MAPPINGS OF FINITE DISTORTION: FORMATION OF CUSPS II

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ABSTRACT. For $s > 0$ given, we consider a planar domain Ω_s with a rectifiable boundary but containing a cusp of degree s , and show that there is no homeomorphism $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ of finite distortion with $\exp(\lambda K) \in L^1_{\text{loc}}(\mathbb{R}^2)$ so that $f(B) = \Omega_s$ when $\lambda > 4/s$ and B is the unit disc. On the other hand, for $\lambda < 2/s$ such an f exists. The critical value for λ remains open.

1. INTRODUCTION

It is well known that only the Jordan curves that satisfy the so-called three point condition can be images of the unit circle under a quasiconformal homeomorphism of the entire plane. There has been considerable recent interest in mappings of locally exponentially integrable distortion [1, 3, 5, 6, 7, 10]. In this setting, the image of the unit circle necessarily has zero area [2], and, in fact, the generalized gauge dimension of the image is bounded away from 2 [4, 12]. For rectifiable images these two conditions are automatically satisfied and in this category no necessary conditions for the image Jordan curve appear to be known. In this paper, we consider the model domains Ω_s that have a cusp of degree $s > 0$ but are otherwise smooth. By a cusp of degree s we mean the curve $|y| = x^{1+s}$, $0 < x < 1/100$. Specifically, these model domains fail the three point condition.

Regarding these domains, we have the following theorem:

Theorem 1. *For $s > 0$ given, there is no homeomorphism $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ of finite distortion with $\exp(\lambda K) \in L^1_{\text{loc}}(\mathbb{R}^2)$ so that $f(B) = \Omega_s$ when $\lambda > 4/s$ and B is the unit disc. On the other hand, for $\lambda < 2/s$ such an f exists.*

If f is additionally quasiconformal in the unit disc B , the critical λ is $1/s$ (see [8]). Based on the constructions we have attempted to make, we expect that the critical λ in Theorem 1 should be $2/s$, but we have not been able to prove this.

2. DEFINITIONS

In this paper we consider the mappings of finite distortion on the plane. By plane we mean either $\mathbb{R}^2$ or the complex plane $\mathbb{C}$, depending on which is more convenient for the situation. In general, we will perform the computations in $\mathbb{R}^2$; the complex plane $\mathbb{C}$ and, moreover, the extended complex plane (i.e. the Riemann sphere) $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is mainly used to simplify the definition of certain mappings.

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Let $\Omega \subset \mathbb{R}^2$ be a domain, i.e. a connected and open subset of $\mathbb{R}^2$. We say that a homeomorphism $f: \Omega \rightarrow f(\Omega) \subset \mathbb{R}^2$ has *finite distortion* if the following conditions are satisfied:

- (1) $f \in W_{\text{loc}}^{1,1}(\Omega; \mathbb{R}^2)$,
- (2) $\|Df(x)\|^2 \leq K(x)J_f(x)$ a.e. $x \in \Omega$,

for some measurable function $K(x) \geq 1$ which is finite almost everywhere. The function $K(x)$ is referred to as a distortion (function) of f and the phrase *exponentially integrable distortion* means that $\exp(\lambda K) \in L_{\text{loc}}^1(\Omega)$ for some $\lambda > 0$.

In the text above, $Df(x)$ denotes the differential matrix of f at the point x (which for homeomorphic $f \in W_{\text{loc}}^{1,1}(\Omega; \mathbb{R}^2)$ exists a.e.) and $J_f(x) := \det Df(x)$ is the Jacobian. The norm of $Df(x)$ is defined as

$$\|Df(x)\| := \max\{|Df(x)e| : e \in \mathbb{R}^2, |e| = 1\}.$$

If $E \subset \mathbb{R}^2$ is a measurable set of positive and finite area, $0 < |E| < \infty$, we denote the mean integral of a measurable function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ over E by

$$f_E := \int_E f(x) dx = \frac{1}{|E|} \int_E f(x) dx.$$

We denote by $B(x, r)$ the open disc of radius $r > 0$ centered at $x \in \mathbb{R}^2$ and write $B := B(0, 1)$ for the unit disc. The boundary of a set A is denoted by ∂A and the closure by $\overline{A}$.

Let E and F be disjoint compact sets in a domain Ω . Let ω be measurable with $0 \leq \omega(x) \leq 1$ almost everywhere. The ω -weighted (2-)capacity of the pair (F, E) w.r.t. Ω is defined to be

$$\text{cap}_\omega(F, E; \Omega) = \inf \left\{ \int_\Omega |\nabla u(x)|^2 \omega(x) dx : u \in C(\Omega) \cap W_{\text{loc}}^{1,1}(\Omega), \right. \\ \left. u \leq 0 \text{ on } F \text{ and } u \geq 1 \text{ on } E \right\}.$$

Finally, we define the model domains $\Omega_s \subset \mathbb{R}^2$ for $s > 0$ by setting

$$(1) \quad \Omega_s = \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < 1, |x_2| < x_1^{1+s}\} \cup B(x_s, r_s),$$

where $x_s = (s+2, 0)$ and $r_s = \sqrt{(s+1)^2 + 1}$. Then Ω_s is domain with a cusp, where the parameter s determines the degree of the cusp, and omitting the origin (tip of the cusp) the boundary of Ω_s is smooth. Notice that Ω_s is clearly not a quasidisc, because the three point condition fails (only) at the tip of the cusp. Thus if we have a conformal or quasiconformal mapping $f: B \rightarrow \Omega_s$, it does not extend to a quasiconformal mapping of the entire plane. In [8] it was shown that if the extension is allowed to be a homeomorphism of finite distortion, then such extension may exist, but in such a case $\lambda \leq 1/s$ if $\exp(\lambda K) \in L_{\text{loc}}^1(\mathbb{R}^2)$. In this paper we will consider the situation when $f: B \rightarrow \Omega_s$ is only a homeomorphism of finite distortion to begin with.

