CONSTRUCTION OF VECTOR FIELDS AND RICCATI FOLIATIONS ASSOCIATED TO GROUPS OF PROJECTIVE AUTOMORPHISMS

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ABSTRACT. Our main result states that given a finitely generated subgroup G of $\text{Aut}(\mathbb{CP}(2))$, there is an algebraic foliation $\mathcal{F}$ on a complex projective 3-manifold M^3 with a bundle structure over $\mathbb{CP}(1)$ and fiber $\mathbb{CP}(2)$, such that $\mathcal{F}$ is transverse to almost every fiber of the bundle and with global holonomy conjugate to G .

1. INTRODUCTION

Foliations transverse to fibrations are among the simplest (constructive) examples of foliated manifolds, once regarded as suspensions of groups of diffeomorphisms (see [1] and [2]). Thus one expects to perform a nice study of them in the global theoretic aspect. In the complex algebraic setting, foliations usually exhibit singularities so this possibility cannot be excluded. Very representative examples of either situation are given by the class of *Riccati foliations* (see [5]) in dimension two. A very interesting study is performed in [7] and a complete reference on the two dimensional case is [6]. In this paper we study one-dimensional holomorphic foliations with singularities which are transverse to a given holomorphic fibration outside a certain subset called the exceptional set. Let us first recall the classical notion. Let $\eta = (E, \pi, B, F)$ be a fibre bundle with total space E , fiber F , basis B and projection $\pi: E \rightarrow B$. A foliation $\mathcal{F}$ on E is *transverse to η* if:

(1) for each $p \in E$, the leaf L_p of $\mathcal{F}$ with $p \in L_p$ is transverse to the fiber $\pi^{-1}(q)$, $q = \pi(p)$;

(2) $\dim(\mathcal{F}) + \dim(F) = \dim(E)$; and

(3) for each leaf L of $\mathcal{F}$, the restriction $\pi|_L: L \rightarrow B$ is a covering map. According to Ehresmann (see [2]) if the fiber F is compact, then the conditions (1) and (2) already imply (3). In the complex setting, all the objects above are holomorphic by hypothesis and several are the interesting frameworks (see [9] for the complex hyperbolic case and [8] for the codimension one case). Under the presence of singularities, a weaker notion must be introduced. We shall say that $\mathcal{F}$ is *transverse to almost every fiber* of the fibre bundle η if there is an analytic subset $\Lambda(\mathcal{F}) \subset E$ which is the union of fibers of η , such that the restriction $\mathcal{F}_0$ of $\mathcal{F}$ to $E_0 = E \setminus \Lambda$ is transverse to the natural subbundle η_0 of η having E_0 as total space. If $\Lambda(\mathcal{F})$ is minimal with this property, then $\Lambda(\mathcal{F})$ is called the *exceptional set* of $\mathcal{F}$. By

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a *Riccati foliation* we mean a foliation $\mathcal{F}$ as above, for which the exceptional set $\Lambda(\mathcal{F})$ is $\mathcal{F}$ -invariant. In particular, we shall consider the *global holonomy* of $\mathcal{F}$ as the global holonomy of the restriction $\mathcal{F}_0$ on $E_0 = E \setminus \Lambda(\mathcal{F})$. In the classical situation of Riccati foliations in $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$, the global holonomy is a finitely generated group of Möbius transformations, *i.e.*, a finitely generated subgroup of $PSL(2, \mathbb{C})$. In [5] the author proves that given a finitely generated subgroup $G < PSL(2, \mathbb{C})$ there is a Riccati foliation in $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$ for which the global holonomy is conjugated to G . In this work we consider the problem of synthesis of Riccati foliations on $\overline{\mathbb{C}} \times \mathbb{CP}(2)$. Our main result below may apply to the study of (singular) projective foliations with *homogeneous transverse structure* (see [2]) in codimension ≥ 2 .

Theorem 1.1 (Synthesis theorem). *Let $f_1, \dots, f_r \in \text{Aut}(\mathbb{CP}(2))$ be holomorphic diffeomorphisms of $\mathbb{CP}(2)$ and let $x_0^0, x_1^0, \dots, x_r^0 \in \overline{\mathbb{C}}$ be $r+1$ points in $\mathbb{CP}(2)$. There is an algebraic (Riccati) foliation $\mathcal{F}$ on a complex projective 3-manifold M^3 , admitting a holomorphic fibre bundle structure $\eta = (M^3, \pi, \overline{\mathbb{C}}, \mathbb{CP}(2))$, over $\mathbb{CP}(2)$ with fiber $\overline{\mathbb{C}}$, with the following properties:*

- (1) $\mathcal{F}$ is transverse to almost every fiber of η .
- (2) The invariant fibers of $\mathcal{F}$ are $\pi^{-1}(x_0^0), \dots, \pi^{-1}(x_r^0)$.
- (3) The global holonomy of $\mathcal{F}$ is conjugate to the subgroup of $\text{Aut}(\mathbb{CP}(2))$ generated by $f_1, \dots, f_r$.

2. FIXED POINTS OF AUTOMORPHISMS OF $\mathbb{CP}(2)$

The group of automorphisms of $\mathbb{CP}(n)$ is induced by the general linear group; that is, $\text{Aut}(\mathbb{CP}(n)) \cong \text{PGL}(n+1, \mathbb{C})$ (see [3]), it identifies an isomorphism $T: \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{n+1}$ with the biholomorphism of the complex projective space $[T]$ defined by: if $r \subset \mathbb{C}^{n+1}$ is a complex line containing $0 \in \mathbb{C}^{n+1}$, then $s = T(r)$ is a complex line containing $0 \in \mathbb{C}^{n+1}$ and we consider $[T]: \mathbb{CP}(n) \rightarrow \mathbb{CP}(n)$ given by $[T](r \setminus \{0\}) = s \setminus \{0\}$.

Aiming the study of Riccati foliations on $\overline{\mathbb{C}} \times \mathbb{CP}(2)$ through the global holonomy we perform the classification of holomorphic diffeomorphisms of $\mathbb{CP}(2)$ by the set of fixed points. Indeed, we use the Jordan canonical forms to obtain the following.

Proposition 2.1. *If $f: \mathbb{CP}(2) \rightarrow \mathbb{CP}(2)$ is a biholomorphism and $\Sigma(f) \subset \mathbb{CP}(2)$ denotes its set of fixed points, then we have the following six possibilities:*

- (1) $\Sigma(f)$ is a set of one, two or three points.
- (2) $\Sigma(f)$ consists of a projective line.
- (3) $\Sigma(f)$ consists of one projective line plus one point.
- (4) $\Sigma(f)$ has dimension two and $\Sigma(f) = \mathbb{CP}(2)$.

In particular, f is conjugate in $\text{Aut}(\mathbb{CP}(2))$ to a map $g \in \text{Aut}(\mathbb{CP}(2))$ of the form,

$$\begin{aligned} g(x:y:z) &= (\lambda_0 x + y : \lambda_0 y + z : \lambda_0 z), & g(x:y:z) &= (\lambda_0 x + y : \lambda_0 y : \lambda_1 z), \\ g(x:y:z) &= (\lambda_0 x : \lambda_1 y : \lambda_2 z), & g(x:y:z) &= (\lambda_0 x : \lambda_0 y : \lambda_1 z), \\ g(x:y:z) &= (\lambda_0 x + y : \lambda_0 y : \lambda_0 z), & g(x:y:z) &= (x : y : z), \end{aligned}$$

where $\lambda_0, \lambda_1, \lambda_2 \in \mathbb{C} \setminus \{0\}$, respectively.

