COMPACT NON-ORIENTABLE HYPERBOLIC SURFACES WITH AN EXTREMAL METRIC DISC

ERNESTO GIRONDO AND GOU NAKAMURA

ABSTRACT. The size of a metric disc embedded in a compact non-orientable hyperbolic surface is bounded by some constant depending only on the genus $g \geq 3$. We show that a surface of genus greater than six contains at most one metric disc of the largest radius. For the case $g = 3$, we carry out an exhaustive study of all the extremal surfaces, finding the location of every extremal disc inside them.

1. INTRODUCTION

Compact hyperbolic surfaces have the unit disc $\mathbb{D}$ (or, equivalently, the upper half-plane $\mathbb{H}$) as universal covering space. The hyperbolic metric on the surface is the one induced by the hyperbolic metric of $\mathbb{D}$.

It is rather obvious that a metric disc inside a compact hyperbolic surface X is no longer a topological disc when the radius is large enough (the diameter of X being finite). In [Ba], Bavard showed that both in the orientable or in the non-orientable case, the radius R of a metric disc properly embedded in the surface verifies the inequality

$$(1.1) \quad \cosh R \leq \frac{1}{2 \sin \frac{\pi}{6-6\chi}},$$

where χ denotes the Euler characteristic, given by $2 - 2g$ in the orientable case and by $2 - g$ in the non-orientable case (the *genus* g denotes either the number of handles or the number of cross caps).

Compact surfaces containing a metric topological disc of the largest radius possible for its genus are called *extremal surfaces*, the disc being also termed extremal. Despite the fact that the theory of extremal hyperbolic *orientable* surfaces has been widely studied (see [Gi-Go1], [Gi-Go2], [Gi-Go3], [Na1], [Na2], [Na3]), no progress has been made for the non-orientable case, as far as we know, since Bavard's work ([Ba]), where the existence of non-orientable extremal surfaces of every $g \geq 2$ was shown.

2. EXTREMALITY OF A NON-ORIENTABLE SURFACE

A compact non-orientable hyperbolic surface X , that is a compact non-orientable surface of genus $g \geq 3$, can be realized as the quotient of $\mathbb{D}$ by a discrete subgroup

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K of isometries containing some orientation reversing elements (a so-called *NEC group*). A metric disc in X centered at p corresponds to a hyperbolic disc centered at a representative z of p in $\mathbb{D}$. Moreover, the metric disc is embedded in X exactly when the whole of its counterpart in $\mathbb{D}$ lies inside the *Dirichlet fundamental polygon* for K centered at z , namely

$$D(K, z) = \{w \in \mathbb{D} \mid d(w, z) \leq d(w, \alpha(z)) \ \forall \alpha \in K\}.$$

Bavard showed in [Ba] that the radius R of such a disc verifies equation (1.1), the equality being attained exactly when $D(K, z)$ is a regular polygon with angle $\frac{2\pi}{3}$. The area of X equals $2\pi(g-2)$, whereas the area of a regular N -gon with angle $\frac{2\pi}{3}$ is

$$(N-2)\pi - \frac{2\pi}{3}N = (N-6)\frac{\pi}{3}.$$

It follows that $N = 6g - 6$.

A non-orientable extremal surface is then obtained by side-pairing the sides of a regular $(6g-6)$ -gon P of angle $\frac{2\pi}{3}$ such that there are $2g-2$ classes of identified vertices, each class having three representatives. Of course, some of the side-pairings have to reverse the orientation, and then the group K they generate is not Fuchsian, but an NEC group.

Now, such a group produces a tessellation $\{\gamma(P)\}_{\gamma \in K}$ of $\mathbb{D}$ by copies of P . But such a tessellation is obviously contained in the one corresponding to an extended triangle group $\Delta^\pm$ of type $(2, 3, 6g-6)$, the NEC group generated by the reflections across the sides of the triangle T with vertices at the center, a vertex and the midpoint of an edge of P , this meaning $K \subset \Delta^\pm$. The index of the inclusion equals $12g-12$, since this is the number of copies of T needed to form P .

Reciprocally, the fundamental domain for an index $12g-12$ torsion free subgroup of $\Delta^\pm(2, 3, 6g-6)$ can be obtained as the union of the $12g-12$ copies of T surrounding its vertex of angle $\frac{\pi}{6g-6}$, yielding the polygon P . Thus we have the following.

Theorem 2.1. *A compact non-orientable surface of genus g contains an extremal disc if and only if it can be uniformized as $\mathbb{D}/K$, where K is a torsion free subgroup of the extended triangle group $\Delta^\pm(2, 3, 6g-6)$ with index $12g-12$.*

3. COMMENSURATORS AND EXTREMAL SURFACES

In the orientable case, it was shown in [Gi-Go1] that it is not possible to find more than one extremal disc in the same genus g surface when the associate, $\Delta(2, 3, 12g-6)$ triangle group is not arithmetic, which in turn happens for $g \geq 4$. The proof goes roughly as follows.

Suppose S is a compact orientable surface that contains two extremal discs D_1 and D_2 . Let K be the Fuchsian group uniformizing S ; thus, D_i corresponds to the inclusion of K in a triangle group Δ_i of type $(2, 3, 12g-6)$. Now there is an orientation preserving isometry γ of $\mathbb{H}$ conjugating Δ_1 to Δ_2 , i.e. such that $\gamma^{-1}\Delta_1\gamma = \Delta_2$. As $K \subset \Delta_1 \cap \Delta_2$, it follows that γ belongs to the (orientable)

commensurator of Δ_1 , which is the group

$$\text{Comm}(\Delta_1) = \left\{ \begin{array}{l} \alpha \in \text{Aut}(\mathbb{H}) \text{ such that } \Delta_1 \cap \alpha^{-1}\Delta_1\alpha \\ \text{has finite index in } \Delta_1 \text{ and in } \alpha^{-1}\Delta_1\alpha \end{array} \right\},$$

where $\text{Aut}(\mathbb{H}) = \mathbb{PSL}_2(\mathbb{R})$ is the group of orientation preserving isometries of $\mathbb{H}$.