3. RESULTS

Theorem 1 follows from Theorem 3 and from the construction presented in Section 4. As a preliminary result, we need the following result from [9], Theorem 1. This result concerns the modulus of continuity of the mappings in question and will be used in the proof of Theorem 3.

Theorem 2. *Let $f: \Omega \rightarrow \mathbb{R}^2$ be a mapping of finite distortion whose distortion function $K(x)$ satisfies for some $\lambda > 0$,*

$$I := \int_{B'} \exp(\lambda K(x)) dx < \infty,$$

where $B' = B(x_0, R) \Subset \Omega$. Then for all $x, y \in B(x_0, (\frac{R}{40})^e [\frac{1}{\pi} I]^{\frac{1-e}{2}})$ we have the estimate

$$(2) \quad |f(x) - f(y)| \leq \frac{C_{I,R,\lambda}}{\log^{\frac{\lambda}{2}} \left(\frac{I}{\pi |x-y|^2} \right)} \left(\int_{B'} J(z, f) dz \right)^{\frac{1}{2}}.$$

We start by proving some lemmas concerning capacity estimates needed in the proof of Theorem 3.

Lemma 1. *Let Ω_s be a cusp domain as defined in (1) and $F \subset \Omega_s$ a compact set. Set $c := \min\{1, \text{dist}(0, F)\}$ and let $0 < r < c/2$. If $E \subset \{(x_1, 0) \in \Omega_s : 0 < x_1 \leq r\}$, then there is a Lipschitz function u on Ω_s such that $u = 1$ on E , $u = 0$ on F and*

$$(3) \quad \int_{\Omega_s} |\nabla u|^2 dx \leq \frac{2s}{r^{-s} - 2^s c^{-s}}.$$

Proof. Define

$$u(x_1, x_2) = \begin{cases} 1 & \text{if } x_1 \leq r, \\ 1 - \int_r^{x_1} \frac{dt}{2t^{1+s}} \left(\int_r^{c/2} \frac{dt}{2t^{1+s}} \right)^{-1} & \text{if } r < x_1 \leq c/2, \\ 0 & \text{if } x_1 > c/2, \end{cases}$$

and compute

$$\begin{aligned} \int_{\Omega_s} |\nabla u(x)|^2 dx &= \left(\int_r^{c/2} \frac{dt}{2t^{1+s}} \right)^{-2} \int_r^{c/2} \int_{-x_1^{1+s}}^{x_1^{1+s}} \frac{1}{4x_1^{2+2s}} dx_2 dx_1 \\ &= \left(\int_r^{c/2} \frac{dt}{2t^{1+s}} \right)^{-1} = \frac{2s}{r^{-s} - 2^s c^{-s}}. \quad \square \end{aligned}$$

Lemma 2. *Let $r \in (0, 3/4)$, $x_0 \in B \subset \mathbb{R}^2$, $D_r := B(x_0, r) \cap B$ and E be a continuum joining x_0 to $A_r := \partial B(x_0, r) \cap B$ in D_r . Suppose that $u \in W^{1,1}(D_r)$ is continuous and satisfies: $u \geq 1$ on E and $u_{D_r} \leq 1/8$. Then for some constant $C(p) > 0$,*

$$(4) \quad \int_{D_r} |\nabla u|^p \geq C(p) r^{-p},$$

for all $1 < p < 2$.

Proof. The proof of this lemma goes through as the proof of Lemma 2.4 in [4] with slight modifications. We represent the proof here for completeness.

The assumptions on u guarantee that the set T of $t \in (0, r)$ with $\max_{x \in A_t} u - \min_{x \in A_t} u \geq 1/4$ has the measure $|T| \geq (1 - \sqrt{2/3})r$. Indeed, as $|A_t| \geq (\pi/2)t$ for

$0 < t < 3/4$, we obtain

$$\begin{aligned} \frac{1}{8} &\geq u_{D_r} = \frac{1}{|D_r|} \int_0^r \int_{A_t} u(\omega) d\omega dt \geq \frac{3}{4} \frac{1}{|D_r|} \int_{[0,r] \setminus T} |A_t| dt \\ &\geq \frac{3\pi}{8} \frac{1}{|D_r|} \int_{[0,r] \setminus T} t dt, \end{aligned}$$

so

$$|[0, r] \setminus T|^2 \leq 2 \int_{[0,r] \setminus T} t dt \leq \frac{2}{3} r^2.$$

