SHAPES OF TETRAHEDRA WITH PRESCRIBED CONE ANGLES

AHTZIRI GONZÁLEZ AND JORGE L. LÓPEZ-LÓPEZ

ABSTRACT. Given real numbers $4\pi > \theta_0 \geq \theta_1 \geq \theta_2 \geq \theta_3 > 0$ so that $\sum_{j=0}^3 \theta_j = 4\pi$, we provide a detailed description of the space of flat metrics on the 2-sphere with 4 conical points of cone angles $\theta_0, \theta_1, \theta_2, \theta_3$, endowed with a geometric structure arising from the area function.

1. INTRODUCTION

Using different techniques, Deligne and Mostow [2] and Thurston [9] studied the space of orientation-preserving similarity classes of flat metrics on the sphere S^2 with n conical singularities of prescribed conical angles (see also [7], [10], and [3]). A geometric structure was imposed on this space using different versions of the area function viewed as a Hermitian form.

The simplest nontrivial examples of such spaces considered in [2] and [9] are the spaces $\mathcal{T}(\theta_1, \theta_2, \theta_3)$ of orientation-preserving similarity classes of flat metrics on the sphere S^2 with 4 conical singularities of prescribed conical angles $4\pi > \theta_0, \theta_1, \theta_2, \theta_3 > 0$ so that $\sum_{j=0}^3 \theta_j = 4\pi$, but it is difficult to obtain a detailed description of these examples from the cited papers. Our main goal herein is to understand these examples quite concretely. Moreover, [9] does not address the case $\theta_0 \geq 2\pi$, which is merely sketched in [2, Section 13]. In this paper, all the possibilities for θ_0 are explored. We also provide a description of the quotient of $\mathcal{T}(\theta_1, \theta_2, \theta_3)$ upon the action of relabeling conical singularities with the same cone angles. Dealing the particular case of the sphere S^2 with 4 conical singularities allows us to give simple proofs.

While preparing the manuscript, the authors were informed by Muciño-Raymundo that a description of the space of similarity classes of flat metrics on S^2 with 4 conical singularities of cone angles $\theta_0 = \theta_1 = 4\pi/3$ and $\theta_2 = \theta_3 = 2\pi/3$ was announced by Cruz-Cota in a talk in July, 2010 (see also [1]).

1.1. Terminology, notation and overview. We define a *tetrahedron* as a 2-sphere with a flat metric having four conical singularities, the so-called vertices. Let $\theta_k > 0$ ($k = 0, 1, 2, 3$) be the cone angles at the vertices. The cone angles must satisfy the so-called Gauss-Bonnet condition, namely $\theta_0 + \theta_1 + \theta_2 + \theta_3 = 4\pi$ (see [4], for example).

Received by the editors December 7, 2010.

2010 *Mathematics Subject Classification.* Primary 51M20; Secondary 58D17, 51M10, 51M25.

The study was partially supported by funding from the UMSNH (by means of a project of the CIC) and the SEP (by means of the Red Temática de Colaboración “Álgebra, topología y análisis”).

©2011 American Mathematical Society
 Reverts to public domain 28 years from publication

A hexagon in the plane $\mathbb{C}$ is determined by a point $(z_1, \dots, z_6) \in \mathbb{C}^6$ where $z_1, \dots, z_6$ denote its consecutive vertices. Notice that degenerated and self-crossing hexagons are allowed. A polygon is called *simple* if it satisfies two conditions: it does not have repeated vertices, and the interior of any edge does not intersect to another edge.

Let $\theta_1, \theta_2, \theta_3$ be real numbers in the interval $(0, 2\pi)$. We denote $V(\theta_1, \theta_2, \theta_3)$ the 3-dimensional subspace of $\mathbb{C}^6$ satisfying the constraint

$$\begin{aligned} z_1 - z_2 &= e^{i\theta_1}(z_3 - z_2), \\ z_3 - z_4 &= e^{i\theta_2}(z_5 - z_4), \\ z_5 - z_6 &= e^{i\theta_3}(z_1 - z_6); \end{aligned}$$

that is, any element $Z \in V(\theta_1, \theta_2, \theta_3)$ has the form

$$(1.1) \quad Z = \left(z_1, \frac{e^{i\theta_1}z_3 - z_1}{e^{i\theta_1} - 1}, z_3, \frac{e^{i\theta_2}z_5 - z_3}{e^{i\theta_2} - 1}, z_5, \frac{e^{i\theta_3}z_1 - z_5}{e^{i\theta_3} - 1} \right).$$

Given a hexagon $Z \in V(\theta_1, \theta_2, \theta_3)$ which is simple, a tetrahedron $\diamond(Z)$ is obtained by gluing the edges $[z_{2k+1}, z_{2k}]$ and $[z_{2k-1}, z_{2k}]$ with a rotation of angle θ_k around the point z_{2k} . The resulting tetrahedron $\diamond(Z)$ has cone points of cone angles $\theta_1, \theta_2, \theta_3, \theta_0 = 4\pi - (\theta_1 + \theta_2 + \theta_3)$. In Section 2 it is shown that any tetrahedron with angles $\theta_1, \theta_2, \theta_3, \theta_0 = 4\pi - (\theta_1 + \theta_2 + \theta_3)$ can be obtained by gluing the sides of a hexagon $Z \in V(\theta_1, \theta_2, \theta_3) \cup V(\theta_1, \theta_3, \theta_2)$ which is simple, where $V(\theta_1, \theta_3, \theta_2)$ denotes the 3-dimensional subspace of $\mathbb{C}^6$ satisfying the constraint

$$\begin{aligned} z_1 - z_2 &= e^{i\theta_1}(z_3 - z_2), \\ z_3 - z_4 &= e^{i\theta_3}(z_5 - z_4), \\ z_5 - z_6 &= e^{i\theta_2}(z_1 - z_6). \end{aligned}$$

In order to parametrize the space of orientation-preserving similarity classes of tetrahedra we must consider the space of hexagons up to orientation-preserving similarity; that is, two hexagons $(z_1, \dots, z_6)$ and $(w_1, \dots, w_6)$ are identified if and only if there is a complex affine transformation $f(z) = az + b$ of $\mathbb{C}$, $a \neq 0$, so that $f(z_k) = w_k$ for all $k = 1, \dots, 6$. Up to a translation, each class of hexagons has a representative $(z_1, \dots, z_6)$ such that $z_1 = 0$. Therefore the space of orientation-preserving similarity classes of hexagons coming from tetrahedra with cone angles $\theta_1, \theta_2, \theta_3$ is a subset of $\mathbb{P}(V_0(\theta_1, \theta_2, \theta_3)) \cup \mathbb{P}(V_0(\theta_1, \theta_3, \theta_2))$, where $\mathbb{P}(V_0(\theta_j, \theta_k, \theta_l))$ denotes the complex projectivization of the 2-dimensional vector space $V_0(\theta_j, \theta_k, \theta_l) = \{Z \in V(\theta_j, \theta_k, \theta_l) : z_1 = 0\}$.

Let $\mathcal{T}(\theta_1, \theta_2, \theta_3)$ be the space of orientation-preserving similarity classes of tetrahedra with cone angles $\theta_1, \theta_2, \theta_3, \theta_0 = 4\pi - (\theta_1 + \theta_2 + \theta_3)$ and labeled vertices. In Section 3 we shall construct triangular regions $\triangle(\theta_1, \theta_2, \theta_3)$ and $\triangle(\theta_1, \theta_3, \theta_2)$ contained in $\mathbb{P}(V_0(\theta_1, \theta_2, \theta_3))$ and $\mathbb{P}(V_0(\theta_1, \theta_3, \theta_2))$, respectively, and we shall see that $\mathcal{T}(\theta_1, \theta_2, \theta_3)$ is obtained by pasting both triangular regions along its boundary.

