A UNIFORM CONTRACTION PRINCIPLE FOR BOUNDED APOLLONIAN EMBEDDINGS

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ABSTRACT. Let $\widehat{H} = H \cup \{\infty\}$ denote the standard one-point completion of a real Hilbert space H . Given any non-trivial proper subset $U \subset \widehat{H}$ one may define the so-called “Apollonian” metric d_U on U . When $U \subset V \subset \widehat{H}$ are nested proper subsets we show that their associated Apollonian metrics satisfy the following uniform contraction principle: Let $\Delta = \text{diam}_V(U) \in [0, +\infty]$ be the diameter of the smaller subsets with respect to the large. Then for every $x, y \in U$ we have

$$d_V(x, y) \leq \tanh \frac{\Delta}{4} d_U(x, y).$$

In dimension one, this contraction principle was established by Birkhoff [Bir57] for the Hilbert metric of finite segments on $\mathbb{RP}^1$. In dimension two it was shown by Dubois in [Dub09] for subsets of the Riemann sphere $\widehat{\mathbb{C}} \sim \widehat{\mathbb{R}^2}$. It is new in the generality stated here.

1. INTRODUCTION AND RESULTS

There are striking similarities between the projective group for the real or complex projective lines and the conformal group of the one-point completion of a real Hilbert space of dimension at least 3. In the first case, the group consists of Möbius maps of the form $z \mapsto \frac{az+b}{cz+d}$ and in the second it is generated by linear isometries, homotheties and the inversion, corresponding to Möbius transformations supplemented with a complex conjugation. In both cases one needs at least 4 points to define a group invariant quantity, i.e. the cross-ratio. Fixing a subset U whose complement contains at least 2 points, the logarithm of cross-ratios may then be used to construct a (semi-)metric on U . On the interval $I = (-1, 1)$, there is a unique (up to a constant) distance invariant under Möbius transformations preserving I . This is precisely the restriction of the Poincaré metric $2|dz|(1 - |z|^2)^{-1}$ on the unit disk in the complex plane. In the case of Hilbert spaces of higher dimensions one may derive the so-called ‘Apollonian metric’ (see below). This latter metric was first introduced for $\widehat{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$ by Barbilian [Bar34] and later rediscovered by Beardon [Bea98].

From a dynamical point of view, it is of interest to know how a subset U metrically embed into a larger subset V with respect to the associated metrics d_U

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and d_V (see below for more precise statements). It is straight-forward from definitions that the injection $i : (U, d_U) \hookrightarrow (V, d_V)$ is non-expanding. More interesting, however, is that it verifies a general uniform contraction principle (UCP): If $\Delta = \text{diam}_V(U) < +\infty$, i.e. the embedding of U has bounded diameter in the larger domain V , then the injection is a strict contraction with a Lipschitz constant bounded by $\tanh \frac{\Delta}{4}$. This is the same formula which appears in Birkhoff's work on the Hilbert metric [Bir57]. We give below the (surprisingly simple) proof of the UCP for the general case and in section 2 some simple dynamical systems applications.

There is no particular reason for sticking to finite dimension, so in the following let H be any real Hilbert space. We write $\langle \cdot, \cdot \rangle$ for the scalar product and $\| \cdot \|$ for the norm on H . Let $\hat{H} = H \cup \{\infty\}$ be a one point completion of H in which the open sets containing ∞ are of the form $\{\infty\} \cup F^c$ with F a bounded closed set. With this convention $\hat{H}$ is compact iff H is finite dimensional. The space $(\hat{H}, \hat{d})$ is a complete metric space of diameter one with respect to the metric:

$$(1.1) \quad \hat{d}(x, y) = \frac{\|x - y\|}{\sqrt{1 + \langle x, x \rangle} \sqrt{1 + \langle y, y \rangle}}, \quad \hat{d}(\infty, y) = \frac{1}{\sqrt{1 + \langle y, y \rangle}}.$$

Definition 1.1. Given four points $\{x_1, x_2, u_1, u_2\} \in \hat{H}$ such that $\{x_1, x_2\}$ and $\{u_1, u_2\}$ are disjoint, we define their cross-ratio to be :

$$(1.2) \quad [x_1, x_2; u_1, u_2] \equiv \frac{\|x_2 - u_1\| \|x_1 - u_2\|}{\|x_1 - u_1\| \|x_2 - u_2\|}.$$