Fix $1 < p < 2$ and assume that $u \in W^{1,p}(D_r)$. Then by Fubini's theorem, $u \in W^{1,p}(A_t)$ for almost every $t \in (0, r)$. It follows from the fundamental theorem of calculus and Hölders inequality that for almost every $t \in (0, r)$ and all $x, y \in A_t$,

$$|u(x) - u(y)| \leq C_1 t \left(\int_{A_t} |\nabla u(\omega)|^p d\omega \right)^{1/p}$$

and thus we have

$$(5) \quad \int_{A_t} |\nabla u(\omega)|^p d\omega \geq \frac{|A_t|}{(4C_1)^p} t^{-p} \quad \text{for a.e. } t \in T.$$

As $|D_r| \leq \pi r^2$ and $|A_t| \geq (\pi/2)t$ for $0 < t < 3/4$, the desired conclusion follows readily from (5):

$$\begin{aligned} \int_{D_r} |\nabla u|^p &\geq \frac{1}{|D_r|(4C_1)^p} \int_T |A_t| t^{-p} dt \geq \frac{1}{(4C_1)^p 2r^2} \int_T t^{1-p} dt \\ &\geq \frac{1}{(4C_1)^p 2r^2} \int_{r-|T|}^r t^{1-p} dt \geq \frac{1 - (2/3)^{1-p/2}}{2(2-p)(4C_1)^p} r^{-p} = C(p)r^{-p}. \quad \square \end{aligned}$$

We continue with a capacity-type estimate relying on ideas from [4]. It will be crucial for us that the integration is over B .

Lemma 3. *Let $E \subset B$ be a continuum such that $E \subset B(x_0, 1/6)$ for some $x_0 \in \partial B$ and $F := \overline{B}(0, 1/4)$. Suppose that $v \in W^{1,1}(B)$ is continuous and satisfies: $v \geq 1$ on E and $v = 0$ on F . If*

$$L := \int_B \exp(\lambda K) < \infty,$$

for some measurable function $K(x) \geq 1$, then

$$(6) \quad \int_B \frac{|\nabla v|^2}{K} \geq C \lambda \left(\log \frac{2\Lambda}{\text{diam } E} \right)^{-2},$$

where

$$\Lambda := \left(\int_B \exp(\lambda K(x)) \right)^{1/2}.$$

Proof. Set $d := \text{diam } E < 1/3$ and pick $a, b \in E$ such that $\text{dist}(a, b) = d$. For $t > 0$ denote $A_t := \partial B(a, t) \cap B$ and $D_t := B(a, t) \cap B$. Notice that

$$(7) \quad e^2 \leq \frac{L}{\pi d^2} \leq \frac{L}{|D_d|} \leq \frac{4L}{\pi d^2}.$$

Next we consider two cases, that either $v_{D_d} \leq 1/8$ or $v_{D_d} > 1/8$.

The Case $v_{D_d} \leq 1/8$. From Lemma 2 we obtain

$$(8) \quad \int_{D_d} |\nabla v|^p \geq C/d^p,$$

for any $1 < p < 2$. Take $p = 4/3$ and apply Hölder's inequality to obtain

$$(9) \quad \frac{C}{d^{4/3}} \leq \left(\int_{D_d} \frac{|\nabla v|^2}{K} \right)^{2/3} \left(\int_{D_d} K^2 \right)^{1/3}.$$

Next we need to have an upper bound for $\int_{D_d} K^2$. We start by considering the auxiliary function

$$\varphi(t) = \exp(\lambda t^{1/2}).$$

One readily observes that φ and $\varphi(t)/t$ are increasing for all $t \geq (2/\lambda)^2$. As $L/|D_d| \geq e^2$, we get that

$$\tau := \left(\frac{1}{\lambda} \log \frac{L}{|D_d|} \right)^2 \geq \left(\frac{2}{\lambda} \right)^2.$$

Thus $\varphi(t)$ and $\varphi(t)/t$ are increasing for all $t \geq \tau$ and

$$\begin{aligned} \int_{D_d} K^2 &\leq \int_{\{K^2 \geq \tau\}} K^2 + \tau |D_d| \leq \frac{\tau}{\varphi(\tau)} \int_{\{K^2 \geq \tau\}} \varphi(K^2) + \tau |D_d| \\ &\leq \frac{\tau}{\varphi(\tau)} \int_B \exp(\lambda K) + \tau |D_d| = 2\tau |D_d|. \end{aligned}$$

The last equality is a consequence of the fact that $\varphi(\tau) = L/|D_d|$. This estimate can be written as

$$\left(\int_{D_d} K^2 \right)^{1/3} \leq C \left(\frac{1}{\lambda} \log \frac{L}{|D_d|} \right)^{2/3}.$$

Applying this inequality in (9) yields

$$\int_{D_d} \frac{|\nabla v|^2}{K} \geq \tilde{C} \lambda \left(\log \frac{L}{|D_d|} \right)^{-1},$$

which in conjunction with (7) implies (6).