The *area Hermitian product*

$$(1.2) \quad \langle (z_1, \dots, z_6), (w_1, \dots, w_6) \rangle_{\mathcal{A}} = \frac{i}{4} \sum_{k=1}^6 (z_k \bar{w}_{k+1} - z_{k+1} \bar{w}_k)$$

determines a geometric structure on $\mathbb{P}(V_0(\theta_1, \theta_j, \theta_k))$, where $\{j, k\} = \{2, 3\}$. The real number $\langle Z, Z \rangle_{\mathcal{A}}$ is equal to the signed area of the hexagon Z . In Section 4 we shall prove that $\triangle(\theta_1, \theta_j, \theta_k)$, $\{j, k\} = \{2, 3\}$, is in fact an actual triangle

(i.e. its edges are geodesics) with respect to the geometric structure determined in $\mathbb{P}(V_0(\theta_1, \theta_j, \theta_k))$ by the area Hermitian product, and the edges of the pair of triangles $\triangle(\theta_1, \theta_2, \theta_3)$ and $\triangle(\theta_1, \theta_3, \theta_2)$ are glued by isometries of this geometric structure to obtain the space $\mathcal{T}(\theta_1, \theta_2, \theta_3)$. So we shall conclude that $\mathcal{T}(\theta_1, \theta_2, \theta_3)$ is a geometric sphere with exactly 3 singularities (Theorem 13). When at least two cone angles are equal, vertices with equal cone angles can be interchanged by an orientation-preserving similarity. We also consider the quotient of the singular geometric sphere $\mathcal{T}(\theta_1, \theta_2, \theta_3)$ upon the action of relabeling vertices with the same cone angles, and we prove that the quotient is also a geometric sphere with exactly 3 singularities (Theorem 14).

Finally, we describe the labeling of vertices with whom we work. Given an oriented tetrahedron $\diamond$, let v_0 be a vertex with a cone angle which is greater than or equal to other three. For $p, q \in \diamond$, $[p, q]$ will be used henceforth to denote a fix length minimizing path from p to q (for the existence of length minimizing paths see [5, Section 1.12], for example). Choose length minimizing paths $[v_0, v_k]$, for $k = 1, 2, 3$, where v_1, v_2, v_3 are the other vertices labeled so that the oriented path $\gamma = [v_0, v_1] \cup [v_1, v_0] \cup [v_0, v_2] \cup [v_2, v_0] \cup [v_0, v_3] \cup [v_3, v_0]$ is compatible with the orientation of the open disc $\diamond \setminus \gamma$; that is, if ω is the unit tangent vector of γ , and $i\omega$ is the normal vector such that the pair $(\omega, i\omega)$ is positively oriented, then $i\omega$ is incoming to $\diamond \setminus \gamma$. There are three possibilities to make that marking of vertices, but we will pick v_1 to be a vertex v_k with a greater cone angle; that is, if θ_k denotes the cone angle at the vertex v_k for $k = 0, 1, 2, 3$, we have $4\pi > \theta_0 \geq \theta_1 \geq \theta_2, \theta_3 > 0$. In this way, the labels of vertices are uniquely determined when the cone angles are pairwise distinct.

Remark 1. $\theta_2 + \theta_3 = 2\pi$ implies $\theta_1 = \theta_2 = \theta_3 = \theta_4 = \pi$.

2. TETRAHEDRA CAN BE UNFOLDED IN THE PLANE AS SIMPLE HEXAGONS

Lemma 2. (1) $[v_0, v_j] \cap [v_0, v_k] = \{v_0\}$ for $j \neq k$.
 (2) Fix $k \in \{1, 2, 3\}$. At most two geodesic length minimizing paths from v_0 to v_k may exist. If two of such paths exist, then there is exactly one geodesic length minimizing path from v_0 to v_j for any $j \neq k$.

Proof. (1) Suppose $[v_0, v_j] \cap [v_0, v_k] \neq \{v_0\}$. Consideration of the local geometry around a vertex yields to observe that the geodesic path $[v_0, v_k]$ cannot visit a vertex v_l of $\diamond$, unless $l \in \{0, k\}$. We take $p \in [v_0, v_k]$ to be the supreme of the set $[v_0, v_j] \cap [v_0, v_k]$. We have $p \in [v_0, v_j] \cap [v_0, v_k]$, $p \neq v_0$ and p is not a vertex of $\diamond$. The length of $[v_0, v_k]$ between v_0 and p is equal to the length of $[v_0, v_j]$ between v_0 and p because $[v_0, v_j]$ and $[v_0, v_k]$ are length minimizing paths. Consideration of the local flat metric around p shows that there exist paths from v_0 to v_k whose lengths are less than the length of $[v_0, v_k]$; a contradiction.

(2) Let I and J be two geodesic length minimizing paths from v_0 to v_k . By part (1) of Lemma 2, the union $I \cup J$ is a topological circle which separates $\diamond$ into two components. It is not difficult to see that each homotopy class of paths in $\diamond \setminus \{v_0, v_1, v_2, v_3\}$ has at most one geodesic representative (see [5, Section 1.13]). The result follows easily from this property and part (1) of Lemma 2. $\square$

The purpose of this section is to prove that any tetrahedron can be unfolded in the plane:

Proposition 3. *There is an orientation-preserving isometric embedding $\Phi : \diamond \setminus \gamma \rightarrow \mathbb{C}$ of the disk $\diamond \setminus \gamma$ into $\mathbb{C}$, both equipped with its Euclidean metric. The boundary $\partial(\Phi(\diamond \setminus \gamma))$ is a simple hexagon.*

Such a map is called a developing map (see [8, Section 3.4]). We obtain Proposition 3 as an immediate consequence of Lemma 4.

Lemma 4. *Let $\tilde{\Phi} : \diamond \rightarrow \mathbb{C}$ be the multivalued extension of the developing map $\Phi : \diamond \setminus \gamma \rightarrow \mathbb{C}$ that arises by continuity. Then $\tilde{\Phi}(p) \neq \tilde{\Phi}(q)$ for any pair of different points $p, q \in \gamma$.*

Proof. We will obtain a contradiction supposing that $p, q \in \gamma$ are different points so that $\tilde{\Phi}(p) = \tilde{\Phi}(q)$. For such points p, q with $\tilde{\Phi}(p) = \tilde{\Phi}(q)$, we have $p \in [v_0, v_j]$ and $q \in [v_0, v_k]$ where $j \neq k$ because $\tilde{\Phi}$ maps injectively geodesics into geodesics. Let $\alpha : [0, 1] \rightarrow \diamond$ be a length minimizing path from p to q . The intersection of γ with the image $\alpha(0, 1)$ of the open interval $(0, 1)$ is nonempty because $\tilde{\Phi}$ maps injectively geodesics into geodesics. Moreover, $\gamma \cap \alpha(0, 1)$ consists exactly of one point because otherwise there is $l \in \{1, 2, 3\}$ such that $[v_0, v_l] \cap \alpha([0, 1])$ has two points, but this cannot happen by the proof of part (1) of Lemma 2. Therefore $\gamma \cap \alpha(0, 1) = \{r\}$ where $r \in [v_0, v_l]$ with $l \neq j, k$. Let us suppose $r = \alpha(t_0)$. Replacing a small enough portion $[\alpha(t_0 - \epsilon), \alpha(t_0 + \epsilon)]$ of $\alpha[0, 1]$ by a wedge shaped polygonal $[\alpha(t_0 - \epsilon), u] \cup [u, \alpha(t_0 + \epsilon)]$ with u close enough to the vertex v_l , we construct a polygonal path $\mathcal{L} = [p, \alpha(t_0 - \epsilon)] \cup [\alpha(t_0 - \epsilon), u] \cup [u, \alpha(t_0 + \epsilon)] \cup [\alpha(t_0 + \epsilon), q] \subset \diamond \setminus \gamma$ close to the polygonal path $[p, \alpha(t_0)] \cup [\alpha(t_0), v_l] \cup [v_l, \alpha(t_0)] \cup [\alpha(t_0), q] \subset [v_0, v_l] \cup \alpha[0, 1]$. Since $\tilde{\Phi}(p) = \tilde{\Phi}(q)$ we may assume that $\tilde{\Phi}(\mathcal{L})$ is a simple polygon in $\mathbb{C}$. Otherwise we apply the argument to another polygonal path $\mathcal{L}' \subset \mathcal{L}$ which $\tilde{\Phi}(\mathcal{L}')$ is a simple polygon. Hence, on the one hand, the sum of the exterior angles of $\tilde{\Phi}(\mathcal{L})$ should be equal to $\pm 2\pi$ by part (1) of Lemma 16. On the other hand, the sum of the exterior angles of $\tilde{\Phi}(\mathcal{L})$ should be equal to 0 by part (2) of Lemma 16 because Φ is angle-preserving and it injectively maps geodesics into geodesics. This is a contradiction. $\square$