Motivated by Proposition 2.1, we introduce the following auxiliary definition.

Definition 2.2. We shall say that a biholomorphism $f : \mathbb{CP}(2) \rightarrow \mathbb{CP}(2)$ different from the identity is:

- (i) of type Pk with $k \in \{1, 2, 3\}$, if $\Sigma(f)$ consists of k points,
- (ii) of type $SL1$, if $\Sigma(f)$ consists of one projective line,
- (iii) of type $P1SL1$ if $\Sigma(f)$ consists of one point and one projective line.

3. CONSTRUCTION OF FOLIATIONS, PROOF OF THE SYNTHESIS THEOREM

In this section we prove Theorem 1.1. We proceed as in [5]; nevertheless, some differences and difficulties arise.

Proof of Theorem 1.1. Let $x_0^0 = 0, x_1^0, \dots, x_r^0$ be points in $\mathbb{C}$. For each $j \in \{0, 1, \dots, r\}$ we take D_j to be a disk of center x_j^0 and radius $\rho > 0$ such that $|x_j^0 - x_k^0| > 2\rho$ for all $j \neq k$, $0 \leq j, k \leq r$. For each $j \in \{1, \dots, r\}$ we choose $x_j' = x_j^0 + \frac{\rho}{2} \in D_j \setminus \{x_j^0\}$ and $x_j'' = \frac{\rho}{2} \exp(\frac{2\pi\sqrt{-1}(j-1)}{r}) \in D_0 \setminus \{0\}$. For each $j \in \{1, \dots, r\}$ we take $\alpha_j : [0, 1] \rightarrow \mathbb{C}$ to be simple curves such that:

- (i) $\alpha_j(0) = x_j''$ and $\alpha_j(1) = x_j'$;
- (ii) $\alpha_j([0, 1]) \cap D_k = \emptyset$ if $j \neq k$, $k \neq 0$;
- (iii) $\alpha_j([0, 1]) \cap \alpha_k([0, 1]) = \emptyset$ if $j \neq k$;
- (iv) $\alpha_j([0, 1]) \cap D_0$ and $\alpha_j([0, 1]) \cap D_j$ are segments of straight lines contained in diameters of D_0 and D_j respectively.

Let $A_1, \dots, A_r$ be strips around $\alpha_1, \dots, \alpha_r$, respectively, such that:

- (v) $A_j \cap D_k = \emptyset$ if $j \neq k$, $k \neq 0$;
- (vi) $A_j \cap A_k = \emptyset$ if $j \neq k$;
- (vii) $A_j \cap D_0$ and $A_j \cap D_j$ are contained in sectors of D_0 and D_j , $1 \leq j \leq r$, respectively.

Let $U = (\bigcup_{j=1}^r A_j) \cup (\bigcup_{j=0}^r D_j)$ be a set and let $\gamma = \partial U$ be a simple curve. Let T be a tubular neighborhood of γ and let $V = (\overline{\mathbb{C}} \setminus U) \cup T$ be a set. Then $\{A_1, \dots, A_r, D_0, \dots, D_r, V\}$ is a covering of $\overline{\mathbb{C}}$ by open sets. For every $k \in \{1, \dots, r\}$ we consider affine coordinates $(x, \tilde{y}_k, \tilde{z}_k)$, $(x, \tilde{u}_k = \frac{1}{\tilde{y}_k}, \tilde{v}_k = \frac{\tilde{z}_k}{\tilde{y}_k})$, $(x, \tilde{s}_k = \frac{1}{\tilde{z}_k}, \tilde{t}_k = \frac{\tilde{y}_k}{\tilde{z}_k})$ in $A_k \times \mathbb{C}^2 \hookrightarrow A_k \times \mathbb{CP}(2)$, and for each $j \in \{0, 1, \dots, r\}$ we consider affine coordinates (x, y_j, z_j) , $(x, u_j = \frac{1}{y_j}, v_j = \frac{z_j}{y_j})$, $(x, s_j = \frac{1}{z_j}, t_j = \frac{y_j}{z_j})$ in $D_j \times \mathbb{C}^2$. Put affine coordinates (w, y, z) in $V \times \mathbb{C}^2$ where $w = \frac{1}{x} \in V$ and $(y, z) \in \mathbb{C}^2$.

We take in each set of the form $A_k \times \mathbb{CP}(2)$, $V \times \mathbb{CP}(2)$ and $D_j \times \mathbb{CP}(2)$ a local model of foliation and glue them together. The local models are as follows:

- (1) In $A_k \times \mathbb{CP}(2)$ we consider the horizontal foliation whose leaves are of the form $A_k \times \{p\}$, $p \in \mathbb{CP}(2)$ for each $j \in \{1, \dots, r\}$.
- (2) In $V \times \mathbb{CP}(2)$ we consider the horizontal foliation whose leaves are of the form $V \times \{p\}$, $p \in \mathbb{CP}(2)$.
- (3) In $D_j \times \mathbb{C}^2$ we consider the singular holomorphic foliation $\mathcal{F}_j$ induced by the vector field X_j in $D_j \times \mathbb{C}^2$ for every $j \in \{0, 1, \dots, r\}$. In fact, for each $j \in \{0, 1, \dots, r\}$ we consider f_j the global holonomy of $\mathcal{F}_j$ on $D_j \times \mathbb{CP}(2)$, where $f_0 = (f_1 \circ \dots \circ f_r)^{-1}$. In this case we note that (Proposition 2.1): for each j there exists an affine coordinate such that it identifies f_j with the

map $F_j : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ and F_j can be written in one of the following forms:

- (a) $F_j(u, v) = (u + \mu v, v + \mu)$ if f_j is of type P1.
- (b) $F_j(u, v) = (\mu u + \nu v, \mu v)$ if f_j is of type P2.
- (c) $F_j(u, v) = (\lambda' u, \lambda'' v)$ if f_j is of type P3.
- (d) $F_j(u, v) = (\lambda'' u, \lambda'' v)$ if f_j is SL1.
- (e) $F_j(u, v) = (u + \nu v, v)$ if f_j is of type P1SL1, where $\lambda', \lambda'', \mu, \nu \in \mathbb{C} \setminus \{0\}$ are different.

• In case (c) or case (d), we consider the singular holomorphic foliation $\mathcal{F}_j$ on $D_j \times \mathbb{C}^2$ given by the vector field

$$(3.1) \quad X_j(x, y_j, z_j) = (x - x_j^0) \frac{\partial}{\partial x} + \alpha'_j y_j \frac{\partial}{\partial y_j} + \alpha''_j z_j \frac{\partial}{\partial z_j}$$

where $\exp(2\pi\sqrt{-1}\alpha'_j) = \lambda'_j$ and $\exp(2\pi\sqrt{-1}\alpha''_j) = \lambda''_j$. However, in case (d) we also have $\alpha''_j = \alpha'_j$ and $\lambda''_j = \lambda'_j$. Let $\gamma_j(\theta) = (x_j^0 + \rho_j \exp(\sqrt{-1}\theta), 0, 0)$, $0 \leq \theta \leq 2\pi$ be a curve where $0 < \rho_j < \rho$. Let $\Sigma_j = \{p_j\} \times \mathbb{C}^2$, $p_j \in \gamma_j([0, 2\pi])$.