Here, $\| \cdot \|$ denotes the Hilbert norm in H and we adapt the usual conventions for dealing with the point at ∞ . When $U \subset \hat{H}$ is a proper subset (by proper we mean that U and U^c are both non-empty) one defines the Apollonian (semi-, pseudo-)distance between points $x_1, x_2 \in U$:

$$(1.3) \quad d_U(x_1, x_2) = \sup_{u_1, u_2 \in U^c} \log[x_1, x_2; u_1, u_2] \in [0, +\infty].$$

We denote by $GM(\hat{H})$ the general conformal group which acts continuously upon $(\hat{H}, \hat{d})$ and is generated by the set of isometries, homotheties (both fixing ∞) and the inversion (which exchanges the origin and ∞):

$$(1.4) \quad I(x) = \frac{x}{\langle x, x \rangle}, \quad I(0) = \infty \quad \text{and} \quad I(\infty) = 0.$$

When $\dim H \geq 3$, the Liouville theorem (see e.g. [Nev60]) shows that any conformal map is in $GM(\hat{H})$. In dimensions 1 or 2, it is the Möbius group (supplemented with complex conjugation in the 2 dimensional case). That d_U is $GM(\hat{H})$ invariant is trivial for isometries and homotheties and in the case of inversions it follows easily from the formula $\|I(x) - I(y)\| = \frac{\|x - y\|}{\|x\| \|y\|}$ (with some care taken with respect to the point at infinity). From the cross-ratio identity $[x, z; u, v] = [x, y; u, v] [y, z; u, v]$ and taking sup in the right order, one also sees that d_U verifies the triangular inequality. When U^c has non-empty interior, d_U is a genuine metric, but in the general case

it need not distinguish points. We refer to e.g. [Bea98, Chapter 3] and [Has04] for further details on the geometry of this metric. Our main result is the following:

Theorem 1.2 (Main Theorem). *Let $U \subset V \subset \widehat{H}$ be non-empty proper subsets with d_U and d_V being the associated Apollonian metrics. Let $\Delta = \sup_{u_1, u_2 \in U} d_V(u_1, u_2)$ be the diameter of the smaller subset within the larger. Then $\forall x_1, x_2 \in U$:*

$$(1.5) \quad d_V(x_1, x_2) \leq \left(\tanh \frac{\Delta}{4} \right) d_U(x_1, x_2).$$

If $\text{diam}_V(U) < +\infty$, the embedding $i : (U, d_V) \hookrightarrow (U, d_U)$ is a uniform contraction.

Proof. We will base our proof upon Birkhoff's inequality [Bir57] for cross-ratios on the projective real line. It is, in fact, a special case of our main theorem when $n = 1$. We will use it in the following version: Let $K = (a_1, a_2)$ be a non-empty open sub-interval of $J = (0, +\infty)$. The Hilbert distance of $s_1, s_2 \in K$ relative to K and J are given by:

$$d_K(s_1, s_2) = \left| \log[s_1, s_2; a_1, a_2] \right| \quad \text{and} \quad d_J(s_1, s_2) = \left| \log \frac{s_2}{s_1} \right|.$$

The quantity $\Delta = \text{diam}_J(K) = \log \frac{a_2}{a_1} \in (0, +\infty]$ measures the diameter of K for the J -metric. Birkhoff [Bir57, p. 220] showed the fundamental inequality :

$$(1.6) \quad d_J(s_1, s_2) \leq \left(\tanh \frac{\Delta}{4} \right) d_K(s_1, s_2), \quad \forall s_1, s_2 \in K.$$

□

Proof of (1.6). It suffices to show this for s_1 and s_2 infinitesimally close. So we differentiate with respect to s_2 at $s_2 = s_1 = s \in (a_1, a_2)$ and search for the optimal value of $\theta > 0$ so that for every $a_1 < s < a_2$: $\frac{1}{s} \leq \theta \frac{a_1 - a_2}{(s - a_1)(a_2 - s)}$, or equivalently

$$(1.7) \quad \theta \geq \inf_{a_1 < s < a_2} \frac{(s - a_1)(a_2 - s)}{s(a_2 - a_1)}.$$

The minimum value at $s = \sqrt{a_1 a_2}$ is $\theta_{\min} = \frac{\sqrt{a_2} - \sqrt{a_1}}{\sqrt{a_2} + \sqrt{a_1}} = \tanh \frac{\log(a_2/a_1)}{4}$ which is therefore the desired contraction constant.