3. DESCRIPTION OF THE SPACE $\mathcal{T}(\theta_1, \theta_2, \theta_3)$

By Lemma 4, we can view $\tilde{\Phi}(\gamma)$ as a simple hexagon. Let us enumerate the vertices of $\tilde{\Phi}(\gamma)$ so that $\tilde{\Phi}(v_1) = z_2$. Since $\tilde{\Phi}(\gamma)$ is simple and positively oriented, the vertex z_2 uniquely determines the enumeration of the other vertices of $\tilde{\Phi}(\gamma)$. Therefore $Z = (z_1, z_2, z_3, z_4, z_5, z_6) \in V(\theta_1, \theta_2, \theta_3) \cup V(\theta_1, \theta_3, \theta_2)$. Up to an orientation-preserving similarity, Z either is equivalent to

$$(3.1) \quad \left(0, \frac{e^{i\theta_1}}{e^{i\theta_1} - 1}, 1, \frac{e^{i\theta_2}\zeta - 1}{e^{i\theta_2} - 1}, \zeta, \frac{-\zeta}{e^{i\theta_3} - 1}\right) \in V_0(\theta_1, \theta_2, \theta_3)$$

when Z is of the form of equation (1.1), or it is equivalent to

$$\left(0, \frac{e^{i\theta_1}}{e^{i\theta_1} - 1}, 1, \frac{e^{i\theta_3}\zeta - 1}{e^{i\theta_3} - 1}, \zeta, \frac{-\zeta}{e^{i\theta_2} - 1}\right) \in V_0(\theta_1, \theta_3, \theta_2)$$

when $Z \in V(\theta_1, \theta_3, \theta_2)$.

Define $\triangle(\theta_1, \theta_2, \theta_3) \subset \mathbb{C}$ to be the triangular region

$$\triangle(\theta_1, \theta_2, \theta_3) = \left\{ \zeta = x + iy : \begin{array}{l} y \geq 0, \\ x \sin(\theta_2/2) + y \cos(\theta_2/2) \geq 0, \\ (1-x) \sin(\theta_3/2) + y \cos(\theta_3/2) \geq 0, \\ (x - \frac{1}{2})^2 + (y + \frac{1}{2} \cot(\theta_1/2))^2 \geq \frac{1}{4} \csc^2(\theta_1/2) \end{array} \right\}.$$

Here

- $x \sin(\theta_2/2) + y \cos(\theta_2/2) = 0$ is a line through $(0,0)$ and angle $-\theta_2/2$ with the positive x -axis.
- $(1-x) \sin(\theta_3/2) + y \cos(\theta_3/2) = 0$ is a line through $(1,0)$ and angle $\theta_3/2$ with the positive x -axis.
- $(x - \frac{1}{2})^2 + (y + \frac{1}{2} \cot(\theta_1/2))^2 = \frac{1}{4} \csc^2(\theta_1/2)$ is a circle through $(0,0)$ and $(1,0)$.

Similarly, we define

$$\triangle(\theta_1, \theta_3, \theta_2) = \left\{ \zeta = x + iy : \begin{array}{l} y \geq 0, \\ x \sin(\theta_3/2) + y \cos(\theta_3/2) \geq 0, \\ (1-x) \sin(\theta_2/2) + y \cos(\theta_2/2) \geq 0, \\ (x - \frac{1}{2})^2 + (y + \frac{1}{2} \cot(\theta_1/2))^2 \geq \frac{1}{4} \csc^2(\theta_1/2) \end{array} \right\}.$$

- Lemma 5.** (1) *Each point on the boundary of $\triangle(\theta_1, \theta_2, \theta_3)$, or on the boundary of $\triangle(\theta_1, \theta_3, \theta_2)$, encodes a tetrahedron $\diamond$ with two shortest paths from v_0 to another vertex.*
- (2) *For each point on the boundary of $\triangle(\theta_1, \theta_2, \theta_3)$ there is exactly one point on the boundary of $\triangle(\theta_1, \theta_3, \theta_2)$ encoding the same tetrahedron.*

Proof. (1) In the hexagon $\tilde{\Phi}(\gamma)$, the condition that a tetrahedron $\diamond$ has two shortest paths from v_0 to another vertex means one vertex z_{2k} has the same distance from z_1, z_3, z_5 . Two of the segments $[z_1, z_{2k}]$, $[z_3, z_{2k}]$, and $[z_5, z_{2k}]$ are edges of the hexagon $\tilde{\Phi}(\gamma)$ and the other segment is interior to the hexagon $\tilde{\Phi}(\gamma)$.

Claim. If the hexagon Z is normalized taking $z_1 = 0$, $z_3 = 1$ as in equation (3.1), and $z_5 = \zeta = z + iy$, then

- (a) $(x - \frac{1}{2})^2 + (y + \frac{1}{2} \cot(\theta_1/2))^2 = \frac{1}{4} \csc^2(\theta_1/2)$ if and only if the segments $[z_1, z_2]$, $[z_3, z_2]$, and $[z_5, z_2]$ have equal length,
- (b) $x \sin(\theta_2/2) + y \cos(\theta_2/2) = 0$ if and only if the segments $[z_1, z_4]$, $[z_3, z_4]$, and $[z_5, z_4]$ have equal length,
- (c) $(1-x) \sin(\theta_3/2) + y \cos(\theta_3/2) = 0$ if and only if the segments $[z_1, z_6]$, $[z_3, z_6]$, and $[z_5, z_6]$ have equal length.

The Claim proves that each point on the boundary of $\triangle(\theta_1, \theta_2, \theta_3)$ encodes a tetrahedron $\diamond$ with two shortest paths from v_0 to another vertex. The proof for $\triangle(\theta_1, \theta_3, \theta_2)$ is similar.

Proof of the Claim. (a) The segments $[z_1, z_2]$, $[z_3, z_2]$, and $[z_5, z_2]$ have equal length if and only if z_2 is center of a circle which pass through $z_1 = 0$, $z_3 = 1$ and $z_5 = \zeta$. We have $z_2 = (1 - i \cot(\theta_1/2))/2$ because $\angle z_1 z_2 z_3 = \theta_1$. The equation of such a circle is $(x - \frac{1}{2})^2 + (y + \frac{1}{2} \cot(\theta_1/2))^2 = \frac{1}{4} \csc^2(\theta_1/2)$.

FIGURE 1. In the proof of the second part of the Claim in Lemma 5, we take a in the upper half-plane for $\theta_2 < \pi$, and a in the lower half-plane for $\theta_2 > \pi$. So the statements in the proof cover both cases.

(b) It is sufficient to prove that angle $\angle z_3 z_1 z_5 = \pi - \theta_2/2$ when the segments $[z_1, z_4]$, $[z_3, z_4]$, and $[z_5, z_4]$ have equal length. Let a be a point such that $\angle z_1 a z_3 = \theta_2$ and the segments $[z_1, a]$ and $[z_3, a]$ have equal length (see Figure 1). Let b be the intersection of the line through z_1 and a with the line through z_4 and z_5 . Since $\angle z_3 z_1 b = \angle z_3 z_5 b$, then the quadrilateral $z_1 z_3 b z_5$ is cyclic and $\angle z_5 b z_1 = \angle z_5 z_3 z_1$. Since $\angle z_5 z_3 z_1 = \angle z_4 z_3 a$, it follows that quadrilateral $a z_3 b z_4$ is also cyclic. Therefore $\angle z_3 z_1 z_5 = \pi - \angle z_5 b z_3 = \angle z_3 a z_4 = \pi - \theta_2/2$.

