{pmatrix} au^s & \gamma \\ 0 & bu^t \end{pmatrix}$$

for some $a, b \in \mathbb{F}'^\times$, $\gamma \in \mathbb{F}'((u))$ and $s, t \in \mathbb{Z}$ with $0 \leqslant s, t < p-1$. We choose this basis to be the standard basis.

Lemma 4.7. (i) *The map Φ extends to a map $\bar{\mathcal{B}} \to \bar{\mathcal{B}}$ also denoted by Φ.*

(ii) *For $q \in \bar{\mathbb{F}}((u))$ and $[x, y]_q \in \mathcal{A}_q$ the map Φ is given by*

$$\Phi([x, y]_q) = [px + s - t, py + s + t]_{q'}$$

with $q' = b^{-1}u^{-t}(au^s\phi(q) + \gamma)$.

Proof. (i) We can use the expressions in (ii) to extend Φ.

(ii) This is an easy computation (see [He, Lemma 4.7] for details). □

Corollary 4.8. *The scheme $\mathcal{GR}^{\boldsymbol{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$ is empty if $2e - d' \not\equiv s + t \mod (p-1)$.*

Proof. This follows from Lemma 2.4 and Lemma 4.7. □

We assume that the scheme is non empty and define

$$P_{\mathrm{red}} = \left[\frac{t-s}{p-1}, -\frac{t+s}{p-1}\right] \in \mathcal{A}_0 \subset \bar{\mathcal{B}}, \tag{4.0.6}$$

$$m(\boldsymbol{v}) = \frac{2e - d' - (s+t)}{p-1} \in \mathbb{Z}. \tag{4.0.7}$$

These definitions imply $\mathcal{GR}^{\boldsymbol{v},\mathrm{loc}}_{V_{\mathbb{F}},0}(\bar{\mathbb{F}}) \subset \bar{\mathcal{B}}(m(\boldsymbol{v}))$.

There are three different cases which we have to study in order to determine the set of $\boldsymbol{v}$-admissible lattices. It makes a difference whether $(M_{\bar{\mathbb{F}}}, \Phi)$ is a split or a non-split extension of two one-dimensional objects. In the split case there are two possibilities: Either the direct summands are isomorphic or non isomorphic.

4.1. The case $(M_{\bar{\mathbb{F}}}, \Phi) \cong (M_1, \Phi_1) \oplus (M_1, \Phi_1)$. In this section we want to analyse the case where $(M_{\mathbb{F}}, \Phi)$ becomes isomorphic to a direct sum of two isomorphic one-dimensional objects after possibly extending the scalars to some finite extension of $\mathbb{F}$, i.e., we want to assume that there exists $\mathbb{F}'/\mathbb{F}$ and an $\mathbb{F}'((u))$-basis e_1, e_2 of $M_{\mathbb{F}'}$ such that

$$M_{\mathbb{F}'} \sim \begin{pmatrix} au^s & 0 \\ 0 & au^s \end{pmatrix} \tag{4.1.1}$$

with $a \in \mathbb{F}'^{\times}$ and $0 \leqslant s < p-1$. We immediately find $\Phi(P_{\mathrm{red}}) = P_{\mathrm{red}}$.

For each $z \in \mathbb{P}^1(\bar{\mathbb{F}})$ we define a (half)-line $\mathcal{L}_z \subset \bar{\mathcal{B}}(m(\boldsymbol{v}))$ by

$$\mathcal{L}_z = \{[x, m(\boldsymbol{v})]_z : x \geqslant 0\} \subset \bar{\mathcal{B}}(m(\boldsymbol{v})) \quad \text{if } z \in \bar{\mathbb{F}} = \mathbb{A}^1(\bar{\mathbb{F}}),$$
$$\mathcal{L}_\infty = \{[x, m(\boldsymbol{v})]_0 : x \leqslant 0\} \subset \bar{\mathcal{B}}(m(\boldsymbol{v})).$$

These lines are defined in such a way that

$$\bigcup_{z \in \mathbb{P}^1(\bar{\mathbb{F}})} \mathcal{L}_z = \bigcup_{z \in \bar{\mathbb{F}}} \mathcal{A}_z \cap \bar{\mathcal{B}}(m(\boldsymbol{v})). \tag{4.1.2}$$

In the sequel we will write $\mathcal{T}$ for the set in (4.1.2). The apartments on the right hand side are given by the basis e_1, $ze_1 + e_2$ and in this basis the semi-linear endomorphism x Φ is of the form (4.1.1).

Lemma 4.9. *Let $Q \in \bar{\mathcal{B}}(m(\boldsymbol{v}))$ be an arbitrary point. Let $Q' \in \mathcal{T}$ be the unique point satisfying $d_1(Q, Q') = d_1(Q, \mathcal{T})$. Then*

$$d_1(Q, \Phi(Q)) = (p+1)d_1(Q, P_{\mathrm{red}}) - 2d_1(Q', P_{\mathrm{red}}),$$
$$d_2(Q, \Phi(Q)) = (p-1)d_2(Q, P_{\mathrm{red}}).$$

Proof. The statement on d_2 follows immediately from Lemma 4.7. For the statement on d_1 we assume $Q' \in \mathcal{L}_0$. The cases $Q' \in \mathcal{L}_z$ for $z \in \bar{\mathbb{F}}$ are analogous and the case $Q' \in \mathcal{L}_\infty$ is obtained by interchanging e_1 and e_2.

First assume $Q = Q'$, i.e., $Q = [x, m(\boldsymbol{v})]_0 \in \mathcal{L}_0$. Then Lemma 4.7 implies $\Phi(Q) = [px, pm(\boldsymbol{v}) + 2s]_0$ and hence $d_1(Q, \Phi(Q)) = (p-1)x = (p-1)d_1(Q, P_{\mathrm{red}})$.

Now assume $Q \neq Q'$. We write

$$Q = [x, m(\boldsymbol{v})]_q, \quad Q' = [x', m(\boldsymbol{v})]_0$$

with $x > x' = v_u(q) \in \mathbb{Z}_{>0}$. Then $\Phi(Q) = [px, pm(\boldsymbol{v}) + 2s]_{\phi(q)}$ by Lemma 4.7. Using $v_u(\phi(q)) = px'$, we find

$$\begin{aligned} d_1(Q, \Phi(Q)) &= (x - x') + (px' - x') + (px - px') \\ &= (p+1)x - 2x' = (p+1)d_1(Q, P_{\mathrm{red}}) - 2d_1(Q', P_{\mathrm{red}}). \end{aligned}$$ □

Remark 4.10. This lemma shows that the case of a direct sum of two isomorphic objects corresponds to the case B 2 in [PR2, 6.d]:

The unique point fixed by Φ is the point P_{red} and the projection of this point to the building for $\mathrm{PGL}_2(\bar{\mathbb{F}}((u)))$ is a vertex. The link of this vertex is the projection of $\mathcal{T}$ and all the half-lines $\mathcal{L}_z$ of $\mathcal{T}$ (for $z \in \mathbb{P}^1(\bar{\mathbb{F}})$) are fixed by Φ.