A well-known result by Margulis shows that $\text{Comm}(G)$ is again discrete exactly when G is not *arithmetic* (see [Mar]). On the other hand, results by Takeuchi ([Ta1] and [Ta2]) show that $\Delta_1(2, 3, 12g - 6)$ is not arithmetic for $g > 3$. The last step in the proof comes from the knowledge (Singerman [Si]) that the group $\Delta_1(2, 3, 12g - 6)$ is *maximal*, in the sense that it cannot be contained with finite index in another discrete group.

We would like to check to what extent these ideas apply to the non-orientable case. For this purpose we have to work a bit with commensurators, arithmeticity and maximality for extended triangle groups.

3.1. Discreteness of commensurators of NEC groups. We will denote by $\text{Aut}^\pm(\mathbb{H}) = \mathbb{PGL}_2(\mathbb{R})$ the extended group of isometries of $\mathbb{H}$ (all the isometries, including those reversing the orientation), and

$$\text{Comm}^\pm(G) = \left\{ \begin{array}{l} \alpha \in \text{Aut}^\pm(\mathbb{H}) \text{ such that } G \cap \alpha^{-1}G\alpha \\ \text{has finite index in } G \text{ and in } \alpha^{-1}G\alpha \end{array} \right\},$$

the extended commensurator group of G .

We shall prove the following.

Theorem 3.1. *Let G be an NEC group, such that the canonical Fuchsian subgroup G^+ is not arithmetic. Then $\text{Comm}^\pm(G)$ is discrete.*

The result follows from two simple lemmas:

Lemma 3.2. *Let G^+ be the canonical index two Fuchsian subgroup of G , so that $G = G^+ \cup \gamma G^+$. Then $\text{Comm}^\pm(G^+) = \text{Comm}(G^+) \cup \gamma \text{Comm}(G^+)$.*

Proof. Let $\alpha \in \text{Comm}^\pm(G^+)$. If $\alpha \in \text{Aut}(\mathbb{H})$, then obviously $\alpha \in \text{Comm}(G^+)$. If not, then $\gamma^{-1}\alpha \in \text{Aut}(\mathbb{H})$, but $\gamma G^+ \gamma^{-1} = G^+$ and so $\gamma^{-1}\alpha \in \text{Comm}(G^+)$, which in turn gives $\alpha \in \gamma \text{Comm}(G^+)$. The opposite inclusion follows since $\gamma \in \text{Comm}^\pm(G^+)$. $\square$

Lemma 3.3. $\text{Comm}^\pm(G^+) = \text{Comm}^\pm(G)$.

Proof. Suppose, as before, $G = G^+ \cup \gamma G^+$, and assume first that $\alpha \in \text{Comm}^\pm(G^+)$. Then $G^+ \cap \alpha^{-1}G^+\alpha$ has finite index in G^+ and in $\alpha^{-1}G^+\alpha$ and therefore $\alpha \in \text{Comm}^\pm(G)$.

Now assume that $\alpha \in \text{Comm}^\pm(G)$. Note that the representatives $a_1, \dots, a_n$ of the cosets $G/(G \cap \alpha^{-1}G\alpha)$ can be chosen in G^+ . Now $(\alpha^{-1}G^+\alpha) \cap (\gamma G^+) = \emptyset$ and $(\alpha^{-1}(\gamma G^+)\alpha) \cap G^+ = \emptyset$. Then

$$\bigcup_{i=1}^n a_i(G^+ \cap \alpha^{-1}G^+\alpha) = G^+$$

since on the one hand we have

$$\bigcup_{i=1}^n a_i(G^+ \cap \alpha^{-1}G^+\alpha) \subseteq G^+, \quad \bigcup_{i=1}^n a_i(\gamma G^+ \cap \alpha^{-1}(\gamma G^+)\alpha) \subseteq \gamma G^+$$

and on the other one

$$\bigcup_{i=1}^n a_i(G^+ \cap \alpha^{-1}G^+\alpha) \cup \bigcup_{i=1}^n a_i(\gamma G^+ \cap \alpha^{-1}(\gamma G^+)\alpha) = \bigcup_{i=1}^n a_i(G \cap \alpha^{-1}G\alpha) = G.$$

It can be shown in a similar way that $[\alpha^{-1}G^+\alpha : G^+ \cap \alpha^{-1}G^+\alpha] < \infty$, and so $\alpha \in \text{Comm}^\pm(G^+)$. $\square$

Proof of Theorem 3.1. If G^+ is non-arithmetic, then $\text{Comm}(G^+)$ is discrete by Margulis' theorem. Now Lemma 3.2 shows that $\text{Comm}^\pm(G^+)$ is discrete. But $\text{Comm}^\pm(G^+) = \text{Comm}^\pm(G)$ by Lemma 3.3, and we are done. $\square$

4. UNIQUENESS OF DISCS FOR HIGH GENERA

Theorem 4.1. *Compact non-orientable hyperbolic surfaces of genus greater than six can contain at most one extremal disc.*

Proof. Suppose S is a compact non-orientable surface that contains two extremal discs D_1 and D_2 . Let K be the NEC group uniformizing S . We know that D_i corresponds to the inclusion of K in an extended triangle group $\Delta_i^\pm$ of type $(2, 3, 6g-6)$. Now, there is an isometry γ of $\mathbb{D}$ conjugating $\Delta_1^\pm$ to $\Delta_2^\pm$, this is $\gamma^{-1}\Delta_1^\pm\gamma = \Delta_2^\pm$. We will show that necessarily γ normalizes $\Delta_1^\pm$, thus $\Delta_2^\pm = \Delta_1^\pm$ and in fact $D_1 = D_2$.