(c) Is similar to the later case. $\square$

- (2) Let $Z = (z_1, z_2, z_3, z_4, z_5, z_6)$ be a hexagon of the form of equation (3.1) such that $[z_1, z_{2k}] = [z_3, z_{2k}] = [z_5, z_{2k}]$ for some $k \in \{1, 2, 3\}$. Hence ζ lies in the boundary of $\triangle(\theta_1, \theta_2, \theta_3)$. Two of the segments $[z_1, z_{2k}]$, $[z_3, z_{2k}]$, and $[z_5, z_{2k}]$ are edges of the hexagon Z and the other is interior to the hexagon Z . If one cuts Z along such interior segment and glues the two edges one obtains, after re-enumerating vertices in the appropriate way, a hexagon in $V(\theta_1, \theta_3, \theta_2)$ representing a point in the boundary of $\triangle(\theta_1, \theta_3, \theta_2)$. $\square$

We have proved the following corollary.

Corollary 6. $\mathcal{T}(\theta_1, \theta_2, \theta_3)$ is the disjoint union of $\triangle(\theta_1, \theta_2, \theta_3)$ and $\triangle(\theta_1, \theta_3, \theta_2)$ upon the identification of the boundaries $\partial\triangle(\theta_1, \theta_2, \theta_3)$ and $\partial\triangle(\theta_1, \theta_3, \theta_2)$ given by the reflection over the line $x = 1/2$.

Remark 7. The corners of the triangular regions $\triangle(\theta_1, \theta_2, \theta_3)$ and $\triangle(\theta_1, \theta_3, \theta_2)$ encode the collision of two vertices whose cone angles add more than 2π of the tetrahedron, namely,

- the corner of $\triangle(\theta_1, \theta_2, \theta_3)$ and $\triangle(\theta_1, \theta_3, \theta_2)$ between the sides $x \sin(\theta_2/2) + y \cos(\theta_2/2) = 0$ and $(1-x) \sin(\theta_3/2) + y \cos(\theta_3/2) = 0$ encodes the collision between the vertices v_0 and v_1 .

- the other two corners of $\triangle(\theta_1, \theta_2, \theta_3)$ and $\triangle(\theta_1, \theta_3, \theta_2)$ encode the collision between an element of $\{v_0, v_1\}$ and an element of $\{v_2, v_3\}$.

4. THE GEOMETRY INDUCED BY THE AREA HERMITIAN PRODUCT

4.1. The signature. Let $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ be the area Hermitian product given by equation (1.2).

Proposition 8. *The signature of the restriction of $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ to the 2-dimensional vector spaces $V_0(\theta_1, \theta_2, \theta_3)$ and $V_0(\theta_1, \theta_3, \theta_2)$ is*

$$\begin{aligned} (0, 2) & \text{ for } \theta_0 > 2\pi, \\ (0, 1) & \text{ for } \theta_0 = 2\pi, \\ (1, 1) & \text{ for } \theta_0 < 2\pi. \end{aligned}$$

This proposition is a particular case of [2, Corollaries 2.21, 13.1.2, 13.2.2] and [11, Theorem 14.3]. Also in [9, Proposition 3.3] a generalization of the case $\theta_0 < 2\pi$ was proved. Consider that this particular case of tetrahedra allows us to give a different proof.

Proof. Two elements $Z_0, W_0 \in V_0(\theta_1, \theta_2, \theta_3)$ are given by setting

$$\begin{aligned} W_0 &= \left(0, 0, 0, \frac{e^{i\theta_2}}{e^{i\theta_2} - 1}, 1, \frac{-1}{e^{i\theta_3} - 1} \right), \\ Z_0 &= \left(0, \frac{e^{i\theta_1}}{e^{i\theta_1} - 1}, 1, \frac{e^{i\theta_2}\zeta - 1}{e^{i\theta_2} - 1}, \zeta, \frac{-\zeta}{e^{i\theta_3} - 1} \right), \end{aligned}$$

where $\zeta = \frac{\sin \frac{\theta_3}{2}}{\sin \frac{\theta_2 + \theta_3}{2}} e^{-i\theta_2/2}$. We can easily compute

$$\langle W_0, W_0 \rangle_{\mathcal{A}} = \frac{\sin \frac{\theta_2 + \theta_3}{2}}{4 \sin \frac{\theta_2}{2} \sin \frac{\theta_3}{2}}, \quad \langle Z_0, W_0 \rangle_{\mathcal{A}} = 0, \quad \langle Z_0, Z_0 \rangle_{\mathcal{A}} = \frac{\sin \frac{\theta_1 + \theta_2 + \theta_3}{2}}{4 \sin \frac{\theta_1}{2} \sin \frac{\theta_2 + \theta_3}{2}}.$$

If $\langle W_0, W_0 \rangle_{\mathcal{A}} \neq 0$ (i.e. $\theta_2 + \theta_3 \neq 2\pi$), then we have, recalling that θ_0 is a maximal angle,

$$\begin{aligned} \langle W_0, W_0 \rangle_{\mathcal{A}} &> 0 \text{ and } \langle Z_0, Z_0 \rangle_{\mathcal{A}} > 0 \text{ for } \theta_1 + \theta_2 + \theta_3 < 2\pi, \\ \langle W_0, W_0 \rangle_{\mathcal{A}} &> 0 \text{ and } \langle Z_0, Z_0 \rangle_{\mathcal{A}} = 0 \text{ for } \theta_1 + \theta_2 + \theta_3 = 2\pi, \\ \langle W_0, W_0 \rangle_{\mathcal{A}} &> 0 \text{ and } \langle Z_0, Z_0 \rangle_{\mathcal{A}} < 0 \text{ for } \theta_1 + \theta_2 + \theta_3 > 2\pi. \end{aligned}$$

Since $\{W_0, Z_0\}$ is an orthogonal basis for the vector space $V_0(\theta_1, \theta_2, \theta_3)$, the proposition follows for the case $\theta_2 + \theta_3 \neq 2\pi$.

If $\theta_2 + \theta_3 = 2\pi$, then $\theta_1 = \theta_2 = \theta_3 = \theta_4 = \pi$ (Remark 1). In this case we can choose equilateral triangles

$$\begin{aligned} W &= \left(1, \frac{1+\rho}{2}, \rho, \frac{\rho+\rho^2}{2}, \rho^2, \frac{1+\rho^2}{2} \right), \\ Z &= \left(1, \frac{1+\rho^2}{2}, \rho^2, \frac{\rho+\rho^2}{2}, \rho, \frac{1+\rho}{2} \right), \end{aligned}$$

where $\rho = e^{2\pi i/3}$. Then we have

$$\langle W, W \rangle_{\mathcal{A}} = \frac{3\sqrt{3}}{4}, \quad \langle W, Z \rangle_{\mathcal{A}} = 0, \quad \langle Z, Z \rangle_{\mathcal{A}} = -\frac{3\sqrt{3}}{4}.$$

Next we translate the hexagons w and Z to get

$$Z_0 = Z - (1, 1, 1, 1, 1, 1), \quad W_0 = W - (1, 1, 1, 1, 1, 1) \in V_0(\theta_1, \theta_2, \theta_3).$$

It is easy to see that the area Hermitian product is invariant up to translations; that is

$$\langle (z_1 - \zeta, \dots, z_6 - \zeta), (w_1 - \xi, \dots, w_6 - \xi) \rangle_{\mathcal{A}} = \langle (z_1, \dots, z_6), (w_1, \dots, w_6) \rangle_{\mathcal{A}}$$

for all $\zeta, \xi \in \mathbb{C}$. Therefore $\{W_0, Z_0\}$ is an orthogonal basis for the vector space $V_0(\pi, \pi, \pi)$ and the proposition follows for the case $\theta_2 + \theta_3 = 2\pi$.

The proof for $V_0(\theta_1, \theta_3, \theta_2)$ is similar. $\square$

4.2. The geometric form. Now we recall how the projectivization of a Hermitian vector space can be geometrized. That is not another thing but the definition of the Bergman and Fubini-Study metrics.

