CONFORMAL GEOMETRY AND DYNAMICS
An Electronic Journal of the American Mathematical Society
Volume 16, Pages 256–268 (August 28, 2012)
S 1088-4173(2012)00242-1

A FATOU TYPE THEOREM FOR COMPLEX MAP GERMS

LEONARDO CÂMARA AND BRUNO SCÁRDUA

ABSTRACT. In this paper we prove a Fatou type theorem for complex map germs. More precisely, we give (generic) conditions assuring the existence of parabolic curves for complex map germs tangent to the identity, in terms of existence of suitable formal separatrices. Such a map cannot have finite orbits.

1. INTRODUCTION

Let $\mathrm{Diff}(\mathbb{C}^n, 0)$ denote the group of germs of holomorphic diffeomorphisms of $\mathbb{C}^n$ fixing the origin and denote by $\mathrm{Diff}_1(\mathbb{C}^n, 0)$ the subgroup of $\mathrm{Diff}(\mathbb{C}^n, 0)$ given by those diffeomorphisms tangent to the identity. Such a map is also called *flat*. The dynamics of flat germs of diffeomorphisms is now beginning to be understood in dimension $n \geq 2$. Let us recall some results in this direction. Given a flat germ $G \in \mathrm{Diff}_1(\mathbb{C}^2, 0)$, write $G = (G_1, ..., G_n) \in \mathrm{Diff}_1(\mathbb{C}^n, 0)$ with $G_j = z_j + P_{j,\nu_j} + \cdots$ expanded in a series of homogeneous polynomials, where $P_{j,k} \equiv 0$ or $\deg(P_j, k) = k$ and $P_{j,\nu_j} \not\equiv 0$. Then one says that $\nu(G) = \min\{\nu_1, ..., \nu_n\}$ is the *order* of G. Let $G \in \mathrm{Diff}_1(\mathbb{C}^2, 0)$, then one says that $\varphi : \Omega \longrightarrow \mathbb{C}^n$ is a *parabolic curve* for G at the origin if it is an injective holomorphic map satisfying the following properties:

(i) $\Omega \subset \mathbb{C}$ is a simply connected domain with $0 \in \partial\Omega$;
(ii) φ is continuous at the origin and $\varphi(0) = 0 \in \mathbb{C}^n$;
(iii) $\varphi(\Omega)$ is invariant under G and $(G|_{\varphi(\Omega)})^{\circ(n)} \to 0 \in \mathbb{C}^n$ as $n \to \infty$.

Moreover, if $[\varphi(\zeta)] \to [v] \in \mathbb{P}^{n-1}$ as $\zeta \to 0$ (where $[\cdot]$ denotes the canonical projection of $\mathbb{C}^n \backslash \{O\}$ onto $\mathbb{P}^{n-1}$), we say that φ is *tangent* to $[v]$ at the origin. A *characteristic direction* for G is a vector $[v] = [v_1 : \cdots : v_n] \in \mathbb{P}^{n-1}$ such that there is $\lambda \in \mathbb{C}$ satisfying $P_{j,\nu_j}(G)(v_1, \cdots, v_n) = \lambda v_j$ for $j = 1, \ldots, n$. If $\lambda \neq 0$, then $[v]$ is called *non-degenerate*; otherwise, it is called *degenerate*.

The existence of parabolic curves is studied by Écalle (see [9]) and Hakim (see [11]) in connection with the existence of non-degenerate characteristic directions, which in a certain sense corresponds to a generic condition.

A germ admitting a parabolic curve has some non-finite orbits. In this sense, the next results are in the same framework of one-dimensional Camacho's theorem (see [5]) (see also Leau-Fatou theorem [8]), and apply to the characterization of germs of foliations admitting holomorphic first integral. From this viewpoint, our current work may be applied to correctly pursue the investigation started in [7].

The next results assure the existence of parabolic curves for germs of flat maps with non-degenerate characteristic directions.

Received by the editors September 20, 2011.
2010 *Mathematics Subject Classification.* Primary 32S65, 37F99; Secondary 32H50, 37F75, 37F10.
Key words and phrases. Complex diffeomorphism germ, parabolic curve, formal separatrix.

Theorem 1.1 (Écalle [9] and Hakim [11]). *Let G be a germ of a holomorphic self-map of $\mathbb{C}^n$ fixing the origin and tangent to the identity. Then for every non-degenerate characteristic direction $[v]$ of G there are $\nu(G)-1$ parabolic curves tangent to $[v]$ at the origin.*

Later on, Abate obtained the following result for a map with an isolated fixed point.

Theorem 1.2 (Abate [1]). *Let $G \in \mathrm{Diff}_1(\mathbb{C}^2, 0)$ and suppose that the origin is an isolated fixed point for G. Then G admits at least $\nu(G)-1$ parabolic curves.*

In [4], Brochero Martínez, Cano and López-Hernanz showed that Abate's theorem can be obtained as a consequence of Theorem 1.1.

Now consider a germ of diffeomorphism $G \in \mathrm{Diff}_1(\mathbb{C}^2, 0)$ tangent to the identity whose *set of fixed points* (denoted by $\mathrm{Fix}(G)$) contains a smooth curve S. Under this hypothesis Abate introduced an index for G at the origin with respect to this smooth curve S as follows. Let $S = \{x_1 = 0\}$, $G(x) = (G_1(x), G_2(x))$, and

$$k(x_2) := \lim_{x_1 \to 0} \frac{G_1(x) - x_1}{x_1(G_2(x) - x_2)},$$

thus $\mathrm{ind}_0(G, S) := \mathrm{Res}_{x_2=0}\, k(x_2)dx_2$, whenever defined. As a straightforward consequence of the proof of Corollary 3.1 in [1] we have the following result.

Theorem 1.3 (Abate [1]). *Let $G \in \mathrm{Diff}_1(\mathbb{C}^2, 0)$ and suppose $S = \mathrm{Fix}(G)$ is a smooth curve through the origin such that $\mathrm{ind}_0(G, S) \notin \mathbb{Q}^+$. Then G admits $\nu(G)-1$ parabolic curves.*

Recall that any map germ $G \in \mathrm{Diff}_1(\mathbb{C}^2, 0)$ can be uniquely written in the form $G(x) = x + \ell G^o(x)$ where $G^o(x)$ is not divisible by the irreducible components of ℓ and $\mathrm{Fix}(G) = \{\ell = 0\}$ is the set of fixed points of G. If $G^o = P^o_{\nu(G^o)} + P^o_{\nu(G^o)+1} + \cdots$ is the homogeneous decomposition of G^o , i.e., where P^o_ν are pairs of homogeneous polynomials of degree $\nu \geq 1$, then one says that G has *pure order* $\nu(G^o)$. Furthermore, following [1], we may suppose that after a finite number of blowups, $\nu(G^o) = 1$; in this case we say that G has pure order 1.

Recently, in [13], L. Molino obtained a result similar to Theorem 1.3 for maps of pure order 1.