Proposition 4.11. *With the notations of Definition 4.1 and (4.0.3), (4.0.7) assume that*

$$(M_{\bar{\mathbb{F}}}, \Phi) \cong (\mathfrak{M}^s(a)[\tfrac{1}{u}], \Phi_a^s) \oplus (\mathfrak{M}^s(a)[\tfrac{1}{u}], \Phi_a^s)$$

for some $a \in \bar{\mathbb{F}}^\times$ *and* $0 \leqslant s < p-1$.

(i) *The schemes* $X^{\boldsymbol{v}}_{[M']}$ *are empty for all* $[M'] \in \mathcal{S}(\boldsymbol{v})\backslash\{[M_a]\}$.

(ii) *The scheme* $X^{\boldsymbol{v}}_{[M_a]}$ *is given by*

$$X^{\boldsymbol{v}}_{[M_a]} \cong \begin{cases} \varnothing & \text{if } m(\boldsymbol{v}) + \dfrac{r_1 - r_2}{p-1} \notin 2\mathbb{Z}, \\ \{*\} & \text{if } 0 = \dfrac{r_1 - r_2}{p-1} \in \mathbb{Z} \text{ and } \dfrac{r_1 - r_2}{p-1} \equiv m(\boldsymbol{v}) \mod 2, \\ \mathbb{P}^1_{\bar{\mathbb{F}}} & \text{if } 0 \neq \dfrac{r_1 - r_2}{p-1} \in \mathbb{Z} \text{ and } \dfrac{r_1 - r_2}{p-1} \equiv m(\boldsymbol{v}) \mod 2. \end{cases}$$

(iii) *If non empty, the scheme* $X_0^{\boldsymbol{v}}$ *is connected.*

Proof. We first claim that every $\boldsymbol{v}$-admissible lattice $\mathfrak{M}$ can be linked to a $\boldsymbol{v}$-admissible lattice $\mathfrak{M}' \in \mathcal{T}$ by a chain of $\mathbb{P}^1$.

Assume $\mathfrak{M} = [x, m(\boldsymbol{v})]_q \notin \mathcal{T}$ and let $Q' \in \mathcal{T}$ be the unique point satisfying $d_1(\mathfrak{M}, Q') = d_1(\mathfrak{M}, \mathcal{T})$. Without loss of generality, we may again assume that $Q' = [x', m(\boldsymbol{v})]_0 \in \mathcal{L}_0$. By construction we have $\mathfrak{M}, Q' \in \mathcal{A}_q$ and we choose the following basis b_1, b_2 of $\mathfrak{M}$:

$$b_1 = u^{(x+m(\boldsymbol{v}))/2} e_1, \quad b_2 = u^{(m(\boldsymbol{v})-x)/2}(qe_1 + e_2).$$

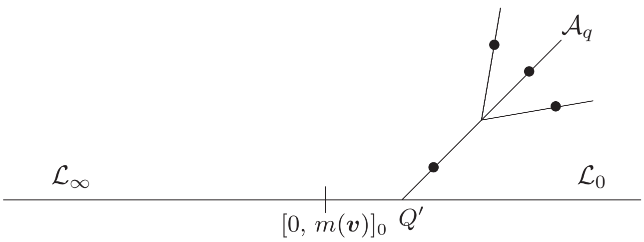


FIGURE 4. The fat points mark the image of the morphism $\bar{\chi}$ in the building in the case $p = 3$ and $\mathbb{F} = \mathbb{F}_3$.

Applying Lemma 3.7 with this basis yields a morphism $\bar{\chi} \colon \mathbb{P}^1_{\bar{\mathbb{F}}} \to \operatorname{Grass} M_{\bar{\mathbb{F}}}$ with $\bar{\chi}(z) = [x, m(\boldsymbol{v})]_{q+zu^{x-1}}$ for $z \in \bar{\mathbb{F}} = \mathbb{A}^1(\bar{\mathbb{F}})$ and $\bar{\chi}(\infty) = [x-2, m(\boldsymbol{v})]_q$. We have

$$d_1(\bar{\chi}(\infty), P_{\mathrm{red}}) < d_1(\bar{\chi}(z), P_{\mathrm{red}}) = d_1(\mathfrak{M}, P_{\mathrm{red}})$$

for all $z \in \bar{\mathbb{F}}$, while $d_1(\bar{\chi}(z), \mathcal{T}) \leqslant d_1(\mathfrak{M}, \mathcal{T})$ for all $z \in \mathbb{P}^1(\bar{\mathbb{F}})$ and, by construction, $d_1(\bar{\chi}(\infty), \mathcal{T}) < d_1(\mathfrak{M}, \mathcal{T})$.

By Lemma 4.9 and Lemma 2.4, the morphism $\bar{\chi}$ factors through $\mathcal{GR}^{\boldsymbol{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$ and the claim follows by induction on the distance $d_1(\mathfrak{M}, \mathcal{T})$.

Now we assume that $\mathfrak{M} \in \mathcal{T}$ is a $\boldsymbol{v}$-admissible lattice and we are looking for a $\boldsymbol{v}$-admissible lattice $\mathfrak{M}'$ that can be linked with $\mathfrak{M}$ by a $\mathbb{P}^1$ and that has strictly smaller distance d_1 from $P_{\mathrm{red}} = [0, \frac{-2s}{p-1}]_0$ than $\mathfrak{M}$, i.e., $d_1(\mathfrak{M}', P_{\mathrm{red}}) < d_1(\mathfrak{M}, P_{\mathrm{red}})$.