Claim 3.1. The holonomy transformation of $\mathcal{F}_j$ associated to Σ_j and γ_j is of the form $(y_j, z_j) \mapsto (\lambda'_j y_j, \lambda''_j z_j)$ where the foliation $\mathcal{F}_j$ on $D_j \times \mathbb{C}^2$ is induced by X_j (equation (3.1)).

In fact, suppose $\Sigma_j = \{x_j^0 + \rho_j\} \times \mathbb{C}^2$, a local transverse section, and $p_j = (x_j^0 + \rho_j, 0, 0) \in \Sigma_j$. Suppose $\pi_1 : D_j \times \mathbb{C}^2 \rightarrow D_j$, $\pi_1(x, y, z) = x$. Observe that the fibers $\pi_1^{-1}(x)$, $x \neq x_j^0$, are transverse to $\mathcal{F}_j$. We consider $q = (x_j^0 + \rho_j, y_j, z_j) \in \Sigma_j$ and $\gamma_q(\theta) = (x(\theta), y_j(\theta), z_j(\theta))$ the lifting of γ_j by π_1 with base point q . Therefore $x'(\theta) = \pi_1(\gamma'_q(\theta)) = \pi_1(\gamma'_j(\theta)) = \sqrt{-1}\rho_j \exp(\sqrt{-1}\theta)$, and, if $Y_j = (y_j, z_j) \in \mathbb{C}^2$, then

$$\frac{Y'_j}{x'} = \frac{Y'_j}{\sqrt{-1}\rho_j \exp(\sqrt{-1}\theta)}.$$

On the other hand, by equation (3.1) we have $\frac{dx}{dT} = x - x_j^0$ and

$$\frac{dY_j}{dT} = \begin{bmatrix} \alpha'_j & 0 \\ 0 & \alpha''_j \end{bmatrix} \cdot \begin{bmatrix} y_j \\ z_j \end{bmatrix} = AY_j$$

so we obtain

$$\frac{dY_j}{dx} = \frac{dY_j}{dT} \cdot \frac{dT}{dx} = \frac{\frac{dY_j}{dT}}{\frac{dx}{dT}} = \frac{AY_j}{x - x_j^0}$$

and we have

$$\frac{Y'_j}{\sqrt{-1}\rho_j \exp(\sqrt{-1}\theta)} = \frac{Y'_j}{x'} = \frac{AY_j}{\rho_j \exp(\sqrt{-1}\theta)},$$

then $Y'_j = \sqrt{-1}AY_j$. Notice that the solution of last equation such that $Y_j(0) = (y_j, z_j)$ is $Y_j(\theta) = \exp(\sqrt{-1}\theta A) \cdot Y_j(0)$. Therefore the holonomy is the biholomorphism $f : \Sigma_j \rightarrow \Sigma_j$ defined by

$$\begin{aligned}
 f(y_j, z_j) &= Y_j(2\pi) \\
 &= \exp\left(2\pi\sqrt{-1}\begin{bmatrix} \alpha'_j & 0 \\ 0 & \alpha''_j \end{bmatrix}\right) \cdot \begin{bmatrix} y_j \\ z_j \end{bmatrix} \\
 &= \begin{bmatrix} \exp(2\pi\sqrt{-1}\alpha'_j) & 0 \\ 0 & \exp(2\pi\sqrt{-1}\alpha''_j) \end{bmatrix} \cdot \begin{bmatrix} y_j \\ z_j \end{bmatrix} \\
 &= (\exp(2\pi\sqrt{-1}\alpha'_j)y_j, \exp(2\pi\sqrt{-1}\alpha''_j)z_j) \\
 &= (\lambda'_j y_j, \lambda''_j z_j)
 \end{aligned}$$

and this proves the claim.

- In case (e), we consider the foliation $\mathcal{F}_j$ on $D_j \times \mathbb{C}^2$ given by

$$(3.2) \quad X_j(x, y_j, z_j) = (x - x_j^0) \frac{\partial}{\partial x} + \frac{\nu}{2\pi\sqrt{-1}} z_j \frac{\partial}{\partial y_j}.$$

Claim 3.2. The holonomy transformation of $\mathcal{F}_j$ associated to Σ_j and γ_j is of the following form $(y_j, z_j) \mapsto (y_j + \nu z_j, z_j)$, where the vector field X_j is given by equation (3.2).

In fact, suppose $\gamma_j(\theta) = (x_j^0 + \rho_j \exp(\sqrt{-1}\theta), 0, 0)$; $0 \leq \theta \leq 2\pi$ is a curve where $0 < \rho_j < \rho$. We consider $\pi_1 : D_j \times \mathbb{C}^2 \rightarrow D_j$, $\pi_1(x, y, z) = x$. Notice that the fibers $\pi_1^{-1}(x)$, $x \neq x_j^0$ are transverse to $\mathcal{F}_j$. Let $\Sigma_j = \{x_j^0 + \rho_j\} \times \mathbb{C}^2$ be a local transverse section and $q = (x_j^0 + \rho_j, y_j, z_j) \in \Sigma_j$. Let $\gamma_q(\theta) = (x(\theta), y_j(\theta), z_j(\theta))$ be the lifting of γ_j by π_1 with base point q . Therefore $x'(\theta) = \pi_1(\gamma'_q(\theta)) = \pi_1(\gamma'_j(\theta)) = \sqrt{-1}\rho_j \exp(\sqrt{-1}\theta)$, and, if $Y_j = (y_j, z_j) \in \mathbb{C}^2$, then

$$\frac{Y'_j}{x'} = \frac{Y'_j}{\sqrt{-1}\rho_j \exp(\sqrt{-1}\theta)}.$$

On the other hand, by equation (3.2) we have $\frac{dx}{dT} = x - x_j^0$ and

$$\frac{dY_j}{dT} = \begin{bmatrix} 0 & \frac{\nu}{2\pi\sqrt{-1}} \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} y_j \\ z_j \end{bmatrix} = AY_j,$$

so we obtain

$$\frac{dY_j}{dx} = \frac{dY_j}{dT} \cdot \frac{dT}{dx} = \frac{\frac{dY_j}{dT}}{\frac{dx}{dT}} = \frac{AY_j}{x - x_j^0}$$

and we have

$$\frac{Y'_j}{\sqrt{-1}\rho_j \exp(\sqrt{-1}\theta)} = \frac{Y'_j}{x'} = \frac{AY_j}{\rho_j \exp(\sqrt{-1}\theta)}$$

therefore $Y'_j = \sqrt{-1}AY_j$. Observe that the solution of $Y'_j = \sqrt{-1}AY_j$, $Y_j(0) = (y_j, z_j)$ is $Y_j(\theta) = \exp(\sqrt{-1}\theta A) \cdot Y_j(0)$. Therefore the holonomy is the biholomorphism $f : \Sigma_j \rightarrow \Sigma_j$ defined by

$$\begin{aligned}
 f(y_j, z_j) &= Y_j(2\pi) \\
 &= \exp\left(2\pi\sqrt{-1}\begin{bmatrix} 0 & \frac{\nu}{2\pi\sqrt{-1}\mu} \\ 0 & 0 \end{bmatrix}\right) \cdot \begin{bmatrix} y_j \\ z_j \end{bmatrix} \\
 &= \exp\left(\begin{bmatrix} 0 & \nu \\ 0 & 0 \end{bmatrix}\right) \cdot \begin{bmatrix} y_j \\ z_j \end{bmatrix} \\
 &= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & \nu \\ 0 & 0 \end{bmatrix}\right) \cdot \begin{bmatrix} y_j \\ z_j \end{bmatrix} \\
 &= \begin{bmatrix} 1 & \nu \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} y_j \\ z_j \end{bmatrix} \\
 &= (y_j + \nu z_j, z_j)
 \end{aligned}$$

and this proves the claim.