The Case $v_{D_d} > 1/8$. Pick a positive integer m such that $1/2^{m+1} < d \leq 1/2^m$ and set $\bar{a} := -a/|a|$. For $i \geq 1$, set $B_i := B(b_i, r_i)$ and $B'_i := B(b'_i, r_i/2)$ where $r_i = 1/2^{m-i+1}$, $b_i = a + 2r_i \bar{a}$ and $b'_i = b_i + (r_i/2) \bar{a}$. In addition, set $B_0 := D_{3/2^{m+1}}$ and $B'_0 := B(b'_0, r_1/4)$ where $b'_0 = a + (5r_1/4) \bar{a}$. Now for $i \geq 1$ we have B'_{i-1} , $B'_i \subset B_i$ with each of $\partial B'_{i-1}$, $\partial B'_i$ being tangent to ∂B_i and $2 \text{diam } B'_{i-1} = \text{diam } B'_i = (1/2) \text{diam } B_i$. Also, D_d , $B'_0 \subset B_0$. From the initial assumption $E \subset B(x_0, 1/6)$ it follows that $5/6 < |a| < 1$ which in turn implies that for $i = m$ we have $F \subset B_m = B(a + \bar{a}, 1/2) \subset B$. Figure 1 presents an example of the case when $m = 2$.

FIGURE 1. Ball chain example for $m = 2$.

Now, as $v_F = 0$, we can write

$$1/8 \leq v_{D_d} = (v_{D_d} - v_{B'_0}) + (v_{B'_0} - v_{B'_1}) + \cdots + (v_{B'_{m-1}} - v_F).$$

By applying a Poincaré inequality, we can estimate the absolute value of the terms with $i = 1, \dots, m-2$ as follows:

$$\begin{aligned} |v_{B'_{i-1}} - v_{B'_i}| &\leq |v_{B'_{i-1}} - v_{B_i}| + |v_{B_i} - v_{B'_i}| \\ &\leq \frac{1}{|B'_{i-1}|} \int_{B_i} |v - v_{B_i}| + \frac{1}{|B'_i|} \int_{B_i} |v - v_{B_i}| \\ &\leq \left(\frac{|B_i|}{|B'_{i-1}|} + \frac{|B_i|}{|B'_i|} \right) C_0 \operatorname{diam} B_i \int_{B_i} |\nabla v| \\ &\leq 20C_0 \operatorname{diam} B_i \int_{B_i} |\nabla v|, \end{aligned}$$

as $|B_i|/|B'_{i-1}| = 16$ and $|B_i|/|B'_i| = 4$. The terms $|v_{D_d} - v_{B'_0}|$ and $|v_{B'_{m-1}} - v_F|$ are dealt in a similar manner, only the constant changes. Notice that B_0 is not a ball but still L -bi-Lipschitz equivalent to a ball, where L is independent of E . Thus we obtain

$$C \leq \sum_{i=0}^m \operatorname{diam} B_i \int_{B_i} |\nabla v|.$$

Next we use Hölder's inequality twice, first to the integrals and then to the sum, resulting in

$$(10) \quad C \leq \left(\sum_{i=0}^m (\operatorname{diam} B_i)^2 \int_{B_i} \frac{|\nabla v|^2}{K} \right)^{1/2} \left(\sum_{i=0}^m \int_{B_i} K \right)^{1/2}.$$

Because $B_0, \dots, B_m \subset B$ and the sets B_i have bounded overlap, we have the estimate

$$\left(\sum_{i=0}^m (\operatorname{diam} B_i)^2 \int_{B_i} \frac{|\nabla v|^2}{K} \right)^{1/2} \leq \tilde{C} \left(\int_B \frac{|\nabla v|^2}{K} \right)^{1/2},$$

so applying this to (10) and raising to the power 2 yields

$$(11) \quad C \leq \left(\int_B \frac{|\nabla v|^2}{K} \right) \left(\sum_{i=0}^m \int_{B_i} K \right).$$

If we are able show that

$$(12) \quad \sum_{i=0}^m \int_{B_i} K \leq \frac{C}{\lambda} m^2,$$

then, by using the fact that $m \simeq \log(1/d)$, the claim (6) is an immediate consequence of (11) and (12).

Notice that $B_i \subset B$ for $0 \leq i \leq m$ and define the auxiliary function $\varphi(t) = \exp(\lambda t)$. Jensen's inequality gives us

$$\varphi \left(\int_{B_i} K \right) \leq \int_{B_i} \exp(\lambda K) \leq \frac{1}{|B_i|} \int_B \exp(\lambda K) = \frac{L}{|B_i|} = C_1 r_i^{-2}.$$

As $\varphi^{-1}(s) = (1/\lambda) \log(s)$ and $r_i = 1/2^{m-i+1}$, we have that

$$(13) \quad \int_{B_i} K \leq \frac{1}{\lambda} \log \frac{C_1}{r_i^2} \leq \frac{C_2}{\lambda} (m - i + 1).$$

Now, as $m - i + 1 \geq 0$ for all $i \in \{0, 1, \dots, m\}$, we have the estimate

$$(14) \quad \sum_{i=0}^m (m - i + 1) = \frac{1}{2} (m + 1)(m + 2) \leq C_3 m^2.$$

Combining inequalities (13) and (14) gives (12), thus completing our proof. $\square$

Now we are ready to proceed with the proof of the following theorem that will imply the first part of Theorem 1.

