CONFORMAL GEOMETRY AND DYNAMICS
An Electronic Journal of the American Mathematical Society
Volume 2, Pages 29–44 (March 25, 1998)
S 1088-4173(98)00019-8

ERGODICITY OF CONFORMAL MEASURES FOR UNIMODAL POLYNOMIALS

EDUARDO A. PRADO

ABSTRACT. Let f be a polynomial and μ a conformal measure for f, i.e., a Borel probability measure μ with Jacobian equal to $|Df(z)|^{\delta}$. We show that if f is a real unimodal polynomial (a polynomial with just one critical point), then μ is ergodic. We also show that μ is ergodic if f is a complex unimodal polynomial with one parabolic periodic point or a quadratic polynomial in the $\mathcal{SL}$ class with a priori bounds (as defined in Lyubich (1997)).

1. INTRODUCTION

Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial. Sullivan showed in [Sul80] that it is possible to construct a conformal measure for f with support on $J(f)$, the Julia set of f, for at least one positive exponent δ. By a conformal measure (or δ-conformal measure, to be more precise) we understand a Borel probability measure μ satisfying the following condition:

$$\mu(f(A)) = \int_A |Df(z)|^{\delta} d\mu(z),$$

whenever f restricted to the set A is one to one.

Conformal measures are natural geometric measures. For example, if $J(f)$ is a one-dimensional manifold, then the one-dimensional Lebesgue measure is a conformal measure for f. If $J(f)$ has positive (two-dimensional) Lebesgue measure, then the Lebesgue measure of $J(f)$ is a conformal measure for f (with $\delta = 2$). If $f : J(f) \to J(f)$ is a hyperbolic system, then $J(f)$ has zero Lebesgue measure. In that case, the Hausdorff measure of $J(f)$ is finite and non-zero and equivalent to a δ-conformal measure, where $\delta = \mathrm{HD}(J(f))$ = Hausdorff Dimension of $J(f)$ (see [Bow75], [Sul80] and [Wal78]). In more general cases, conformal measures still reflect geometric properties of $J(f)$ (see for example [DU91a], [DU91b], [Pra95], [Prz] and [U]).

We say that μ is ergodic if $\mu(X) = 0$ or $\mu(X) = 1$ whenever we have $X = f^{-1}(X)$. Notice that usually when one talks about ergodicity of a measure it is assumed that the measure is invariant. In our case, due to the definition of conformal measure we are not dealing with an invariant measure but rather a quasi-invariant measure.

Received by the editors September 1, 1997 and, in revised form, December 15, 1997.
1991 *Mathematics Subject Classification.* Primary 58F03, 58F23.
Key words and phrases. Holomorphic dynamics, conformal measures.
Supported in part by CNPq-Brazil and S.U.N.Y. at Stony Brook.

One way to study chaotic dynamical systems is to find a natural ergodic measure with respect to this system. Our goal in this work is to study ergodic properties of polynomial dynamical systems using conformal measures.

We will show that a conformal measure is ergodic if f is a polynomial with certain properties. Let us introduce some definitions in order to state our main result.

Let f be a quadratic polynomial with only repelling periodic points. Following [Lyu97], we will say that f satisfies the *secondary limb condition* if there is a finite family of truncated secondary limbs L_i of the Mandelbrot set such that the hybrid classes of all renormalizations $R^m(f)$ belong to $\bigcup L_i$. Let $\mathcal{SL}$ stand for the class of quadratic polynomials satisfying the secondary limb condition.

Some examples of polynomials of class $\mathcal{SL}$ are: Yoccoz and Lyubich polynomials, and also infinitely many times renormalizable real polynomials of degree two. A quadratic polynomial is a Yoccoz polynomial if it is at most finitely many times renormalizable, with only repelling periodic points. A Lyubich polynomial is an infinitely many times renormalizable quadratic polynomial in $\mathcal{SL}$ with a priori bounds as described in [Lyu97] (see the definition of a priori bounds below).

If $f : U \to V$ is a polynomial-like map, then we say that $\mathrm{mod}(f)$ is equal to the modulus of the topological annulus $A = V \setminus \overline{U}$. There exists a conformal mapping between the annulus $A = V \setminus \overline{U}$ and a standard annulus $\{z : 1 < |z| < r\}$ for some $r > 1$. The *modulus* of A is defined as $\frac{1}{\pi} \log r$.

We say that an infinitely renormalizable polynomial f has *a priori bounds* if there exists an $\epsilon > 0$ such that $\mathrm{mod}(R^m(f)) > \epsilon$, where $R^m(f)$ is the m^{th} renormalization of f, for infinitely many m. According to [GS], [LvS95] and [LyuY95], all infinitely renormalizable real unimodal polynomials have a priori bounds. By *unimodal polynomial*, we mean an even degree polynomial with just one critical point.

We will show the following:

Theorem 1. *Let f be a polynomial with just one critical point and $J(f)$ connected. Let μ be a conformal measure for f. Suppose that f is either:*

1. *of class $\mathcal{SL}$, finitely many times renormalizable, or infinitely many times renormalizable with a priori bounds, or*
2. *a polynomial with a parabolic periodic point, or*
3. *any unimodal polynomial with real coefficients.*

Then μ is ergodic.

Ergodicity of conformal measures is known if f is expanding on $J(f)$ (see [Bow75], [Sul80] and [Wal78]). Suppose that f has just one critical point and an attracting cycle. Then f is expanding on $J(f)$. If $J(f)$ is disconnected (and f has just one critical point), then f is again expanding when restricted to $J(f)$. From this fact and Theorem 1 we conclude the following:

Corollary 2. *If f is any unimodal polynomial with real coefficients and μ a conformal measure for f, then μ is ergodic.*

Suppose that the Lebesgue measure of $J(f)$ is positive (as we mentioned before, in this case the Lebesgue measure is a conformal measure for f). In [McM95], McMullen asked how many ergodic components would $f : J(f) \to J(f)$ have with respect to the Lebesgue measure. The previous Corollary gives an answer for this question in the real unimodal case.

If $f : J(f) \to J(f)$ is hyperbolic, then there is only one number δ, namely the Hausdorff dimension of $J(f)$, for which there exists a δ-conformal measure for f. Moreover there is only one conformal measure with such exponent. For more general examples, it is not true that there is only one δ for which there exists a δ-conformal measure for f (see [DU91b]). In general we can ask whether there exists an exponent δ such that there is more than one δ-conformal measure for f. The following is an immediate consequence of the ergodicity of conformal measures:

Corollary 3. *Let f be as in Theorem 1. Then for any $\delta > 0$, there exists at most one δ-conformal measure for f.*

The situation studied in this paper is the complex counterpart for the ergodicity result in [BL91] where the Lebesgue measure is shown to be ergodic under S-unimodal maps.

2. Renormalization and combinatorics

2.1. **Non-renormalizable polynomials.** We will briefly describe how to construct the Yoccoz puzzle pieces for a quadratic polynomial. See [Hub] and [Mil91] for a complete exposition of such construction. See [Mil90] for background material concerning one-dimensional complex dynamics.