Now, returning to the general case, let $x_1, x_2 \in U$ be distinct points. We have $d_V(x_1, x_2) \leq d_U(x_1, x_2)$ since the sup in the latter case is over a larger set. We may assume that $\Delta = \text{diam}_V(U) < +\infty$ and $0 < d_V(x_1, x_2) \leq d_U(x_1, x_2) < +\infty$ (or else the statement is trivial). Let $\epsilon > 0$ and pick $v_1, v_2 \in V^c$ so that

$$d_V(x_1, x_2) \leq (1 + \epsilon) \log[x_1, x_2; v_1, v_2].$$

To simplify calculations, we choose a transformation in $GM(\widehat{H})$ which maps v_1 to zero and v_2 to infinity. We recall that this preserves cross-ratios. By a slight abuse of notation we still write x_1, x_2 for the images in $\widehat{H}$ of the corresponding points. We have then $0 < d_V(x_1, x_2) \leq (1 + \epsilon) \log \frac{\|x_2\|}{\|x_1\|}$ so, in particular, $\|x_1\| < \|x_2\|$. When

FIGURE 1. Construction of cross-ratios. $v_1 = 0$ and $v_2 = \infty$.

$u_1, u_2 \in U$ we have, in these new coordinates,

$$\left| \log \frac{\|u_2\|}{\|u_1\|} \right| = |\log[u_1, u_2, 0, \infty]| \leq d_V(u_1, u_2) \leq \Delta < +\infty.$$

In other words, U is bounded away from the origin and infinity.

Consider now the formula for the distance of x_1, x_2 relative to U . It splits into a sum of two supremums (this splitting is one of the deeper reasons why the Apollonian metric is easy to handle):

$$d_U(x_1, x_2) = \sup_{u_1 \in U^c} \log \frac{\|x_2 - u_1\|}{\|x_1 - u_1\|} + \sup_{u_2 \in U^c} \log \frac{\|x_1 - u_2\|}{\|x_2 - u_2\|}.$$

The suprema of these two terms are denoted α_1 and α_2 . They are both finite. We define the Apollonian ball

$$B_1 = B_{\alpha_1}(x_1, x_2) = \left\{ u \in \widehat{H} : \frac{\|x_1 - u\|}{\|x_2 - u\|} < \alpha_1 \right\} \subset U$$

and similarly for the ball $B_2 = B_{\alpha_2}(x_2, x_1) \subset U$ (see Figure 1).

A priori B_1 is a generalized open ball containing x_1 but as U is bounded, B_1 must be an open ball in the usual bounded sense (and α_1 must be greater than one). Let $t_1 x_1$ (with $0 < t_1 < 1$) be the intersection between the segment $\{t x_1 : 0 \leq t \leq 1\}$ and the sphere $\partial B_{u_1}(x_1, x_2)$. Similarly, let $t_2 x_2$ (with $1 < t_2 < +\infty$) be the intersection between the segment $\{t x_2 : 1 \leq t \leq +\infty\}$ and $\partial B_{u_2}(x_2, x_1)$ (see Figure 1). Then $t_1 x_1, t_2 x_2 \in \text{Cl } U$ and $\|t_1 x_1\| < \|x_1\| < \|x_2\| < \|t_2 x_2\|$. From the way we defined t_1 and t_2 we have the following lower bound

$$\begin{aligned} d_U(x_1, x_2) = \alpha_1 + \alpha_2 &= \log \frac{\|x_2 - t_1 x_1\|}{\|x_1 - t_1 x_1\|} \times \frac{\|x_1 - t_2 x_2\|}{\|x_2 - t_2 x_2\|} \\ &= \log \frac{\|x_2 - t_1 x_1\|}{\|x_1\| - \|t_1 x_1\|} \times \frac{\|x_1 - t_2 x_2\|}{\|t_2 x_2\| - \|x_2\|} \\ (1.8) \quad &\geq \log \frac{\|x_2\| - \|t_1 x_1\|}{\|x_1\| - \|t_1 x_1\|} \times \frac{\|t_2 x_2\| - \|x_1\|}{\|t_2 x_2\| - \|x_2\|}. \end{aligned}$$

