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THE SPACE-LIKE SURFACES WITH VANISHING CONFORMAL FORM IN THE CONFORMAL SPACE

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ABSTRACT. The conformal geometry of surfaces in the conformal space $\mathbf{Q}_1^n$ is studied. We classify the space-like surfaces in $\mathbf{Q}_1^n$ with vanishing conformal form up to conformal equivalence.

1. INTRODUCTION

In [1] Wang gave the structure equations for Möbius geometry of submanifolds in the unit sphere. Three fundamental tensors $\mathbb{A}$, $\mathbb{B}$ and Φ arise naturally in the structure equations. In [1] $\mathbb{A}$ is called the Blaschke tensor, $\mathbb{B}$ the Möbius second fundamental form, and Φ the Möbius form. Together with Möbius metric g, these tensors determine the submanifold up to Möbius transformations of the unit sphere.

Li and Wang [2] classified surfaces with vanishing Möbius form in sphere space $\mathbb{S}^{n+1}$. Readers should be reminded that Bryant [3] have classified all minimal surfaces with constant curvature in the unit sphere $\mathbf{S}^n$, the hyperbolic space $\mathbf{H}^n$ and Euclidean space $\mathbf{R}^n$. Li and Wang used Bryant's results. It is interesting to classify stationary surfaces with constant curvature in $\mathbf{R}_1^n, \mathbf{S}_1^n$ or $\mathbf{H}_1^n$. For some other results about Lorentz conformal geometry, see [4]–[7]. Further relative knowledge refers to [8]–[10].

We conclude this paper with the following.

The Main Theorem. *Let $x : \mathbf{M} \to \mathbf{Q}_1^n$ be a regular spacelike full surface with vanishing form. Then x is one of the following four alternatives:*

(i) *x is a stationary surface with constant curvature in $\mathbf{R}_1^n, \mathbf{S}_1^n$ or $\mathbf{H}_1^n$.*

(ii) *x is a hyperbolic cylinder $\mathbf{H}^1 \times \mathbf{R}$ in $\mathbf{R}_1^3$.*

(iii) *x is a surface $\mathbf{H}^1(\sqrt{r}) \times \mathbf{S}^1(\sqrt{1+r})$ in $\mathbf{S}_1^3$, $r > 0$.*

(iv) *x is a hyperbolic torus $\mathbf{H}^1(\sqrt{1+r}) \times \mathbf{H}^1(\sqrt{-r})$ in $\mathbf{H}_1^3$, $-\frac{1}{2} \le r < 0$.*

2. THE FUNDAMENTAL EQUATIONS

Let $\mathbb{R}_s^n$ be the real vector space $\mathbb{R}^n$ with the Lorentzian inner product $\langle , \rangle$ given by

$$\langle X, Y \rangle := \sum_{i=1}^{n-s} x_i y_i - \sum_{i=n-s+1}^{n} x_i y_i,$$

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where $X = (x_1, \cdots x_n), Y = (y_1, \cdots, y_n) \in \mathbb{R}^n$. We define the de Sitter sphere $\mathbf{S}_1^n$ and anti-de Sitter sphere $\mathbf{H}_1^n$ by

$$\mathbf{S}_1^n = \{u \in \mathbf{R}_1^{n+1} | \langle u, u \rangle = 1\}, \quad \mathbf{H}_1^n = \{u \in \mathbf{R}_2^{n+1} | \langle u, u \rangle = -1\}.$$

We call Lorentzian space $\mathbf{R}_1^n$, de Sitter sphere $\mathbf{S}_1^n$ and anti-de Sitter sphere $\mathbf{H}_1^n$ Lorentzian space forms. Denote

$$\mathbf{Q}_1^n = \{[x] \text{ is the projective coordinates} | \langle x, x \rangle = 0, x \in \mathbf{R}_2^{n+2}\}.$$

By some conformal diffeomorphisms we may regard $\mathbf{Q}_1^n$ as the common compactified space of $\mathbf{R}_1^n, \mathbf{S}_1^n$ and $\mathbf{H}_1^n$. In fact the conformal space $\mathbf{Q}_1^n$ has a standard Lorentzian metric. We research the conformal geometry of surfaces under the conformal group of this Lorentzian metric. We refer the reader to [4] and [7] for further details.