Given $\theta_1, \theta_2, \theta_3$, we put $\mathcal{E} = \{Z \in V_0(\theta_1, \theta_2, \theta_3) : \langle Z, Z \rangle_{\mathcal{A}} > 0\}$. The complex projectivization $\mathbb{P}(\mathcal{E})$ can be seen as a space of hexagons with positive area up to equivalence by orientation-preserving similarities. By $\pi : \mathcal{E} \rightarrow \mathbb{P}(\mathcal{E})$ we denote the canonical projection. The Hermitian structure on $\mathcal{E}$ defined by

$$\langle W_1, W_2 \rangle_Z = \frac{\langle W_1, W_2 \rangle_{\mathcal{A}}}{\langle Z, Z \rangle_{\mathcal{A}}}, \quad \text{for } Z \in \mathcal{E} \text{ and } W_1, W_2 \in T_Z \mathcal{E},$$

is invariant up to multiplication by a complex scalar. We define on $\mathbb{P}(\mathcal{E})$ a pseudo-Hermitian structure $\langle \cdot, \cdot \rangle$ so as to make the canonical projection $\pi : \mathcal{E} \rightarrow \mathbb{P}(\mathcal{E})$ a submersion (see [6, Example 1.11.14]): if $W_1, W_2 \in T_Z \mathcal{E}$ we put $\langle \pi_* W_1, \pi_* W_2 \rangle_{\pi(Z)} = \langle W_1, W_2 \rangle_Z$. For each $Z \in \mathcal{E}$ so that $\langle Z, Z \rangle_{\mathcal{A}} = 1$ one have a linear isomorphism onto $T_{\pi(Z)} \mathbb{P}(\mathcal{E})$ by restricting π_* to the horizontal tangent subspace $T_Z^h \mathcal{E} = \{W \in T_Z \mathcal{E} : \langle W, Z \rangle_{\mathcal{A}} = 0\}$. With this presentation of $T_{\pi(Z)} \mathbb{P}(\mathcal{E})$ one verifies that

$$\pi_*(W) = \frac{1}{\langle Z, Z \rangle_{\mathcal{A}}^{1/2}} \left(W - \frac{\langle W, Z \rangle_{\mathcal{A}}}{\langle Z, Z \rangle_{\mathcal{A}}} Z \right)$$

for any $Z \in \mathcal{E}$ and $W \in T_Z \mathcal{E}$. Hence,

$$(4.1) \quad \langle \pi_* W_1, \pi_* W_2 \rangle_{\pi(Z)} = \frac{\langle W_1, W_2 \rangle_{\mathcal{A}} \langle Z, Z \rangle_{\mathcal{A}} - \langle W_1, Z \rangle_{\mathcal{A}} \langle Z, W_2 \rangle_{\mathcal{A}}}{\langle Z, Z \rangle_{\mathcal{A}}^2}$$

for any $Z \in \mathcal{E}$ and $W_1, W_2 \in T_Z \mathcal{E}$. Formula (4.1) also applies to $\mathbb{P}(\mathcal{E}')$, where $\mathcal{E}' = \{Z \in V_0(\theta_1, \theta_3, \theta_2) : \langle Z, Z \rangle_{\mathcal{A}} > 0\}$.

Proposition 9. *Endowed with the metric $\langle \cdot, \cdot \rangle$, $\mathbb{P}(\mathcal{E})$ and $\mathbb{P}(\mathcal{E}')$ are*

- (1) *2-spheres of radius 1/2 for $\theta_0 > 2\pi$,*
- (2) *planes with the zero metric for $\theta_0 = 2\pi$,*
- (3) *pseudo-Riemannian versions of the hyperbolic plane for $\theta_0 < 2\pi$.*

Proof. We prove the proposition for $\mathbb{P}(\mathcal{E})$. The proof for $\mathbb{P}(\mathcal{E}')$ is the same.

First suppose that $\theta_0 > 2\pi$. The restriction of the area Hermitian product to $V_0(\theta_1, \theta_2, \theta_3)$ is given by $\langle (\zeta_1, \zeta_2), (\xi_1, \xi_2) \rangle_{\mathcal{A}} = \zeta_1 \bar{\xi}_1 + \zeta_2 \bar{\xi}_2$ with respect to an $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ -orthonormal basis for $V_0(\theta_1, \theta_2, \theta_3)$ (Proposition 8). The projectivization $\mathbb{P}(\mathcal{E})$ is the complex projective line. If we take an affine chart $(\zeta_1 : \zeta_2) \mapsto \zeta = \zeta_2 / \zeta_1$ of $\mathbb{P}(\mathcal{E})$ and we regard the tangent vectors $\xi_1, \xi_2 \in T_{\zeta} \mathbb{C}$ as the vectors $(0, \xi_1), (0, \xi_2) \in T_{(1, \zeta)} \mathcal{E}$ we get, applying the formula (4.1),

$$(4.2) \quad \langle \pi_*(0, \xi_1), \pi_*(0, \xi_2) \rangle_{\pi(1, \zeta)} = \frac{\xi_1 \bar{\xi}_2}{(1 + |\zeta|^2)^2}.$$

This is the metric that comes from the stereographic projection from the north pole $(0, 0, 1)$ of the sphere $\{x^2 + y^2 + (z - 1/2)^2 = 1/4\}$ onto the xy -plane.

Now assume that $\theta_0 = 2\pi$. There is a basis so that the restriction of the area Hermitian product to $V_0(\theta_1, \theta_2, \theta_3)$ takes the form $\langle(\zeta_1, \zeta_2), (\xi_1, \xi_2)\rangle_{\mathcal{A}} = \zeta_2 \bar{\xi}_2$ (Proposition 8). The image of $\mathcal{E}$ under the affine chart $(\zeta_1 : \zeta_2) \mapsto \zeta = \zeta_2/\zeta_1$ of $\mathbb{P}(V_0(\theta_1, \theta_2, \theta_3))$ is the punctured plane $\mathbb{C} \setminus \{0\}$. We regard the tangent vectors $\xi_1, \xi_2 \in T_{\zeta}\mathbb{C}$ as the vectors $(0, \xi_1), (0, \xi_2) \in T_{(1, \zeta)}\mathcal{E}$. Applying the formula (4.1) we get

$$\langle\pi_*(0, \xi_1), \pi_*(0, \xi_2)\rangle_{\pi(1, \zeta)} = 0.$$

Finally suppose that $\theta_0 < 2\pi$. The restriction of the area Hermitian product to $V_0(\theta_1, \theta_2, \theta_3)$ is given by $\langle(\zeta_1, \zeta_2), (\xi_1, \xi_2)\rangle_{\mathcal{A}} = -\zeta_1 \bar{\xi}_1 + \zeta_2 \bar{\xi}_2$ with respect to an $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ -orthonormal basis for $V_0(\theta_1, \theta_2, \theta_3)$ (Proposition 8). The image of $\mathcal{E}$ under the affine chart $(\zeta_1 : \zeta_2) \mapsto \zeta = \zeta_2/\zeta_1$ of $\mathbb{P}(V_0(\theta_1, \theta_2, \theta_3))$ is $B = \{|\zeta| > 1\}$. We regard the tangent vectors $\xi_1, \xi_2 \in T_{\zeta}B$ as the vectors $(0, \xi_1), (0, \xi_2) \in T_{(1, \zeta)}\mathcal{E}$. Applying the formula (4.1) we get

$$(4.3) \quad \langle\pi_*(0, \xi_1), \pi_*(0, \xi_2)\rangle_{\pi(1, \zeta)} = -\frac{\xi_1 \bar{\xi}_2}{(1 - |\zeta|^2)^2}.$$

□

Remark 10. Formula (4.3) determines a negative definite pseudo-Riemannian metric. In order to obtain a positive definite Riemannian metric, it is necessary to take the projectivization $\mathbb{P}(\mathcal{N})$ of hexagons with negative area $\mathcal{N} = \{Z \in V_0(\theta_1, \theta_2, \theta_3) : \langle Z, Z \rangle_{\mathcal{A}} < 0\}$, instead of taking $\mathbb{P}(\mathcal{E})$, to apply the formula (4.1) to get the positive definite hyperbolic metric $\frac{\xi_1 \bar{\xi}_2}{(1 - |\zeta|^2)^2}$ on the unitary ball $\{|\zeta_2/\zeta_1| < 1\}$. Formula (4.1) changes to

$$\langle\pi_* W_1, \pi_* W_2\rangle_{\pi(Z)} = \frac{\langle W_1, Z \rangle_{\mathcal{A}} \langle Z, W_2 \rangle_{\mathcal{A}} - \langle W_1, W_2 \rangle_{\mathcal{A}} \langle Z, Z \rangle_{\mathcal{A}}}{\langle Z, Z \rangle_{\mathcal{A}}^2}$$

and the positive definite hyperbolic metric $\frac{\xi_1 \bar{\xi}_2}{(1 - |\zeta|^2)^2}$ of constant curvature 4 on the unitary ball $\{|\zeta| < 1\}$ is obtained.