Theorem 1.4 (L. Molino [13]). *Let $G \in \mathrm{Diff}_1(\mathbb{C}^2, 0)$ be a map of pure order 1 and suppose $S = \mathrm{Fix}(G)$ is a smooth curve through the origin such that $\mathrm{ind}_0(G, S) \neq 0$. Then G admits $\nu(G)-1$ parabolic curves.*

Let $G(x) = x + P_\nu(x) + O(\|x\|^{\nu+1})$ where $P_\nu(x) = (P_{1,\nu}(x), P_{2,\nu}(x)) \not\equiv 0$ with $P_{j,\nu}$ being a homogeneous polynomial of degree ν. Then we say that G is *dicritical* if $x_2 P_{1,\nu}(x) - x_1 P_{2,\nu}(x) \equiv 0$ and *non-dicritical* otherwise. The dynamics of dicritical maps is studied in [3]. Nevertheless, for our purposes, it suffices to use the relationship between parabolic curves and dicritical fixed points given by the following result.

Theorem 1.5 (Abate [1]). *Let $G \in \mathrm{Diff}_1(\mathbb{C}^2, 0)$ be a dicritical germ of a holomorphic map tangent to the identity, then G admits infinitely many parabolic curves.*

In this article we develop the viewpoint introduced in [4] and study conditions over the infinitesimal generator of G in order to determine the existence of parabolic curves.

Let $G \in \mathrm{Diff}_1(\mathbb{C}^2, 0)$ and $\hat{C}$ be an *algebroid curve* given by $\hat{f} \in \mathbb{C}[[x, y]]$ (see Remarque 1.4, page 2 [16] for the precise definition). Then $\hat{C}$ is said to be *invariant* by G whenever it induces an action in the ring $\mathbb{C}[[x, y]]/(\hat{f})$, i.e., if $\hat{f} \circ G \in (\hat{f})$, the principal ideal generated by $\hat{f}$ in $\mathbb{C}[[x, y]]$. In this case $\hat{C}$ is said to be a *formal separatrix* of G. The formal separatrix $\hat{C}$ is said to be *purely formal* if it does not admit any analytic divisor. Moreover, $\hat{C}$ is said to be *dynamically trivial with respect to* G, or also *completely fixed* by G, if the action of G on $\mathbb{C}[[x, y]]/(\hat{f})$ is given by the identity. Equivalently, G can be written in the form $G = \mathrm{Id} + \hat{f}\ \hat{G}_1$, where $\hat{G}_1$ has as coordinate functions, elements in $\mathbb{C}[[x, y]]$. The set of formal separatrices completely fixed by G will be denoted by $\hat{\mathrm{Fix}}(G)$.

Let $G \in \mathrm{Diff}_1(\mathbb{C}^2, 0)$, then after a finite number of blow ups, one has a germ of morphism $\pi : (\mathcal{M}, D) \longrightarrow (\mathbb{C}^2, 0)$ such that the formal separatrices of $\tilde{G} := \pi^* G$ are resolved and $\tilde{G}$ is locally written in the form $\tilde{G} = \exp[1]\ell \cdot \hat{X}_0$, where $\ell \in \mathcal{O}_2$ and the first jet $J^1(\hat{X}_0)$ is reduced in the sense of [6]. The germ of morphism so obtained, with the minimum number of blowing ups, is called the *minimal resolution* of G. Further let $\hat{C}$ be an algebroid curve given by $\hat{f} \in \mathbb{C}[x, y]$, then denote by $\tilde{C} := \pi^* C$ the algebroid curve given by $\tilde{\hat{f}} := \pi^* \hat{f} = \hat{f} \circ \pi$. Moreover, let $\mathrm{Sing}(\tilde{C}) \subset D$ be the singular set of $\tilde{C}$. Our main result is the following:

Theorem A. *Let $G \in \mathrm{Diff}_1(\mathbb{C}^2, 0)$ be a germ of a flat diffeomorphism admitting a non-dynamically trivial formal separatrix $\hat{C}$. Then G admits*

$$\sum_{p \in \mathrm{Sing}(\tilde{C})} (\nu(\tilde{G}_p) - 1)$$

parabolic curves tangent to the irreducible components of $\hat{C}$.

Let $G \in \mathrm{Diff}_1(\mathbb{C}^2, 0)$ be of pure order one with $S := \mathrm{Fix}(G)$ smooth. Recall from [4] that Abate's residual index of G with respect to S at the origin coincides with the Camacho-Sad index of its infinitesimal generator. Further, recall that generically $J^1(G^o)$ has two distinct eigenvalues, and thus is diagonalizable. Therefore, the following is a generalization of Theorem 1.3.

Theorem B. *Let $G \in \mathrm{Diff}_1(\mathbb{C}^2, 0)$ be a flat germ of a complex diffeomorphism and suppose that:*

(i) $\mathrm{Fix}(G)$ *is smooth;*
(ii) G *is of pure order one;*
(iii) $J^1(G^o)$ *is diagonal.*

Then G has an irreducible purely formal (non-dynamically trivial) separatrix $\hat{C}$. In particular, G admits $\nu(G) - 1$ parabolic curves tangent to $\hat{C}$.

At this point some comments are worthwhile. First remark that the above result gives an affirmative answer for the question of existence of parabolic curves for some cases not dealt with by the results of Hakim, Abate and Molino (cf. e.g. Example 2.14). Anyway this is not a full generalization of Molino's result because there is $G \in \mathrm{Diff}(\mathbb{C}^2, 0)$ such that $J^1 G_0$ is a Jordan cell type, G has a parabolic curve and does not admit a non-dynamically trivial invariant algebroid curve (cf. e.g. Example 3.1).

Regarding the main hypothesis in Theorem A, we observe that the existence of a non-dynamically trivial invariant curve is generic in the class of flat map germs, in the natural topology induced by the convergence of the truncations in n-th jet space for all non-negative integer n (cf. Definition 3.4, page 27 in [12], and [2], page A.IV.25). Indeed, the non-existence of such a (non-dynamically trivial) curve for a germ map $G \in \mathrm{Diff}(\mathbb{C}^2, 0)$ is equivalent to the fact that the sets $\mathrm{Fix}(G)$ and $\mathrm{Sep}(\hat{X}_0)$ are the same, i.e., the set $\mathrm{Fix}(G)$ is the set of separatrices of the pure infinitesimal generator $\hat{X}_0$ of G.

2. Maps with non-dynamically trivial formal separatrix

In this section we show how the presence of a non-dynamically trivial separatrix for a flat map germ, induces the existence of a parabolic curve for this map. We begin with an analysis of the relation between such curves and the infinitesimal generator of the corresponding map.

2.1. **Infinitesimal generators and fixed curves.** Let $\hat{X} \in \hat{\mathfrak{X}}_2(\mathbb{C}^2, 0)$, then one says that $\hat{f} \in \mathbb{C}[[x, y]]$ is a singular algebroid curve of $\hat{X}$ if the last one can be written in the form $\hat{X} = \hat{f}\ \hat{X}_1$. The set of singular algebroid curves of $\hat{X}$ will be denoted by $\widehat{\mathrm{Sing}}(\hat{X})$.