We may assume $\mathfrak{M} = [x, m(\boldsymbol{v})]_0 \in \mathcal{L}_0$. Assuming $x > 1$, our candidate for $\mathfrak{M}'$ is $[x-2, m(\boldsymbol{v})]_0$. Fixing a basis

$$b_1 = u^{(x+m(\boldsymbol{v}))/2} e_1, \quad b_2 = u^{(m(\boldsymbol{v})-x)/2} e_2$$

of $\mathfrak{M}$ so that $\mathfrak{M}' = \langle u^{-1} b_1, u b_2 \rangle$ yields a morphism $\bar{\chi}\colon \mathbb{P}^1_{\bar{\mathbb{F}}} \to \mathrm{Grass}\, M_{\bar{\mathbb{F}}}$ with $\bar{\chi}(0) = \mathfrak{M}$ and $\bar{\chi}(\infty) = \mathfrak{M}'$. This morphism factors through $\mathcal{GR}^{\boldsymbol{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$ if and only if the lattices $\bar{\chi}(z) = [x, m(\boldsymbol{v})]_{zu^{x-1}}$ are $\boldsymbol{v}$-admissible for all $z \in \bar{\mathbb{F}} \backslash \{0\}$. This is the case if and only if

$$\begin{aligned} d_1(\bar{\chi}(z), \Phi(\bar{\chi}(z))) &= (p+1) d_1(\bar{\chi}(z), P_{\mathrm{red}}) - 2 d_1([x-1, m(\boldsymbol{v})]_0, P_{\mathrm{red}}) \\ &= (p+1)x - 2(x-1) = (p-1)x + 2 \leqslant r_1 - r_2. \end{aligned}$$

Consider the following subset of $\boldsymbol{v}$-admissible lattices

$$\mathcal{N} = \Big\{ \mathfrak{M} \in \mathcal{GR}^{\boldsymbol{v},\mathrm{loc}}_{V_{\mathbb{F}},0}(\bar{\mathbb{F}}) \colon \mathfrak{M} \notin \mathcal{T} \text{ or } \Big(\mathfrak{M} \in \mathcal{T} \text{ and } d_1(\mathfrak{M}, P_{\mathrm{red}}) \leqslant \frac{r_1 - r_2 - 2}{p-1} \Big) \Big\}.$$

So far, we have shown that all $\boldsymbol{v}$-admissible lattices $\mathfrak{M} \in \mathcal{N}$ can either be linked to the lattice $[0, m(\boldsymbol{v})]_0$ or to one of the lattices

$$\{[1, m(\boldsymbol{v})]_z \colon z \in \bar{\mathbb{F}}\} \cup \{[-1, m(\boldsymbol{v})]_0\} = \{\mathfrak{M} \in \bar{\mathcal{B}}(m(\boldsymbol{v})) \colon d_1(\mathfrak{M}, P_{\mathrm{red}}) = 1\} \quad (4.1.3)$$

by a chain of $\mathbb{P}^1$. Here, the two different cases depend on $m(\boldsymbol{v})$ mod 2. Hence the subset of $\mathcal{GR}^{\boldsymbol{v},\mathrm{loc}}_{V_{\mathbb{F}},0} \otimes_{\mathbb{F}} \bar{\mathbb{F}}$ given by the lattices in $\mathcal{N}$ is connected: using Remark 3.8 again, the set in (4.1.3) forms a $\mathbb{P}^1$. The proposition now follows from the following two facts:

(a) *If $\mathfrak{M} \in \mathcal{N}$, then $\mathfrak{M}$ is not $\boldsymbol{v}$-ordinary, i.e., $\mathcal{N} \subset X_0^{\boldsymbol{v}}(\bar{\mathbb{F}})$.*

(b) *If $\mathfrak{M} \notin \mathcal{N}$ is $\boldsymbol{v}$-admissible, then $\mathfrak{M} \in X^{\boldsymbol{v}}_{[M_a]}(\bar{\mathbb{F}})$ and*

$$\mathfrak{M} \in \Big\{ \Big[\frac{r_1 - r_2}{p-1}, m(\boldsymbol{v}) \Big]_z \colon z \in \bar{\mathbb{F}} \Big\} \cup \Big\{ \Big[-\frac{r_1 - r_2}{p-1}, m(\boldsymbol{v}) \Big]_0 \Big\}. \quad (4.1.4)$$

By Remark 3.8, this set forms a $\mathbb{P}^1$ if $r_1 \neq r_2$. Otherwise it is a single point.

Proof of (a). If $\mathfrak{M} \in \mathcal{T}$, then $d_1(\mathfrak{M}, \langle \Phi(\mathfrak{M}) \rangle) \leqslant r_1 - r_2 - 2 < r_1 - r_2$ and hence the elementary divisors of $\langle \Phi(\mathfrak{M}) \rangle$ with respect to $\mathfrak{M}$ are not given by $(e - r_2, e - r_1)$.

If $\mathfrak{M} \notin \mathcal{T}$, say $\mathfrak{M} = [x, m(\boldsymbol{v})]_q$ with $x > v_u(q) > 0$ for example, then $\mathfrak{M} = \langle b_1, b_2 \rangle$ with

$$b_1 = u^{(x+m(\boldsymbol{v}))/2} e_1, \quad b_2 = u^{(m(\boldsymbol{v})-x)/2} (q e_1 + e_2)$$

and one finds

$$\mathfrak{M} \sim (a_{ij})_{ij} = \begin{pmatrix} a u^{\frac{p-1}{2}(x+m(\boldsymbol{v}))+s} & a\phi(q) u^{\frac{p-1}{2} m(\boldsymbol{v}) - \frac{p+1}{2} x + s} \\ 0 & a u^{\frac{p-1}{2}(m(\boldsymbol{v})-x)+s} \end{pmatrix}$$

with $v_u(a_{12}) < v_u(a_{11})$, because $v_u(q) < x$, and hence the minimal elementary divisor of $\langle \Phi(\mathfrak{M}) \rangle$ with respect to $\mathfrak{M}$ is not given by a Φ-stable subspace.

Proof of (b). Let $\mathfrak{M} \notin \mathcal{N}$ be $\boldsymbol{v}$-admissible. Then $\mathfrak{M} \in \mathcal{T}$ and

$$\frac{r_1 - r_2 - 2}{p-1} < d_1(\mathfrak{M}, P_{\mathrm{red}}) \leqslant \frac{r_1 - r_2}{p-1}.$$

We show that $d_1(\mathfrak{M}, P_{\mathrm{red}}) = \frac{r_1 - r_2}{p-1}$ which implies (4.1.4).