The last expression is the cross-ratio of the four (ordered) points on the positive real line $0 < \|t_1 x_1\| < \|x_1\| < \|x_2\| < \|t_2 x_2\| < +\infty$. Let us write $J = (0, \infty)$, $K = (\|t_1 x_1\|, \|t_2 x_2\|)$ and $s_1 = \|x_1\|$, $s_2 = \|x_2\|$. By our construction

$$\text{diam}_J(K) = \log \frac{\|t_2 x_2\|}{\|t_1 x_1\|} \leq d_V(t_1 x_1, t_2 x_2) \leq \text{diam}_V(U) = \Delta,$$

where we used that $t_1x_1, t_2x_2 \in \text{Cl } U$ and that $v_1 = 0, v_2 = \infty \in V$. Also, $d_K(\|x_1\|, \|x_2\|) \leq d_U(x_1, x_2)$ by the above bound (1.8). So using Birkhoff's inequality (1.6) we get

$$\frac{d_V(x_1, x_2)}{1 + \epsilon} \leq d_J(s_1, s_2) \leq \left(\tanh \frac{\text{diam}_J(K)}{4} \right) d_K(s_1, s_2) \leq \left(\tanh \frac{\Delta}{4} \right) d_U(x_1, x_2),$$

and since $\epsilon > 0$ was arbitrary we see that $d_V(x_1, x_2) \leq \left(\tanh \frac{\Delta}{4} \right) d_U(x_1, x_2)$, which is what we aimed to show. $\square$

2. SOME APPLICATIONS

In the one dimensional case, the result of Birkhoff [Bir57] has a vast variety of applications related to the Perron-Frobenius type of results and the presence of spectral gaps of real operators contracting a real convex cone; see e.g. [Bal00]. In the case of complex operators similar spectral gap results were obtained first in [Rug10] and then simplified in [Dub09] using a complex Hilbert metric and the 2-dimensional version of the UCP for the Apollonian metric. We discuss in the following some possible applications in the case of arbitrary dimension.

Corollary 2.1. *Let $U \subset V$ and Δ be as in the main theorem and write $\Gamma(V, U) = \{\gamma \in GM(\widehat{H}) : \gamma(V) \subset U\}$ for the elements of the conformal group that map V into U . Then for every $\gamma \in \Gamma(V, U)$ we have $\gamma^{-1} \in \Gamma(U^c, V^c)$ and the mappings $\gamma : (V, d_V) \rightarrow (V, d_V)$ and $\gamma^{-1} : (U^c, d_{U^c}) \rightarrow (U^c, d_{U^c})$ are $(\tanh \frac{\Delta}{4})$ -Lipschitz.*

Proof. $\gamma \in \Gamma(U, V)$ preserves cross-ratios, and $\gamma(V) \subset U$ so writing $\theta = \tanh \Delta/4$ we have for $v_1, v_2 \in V$:

$$d_V(\gamma(v_1), \gamma(v_2)) \leq \theta d_U(\gamma(v_1), \gamma(v_2)) \leq \theta d_{\gamma(V)}(\gamma(v_1), \gamma(v_2)) \leq \theta d_V(v_1, v_2).$$

The inverse map is bijective so it maps U^c into V^c . We have the same bound for its contraction rate since

$$\text{diam}_V(U) = \text{diam}_{U^c}(V^c) = \sup_{v_1, v_2 \in V} \sup_{u_1, u_2 \in U^c} \log[u_1, u_2; v_1, v_2].$$

$\square$

Corollary 2.2. *In finite dimension when $\text{Cl } U \subset \text{Int } V$ for the topology of $(\widehat{H}, \widehat{d})$, compactness shows that $\text{diam}_V(U) < +\infty$ so the embedding $(U, d_U) \hookrightarrow (V, d_V)$ is a strict Lipschitz contraction.*