Remark 11. Since

$$V_0(\theta_1, \theta_2, \theta_3) = \left\{ \left(0, \frac{e^{i\theta_1} \xi}{e^{i\theta_1} - 1}, \xi, \frac{e^{i\theta_2} \zeta - \xi}{e^{i\theta_2} - 1}, \zeta, \frac{-\zeta}{e^{i\theta_3} - 1} \right) : \xi, \zeta \in \mathbb{C} \right\}$$

we may regard from now on the set

$$(4.4) \quad \left\{ \left(0, \frac{e^{i\theta_1}}{e^{i\theta_1} - 1}, 1, \frac{e^{i\theta_2} \zeta - 1}{e^{i\theta_2} - 1}, \zeta, \frac{-\zeta}{e^{i\theta_3} - 1} \right) : \zeta \in \mathbb{C} \right\}$$

as an affine chart of $\mathbb{P}(V_0(\theta_1, \theta_2, \theta_3))$. In the proof of Proposition 8 we get an orthogonal basis $\{Z_0, W_0\}$ for $V_0(\theta_1, \theta_2, \theta_3)$. Notice that the line through W_0 is the point at infinity for the above affine chart (4.4). The normalizations of Z_0 and W_0 form an orthonormal basis for $V_0(\theta_1, \theta_2, \theta_3)$. Since any linear change of coordinates of a 2-dimensional vector space V can be seen as a Möbius transformation of the projectivization $\mathbb{P}(V)$, we have that equation (4.1) determines in the above affine chart (4.4) a conformal model of the respective geometric model given in Proposition 9.

Lemma 12. *If $\theta_0 \neq 2\pi$, then the boundary of $\triangle(\theta_1, \theta_2, \theta_3)$ and $\triangle(\theta_1, \theta_3, \theta_2)$ is composed by three geodesics of the corresponding geometric structure, and the angles between these geodesics are $|2\pi - \theta_1 - \theta_2|/2$, $|2\pi - \theta_2 - \theta_3|/2$, $|2\pi - \theta_3 - \theta_1|/2$.*

Proof. We prove the lemma for $\triangle(\theta_1, \theta_2, \theta_3)$ with $\theta_2 + \theta_3 \neq 2\pi$. The proofs for $\triangle(\theta_1, \theta_3, \theta_2)$ and $\triangle(\pi, \pi, \pi)$ are similar.

Case $\theta_0 > 2\pi$. Recall the deduction of equation (4.2). The geodesic paths of the sphere $\{x^2 + y^2 + (z - 1/2)^2 = 1/4\}$ are circles through antipodal points. If the xy -plane is viewed as the $\mathbb{C}$ -plane, the points ζ_1 and ζ_2 represent antipodal points if and only if $\zeta_2 = -1/\bar{\zeta}_1$. Therefore two points Z and W in the affine chart (4.4) represent antipodal points in $\mathbb{P}(V_0(\theta_1, \theta_2, \theta_3))$ if and only if $\langle Z, W \rangle_{\mathcal{A}} = 0$.

Both lines $x \sin(\theta_2/2) + y \cos(\theta_2/2) = 0$ and $(1-x) \sin(\theta_3/2) + y \cos(\theta_3/2) = 0$ passing through the antipodal points

$$\left(0, 0, 0, \frac{e^{i\theta_2}}{e^{i\theta_2} - 1}, 1, \frac{-1}{e^{i\theta_3} - 1}\right), \quad \left(0, \frac{e^{i\theta_1}}{e^{i\theta_1} - 1}, 1, \frac{e^{i\theta_2}\zeta - 1}{e^{i\theta_2} - 1}, \zeta, \frac{-\zeta}{e^{i\theta_3} - 1}\right),$$

for $\zeta = \frac{\sin \frac{\theta_3}{2}}{\sin \frac{\theta_2 + \theta_3}{2}} e^{-i\theta_2/2}$. Since the metric is conformal (Remark 11), both lines represent circles in the sphere and the angle between them is equal to $\pi - (\theta_2 + \theta_3)/2$.

$(x - \frac{1}{2})^2 + (y + \frac{1}{2} \cot(\theta_1/2))^2 = \frac{1}{4} \csc^2(\theta_1/2)$ passing through the antipodal points

$$\left(0, \frac{e^{i\theta_1}}{e^{i\theta_1} - 1}, 1, \frac{-1}{e^{i\theta_2} - 1}, 0, 0\right), \quad \left(0, \frac{e^{i\theta_1}}{e^{i\theta_1} - 1}, 1, \frac{e^{i\theta_2}\zeta - 1}{e^{i\theta_2} - 1}, \zeta, \frac{-\zeta}{e^{i\theta_3} - 1}\right)$$

where $\zeta = \frac{\sin \frac{\theta_1 + \theta_2}{2}}{\sin \frac{\theta_1}{2}} e^{-i\theta_2/2}$. Since the metric is conformal (Remark 11), this circle represents a circle in the sphere forming angles equal to $|2\pi - \theta_1 - \theta_2|/2$ and $|2\pi - \theta_3 - \theta_1|/2$ with $x \sin(\theta_2/2) + y \cos(\theta_2/2) = 0$ and $(1-x) \sin(\theta_3/2) + y \cos(\theta_3/2) = 0$, respectively.

Case $\theta_0 < 2\pi$. The geodesics of the metric given by equation (4.3) are arcs of circle orthogonal to the unit circle because it can be seen as a conformal model of the hyperbolic geometry (Remarks 10 and 11). The boundary of $\mathcal{E} \subset V_0(\theta_1, \theta_2, \theta_3)$ is the set $\{Z \in V_0(\theta_1, \theta_2, \theta_3) : \langle Z, Z \rangle_{\mathcal{A}} = 0\}$. For Z of the form

$$Z = \left(0, \frac{e^{i\theta_1}}{e^{i\theta_1} - 1}, 1, \frac{e^{i\theta_2}\zeta - 1}{e^{i\theta_2} - 1}, \zeta, \frac{-\zeta}{e^{i\theta_3} - 1}\right)$$

equation $\langle Z, Z \rangle_{\mathcal{A}} = 0$ can be rewritten as

$$\left(x - \frac{\cos \frac{\theta_2}{2} \sin \frac{\theta_3}{2}}{\sin \left(\frac{\theta_2 + \theta_3}{2}\right)}\right)^2 + \left(y + \frac{\sin \frac{\theta_2}{2} \sin \frac{\theta_3}{2}}{\sin \left(\frac{\theta_2 + \theta_3}{2}\right)}\right)^2 = -\frac{\sin \frac{\theta_2}{2} \sin \frac{\theta_3}{2} \sin \left(\frac{\theta_1 + \theta_2 + \theta_3}{2}\right)}{\sin \frac{\theta_1}{2} \sin^2 \left(\frac{\theta_2 + \theta_3}{2}\right)}$$

for $\zeta = x + iy$. This is a circle orthogonal to the paths

$$\begin{aligned} x \sin(\theta_2/2) + y \cos(\theta_2/2) &= 0, \\ (1-x) \sin(\theta_3/2) + y \cos(\theta_3/2) &= 0, \\ (x - \frac{1}{2})^2 + (y + \frac{1}{2} \cot(\theta_1/2))^2 &= \frac{1}{4} \csc^2(\theta_1/2), \end{aligned}$$

which form the mentioned angles. $\square$

5. THE MAIN THEOREM

Theorem 13. $\mathcal{T}(\theta_1, \theta_2, \theta_3)$ is a thrice punctured 2-sphere.