Let $\mathbb{C}[[z]]_i$ denote the set of vectors of $\mathbb{C}[[z]]$ whose coordinates are homogeneous polynomials of degree i (in the variable $z = (z_1, z_2)$). The subgroup of formal diffeomorphisms of two variables tangent to the identity with order k is defined as $\widehat{\mathrm{Diff}}_k(\mathbb{C}^2, 0) = \{\hat{h}(z) = z + P_k(z) + \cdots \mid \hat{h} \in \widehat{\mathrm{Diff}}(\mathbb{C}^2, 0)\}$. Similarly, the group of germs of holomorphic diffeomorphisms at the origin $0 \in \mathbb{C}^2$ tangent to the identity with order k is defined as $\mathrm{Diff}_k(\mathbb{C}^2, 0) = \widehat{\mathrm{Diff}}_k(\mathbb{C}^2, 0) \cap \mathrm{Diff}(\mathbb{C}^2, 0)$. The Lie algebra of formal vector fields of $\mathbb{C}^2$ of order k is defined by $\hat{\mathfrak{X}}_k(\mathbb{C}^2, 0) = \{\hat{f}_1(z)\frac{\partial}{\partial z_1} + \hat{f}_2(z)\frac{\partial}{\partial z_2} \mid \hat{f}_1, \hat{f}_2 \in \bigoplus_{i=k}^{\infty} \mathbb{C}[[z]]_i\}$.

The following result is found in [12] and also in Proposition 2.1 in [3].

Proposition 2.1. *The exponential map* $\exp\colon \hat{\mathfrak{X}}_k(\mathbb{C}^2, 0) \to \widehat{\mathrm{Diff}}_k(\mathbb{C}^2, 0)$ *is a bijection.*

In particular, for any flat map germ $G \in \mathrm{Diff}_1(\mathbb{C}^2, 0)$ there is a unique formal vector field of order at least two, say $\hat{X} \in \hat{\mathfrak{X}}_2(\mathbb{C}^2, 0)$, such that $G = \exp[1]\ \hat{X}$. We call $\hat{X}$ the *infinitesimal generator* of G.

Lemma 2.2. *Let* $G \in \mathrm{Diff}_1(\mathbb{C}^2, 0)$ *and* $\hat{X} \in \hat{\mathfrak{X}}_2(\mathbb{C}^2, 0)$ *be its infinitesimal generator. Then*

$$\widehat{\mathrm{Fix}}(G) = \widehat{\mathrm{Sing}}(\hat{X}).$$

Proof. First let $G = (x+p(x, y), y+q(x, y))$ and $\hat{X} = \hat{a}(x, y)\frac{\partial}{\partial x} + \hat{b}(x, y)\frac{\partial}{\partial y}$. Suppose $\hat{X} = \hat{f}\hat{X}_1$ with $\hat{f} \in \mathbb{C}[[x, y]]$ and $\hat{X}_1 \in \hat{\mathfrak{X}}(\mathbb{C}^2, 0)$. Since $\hat{f}$ divides$(\hat{f}\hat{X}_1)^{\circ j}(h)$ for any $h \in \mathcal{O}_2$, and

$$x + p(x, y) = \exp[1]\hat{f}\hat{X}_1(x) = x + \hat{f}\hat{X}_1(x) + \frac{1}{2!}(\hat{f}\hat{X}_1)^{\circ 2}(x) + \cdots,$$

$$y + q(x, y) = \exp[1]\hat{f}\hat{X}_1(y) = y + \hat{f}\hat{X}_1(y) + \frac{1}{2!}(\hat{f}\hat{X}_1)^{\circ 2}(y) + \cdots,$$

then $\hat{f}$ divides both p and q. Conversely, suppose that $\hat{f}$ divides p and q, then the formal logarithm formula (cf. e.g. equation (3.10), p. 34 in [12]) says that

$$\hat{X} = \ln G = (G - \mathrm{Id}) - \frac{1}{2}(G - \mathrm{Id})^2 + \frac{1}{3}(G - \mathrm{Id})^3 \mp \cdots ,$$

where Id is the identity map. The result thus follows immediately. □

The set $\hat{\mathrm{Fix}}(G)$ can be characterized as follows.

Proposition 2.3. *Let $G \in \mathrm{Diff}_1(\mathbb{C}^2, 0)$ and $\hat{f} \in \hat{\mathrm{Fix}}(G)$, then $\hat{f}$ can be written in the form $\hat{f} = \hat{u} \cdot f$ where $f \in \mathcal{O}_2$ and $\hat{u} \in \mathbb{C}[[x, y]]$ is a unity. In particular, if $\hat{X}$ is the infinitesimal generator of G, then it can be written in the form*

$$\hat{X} = \ell \cdot \hat{X}^o$$

where $\ell \in \mathcal{O}_2$ and the coefficients of $\hat{X}^o$ are relatively prime (in $\mathbb{C}[[x, y]]$).

Proof. Let $G(x, y) = (x + p(x, y), y + q(x, y))$ and $\hat{f} \in \hat{\mathrm{Fix}}(G)$, then $\hat{f}$ divides $(G - \mathrm{Id})(x, y) = (p(x, y), q(x, y))$. Since $\mathbb{C}[[x, y]]$ and $\mathcal{O}_2$ are unique factorization domains, and since $g \in \mathcal{O}_2$ is irreducible in $\mathcal{O}_2$ if and only if it is irreducible in $\mathbb{C}[[x, y]]$, then $\hat{f}$ must be written in the form $\hat{f} = \hat{u} \cdot f$ where $\hat{u} \in \mathbb{C}[[x, y]]$ is a unity and $f \in \mathcal{O}_2$ is a common divisor of $p, q \in \mathcal{O}_2$. From Lemma 2.2 one has that $\hat{\mathrm{Fix}}(G) = \hat{\mathrm{Sing}}(\hat{X})$; the result follows immediately. □

The writing $\hat{f} = \hat{u} \cdot f$ is not unique, but if we also have $\hat{f} = \hat{u}_1 \cdot f_1$ with $\hat{u}_1$ a formal unity and $\hat{f}_1 \in \mathcal{O}_2$, then we have $\hat{u} = v \cdot \hat{u}_1$, $f_1 = v \cdot f$ for some (holomorphic) unity $v \in \mathcal{O}_2$.

The previous result, although of a simple nature, has two important consequences. The first is that any element of $\hat{\mathrm{Fix}}(G)$ has in fact an analytic realization as the zero set of f (where $\hat{f} = \hat{u} f$). The second one is that the resolution of G can be obtained automatically from the formal version of Seidenberg's resolution of $\hat{X}^o$ (as in the resolution of holomorphic foliations in $(\mathbb{C}^2, 0)$), since $G = \exp[1]\ell \cdot \hat{X}^o$. From now on we shall refer to $\hat{X}^o$ as the *pure infinitesimal generator* of G.

Let $\mathrm{Sing}(\hat{X}) := \hat{\mathrm{Sing}}(\hat{X}) \cap \mathcal{O}_2$, then one has the following straightforward consequence (whose first part is stated without proof in [4]).

