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THE ASYMPTOTIC BEHAVIOR OF JENKINS-STREBEL RAYS

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ABSTRACT. In this paper, we consider the asymptotic behavior of two Teichmüller geodesic rays determined by Jenkins-Strebel differentials, and we obtain a generalization of a theorem by the author in *On behavior of pairs of Teichmüller geodesic rays*, 2014 . We also consider the infimum of the asymptotic distance up to choice of base points of the rays along the geodesics. We show that the infimum is represented by two quantities. One is the detour metric between the end points of the rays on the Gardiner-Masur boundary of the Teichmüller space, and the other is the Teichmüller distance between the end points of the rays on the augmented Teichmüller space.

1. INTRODUCTION

Let X be a Riemann surface of genus g with n punctures such that $3g-3+n>0$, and let $T(X)$ be the Teichmüller space of X. Any Teichmüller geodesic ray on $T(X)$ is determined by a holomorphic quadratic differential on a base point of the ray. A geodesic ray is called a Jenkins-Strebel ray if it is given by a Jenkins-Strebel differential. In [Ama14], we obtain a condition for two Jenkins-Strebel rays to be asymptotic (Corollary 1.2 in [Ama14]). To obtain this condition, we use Theorem 1.1 in [Ama14] which gives the explicit asymptotic value of the Teichmüller distance between two similar Jenkins-Strebel rays with the same end point in the augmented Teichmüller space. In this paper, we improve this theorem, and obtain the asymptotic value of the distance between any two Jenkins-Strebel rays.

Let r, r' be Jenkins-Strebel rays on $T(X)$ from $r(0)=[Y,f]$, $r'(0)=[Y',f']$ determined by Jenkins-Strebel differentials q, q' with unit norm on Y, Y' respectively. It is known (cf. [HS07]) that the Jenkins-Strebel rays r, r' have limits, say $r(\infty)$, $r'(\infty)$, on the boundary of the augmented Teichmüller space $\hat{T}(X)$. Suppose that r, r' are similar, that is, there exist mutually disjoint simple closed curves $\gamma_1,\cdots,\gamma_k$ on X such that the set of homotopy classes of core curves of the annuli corresponding to q, q' are represented by $f(\gamma_1),\cdots,f(\gamma_k)$ on Y and $f'(\gamma_1),\cdots,f'(\gamma_k)$ on Y' respectively. We denote by m_j, m'_j the moduli of the annuli on Y, Y' with core curves homotopic to $f(\gamma_j)$, $f'(\gamma_j)$ respectively. We can define the Teichmüller distance $d_{\hat{T}(X)}(r(\infty),r'(\infty))$ between the end points $r(\infty)$, $r'(\infty)$.

Our main result is the following:

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Theorem 1.1. *For any two Jenkins-Strebel rays r, r',*

$$\lim_{t\to\infty} d_{T(X)}(r(t), r'(t)) = \tag{1}$$
$$= \begin{cases} \max\left\{\dfrac{1}{2}\log \max\limits_{j=1,\cdots,k}\left\{\dfrac{m'_j}{m_j}, \dfrac{m_j}{m'_j}\right\}, d_{\hat{T}(X)}(r(\infty), r'(\infty))\right\} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(if } r,\ r' \text{ are similar)}, \\ +\infty \ \text{(otherwise)}. \end{cases}$$

Corollary 1.2. *If r, r' are similar, the minimum value of equation* (1) *when we shift the base points of r, r' along the rays is given by*

$$\max\left\{\frac{1}{2}\delta, d_{\hat{T}(X)}(r(\infty), r'(\infty))\right\},$$

where $\delta = \dfrac{1}{2}\log \max\limits_{j=1,\cdots,k} \dfrac{m'_j}{m_j} + \dfrac{1}{2}\log \max\limits_{j=1,\cdots,k} \dfrac{m_j}{m'_j}$.

Remark. The quantity δ is known as the *detour metric* between end points of the rays r, r' in the Gardiner-Masur boundary of $T(X)$. We refer to [Wal12], and also [Ama14].

2. Preliminaries

2.1. **Teichmüller spaces.** Let X be an analytically finite Riemann surface which has genus g and n punctures; briefly, we say it is of type (g,n). We assume that $3g-3+n>0$. Let $T(X)$ be the *Teichmüller space* of X. It is the set of equivalence classes of pairs of a Riemann surface Y and a quasiconformal mapping $f: X\to Y$. Two pairs (Y,f) and (Y',f') are equivalent if there is a conformal mapping $h: Y\to Y'$ such that $h\circ f$ is homotopic to f'. We denote by $[Y,f]$ the equivalence class of a pair (Y,f). The *Teichmüller distance* $d_{T(X)}$ is a complete distance on $T(X)$ which is defined by the following. For any $[Y,f]$, $[Y',f']$ in $T(X)$, $d_{T(X)}([Y,f],[Y',f']) = \frac{1}{2}\log\inf K(h)$, where the infimum ranges over all quasiconformal mappings $h: Y\to Y'$ such that $h\circ f$ is homotopic to f', and $K(h)$ is the maximal quasiconformal dilatation of h.