Theorem 3. *Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a homeomorphic mapping of finite distortion with $\exp(\lambda K) \in L^1(2B)$ for some $\lambda > 0$. If $f(B) = \Omega_s$, then $\lambda \leq 4/s$.*

Proof. Let $F := \overline{B}(0, 1/4)$ and set $F' := f(F)$. As f is a homeomorphism, the set $F' \subset \Omega_s$ will be connected, compact and $c := \min\{1, \text{dist}(0, F')\} > 0$. Thus we may pick $0 < r < c/2$ and define continua $E' := \{(x_1, 0) \in \Omega_s : r/2 \leq x_1 \leq r\}$ and $E := f^{-1}(E')$ such that $E \subset B(f^{-1}(0), 1/6)$. By applying Lemma 1 we obtain a Lipschitz function u on Ω_s such that $u = 1$ on E' , $u = 0$ on F' , and

$$(15) \quad \int_{\Omega_s} |\nabla u|^2 dx \leq \frac{2s}{r^{-s} - 2^s c^{-s}}.$$

Set $v = u \circ f$ and recall that if $f \in W_{\text{loc}}^{1,1}(B, \mathbb{R}^2)$ is a homeomorphism, then for each non-negative measurable function $w: \mathbb{R}^2 \rightarrow \mathbb{R}$,

$$(16) \quad \int_B w \circ f |J_f| \leq \int_{f(B)} w$$

(see [6]). Now, as f is a homeomorphism and $f \in W_{\text{loc}}^{1,1}(\mathbb{R}^2, \mathbb{R}^2)$ we know that ∇v exists almost everywhere, $|\nabla v| = |Df| |\nabla u \circ f|$ is locally integrable, and thus $v \in C(B) \cap W_{\text{loc}}^{1,1}(B)$. Also, $v = 1$ on E and $v = 0$ on F , so v is clearly admissible

for $\text{cap}_{1/K}(F, E; B)$. Since f has finite exponentially integrable distortion, we can use the distortion inequality together with (16) to obtain the estimate

$$\int_B \frac{|\nabla v|^2}{K} \leq \int_B |\nabla u \circ f|^2 J_f \leq \int_{\Omega_s} |\nabla u|^2$$

and thus with (15) we readily have the inequality

$$(17) \quad \text{cap}_{1/K}(F, E; B) \leq \frac{2s}{r^{-s} - 2^s c^{-s}}.$$

Applying Lemma 3 we obtain

$$C_1 \lambda \left(\log \frac{C_2}{\text{diam } E} \right)^{-2} \leq \text{cap}_{1/K}(F, E; B)$$

and combining with (17) and applying the modulus of continuity estimate (2) gives

$$(18) \quad C_1 \lambda \left(\log \frac{C_2}{\text{diam } E} \right)^{-2} \leq C_3 (\text{diam } E')^s \leq \tilde{C}_1 \left(\log \frac{\tilde{C}_2}{\text{diam } E} \right)^{-\lambda s/2}.$$

The fact that f is a homeomorphism implies that when $r \rightarrow 0$, then $\text{diam } E = \text{diam } f^{-1}(E') \rightarrow 0$. As the inequality (18) must hold for all $0 < r < c/2$, it follows that $2 \geq \lambda s/2$, which gives the desired result. $\square$

4. CONSTRUCTION

This construction follows the method presented in [8], Theorem 1; we only need to fine-tune the definitions of $g(r)$ and $G_s(r)$ there to give us the correct result. For the convenience of the reader we represent the entire construction.

Let $s > 0$. Our goal is to construct a homeomorphism $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ of finite distortion for which $f(B) = \Omega_s$ and $\exp(\lambda K_f) \in L^1_{\text{loc}}(\mathbb{R}^2)$ for all $\lambda < 2/s$. We begin by mapping the unit disc B conformally onto the open right half plane $H_R := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0\}$ so that the point $(-1, 0) \in \partial B$ maps to the origin. Using the complex notation, we define this mapping $f_1: \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ as $f_1(z) = (z+1)/(1-z)$.

Next, we set

$$g(r) = \frac{\sqrt{\pi}}{s} \log^{-1}(2/r)$$

and define linear functions $L_r^i:]-\frac{\pi}{2}, \frac{\pi}{2}[\rightarrow \mathbb{R}$ and $L_r^o: [\frac{\pi}{2}, \frac{3\pi}{2}] \rightarrow \mathbb{R}$ by setting

$$L_r^i(\theta) = \frac{2\theta}{\pi} \arctan g^2(r)$$

and

$$L_r^o(\theta) = 2\theta - \pi + \left(2 - \frac{2\theta}{\pi}\right) \arctan g^2(r).$$

Let $x \in \mathbb{R}^2$ satisfy $|x| \leq 1$. We may represent x in polar coordinates, $x = (r \cos \theta, r \sin \theta)$, so that $0 \leq r \leq 1$ and either $\theta \in]-\frac{\pi}{2}, \frac{\pi}{2}[$ or $\theta \in [\frac{\pi}{2}, \frac{3\pi}{2}]$. Using this notation, we define the mapping $f_2: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ in polar coordinates on $\overline{B}$ by setting

$$(19) \quad f_2(r, \theta) = \begin{cases} (G_s(r), L_r^i(\theta)) & \text{if } 0 < r \leq 1 \text{ and } \theta \in]-\frac{\pi}{2}, \frac{\pi}{2}[\\ (G_s(r), L_r^o(\theta)) & \text{if } 0 < r \leq 1 \text{ and } \theta \in [\frac{\pi}{2}, \frac{3\pi}{2}] \\ 0 & \text{if } r = 0, \end{cases}$$

where $G_s(r) = g^{2/s}(r)\sqrt{1+g^4(r)}$. Outside the closed unit disc $\overline{B}$, the mapping f_2 will be defined in a bi-Lipschitz manner. First we define $h: S(0, 1) \rightarrow S(0, G_s(1))$ by setting $h(x) = f_2(x)$ on the unit circle $S := S(0, 1) = \partial\overline{B}(0, 1)$. The mapping h is clearly a bi-Lipschitz mapping on S . Next we set

$$(20) \quad f_2(x) = |x|h\left(\frac{x}{|x|}\right) \quad \text{if } x \in \mathbb{R}^2 \setminus \overline{B}.$$