Corollary 2.4. *Let $G \in \mathrm{Diff}_1(\mathbb{C}^2, 0)$ and $\hat{X} \in \hat{\mathfrak{X}}_2(\mathbb{C}^2, 0)$ be its infinitesimal generator, then $\mathrm{Fix}(G) = \mathrm{Sing}(\hat{X})$. In particular, each purely formal separatrix of G is automatically non-dynamically trivial.*

2.2. **Infinitesimal generators and characteristic directions.** In this paragraph we show how to detect the characteristic directions of a flat map germ in its infinitesimal generator. The first step is the following result stated in [4].

Lemma 2.5. *Let $G \in \mathrm{Diff}_1(\mathbb{C}^2, 0)$ and $\hat{X} \in \hat{\mathfrak{X}}_2(\mathbb{C}^2, 0)$ be its infinitesimal generator. Then $\nu := \mathrm{ord}(G) = \mathrm{ord}(\hat{X})$ and*

$$G = \mathrm{Id} + \hat{X} \bmod \{x^i y^{\nu - i} : i = 0, \ldots, \nu\}.$$

In particular, the characteristic directions of G correspond to the points of the tangent cone of $\hat{X}$.

Proof. Let $G(x,y)=(x+\sum_{n=2}^{\infty}p_n(x,y),y+\sum_{n=2}^{\infty}q_n(x,y))$ and

$$\hat{X}=\sum_{n=2}^{\infty}\left(\hat{a}_n(x,y)\frac{\partial}{\partial x}+\hat{b}_n(x,y)\frac{\partial}{\partial y}\right)$$

be the homogeneous expansions of G and $\hat{X}$ respectively. Since $G=\exp[1]\hat{X}$, then

$$\begin{cases} p_{m+1}=\hat{a}_{m+1}+HT_{m+1}\left(\sum_{j=2}^{m}\frac{1}{j!}\hat{X}_m^{\circ j}(x)\right), \\ q_{m+1}=\hat{b}_{m+1}+HT_{m+1}\left(\sum_{j=2}^{m}\frac{1}{j!}\hat{X}_m^{\circ j}(y)\right), \end{cases} \tag{2.1}$$

where $\hat{X}_m=\sum_{n=2}^{m}\left(\hat{a}_n(x,y)\frac{\partial}{\partial x}+\hat{b}_n(x,y)\frac{\partial}{\partial y}\right)$ and $HT_{m+1}(h)$ is the homogeneous term of h of order $m+1$. Clearly, $\nu:=\operatorname{ord}(G)=\operatorname{ord}(\hat{X})$, $p_\nu=\hat{a}_\nu$ and $q_\nu=\hat{a}_\nu$. $\square$

The above result supports the following definition. Let $\hat{X}\in\hat{\mathfrak{X}}_2(\mathbb{C}^2,0)$ be given by $\hat{X}(x,y)=\hat{a}(x,y)\frac{\partial}{\partial x}+\hat{b}(x,y)\frac{\partial}{\partial y}$, where $\hat{a}=\sum_{j\geq\nu}\hat{a}_j$, $\hat{b}=\sum_{j\geq\nu}\hat{b}_j$ are the expansions in terms of homogeneous polynomials and let $P_\nu=(\hat{a}_\nu,\hat{b}_\nu)$. Then one says that $[v]\in\mathbb{P}^1$ is a *characteristic direction* for $\hat{X}$ if $P_\nu(v)=\lambda v$. In particular $[v]$ is said to be a non-degenerate characteristic direction if $\lambda\neq 0$.

The next result shows that this definition is invariant under formal coordinate change.

Proposition 2.6. *Let $\hat{X},\hat{Y}\in\hat{\mathfrak{X}}_2(\mathbb{C}^2,0)$ and suppose $\Phi\in\widehat{\mathrm{Diff}}(\mathbb{C}^2,0)$ is such that $\hat{Y}=\Phi_*\hat{X}$. Then $\Phi'(0)$ takes the (non-degenerate) characteristic directions of $\hat{X}$ onto the (non-degenerate) characteristic directions of Y.*

Proof. Let $\hat{X}:=P\frac{\partial}{\partial x}+Q\frac{\partial}{\partial y}$, $\hat{Y}:=R\frac{\partial}{\partial x}+S\frac{\partial}{\partial y}$, $\omega_{\hat{X}}=Qdx-Pdy$, $\omega_Y=Sdx-Rdy$ and $\Phi'(0)=(a_{ij})_{2\times 2}$. Then the characteristic directions of $\hat{X}$ in the coordinates containing the direction $(0:1)\in\mathbb{P}^1$ are given by $P_\nu(u,1)-uQ_\nu(u,1)=0$ and the degenerate ones are precisely those points such that $Q_\nu(u,1)=0$. Analogously, the characteristic directions of $\hat{Y}$ in the same coordinates are given by $R_\nu(u,1)-uS_\nu(u,1)=0$ and the degenerate ones are precisely those points such that $S_\nu(u,1)=0$. On the other hand, $\Phi'(0)$ maps the direction $(u:1)$ in the direction $(a_{11}u+a_{12}:a_{21}u+a_{22})=(\frac{a_{11}u+a_{12}}{a_{21}u+a_{22}}:1)$. Since

$$\begin{aligned}\Phi^*\omega_{\hat{Y}}&=S_\nu(a_{11}x+a_{12}y,a_{21}x+a_{22}y)d(a_{11}x+a_{12}y)\\&\quad-R_\nu(a_{11}x+a_{12}y,a_{21}x+a_{22}y)d(a_{21}x+a_{22}y)\operatorname{mod}(\{x^iy^{\nu-i+1}\}_{i=0}^{\nu+1})\\&=[a_{11}S_\nu(a_{11}x+a_{12}y,a_{21}x+a_{22}y)-a_{21}R_\nu(a_{11}x+a_{12}y,a_{21}x+a_{22}y)]dx\\&\quad-[a_{22}R_\nu(a_{11}x+a_{12}y,a_{21}x+a_{22}y)-a_{12}S_\nu(a_{11}x+a_{12}y,a_{21}x+a_{22}y)]dy\\&\quad\operatorname{mod}(\{x^iy^{\nu-i+1}\}_{i=0}^{\nu+1}),\end{aligned}$$

then

$$\begin{aligned}Q_\nu(u,1)&=[a_{21}u+a_{22}]^\nu\left[a_{11}S_\nu(\frac{a_{11}u+a_{12}}{a_{21}u+a_{22}},1)-a_{21}R_\nu(\frac{a_{11}u+a_{12}}{a_{21}u+a_{22}},1)\right],\\P_\nu(u,1)&=[a_{21}u+a_{22}]^\nu\left[a_{22}R_\nu(\frac{a_{11}u+a_{12}}{a_{21}u+a_{22}},1)-a_{12}S_\nu(\frac{a_{11}u+a_{12}}{a_{21}u+a_{22}},1)\right].\end{aligned}$$