Clearly $\gamma \in \text{Comm}^\pm(\Delta_1^\pm)$, the (extended) commensurator of $\Delta_1^\pm$. Now, Theorem 3.1 shows that $\text{Comm}^\pm(\Delta_1^\pm)$ is discrete when the Fuchsian triangle subgroup Δ_1^+ of $\Delta_1^\pm$ is nonarithmetic. By Takeuchi's results ([Ta1]), this happens for $g > 6$.

Now, the group $\Delta_1^\pm(2, 3, 6g-6)$ is maximal for every $g > 2$ (see [Es-Iz]), thus for $g > 6$ we have $\text{Comm}^\pm(\Delta_1^\pm) = \Delta_1^\pm$, which proves $\Delta_2^\pm = \Delta_1^\pm$ and therefore $D_1 = D_2$. $\square$

5. THE CASE OF GENUS 3

The lowest genus for which a non-orientable compact surface is *hyperbolic* is $g = 3$, and Theorem 4.1 indicates that extremal discs are unique in genera $g > 6$. Therefore, $g = 3$ is the lowest genus in which extremal surfaces with more than one extremal disc may occur. We shall study this case carefully.

5.1. Side-pairing patterns of the 12-gon. Since a non-orientable extremal surface of genus 3 has the regular 12-gon P as its fundamental region, we shall find the side-pairing patterns of P . We use the same kind of arguments as Jørgensen and Näätänen in [Jo-Nä]: Identifying the equivalent sides of P , we have a trivalent graph with 6 edges and 4 vertices on the surface S derived from P . It is quite easy to show that there are 5 graphs fulfilling these conditions (Figure 1).

FIGURE 1. Trivalent graphs with 4 vertices

On each graph we consider the closed walks satisfying the following.

- (i) We walk on each edge of the graph exactly twice, not necessarily in opposite directions.
- (ii) We do not walk on each edge consecutively in both directions without walking on other edges.

A careful case-by-case analysis shows that the closed walks on the graphs A , B , C , D , and E derive 11 side-pairing patterns of the 12-gon (Figure 2). Here, a dotted line indicates a side-pairing such that the sides are pasted by the same direction. We consider the side-pairing pattern obtained from the mirror image of P_2 as P_2 itself.

FIGURE 2. Side-pairing patterns

Theorem 5.1. *There exist 11 side-pairing patterns $P_1, \dots, P_{11}$ for non-orientable extremal surfaces of genus 3. P_1 is obtained from A while P_2, P_3, P_4 come from B , P_5, P_6, P_7 from C , P_8, P_9, P_{10} from D , and P_{11} is obtained from E .*

5.2. Extremal discs for the 11 surfaces. We shall normalize the hyperbolic regular 12-gon P such that the center is the origin and such that the vertices v_n satisfy

$$\arg v_n = \frac{(2n-1)\pi}{12} \quad (n = 1, \dots, 12).$$

We denote by C_n the sides of P (Figure 3), and by w_n the middle point of C_n , where subscripts are regarded as modulo 12. Put $l = d(0, v_1)$, $s = d(v_1, v_2)$ and

$\beta = \frac{\pi}{12}$. Then it is easy to show that

$$s = 2 \sinh^{-1} \left(\frac{2}{\sqrt{3}} \sin \beta \sinh R \right) \approx 0.951$$

and

$$l = \sinh^{-1} \left(\frac{2}{\sqrt{3}} \sinh R \right) \approx 1.401,$$

where R denotes the maximal radius satisfying (1.1) in Section 1.

FIGURE 3. A regular 12-gon

A direct computation yields:

Lemma 5.2. *Let $\alpha_{n,m}$ be an orientation preserving side-pairing mapping from C_n onto C_m . Then $\alpha_{n,m}$ is of the form*

$$\alpha_{n,m}(z) = \frac{i \cosh R e^{i(m-n)\beta} z - i \sinh R e^{i(m+n)\beta}}{i \sinh R e^{-i(m+n)\beta} z - i \cosh R e^{-i(m-n)\beta}},$$

where $\beta = \pi/12$. Let $\gamma_{n,m}$ be an orientation reversing side-pairing mapping from C_n onto C_m . Then $\gamma_{n,m}(z) = \alpha_{-n,m}(\bar{z})$.

Now, using the trigonometry of hyperbolic geometry, we have the following lemma.