Suppose that $x : \mathbf{M} \to \mathbf{Q}_1^n$ is a space-like surface. That is, $x_*(\mathrm{T}\mathbf{M})$ is a non-degenerated subbundle of $\mathrm{T}\mathbf{Q}_1^n$. Let $y : U \to \mathbf{R}_2^{n+2}$ be a lift of $x : \mathbf{M} \to \mathbf{Q}_1^n$ defined in an open subset U of $\mathbf{M}$. We denote by Δ and κ, Laplacian and the normalized scalar curvature of the local positive definite metric $\langle dy, dy \rangle$. Then we have the following theorem.

Theorem 2.1 (see [1], Theorem 1.2)**.** *On* $\mathbf{M}$ *the* 2*-form* $g = -(\langle \Delta y, \Delta y \rangle - 4\kappa)\langle dy, dy \rangle$ *is a globally defined invariant of* $x : \mathbf{M} \to \mathbf{Q}_1^n$ *under the Lorentz group transformations of* $\mathbf{Q}_1^n$.

Let $x : \mathbf{M}^2 \to \mathbf{Q}_1^n$ be a *regular* space-like surface. That is, the 2-form $g = -(\langle \Delta y, \Delta y \rangle - 4\kappa)\langle dy, dy \rangle$, which is called a conformal metric, is non-degenerated. Let $Y = \sqrt{-(\langle \Delta y, \Delta y \rangle - 4\kappa)}y$ be the canonical lift of x and define $N : \mathbf{M} \to \mathbf{R}_2^{n+2}$ by $N = -\frac{1}{2}\Delta Y - \frac{1}{8}\langle \Delta Y, \Delta Y \rangle Y$. Let $\{E_\alpha\}$ be a local basis of the conformal normal bundle $\mathbf{V}$ of x. If $z = u + \mathrm{i}v$ is a local isothermal coordinate on $\mathbf{M}$ for g, we can write $g = e^{2\omega}|dz|^2 = \frac{1}{2}e^{2\omega}(dz \otimes d\bar{z} + d\bar{z} \otimes dz)$ for some local smooth function ω. Denote by K the Gauss curvature of g; we have $\Delta Y = 4e^{-2\omega}Y_{z\bar{z}}, K = -4e^{-2\omega}\omega_{z\bar{z}}$. Since $\{Y, N, \mathrm{Re}(Y_z), \mathrm{Im}(Y_z), E_\alpha\}$ is a moving frame in $\mathbf{R}_2^{n+2}$ along $\mathbf{M}$, one can write the structure equations and the fundamental equations as in [2]. Define

$$\psi = 2\langle N_z, Y_z \rangle, \quad \phi_\alpha = \langle N_z, E_\alpha \rangle, \quad \Omega_\alpha = 2\langle Y_{zz}, E_\alpha \rangle, \quad A_{\alpha\beta} = \langle (E_\alpha)_z, E_\beta \rangle, \tag{2.1}$$

and $\phi^\alpha = \sum_\beta g^{\alpha\beta}\phi_\beta, \quad \Omega^\alpha = \sum_\beta g^{\alpha\beta}\Omega_\beta, \quad A_\alpha^\beta = \sum_\gamma g^{\beta\gamma}A_{\alpha\gamma}$. The structure equations are

$$N_z = \frac{1}{8}(4K-1)Y_z + e^{-2\omega}\psi Y_{\bar{z}} + \sum_\alpha \phi^\alpha E_\alpha, \tag{2.2}$$

$$Y_{zz} = -\frac{1}{2}\psi Y + 2\omega_z Y_z + \frac{1}{2}\sum_\alpha \Omega^\alpha E_\alpha, \tag{2.3}$$

$$Y_{z\bar{z}} = -\frac{1}{16}e^{2\omega}(4K-1)Y - \frac{1}{2}e^{2\omega}N,$$

$$(E_\alpha)_z = -\phi_\alpha Y - e^{-2\omega}\Omega_\alpha Y_{\bar{z}} + \sum_\beta A_\alpha^\beta E_\beta. \tag{2.4}$$