In this section we will consider quadratic polynomials f with repelling periodic points. We shall keep in mind though that the construction of Yoccoz puzzles that will be described in this section can be repeated for polynomials with degree greater than two.

We say that $g : U \to U'$ is a *quadratic-like map* if it is a double branched covering; U and U' are open topological disks with U compactly contained in U'. The *filled Julia set* of g is the set $\{z \in U : g^n(z)$ is defined for all natural numbers $n\}$. There are two fixed points of g inside its filled Julia set. If the filled Julia set is connected and both fixed points are repelling, one of them, the dividing fixed point, disconnects the filled Julia set of g in more than one connected component. The other does not. Usually the dividing fixed point is denoted by α and the other is denoted by β. Quadratic-like maps were first introduced and studied in [DH85].

Remember that a polynomial or a polynomial-like map f with connected filled Julia set is *renormalizable* if there exist open topological disks $U \subset U'$ with $0 \in U$ and $R(f) : U \to U'$ being a quadratic-like map with connected filled Julia set. We define $R(f) = f^k|_U$, with k the smallest natural number larger than 1 satisfying the previous conditions. We call k the *period of renormalization*. Here $R(f)$ stands for the renormalization of f. We also require as part of the definition of renormalization that the sets $f^i(J(R(f)))$, $i = 0, 1, \dots, k-1$ (called *little Julia sets*) are pairwise disjoint, except perhaps when they touch at their β fixed points.

We can ask whether $R(f)$ is renormalizable or not and then define renormalizations of f of higher orders. So, each renormalization of f defines a quadratic polynomial-like map. We refer the reader to [McM94] for more details concerning renormalization.

Let f be a degree two non-renormalizable polynomial with both fixed points repelling and let G be the Green function of the filled Julia set of f. There are q external rays landing at the dividing fixed point of f, where $q \geq 2$. The q *Yoccoz puzzle pieces of depth zero* are the components of the topological disk defined by $G(z) < G_0$, where G_0 is any fixed positive constant, cut along the q external rays landing at the dividing fixed points. We denote $Y^0(x)$ the puzzle piece of depth zero containing x. We define the *puzzle pieces of depth n* as being the connected components of the pre-images of any puzzle piece of depth zero under f^n. Again, if x is an element of a given puzzle piece of depth n, then we denote such a puzzle piece by $Y^n(x)$.

Suppose now that f is at most finitely renormalizable with only repelling periodic points. Let α be the dividing fixed point of the last renormalization of f. Let G be the Green function of the filled Julia set of f. In that case we define the puzzle pieces of depth zero as being the components of the topological disk $G(z) < G_0$, G_0 a positive constant, cut along the rays landing at all points of the f-periodic orbit of α. As before we define the puzzle pieces of depth n as being the connected components of the pre-images under f^n of the puzzle pieces of depth zero. The puzzle piece at depth n containing x is denoted by $Y^n(x)$.

We will consider the Yoccoz puzzle pieces as open topological disks. Under this consideration the Yoccoz partition will be well defined over the Julia set of the polynomial f minus the set of pre-images of the dividing fixed point of the last renormalization of f (which is f itself in the non-renormalizable case).

A quadratic polynomial is a *Yoccoz polynomial* if it is at most finitely renormalizable with only repelling periodic points. We will need the following result:

Theorem 2.1 (Yoccoz). *If f is a Yoccoz polynomial, then $\bigcap_{n\geq 0} Y^n(x) = \{x\}$ for any x where the Yoccoz partition is defined.*

The following is an analogous Theorem for higher degree real unimodal polynomials:

Theorem 2.2 ([LvS95]). *Let $f(z) = z^l + c$, l even, c real and f finitely many times renormalizable with only repelling periodic points. Then for any x where the Yoccoz partition is defined we have: $\bigcap_{n\geq 0} Y^n(x) = \{x\}$.*

2.2. **The $\mathcal{SL}$ class and the principal nest.** Here we will describe the secondary limb class of quadratic polynomials. See [Lyu97] for a detailed exposition on this matter. We will need some technical definitions.

Let us start with a quadratic polynomial f with only repelling periodic points. Given a Yoccoz puzzle piece Y_i^n of f and a point x such that $f^j(x)$ belongs to Y_i^n, we define the *pull back of Y_i^n along the orbit of x* as being the only connected component of $f^{-j}(Y_i^n)$ containing x. If moreover x belongs to Y_i^n and j is the minimal positive moment with the above property, then we say that j is *the first return time of x to Y_i^n*. A puzzle piece is said to be a *critical puzzle piece* if it contains the critical point. Notice that if we pull back a critical puzzle piece $Y^n(0)$ along the first return of the critical point to $Y^n(0)$ we get a new critical puzzle piece.

Suppose that f is not Douady-Hubbard immediately renormalizable (see [Lyu97]). Then it is possible to find a critical puzzle piece (that will be denoted by $V^{0,0}$) satisfying the following: if the pull back of $V^{0,0}$ along the first return of the critical point to $V^{0,0}$ is denoted by $V^{0,1}$, then the closure of $V^{0,1}$ is properly contained in $V^{0,0}$. We keep repeating this procedure: define $V^{0,t+1}$, the puzzle piece of level $t+1$, as being the pull back of $V^{0,t}$, the puzzle piece of level t, along the first return of the critical point to $V^{0,t}$. This procedure stops if the critical point does not return to a certain critical puzzle piece. If we assume that the critical point is combinatorially recurrent, then we can repeat this procedure forever. So let us assume this is the case. The collection $V^{0,t}$ for t being a natural number is the *principal nest on the first renormalization level*.

Now we have a sequence of first return maps $f^{l(t)} : V^{0,t+1} \to V^{0,t}$. By definition $V^{0,0}$ properly contains $V^{0,1}$. This implies that each $V^{0,t}$ properly contains $V^{0,t+1}$. It is also easy to see that each $f^{l(t)} : V^{0,t+1} \to V^{0,t}$ is a quadratic-like map.

We say that $f^{l(t)} : V^{0,t+1} \to V^{0,t}$ is a *central return* or that t is a *central return level* if $f^{l(t)}(0)$ belongs to $V^{0,t+1}$. A *cascade of central returns* is a set of consecutive central return levels. More precisely, a cascade of central returns is a collection of central return levels $t = t_0, \dots, t_0 + (N-1)$ followed by a non-central return at level $t_0 + N$. In this case we say that the above cascade of central returns has length N. We could also have an infinite cascade of central returns. Notice that with the above terminology a non-central return level is a cascade of central return of length zero.