Suppose that $\mathfrak{M} = [x,\, m(\boldsymbol{v})]_z$ with $z \in \bar{\mathbb{F}}$ and

$$x = \pm\frac{r_1 - r_2 - 1}{p-1} \in \mathbb{Z}, \qquad m(\boldsymbol{v}) = \frac{2e - d' - 2s}{p-1} = \frac{2e - r_1 - r_2 - 2s}{p-1}.$$

In this case we find

$$x + m(\boldsymbol{v}) = \frac{2e - 2s - (r_1 + r_2) \pm (r_1 - r_2) \mp 1}{p-1} \notin 2\mathbb{Z},$$

contradiction. We are left to show that $\mathfrak{M} \in X^{\boldsymbol{v}}_{[M_a]}(\bar{\mathbb{F}})$, i.e., that there exists a vector $e_{\mathfrak{M}} \in \mathfrak{M}$ and a Φ-stable subspace $\bar{\mathbb{F}}[\![u]\!]e_{\mathfrak{M}} \subset \mathfrak{M}$ with $\Phi(e_{\mathfrak{M}}) = au^{e-r_1}e_{\mathfrak{M}}$. An easy computation shows that we may choose

$$\begin{aligned} e_{\mathfrak{M}} &= u^{\frac{e-r_1-s}{p-1}}(ze_1 + e_2) \quad &&\text{if } \mathfrak{M} = \Big[\frac{r_1 - r_2}{p-1},\, m(\boldsymbol{v})\Big]_z, \ z \in \bar{\mathbb{F}}, \\ e_{\mathfrak{M}} &= u^{\frac{e-r_1-s}{p-1}} e_1 &&\text{if } \mathfrak{M} = \Big[-\frac{r_1 - r_2}{p-1},\, m(\boldsymbol{v})\Big]_0. \end{aligned}$$

□

4.2. The case $(M_{\bar{\mathbb{F}}},\, \Phi) \cong (M_1,\, \Phi_1) \oplus (M_2,\, \Phi_2)$. In this section we treat the case where $(M_{\mathbb{F}},\, \Phi)$ becomes isomorphic to the direct sum of two non-isomorphic one-dimensional objects after extending the scalars to some finite extension. Most of the proofs are similar to the above case and are omitted. For details we refer to [He]. The situation is the following: There exists a finite extension $\mathbb{F}'$ of $\mathbb{F}$ and a basis $e_1,\, e_2$ of $M_{\mathbb{F}'}$ such that

$$M_{\mathbb{F}'} \sim \begin{pmatrix} au^s & 0 \\ 0 & bu^t \end{pmatrix}$$

with $a,\, b \in \mathbb{F}'^{\times}$ and $0 \leqslant s,\, t < p-1$. As we are assuming that the direct summands are not isomorphic, we further have $s \neq t$ or $a \neq b$. Again we find $\Phi(P_{\text{red}}) = P_{\text{red}}$.

Lemma 4.12. *Let $Q \in \bar{\mathcal{B}}(m(\boldsymbol{v}))$ be an arbitrary point. Let $Q' \in \mathcal{A}_0 \cap \bar{\mathcal{B}}(m(\boldsymbol{v}))$ be the unique point satisfying $d_1(Q,\, Q') = d_1(Q,\, \mathcal{A}_0)$. Then*

$$\begin{aligned} d_1(Q,\, \Phi(Q)) &= (p+1)d_1(Q,\, P_{\text{red}}) - 2d_1(Q',\, P_{\text{red}}) \\ d_2(Q,\, \Phi(Q)) &= (p-1)d_2(Q,\, P_{\text{red}}). \end{aligned}$$

Proof. This is similar to Lemma 4.9. See [He, Lemma 4.13] for details. □

Remark 4.13. Again, this Lemma shows the connection to [PR2, 6.d]. The point fixed by Φ is again the point P_{red}.

If $s = t$, then we are in the case B 2 of loc. cit.: The projection of the fixed point to the building for $\mathrm{PGL}_2(\bar{\mathbb{F}}((u)))$ is a vertex. Exactly two of the half-lines of the link of this vertex are fixed by Φ.

If $s \neq t$ we are in the case A 2 of loc. cit.: The projection of the fixed point P_{red} is not a vertex but it lies on an edge and the projections of the two half-lines $\{[x,\, m(\boldsymbol{v})]_0 \colon x \leqslant \frac{t-s}{p-1}\}$ and $\{[x,\, m(\boldsymbol{v})]_0 \colon x \geqslant \frac{t-s}{p-1}\}$ to the building for $\mathrm{PGL}_2(\bar{\mathbb{F}}((u)))$ are fixed by Φ.

Proposition 4.14. *With the notations of Definition 4.1 and (4.0.3), (4.0.7) assume that*

$$(M_{\bar{\mathbb{F}}},\,\Phi)\cong(\mathfrak{M}^s(a)[\tfrac{1}{u}],\,\Phi_a^s)\oplus(\mathfrak{M}^t(b)[\tfrac{1}{u}],\,\Phi_b^t)$$

with $a,\,b\in\bar{\mathbb{F}}^\times$ and $0\leqslant s,\,t<p-1$. Further assume $a\neq b$ or $s\neq t$.

(i) *The schemes $X^{\boldsymbol{v}}_{[M']}$ are empty for all $[M']\in\mathcal{S}(\boldsymbol{v})\backslash\{[M_a],\,[M_b]\}$.*

(ii) *If $s=t$, then*

$$X^{\boldsymbol{v}}_{[M_a]}\cong X^{\boldsymbol{v}}_{[M_b]}=\begin{cases}\varnothing & \text{if } m(\boldsymbol{v})+\dfrac{r_1-r_2}{p-1}\notin 2\mathbb{Z},\\ \{*\} & \text{if } m(\boldsymbol{v})+\dfrac{r_1-r_2}{p-1}\in 2\mathbb{Z},\end{cases}$$

further $X^{\boldsymbol{v}}_{[M_a]}=X^{\boldsymbol{v}}_{[M_b]}$ if and only if $r_1=r_2$.