Lemma 2.3. *Suppose that $U \subset B(x_0, R)$, $R < \infty$. Then*

$$(2.1) \quad \|u_1 - u_2\| \leq \frac{R}{2} d_U(u_1, u_2), \quad \forall u_1, u_2 \in U.$$

Suppose that $U \subset V$ and that $r = \text{dist}(U, V^c) = \sup_{u \in U, w \in V^c} \|u - w\| > 0$. Then

$$(2.2) \quad d_V(u_1, u_2) \leq \frac{2}{r} \|u_1 - u_2\|, \quad \forall u_1, u_2 \in U.$$

Proof. When $x \in B(x_0, R)$ and h is small we get from a straight-forward calculation:

$$d_B(x, x+h) = \frac{2R}{R^2 - \|x - x_0\|^2} \|h\| + o(h).$$

Thus, $ds = \frac{2R}{R^2 - \|x - x_0\|^2} \|dx\| \geq \frac{2}{R} \|dx\|$ and $\|v_1 - v_2\| \leq d_B(v_1, v_2) \leq d_V(v_1, v_2)$ (since $V \subset B$).

If $B(u_1, r), B(u_2, r) \subset V$, then we get for $w \in V^c$: $\frac{\|u_2 - w\|}{\|u_1 - w\|} \leq 1 + \frac{\|u_2 - u_1\|}{r}$
and $d_V(u_1, u_2) \leq 2 \log \left(1 + \frac{\|u_2 - u_1\|}{r} \right) \leq \frac{2}{r} \|u_2 - u_1\|$. $\square$

Theorem 2.4. *Let $U \subset V$ be non-empty proper subsets of $(\widehat{H}, \widehat{d})$ such that $\text{Cl } V \neq \widehat{H}$ and $\Delta = \text{diam}_V(U) < +\infty$. Let $\gamma_1, \dots, \gamma_k \in \Gamma(V, U)$ and write*

$$\Lambda \equiv \Lambda(\gamma_1, \dots, \gamma_k) = \bigcap_{n \geq 1} \text{Cl} \bigcup_{1 \leq i_1, \dots, i_n \leq k} \gamma_{i_1} \circ \dots \circ \gamma_{i_n}(V)$$

for the associated limit set. Then Λ is compact and has Hausdorff and Box dimensions not greater than $-\log k / \log \tanh \frac{\Delta}{4}$.

Proof. Pick $q \in \widehat{H} \setminus \text{Cl } V$ and map q to infinity by an inversion in q . In the new coordinates, V is bounded, so by the previous lemma, Hilbert distances are bounded by Apollonian distances. At level $n \geq 1$ each set in the finite union has diameter not greater than $r = \Delta(\tanh \frac{\Delta}{4})^{n-1}$ which becomes arbitrarily small as $n \rightarrow \infty$. There are $N_r = k^n$ elements in the union. As Λ is closed and has finite covers of arbitrarily small diameters it is compact and we have the bound

$$\dim_H(\Lambda) \leq \limsup_n \frac{\log N_r}{\log 1/r} = \frac{\log k}{\log \tanh \frac{\Delta}{4}}.$$

$\square$

When the images $\text{Cl}(\gamma_i(V))$, $1 \leq i \leq k$ are pairwise disjoint, the Hausdorff dimension may also be obtained from a Bowen-like formula as in [Rug08] or [MU98]. We omit the details. Note that we do not assume here that H is finite dimensional.

Remark 2.5. In finite dimension $d \geq 2$ the Apollonian metric for an open ball $V = B(0, R)$ is the same as the hyperbolic metric for the ball, i.e. $ds = \frac{2r\|dx\|}{r^2 - \|x\|^2}$. In this case it is well known that if γ maps V inside V and $\gamma(V)$ has bounded diameter, then γ is a uniform contraction.