It is possible to show that the principal nest on the first renormalization level ends with an infinite cascade of central returns if and only if f is renormalizable (see [Lyu97]). In that case, denote the first level of this infinite cascade of central returns by $t(0)$. Then we define the first renormalization $R(f)$ of f as being the quadratic-like map $f^{l(t(0))} : V^{0,t(0)+1} \to V^{0,t(0)}$. The filled Julia set of $R(f)$ is connected. It is also possible to show that $\bigcap V^{0,n} = J(R(f))$. Again we can find the dividing fixed point of the Julia set of $R(f)$, some external rays landing at it and define new puzzle pieces over the Julia set of $R(f)$. The rays landing at the new dividing fixed point are not canonically defined (remember that $R(f)$ is a polynomial-like map). We are not taking the external rays of the original polynomial. Instead we need to make a proper selection of those rays (see [Lyu97]). As before we can construct the principal nest for $R(f)$, provided that $R(f)$ is not Douady-Hubbard immediately renormalizable. The elements of this new principal nest are denoted by $V^{1,0}, V^{1,1}, \dots, V^{1,t}, \dots$ and the nest is called the *principal nest on the second renormalization level*. If this new principal nest also ends in an infinite cascade of central returns, we repeat the procedure just described and construct a third principal nest. We repeat this process as many times as we can.

Now we define the *principal nest* of the polynomial f as being the set of critical puzzle pieces

$$
\begin{aligned}
&V^{0,0} \supset V^{0,1} \supset \cdots \supset V^{0,t(0)} \supset V^{0,t(0)+1} \supset V^{1,0} \supset V^{1,1} \supset \cdots \supset V^{1,t(1)} \\
&\qquad \supset V^{1,t(1)+1} \supset \cdots \supset V^{m,0} \supset V^{m,1} \supset \cdots \supset V^{m,t(m)} \supset V^{m,t(m)+1} \supset \cdots .
\end{aligned} \tag{2.1}
$$

In order to go ahead with the definition of the class of polynomials we are interested in, we need the notion of a truncated secondary limb. A *limb* in the Mandelbrot set M is the connected component of $M \setminus \{c_0\}$ not containing 0, where c_0 is a bifurcation point on the main cardioid. If we remove from the limb a neighborhood of its root c_0, we get a *truncated limb*. A similar object corresponding to the second bifurcation from the main cardioid is a *truncated secondary limb*.

We say that a quadratic polynomial with only repelling periodic points satisfies *the secondary limb condition* if there is a finite family of truncated secondary limbs L_i of the Mandelbrot set such that the hybrid class of all renormalizations $R^m(f)$ belongs to $\bigcup L_i$. Let $\mathcal{SL}$ stand for the class of quadratic polynomials satisfying the secondary limb condition.

2.3. **Complex bounds.** We say that f has *a priori bounds or complex bounds* if there exists an $\epsilon > 0$ such that $\mathrm{mod}(R^m(f)) > \epsilon$, for infinitely many renormalizations of f. A priori bounds is, as we shall see, one of the main properties that we will be using in this paper. In [Lyu97] it was conjectured that the secondary limb condition described above implies a priori bounds. Also, a large class of infinitely many times renormalizable quadratic polynomials satisfying the secondary limb condition with a priori bounds was constructed (see [Lyu97]). The next Theorem follows from Theorem II in [Lyu97].

Theorem 2.3 (Lyubich)**.** *Let f be a Yoccoz polynomial. There exists a constant $c > 0$ such that $\mathrm{mod}(V^{0,t} \setminus V^{0,t+1}) > c$, for all t.*

Complex bounds were proved first by Sullivan (see [MvS93]) for real infinitely renormalizable quadratic maps with bounded combinatorics. Later in [GS], [LvS95] and [LyuY95] the restriction on the combinatorics was removed. Also [LvS95] provides complex bounds for infinitely renormalizable polynomials of the form $f(z) = z^l + c$, where l is even and c is real.

Theorem 2.4. *Let $f(z) = z^l + c$ be an infinitely renormalizable real polynomial of even degree l. If a_n is the period of the n^{th} renormalization of f, then there exist topological disks $V^{n,0}$ and $V^{n,1}$ such that:*

1. $0 \in V^{n,1}$*;*
2. $\mathrm{cl}(V^{n,1}) \subset V^{n,0}$*;*
3. $\mathrm{mod}(V^{n,0} \setminus V^{n,1}) \geq c > 0$*;*
4. $f^{a_n} : V^{n,1} \to V^{n,0}$ *is a polynomial-like map of degree l with connected filled Julia set;*
5. $\mathrm{diam}(V^{n,0}) \to 0$ *as $n \to \infty$.*

2.4. **Unbranched maps.** The unbranched condition is, as we shall see, a property that should go along with complex bounds so that we can have control of certain pull-backs. Before going into the definition of the unbranched condition let us define generalized polynomial-like maps:

Definition 2.5 ([Lyu91])**.** Let U and U_i be open topological disks, $i = 0, 1, \ldots, n$. Suppose that $\mathrm{cl}(U_i) \subset U$ and $U_i \cap U_j = \emptyset$ if i is different from j. A generalized polynomial-like map is a map $f : \bigcup U_i \to U$ such that the restriction $f|U_i$ is a branched covering of degree $d_i, d_i \geq 1$.

We will not use the above Definition in full generality. From now on, all generalized polynomial-like maps in this work will have just one critical point. We will fix our notation as follows: $f|U_0$ is a branched covering of degree d onto U (with zero being the only critical point) and $f|U_i$ is an isomorphism onto U, if $i = 1, \ldots, n$.

For the next definition we will consider a polynomial f with just one critical point and $g : \bigcup V_i \to V$ a generalized polynomial-like map. Assume that g is defined in each V_i as the first return map to V under f. In particular we assume that V_0 is the pull back of V along the first return of the critical point of f to V.

Definition 2.6 ([LvS95] and [McM94])**.** We say that g is unbranched if whenever $f^i(0)$ belongs to V, then $f^i(0)$ is an iterate of 0 under g.

Let g be as in the above definition. If we assume that the critical point is recurrent, then the intersection of the critical set of f with V is contained in the domain of g. Notice that if f is renormalizable and if g is a renormalization of f, then the above definition coincides with the unbranched renormalization definition from [McM94].