(iii) *If $s\neq t$, then*

$$X^{\boldsymbol{v}}_{[M_a]}=\begin{cases}\varnothing & \text{if } \dfrac{t-s}{p-1}-\dfrac{r_1-r_2}{p-1}+m(\boldsymbol{v})\notin 2\mathbb{Z},\\ \{*\} & \text{if } \dfrac{t-s}{p-1}-\dfrac{r_1-r_2}{p-1}+m(\boldsymbol{v})\in 2\mathbb{Z};\end{cases}$$

$$X^{\boldsymbol{v}}_{[M_b]}=\begin{cases}\varnothing & \text{if } \dfrac{t-s}{p-1}+\dfrac{r_1-r_2}{p-1}+m(\boldsymbol{v})\notin 2\mathbb{Z},\\ \{*\} & \text{if } \dfrac{t-s}{p-1}+\dfrac{r_1-r_2}{p-1}+m(\boldsymbol{v})\in 2\mathbb{Z}.\end{cases}$$

(iv) *If non empty the scheme $X^{\boldsymbol{v}}_0$ is connected.*

Proof. Again, this is similar to the proof of Proposition 4.11. We obtain a big connected component (if non empty) corresponding to the non-$\boldsymbol{v}$-ordinary lattices and at most one or two additional points. Consider the following two points:

$$Q_+=\Big[\frac{t-s}{p-1}+\frac{r_1-r_2}{p-1},\,m(\boldsymbol{v})\Big]_0,$$
$$Q_-=\Big[\frac{t-s}{p-1}-\frac{r_1-r_2}{p-1},\,m(\boldsymbol{v})\Big]_0.$$

If $s=t$ and $\frac{r_1-r_2}{p-1}+m(\boldsymbol{v})\in 2\mathbb{Z}$, then the lattices $\mathfrak{M}_\pm=[\pm\frac{r_1-r_2}{p-1},\,m(\boldsymbol{v})]_0$ define points $Q_-=\mathfrak{M}_-\in X^{\boldsymbol{v}}_{[M_a]}$ and $Q_+=\mathfrak{M}_+\in X^{\boldsymbol{v}}_{[M_b]}$ which coincide if and only if $r_1=r_2$.

If $s\neq t$, then

$$Q_-\text{ defines an isolated point in }X^{\boldsymbol{v}}_{[M_a]}\ \Leftrightarrow\ \frac{t-s-(r_1-r_2)}{p-1}+m(\boldsymbol{v})\in 2\mathbb{Z},$$

$$Q_+\text{ defines an isolated point in }X^{\boldsymbol{v}}_{[M_a]}\ \Leftrightarrow\ \frac{t-s+(r_1-r_2)}{p-1}+m(\boldsymbol{v})\in 2\mathbb{Z}.$$

This cannot happen at the same time, as $\frac{t-s}{p-1}\notin\mathbb{Z}$. See [He, Prop. 4.15] for details. □

4.3. The case of a non split extension. Finally, we analyse the case where $(M_{\bar{\mathbb{F}}}, \Phi)$ is a non split extension of two one dimensional objects. There is a basis e_1, e_2 such that

$$M_{\bar{\mathbb{F}}} \sim \begin{pmatrix} au^s & \gamma \\ 0 & bu^t \end{pmatrix}$$

with $0 \leqslant s, t < p-1$ and $a, b \in \bar{\mathbb{F}}^\times$, $\gamma \in \bar{\mathbb{F}}((u))$. In any basis of the form e_1, qe_1+e_2 defining the apartment $\mathcal{A}_q$, the endomorphism Φ is upper triangular with diagonal entries au^s and bu^t, and we fix the basis such that the valuation of the upper right entry $k := v_u(\gamma)$ is maximal.

Lemma 4.15. (i) *The integer* $k = v_u(\gamma)$ *satisfies*

$$k \leqslant \frac{pt-s}{p-1}.$$

(ii) *If* $\mathfrak{M} = [x, y]_q$ *with* $\min\{x, v_u(q)\} \geqslant \frac{k-s}{p}$, *then*

$$d_1(\mathfrak{M}, \langle\Phi(\mathfrak{M})\rangle) = (p+1)x + s + t - 2k,$$
$$d_2(\mathfrak{M}, \langle\Phi(\mathfrak{M})\rangle) = (p-1)d_2(\mathfrak{M}, P_{\mathrm{red}}).$$

(iii) *If* $\mathfrak{M} = [x, y]_q$ *with* $x < \frac{k-s}{p}$ *or* $v_u(q) < \frac{k-s}{p}$, *let* $Q' \in \mathcal{A}_0 \cap \bar{\mathcal{B}}(y)$ *be the unique point such that* $d_1(\mathfrak{M}, Q') = d_1(\mathfrak{M}, \mathcal{A}_0)$. *Then*

$$d_1(\mathfrak{M}, \langle\Phi(\mathfrak{M})\rangle) = (p+1)d_1(\mathfrak{M}, P_{\mathrm{red}}) - 2d_1(Q', P_{\mathrm{red}}),$$
$$d_2(\mathfrak{M}, \langle\Phi(\mathfrak{M})\rangle) = (p-1)d_2(\mathfrak{M}, P_{\mathrm{red}}).$$

Proof. (i) This follows from the maximality of $k = v_u(\gamma)$: We have

$$\Phi(qe_1 + e_2) = (\gamma + au^s\phi(q) - bu^tq)e_1 + bu^t(qe_1 + e_2). \tag{4.3.1}$$

And

$$v_u(au^s\phi(q) - bu^tq) = \begin{cases} v_u(q) + t & \text{if } v_u(q) > \dfrac{t-s}{p-1}, \\ pv_u(q) + s & \text{if } v_u(q) < \dfrac{t-s}{p-1}. \end{cases}$$

If we had $k = v_u(\gamma) = v_u(q) + t$ for any q with $v_u(q) > \frac{t-s}{p-1}$, we could delete the leading coefficient of γ in (4.3.1) which contradicts the maximality of $v_u(\gamma)$. Hence we have $k < v_u(q) + t$ for all q with $v_u(q) > \frac{t-s}{p-1}$ which yields the first claim.

(ii) The first part of the lemma implies $\frac{k-s}{p} \geqslant k-t$ and hence our assumptions on $v_u(q)$ imply $k \leqslant \min\{v_u(q)+t, pv_u(q)+s\}$. We find $v_u(\gamma + au^s\phi(q) - bu^tq) = v_u(\gamma)$ and we may assume $q = 0$, i.e., $\mathfrak{M} \in \mathcal{A}_0$, as the situation is the same as in the standard apartment. Now we have $\langle\Phi(\mathfrak{M})\rangle = [px + s - t, py + s + t]_{b^{-1}u^{-t}\gamma}$ and $x \geqslant \frac{k-s}{p}$ implies

$$px + s - t \geqslant k - t, \quad x \geqslant k - t.$$

Thus $d_1(\mathfrak{M}, \langle\Phi(\mathfrak{M})\rangle) = (px+s-t-(k-t)) + (x-(k-t)) = (p+1)x+s+t-2k$. The statement on d_2 is easy.