For $\theta_0 > 2\pi$: The area Hermitian product endows $\mathcal{T}(\theta_1, \theta_2, \theta_3)$ with a metric of constant curvature 4. The completion of this punctured geometric sphere is a sphere with three conical singularities with cone angles $|2\pi - \theta_1 - \theta_2|$, $|2\pi - \theta_2 - \theta_3|$, and $|2\pi - \theta_3 - \theta_1|$.

For $\theta_0 < 2\pi$: The area Hermitian product endows $\mathcal{T}(\theta_1, \theta_2, \theta_3)$ with a metric of constant curvature -4 . The completion of this punctured geometric sphere is a sphere with three singularities which can be conical singularities or cusps. The singularities have cone angles $|2\pi - \theta_1 - \theta_2|$, $|2\pi - \theta_2 - \theta_3|$, and $|2\pi - \theta_3 - \theta_1|$. When $2\pi - \theta_j - \theta_k = 0$, the singularity is a cusp.

The collision of vertices with cone angle θ_k and θ_l gives rise to the conical point in $\mathcal{T}(\theta_1, \theta_2, \theta_3)$ with cone angle $|2\pi - \theta_k - \theta_l| = |2\pi - \theta_i - \theta_j|$, where $\{i, j, k, l\} = \{0, 1, 2, 3\}$.

For $\theta_0 < 2\pi$, Theorem 13 can be deduced from [9, Proposition 3.5] (see also [7, Section 5]).

Proof. To get the theorem as a consequence of Corollary 6, Remark 7, Proposition 9 and Lemma 12, we only have to verify that edges of $\triangle(\theta_1, \theta_2, \theta_3)$ and $\triangle(\theta_1, \theta_3, \theta_2)$ are pasted by isometries.

Regions $\triangle(\theta_1, \theta_2, \theta_3)$ and $\triangle(\theta_1, \theta_3, \theta_2)$ are contained in the affine charts

$$\left\{ \left(0, \frac{e^{i\theta_1}}{e^{i\theta_1} - 1}, 1, \frac{e^{i\theta_2}\zeta - 1}{e^{i\theta_2} - 1}, \zeta, \frac{-\zeta}{e^{i\theta_3} - 1} \right) : \zeta \in \mathbb{C} \right\}$$

$$\left\{ \left(0, \frac{e^{i\theta_1}}{e^{i\theta_1} - 1}, 1, \frac{e^{i\theta_3}\zeta - 1}{e^{i\theta_3} - 1}, \zeta, \frac{-\zeta}{e^{i\theta_2} - 1} \right) : \zeta \in \mathbb{C} \right\}$$

of $\mathbb{P}(V_0(\theta_1, \theta_2, \theta_3))$ and $\mathbb{P}(V_0(\theta_1, \theta_3, \theta_2))$, respectively. Written in these charts, the function $\phi_1 : \mathbb{P}(V_0(\theta_1, \theta_2, \theta_3)) \rightarrow \mathbb{P}(V_0(\theta_1, \theta_3, \theta_2))$ given by the translation

$$\zeta \mapsto \zeta + \left(\sin(\theta_2/2)e^{-i\theta_3/2} - \sin(\theta_3/2)e^{-i\theta_2/2} \right) / \sin((\theta_2 + \theta_3)/2)$$

and the function $\phi_2 : \mathbb{P}(V_0(\theta_1, \theta_3, \theta_2)) \rightarrow \mathbb{P}(V_0(\theta_1, \theta_2, \theta_3))$ given by the reflection on the vertical line through the point $\sin(\theta_2/2)e^{-i\theta_3/2} / \sin((\theta_2 + \theta_3)/2)$ are isometries. Here $\sin(\theta_3/2)e^{-i\theta_2/2} / \sin((\theta_2 + \theta_3)/2)$ is the intersection of lines $x \sin(\theta_2/2) + y \cos(\theta_2/2) = 0$ and $(1 - x) \sin(\theta_3/2) + y \cos(\theta_3/2) = 0$ compounding the boundary of $\triangle(\theta_1, \theta_2, \theta_3)$; and $\sin(\theta_2/2)e^{-i\theta_3/2} / \sin((\theta_2 + \theta_3)/2)$ is the intersection of lines $x \sin(\theta_3/2) + y \cos(\theta_3/2) = 0$ and $(1 - x) \sin(\theta_2/2) + y \cos(\theta_2/2) = 0$ compounding the boundary of $\triangle(\theta_1, \theta_3, \theta_2)$. The isometry $\phi_2 \circ \phi_1 : \mathbb{P}(V_0(\theta_1, \theta_2, \theta_3)) \rightarrow \mathbb{P}(V_0(\theta_1, \theta_3, \theta_2))$ is a realization of the gluing of the edges of $\triangle(\theta_1, \theta_2, \theta_3)$ and $\triangle(\theta_1, \theta_3, \theta_2)$ to get $\mathcal{T}(\theta_1, \theta_2, \theta_3)$. $\square$

6. RELABELING VERTICES WITH THE SAME CONE ANGLE

Now we focus on the space of tetrahedra $\mathcal{T}(\theta_1, \theta_2, \theta_3)$ when two or more cone angles of the tetrahedra are equal. Theorem 13 establishes that $\mathcal{T}(\theta_1, \theta_2, \theta_3)$ is a 2-sphere with a Riemannian metric of constant curvature ± 4 which has conical singularities with cone angles $\Theta_j = |2\pi - \theta_k - \theta_l|$, where $\{j, k, l\} = \{1, 2, 3\}$. Equality between cone angles of the tetrahedra implies equality between cone angles of the

space of tetrahedra $\mathcal{T}(\theta_1, \theta_2, \theta_3)$, namely:

- (1) $\theta_j = \theta_k \Rightarrow \Theta_j = \Theta_k$ for $j, k \in \{1, 2, 3\}$.
- (2) $\theta_0 = \theta_j \Rightarrow \Theta_k = \Theta_l$ for $\{j, k, l\} = \{1, 2, 3\}$.

It is clear that:

- (1) Unless $\Theta_j = \Theta_k$ for some pair $j \neq k$, the orientation-preserving symmetry group of $\mathcal{T}(\theta_1, \theta_2, \theta_3)$ is trivial.
- (2) When exactly two angles Θ_j and Θ_k are equal, the orientation-preserving symmetry group of $\mathcal{T}(\theta_1, \theta_2, \theta_3)$ is the order two group.
- (3) When $\Theta_1 = \Theta_2 = \Theta_3$, the orientation-preserving symmetry group of $\mathcal{T}(\theta_1, \theta_2, \theta_3)$ is the dihedral group D_3 .

Let $\mathcal{S} = \mathcal{S}(\theta_0, \theta_1, \theta_2, \theta_3)$ denote the space of orientation-preserving similarity classes of tetrahedra with cone angles $2\pi > \theta_0, \theta_1, \theta_2, \theta_3 > 0$ *without labeled vertices*. Let $\text{Sym}(\mathcal{T}(\theta_1, \theta_2, \theta_3))$ be the orientation-preserving symmetry group of $\mathcal{T}(\theta_1, \theta_2, \theta_3)$.

Theorem 14. *$\mathcal{S}$ is the quotient $\mathcal{T}(\theta_1, \theta_2, \theta_3)/\text{Sym}(\mathcal{T}(\theta_1, \theta_2, \theta_3))$. It follows that $\mathcal{S}$ is a 2-sphere with a Riemannian metric of constant curvature ± 4 which has three conical singularities.*

- (1) *The tern of cone angles of $\mathcal{S}$ is $(\pi, \Theta_j/2, \Theta_k) = (\pi, \Theta_j/2, \Theta_l)$ when the angles θ_k and θ_l are the only ones that are equal, where $\{j, k, l\} = \{1, 2, 3\}$.*
- (2) *The tern of cone angles of $\mathcal{S}$ is $(\pi, \Theta_1/2, \Theta_2) = (\pi, \Theta_1/2, \Theta_3)$ when either $\theta_0 = \theta_1 \neq \theta_2 = \theta_3$ or the angles θ_0 and θ_1 are the only ones that are equal.*
- (3) *The tern of cone angles of $\mathcal{S}$ is $(2\pi/3, \pi, \Theta_1) = (2\pi/3, \pi, \Theta_2) = (2\pi/3, \pi, \Theta_3)$ when at least three cone angles $\theta_j, \theta_k, \theta_l$ are equal, where $j, k, l \in \{0, 1, 2, 3\}$.*

Proof. On the one hand, it is easy to see that $\text{Sym}(\mathcal{T}(\theta_1, \theta_2, \theta_3))$ acts on $\mathcal{T}(\theta_1, \theta_2, \theta_3)$ identifying points which represent the same tetrahedron but with different labeled vertices.