- In case (b), we consider the singular foliation $\mathcal{F}_j$ on $D_j \times \mathbb{C}^2$ given by

$$(3.3) \quad X_j(x, y_j, z_j) = (x - x_j^0) \frac{\partial}{\partial x} + (\lambda y_j + \frac{\nu}{2\pi\sqrt{-1}\mu} z_j) \frac{\partial}{\partial y_j} + \lambda z_j \frac{\partial}{\partial z_j}$$

where $\exp(2\pi\sqrt{-1}\lambda) = \mu$.

Let $\gamma_j(\theta) = (x_j^0 + \rho_j \exp(\sqrt{-1}\theta), 0, 0)$, $0 \leq \theta \leq 2\pi$ be a curve where $0 < \rho_j < \rho$.

Claim 3.3. The holonomy transformation of $\mathcal{F}_j$ associated to Σ_j and γ_j is of the form $(y_j, z_j) \mapsto (\mu y_j + \nu z_j, \mu z_j)$ where $\mathcal{F}_j$ is induced by X_j given by (3.3).

In fact, let $\Sigma_j = \{x_j^0 + \rho_j\} \times \mathbb{C}^2$ be a local transverse section and let $p_j = (x_j^0 + \rho_j, 0, 0) \in \Sigma_j$. Suppose $\pi_1 : D_j \times \mathbb{C}^2 \rightarrow D_j$, $\pi_1(x, y, z) = x$. Notice that the fibers $\pi_1^{-1}(x)$, $x \neq x_j^0$ are transverse to $\mathcal{F}_j$. Let $q = (x_j^0 + \rho_j, y_j, z_j) \in \Sigma_j$ and let $\gamma_q(\theta) = (x(\theta), y_j(\theta), z_j(\theta))$ be the lifting of γ_j by π_1 with base point q . Therefore $x'(\theta) = \pi_1(\gamma'_q(\theta)) = \pi_1(\gamma'_j(\theta)) = \sqrt{-1}\rho_j \exp(\sqrt{-1}\theta)$, and, if $Y_j = (y_j, z_j) \in \mathbb{C}^2$, then $\frac{Y'_j}{x'} = \frac{Y'_j}{\sqrt{-1}\rho_j \exp(\sqrt{-1}\theta)}$. On the other hand, by equation (3.3) we have $\frac{dx}{dT} = x - x_j^0$ and

$$\frac{dY_j}{dT} = \begin{bmatrix} \lambda & \frac{\nu}{2\pi\sqrt{-1}\mu} \\ 0 & \lambda \end{bmatrix} \cdot \begin{bmatrix} y_j \\ z_j \end{bmatrix} = AY_j,$$

so we obtain

$$\frac{dY_j}{dx} = \frac{dY_j}{dT} \cdot \frac{dT}{dx} = \frac{\frac{dY_j}{dT}}{\frac{dx}{dT}} = \frac{AY_j}{x - x_j^0}$$

and we have

$$\frac{Y'_j}{\sqrt{-1}\rho_j \exp(\sqrt{-1}\theta)} = \frac{Y'_j}{x'} = \frac{AY_j}{\rho_j \exp(\sqrt{-1}\theta)}$$

therefore $Y'_j = \sqrt{-1}AY_j$. Observe that the solution of $Y'_j = \sqrt{-1}AY_j$ such that $Y_j(0) = (y_j, z_j)$ is $Y_j(\theta) = \exp(\sqrt{-1}\theta A) \cdot Y_j(0)$. Therefore the holonomy is the

biholomorphism $f : \Sigma_j \rightarrow \Sigma_j$ defined by

$$\begin{aligned}
 f(y_j, z_j) &= Y_j(2\pi) = \exp \left\{ 2\pi\sqrt{-1} \begin{pmatrix} \lambda & \frac{\nu}{2\pi\sqrt{-1}\mu} \\ 0 & \lambda \end{pmatrix} \right\} \cdot \begin{pmatrix} y_j \\ z_j \end{pmatrix} \\
 &= \exp \left\{ \begin{pmatrix} 2\pi\sqrt{-1}\lambda & 0 \\ 0 & 2\pi\sqrt{-1}\lambda \end{pmatrix} + \begin{pmatrix} 0 & \frac{\nu}{\mu} \\ 0 & 0 \end{pmatrix} \right\} \cdot \begin{pmatrix} cy_j \\ z_j \end{pmatrix} \\
 &= \exp \left\{ \begin{pmatrix} 2\pi\sqrt{-1}\lambda & 0 \\ 0 & 2\pi\sqrt{-1}\lambda \end{pmatrix} \right\} \cdot \exp \left\{ \begin{pmatrix} 0 & \frac{\nu}{\mu} \\ 0 & 0 \end{pmatrix} \right\} \begin{pmatrix} y_j \\ z_j \end{pmatrix} \\
 &= \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\nu}{\mu} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} cy_j \\ z_j \end{pmatrix} = \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} 1 & \frac{\nu}{\mu} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_j \\ z_j \end{pmatrix} \\
 &= \begin{pmatrix} \mu & \nu \\ 0 & \mu \end{pmatrix} \cdot \begin{pmatrix} cy_j \\ z_j \end{pmatrix} = (\mu y_j + \nu z_j, \mu z_j)
 \end{aligned}$$

and this proves the claim.

- In case (a), we consider the foliation $\mathcal{F}_j$ on $D_j \times \mathbb{C}^2$ given by

$$(3.4) \quad X_j(x, y_j, z_j) = (x - x_j^0) \frac{\partial}{\partial x} + \left(\frac{\mu}{2\pi\sqrt{-1}} z_j - \frac{\mu^2}{4\pi\sqrt{-1}} \right) \frac{\partial}{\partial y_j} + \frac{\mu}{2\pi\sqrt{-1}} \frac{\partial}{\partial z_j}.$$

Claim 3.4. The holonomy transformation of $\mathcal{F}_j$ associated to Σ_j and γ_j is of the form $(y_j, z_j) \mapsto (y_j + \mu z_j, z_j + \mu)$.