Lemma 2.7. *Suppose that $g : \bigcup V_i \to V$ is unbranched. Let z be a point in $\mathbb{C}$ such that its f-orbit hits V_0. Let k be the smallest non-negative number such that $f^k(z)$ belongs to V_0. Then we can pull V back univalently along the orbit $z, f(z), \ldots, f^k(z)$.*

Proof. Suppose this is not the case. Then for some positive r smaller than k, $f^{-r}(V)$ hits the critical value $f(0)$. By $f^{-r}(V)$ we understand the pull back of V along the orbit $f^{k-r}(z), \ldots, f^k(z)$. In that case, $f^{i+1}(0) \in f^i(f^{-r}(V))$, for $i = 0, \ldots, r$. Suppose that there exists $0 \leq i_{\max} < r$ maximal such that $f^{i_{\max}+1}(0)$ belongs to V. In that case, by the unbranched property there exists a component V_j of the domain of g containing $f^{i_{\max}+1}(0)$. As V_j is the pull back of V under the first return of $f^{i_{\max}+1}(0)$ to V, we conclude that $f^{i_{\max}}(f^{-r}(V)) \subset V_j$. Now, as the pull back of V_j under f along $0, \ldots, f^{i_{\max}+1}(0)$ is contained in V_0, it follows that $f^{k-r-1}(z) \in f^{-r-1}(V) \subset V_0$, contradicting the minimality of k. Suppose now that there is no $i_{\max}$, i.e., if the first return time of 0 to V is $r+1$. Then $f^{r+1} : f^{-r-1}(V) \to V$ coincides with the first return map $g : V_0 \to V$. This also contradicts the minimality of k. $\square$

Lemma 2.8. *Let f be an $\mathcal{SL}$ polynomial with a priori bounds. Then for infinitely many n we can find U_n and V_n such that the n^{th}-renormalization of f is given by $R^n(f) : U_n \to V_n$ and both the unbranched condition and the a priori bound condition are verified for U_n and V_n.*

Proof. See Lemma 9.3 in [Lyu97] for the proof. $\square$

Let $f^{a_n} : V^{n,1} \to V^{n,0}$ be the polynomial-like maps introduced on Theorem 2.4 (the n^{th} renormalization of $f(z) = z^l + c$). We have the following:

Lemma 2.9. *Let $f(z) = z^l + c$ be infinitely many times renormalizable with l even and c real. Then the polynomial-like maps $f^{a_n} : V^{n,1} \to V^{n,0}$ are unbranched, for infinitely many n.*

Proof. This is due to the construction of the set $V^{n,1}$ and $V^{n,0}$ in [LvS95]. $\square$

For the next Lemma, let f be either a Yoccoz polynomial or any finitely many times renormalizable real polynomial with only repelling periodic points and just one critical point. Suppose that this critical point is recurrent. Let $Y^n(0)$ be a critical Yoccoz piece and $Y^{n+k}(0)$ be the pull back of $Y^n(0)$ corresponding to the first return of the critical point to $Y^n(0)$.

Lemma 2.10. *Let z be a point in $\mathbb{C}$ such that its f-orbit hits $Y^{n+k}(0)$. Let m be the smallest non-negative time such that $f^m(z) = Y^{n+k}(0)$. Then we can univalently pull $Y^n(0)$ back along the orbit $z, \ldots, f^m(z)$.*

Proof. If not, $f^{-t}(Y^n(0))$ would contain the critical point, for some t less than m (here f^{-t} means the branch of f^{-t} along the orbit of x). That would mean that t is greater or equal to the first return time of 0 to $Y^n(0)$. That would imply $f^{-t}(Y^n(0)) \subset Y^{n+k}(0)$ by the Markov property of puzzle pieces. In other words, z would hit $Y^{n+k}(0)$ on a time strictly less than m, contradicting the definition of m. $\square$

Lemma 2.11. *Let f be either a Yoccoz polynomial or any finitely many times renormalizable real polynomial with only repelling periodic points. Then for infinitely many critical puzzles $Y^j(0)$, we can find a topological disk D_j such that:*

1. $Y^j(0) \subset D_j$;
2. $\mathrm{mod}(D_j \setminus Y^j(0)) > c(f) > 0$;
3. *If z is a point in $\mathbb{C}$ such that its f-orbit hits $Y^j(0)$ and m is the smallest non-negative time such that $f^m(z) \in Y^j(0)$, then we can univalently pull D_j back along the orbit $z, \dots, f^m(z)$.*

Proof. For a Yoccoz polynomial this follows from Lemma 2.10 and Theorem 2.3. If f is real but not of degree two, then the Lemma follows from [LvS95]. □

3. Density estimates

From now on f will be a polynomial of even degree with just one critical point and μ will denote a δ-conformal measure concentrated on the Julia set of f.

The analytic tool that we will use is the well known Koebe distortion Theorem:

Theorem 3.1 (Koebe)**.** *Let $A \subset B$ be two topological disks contained in the complex plane. Suppose that f is univalent when restricted to B. Also suppose that $B \setminus A$ is a topological annulus with positive modulus m. Then*

$$\frac{1}{K} \le \frac{|Df(z_1)|}{|Df(z_2)|} \le K$$

for all z_1 and z_2 in A, where the constant K depends only on the number m.

The constant K that appears in the Lemma is called the Koebe constant. Under the conditions of the above Lemma we say that f has bounded distortion inside the set A.

Let f be either a Yoccoz polynomial or a finitely many times renormalizable real polynomial with only repelling periodic points. Notice that if a periodic point of f in $J(f)$ is expanding, then the set of all its pre-images has zero μ-measure. As we used just expanding periodic points to construct puzzle pieces, given any closed subset X of $J(f)$, we can create a cover K_i of X (up to a set of zero measure) built up by puzzles pieces and with $\lim \mu(K_i) = \mu(X)$. This follows from Theorem 2.1 and Theorem 2.2.

Definition 3.2. The density of a set X inside a set Y is defined as the following ratio: $\mathrm{dens}(X|Y) = \frac{\mu(X \cap Y)}{\mu(Y)}$.

Lemma 3.3. *Let f be either a Yoccoz polynomial or a finitely many times renormalizable real polynomial of even degree with only repelling periodic points. Let $X \subset J(f)$ be any measurable subset. If $\mu(X) > 0$, there is x in X such that $\limsup(\mathrm{dens}(X|Y^n(x))) = 1$.*

Proof. Assume $\mu(X) > 0$. If X is not closed, take $W \subset X$ compact with $\mu(X \setminus W)$ small. Notice that $\mathrm{dens}(X|Y^n(x)) \ge \mathrm{dens}(W|Y^n(x))$ for any $Y^n(x)$. For all $\varepsilon > 0$, there exists $i(\varepsilon)$, such that $1 - \varepsilon \le \frac{\mu(W \cap K_i)}{\mu(K_i)} \le 1$ if $i > i(\varepsilon)$ (remember that K_i are the covers of X made out of puzzle pieces). So we have for i big $\mathrm{dens}(W|K_i) = \frac{\mu(W \cap K_i)}{\mu(K_i)} \ge 1 - \varepsilon$. As K_i is the union of puzzle pieces we can certainly find a puzzle piece in K_i, say $Y^{n(i)}(x_i)$ such that $\mathrm{dens}(W|Y^{n(i)}(x_i)) \ge 1 - \varepsilon$. Now replacing X by $X \cap Y^{n(i)}(x_i)$ and repeating this argument we will end up with the desired result. □

Definition 3.4. The point $x \in X$ obtained in the previous Lemma is called a weak density point of X.