The fundamental equations are

$$\psi_{\bar z} = \frac{1}{2}e^{2\omega}K_z - \sum_\alpha \Omega^\alpha \bar\phi_\alpha, \ \sum_\alpha \Omega^\alpha\Omega_\alpha \tag{2.5}$$

$$= -\frac{1}{4}e^{4\omega}, \ (\Omega_\alpha)_{\bar z} = -\sum_\beta \Omega^\beta \bar A_{\beta\alpha} - e^{2\omega}\phi_\alpha, \tag{2.6}$$

$$(\phi_\alpha)_{\bar z} - \frac{1}{2}e^{-2\omega}\bar\psi\Omega_\alpha + \sum_\beta \phi^\beta \bar A_{\beta\alpha} = (\bar\phi_\alpha)_z - \frac{1}{2}e^{-2\omega}\psi\bar\Omega_\alpha + \sum_\beta \bar\phi^\beta A_{\beta\alpha}, \tag{2.7}$$

$$(A_{\alpha\beta})_{\bar z} - (\bar A_{\alpha\beta})_z = \frac{1}{2}e^{-2\omega}(\Omega_\alpha\bar\Omega_\beta - \bar\Omega_\alpha\Omega_\beta) + \sum_\gamma(\bar A_{\alpha\gamma}A^\gamma_\beta - A_{\alpha\gamma}\bar A^\gamma_\beta). \tag{2.8}$$

Remark 2.1. $\Psi = \psi dz\otimes dz, \Phi = \sum_\alpha(\phi^\alpha dz + \bar\phi^\alpha d\bar z)\otimes E_\alpha$ and $\Omega = \sum_\alpha \Omega^\alpha dz\otimes dz\otimes E_\alpha$ are globally defined conformal invariants.

Remark 2.2. The Willmore equations are

$$(\phi_\alpha)_{\bar z} - \frac{1}{2}e^{-2\omega}\bar\psi\Omega_\alpha + \sum_\beta \phi^\beta \bar A_{\beta\alpha} = 0, \forall\alpha. \tag{2.9}$$

3. The classification of space-like surfaces in $\mathbf{Q}^n_1$ with $\Phi = 0$

Let $x : \mathbf{M} \to \mathbf{Q}^n_1$ be a space-like surface in $\mathbf{Q}^n_1$ with vanishing conformal form, i.e., $\phi_\alpha = 0, 3 \le \alpha \le n$. Then the fundamental equations come into

$$\psi_{\bar z} = \frac{1}{2}e^{2\omega}K_z, \quad \bar\psi\Omega_\alpha = \psi\bar\Omega_\alpha, \quad e^{-4\omega}\sum_\alpha \Omega^\alpha\Omega_\alpha = -\frac{1}{4}, \tag{3.1}$$

$$(\Omega_\alpha)_{\bar z} = -\sum_\beta \Omega^\beta \bar A_{\beta\alpha}, \quad A_{\alpha\beta} = -A_{\beta\alpha}. \tag{3.2}$$

It follows from (3.2) that

$$(\sum_\alpha \Omega^\alpha\Omega_\alpha)_{\bar z} = 0, \tag{3.3}$$

thus the globally defined 4-form $\sum_\alpha \Omega^\alpha\Omega_\alpha dz^4$ is holomorphic on $\mathbf{M}$. From (3.1) we get

$$\bar\psi\sum_\alpha \Omega^\alpha\Omega_\alpha = \sum_\alpha \Omega^\alpha\bar\Omega_\alpha = -\frac{1}{4}e^{4\omega}\psi. \tag{3.4}$$

Immediately we have the following lemma.

Lemma 3.1. *Let $x : \mathbf{M} \to \mathbf{Q}^n_1$ be a surface in $\mathbf{Q}^n_1$ with vanishing conformal form. Then the conformal invariant $\Psi = \psi dz^2$ is holomorphic on $\mathbf{M}$. Then Ψ vanishes identically or the zero points of Ψ are isolated.*

First we consider the case that $\Psi \equiv 0$. Thus from (3.1)*, K must be a constant. Then we get from* (2.3) *that*

$$N = \frac{1}{8}(4K-1)Y + \mathbf{c},$$

for some constant vector $\mathbf{c} \neq 0$ in $\mathbf{R}^{n+2}_2$. Therefore x is a conformal isotropic surface, i.e., x is a stationary surface with constant curvature in $\mathbf{R}^n_1, \mathbf{S}^n_1$, or $\mathbf{H}^n_1$ (see [7]*).*