Lemma 5.3. *Let S be a non-orientable extremal surface of genus 3 and $\pi : \mathbb{D} \rightarrow S$ the natural projection. Let p be the center of an extremal disc of S . Then the list of possible hyperbolic distances between two points in the set $\pi^{-1}(p)$, in increasing order, starts with $d_1 = 2R \approx 2.553$, $d_2 \approx 3.755$, $d_3 \approx 4.425$.*

Proof. Consider a tessellation of the hyperbolic plane by 12-gons. The discs inscribed in the polygons have radius R , and we must just compute the possible distances between the centers of them. It is clear that $d_1 = 2R$, while d_2 is determined by the circles A and C , d_3 by A and D (Figure 4), etc. It then follows that $d_2 = 2 \sinh^{-1}(\sinh 2R \sin 2\beta) \approx 3.755$ and $d_3 = 2 \sinh^{-1}(\sinh 2R \sin 3\beta) \approx 4.425$. $\square$

FIGURE 4. Discs in the hyperbolic plane

Let $K_n \subset P$ ($n = 1, \dots, 12$) be the pentagon with vertices at w_{n-1} , v_n , v_{n+1} , w_{n+1} , and the origin (Figure 5).

FIGURE 5. K_6

Lemma 5.4. *Let n be fixed. If $z \in K_n$ is projected to the center of an extremal disc, then $d(z, t_n(z)) = d_1$, where t_n denotes an orientation preserving or reversing side-pairing mapping from C_n onto some C_m .*

Proof. Our proof is similar to that of Lemma 3 in [Na3]. Since Lemma 4 in [Gi-Go3] holds for an orientation reversing isometry, $d(z, t_n(z))$ is less than or equal to the maximum of the lengths $d(0, t_n(0))$, $d(w_{n-1}, t_n(w_{n-1}))$, $d(v_n, t_n(v_n))$, $d(v_{n+1}, t_n(v_{n+1}))$, and $d(w_{n+1}, t_n(w_{n+1}))$.

Since $d(0, t_n(0)) = d_1$, $d(v_n, t_n(v_n)) \leq d(v_n, v_{n+6}) = 2l \approx 2.803$, and

$$\begin{aligned} d(w_{n-1}, t_n(w_{n-1})) &\leq d(w_{n-1}, 0) + d(0, t_n(v_n)) + d(t_n(v_n), t_n(w_{n-1})) \\ &\leq R + l + s/2 \approx 3.154, \end{aligned}$$

it follows that $d(z, t_n(z)) = d_1$ from Lemma 5.3. $\square$

Lemma 5.5. *Suppose that P has a side-pairing $t_{n,m} : C_n \rightarrow C_m$. If $z \in K_n$ is projected to the center of an extremal disc, then z lies on one of the following curves:*

(1) *If $t_{n,m} = \alpha_{n,m}$, then*

$$\begin{aligned} L_{n,m} &: \left| z - \frac{\tanh R e^{i(n+m)\beta}}{2 \cos(n-m)\beta} \right| \\ &= \frac{\tanh R}{2 |\cos(n-m)\beta|} \quad (n-m \not\equiv 6 \pmod{12}), \\ M_{n,m} &: z = \frac{e^{2in\beta}}{\tanh R} - t e^{i(n+m+6)\beta} \quad (t \in \mathbb{R}). \end{aligned}$$

(2) *If $t_{n,m} = \gamma_{n,m}$, then*

$$\begin{aligned} L'_{n,m} &: \left| z - \frac{e^{i(n+m)\beta}}{2 \tanh R \cos(n-m)\beta} \right| \\ &= \frac{1}{2 \tanh R |\cos(n-m)\beta|} \quad (n-m \not\equiv 6 \pmod{12}), \\ M'_{n,m} &: z = \tanh R e^{2in\beta} - t e^{i(n+m+6)\beta} \quad (t \in \mathbb{R}). \end{aligned}$$

Proof. From Lemma 5.4 it follows that $d(z, t_{n,m}(z)) = d_1$.

(1) When $t_{n,m} = \alpha_{n,m}$, we obtain $L_{n,m}$ and $M_{n,m}$ from the above equation (see [Na1], Theorem 3.4).

(2) Since $\gamma_{n,m}(z) = \alpha_{-n,m}(\bar{z})$, we have $d(z, \alpha_{-n,m}(\bar{z})) = d_1$, so that

$$\frac{|z - \alpha_{-n,m}(\bar{z})|^2}{(1 - |z|^2)(1 - |\alpha_{-n,m}(\bar{z})|^2)} = \sinh^2 \frac{d_1}{2} = \sinh^2 R.$$

By a direct calculation, we see that this equation is equal to

$$\begin{aligned} &\{x \cos(m+n)\beta + y \sin(m+n)\beta - (x^2 + y^2) \tanh R \cos(m-n)\beta\} \\ &\times \{x \cos(m+n)\beta + y \sin(m+n)\beta - \tanh R \cos(m-n)\beta\} = 0. \end{aligned}$$

From the first and the second factor of the left-hand side we have the circle $L'_{n,m}$ and the Euclidean line $M'_{n,m}$, respectively. $\square$

Remark 5.6. Let n be fixed. $L_{n,m}$ intersects K_n if $m = n \pm 5, n \pm 4, n \pm 3$. $M_{n,m}$ intersects K_n if $m = n + 6, n \pm 5$. $L'_{n,m}$ intersects K_n if $m = n \pm 5, n \pm 4, n \pm 3$. $M'_{n,m}$ intersects K_n if $m = n + 6, n \pm 5, n \pm 4$.

Example 5.7. Put $n = 6$. Figures 6 and 7 show $M_{6,0}$, $L_{6,1}$, $M_{6,1}$, $L_{6,2}$, $L_{6,3}$, $M'_{6,0}$, $L'_{6,1}$, $M'_{6,1}$, $L'_{6,2}$, $M'_{6,2}$, and $L'_{6,3}$ in K_6 .