In fact, let $\gamma_j(\theta) = (x_j^0 + \rho_j \exp(\sqrt{-1}\theta), 0, 0)$, $0 \leq \theta \leq 2\pi$ be a curve where $0 < \rho_j < \rho$, let $\Sigma_j = \{x_j^0 + \rho_j\} \times \mathbb{C}^2$ be a local transverse section and let $p_j = (x_j^0 + \rho_j, 0, 0) \in \Sigma_j$. Suppose $\pi_1 : D_j \times \mathbb{C}^2 \rightarrow D_j$, $\pi_1(x, y, z) = x$. Notice that the fibers $\pi_1^{-1}(x)$, $x \neq x_j^0$ are transverse to $\mathcal{F}_j$. Let $q = (x_j^0 + \rho_j, y_j, z_j) \in \Sigma_j$ and let $\gamma_q(\theta) = (x(\theta), y_j(\theta), z_j(\theta))$ be the lifting of γ_j by π_1 with base point q . Therefore $x'(\theta) = \pi_1(\gamma'_q(\theta)) = \pi_1(\gamma'_j(\theta)) = \sqrt{-1}\rho_j \exp(\sqrt{-1}\theta)$, and, if $Y_j = (y_j, z_j) \in \mathbb{C}^2$, then $\frac{Y'_j}{x'} = \frac{Y'_j}{\sqrt{-1}\rho_j \exp(\sqrt{-1}\theta)}$. On the other hand, by equation (3.4) we have $\frac{dx}{dT} = x - x_j^0$ and

$$\frac{dY_j}{dT} = \left[\begin{pmatrix} 0 & \frac{\mu}{2\pi\sqrt{-1}} \\ 0 & 0 \end{pmatrix} \right] \cdot \begin{pmatrix} cy_j \\ z_j \end{pmatrix} + \left[\begin{pmatrix} c - \frac{\mu^2}{4\pi\sqrt{-1}} \\ \frac{\mu}{2\pi\sqrt{-1}} \end{pmatrix} \right] = AY_j + B,$$

so we obtain

$$\frac{dY_j}{dx} = \frac{dY_j}{dT} \cdot \frac{dT}{dx} = \frac{\frac{dY_j}{dT}}{\frac{dx}{dT}} = \frac{AY_j + B}{x - x_j^0}$$

and we have

$$\frac{Y'_j}{\sqrt{-1}\rho_j \exp(\sqrt{-1}\theta)} = \frac{Y'_j}{x'} = \frac{AY_j + B}{\rho_j \exp(\sqrt{-1}\theta)}$$

therefore $Y'_j = \sqrt{-1}AY_j + \sqrt{-1}B$. Observe that the solution of

$$Y'_j = \sqrt{-1}AY_j + \sqrt{-1}B$$

with $Y_j(0) = (y_j, z_j)$ is

$$Y_j(\theta) = \exp(\sqrt{-1}\theta A) \cdot \left[Y_j(0) + \int_0^\theta \exp(\sqrt{-1}sA) \cdot B(s) ds \right],$$

that is,

$$Y_j(\theta) = \left(-\frac{\mu^2}{4\pi}\theta - \frac{1}{2}\frac{\mu^2}{4\pi^2}\theta^2 + y_j + \frac{\mu^2}{4\pi^2}\theta^2 + \frac{\mu}{2\pi}\theta z_j, \frac{\mu}{2\pi}\theta + z_j \right).$$

Therefore the holonomy is the biholomorphism $f : \Sigma_j \rightarrow \Sigma_j$ defined by

$$f(y_j, z_j) = Y_j(2\pi) = (y_j + \mu z_j, z_j + \mu)$$

and this proves the claim.

Let us glue together the foliation on $A_j \times \mathbb{C}^2$ and the foliations on $D_j \times \mathbb{C}^2$. First we consider the coordinate systems $(x, y_j, z_j) \in D_j \times \mathbb{CP}(2)$ and $(x, \tilde{y}_j, \tilde{z}_j) \in A_j \times \mathbb{CP}(2)$. Suppose f_j is of type P3 or SL1; then we are in case (c) or case (d). Observe that $A_j \cap D_j$ is simply connected and $x_j^0 \notin A_j \cap D_j$, and then we consider the coordinate system $(x, \bar{y}_j, \bar{z}_j)$ in $(A_j \cap D_j) \times \mathbb{C}^2$ where

$$(3.5) \quad \bar{y}_j = y_j \exp(-\alpha'_j \log(\frac{x - x_j^0}{\frac{\rho}{2}})), \quad \bar{z}_j = z_j \exp(-\alpha''_j \log(\frac{x - x_j^0}{\frac{\rho}{2}})).$$

Here $\log$ is the branch of the logarithm in $\mathbb{C} \setminus \{x + \sqrt{-1}y; x \leq 0\}$ such that $\log(1) = 0$. Observe that $x'_j = x_j^0 + \frac{\rho}{2}$ implies that $\bar{y}_j(x'_j, y_j) = y_j$, $\bar{z}_j(x'_j, z_j) = z_j$ and $\bar{y}_j(x, 0) = 0$, $\bar{z}_j(x, 0) = 0$. Then the leaves of the foliation $\mathcal{F}_j$ on $D_j \times \mathbb{C}^2$ restricted to $(A_j \cap D_j) \times \mathbb{C}^2$ are $(\bar{y}_j, \bar{z}_j) = \text{constant}$. Let us identify the point $(x, \tilde{y}_j, \tilde{z}_j) \in (A_j \cap D_j) \times \mathbb{C}^2 \subset A_j \times \mathbb{C}^2$ with the point $(x, y_j, z_j) \in (A_j \cap D_j) \times \mathbb{C}^2 \subset D_j \times \mathbb{C}^2$, where

$$(3.6) \quad y_j = \tilde{y}_j \exp(\alpha'_j \log(\frac{x - x_j^0}{\frac{\rho}{2}})), \quad z_j = \tilde{z}_j \exp(\alpha''_j \log(\frac{x - x_j^0}{\frac{\rho}{2}})).$$

Notice that (3.6) is equivalent to identifying $(x, \tilde{y}_j, \tilde{z}_j)$ with (x, y_j, z_j) and then, with (3.6) we are gluing together plaques of the horizontal foliation $\tilde{\mathcal{F}}_j$ on $(A_j \cap D_j) \times \mathbb{C}^2 \subset A_j \times \mathbb{C}^2$ with plaques of $\mathcal{F}_j$ on $(A_j \cap D_j) \times \mathbb{C}^2 \subset D_j \times \mathbb{C}^2$. Observe that this identification sends the fiber $\{x = c\} \subset A_j \times \mathbb{C}^2$, $c \in A_j \cap D_j$, in the fiber $\{x = c\} \subset D_j \times \mathbb{C}^2$, and the holonomy of the curve $\beta_j = \alpha_j * \gamma_j * \alpha_j^{-1}$ in the section $\Sigma''_j = \{x''_j\} \times \mathbb{C}^2 \subset A_j \times \mathbb{C}^2$ with respect to the foliation obtained by gluing together the $\tilde{\mathcal{F}}_j$ and $\mathcal{F}_j$ is of the form $(\tilde{y}_j, \tilde{z}_j) \mapsto (\lambda'_j \tilde{y}_j, \lambda''_j \tilde{z}_j)$. Let us call this foliation $\tilde{\mathcal{F}}_j$ also.