(iii) If $\mathfrak{M} \notin \mathcal{A}_0$, then $v_u(q) < \frac{k-s}{p} \leqslant \frac{t-s}{p-1}$ and hence

$$v_u(\gamma + au^s\phi(q) - bu^tq) = v_u(au^s\phi(q) - bu^tq)$$

and the situation is the same as in the split case, i.e., the case $\gamma = 0$. If $\mathfrak{M} \in \mathcal{A}_0$, then $\langle \Phi(\mathfrak{M})\rangle \in \mathcal{A}_0$ and the statement is easy. □

Remark 4.16. In the case of a non split extension we are in the case B 2 or A 3 of [PR2, 6.d]. More precisely, if $\frac{k-s}{p} \notin \mathbb{Z}$, then the unique fixed point of [PR2, Prop. 6.1] is not in the building $\bar{\mathcal{B}}$. It is only visible after extending $\bar{\mathbb{F}}((u))$ to some separable wildly ramified extension. The x and y-coordinates of the fixed point will coincide with the coordinates of P_{red} but it lies in an apartment that is only visible after extending the scalars: by Lemma 4.7 and Remark 2.1 (i), we have to find some q satisfying $\frac{t-s}{p-1} \leqslant v_u(q - b^{-1}u^{-t}(au^s\phi(q)+\gamma))$ in order to obtain a fixed point. The apartment $\mathcal{A}_q$ defined by q will branch off from the standard apartment at the line $x = v_u(q) = \frac{k-s}{p}$. The image of the half-line $\{[x,\, m(\boldsymbol{v})]_0 \colon x \leqslant \frac{k-s}{p}\}$ in the building for $\mathrm{PGL}_2(\bar{\mathbb{F}}((u)))$ is stable under Φ and the geodesic between $[\lfloor\frac{k-s}{p}\rfloor + 1,\, m(\boldsymbol{v})]_0$ and its image under Φ contains the (projection of the) point $[x_0,\, m(\boldsymbol{v})]_0$ in the building for $\mathrm{PGL}_2(\bar{\mathbb{F}}((u)))$. This is the case A 3 of [PR2, 6.d].

If $\frac{k-s}{p} \in \mathbb{Z}$, then we are in the case B 2 of [PR2, 6.d]: In this case the maximality of $k = v_u(q)$ implies $k - t = \frac{k-s}{p} = \frac{t-s}{p-1} = 0$ (otherwise we could delete the leading coefficient of γ) and we find that P_{red} is the fixed point in the building. In this case there is a unique half-line in the building for $\mathrm{PGL}_2(\bar{\mathbb{F}}((u)))$ that is fixed by Φ, namely the image of the half-line $\{[x,\, m(\boldsymbol{v})]_0 \colon x \leqslant 0\}$ under the projection.

Proposition 4.17. *With the notations of Definition 4.1 and (4.0.3), (4.0.7), assume that $(M_{\bar{\mathbb{F}}},\, \Phi)$ is a non split extension*

$$0 \to (\mathfrak{M}^s(a)[\tfrac{1}{u}],\, \Phi_a^s) \to (M_{\bar{\mathbb{F}}},\, \Phi) \to (\mathfrak{M}^t(b)[\tfrac{1}{u}],\, \Phi_b^t) \to 0$$

for some $a,\, b \in \bar{\mathbb{F}}^\times$ and $0 \leqslant s,\, t < p-1$.

(i) *The schemes $X^{\boldsymbol{v}}_{[M']}$ are empty for all $[M'] \in \mathcal{S}(\boldsymbol{v})\backslash\{[M_a]\}$.*

(ii) *For $X^{\boldsymbol{v}}_{[M_a]}$ the following holds:*

$$X^{\boldsymbol{v}}_{[M_a]} = \begin{cases} \varnothing & \text{if } \dfrac{t-s}{p-1} - \dfrac{r_1-r_2}{p-1} + m(\boldsymbol{v}) \notin 2\mathbb{Z}, \\ \{*\} & \text{if } \dfrac{t-s}{p-1} - \dfrac{r_1-r_2}{p-1} + m(\boldsymbol{v}) \in 2\mathbb{Z}. \end{cases}$$

(iii) *If non empty, the scheme $X^{\boldsymbol{v}}_0$ is connected.*

Proof. Lemma 4.15 (i) implies $\frac{k-s}{p} \leqslant \frac{t-s}{p-1}$ and an easy computation using the same inequality shows that

$$\frac{t-s}{p-1} - \frac{r_1-r_2}{p-1} \leqslant \frac{k-s}{p} \Leftrightarrow \frac{k-s}{p} \leqslant \frac{1}{p+1}(r_1 - r_2 - s - t + 2k),$$

and hence

$$\frac{t-s}{p-1} - \frac{r_1-r_2}{p-1} \leqslant \frac{k-s}{p} \leqslant \frac{1}{p+1}(r_1 - r_2 - s - t + 2k),$$

if $\mathcal{GR}^{\boldsymbol{v},\mathrm{loc}}_{V_{\mathbb{F}},0} \neq \varnothing$. Further denote by $\tilde{\mathcal{N}}$ the set of $\boldsymbol{v}$-admissible lattices $\mathfrak{M} = [x,\, m(\boldsymbol{v})]_q$ with $\frac{k-s}{p} \leqslant \min\{x,\, v_u(q)\}$.

As we have seen above, the situation for the $\boldsymbol{v}$-admissible lattices $\mathfrak{M} \notin \widetilde{\mathcal{N}}$ is the same as in the split case. Hence we can link all $\boldsymbol{v}$-admissible lattices to $\boldsymbol{v}$-admissible lattices in $\mathcal{A}_0$ by a chain of $\mathbb{P}^1$. If $\mathfrak{M} = [x,\, m(\boldsymbol{v})]_0$ is a $\boldsymbol{v}$-admissible lattice in $\mathcal{A}_0$ with $x + 2 < \frac{k-s}{p}$, then there is a $\mathbb{P}^1$ in $\mathcal{GR}^{\boldsymbol{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$ containing $\mathfrak{M} = [x,\, m(\boldsymbol{v})]_0$ and $[x+2,\, m(\boldsymbol{v})]_0$, except if $\mathfrak{M} = \mathfrak{M}_- = [\frac{t-s-(r_1-r_2)}{p-1},\, m(\boldsymbol{v})]_0$ which defines an isolated point in $X^{\boldsymbol{v}}_{[M_a]}$ if $\frac{t-s}{p-1} - \frac{r_1-r_2}{p-1} + m(\boldsymbol{v}) \in 2\mathbb{Z}$ (compare Proposition 4.14).