FIGURE 6. $M_{6,0}; L_{6,1}$ and $M_{6,1}; L_{6,2}; L_{6,3}$ FIGURE 7. $M'_{6,0}; L'_{6,1}$ and $M'_{6,1}; L'_{6,2}; L'_{6,3}$

For every n , we shall draw curves $L_{n,m}$, $M_{n,m}$ (or $L'_{n,m}$, $M'_{n,m}$) on K_n according to the side-pairing pattern P_j ($j = 1, \dots, 10$). Then the intersections of these curves are candidates to be projected to the center of extremal discs.

P_5 has a candidate $\zeta = -\frac{1}{2}(-1 + \sqrt{3})^{\frac{3}{2}}$ appart from the origin (Figure 8). Then for the side-pairing $\gamma_{3,9}$, we have $d(\gamma_{3,9}(\zeta), \zeta) \approx 2.999$, which is between $d_1(\approx 2.553)$ and $d_2(\approx 3.755)$. Hence $\pi(\zeta)$ is not the center of an extremal disc.

FIGURE 8. P_5

In the case of $P_1, P_2, P_3, P_4, P_{10}$ and P_{11} we also have only one candidate. For P_6, P_7, P_8 and P_9 there are, at first sight, three possibilities (see Figure 9), but two of them are discarded in the same way as the candidate of P_5 :

P_6 : Put

$$\zeta_{\pm} = \frac{-3 - \sqrt{3} + \sqrt{12 - 2\sqrt{3}} \pm i(-7 + 3\sqrt{3} + \sqrt{12 - 2\sqrt{3}})}{4\sqrt{-1 + \sqrt{3}}(1 + \sqrt{3})},$$

that equals approximately $-0.1936 \pm 0.4461i$. Then we have $d(\gamma_{0,6}(\zeta_{\pm}), \zeta_{\pm}) \approx 3.505$.

P_7 : Put

$$\zeta_{\pm} = \frac{-(-1 + \sqrt{3})^{\frac{3}{2}} \pm i(\sqrt{22 - 10\sqrt{3}} - 2\sqrt{-1 + \sqrt{3}})}{2(-1 + \sqrt{3})},$$

approximately $-0.4277 \pm 0.3087i$. Then we have $d(\gamma_{0,6}(\zeta_{\pm}), \zeta_{\pm}) \approx 3.170$.

P_8 : Put

$$\zeta_{\pm} = \frac{\sqrt{-1 + \sqrt{3}}}{74}(1 + 5\sqrt{3} \pm i(31 + 7\sqrt{3})) (\approx 0.1116 \pm 0.4986i).$$

Then we have $d(\gamma_{0,6}(\zeta_{\pm}), \zeta_{\pm}) \approx 3.691$.

P_9 : Put

$$\zeta_1 = \frac{\sqrt{-1 + \sqrt{3}}}{74}(23 + 4\sqrt{3} - i(10 + 13\sqrt{3})) (\approx 0.3460 - 0.3759i)$$

and

$$\zeta_2 = -\frac{\sqrt{-1 + \sqrt{3}}}{74}(1 + 5\sqrt{3} + i(31 + 7\sqrt{3})) (\approx -0.1116 - 0.4986i).$$

Then we have $d(\alpha_{0,7}(\zeta_j), \zeta_j) \approx 2.854$ ($j = 1, 2$).

We shall show that the remaining candidate of each polygon (except P_5) is indeed the center of an extremal disc.

FIGURE 9. Points projected to the center of an extremal disc

For each side-pairing pattern P_j and the candidate $\zeta \in P_j$ ($\zeta \neq 0$), we shall find an isometry $f \in \text{Aut}^{\pm}(\mathbb{D})$ satisfying $f(\zeta) = 0$ and $ftf^{-1} \in K$ for every side-pairing t of P_j , where K denotes the group generated by the side-pairings of P_j . This f then induces an automorphism of the surface derived from the side-pairing of P_j , and this automorphism sends the obvious extremal disc to a second one centered at $\pi(\zeta)$.

P_1 : Put $\zeta = \frac{1}{4}\sqrt{-1+\sqrt{3}}(1-\sqrt{3}+i(1+\sqrt{3}))$ ($\approx -0.1565 + 0.5843i$), and $f(z) = (\zeta - z)/(1 - \bar{\zeta}z)$. Then it follows that

$$\begin{aligned} f\alpha_{0,3}f^{-1} &= \alpha_{4,7}, & f\gamma_{1,2}f^{-1} &= \gamma_{5,6}, \\ f\alpha_{8,11}f^{-1} &= \alpha_{7,4}\alpha_{3,0}, & f\gamma_{9,10}f^{-1} &= \alpha_{7,4}\gamma_{9,10}\alpha_{4,7}. \end{aligned}$$

Therefore ζ is projected to the center of an extremal disc.

P_2 : Put

$$\zeta = \frac{\sqrt{-1+\sqrt{3}}}{10-4\sqrt{3}}(\sqrt{3}+i(-2+\sqrt{3}))(\approx 0.4824 - 0.07463i),$$

and $f(z) = e^{2\theta i}(\bar{\zeta} - \bar{z})/(1 - \zeta\bar{z})$, where $\theta = \arg \zeta$. Then it follows that

$$\begin{aligned} f\gamma_{0,5}f^{-1} &= \gamma_{5,0}, & f\gamma_{1,3}f^{-1} &= \gamma_{3,1}, \\ f\alpha_{2,6}f^{-1} &= \gamma_{5,0}\gamma_{1,3}, & f\gamma_{4,7}f^{-1} &= \alpha_{8,11}\gamma_{0,5}, \\ f\alpha_{8,11}f^{-1} &= \alpha_{11,8}, & f\gamma_{9,10}f^{-1} &= \gamma_{10,9}. \end{aligned}$$