Now we come to discuss the case that the zero points of Ψ are isolated. In this case we can cut $\mathbf{M}$ by some disjoint curves C_i to get a simply connected domain $\mathbf{U} = \mathbf{M}\backslash \sum_i C_i$ such that $x : \mathbf{U} \to \mathbf{Q}_1^n$ is a surface with $\Psi \neq 0$ on U. By choosing a complex coordinate if necessary, we may assume that $\psi \equiv 1$ on $\mathbf{U}$. It follows from (2.1) and (3.1) that $K = -4e^{-2\omega}\omega_{z\bar{z}} = 0$. By (3.3) we know that $\{\Omega_\alpha\}$ are real functions. We define a global real vector field $E \in \mathbf{V}$ by

$$E = 2e^{-2\omega} \sum_a \Omega^\alpha E_\alpha, \tag{3.5}$$

then by (3.2) we have $\langle E, E\rangle = -1$. Choosing $\tilde{E}_3 = E$ and expanding it to a local orthonormal basis of the conformal normal bundle $\{\tilde{E}_3, \tilde{E}_4, \cdots, \tilde{E}_n\}$, one can easily verify that

$$\tilde{\Omega}_3 = -\frac{1}{2}e^{2\omega}, \tilde{\Omega}_4 = \cdots = \tilde{\Omega}_n = 0. \tag{3.6}$$

Using (3.3), we get

$$(\tilde{\Omega}_3)_{\bar{z}} = -\sum_{\beta=3}^{n} \tilde{\Omega}^\beta \bar{A}_{\beta 3} = -\tilde{\Omega}_3 \bar{A}_{33} = 0, \tag{3.7}$$

$$0 = (\tilde{\Omega}_\alpha)_{\bar{z}} = -\sum_{\beta\neq\alpha} \tilde{\Omega}^\beta \bar{A}_{\beta\alpha} = -\tilde{\Omega}_3 \bar{A}_{3\alpha}, \alpha > 3. \tag{3.8}$$

Thus ω is a constant and $A_{3\alpha} = 0, \forall\alpha$. Now the structure equations read

$$N_z = \frac{1}{8}Y_z + e^{-2\omega}Y_{\bar{z}}, \tag{3.9}$$

$$Y_{zz} = -\frac{1}{2}Y - \frac{1}{4}e^{2\omega}\tilde{E}_3, \quad Y_{z\bar{z}} = \frac{1}{16}e^{2\omega}Y - \frac{1}{2}e^{2\omega}N, \tag{3.10}$$

$$(\tilde{E}_3)_z = \frac{1}{2}Y_{\bar{z}}, \quad (\tilde{E}_\alpha)_z = \sum_\beta A_\alpha^\beta \tilde{E}_\beta, \quad \alpha > 3. \tag{3.11}$$

A surface is said to be full in $\mathbf{Q}_1^n$ if $x(\mathbf{M})$ does not lie in any totally umbilic $\mathbf{Q}^{n-1}$ of $\mathbf{Q}_1^n$. We assume that $n \geq 4$. Then fixing a point $p \in U$, we can find a constant vector $\xi \in \mathbf{R}_2^{n+2}$ with $\langle \xi, \xi\rangle = 1$ such that

$$\begin{aligned} \langle Y(p), \xi\rangle = 0, \quad \langle N(p), \xi\rangle = 0, \quad \langle Y_u(p), \xi\rangle = 0,& \\ \langle Y_v(p), \xi\rangle = 0, \quad \langle \tilde{E}_3(p), \xi\rangle = 0.& \end{aligned} \tag{3.12}$$

We define real functions

$$\begin{aligned} f_1 = \langle Y, \xi\rangle, \quad f_2 = \langle N, \xi\rangle, \quad f_3 = \langle Y_u, \xi\rangle,& \\ f_4 = \langle Y_v, \xi\rangle, f_5 = \langle \tilde{E}_3, \xi\rangle.& \end{aligned} \tag{3.13}$$

Then by (3.11)–(3.13) we can find constants $\{a_{\lambda\mu}\}$ and $\{b_{\lambda\mu}\}$ such that

$$(f_\lambda)_u = \sum_\mu a_{\lambda\mu} f_\mu, \quad (f_\lambda)_v = \sum_\mu b_{\lambda\mu} f_\mu, \quad 1 \leq \lambda, \mu \leq 5. \tag{3.14}$$

By (3.12) and the uniqueness of the linear PDE (3.14) we get $f_\lambda \equiv 0$. In particular, $f_1 = \langle Y, \xi\rangle = 0$ on U, which implies that $\langle Y, \xi\rangle = 0$ on $\mathbf{M}$. If x is full, then n must be 3.

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