Proposition 3.5. *Let $A \subset B$ be two μ-measurable subsets of the complex plane. Suppose that f restricted to an open neighborhood of B is one to one. Also suppose that there exists a positive constant K such that*

$$\frac{1}{K} \leq \frac{|Df(z_1)|}{|Df(z_2)|} \leq K$$

for all z_1 and z_2 in B, then

$$\frac{1}{K^\delta} \operatorname{dens}(A|B) \leq \operatorname{dens}(f(A)|f(B)) \leq K^\delta \operatorname{dens}(A|B).$$

Proof. Follows from the definitions of conformal measure and $\operatorname{dens}(A|B)$. □

If U is a subset of the complex plane, we will denote by U^c the complement of U inside the complex plane.

Lemma 3.6. *Let f be either a Yoccoz polynomial or a finitely many times renormalizable real polynomial of even degree with only repelling periodic points. Let μ be a conformal measure for f. Let U be any neighborhood of the critical point. Then the set*

$$\{x \in \mathbb{C} : f^n(x) \in U^c, \text{ for all positive } n\}$$

has zero μ-measure.

Proof. It is enough to show this Lemma for $U = Y^i(0)$ because by Theorem 2.1 and Theorem 2.2 any neighborhood of the critical point contains some $Y^i(0)$, for i sufficiently big. Suppose that the set $A = \{x \in \mathbb{C} : f^n(x) \in Y^i(0)^c$, for all n positive$\}$ has positive measure, for some i fixed. Then this set has a point of weak density x, according to Lemma 3.3. So we can find some sequence $n(j) \to \infty$ such that $\operatorname{dens}(A|Y^{n(j)}(x)) \to 1$.

Notice that $f^{n(j)-i}(Y^{n(j)}(x))$ is a puzzle piece of depth i and none of the puzzle pieces $Y^{n(j)}(x), f(Y^{n(j)}(x)), \ldots, f^{n(j)-i}(Y^{n(j)}(x))$ contains the critical point. That is because of the Markov property of puzzle pieces and the fact that $Y^{n(j)}(x)$ contains elements of the set A. So for all $Y^{n(j)}(x)$, $f^{n(j)-i}(Y^{n(j)}(x))$ is a puzzle piece of depth i distinct from $Y^i(0)$ and the restriction $f^{n(j)-i} : Y^{n(j)}(x) \to Y^i(f^{n(j)-i}(x))$ is an isomorphism. As there exist just finitely many puzzle pieces of depth i, then there is a fixed puzzle piece $Y^i(y)$ (distinct from the one containing the critical point) such that $f^{n(j)-i}(Y^{n(j)}(x)) = Y^i(y)$ for infinitely many $n(j)$. Passing to a subsequence and keeping the same notation we will assume that $f^{n(j)-i} : Y^{n(j)}(x) \to Y^i(y)$ is an isomorphism for all $n(j)$.

We will construct a neighborhood of $Y^i(y)$ where the inverse branch $f^{-(n(j)-i)}$ along the orbit $x, f(x), \ldots, f^{n(j)-i}(x)$ is defined as an isomorphism.

Let $i_1 > i$ such that $\operatorname{mod}(Y^i(0) \setminus Y^{i_1}(0))$ is positive. This is possible by Theorem 2.1 and Theorem 2.2.

The boundary of $Y^i(y)$ is composed of pairs of external rays landing at points in the Julia set and equipotentials. The intersection of this boundary with the Julia set is finite. Let z be a point in this finite intersection. Consider all puzzle pieces of depth i_1 containing z in its boundary. The closure of the union of those puzzle pieces is a neighborhood of z in the plane. Let us call such neighborhood V_z. Notice that each equipotential and the pieces of external rays landing at z

outside V_z are at some definite distance from the Julia set. Take a small tubular neighborhood (not intersecting the Julia set) of each one of the equipotentials and pieces of external rays contained in the boundary of $Y^i(y)$, but outside V_z. Now we define the neighborhood N of $Y^i(y)$ as being the union of each V_z with all tubular neighborhoods described above and $Y^i(y)$ itself (see Figure 1). Notice that we can make N into a topological disk if i_1 is big and the tubular neighborhoods small. Also notice that since the distance between the boundaries of $Y^i(y)$ and N is strictly positive, we get that $\operatorname{mod}(N \setminus Y^i(y))$ is strictly positive.

Now let us prove that for any $n(j)$ we can pull N back isomorphically along the orbit $x, \dots, f^{n(j)-i}(x)$.

The pull back of $Y^i(y)$ along $x, \dots, f^{n(j)-i}(x)$ cannot hit the critical point (this follows from the way we chose the puzzle pieces $Y^i(y)$). None of the pull backs of the tubular neighborhoods can hit the critical point because those neighborhoods are outside the Julia set. The pull back of V_z along $x, \dots, f^{n(j)-i}(x)$ can not touch the critical point. If the pull back of V_z would hit the critical point, then it would be inside $Y^{i_1}(0)$ (because V_z is made out of puzzle pieces of depth i_1). By the choice of i_1 ($\operatorname{mod}(Y^i(0) \setminus Y^{i_1}(0)) > 0$) and because all the puzzle pieces of V_z have a common boundary point with $Y^i(y)$ we would conclude that some pre-image of $Y^i(y)$ along $x, \dots, f^{n(j)-i}(x)$ would intersect $Y^i(0)$. Contradiction!

So we can pull N back isomorphically along the orbit $x, \dots, f^{n(j)-i}(x)$ for any $n(j)$. By the construction of N we have: $\operatorname{mod}(N \setminus Y^i(y)) > 0$. So we conclude that $f^{n(j)-i} : Y^{n(j)}(x) \to Y^i(y)$ has bounded distortion with the Koebe constant not depending on $n(j)$.

Using this bounded distortion property, Proposition 3.5 and the fact that x is a density point for A, we conclude that $\operatorname{dens}(A|Y^i(y))$ is arbitrarily close to one. On the other hand there exists some pre-image of $Y^i(0)$ inside $Y^i(y)$, so $\operatorname{dens}(A|Y^i(y))$ is bounded away from 1. Contradiction! □

Let us prove a similar result for the classes of infinitely renormalizable polynomials that we are dealing with:

Lemma 3.7. *Let f be any $\mathcal{SL}$ quadratic polynomial with a priori bounds and let μ be a conformal measure for f. Let U be any neighborhood of the critical point. Then the set*

$$\{x \in \mathbb{C} : f^n(x) \in U^c, \text{ for all positive } n\}$$

has zero μ-measure.