P_3 : Put $\zeta = \frac{1}{4}\sqrt{-1+\sqrt{3}}(1-\sqrt{3}+i(1+\sqrt{3}))$, and $f(z) = (\zeta - z)/(1 - \bar{\zeta}z)$. Then it follows that

$$\begin{aligned} f\alpha_{0,3}f^{-1} &= \alpha_{4,7}, & f\gamma_{1,6}f^{-1} &= \gamma_{5,2}\alpha_{4,7}, \\ f\gamma_{2,5}f^{-1} &= \gamma_{5,2}, & f\alpha_{8,11}f^{-1} &= \alpha_{7,4}\alpha_{3,0}, \\ f\gamma_{9,10}f^{-1} &= \alpha_{7,4}\gamma_{9,10}\alpha_{4,7}. \end{aligned}$$

P_4 : Put $\zeta = \frac{1}{4}\sqrt{-1+\sqrt{3}}(1-\sqrt{3}+i(1+\sqrt{3}))$, and $f(z) = (\zeta - z)/(1 - \bar{\zeta}z)$. Then it follows that

$$\begin{aligned} f\alpha_{0,3}f^{-1} &= \alpha_{4,7}, & f\alpha_{1,5}f^{-1} &= \alpha_{5,1}, \\ f\alpha_{2,6}f^{-1} &= \alpha_{6,2}, & f\alpha_{8,11}f^{-1} &= \alpha_{7,4}\alpha_{3,0}, \\ f\gamma_{9,10}f^{-1} &= \alpha_{7,4}\gamma_{9,10}\alpha_{4,7}. \end{aligned}$$

P_6 : Put $\zeta = -\frac{1}{2}(-1+\sqrt{3})^{\frac{3}{2}}(\approx -0.3131)$, and $f(z) = (\zeta - z)/(1 - \bar{\zeta}z)$. Then it follows that

$$\begin{aligned} f\gamma_{0,6}f^{-1} &= \gamma_{6,0}, & f\gamma_{1,11}f^{-1} &= \gamma_{0,6}\alpha_{7,10}, \\ f\alpha_{2,5}f^{-1} &= \alpha_{7,10}, & f\alpha_{3,8}f^{-1} &= \alpha_{8,3}, \\ f\alpha_{4,9}f^{-1} &= \alpha_{9,4}. \end{aligned}$$

P_7 : Put $\zeta = -\frac{1}{2}\sqrt{-1+\sqrt{3}}(\approx -0.4277)$, and $f(z) = (\zeta - z)/(1 - \bar{\zeta}z)$. Then it follows that

$$\begin{aligned} f\gamma_{0,6}f^{-1} &= \gamma_{6,0}, & f\gamma_{1,3}f^{-1} &= \gamma_{4,8}\alpha_{7,2}, \\ f\alpha_{2,7}f^{-1} &= \alpha_{7,2}, & f\gamma_{4,8}f^{-1} &= \gamma_{8,4}, \\ f\alpha_{5,10}f^{-1} &= \alpha_{10,5}, & f\gamma_{9,11}f^{-1} &= \gamma_{0,6}\alpha_{5,10}. \end{aligned}$$

P_8 : Put $\zeta = \frac{1}{2}\sqrt{-1 + \sqrt{3}}$, and $f(z) = (\zeta - z)/(1 - \bar{\zeta}z)$. Then it follows that

$$\begin{aligned} f\gamma_{0,6}f^{-1} &= \gamma_{6,0}, & f\gamma_{1,4}f^{-1} &= \gamma_{8,11}, \\ f\gamma_{2,10}f^{-1} &= \gamma_{10,2}, & f\alpha_{3,7}f^{-1} &= \gamma_{6,0}\gamma_{11,8}, \\ f\alpha_{5,9}f^{-1} &= \gamma_{4,1}\gamma_{0,6}. \end{aligned}$$

P_9 : Put $\zeta = \frac{1}{4}\sqrt{-1 + \sqrt{3}}(1 - \sqrt{3} + i(1 + \sqrt{3}))$, and $f(z) = (\zeta - z)/(1 - \bar{\zeta}z)$. Then it follows that

$$\begin{aligned} f\alpha_{0,7}f^{-1} &= \gamma_{11,3}\gamma_{4,8}, & f\alpha_{1,9}f^{-1} &= \gamma_{11,3}\gamma_{5,2}, \\ f\gamma_{2,5}f^{-1} &= \gamma_{5,2}, & f\gamma_{3,11}f^{-1} &= \gamma_{8,4}, \\ f\alpha_{6,10}f^{-1} &= \gamma_{8,4}\gamma_{2,5}. \end{aligned}$$

P_{10} : Put $\zeta = \frac{1}{4}\sqrt{-1 + \sqrt{3}}(1 - \sqrt{3} + i(1 + \sqrt{3}))$, and $f(z) = (\zeta - z)/(1 - \bar{\zeta}z)$. Then it follows that

$$\begin{aligned} f\alpha_{0,7}f^{-1} &= \alpha_{8,3}\alpha_{4,11}, & f\gamma_{1,10}f^{-1} &= \alpha_{11,4}\gamma_{5,2}, \\ f\gamma_{2,5}f^{-1} &= \gamma_{5,2}, & f\alpha_{3,8}f^{-1} &= \alpha_{8,3}, \\ f\alpha_{4,11}f^{-1} &= \alpha_{11,4}, & f\gamma_{6,9}f^{-1} &= \alpha_{8,3}\gamma_{2,5}. \end{aligned}$$