By analogy with these ideas, if f_j is of type $P1SL1$, then we are in case (e) and we consider the coordinate system $(x, \bar{y}_j, \bar{z}_j)$ in $(A_j \cap D_j) \times \mathbb{C}^2$ where

$$(3.7) \quad \bar{y}_j = y_j - \frac{\nu}{2\pi\sqrt{-1}} \log\left(\frac{x - x_j}{\frac{\rho}{2}}\right) z_j, \quad \bar{z}_j = z_j,$$

i.e.,

$$\begin{pmatrix} c\bar{y}_j \\ \bar{z}_j \end{pmatrix} = \begin{bmatrix} 1 & -\frac{\nu}{2\pi\sqrt{-1}} \log(\frac{x - x_j}{\frac{\rho}{2}}) \\ 0 & 1 \end{bmatrix} \cdot \begin{pmatrix} cy_j \\ z_j \end{pmatrix}.$$

The leaves of the foliation $\mathcal{F}_j$ on $D_j \times \mathbb{C}^2$ restricted to $(A_j \cap D_j) \times \mathbb{C}^2$ are $(\bar{y}_j, \bar{z}_j) = \text{the constant vector again}$. We identify the point $(x, \tilde{y}_j, \tilde{z}_j) \in (A_j \cap D_j) \times \mathbb{C}^2 \subset A_j \times \mathbb{C}^2$ with the point $(x, y_j, z_j) \in (A_j \cap D_j) \times \mathbb{C}^2 \subset D_j \times \mathbb{C}^2$, where

$$(3.8) \quad y_j = \tilde{y}_j + \frac{\nu}{2\pi\sqrt{-1}} \log\left(\frac{x - x_j}{\frac{\rho}{2}}\right) \tilde{z}_j, \quad z_j = \tilde{z}_j,$$

i.e.,

$$\begin{bmatrix} \bar{y}_j \\ \bar{z}_j \end{bmatrix} = \begin{bmatrix} 1 & \frac{\nu}{2\pi\sqrt{-1}} \log(\frac{x - x_j}{\frac{\rho}{2}}) \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} y_j \\ z_j \end{bmatrix}.$$

Notice that (3.8) is equivalent to identify $(x, \tilde{y}_j, \tilde{z}_j)$ with (x, y_j, z_j) .

And on the one hand, if f_j is of type P2, we are in case (b) and we change the identifications (3.5) and (3.6) by

$$(3.9) \quad \begin{aligned} \bar{y}_j &= y_j \exp(-\lambda \log(\frac{x-x_j^0}{\frac{\rho}{2}})) - z_j \frac{\nu}{2\pi\sqrt{-1}\mu} \log(\frac{x-x_j^0}{\frac{\rho}{2}}) \exp(-\lambda \log(\frac{x-x_j^0}{\frac{\rho}{2}})), \\ \bar{z}_j &= z_j \exp(-\lambda \log(\frac{x-x_j^0}{\frac{\rho}{2}})), \end{aligned}$$

i.e.,

$$\begin{bmatrix} \bar{y}_j \\ \bar{z}_j \end{bmatrix} = \begin{bmatrix} \exp(-\lambda \log(\frac{x-x_j^0}{\frac{\rho}{2}})) & -\exp(-\lambda \log(\frac{x-x_j^0}{\frac{\rho}{2}})) \frac{\nu}{2\pi\sqrt{-1}} \log(\frac{x-x_j^0}{\frac{\rho}{2}}) \\ 0 & \exp(-\lambda \log(\frac{x-x_j^0}{\frac{\rho}{2}})) \end{bmatrix} \cdot \begin{bmatrix} y_j \\ z_j \end{bmatrix}$$

and

$$(3.11) \quad y_j = \tilde{y}_j \exp(\lambda \log(\frac{x-x_j^0}{\frac{\rho}{2}})) + z_j \frac{\nu}{2\pi\sqrt{-1}\mu} \log(\frac{x-x_j^0}{\frac{\rho}{2}}) \exp(\lambda \log(\frac{x-x_j^0}{\frac{\rho}{2}})),$$

$$(3.12) \quad z_j = \tilde{z}_j \exp(\lambda \log(\frac{x-x_j^0}{\frac{\rho}{2}})),$$

i.e.,

$$\begin{bmatrix} y_j \\ z_j \end{bmatrix} = \begin{bmatrix} \exp(\lambda \log(\frac{x-x_j^0}{\frac{\rho}{2}})) & \exp(\lambda \log(\frac{x-x_j^0}{\frac{\rho}{2}})) \frac{\nu}{2\pi\sqrt{-1}} \log(\frac{x-x_j^0}{\frac{\rho}{2}}) \\ 0 & \exp(\lambda \log(\frac{x-x_j^0}{\frac{\rho}{2}})) \end{bmatrix} \cdot \begin{bmatrix} \tilde{y}_j \\ \tilde{z}_j \end{bmatrix}$$

and we repeat these ideas. On the other hand, if f_j is of type P1 we are in case (a) and the identifications are

$$(3.13) \quad \bar{y}_j = y_j + (z_j \frac{\mu}{2\pi\sqrt{-1}} + \frac{1}{2} \frac{\mu^2}{(2\pi\sqrt{-1})^2} \log(\frac{x-x_j^0}{\frac{\rho}{2}}) + \frac{\mu^2}{4\pi\sqrt{-1}}) \cdot \log(\frac{x-x_j^0}{\frac{\rho}{2}}),$$

$$(3.14) \quad \bar{z}_j = z_j + \frac{\mu}{2\pi\sqrt{-1}} \log(\frac{x-x_j^0}{\frac{\rho}{2}}),$$

i.e.,

$$\begin{aligned} \begin{bmatrix} \bar{y}_j \\ \bar{z}_j \end{bmatrix} &= \begin{bmatrix} 1 & -\frac{\mu}{2\pi\sqrt{-1}} \log(\frac{x-x_j^0}{\frac{\rho}{2}}) \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} y_j \\ z_j \end{bmatrix} \\ &+ \begin{bmatrix} (\frac{1}{2} \frac{\mu^2}{(2\pi\sqrt{-1})^2} \log(\frac{x-x_j^0}{\frac{\rho}{2}}) + \frac{\mu^2}{4\pi\sqrt{-1}}) \cdot \log(\frac{x-x_j^0}{\frac{\rho}{2}}) \\ \frac{\mu}{2\pi\sqrt{-1}} \log(\frac{x-x_j^0}{\frac{\rho}{2}}) \end{bmatrix} \end{aligned}$$

and

$$(3.15) \quad y_j = \tilde{y}_j + (\tilde{z}_j \frac{\mu}{2\pi\sqrt{-1}} + \frac{1}{2} \frac{\mu^2}{(2\pi\sqrt{-1})^2} \log(\frac{x-x_j^0}{\frac{\rho}{2}}) - \frac{\mu^2}{4\pi\sqrt{-1}}) \cdot \log(\frac{x-x_j^0}{\frac{\rho}{2}}),$$

$$(3.16) \quad z_j = \tilde{z}_j + \frac{\mu}{2\pi\sqrt{-1}} \log(\frac{x-x_j^0}{\frac{\rho}{2}}),$$

i.e.,

$$\begin{bmatrix} y_j \\ z_j \end{bmatrix} = \begin{bmatrix} 1 & \frac{\mu}{2\pi\sqrt{-1}} \log(\frac{x-x_j^0}{\frac{\mu}{2}}) \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \tilde{y}_j \\ \tilde{z}_j \end{bmatrix} + \begin{bmatrix} (\frac{1}{2} \frac{\mu^2}{(2\pi\sqrt{-1})^2} \log(\frac{x-x_j^0}{\frac{\mu}{2}}) - \frac{\mu^2}{4\pi\sqrt{-1}}) \cdot \log(\frac{x-x_j^0}{\frac{\mu}{2}}) \\ \frac{\mu}{2\pi\sqrt{-1}} \log(\frac{x-x_j^0}{\frac{\mu}{2}}) \end{bmatrix}.$$