A simple calculation shows that if h is an L -bi-Lipschitz mapping, then f_2 will also be an L -bi-Lipschitz mapping on $S(0, R)$ for all $R \geq 1$. This and the fact that $|f_2(x)| = |x|G_s(1)$ for all $|x| \geq 1$ assures that f_2 will be a bi-Lipschitz mapping on $\mathbb{R}^2 \setminus \overline{B}$ and the bi-Lipschitz constant of f_2 depends only on L and $G_s(1)$.

The definition in (19) gives a mapping that maps the line segment $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0, -1 \leq x_2 \leq 1\}$ to the set $\{(x_1, x_2) \in \mathbb{R}^2 : |x_2| = x_1^{1+s}, 0 \leq x_1 \leq g^{2/s}(1)\}$, thus forming a cusp of the correct degree. Also, $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0, |x_2| > 1\}$ gets mapped to the set $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 > g^{2/s}(1), |x_2| = g^2(1)x_1\}$. What we have now is an unbounded domain with a cusp at the origin and whose boundary is a piecewise smooth curve. There are only three points where the smoothness fails, the origin (zero angle) and the points $(g^{2/s}(1), g^{2+2/s}(1))$ and $(g^{2/s}(1), -g^{2+2/s}(1))$ (non-zero angles). These two non-zero angles have the size $\pi - \arctan((1+s)g^2(1)) + \arctan g^2(1)$.

Next, we make this cusp domain bounded by mapping the right half plane onto the disc $B((1/2, 0), 1/2)$ with the mapping $f_3: \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}, f_3(z) = z/(z+1)$ and denote $\tilde{\Omega}_s := f_3(f_2(f_1(B)))$. The mapping f_3 will somewhat alter the shape of the cusp at the origin, but not essentially as it will be seen. On the other hand, being conformal, it will preserve the size of the two aforementioned non-zero angles and will create one more non-zero angle at the point $(1, 0)$, of the size $\pi/2$. Otherwise the boundary curve will still be smooth. Because of these facts, there exists a sense preserving bi-Lipschitz mapping $f_4: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ for which $f_4(\tilde{\Omega}_s) = \Omega_s$; moreover, this mapping f_4 can be chosen so that for some bounded function $L(x, y): \mathbb{R}^2 \rightarrow [1, L]$, $1 < L < \infty$, we have that

$$(21) \quad \frac{1}{L(x, y)}|x - y| \leq |f_4(x) - f_4(y)| \leq L(x, y)|x - y|$$

and $L(x, y) \rightarrow 1$ when $|x| + |y| \rightarrow 0$. To justify this last claim we notice that for $0 < r_0 < G_s(1)$ we have by a straightforward calculation that

$$(22) \quad \begin{aligned} \partial\tilde{\Omega}_s \cap B(0, r_0) = \\ \{p^{-1}(t)(t^{2+2s} + t^2 + t, t^{1+s}) \in \mathbb{R}^2 : 0 \leq t < r_0\} \\ \cup \{p^{-1}(t)(t^{2+2s} + t^2 + t, -t^{1+s}) \in \mathbb{R}^2 : 0 \leq t < r_0\}, \end{aligned}$$

where $p(t) := t^{2+2s} + t^2 + 2t + 1$. If we consider the upper half plane we see from (22) that near the origin the boundary points of $\tilde{\Omega}_s$ are of the form $(t + o(t^2), t^{1+s} + o(t^{2+s}))$ and likewise on the lower half plane. Thus the bi-Lipschitz correction of $\tilde{\Omega}_s$ to Ω_s can be done in the manner explained before.

Finally we will set $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $f(x) = f_4 \circ f_3 \circ \tilde{f}_2 \circ f_1(x)$. Here $\tilde{f}_2$ is the homeomorphic extension of f_2 to $\overline{\mathbb{C}}$ obtained by setting $\tilde{f}_2(x) = f_2(x)$ when x is finite and $\tilde{f}_2(\infty) = \infty$. This definition clearly gives us a sense preserving homeomorphism for which $f(B) = \Omega_s$.

Next we show that the distortion function of f satisfies the required conditions. In fact, it will be enough to compute the distortion of f_2 , because the conformal mappings f_1 and f_3 do not give any contribution to it and the contribution of the bi-Lipschitz mapping f_4 will be in fact a bounded multiplier that goes to one when we approach the origin. The latter claim follows from the facts that the distortion of an L -bi-Lipschitz mapping is L^2 , and f_4 satisfies the inequality (21).