Proof. Let us denote the set in the statement of this Lemma by A. We have $A = J(f) \setminus \bigcup_k f^{-k}(U)$. So A is a nowhere dense forward invariant set. Notice that $A \cap \overline{O}$ is empty (because of the definition of A and because $\overline{O}$ is minimal if f is infinitely many times renormalizable with a priori bounds). In view of the Lebesgue density Theorem (see Theorem 2.9.11 in [Fed69]), the set of density points of A has full measure inside A. Here by density points we mean $x \in A$ such that $\lim_{r\to 0} \operatorname{dens}(A|B(x,r)) = 1$, where $B(x,r)$ is the Euclidean disk with center at x and radius r. Suppose that $\mu(A)$ is positive. Then we conclude that there exists a density point x in A. There also exists y inside A and a sequence of natural numbers $k_j \to \infty$ such that $f^{k_j}(x) \to y$. We can univalently pull back a disk of definite size centered in y along $x, f(x), \dots, f^{k_j}(x)$ (to be more precise, the size of this disk is $\operatorname{dist}(y, \overline{O})$). That implies that we can fix a positive number η and univalently pull

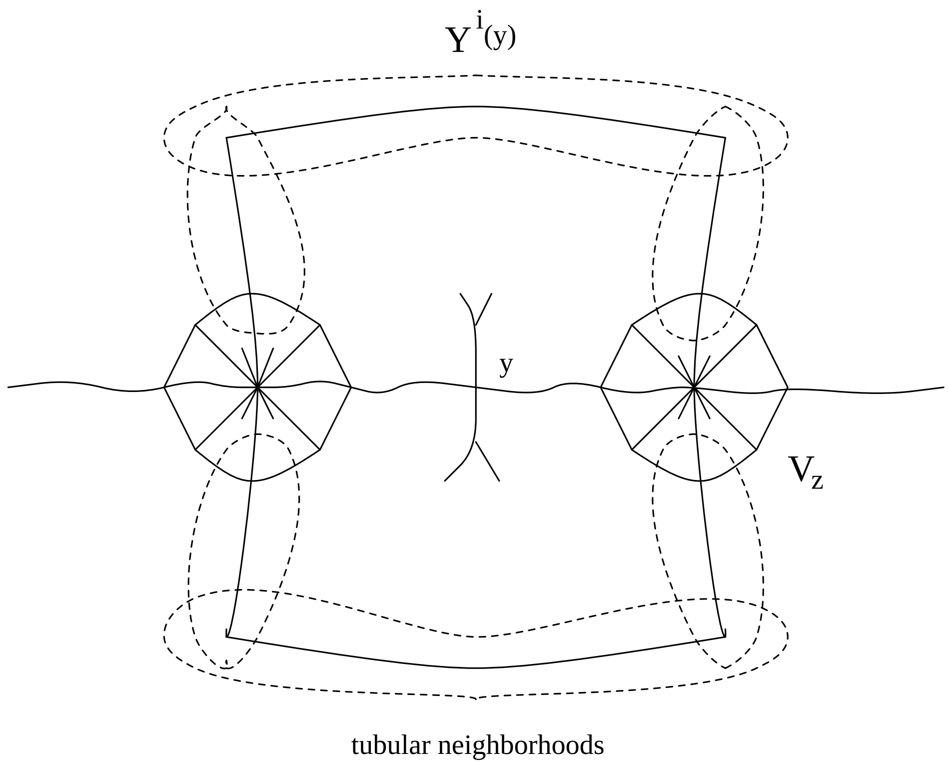

FIGURE 1. Construction of the neighborhood N of $Y^i(y)$

back the disk $B(f^{k_j}(x), \eta)$, along $x, f(x), \ldots, f^{k_j}(x)$. Since A is nowhere dense and μ is positive on non-empty open subsets of the Julia set, for large k_j we have:

$$\mu(B(f^{k_j}(x), \frac{\eta}{2}) \setminus A) \geq \mu(B(y, \frac{\eta}{4}) \setminus A) > 0.$$

As a consequence of Koebe's Theorem, the definition of conformal measure and the invariance of A we have:

$$|Df^{k_j}(x)|^{-\delta} \mu(B(f^{k_j}(x), \frac{\eta}{2}) \setminus A) \leq K \mu(B(x, K\frac{\eta}{2}|Df^{k_j}(x)|^{-1}) \setminus A).$$

Let us denote $r = K\frac{\eta}{2}|Df^{k_j}(x)|^{-1}$. From the above and from the definition of conformal measure we get:

$$\begin{aligned}\frac{\mu(B(x,r) \setminus A)}{\mu(B(x,r))} &\geq \frac{K^{-1}|Df^{k_j}(x)|^{\delta}}{\mu(f^{k_j}(B(x,r)))} |Df^{k_j}(x)|^{-\delta} K^{-1} \mu(B(f^{k_j}(x), \frac{\eta}{2}) \setminus A) \\ &\geq \frac{K^{-2}\mu(B(f^{k_j}(x), \frac{\eta}{2}) \setminus A)}{\mu(f^{k_j}(B(x,r)))} \geq K^{-2}\mu(B(y, \frac{\eta}{4}) \setminus A) \geq c > 0.\end{aligned}$$

As $\lim_{k_j \to \infty} |Df^{k_j}(x)| = \infty$ (because of bounded distortion and lack of normality inside $J(f)$) we get: $\limsup_{r \to 0} \frac{\mu(B(x,r) \setminus A)}{\mu(B(x,r))} > 0$, which contradicts the choice of x as a density point of A. $\square$

Lemma 3.8. *Let $f(z) = z^l + c$, with l even and c real, be an infinitely renormalizable polynomial and μ a conformal measure for f. Let U be any neighborhood of* *the critical point. Then the set*

$$\{x \in \mathbb{C} : f^n(x) \in U^c, \text{ for all positive } n\}$$

has zero μ-measure.

Proof. The proof of this Lemma is identical to the proof of the previous Lemma. The essential information we used in the proof of Lemma 3.7 was the complex bounds. The complex bounds property in the present case is guaranteed by Theorem 2.4. ☐

Note that in Lemma 3.7 and Lemma 3.8 we used the fact that f restricted to $\overline{O}$ is minimal which is not necessarily true for polynomials which are at most finitely many times renormalizable. On the other hand, in Lemma 3.6 we used the fact that we have a partition for the entire Julia set by puzzle pieces whose pre-images shrink to points. We do not have that for the polynomials in Lemma 3.7 and Lemma 3.8.

From the previous lemmas we conclude that the set

$$\mathcal{W} = \{z \in J(f) : 0 \in w(z)\}$$

has full measure, i.e., $\mu(\mathcal{W}) = 1$. Here $w(z)$ denotes the w-limit set of z.

Remember that in the case of a finitely many times renormalizable polynomial (with only repelling periodic points), we constructed the puzzle pieces using the α fixed point of the Julia set of the last renormalization of the polynomial. Because of that we think the principal nest of such polynomials as having just one renormalization level. So the first index of the principal nest is always 0: $V^{0,n}$. This is convenient in order to keep our notation simpler.