Now let us glue together the new foliation $\tilde{\mathcal{F}}_j$ on $(A_j \cup D_j) \times \mathbb{C}^2$ with the foliation $\mathcal{F}_0$ on $(A_j \cap D_0) \times \mathbb{C}^2$. First, if f_0 is of type P3 or SL1, we consider the coordinate system $(x, \bar{y}_0, \bar{z}_0)$ in $(A_j \cap D_0) \times \mathbb{C}^2$ where

$$(3.17) \quad \bar{y}_0 = y_0 \exp(-\alpha'_0 \log(\frac{x}{x_j''})), \quad \bar{z}_0 = z_0 \exp(-\alpha''_0 \log(\frac{x}{x_j''})).$$

Observe that the leaves of the foliation $\tilde{\mathcal{F}}_j$ on $D_0 \times \mathbb{C}^2$ restricted to $(A_j \cap D_0) \times \mathbb{C}^2$ are $(\bar{y}_0, \bar{z}_0) = \text{the constant vector}$. We identify the points $(x, \tilde{y}_j, \tilde{z}_j) \in (A_j \cap D_0) \times \mathbb{C}^2 \subset A_j \times \mathbb{C}^2$ with $(x, y_0, z_0) \in (A_j \cap D_0) \times \mathbb{C}^2$ by

$$(3.18) \quad y_0 = \tilde{y}_j \exp(\alpha'_0 \log(\frac{x}{x_j''})), \quad z_0 = \tilde{z}_j \exp(\alpha''_0 \log(\frac{x}{x_j''})),$$

where $\lambda'_0 = \exp(2\pi\sqrt{-1}\alpha'_0)$ and $\lambda''_0 = \exp(2\pi\sqrt{-1}\alpha''_0)$.

Notice that identification (3.18) glues together plaques of $\tilde{\mathcal{F}}_j$ on $(A_j \cup D_j) \times \mathbb{C}^2$ with plaques of $\mathcal{F}_0$ and this defines a new foliation such that the holonomy of the curve β_j in the section $\{x_j''\} \times \mathbb{C}^2 \subset D_0 \times \mathbb{C}^2$ is given by $(y_0, z_0) \mapsto (\lambda'_j y_0, \lambda''_j z_0)$.

By analogy with this case, if f_0 is of type P1SL1, we consider the coordinate system $(x, \bar{y}_0, \bar{z}_0)$ in $(A_j \cap D_0) \times \mathbb{C}^2$ where

$$(3.19) \quad \bar{y}_0 = y_0 - \frac{\nu}{2\pi\sqrt{-1}} \log\left(\frac{x}{x_j''}\right) z_0, \quad \bar{z}_0 = z_0.$$

We identify the point $(x, \tilde{y}_j, \tilde{z}_j) \in (A_j \cap D_0) \times \mathbb{C}^2 \subset A_j \times \mathbb{C}^2$ with the point $(x, y_0, z_0) \in (A_j \cap D_0) \times \mathbb{C}^2 \subset D_0 \times \mathbb{C}^2$ by

$$(3.20) \quad y_0 = \tilde{y}_j + \frac{\nu}{2\pi\sqrt{-1}} \log\left(\frac{x}{x_j''}\right) \tilde{z}_j, \quad z_0 = \tilde{z}_j,$$

However, if f_0 is of type P2, then we consider the coordinate system $(x, \bar{y}_0, \bar{z}_0)$ in $(A_j \cap D_0) \times \mathbb{C}^2$ where

$$(3.21) \quad \bar{y}_0 = y_0 \exp(-\lambda \log(\frac{x}{x_j''})) - z_0 \frac{\nu}{2\pi\sqrt{-1}\mu} \log(\frac{x}{x_j''}) \exp(-\lambda \log(\frac{x}{x_j''})),$$

$$(3.22) \quad \bar{z}_0 = z_0 \exp(-\lambda \log(\frac{x}{x_j''})).$$

We identify the point $(x, \tilde{y}_j, \tilde{z}_j) \in (A_j \cap D_0) \times \mathbb{C}^2 \subset A_j \times \mathbb{C}^2$ with the point $(x, y_0, z_0) \in (A_j \cap D_0) \times \mathbb{C}^2$ by

$$(3.23) \quad y_0 = \tilde{y}_j \exp(\lambda \log(\frac{x}{x_j''})) + \tilde{z}_j \frac{\nu}{2\pi\sqrt{-1}\mu} \log(\frac{x}{x_j''}) \exp(\lambda \log(\frac{x}{x_j''})),$$

$$(3.24) \quad z_0 = \tilde{z}_j \exp(\lambda \log(\frac{x}{x_j''})).$$

On the other hand, if f_0 is of type P1, then we consider the coordinate system $(x, \bar{y}_0, \bar{z}_0)$ in $(A_j \cap D_0) \times \mathbb{C}^2$ where

$$(3.25) \quad \bar{y}_0 = y_0 + (z_0 \frac{\mu}{2\pi\sqrt{-1}} + \frac{1}{2} \frac{\mu^2}{(2\pi\sqrt{-1})^2} \log(\frac{x}{x_j''}) + \frac{\mu^2}{4\pi\sqrt{-1}}) \cdot \log(\frac{x}{x_j''}),$$

$$(3.26) \quad \bar{z}_0 = z_0 + \frac{\mu}{2\pi\sqrt{-1}} \log(\frac{x}{x_j''}),$$

and we identify the point $(x, \tilde{y}_j, \tilde{z}_j) \in (A_j \cap D_0) \times \mathbb{C}^2 \subset A_j \times \mathbb{C}^2$ with the point $(x, y_0, z_0) \in (A_j \cap D_0) \times \mathbb{C}^2$ by

$$(3.27) \quad y_0 = \tilde{y}_j + (\tilde{z}_j \frac{\mu}{2\pi\sqrt{-1}} + \frac{1}{2} \frac{\mu^2}{(2\pi\sqrt{-1})^2} \log(\frac{x}{x_j''}) - \frac{\mu^2}{4\pi\sqrt{-1}}) \cdot \log(\frac{x}{x_j''}),$$

$$(3.28) \quad z_0 = \tilde{z}_j + \frac{\mu}{2\pi\sqrt{-1}} \log(\frac{x}{x_j''}).$$

Now suppose $\gamma_0(\theta) = \frac{\rho}{2} \exp(\sqrt{-1}\theta)$, $0 \leq \theta \leq 2\pi$, and for every $j = 1, \dots, k$ let μ_j be the segment of γ_0 between x_j'' and $\frac{\rho}{2}$ in the positive sense. Let

$$\delta_j = \mu_j * \beta_j * \mu_j^{-1} = \mu_j * \alpha_j * \gamma_j * \alpha_j^{-1} * \mu_j^{-1},$$

where $\gamma_j(\theta) = \frac{\rho}{2} \exp(\sqrt{-1}\theta) + x_j$, $\theta \in [0, 2\pi]$ and $\Sigma_0 = \{\frac{\rho}{2}\} \times \mathbb{C}^2$. The holonomy of the curve δ_j in Σ_0 is given by conditions in item (3).

Notice that we have a foliation on $(\bigcup A_k \cup \bigcup D_j) \times \mathbb{C}P(2)$ such that the holonomy of the curve β_j is a biholomorphism of $\mathbb{C}P(2)$.