Outside the unit disc B the mapping f_2 is quasiconformal, as it is seen directly from the definition to be sense preserving and bi-Lipschitz there. Hence we need to show that f_2 has an exponentially integrable distortion in B . To show this, we consider a point $x \in B$, which we again represent in the form $x = (r \cos \theta, r \sin \theta)$, such that $0 \leq r < 1$ and either $\theta \in]-\frac{\pi}{2}, \frac{\pi}{2}[$ or $\theta \in [\frac{\pi}{2}, \frac{3\pi}{2}]$. We first consider the case $\theta \in]-\frac{\pi}{2}, \frac{\pi}{2}[$, i.e. the set $B \cap H_R$.

To compute $K_{f_2}(x)$, we choose a local coordinate system at every point $x \in B$ by setting $e_1^x = (\cos \theta, \sin \theta)$ and $e_2^x = (-\sin \theta, \cos \theta)$. Notice that the vector e_1^x points to the radial direction and e_2^x is perpendicular to it, so it points to the angular direction. To this basis on the preimage side we associate a similar basis on the image side, denoting it by $(e_1^{f_2(x)}, e_2^{f_2(x)})$. We now represent the differential matrix of f_2 at the point x by using the bases (e_1^x, e_2^x) and $(e_1^{f_2(x)}, e_2^{f_2(x)})$ which we will refer to from now on as E_x and $E_{f_2(x)}$, respectively. The resulting differential matrix is

$$(23) \quad \begin{bmatrix} \frac{d}{dr} G_s(r) & 0 \\ G_s(r) \frac{d}{dr} L_r^i(\theta) & \frac{G_s(r)}{r} \frac{d}{d\theta} L_r^i(\theta) \end{bmatrix} \\ = \begin{bmatrix} \frac{2}{\sqrt{\pi}r} g^{2/s+1}(r) \sqrt{1+g^4(r)} + \frac{2s}{\sqrt{\pi}r} \frac{g^{2/s+5}(r)}{\sqrt{1+g^4(r)}} & 0 \\ \frac{4s\theta}{\pi\sqrt{\pi}r} \frac{g^{2/s+3}(r)}{\sqrt{1+g^4(r)}} & \frac{2}{\pi r} g^{2/s}(r) \sqrt{1+g^4(r)} \arctan g^2(r) \end{bmatrix}$$

and is computed as follows.

As the radial part of f_2 depends only on r , the partial derivative of the first component of f_2 in the basis $E_{f_2(x)}$ in the e_1^x -direction is $\frac{d}{dr} G_s(r)$. The radial part does not depend on the angle, and so the partial derivative of the first component of f_2 in the basis $E_{f_2(x)}$ in the e_2^x -direction is zero.

Next we observe that for $\varepsilon > 0$ the change in the angular part, and thus in the $e_2^{f_2(x)}$ -direction, is $G_s(r)(L_{r+\varepsilon}^i(\theta) - L_r^i(\theta))$ when the change to the e_1^x direction is ε . Thus the partial derivative of the second component of f_2 in the basis $E_{f_2(x)}$ in the e_1^x -direction is

$$(24) \quad \lim_{\varepsilon \rightarrow 0} \frac{G_s(r)(L_{r+\varepsilon}^i(\theta) - L_r^i(\theta))}{\varepsilon} = G_s(r) \frac{d}{dr} L_r^i(\theta).$$

Similarly, the partial derivative of the second component of f_2 in the basis $E_{f_2(x)}$ in the e_2^x -direction is

$$(25) \quad \lim_{\varepsilon \rightarrow 0} \frac{G_s(r)(L_r^i(\theta + \varepsilon) - L_r^i(\theta))}{r\varepsilon} = \frac{G_s(r)}{r} \frac{d}{d\theta} L_r^i(\theta).$$

To estimate $K_{f_2}(x)$ from (23) we use the following known result (see [11]) which states that for a linear bijection $A: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ the distortion K of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ satisfies

$$K + \frac{1}{K} = \frac{a^2 + b^2 + c^2 + d^2}{|ad - bc|}.$$

This and the fact that $K \geq 1$ gives us a simple upper estimate for K :

$$K \leq \frac{a^2 + b^2 + c^2 + d^2}{|ad - bc|}.$$

Thus by a straightforward calculation we obtain the estimate

$$(26) \quad K_{f_2}(x) \leq \sqrt{\pi} g^{-1}(|x|) \frac{g^2(|x|)}{\arctan g^2(|x|)} + \frac{1}{\sqrt{\pi}} g(|x|) \left(\frac{g^2(|x|)}{\arctan g^2(|x|)} \right)^{-1} \\ + \left(\sqrt{\pi} s^2 g^7(|x|) + 2\sqrt{\pi} s g^3(|x|) + \frac{4\sqrt{\pi} s^2 \theta^2}{\pi^2} g^3(|x|) \right) \frac{g^2(|x|)}{\arctan g^2(|x|)}.$$

Because $g^2(|x|)/(\arctan g^2(|x|)) \rightarrow 1$ and $g^n(|x|) \rightarrow 0$ for all $n \geq 1$ as $|x| \rightarrow 0$, one easily observes from (26) that for all $x \in B \cap H_R$,

$$(27) \quad K_{f_2} \leq s \log(2/|x|) + E(x),$$

where $E: \mathbb{R}^2 \rightarrow \mathbb{R}$ is some bounded function.