P_{11} : Put $\zeta = -\frac{1}{2}(-1 + \sqrt{3})^{\frac{3}{2}}$, and $f(z) = (\zeta - z)/(1 - \bar{\zeta}z)$. Then it follows that

$$\begin{aligned} f\gamma_{0,6}f^{-1} &= \gamma_{6,0}, & f\gamma_{1,11}f^{-1} &= \gamma_{0,6}\alpha_{7,10}, \\ f\alpha_{2,5}f^{-1} &= \alpha_{7,10}, & f\gamma_{3,4}f^{-1} &= \gamma_{8,9}. \end{aligned}$$

We note in passing that all the mappings we have denoted above by f are involutions, and that all of them are orientation preserving except the one corresponding to P_2 .

Let us denote by S_j the surface derived from the side-pairing pattern P_j . The analysis we have just carried out proves the following result.

Theorem 5.8. *Non-orientable extremal surfaces of genus three admit at most two extremal discs. Table 1 shows the centers of extremal discs for all the surfaces $S_1, S_2, \dots, S_{11}$, where π denotes the natural projection from $\mathbb{D}$ onto each surface.*

Remark 5.9. $S_1, S_2, \dots, S_{11}$ are in fact all different (non-isomorphic) surfaces. If there exists an isomorphism T from S_j onto S_k ($j \neq k$), then we can assume that $T(o) = o'$, where $o \in S_j$ and $o' \in S_k$ are the projections of the origin (the reason for this assumption being the fact that, as we already know, whenever a surface contains two extremal discs there is an automorphism sending one to the other). The lift $\tilde{T}$ of T is considered as a mapping of $\mathbb{D}$ fixing the origin, so that $\tilde{T}(z) = e^{i\theta}z$ or $\tilde{T}(z) = e^{i\theta}\bar{z}$ for some $\theta \in \mathbb{R}$. From the side-pairing patterns of $P_1, P_2, \dots, P_{11}$ we see that there is no $\tilde{T}$ for every pair P_j, P_k (cf. [Gi-Go3], Theorem 1).

Corollary 5.10. *For the 11 surfaces, the full group of automorphisms is shown in Table 2.*

The generators in Table 2 denote the following automorphisms: T is induced by $f(z) = (\zeta - z)/(1 - \bar{\zeta}z)$, where ζ is the point, different from the origin, to be projected to the center of an extremal disc; σ, J , and J_0 are induced by $z \mapsto e^{2\pi i/3}z$, $z \mapsto e^{-10\beta i}\bar{z}$, and $z \mapsto \bar{z}$, respectively.

<i>Surface</i>	<i>Centers</i>
S_1	$\pi(0), \pi\left(\frac{1}{4}\sqrt{-1+\sqrt{3}}(1-\sqrt{3}+i(1+\sqrt{3}))\right)$
S_2	$\pi(0), \pi\left(\frac{\sqrt{-1+\sqrt{3}}}{10-4\sqrt{3}}(\sqrt{3}+i(-2+\sqrt{3}))\right)$
S_3	$\pi(0), \pi\left(\frac{1}{4}\sqrt{-1+\sqrt{3}}(1-\sqrt{3}+i(1+\sqrt{3}))\right)$
S_4	$\pi(0), \pi\left(\frac{1}{4}\sqrt{-1+\sqrt{3}}(1-\sqrt{3}+i(1+\sqrt{3}))\right)$
S_5	$\pi(0)$
S_6	$\pi(0), \pi\left(-\frac{1}{2}(-1+\sqrt{3})^{\frac{3}{2}}\right)$
S_7	$\pi(0), \pi\left(-\frac{1}{2}\sqrt{-1+\sqrt{3}}\right)$
S_8	$\pi(0), \pi\left(\frac{1}{2}\sqrt{-1+\sqrt{3}}\right)$
S_9	$\pi(0), \pi\left(\frac{1}{4}\sqrt{-1+\sqrt{3}}(1-\sqrt{3}+i(1+\sqrt{3}))\right)$
S_{10}	$\pi(0), \pi\left(\frac{1}{4}\sqrt{-1+\sqrt{3}}(1-\sqrt{3}+i(1+\sqrt{3}))\right)$
S_{11}	$\pi(0), \pi\left(-\frac{1}{2}(-1+\sqrt{3})^{\frac{3}{2}}\right)$

TABLE 1. Centers of extremal discs in non-orientable surfaces of genus 3.

<i>Surface</i>	<i>Aut[±]</i>
S_1	$D_3 \times \mathbb{Z}_2 = \langle \sigma, J \rangle \times \langle JT \rangle$
S_2	$\mathbb{Z}_2 = \langle T \rangle$
S_3	$\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle T \rangle \times \langle J \rangle$
S_4	$\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle T \rangle \times \langle J \rangle$
S_5	$\mathbb{Z}_2 = \langle J_0 \rangle$
S_6	$\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle T \rangle \times \langle J_0 \rangle$
S_7	$\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle T \rangle \times \langle J_0 \rangle$
S_8	$\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle T \rangle \times \langle J_0 \rangle$
S_9	$\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle T \rangle \times \langle J \rangle$
S_{10}	$D_3 \times \mathbb{Z}_2 = \langle \sigma, J \rangle \times \langle T \rangle$
S_{11}	$\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle T \rangle \times \langle J_0 \rangle$

TABLE 2. Full group of automorphisms of the non-orientable extremal surfaces of genus 3.