Let $X \subset \mathcal{W}$ be any compact set. If f is a finitely many times renormalizable polynomial (real, if the degree is greater than 2) with only repelling periodic points we can create a cover of X by puzzle pieces as follows. Fix $V^{0,n}$. For every $x \in X$ there exists a first time m such that $f^m(x) \in V^{0,n}$. So we can pull $V^{0,n}$ along the orbit of x back to a puzzle piece containing x. Changing $x \in X$ we will obtain the desired cover. Let us call this cover $\mathcal{O}_n$. We can make a similar construction for any $\mathcal{SL}$ polynomial with a priori bounds using the sets $V^{m,t(m)}$ constructed in Subsection 2.2 (we will just consider the renormalization levels where we have the a priori bounds). For any real unimodal infinitely renormalizable polynomial we can also repeat the same construction using the sets $V^{n,1}$ introduced on Theorem 2.4. We have the following properties:

1. $\mathcal{O}_n$ is an open cover;
2. $\mathcal{O}_n \subset \mathcal{O}_{n-1}$;
3. $\bigcap \mathcal{O}_n = X$;
4. $\mu(\mathcal{O}_n) \to \mu(X)$ as $n \to \infty$.

The first and the second properties are trivial. The third one is a consequence of Theorem 2.1 and Theorem 2.2, if f is a finitely many times renormalizable polynomial (real or degree two) with only repelling periodic points. The same fact follows for $\mathcal{SL}$ polynomials with a priori bounds, if we use Lemma 2.8 and the lack of normality inside $J(f)$. If f is an infinitely many times renormalizable real polynomial, the same argument holds if we use Lemma 2.9. The last one follows from the fact that if $\bigcap \mathcal{O}_n = X$ then $\sum \mu(\mathcal{O}_n \setminus \mathcal{O}_{n+1}) < \infty$, hence $\mu(\mathcal{O}_n \setminus X) = \sum_{k \geq n}(\mathcal{O}_{k+1} \setminus \mathcal{O}_k) \to 0$, so that $\mu(\mathcal{O}_n) \to \mu(X)$.

To simplify the notation, elements of $\mathcal{O}_n$ will be denoted by the letter U (indexed in some convenient fashion).

Lemma 3.9. *For all i, there exists U^i in $\mathcal{O}_i$ such that* $\operatorname{dens}(X|U^i) \to 1$, *as $i \to \infty$.*

Proof. Similar to Lemma 3.3. □

We will now finish this section with a Lemma for polynomials with parabolic periodic points:

Lemma 3.10. *Let $f(z) = z^l + c$, with $l \geq 2$ an integer and c complex, be a polynomial with a parabolic periodic point and μ a conformal measure for f. Let U be any neighborhood of the parabolic periodic point. Then the set*

$$\{x \in \mathbb{C} : f^n(x) \in U^c, \text{ for all positive } n\}$$

has zero μ-measure.

Proof. We can assume that f has a fixed point w. If this is not the case, we change f by a convenient power f^n in order to get a fixed point.

The orbit of the critical point of f converges to w. That is because the critical point of f (or f^n) is contained in the union of the attracting petals of w. In that case, if U is any neighborhood of w, there exists an $\varepsilon > 0$ such that all the inverse branches of f^m, for any natural m, are defined in $B(z, \varepsilon)$, for any z outside U. If we denote by A the set in the statement of this Lemma, then the distance from A to the orbit of the critical point is positive. Now we can repeat step by step the proof of Lemma 3.7. □

4. Proof of Theorem 1

4.1. **The non-parabolic cases.** Let $Y \subset \mathcal{W} = \{z \in J(f) : 0 \in w(z)\} \subset J(f)$ be an f-invariant set (remember that $\mathcal{W}$ has full measure). Suppose that $\mu(Y) > 0$.

If f is a Yoccoz polynomial, by Lemma 3.9 we can find U^n in $\mathcal{O}_n$ such that $\operatorname{dens}(Y|U^n) \to 1$. Let $f^{j(n)} : U^n \to V^{0,n}$ be an isomorphism (given by the definition of $\mathcal{O}_n$). Then by Lemma 2.11 and Koebe's Theorem we conclude that $f^{j(n)}$ has bounded distortion, i.e.:

$$\frac{1}{K} \leq \frac{|D(f^{j(n)})(z_1)|}{|D(f^{j(n)})(z_2)|} \leq K$$

for all z_1 and z_2 in U^n, where K depends just on $c(f)$, the constant that appears in the statement of Lemma 2.11.

Now let us apply Proposition 3.5 to the sets $Y^c \cap U^n$ and U^n with respect to the map $f^{j(n)}$. Due to the fact that the set Y is f-invariant and that $f^{j(n)}(U^n) = V^{0,n}$ we get $\frac{1}{K^\delta} \operatorname{dens}(Y^c|U^n) \leq \operatorname{dens}(Y^c|V^{0,n}) \leq K^\delta \operatorname{dens}(Y^c|U^n)$.

We know that $\operatorname{dens}(Y|U^n) \to 1$. Passing to the complement of Y we get $\operatorname{dens}(Y^c|U^n) \to 0$. From this observation and the above inequalities we conclude that $\operatorname{dens}(Y^c|V^{0,n}) \to 0$ and $\operatorname{dens}(Y|V^{0,n}) \to 1$.

Notice that if $\mu(Y^c) > 0$, then we can repeat the argument changing Y by Y^c. Doing this we get $\operatorname{dens}(Y^c|V^{0,n}) \to 1$ and that contradicts the previous statement because $\operatorname{dens}(Y|V^{0,n}) + \operatorname{dens}(Y^c|V^{0,n}) = 1$.

So we conclude that $\mu(Y^c)$ =0, or equivalently, $\mu(Y) = 1$. This finishes the proof of the Theorem if f is a Yoccoz polynomial. For any other finitely many times renormalizable real polynomial of even degree with only repelling periodic points, the proof is identical.

If f is an $\mathcal{SL}$ polynomial with a priori bounds, the proof of Theorem 1 is basically the same. The only difference is that we use Lemma 2.8 instead of Lemma 2.11.

If $f(z) = z^l + c$ is infinitely many times renormalizable, l even and c real, then we need Theorem 2.4 and Lemma 2.9 to carry out the above argument. Again the proof is the same.

4.2. **The parabolic case.** Now, assume that $f(z) = z^l + c$, l even and c complex with a fixed parabolic point w. We know by Lemma 3.10 that if U is a neighborhood of w, then the set $\bigcup_{n=0}^{\infty} f^{-n}(U)$ has full measure. Let us assume that U has a small diameter. Then the set $f^{-1}(U)$ has l connected components: one containing w and the others containing w_i, the pre-images of w, other than w itself. The connected component of $f^{-1}(U)$ containing w_i will be denoted by U_i'. We will denote the union of all the U_i''s by U'. If U is small, then $U \cap J(f)$ is contained in the union of the repelling petals of the parabolic point w. So, $f^{-1}(U) \subset U \cup U'$, up to a set of zero measure (remember that μ is supported on $J(f)$).

Taking into account all the previous observations, we conclude that up to a set of zero measure we have:

$$\bigcup_{n=0}^{\infty} f^{-n}(U) = \bigcup_{n=0}^{\infty} f^{-n}(U') \cup U.$$

1^{st} **case.** Let us assume that $\mu(w) = 0$. Then by the regularity of μ and by the last equality the following is true: for any $\varepsilon > 0$, there exists a sufficiently small neighborhood U of w such that $\mu(\bigcup_{n=0}^{\infty} f^{-n}(U')) > 1 - \varepsilon$ (just take U such that $\mu(U) < \varepsilon$).