In particular,

$$\begin{aligned}P_\nu(u,1)-uQ_\nu(u,1) &= [a_{21}u+a_{22}]^\nu[(a_{21}u+a_{22})R_\nu(v,1)-(a_{11}u+a_{12})S_\nu(v,1)]\\ &= [a_{21}u+a_{22}]^{\nu+1}\left\{R_\nu(v,1)-vS_\nu(v,1)\right\},\end{aligned}$$

where $v=\frac{a_{11}u+a_{12}}{a_{21}u+a_{22}}$. Since Φ is a diffeomorphism, then $a_{21}\cdot a_{22}\neq 0$. Therefore, $\Phi'(0)$ maps the characteristic directions of $\hat{X}$ onto the characteristic directions of $\hat{Y}$. Furthermore, $(u_0,1)$ is a zero for both $P_\nu(u,1)$ and $Q_\nu(u,1)$ if and only if $v_0=\frac{a_{11}u_0+a_{12}}{a_{21}u_0+a_{22}}$ is a zero for both $R_\nu(v,1)$ and $S_\nu(v,1)$. □

2.3. **Tangent cones and invariant curves.** Let $\hat{X}\in\hat{\mathfrak{X}}(\mathbb{C}^2,0)$ be a formal vector field and $\hat{C}$ an algebroid curve given by $\hat{f}\in\mathbb{C}[[x,y]]$. We say that $\hat{C}$ is a *formal separatrix*[1] of $\hat{X}$ if $\hat{X}(\hat{f})\in(\hat{f})$. This notion does not depend on the representative $\hat{f}$ (i.e., on the formal equation) for the curve $\hat{C}$.

Proposition 2.7. *Let $G\in\mathrm{Diff}_1(\mathbb{C}^2,0)$, and $\hat{X}(x,y)=\hat{a}(x,y)\frac{\partial}{\partial x}+\hat{b}(x,y)\frac{\partial}{\partial y}$ be its infinitesimal generator. Then $\hat{f}\in\mathbb{C}[[x,y]]$ defines a formal separatrix of $\hat{X}$ if and only if $\hat{f}$ is invariant by G.*

Proof. Suppose $\hat{f}$ represents a formal separatrix of $\hat{X}$, then $\hat{X}(\hat{f})=\hat{f}\hat{g}$. From Leibniz' rule, one has by induction that $\hat{X}^{\circ(n)}(\hat{f})\in(\hat{f})$ for all $n\geq 1$. Therefore,

$$\begin{aligned}\hat{f}\circ G &= (\mathrm{Id}+\sum_{n\geq 1}\frac{1}{n!}\hat{X}^{\circ(n)})[\hat{f}]\\ &= \hat{f}+\sum_{n\geq 1}\frac{1}{n!}\hat{X}^{\circ(n)}(\hat{f})\in(\hat{f}).\end{aligned}$$

Conversely, suppose $\hat{f}$ is invariant by G, then $\hat{f}\circ G\in(\hat{f})$ and thus $\hat{f}\circ(G-\mathrm{Id})\in(\hat{f})$. More generally, one can obtain by induction that $\hat{f}\circ(G-\mathrm{Id})^{\circ(n)}\in(\hat{f})$. Since $\hat{X}=\ln G=(G-\mathrm{Id})-\frac{1}{2}(G-\mathrm{Id})^{\circ(2)}+\frac{1}{3}(G-\mathrm{Id})^{\circ(3)}\mp\cdots$, then $\hat{X}(\hat{f})\in(\hat{f})$. □

In view of the above result, we sometimes refer to a formal separatrix of $\hat{X}\in\hat{\mathfrak{X}}_2(\mathbb{C}^2,0)$ as a formal separatrix of its time one map of the associated flow, i.e., a *formal separatrix* of the map $G=\exp[1]\hat{X}$.

Proposition 2.8. *Let $G\in\mathrm{Diff}_1(\mathbb{C}^2,0)$ and suppose that $\hat{f}\in\mathbb{C}[[x,y]]$ represents a formal separatrix of G. Then the points of the tangent cone of $\hat{f}$ determine characteristic directions for G. Furthermore, if $\hat{f}\in\hat{\mathrm{Fix}}(G)$, then the tangent cone of $\hat{f}$ determines only degenerate characteristic directions.*

Proof. Let $\hat{X}=\hat{a}(x,y)\frac{\partial}{\partial x}+\hat{b}(x,y)\frac{\partial}{\partial y}$ be the infinitesimal generator of G, then Lemma 2.5 assures that it is enough to prove that the tangent cone of $\hat{f}$ is contained in the tangent cone of $\hat{X}$. If $\hat{X}$ is dicritical, there is nothing to prove, since all directions are automatically characteristic. Thus we suppose that $\hat{X}$ is

[1] Some authors also call it logarithm, cf. [15] p. 152.

non-dicritical. After one blow up we have

$$\begin{aligned}\pi^*\hat{X}(t,x) &= \hat{a}\circ\pi(t,x)\pi^*(\frac{\partial}{\partial x}) + \hat{b}\circ\pi(t,x)\pi^*(\frac{\partial}{\partial y})\\ &= \hat{a}(x,tx)(\frac{\partial}{\partial x} - \frac{y}{x^2}\frac{\partial}{\partial t}) + \hat{b}(x,tx)\frac{1}{x}\frac{\partial}{\partial t}\\ &= (\frac{1}{x}\hat{b}(x,tx) - \frac{t}{x}\hat{a}(x,tx))\frac{\partial}{\partial t} + \hat{a}(x,tx)\frac{\partial}{\partial x}. \end{aligned}\tag{2.2}$$

Thus if $\hat{a} = \sum_{j\geq\nu}\hat{a}_j$ and $\hat{b} = \sum_{j\geq\nu}\hat{b}_j$ are expansions in homogeneous polynomials, then

$$\begin{aligned}\hat{X}^{(1)}(t,x) &= \frac{1}{x^{\nu-1}}\pi^*\hat{X}(x,t)\\ &= \frac{1}{x^{\nu-1}}\Bigg[x^{\nu}(\hat{a}_\nu(1,t) + x(\cdots))\frac{\partial}{\partial x}\\ &\qquad + \frac{x^\nu}{x}(\hat{b}_\nu(1,t) - t\hat{a}_\nu(1,t) + x(\cdots))\frac{\partial}{\partial t}\Bigg]\\ &= x(\hat{a}_\nu(1,t) + x(\cdots))\frac{\partial}{\partial x} + (\hat{b}_\nu(1,t) - t\hat{a}_\nu(1,t) + x(\cdots))\frac{\partial}{\partial t}.\end{aligned}\tag{2.3}$$