Now let $\widetilde{M}$ be a complex manifold obtained by gluing together all the foliations $\widetilde{\mathcal{F}}_j$. Let $\widetilde{\mathcal{F}}$ be the foliation in $\widetilde{M}$ obtained at the end of the process. Then the holonomy of the leaf $U = (\bigcup_{i=0}^r A_i) \cup (\bigcup_{j=0}^r D_j)$ in Σ_0 is generated by $f_1, \dots, f_r$. Notice that $\widetilde{M}$ admits the vertical foliation $\{x = \text{constant}\}$ and it cuts U at a single point; hence, we can define a projection $\tilde{p} : \widetilde{M} \rightarrow U$ such that $\tilde{p}^{-1}(x)$ is the leaf of this new foliation. Observe that this new foliation is transverse to $\widetilde{\mathcal{F}}$ in $\widetilde{M} \setminus \{x = x_0^0, \dots, x = x_r^0\}$.

Now let us glue together the foliation $\widetilde{\mathcal{F}}$ and the foliation on $V \times \mathbb{C}P(2)$. First, we consider the annulus $A = T \cap U$, if δ is a closed curve in A which generates the homotopy of A , then the holonomy of δ with respect to $\widetilde{\mathcal{F}}$ in some transversal section is trivial, because δ is homotopic to the curve $\delta_1 * \dots * \delta_r * \gamma_0$ in $U \setminus \bigcup_{j=0}^k \{x_j^0\}$ and the holonomy of this is the identity. Then we obtain that the restricted foliation $\widetilde{\mathcal{F}}|_{\tilde{p}^{-1}(A)}$ is a product foliation; that is, there is a biholomorphism φ of the same neighborhood W of $\tilde{p}^{-1}(A)$ onto $A \times \Delta$, where $\Delta \subset \mathbb{C}^2$ is a polydisc such that φ sends leaves of $\widetilde{\mathcal{F}}|_W$ onto leaves of trivial foliation. This map φ can be chosen such that $\varphi(p^{-1}(x) \cap W) = \{x\} \times \Delta$. In order to complete the construction of a complex 3-manifold M^3 in the conditions of the theorem it is sufficient to glue together the foliation $\widetilde{\mathcal{F}}$ on $\widetilde{M}$ and the trivial foliation on $V \times \mathbb{C}P(2)$ by φ . This proves the first part of the statement.

Notice that if we assume that f_j ($j = 1, \dots, r$) and f_0 are all of type P3 or SL1, then the vector field X_j in the affine coordinate $(x, u_j = \frac{1}{y_j}, v_j = \frac{z_j}{y_j}) \in D_j \times \mathbb{C}^2$ is given by

$$X_j(x, u_j, v_j) = (x - x_j^0) \frac{\partial}{\partial x} - \alpha_j' u_j \frac{\partial}{\partial u_j} + (\alpha_j'' - \alpha_j') v_j \frac{\partial}{\partial v_j},$$

and, in affine coordinates $(x, s_j = \frac{y_j}{z_j}, t_j = \frac{1}{z_j})$,

$$X_j(x, s_j, t_j) = (x - x_j^0) \frac{\partial}{\partial x} + (\alpha'_j - \alpha''_j) s_j \frac{\partial}{\partial s_j} - \alpha''_j t_j \frac{\partial}{\partial t_j}.$$

We take the coordinate systems $(x, \bar{u}_j, \bar{v}_j)$ and $(x, \bar{s}_j, \bar{t}_j)$ in $(A_j \cap D_j) \times \mathbb{C}^2$ where

$$(3.29) \quad \bar{u}_j = u_j \exp(\alpha'_j \log(\frac{x - x_j^0}{\frac{\rho}{2}})), \quad \bar{v}_j = v_j \exp(-(\alpha''_j - \alpha'_j) \log(\frac{x - x_j^0}{\frac{\rho}{2}})),$$

$$(3.30) \quad \bar{s}_j = s_j \exp(-(\alpha'_j - \alpha''_j) \log(\frac{x - x_j^0}{\frac{\rho}{2}})), \quad \bar{t}_j = t_j \exp(\alpha''_j \log(\frac{x - x_j^0}{\frac{\rho}{2}})).$$

We identify the points $(x, \tilde{u}_j, \tilde{v}_j)$ and $(x, \tilde{s}_j, \tilde{t}_j)$ in $(A_j \cap D_j) \times \mathbb{C}^2 \subset A_j \times \mathbb{C}^2$ with the points (x, u_j, v_j) and (x, s_j, t_j) in $(A_j \cap D_j) \times \mathbb{C}^2 \subset D_j \times \mathbb{C}^2$, respectively, where

$$(3.31) \quad u_j = \tilde{u}_j \exp(-\alpha'_j \log(\frac{x - x_j^0}{\frac{\rho}{2}})), \quad v_j = \tilde{v}_j \exp((\alpha''_j - \alpha'_j) \log(\frac{x - x_j^0}{\frac{\rho}{2}})),$$

$$(3.32) \quad s_j = \tilde{s}_j \exp((\alpha'_j - \alpha''_j) \log(\frac{x - x_j^0}{\frac{\rho}{2}})), \quad t_j = \tilde{t}_j \exp(-\alpha''_j \log(\frac{x - x_j^0}{\frac{\rho}{2}})).$$

When we take the affine coordinates $(x, u_0 = \frac{1}{y_0}, v_0 = \frac{z_0}{y_0})$, $(x, s_0 = \frac{y_0}{z_0}, t_0 = \frac{1}{z_0}) \in D_0 \times \mathbb{C}^2$, we identify the points $(x, \tilde{u}_j, \tilde{v}_j)$, $(x, \tilde{s}_j, \tilde{t}_j) \in (A_j \cap D_0) \times \mathbb{C}^2 \subset A_j \cap \mathbb{C}^2$ with the points (x, u_0, v_0) , $(x, s_0, t_0) \in (A_j \cap D_0) \times \mathbb{C}^2 \subset D_0 \times \mathbb{C}^2$ respectively, where

$$(3.33) \quad u_0 = \tilde{u}_j \exp(-\alpha'_0 \log(\frac{x}{x_j''})), \quad v_0 = \tilde{v}_j \exp((\alpha''_0 - \alpha'_0) \log(\frac{x}{x_j''})),$$

$$(3.34) \quad s_0 = \tilde{s}_j \exp((\alpha'_0 - \alpha''_0) \log(\frac{x}{x_j''})), \quad t_0 = \tilde{t}_j \exp(-\alpha''_0 \log(\frac{x}{x_j''})).$$

This proves the existence of a compact three-manifold M^3 with a bundle structure with fiber $\mathbb{C}P(2)$ over $\mathbb{C}P(1)$, and a (Riccati) holomorphic foliation $\mathcal{F}$ on M^3 , transverse to almost every fiber, with invariant fibers $\pi^{-1}(x_0), \dots, \pi^{-1}(x_r)$, and having global holonomy conjugate to the group $G = \langle f_1, \dots, f_r \rangle$. It remains to observe that M^3 is biholomorphic to a projective manifold because of the bundle structure with fiber $\mathbb{C}P(2)$ over $\mathbb{C}P(1)$. Finally, the foliation $\mathcal{F}$ is conjugate to an algebraic foliation on the projective model because any holomorphic foliation on a projective manifold is algebraic. $\square$

Question 3.5. Assume that f_j ($j = 1, \dots, r$) and f_0 are all of type P3 or SL1. Is it true that from the above construction we have $M^3 = \overline{\mathbb{C}} \times \mathbb{C}P(2)$ and η is trivial?

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