Next we consider the case $\theta \in [\frac{\pi}{2}, \frac{3\pi}{2}]$, i.e. $x \in B \setminus H_R$. Similar reasoning as in the first case gives the differential matrix

$$\begin{bmatrix} \frac{d}{dr} G_s(r) & 0 \\ G_s(r) \frac{d}{dr} L_r^o(\theta) & \frac{G_s(r)}{r} \frac{d}{d\theta} L_r^o(\theta) \end{bmatrix} \\ = \begin{bmatrix} \frac{2}{\sqrt{\pi r}} g^{2/s+1}(r) \sqrt{1+g^4(r)} + \frac{2s}{\sqrt{\pi r}} \frac{g^{2/s+5}(r)}{\sqrt{1+g^4(r)}} & 0 \\ (2 - \frac{2\theta}{\pi}) \frac{2s}{\sqrt{\pi r}} \frac{g^{2/s+3}(r)}{\sqrt{1+g^4(r)}} & (2 - \frac{2}{\pi} \arctan g^2(r)) \frac{1}{r} g^{2/s}(r) \sqrt{1+g^4(r)} \end{bmatrix}.$$

The estimate for $K_{f_2}(x)$ is

$$(28) \quad K_{f_2}(x) \leq \sqrt{\pi} g^{-1}(|x|) - \frac{1}{\sqrt{\pi}} g^{-1}(|x|) \arctan g^2(|x|) \\ + \left(\frac{2}{\sqrt{\pi}} + \left(\frac{4s}{\sqrt{\pi}} + (2 - \frac{2\theta}{\pi})^2 \frac{2s^2}{\sqrt{\pi}} \right) g^4(|x|) + \frac{2s^2}{\sqrt{\pi}} g^8(|x|) \right) \frac{g(|x|)}{2 - \frac{2}{\pi} \arctan g^2(|x|)}.$$

Now, as $|x| \rightarrow 0$ we have for all $n \geq 1$ that $g^n(|x|) \rightarrow 0$, $\arctan g^2(|x|) \rightarrow 0$ and $g^{-1}(|x|) \arctan g^2(|x|) \rightarrow 0$ so (28) clearly implies that for $x \in B \setminus H_R$,

$$(29) \quad K_{f_2} \leq s \log(2/|x|) + E'(x),$$

where $E': \mathbb{R}^2 \rightarrow \mathbb{R}$ is again some bounded function. The estimates (27) and (29) show that $\exp(\lambda K_{f_2}(x)) \in L^1(B)$ for all $\lambda < 2/s$.

As mentioned before, f_1 and f_3 do not give any contribution to the distortion of f and the contribution of f_4 is a bounded multiplier that goes to one when one approaches the origin. Moreover, f_1 is bi-Lipschitz in $f^{-1}(B)$. Thus the distortion $K_f(x)$ of f satisfies $\exp(\lambda K_f(x)) \in L_{\text{loc}}^1(\mathbb{R}^2)$ for all $\lambda < 2/s$. $\square$

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REFERENCES

1. K. Astala, T. Iwaniec, P. Koskela, and G. Martin, *Mappings of BMO-bounded distortion*, Math. Ann. **317** (2000), no. 4, 703–726. MR1777116 (2001i:30016)
2. G. David, *Solutions de l'équation de Beltrami avec $\|\mu\|_\infty = 1$* , Ann. Acad. Sci. Fenn. Ser. A I Math. **13** (1988), no. 1, 25–70. MR975566 (90d:30058)
3. V. Gutlyanskiĭ, O. Martio, T. Sugawa, and M. Vuorinen, *On the degenerate Beltrami equation*, Trans. Amer. Math. Soc. **357** (2005), no. 3, 875–900. MR2110425 (2005i:30026)
4. D.A. Herron and P. Koskela, *Mappings of finite distortion: gauge dimension of generalized quasicircles*, Illinois J. Math. **47** (2003), no. 4, 1243–1259. MR2037001 (2005a:30031)
5. T. Iwaniec, P. Koskela, and G. Martin, *Mappings of BMO-distortion and Beltrami-type operators. Dedicated to the memory of Tom Wolff*, J. Anal. Math. **88** (2002), 337–381. MR1979776 (2004f:30015)
6. T. Iwaniec and G. Martin, *Geometric function theory and non-linear analysis*, Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2001. MR1859913 (2003c:30001)
7. J. Kauhanen, P. Koskela, and J. Malý, *Mappings of finite distortion: condition N*, Michigan Math. J. **49** (2001), no. 1, 169–181. MR1827080 (2002d:30027)
8. P. Koskela and J. Takkinen, *Mappings of finite distortion: formation of cusps*, Publ. Mat. **51** (2007), no. 1, 223–242. MR2307153
9. J. Onninen and X. Zhong, *A note on mappings of finite distortion: the sharp modulus of continuity*, Michigan Math. J. **53** (2005), no. 2, 329–335. MR2152704 (2006c:30025)
10. V. Ryazanov, U. Srebro, and E. Yakubov, *BMO-quasiconformal mappings*, J. Anal. Math. **83** (2001), 1–20. MR1828484 (2002a:30034)
11. J. Väisälä, *Lectures on n -dimensional quasiconformal mappings*, Lecture Notes in Mathematics, no. 229, Springer-Verlag, New York, 1971. MR0454009 (56:12260)
12. S. Zakeri, *David maps and Hausdorff dimension*, Ann. Acad. Sci. Fenn. Math. **29** (2004), no. 1, 121–138. MR2041702 (2005a:30034)

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