Hence the tangent cone of $\hat{X}$ in the (t,x) coordinates is determined by $\hat{b}_\nu(1,t) - t\hat{a}_\nu(1,t) = 0$. Since $d\hat{f}\cdot\hat{X} = \hat{f}k$, for some $k\in\mathbb{C}[[x,y]]$, where $\nu = \mathrm{ord}(\hat{X})$, $\mu = \mathrm{ord}(\hat{f})$, and $\alpha = \mathrm{ord}(k) = \nu - 1$, then

$$\begin{aligned}0 &= d(\pi^*\hat{f})\cdot\pi^*\hat{X} - (\pi^*\hat{f})(\pi^*k)\\ &= d(x^\mu\hat{f}^{(1)})\cdot x^{\nu-1}\hat{X}^{(1)} - x^\mu\hat{f}^{(1)}x^\alpha k^{(1)}\\ &= x^{\mu+\nu-2}(\mu\hat{f}^{(1)}dx + xd\hat{f}^{(1)})\cdot\hat{X}^{(1)} - x^{\mu+\nu-1}\hat{f}^{(1)}k^{(1)},\end{aligned}$$

where $\hat{f}^{(1)}(t,x) := \frac{1}{x^\mu}\pi^*\hat{f}(t,x)$ and $k^{(1)}(t,x) := \frac{1}{x^\alpha}\pi^*k(t,x)$. In particular, $\hat{f}^{(1)}$ divides $d\hat{f}^{(1)}\cdot\hat{X}^{(1)}$, i.e., $\hat{X}^{(1)}(\hat{f}^{(1)})\in(\hat{f}^{(1)})$. Thus, if $\hat{f} = \sum_{j\geq\mu}\hat{f}_j$ is the decomposition of $\hat{f}$ in homogeneous polynomials, then the previous equation restricted to $(x=0)$ leads to $(\hat{b}_\nu(1,t) - t\hat{a}_\nu(1,t))\frac{\partial\hat{f}_\mu(1,t)}{\partial t}\in(\hat{f}_\mu(1,t))$. Since up to a linear change of coordinates we may suppose that $\hat{f}_\mu(1,t)$ and $\frac{\partial\hat{f}_\mu(1,t)}{\partial t}$ are relatively prime, then $\hat{f}_\mu(1,t)$ divides $\hat{b}_\nu(1,t) - t\hat{a}_\nu(1,t)$, and the first statement follows. Finally suppose that $\hat{f}$ divides both $\hat{a}$ and $\hat{b}$, then taking $(x=0)$ one obtains that $\hat{f}_\mu(1,t)$ divides both $\hat{b}_\nu(1,t)$ and $\hat{a}_\nu(1,t)$. Thus its tangent cone determines just degenerate characteristic directions. □

One says that the resolution of $G\in\mathrm{Diff}_1(\mathbb{C}^2,0)$ has a *dicritical component* if $G_p^{(n)} := \left(\pi^{(n)}\right)^{-1}(G)$ is dicritical at some singular point $p\in D^{(n)} := \left(\pi^{(n)}\right)^{-1}(0)$, where $\pi^{(n)}:\mathcal{M}^{(n)}\longrightarrow(\mathbb{C}^2,0)$ is the n-th stage in the resolution process of G.

Remark 2.9. Similar to the analytic case, one can prove that a formal vector field can be resolved after a finite number of blowups. Further, any vector field having diagonalizable linear part has precisely two formal separatrices.

Corollary 2.10. *Suppose that $G\in\mathrm{Diff}_1(\mathbb{C}^2,0)$ has infinitely many separatrices, then its resolution has a dicritical component.*

Proof. From Remark 2.9, we may suppose without loss of generality that the first tangent cone of the formal separatrices of G intersects $D := \left(\pi^{(1)}\right)^{-1}(0)$ at infinitely many distinct directions. Finally, Proposition 2.8 assures that G has infinitely many characteristic directions; however, this is impossible unless G is dicritical. □

Remark 2.11. In [3] it is proved that any germ of resonant map with finite orbits and infinitely many separatrices is periodic. Therefore, one can reprove this result with the application of Corollary 2.10, since dicritical maps always have at least one parabolic curve (cf. Theorem 1.5).

Since we are interested in the dynamics of maps tangent to the identity and the dynamics of dicritical maps are well understood, at least with respect to the existence of parabolic curves, then hereafter we shall deal only with maps having finitely many formal separatrices.

2.4. **Resolution of singularities and invariant curves.** As we have already observed, Corollary 2.4 assures that any completely fixed curve is in fact convergent, i.e., any element of $\hat{\mathrm{Fix}}(G)$ has in fact an analytic realization as the zero set of an analytic function.

The following result gives a geometric characterization for the existence of non-degenerate characteristic directions in terms of the existence of non-dynamically trivial formal separatrices.

Proposition 2.12. *Let $G \in \mathrm{Diff}_1(\mathbb{C}^2, 0)$ be a flat complex map germ and $\hat{C}$ a non-dynamically trivial formal separatrix of G. Denote by $\pi : (\mathcal{M}, D) \longrightarrow (\mathbb{C}^2, 0)$ the minimal resolution of G. Then the singular points of $\widetilde{C} := \pi^*\hat{C}$ correspond exactly to non-degenerate characteristic directions of $\widetilde{G} := \pi^*G$.*

Proof. Suppose that $\hat{f} \notin \hat{\mathrm{Fix}}(G)$ is a non-dynamically trivial formal sepatratrix and let $\hat{X}$ be the infinitesimal generator of G. Let n be the number of steps (blowups) in the minimal resolution process. Since G has no dicritical component, then Corollary 2.4 assures that $\hat{X}$ can be written in the form $\hat{X} = g\hat{X}^o$, where $\mathrm{Fix}(G)$ is determined by $(g = 0)$ with $g \in \mathcal{O}_2$ and the coefficients of $\hat{X}^o$ are relatively prime. After resolving $\mathrm{Fix}(G)$, $\mathrm{Sep}(\hat{X}^o)$ and $\hat{X}^o$, one may suppose that near a singular point $p \in D^{(n)} := \pi^{-1}(0)$ there are local coordinates (x, y) such that $p = (0, 0)$ and the map $G_n := G_p^{(n)}$ has one of the following forms: (i) $G_n = \exp[1]y^\tau x^\kappa \hat{X}_n^o$ with $\tau, \kappa \geq 1$ and $\mathrm{Sep}(\hat{X}_n^o)$ coincides with $\mathrm{Fix}(G_n) \in (xy = 0)$; (ii) $G_n = \exp([1]y^\tau \hat{X}_n^o)$ with $\tau \geq 1$ and $\mathrm{Fix}(G_n) = (y = 0)$. Clearly, in both cases $\hat{X}_n^o$ is reduced. From Remark 2.9 the singular points associated with $\hat{f}$ are of the form (ii). In order to simplify the notation let us omit the index n for a while. From Proposition 2.7 the non-degenerate characteristic directions are formal invariants, thus Corollary 2.4 assures that after a formal change of coordinates we may suppose that $G = \exp[1]y^k \hat{X}^o$ where we have one of the following cases: (a) (the saddle-node normal form) $\hat{X}^o(x, y) = x^{p+1}\frac{\partial}{\partial x} + y(1 + \lambda x^p)\frac{\partial}{\partial y}$ with $\lambda \in \mathbb{C}$, or (b) (the Poincaré-Dulac normal form) $\hat{X}^o(x, y) = \lambda x(1 + \cdots)\frac{\partial}{\partial x} + \mu y(1 + \cdots)\frac{\partial}{\partial y}$, where $\mu, \lambda \neq 0$ and $\frac{\mu}{\lambda} \notin \mathbb{Q}_+$. In both cases the non-dynamically trivial separatrix is written in the form $\hat{f}(x, y) = x$. Let us first consider the case (a): Since $\hat{a}(x, y) = y^k x^{p+1}$ and $\hat{b}(x, y) = y^{k+1}$, then $\hat{f}_1(u, 1) = u$ does not divide $\hat{b}_1(u, 1) = 1$. Now let us consider the case (b): Since $\hat{a}(x, y) = \lambda x$ and $\hat{b}(x, y) = \mu y$, then $\hat{f}_1(u, 1) = u$ does not divide