Let $\mathfrak{M}' = [x_0,\, m(\boldsymbol{v})]_0$ be the lattice where x_0 is the maximal integer smaller than $\frac{k-s}{p}$ that is congruent to $m(\boldsymbol{v}) \mod 2$. We claim:

(a) *If $\mathfrak{M}'$ is $\boldsymbol{v}$-admissible and $x_0 \neq \frac{t-s}{p-1} - \frac{r_1-r_2}{p-1}$, then any lattice in $\widetilde{\mathcal{N}}$ can be linked to $\mathfrak{M}'$ by a chain of $\mathbb{P}^1$.*

(b) *The lattices $\mathfrak{M} \in \widetilde{\mathcal{N}}$ are non-$\boldsymbol{v}$-ordinary.*

This finishes the proof of the proposition.

Proof of (a). Let $\mathfrak{M} = [x,\, m(\boldsymbol{v})]_q \in \widetilde{\mathcal{N}}$ be a lattice. Without loss of generality, we may assume x $\mathfrak{M} \in \mathcal{A}_0$, as the situation is the same in all apartments $\mathcal{A}_q$ with $v_u(q) \geqslant \frac{k-s}{p}$. By Lemma 4.15, we have

$$\frac{k-s}{p} \leqslant x \leqslant \frac{1}{p+1}(r_1 - r_2 - s - t + 2k).$$

We consider the basis

$$b_1 = u^{(x+m(\boldsymbol{v}))/2} e_1, \quad b_2 = u^{(m(\boldsymbol{v})-x)/2} e_2$$

of $\mathfrak{M}$ and by Lemma 3.7, there is a morphism

$$\bar{\chi}\colon \mathbb{P}^1_{\bar{\mathbb{F}}} \to \operatorname{Grass} M_{\bar{\mathbb{F}}}$$

with $\bar{\chi}(z) = [x,\, m(\boldsymbol{v})]_{zx^{-1}}$ for $z \in \bar{\mathbb{F}}$ and $\bar{\chi}(\infty) = [x-2,\, m(\boldsymbol{v})]_0$.

If $x - 1 \geqslant \frac{k-s}{p}$, then the morphism factors over $\mathcal{GR}^{\boldsymbol{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$.

Consider the following two cases:

If $\frac{k-s}{p} \leqslant x_0 + 1$, then this argument shows that we can link all $\mathfrak{M} \in \widetilde{\mathcal{N}}$ to the lattice $[x_0,\, m(\boldsymbol{v})]_0$ by a chain of $\mathbb{P}^1$.

If $\frac{k-s}{p} > x_0 + 1$, then this argument shows that we can link all $\mathfrak{M} \in \widetilde{\mathcal{N}}$ to the lattice $\mathfrak{M}'' = [x_0 + 2,\, m(\boldsymbol{v})]_0$ by a chain of $\mathbb{P}^1$. We can link the lattice $\mathfrak{M}''$ to the lattice $\mathfrak{M}' = [x_0,\, m(\boldsymbol{v})]_0$ if the lattices $\mathfrak{M}_z = [x_0,\, m(\boldsymbol{v})]_{zu^{x-1}}$ are $\boldsymbol{v}$-admissible for all $z \in \bar{\mathbb{F}}$. For $z \neq 0$ we have

$$d_1(\mathfrak{M}_z,\, \langle\Phi(\mathfrak{M}_z)\rangle) = (p+1)d_1(\mathfrak{M}_z,\, P_{\mathrm{red}}) - 2d_1(Q',\, P_{\mathrm{red}}),$$

where $Q' = [x_0 + 1,\, m(\boldsymbol{v})]_0$ is the unique point in $\mathcal{A}_0$ with minimal distance from $\mathfrak{M}_z$. Hence $d_1(\mathfrak{M}_z,\, \langle\Phi(\mathfrak{M}_z)\rangle) = d_1(\mathfrak{M}',\, \langle\Phi(\mathfrak{M}')\rangle) + 2$ and the morphism factors through $\mathcal{GR}^{\boldsymbol{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$ as $x_0 \neq \frac{t-s}{p-1} - \frac{r_1-r_2}{p-1}$. (Otherwise $\mathfrak{M}'$ is the unique isolated point in $X^{\boldsymbol{v}}_{[M_a]}$).

Proof of (b). Let $\mathfrak{M} \in \widetilde{\mathcal{N}}$ be a lattice. Similarly to the proof of Proposition 4.11, we find

$$\mathfrak{M} \sim \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}$$

with $v_u(a_{12}) < v_u(a_{11})$ and hence the minimal elementary divisor of $\langle\Phi(\mathfrak{M})\rangle$ with respect to $\mathfrak{M}$ is not defined by a Φ-stable subspace. □

Summarizing the results on the connected components we find the following theorem.

Theorem 4.18. *Assume that* $(M_{\mathbb{F}}, \Phi)$ *becomes reducible after extending the scalars to some finite extension* $\mathbb{F}'$ *of* $\mathbb{F}$.

(i) *The subschemes* $X_0^{\boldsymbol{v}}$ *and* $X_{[M']}^{\boldsymbol{v}}$ *are open and closed in* $\mathcal{GR}_{V_{\mathbb{F}},0}^{\boldsymbol{v},\mathrm{loc}} \otimes_{\mathbb{F}} \bar{\mathbb{F}}$ *for all isomorphism classes* $[M'] \in \mathcal{S}(\boldsymbol{v})$.
(ii) *If non empty, the scheme* $X_0^{\boldsymbol{v}}$ *is connected.*
(iii) *For each* $[M'] \in \mathcal{S}(\boldsymbol{v})$ *the scheme* $X_{[M']}^{\boldsymbol{v}}$ *is either empty, a single point or isomorphic to* $\mathbb{P}^1_{\bar{\mathbb{F}}}$.
(iv) *There are at most two isomorphism classes* $[M'] \in \mathcal{S}(\boldsymbol{v})$ *such that* $X_{[M']}^{\boldsymbol{v}} \neq \varnothing$.

Proof. This is a summary of the Propositions 4.11, 4.14 and 4.17. □

This theorem implies a modified version of the conjecture of Kisin stated in [Ki, 2.4.16].