Proof. As the centers of extremal discs must be preserved by the automorphisms, the result will follow from Theorem 5.8. Recall that the automorphisms of S_i lift to automorphisms of $\mathbb{D}$ that can reverse or preserve orientation, but all them must preserve the points in $\mathbb{D}$ corresponding to centers of extremal discs.

The case of S_5 is the most simple one, since every automorphism must fix the center of its unique extremal disc. In particular, the orientation preserving automorphisms must be induced by rotations around the origin and, accordingly, the orientation reversing automorphisms have to be induced by hyperbolic reflections across lines passing through 0. It can be easily seen that $z \mapsto \bar{z}$ is the only one of these mappings that is compatible with the side-pairing in P_5 (see Figure 2).

We already know that the rest of the surfaces have an automorphism T that interchanges the two extremal discs. This is the only automorphism of S_2 , since the side-pairing in P_2 is not compatible with rotations or reflections fixing the origin. The same arguments show that $S_3, S_4, S_6, S_7, S_8, S_9$ and S_{11} have only one more automorphism, coming from a reflection in all cases.

For S_1 and S_{10} we have to work a little bit more, due to the fact that the side-pairings of P_1 and P_{10} are compatible both with some reflections and with some rotations. We will show that $\text{Aut}^\pm(S_1) = D_3 \times \mathbb{Z}_2 = \langle \sigma, J \rangle \times \langle JT \rangle$ and $\text{Aut}^\pm(S_{10}) = D_3 \times \mathbb{Z}_2 = \langle \sigma, J \rangle \times \langle T \rangle$.

We recall that σ and J are induced by the rotation of $\frac{2\pi}{3}$ around the origin and the reflection with respect to the line $\overline{v_4 v_{10}}$, respectively. By Remark 5.9, an element of $\text{Aut}^\pm(S_j)$ ($j = 1, 10$) fixing o must be generated by σ or J . It is easy to show that $\langle \sigma, J \rangle = D_3$, the dihedral group of order 6. Let w be the middle point of the line 0ζ ($\zeta = v_4$) with respect to the hyperbolic metric. Put $w_1 = e^{2\pi i/3}w$ and $w_2 = e^{4\pi i/3}w$. Let p, q, q_1 and q_2 be the projections in S_j of ζ, w, w_1 and w_2 , respectively. Since $f(z) = (\zeta - z)/(1 - \bar{\zeta}z)$ fixes w , it follows that $T(q) = q$. And $T(o) = p$, since $f(0) = \zeta$.

• S_1 . – Note that $T(q_1) = q_2$ because $\alpha_{3,0} \circ f(w_1) = w_2$. We shall show that JT commutes with σ and with J . Since

$$JTJT(o) = JTJ(p) = JT(p) = J(o) = o,$$

it follows that $JTJT \in \langle \sigma, J \rangle = \{1, \sigma, \sigma^2, J, \sigma J, \sigma^2 J\}$. Since $JTJT(q) = q$, we have $JTJT = 1$ or $JTJT = J$. But $JTJT = J$ would imply $J = 1$; a contradiction. Therefore $JTJT = 1$ and hence $(JT)J = J(JT)$. Similarly, we have

$$JT\sigma TJ\sigma^{-1}(o) = JT\sigma TJ(o) = JT\sigma T(o) = JT\sigma(p) = JT(p) = J(o) = o,$$

thus $JT\sigma TJ\sigma^{-1} \in \langle \sigma, J \rangle$. Now, since

$$JT\sigma TJ\sigma^{-1}(q) = JT\sigma TJ(q_2) = JT\sigma T(q_1) = JT\sigma(q_2) = JT(q) = J(q) = q,$$

we have $JT\sigma TJ\sigma^{-1} = 1$ or J . But if $JT\sigma TJ\sigma^{-1} = J$, then $T\sigma T = \sigma J$. The left-hand side in this last identity has order 3, whereas the right-hand side has order 2. Hence $JT\sigma TJ\sigma^{-1} = 1$ or, equivalently, $(JT)\sigma = \sigma(JT)$.

• S_{10} . – As $\alpha_{4,11} \circ f(w_2) = w_2$, we have $T(q_2) = q_2$. Also $T(q_1) = q_1$, since $\alpha_{3,8} \circ f(w_1) = w_1$. To see that T commutes with J , argue as for S_1 .

We shall show that T also commutes with σ :

Since

$$\sigma T\sigma^{-1}T(o) = \sigma T\sigma^{-1}(p) = \sigma T(p) = \sigma(o) = o,$$

it follows that $\sigma T \sigma^{-1} T \in \langle \sigma, J \rangle$. Now

$$\sigma T \sigma^{-1} T(q) = \sigma T \sigma^{-1}(q) = \sigma T(q_2) = \sigma(q_2) = q,$$

and therefore $\sigma T \sigma^{-1} T = 1$ or J . If we would have $\sigma T \sigma^{-1} T = J$, it would follow that $T \sigma^{-1} T = \sigma^{-1} J$. Again, the left-hand side has order 3 while the right-hand side has order 2. $\square$

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DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD AUTÓNOMA DE MADRID, 28049 MADRID, SPAIN

E-mail address: ernesto.gironde@uam.es

Current address: Departamento de Matemáticas, IMAFF, CSIC, Madrid, Spain

SCIENCE DIVISION, CENTER FOR GENERAL EDUCATION, AICHI INSTITUTE OF TECHNOLOGY, YAKUSA-CHO, TOYOTA 470-0392, JAPAN

E-mail address: gou@aitech.ac.jp