$\hat{b}_1(u,1) = \mu$. Therefore, in both cases $(0:1)$ is a non-degenerate characteristic direction for G. $\square$

Above we have proved that, for a flat map germ, a non-dynamically trivial invariant formal curve is associated with a non-degenerate characteristic direction. This is also a characterization of (the existence of) non-dynamically trivial formal separatrices. Indeed, with the same notation above, suppose that G_n has a non-degenerate characteristic direction at the point $p \in \{\tilde{\hat{f}} = 0\} \cap D$, then Corollary 2.4 and Proposition 2.8 assure that it can be written locally in the form $G_n = \exp[1]y^\tau \hat{X}_0$ where the coefficients of $\hat{X}_0$ are relatively prime. From Remark 2.9 it follows that G has a non-dynamically trivial formal separatrix.

Notice that the proofs of Propositions 2.8 and 2.12 show that, after at least the minimal resolution, both degenerate and non-degenerate characteristic directions are stable under blow-ups.

Now we are in position to prove our main results:

Proof of Theorem A. First notice that the irreducible components of $\hat{C}$ are in one-to-one correspondence with the points of $\mathrm{Sing}(\tilde{C})$. Now the result comes as a straightforward consequence of Proposition 2.12 and Theorem 1.1. $\square$

Theorem B is now a consequence of Theorem A:

Proof of Theorem B. Since $G = \exp[1]y^\tau \hat{X}^o$ with $\tau \geq 1$, $\mathrm{Fix}(G_n) = (y=0)$, $\hat{X}^o$ has order one, and the linear part of $\hat{X}^o$ is diagonal, then Remark 2.9 assures that it has at least two formal separatrices. Since $\mathrm{Fix}(G_n) = (y=0)$, then at least one of them is a non-dynamically trivial formal separatrix. The result then follows from Theorem A. $\square$

Let $G = \exp[1]\hat{X}$ with $\hat{X} = A(x,y)\frac{\partial}{\partial x} + B(x,y)\frac{\partial}{\partial y}$ and $\widehat{X}_1 = \pi^*\widehat{X}$ be the formal vector field in the chart containing the direction $(0:1) \in \mathbb{P}^1$ obtained from $\widehat{X}$ by a blow up. Then a straightforward calculation shows that

$$\widehat{X}_1(u,y) = y^{\nu-1}(A_\nu(u,1) - uB_\nu(u,1) + \cdots)\frac{\partial}{\partial u} + y^\nu(B_\nu(u,1) + \cdots)\frac{\partial}{\partial y}, \tag{2.4}$$

where $\nu = \nu(G)$. In face of this remark, the following slight generalization of Abate's theorem (Theorem 1.2) is an immediate consequence of Theorem B.

Theorem 2.13. *Let $G \in \mathrm{Diff}_1(\mathbb{C}^2, 0)$ be a germ of a flat diffeomorphism with isolated fixed point at the origin. Then G has a purely formal separatrix $\widehat{C}$. If $\widehat{C}_1, \cdots, \widehat{C}_n$ are the irreducible components of $\widehat{C}$, then G admits* (*at least*) *$\nu(G) - 1$ parabolic curves tangent to $\widehat{C}_i$ for each $i = 1, \ldots, n$.*

Proof. Let $\pi : (\mathcal{M}, D) \longrightarrow (\mathbb{C}^2, 0)$ be the minimal resolution of G and $\widetilde{G} = \pi^* G$. Since $\mathrm{Fix}(G)$ is isolated, then Corollary 2.4 assures that $\widetilde{G}$ is written in local coordinates about its singular points (i.e., the singular points of its pure infinitesimal generator $\hat{X}_0$) in the form $G = \exp[1]y^\tau \hat{X}_0$, where $D = (y=0)$, the coefficients of $\hat{X}_0$ are relatively prime, and $J^1(\hat{X}_0)$ is reduced. From Remark 2.9, it follows that $\widetilde{G}$ has a purely formal separatrix through each of its singular points. Thus the first statement follows by blowing down the purely formal separatrices of $\widetilde{G}$. For the last statement, recall from (2.4) that $\nu(\widetilde{G}_p) \geq \nu(G)$ for each singular point $p \in \widetilde{C} \cap D$ of G. Thus the result follows from Theorem B. $\square$

Notice that this result tells a little bit more than what is stated in Theorem 1.2. For instance, one knows from Theorem A that the parabolic curves are in fact tangent to the irreducible components of the non-dynamically trivial formal separatrix of G. Further, if one knows the order of $\tilde{G}_p$ for all $p \in (\tilde{C})$, then a better estimate for the number of parabolic curves can be performed.

Theorem B also applies to some cases which are not treated by Abate or L. Molino, as for instance:

Example 2.14. Let $G \in \mathrm{Diff}_1(\mathbb{C}^2, 0)$ be given by $G = \exp[1]y^\tau \hat{X}^o$ where $\tau \geq 1$ and $\hat{X}^o(x,y) = [\lambda x + O(x^2, xy, y^2)]\frac{\partial}{\partial x} + yB(x,y)\frac{\partial}{\partial y}$ with $\lambda \neq 0$ and $B(0,0) = 0$. Then $\mathrm{ind}_0(G, S) = 0$ and G admits a parabolic curve.

3. Maps having only dynamically trivial separatrices

Here we study how sharp our main result Theorem A is. Notice that Proposition 2.12 shows that G admits a non-dynamically trivial formal separatrix if and only if there is a non-degenerate characteristic direction associated with this curve along the resolution of G. Écalle's approach to Theorem 1.1 shows that the existence of characteristic directions determine a non-dynamically trivial formal separatrix whose Borel-Laplace sum generates a parabolic curve. On the other hand, in [13] L. Molino calls our attention to the fact that it is possible to have parabolic curve tangent to degenerate characteristic directions. The next example shows that parabolic curves may happen even without the presence of non-dynamically trivial formal separatrices, i.e., tangent to dynamically trivial (analytic) separatrices.