If $U_2 \subset U_1$, then $\bigcup_{n=0}^{\infty} f^{-n}(U_2') \subset \bigcup_{n=0}^{\infty} f^{-n}(U_1')$. So there is an ε_2 depending on U_2 such that $\mu(\bigcup_{n=0}^{\infty} f^{-n}(U_1')) \geq \mu(\bigcup_{n=0}^{\infty} f^{-n}(U_2')) > 1 - \varepsilon_2$. If $\text{diam}(U_2)$ is taken arbitrarily small, then ε_2 will be arbitrarily close to zero. This implies that $\mu(\bigcup_{n=0}^{\infty} f^{-n}(U_1')) = 1$.

As $f^n(0) \to w$, the points w_i are at a positive distance from the critical orbit. This implies that there exists a positive number α such that all the inverse branches of f^n are defined on the disk $B(w_i, \alpha)$.

Let us show that μ is an ergodic measure. Let $Y \subset J(f)$ be an f-invariant set such that $\mu(Y) > 0$. In view of the Lebesgue density Theorem the set of density points of Y has full measure inside Y.

By the previous paragraphs we conclude that there exist $x \in Y$ and a sequence $\{k_j\}$ of numbers such that $\lim_{r \to 0} \text{dens}(Y|B(x,r)) = 1$ and that $\lim_{k_j \to \infty} f^{k_j}(x) = w_{i_0}$, for some fixed i_0.

We will show that w_{i_0} is a density point of Y. If this is not the case, then $\lim_{r_i \to 0} \text{dens}(Y|B(-w, r_i)) < 1$, for some sequence of positive numbers r_i tending to zero. So there exists $\eta < \alpha$ such that $\mu(B(w_{i_0}, \frac{\eta}{4}) \backslash Y) > 0$. Now we can mimic the proof of Lemma 3.7 to conclude that x is not a density point of Y. Contradiction!

As the restriction of f to a small neighborhood of w_{i_0} is an isomorphism onto a neighborhood of w and Y is f-invariant, we conclude that w is a density point of Y.

If the measure of the complement of Y is positive, then for the same reason w is a density point of the complement of Y. So we must have $\mu(Y) = 1$, and μ is ergodic.

2^{nd} **case.** If $\mu(w) > 0$, then Lemma 11 and Theorem 13 from [DU91b] show that μ is supported on the grand-orbit of w, and then it is ergodic.

If f has a parabolic periodic point w of period p, then f^p has a fixed parabolic point. We can prove the Theorem in exactly the same way we did before. So Theorem 1 is proven.

Acknowledgment

I would like to thank Misha Lyubich for many hours of valuable mathematical conversations and for continuous encouragement. I also thank Marco Martens for various helpful discussions. I thank John Milnor, Feliks Przytycki and Mariuzs Urbański for reading a preliminary version of this paper and making important comments. I am grateful to CNPq-Brazil and the Department of Mathematics of SUNY at Stony Brook for financial support. Part of this work is taken from the authors's PhD thesis presented in June, 1995 at SUNY at Stony Brook under the direction of M. Lyubich.

References

[BL91] A. M. Blokh and M. Lyubich, Measurable dynamics of S-unimodal maps of the interval, *Ann. Sci. École Norm. Sup.*, 24, 545–573, 1991. MR **93f:**58132

[Bow75] R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, volume 470 of *Lecture Notes in Mathematics*, Springer-Verlag, 1975. MR **56:**1364

[DH85] A. Douady and J. Hubbard, On the dynamics of polynomial-like mappings, *Ann. Sci. École Norm. Sup.*, 18, 287–343, 1985. MR **87f:**58083

[DU91a] M. Denker and M. Urbański, On Sullivan's conformal measures for rational maps of the Riemann sphere, *Nonlinearity*, 4, 365–384, (1991). MR **92f:**58097

[DU91b] M. Denker and M. Urbański, Hausdorff and conformal measures on Julia sets with a rationally indifferent periodic point, *J. London Math. Soc.*, 2(43), 107–118, 1991. MR **92k:**58153

[Fed69] F. Federer, *Geometric measure theory*, Springer-Verlag, 1969. MR **41:**1976

[GS] J. Graczyk and G. Swiatek, Polynomial-like property for real quadratic polynomials, preprint, 1995.

[Hub] J. Hubbard, Local connectivity of Julia sets and bifurcation loci: three theorems of J.-C. Yoccoz, *Topological methods in modern mathematics, A symposium in honor of John Milnor*, Publish or Perish, 467–511. MR **94c:**58172

[LvS95] G. Levin and S. van Strien, Local connectivity of Julia set of real polynomials, *Ann. of Math.*, to appear.

[Lyu91] M. Lyubich, On the Lebesgue measure of the Julia set of a quadratic polynomial, *IMS-Stony Brook preprint series*, (1991/10), 1991.

[Lyu97] M. Lyubich, Dynamics of quadratic polynomials, I-II, *Acta Math.*, 178, 185–297, 1997. CMP 97:15

[LyuY95] M. Lyubich and M. Yampolski, Complex bounds for real polynomials, *Ann. Inst. Fourier*, 47, 1219–1255, 1997.

[McM94] C. McMullen, *Complex dynamics and renormalization*, Number 135, Princeton Univ. Press, 1994. MR **96b:**58097

[McM95] C. McMullen, The classification of conformal dynamical systems, preprint, 1995. CMP 98:02

[MvS93] W. de Melo and S. van Strien, *One dimensional dynamics*, Springer-Verlag, 1993. MR **95a:**58035

[Mil90] J. Milnor, Dynamics in one complex variable: Introductory lectures, *IMS-Stony Brook preprint series*, (1990/5), 1990.

[Mil91] J. Milnor, Local connectivity of Julia sets: expository lectures, *IMS-Stony Brook preprint series*, (1991/10), 1991.

[Pra95] E. A. Prado, Conformal measures in polynomial dynamics, In *PhD thesis*, SUNY at Stony Brook, 1995.

[Prz] F. Przytycki, Iterations of holomorphic Collet-Eckmann maps: conformal and invariant measures, preprint, 1996. CMP 96:17

[Sul80] D. Sullivan, *Conformal dynamics*, 725–752, volume 1007 of *Lecture Notes in Mathematics*, Springer-Verlag, 1980.

[U] M. Urbański, Rational functions with no recurrent critical points, Ergodic Theory Dynam. Systems 14 (1994), no. 2, 391–414. MR **95g:**58191

[Wal78] P. Walters, Invariant measures and equilibrium states for some mappings which expand distance, *Trans. Amer. Math. Soc.*, 263, 121–153, 1978. MR **57:**6371

Instituto de Matemática e Estatística, Universidade de São Paulo, Caixa Postal 66281 CEP 05315-970, São Paulo, Brazil

E-mail address: prado@ime.usp.br