On the other hand, the number of possible relabels of vertices with equal cone angle is $|\text{Sym}(\mathcal{T}(\theta_1, \theta_2, \theta_3))|$. $\square$

APPENDIX A. CURVATURE OF POLYGONAL PATHS

Let $Z = (z_1, \dots, z_n)$ be a simple polygon. We called the interior of Z to the set of points in the bounded component of the complement of Z . We said that the segment $[z, w]$ is contained in Z if the interior of the segment $[z, w]$ is a subset of the interior of Z . A diagonal of Z is a segment $[z_j, z_k]$ such that $k, j \in \{1, \dots, n\}$ but $k \notin \{j, j \pm 1\}$.

Lemma 15. *Let $Z = (z_1, \dots, z_n)$ be a simple polygon with $n \geq 4$ vertices.*

- (1) *Z has a diagonal contained in Z .*
- (2) *Z admits a triangulation with $n - 2$ triangles whose edges are of the form $[z_j, z_k]$, where $j, k \in \{1, \dots, n\}$.*

Proof. (1) Given $k \in \{2, \dots, n-1\}$, the line through the points z_{k-1}, z_k and the line through the points z_k, z_{k+1} split together any ball centered at z_k into four parts. Since the polygon is simple, one of these parts is contained in the interior of Z for a small enough ball centered at z_k . Let R_0 be this small part contained in the interior of Z . Consider the maximal segments $[z_k, w]$ contained in Z and intersecting R_0 . If the ends w of all these segments

belong to the same edge of Z , then $[z_{k-1}, z_{k+1}]$ is a diagonal contained in Z ; otherwise one of the segments $[z_k, w]$ is of the form $[z_k, z_j]$ with $j \neq k \pm 1$, and $[z_k, z_j]$ is a diagonal contained in Z .

- (2) It can be proved by induction using part (1) of this lemma to reduce to polygons with a smaller number of vertices. $\square$

Let $Z = (z_1, \dots, z_n)$ be a simple polygon. The term *exterior angle* to Z at z_j means the angle $-\pi < \alpha_j \leq \pi$ from the vector $z_j - z_{j-1}$ to the vector $z_{j+1} - z_j$ measured in the positive sense. In this way, the exterior angle at z_j is the turning of the tangent vector at the vertex z_j .

A polygonal path in $\mathbb{C}$ is a union of segments $\bigcup_{j=1}^{n-1} [z_j, z_{j+1}]$. A polygonal path is said to be *simple* if it satisfies two conditions: it does not have repeated vertices, and the interior of any edge does not intersect to another edge. The term *exterior angle* of $\bigcup_{j=1}^{n-1} [z_j, z_{j+1}]$ at z_j , for $j \neq 1, n$, means the angle $-\pi < \alpha_j \leq \pi$ from the vector $z_j - z_{j-1}$ to the vector $z_{j+1} - z_j$ measured in the positive sense. By the exterior angle at z_n we mean the angle $-\pi < \alpha_n \leq \pi$ from the vector $z_n - z_{n-1}$ to the vector $z_2 - z_1$ measured in the positive sense. The definition is consistent with the definition of an exterior angle for a simple polygon (i.e. when $z_n = z_1$). By the sum of exterior angles we mean $\alpha_2 + \alpha_3 + \dots + \alpha_n$.

Lemma 16. (1) *Let $Z = (z_1, \dots, z_n)$ be a simple polygon. The sum of exterior angles of Z is equal to 2π if Z is positively oriented, and it is equal to -2π if Z is negatively oriented.*

- (2) *Let $L = [z_1, z_2] \cup [z_2, z_3] \cup [z_3, z_4] \cup [z_4, z_5]$ be a wedge shaped simple polygonal path with $\alpha_5 = 0$, like in Lemma 4. For any pair $s, t \in L$ the polygonal portion of L which lies between s and t has the sum of exterior angles equal to 0.*

Proof. (1) By the interior angle of Z at z_j is meant $\theta_j = \pi - \alpha_j$. Notice that $0 \leq \theta_j < 2\pi$. From part (2) of Lemma 15 we have $\sum_{j=1}^n \theta_j = (n-2)\pi$. Therefore $\sum_{j=1}^n \alpha_j = 2\pi$ if Z is positively oriented, and $\sum_{j=1}^n \alpha_j = -2\pi$ if Z is negatively oriented.

- (2) There are several cases depending on which segments contain the points s, t . Each one of these cases is easy to verify. $\square$

ACKNOWLEDGEMENTS

The authors thank Aldo-Hilario Cruz-Cota for explaining his doctoral thesis and the preprint [1] to them. They are also grateful to Jesús Muciño-Raymundo for pointing out the work of Cruz-Cota. The authors gratefully acknowledge valuable and stimulating conversations with Alberto Verjovsky and Ferrán Valdez. The authors are grateful to Erika Atienzo for proof-reading the text.

REFERENCES

- [1] A. H. Cruz-Cota, *The moduli space of hex spheres*, <http://arxiv.org/abs/1010.5235>, 2010.
- [2] P. Deligne and G. D. Mostow, *Monodromy of hypergeometric functions and non-lattice integral monodromy*, Publ. Math. Inst. Hautes Études Sci. **63** (1986), 5–89. MR849651 (88a:22023a)
- [3] F. Fillastre, *From spaces of polygons to spaces of polyhedra following Bavard, Ghys and Thurston*, <http://fillastre.u-cergy.fr/articles.html>, 2009.

- [4] H. Gluck, K. Krigelman, and D. Singer, *The converse to the Gauss-Bonnet theorem in pl*, J. Differential Geom. **9** (1974), 601–616. MR0390962 (52:11785)
- [5] M. Gromov, *Metric structures for Riemannian and non-Riemannian spaces*, Birkhäuser, 1998. MR1699320 (2000d:53065)
- [6] W. Klingenberg, *Riemannian geometry*, de Gruyter Stud. Math., vol. 1, de Gruyter, 1982.
- [7] S. Kojima, *Complex hyperbolic cone structures on the configuration spaces*, Rend. Istit. Mat. Univ. Trieste **32** (2001), 149–163. MR1893396 (2003b:55018)
- [8] W. Thurston, *Three-dimensional geometry and topology*, vol. 1, Princeton University Press, 1997. MR1435975 (97m:57016)
- [9] ———, *Shapes of polyhedra and triangulations of the sphere*, The Epstein Birthday Shrift, Geom. Topol. Monogr., vol. 1, 1998, pp. 511–549. MR1668340 (2000b:57026)
- [10] M. Troyanov, *On the moduli space of singular Euclidean surfaces*, Handbook of Teichmüller Theory, Vol. 1, IRMA Lect. Math. Theor. Phys., no. 11, 2007, pp. 507–540. MR2349679 (2009b:57041)
- [11] W. A. Veech, *Flat surfaces*, Amer. J. Math. **115** (1993), no. 3, 589–689. MR1221838 (94g:30043)

CIMAT, MINERAL DE VALENCIANA, C.P. 36240, GUANAJUATO, GTO., MEXICO
E-mail address: ahtziri@cimat.mx

FACULTAD DE CIENCIAS FÍSICO-MATEMÁTICAS, UMSNH, CIUDAD UNIVERSITARIA, C.P. 58040, MORELIA, MICH., MEXICO
E-mail address: jllopez@umich.mx