Other metrics may be constructed from the Apollonian metric (cf. [Has04]). Let $v \in H^*$, $\|h\| \leq \|v\|/4$ and write $x = \frac{\langle h, v \rangle}{\langle v, v \rangle} \in [-1/4, 1/4]$ and $\|h\|^2 = \|v\|^2(x^2 + y^2)$. Calculus shows that $\left| \frac{1}{2} \log((1+x)^2 + y^2) - x \right| \leq x^2 + y^2$ (when $x \geq -1/4$). Therefore,

$$\left| \log \frac{\|v+h\|}{\|v\|} - \langle I(v), h \rangle \right| = \left| \log \frac{\|v+h\|}{\|v\|} - \frac{\langle v, h \rangle}{\langle v, v \rangle} \right| \leq \frac{\langle h, h \rangle}{\langle v, v \rangle}.$$

Assume in the following that U is open. Let $x \in U$ and set $r = \inf_{u \in U^c} d(x, u) > 0$. When $u_1, u_2 \in U^c$ and $\|h\| \leq r/4$ we get:

$$|\log[x, x+h; u_1, u_2] - \langle I(x-u_1) - I(x-u_2), h \rangle| \leq 2\|h\|^2/r^2.$$

It follows that the following limit exists and define a Finsler (pseudo-)norm on the tangent space of U :

$$(2.3) \quad p_{U,x}(h) \equiv \lim_{t \rightarrow 0} \frac{1}{t} d_U(x + th, x) = \sup_{u_1, u_2 \in U^c} |\langle I(x - u_1) - I(x - u_2), h \rangle|.$$

It is only a pseudo-norm when U^c is contained in a generalized ball, since in that case $p_{U,x}$ may vanish in some directions. If $\gamma : [0, 1] \rightarrow U$ is a continuous path, then we may define its (pseudo-)length to be

$$\ell(\gamma) \equiv \limsup_{\delta \rightarrow 0} \sum_{k=0}^n d_U(\gamma(t_{k+1}), \gamma(t_k)),$$

where $0 = t_0 < t_1 < \dots < t_n = 1$ and $t_{k+1} - t_k < \delta$. Then

$$(2.4) \quad d_U^{\text{in}}(x, y) = \inf \{ \ell(\gamma) : \gamma \in C([0, 1], U), \gamma(0) = x, \gamma(1) = y \}$$

defines a (pseudo-)metric which in [Has04] was coined the Apollonian inner metric. When γ is piecewise C^1 , we have $\ell(\gamma) = \int_0^1 p_{U,x}(\dot{\gamma}(t)) dt$. Another possibility is to maximize (2.3) over directions. This leads to a conformal Riemannian metric $ds = g_U(x) \|dx\|$ with

$$(2.5) \quad g_U(x) = \sup_{\|h\|=1} p_{U,x}(h) = \sup_{u_1, u_2 \in U^c} \frac{\|u_1 - u_2\|}{\|x - u_1\| \|x - u_2\|}.$$

An advantage of this metric is perhaps that it distinguishes points when U^c contains at least two points. It is easy to see that $g_U(x)$ is continuous (as we assumed U to be open). We write $d_U^{\text{Rie}}(x, y)$ for the Riemannian distance of x and y with respect to this metric.

Corollary 2.6. *Let $U \subset V \subset \widehat{H}$ (with $\text{Cl } V \neq \widehat{H}$) be non-empty proper subsets and let $\Delta = \sup_{u_1, u_2 \in U} d_V(u_1, u_2)$ be the diameter of the smaller subset within the larger with respect to the Apollonian metric. Then for every $x, y \in U$:*

$$(2.6) \quad p_{V,x}(h) \leq \left(\tanh \frac{\Delta}{4} \right) p_{U,x}(h), \quad h \in E,$$

$$(2.7) \quad d_V^{\text{in}}(x, y) \leq \left(\tanh \frac{\Delta}{4} \right) d_U^{\text{in}}(x, y),$$

$$(2.8) \quad d_V^{\text{Rie}}(x, y) \leq \left(\tanh \frac{\Delta}{4} \right) d_U^{\text{Rie}}(x, y).$$

Proof. For $x, x + th \in U$ we have by the Main Theorem

$$\frac{1}{t} d_V(x, x + th) \leq \tanh \frac{\Delta}{4} \frac{1}{t} d_U(x, x + th).$$

The first inequality follows. The second follows by taking limits in the right order. For the Riemannian metric, one has

$$g_V(x) \leq \sup_{\|h\|=1} p_{V,x}(h) \leq \sup_{\|h\|=1} \left(\tanh \frac{\Delta}{4} \right) p_{U,x}(h) = \left(\tanh \frac{\Delta}{4} \right) g_U(x)$$

which yields the last inequality. $\square$

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