Example 3.1. Here we consider a map $G \in \mathrm{Diff}(\mathbb{C}^2, 0)$ with pure generator of non-trivial Jordan cell type. Let $G = \exp[1]y^\tau \hat{X}^o$ where $\tau \geq 2$ and $\hat{X}^o(x,y) = (y + \lambda x + \cdots)\frac{\partial}{\partial x} + \lambda y(1 + \cdots)\frac{\partial}{\partial y}$ with $\lambda \neq 0$. Since $\tau \geq 2$, then the action of dG in the normal bundle of $S := \mathrm{Fix}(G)$ is trivial. Furthermore,

$$\begin{aligned}\operatorname*{ind}_{x=0}(G,S) = \mathrm{CS}(\hat{X}^o, S, 0) &= \operatorname*{Res}_{x=0} \left.\frac{\lambda y(1+\cdots)}{y(y+\lambda x+\cdots)}\right|_{y=0} \\ &= \operatorname*{Res}_{x=0} \frac{1}{x} = 1.\end{aligned}$$

Then Theorem 1.4 assures that there is a parabolic curve tangent to the S. After one blow up, one obtains a map $\widetilde{G}$ with just an isolated singularity of the form $\widetilde{G}(t,x) = \exp[1]t^\tau x^{\tau+1}\widetilde{X}^o(t,x)$ where $\widetilde{X}^o(t,x) = x(\lambda + t + \cdots)\frac{\partial}{\partial x} - t(t + \cdots)\frac{\partial}{\partial t}$. Since parabolic curves can only appear next to singular points (cf. [1]), then $\widetilde{G}$ has a parabolic curve tangent to $(t = 0)$. Nevertheless, from Remark 2.9, $\widetilde{G}(t,x)$ has no non-dynamically trivial formal separatrix.

Now study a map whose purely infinitesimal generator is of nilpotent Jordan cell type.

Example 3.2. Let $G = \exp[1]y^\tau \hat{X}^o$ where $\tau \geq 2$ and $\hat{X}^o(x,y) = (\lambda y + A(x,y))\frac{\partial}{\partial x} + yB(x,y)\frac{\partial}{\partial y}$ where $\lambda \neq 0$, $A \in (x^2, xy, y^2)$, $B(0,0) = 0$. Since $\tau \geq 2$, then the action of dG in the normal bundle of $S := \mathrm{Fix}(G)$ is trivial. Furthermore,

$$\begin{aligned}\operatorname*{ind}_{x=0}(G,S) = \mathrm{CS}(\hat{X}^o, S, 0) &= \operatorname*{Res}_{x=0} \left.\frac{yB(x,y)}{y(\lambda y + A(x,y))}\right|_{y=0} \\ &= \operatorname*{Res}_{x=0} \left.\frac{B(x,y)}{A(x,y)}\right|.\end{aligned}$$

Then Theorem 1.4 assures that there is a parabolic curve tangent to the S. After one blow up, one obtains a map $\widetilde{G}$ with just an isolated singularity of the form $\widetilde{G}(t,x) = \exp[1]t^{\tau}x^{\tau+1}\widetilde{X}^o(t,x)$ where $\widetilde{X}^o(t,x) = x(\lambda + t + \cdots)\frac{\partial}{\partial x} - t(t + \cdots)\frac{\partial}{\partial t}$. Since parabolic curves can only appear next to singular points (cf. [1]), then $\widetilde{G}$ has a parabolic curve tangent to $(t = 0)$. Nevertheless, from Remark 2.9, $\widetilde{G}(t,x)$ has no non-dynamically trivial formal separatrix.

References

[1] M. Abate. *The residual index and the dynamics of maps tangent to the identity*, Duke Math. J. **107** (2001), 173–207. MR1815255 (2003a:32028)

[2] N. Bourbaki. Elements of Mathematics, Algebra II. Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo, Hong Kong. MR2327161

[3] F. Brochero Martínez. *Groups of germs of analytic diffeomorphism in* $(\mathbb{C}^2, 0)$, J. Dynam. Control Systems **9** (2003), 1–32. MR1956442 (2004a:32047)

[4] F.E. Brochero Martínez, F. Cano and L. López-Hernaz. *Parabolic curves for diffeomorphisms in* $(\mathbb{C}^2, 0)$. Publ. Mat. 52 (2008), no. 1, 189–194. MR2384846 (2009a:37090)

[5] C. Camacho. *On the local structure of conformal maps and holomorphic vector fields in* $\mathbb{C}^2$, Journées Singuliéres de Dijon (Univ. Dijon, Dijon, 1978, 3, 83–94), Astérisque 59-60, Soc. Math. France, Paris, 1978. MR542732 (81d:58016)

[6] C. Camacho and P. Sad. *Invariant varieties through singularities of holomorphic vector fields,* Ann. of Math. (2) **115** (1982), 579–595. MR657239 (83m:58062)

[7] L. Câmara and B. Scárdua. *On the integrability of holomorphic vector fields.* Discrete Contin. Dyn. Syst. 25 (2009), no. 2, 481–493. MR2525187 (2010j:37075)

[8] L. Carleson and T.W. Gamelin, Complex Dynamics, Springer, (1992). MR1230383 (94h:30033)

[9] J. Écalle. Les fonctions résurgentes. Tome III. (French) [Resurgent functions. Vol. III] L'équation du pont et la classification analytique des objects locaux. [The bridge equation and analytic classification of local objects] Publications Mathématiques d'Orsay [Mathematical Publications of Orsay], 85-5. Université de Paris-Sud, Département de Mathématiques, Orsay, 1985. MR852210 (87k:32009)

[10] R.C. Gunning and H. Rossi. Analytic functions of several complex variables, Prentice Hall, Inc., Englewood Cliffs, NJ, 1965. MR0180696 (31:4927)

[11] M. Hakim. *Analytic transformations of* $(\mathbb{C}^p, 0)$ *tangent to the identity*, Duke Math. J. **92** (1998), 403–428. MR1612730 (99a:32036)

[12] Yu. Ilyashenko and S. Yakovenko. Lectures on analytic differential equations. Graduate Studies in Mathematics, 86. American Mathematical Society, Providence, RI, 2008. MR2363178 (2009b:34001)

[13] L. Molino. *The dynamics of maps tangent to the identity and with nonvanishing index.* Trans. Amer. Math. Soc. 361 (2009), 1597–1623. MR2457410 (2010b:32024)

[14] A. Seidenberg. *Reduction of the singularities of the differentiable equation* $Ady = Bdx$, Amer. J. Math. **90** (1968), 248–269. MR0220710 (36:3762)

[15] T. Suwa. *Indices of vector fields and residues of singular holomorphic foliations.* Actualités Mathematiques, Hermann, Paris 1998. MR1649358 (99h:32042)
[16] O. Zariski. Le problème des modules pour les branches planes, Hermann, Paris, 1986. MR861277 (88a:14031)

Departamento de Matemática - CCE, Universidade Federal do Espírito Santo, CP. 68530, Av. Fernando Ferrari 514, 29075-910 - Vitória - ES, Brazil
E-mail address: leonardo.camara@ufes.br

Instituto de Matemática - Universidade Federal do Rio de Janeiro, CP. 68530-Rio de Janeiro-RJ, 21945-970 - Brazil
E-mail address: scardua@im.ufrj.br