Definition 4.19. For an integer s denote by $\mathcal{GR}_{V_{\mathbb{F}},0}^{\boldsymbol{v},\mathrm{loc},s}$ the open and closed subscheme of $\mathcal{GR}_{V_{\mathbb{F}},0}^{\boldsymbol{v},\mathrm{loc}}$ consisting of all $\boldsymbol{v}$-admissible lattices $\mathfrak{M}$, where the rank of the maximal Φ-stable subobject $\mathfrak{M}_1$ satisfying $\langle\Phi(\mathfrak{M}_1)\rangle = u^{e-r_1}\mathfrak{M}_1$ is equal to s.

Corollary 4.20. *Assume* $p \neq 2$ *and let* $\rho\colon G_K \to V_{\mathbb{F}}$ *be any two-dimensional continuous representation of* G_K *that admits a finite flat model after possibly extending the scalars to some finite extension of* $\mathbb{F}$.

Assume that $\mathrm{End}_{\mathbb{F}'[G_K]}(V_{\mathbb{F}'})$ *is a simple algebra for all finite extensions* $\mathbb{F}'$ *of* $\mathbb{F}$. *Then* $\mathcal{GR}_{V_{\mathbb{F}},0}^{\boldsymbol{v},\mathrm{loc},s}$ *is geometrically connected for all* s. *Furthermore*

(i) *If* $s = 1$ *and* $\mathrm{End}_{\mathbb{F}'[G_K]}(V_{\mathbb{F}'}) = \mathbb{F}'$ *for all finite extensions* $\mathbb{F}'$ *of* $\mathbb{F}$, *then* $\mathcal{GR}_{V_{\mathbb{F}},0}^{\boldsymbol{v},\mathrm{loc},s}$ *is either empty or a single point.*
If $s = 1$ *and* $\mathrm{End}_{\mathbb{F}'[G_K]}(V_{\mathbb{F}'}) = M_2(\mathbb{F}')$ *for some finite extension* $\mathbb{F}'$ *of* $\mathbb{F}$, *then* $\mathcal{GR}_{V_{\mathbb{F}},0}^{\boldsymbol{v},\mathrm{loc},s}$ *is either empty or becomes isomorphic to* $\mathbb{P}^1_{\mathbb{F}'}$ *after extending the scalars to* $\mathbb{F}'$.
(ii) *If* $s = 2$, *then* $\mathcal{GR}_{V_{\mathbb{F}},0}^{\boldsymbol{v},\mathrm{loc},s}$ *is either empty or a single point.*

Proof. Our definitions imply

$$\mathcal{GR}_{V_{\mathbb{F}},0}^{\boldsymbol{v},\mathrm{loc},0} \otimes_{\mathbb{F}} \bar{\mathbb{F}} = X_0^{\boldsymbol{v}}.$$

Further

$$\bigcup_{[M'] \in \mathcal{S}(\boldsymbol{v})} X_{[M']}^{\boldsymbol{v}} = \begin{cases} \mathcal{GR}_{V_{\mathbb{F}},0}^{\boldsymbol{v},\mathrm{loc},1} & \text{if } r_1 > r_2, \\ \mathcal{GR}_{V_{\mathbb{F}},0}^{\boldsymbol{v},\mathrm{loc},2} & \text{if } r_1 = r_2. \end{cases}$$

By [Br, Thm. 3.4.3] we have $\mathrm{End}_{\mathbb{F}'[G_K]}(V_{\mathbb{F}'}) = \mathrm{End}_{\mathbb{F}'((u)),\Phi}(M_{\mathbb{F}'})$. The same theorem implies that the image of the category of finite flat G_K-representations on finite length $\mathbb{Z}_p$-algebras under the restriction to G_{K_∞} is closed under subobjects and quotients. Hence $V_{\mathbb{F}'}$ is irreducible (resp. reducible, resp. split reducible) if and only if $(M_{\mathbb{F}'}, \Phi)$ is. An easy computation yields:

$\mathrm{End}_{\mathbb{F}'[G_K]}(V_{\mathbb{F}'}) = \mathbb{F}'$ if $V_{\mathbb{F}'}$ is irreducible or non-split reducible.

$\mathrm{End}_{\mathbb{F}'[G_K]}(V_{\mathbb{F}'}) = \mathbb{F}' \times \mathbb{F}'$ if $V_{\mathbb{F}'}$ is the direct sum of two non-isomorphic one-dimensional representations.

$\mathrm{End}_{\mathbb{F}'[G_K]}(V_{\mathbb{F}'}) = M_2(\mathbb{F}')$ if $V_{\mathbb{F}'}$ is the direct sum of two isomorphic one-dimensional representations.

The corollary now follows from Theorem 4.18 and Propositions 4.11, 4.14 and 4.17. □

Remark 4.21. The structure of the connected component $X_0^{\boldsymbol{v}}$ of non-$\boldsymbol{v}$-ordinary lattices in the reducible case is more complicated than in the absolutely simple case. In the absolutely simple case we have

$$X_0^{\boldsymbol{v}} = \mathcal{GR}^{\boldsymbol{v},\mathrm{loc}}_{V_{\mathbb{F}},0} \otimes_{\mathbb{F}} \bar{\mathbb{F}}$$

and this is isomorphic to a Schubert variety. In the reducible case it turns out that $X_0^{\boldsymbol{v}}$ is in general not irreducible and its irreducible components have varying dimensions. In the reducible case there is always a Φ-stable half-line in the building for $\mathrm{PGL}_2(\bar{\mathbb{F}}((u)))$. There is a unique fixed point P_{fix} which possibly does not lie in $\bar{\mathcal{B}}$ but is visible after extending the scalars to some wildly ramified extension of $\bar{\mathbb{F}}((u))$. If $\mathfrak{M}$ is a lattice such that the geodesic between $\mathfrak{M}$ and the fixed point P_{fix} in the building for $\mathrm{PGL}_2(\bar{\mathbb{F}}((u)))$ runs through a Φ-stable half-line, then

$$d_1(\mathfrak{M}, \langle\Phi(\mathfrak{M})\rangle) = (p+1)d_1(\mathfrak{M}, P_{\mathrm{fix}}) - 2d_1(Q', P_{\mathrm{fix}}),$$

where Q' is the projection of $\mathfrak{M}$ to this Φ-stable half-line. In [PR2, 6.d] the set of $\boldsymbol{v}$-admissible lattices in the building is described as a union of *thinning tubes*. Analysing the irreducible components we find that these "thinning tubes" correspond to Schubert varieties of decreasing dimension along these lines. See [He, 5] for a detailed discussion of the irreducible components of $X_0^{\boldsymbol{v}}$.

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