2.2. **Holomorphic quadratic differentials.** A *holomorphic quadratic differential* q on X is a tensor of the form $q(z)dz^2$ where $q(z)$ is a holomorphic function of a local coordinate z on X. For any $q\neq 0$, a zero of q or a puncture of X is called a *critical point* of q. Then, q has finitely many critical points. We allow q to have poles of order 1 at punctures of X. Then, the L^1-norm $\|q\| = \iint_X |q|$ is finite, where $|q| = |q(z)|dxdy$. If $\|q\|=1$, we call q of unit norm.

Let p_0 be a non-critical point of q and U be a small neighborhood of p_0 which does not contain any other critical points of q. For any point p in U, we can define a new local coordinate $\zeta(p) = \int_{z(p_0)}^{z(p)} q(z)^{\frac{1}{2}}dz$ on X where z is a local coordinate on U. The coordinate ζ is called a *q-coordinate*. By q-coordinates, we see that $q = d\zeta^2$, and in a common neighborhood of two q-coordinates ζ_1, ζ_2, the equation $\zeta_2 = \pm\zeta_1+$constant holds.

Suppose that p_0 is a critical point of q, and its order is $n\geq -1$. In a small neighborhood of p_0 which does not contain any other critical points of q, there exists a local coordinate z on X such that $z(p_0)=0$ and $q = z^n dz^2$. For instance, we refer to [Str84]. For any non-critical point in the neighborhood of p_0, there exists a q-coordinate ζ. By $d\zeta^2 = z^n dz^2$, the transformation $\zeta = \frac{2}{n+2} z^{\frac{n+2}{2}}$ holds. For any $k = 0, \cdots, n+1$, the set $\{\frac{2\pi k}{n+2} \leq \arg z \leq \frac{2\pi(k+1)}{n+2}\}$ on the z-plane is mapped to the half-plane $\{0 \leq \arg \zeta \leq \pi\}$ or $\{\pi \leq \arg \zeta \leq 2\pi\}$ on the ζ-plane. We can see the trajectory flow in the neighborhood of p_0 as the $n+2$ copies of the half-plane with the gluing along each horizontal edge of the planes (Figure 1).

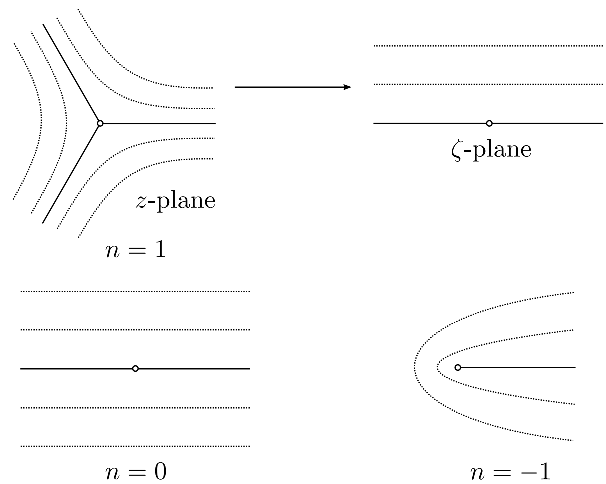

FIGURE 1. The trajectory flow in the neighborhood of p_0 in the case of $n = 1, 0, -1$

A *horizontal trajectory* of q is a maximal smooth arc $z = \gamma(t)$ on X which satisfies $q(\gamma(t))(\frac{d\gamma(t)}{dt})^2 > 0$. By definition, horizontal trajectories of q do not contain critical points of q. All horizontal trajectories of q are Euclidean horizontal arcs in q-coordinates; moreover, by the form of transformations of q-coordinates, "horizontal directions" are preserved. A *saddle connection* of q is a horizontal trajectory which joins critical points of q. We denote by Γ_q the set of all critical points of q and all saddle connections of q. Any component of $X - \Gamma_q$ is classified to the following two cases.

- *Annulus*: It is an annulus which is swept out by simple closed horizontal trajectories of q. These are free homotopic to each other. We call the simple closed horizontal trajectories the *core curves* of the annulus.
- *Minimal domain*: This domain is generated by infinitely many recurrent horizontal trajectories which are dense in the domain.

Since q has finitely many critical points, the number of components of $X - \Gamma_q$ is finite. If $X - \Gamma_q$ has only annuli, we call q a *Jenkins-Strebel differential.*

2.3. **Teichmüller geodesic rays.** For any holomorphic quadratic differential $q \neq 0$ on X, a quasiconformal mapping $f : X \to Y$ whose Beltrami coefficient is of the form $\mu_f = -\frac{K(f)-1}{K(f)+1}\frac{\bar{q}}{|q|}$ is called the *Teichmüller mapping.* For any quasiconformal mapping $g : X \to Y$, there exists a Teichmüller mapping $f : X \to Y$ which is homotopic to g. Furthermore, the Teichmüller mapping satisfies $K(f) \leq K(g)$ where the equality holds if and only if $f = g$.

Remark. More generally, If X and Y have the same genus and punctures, for any orientation preserving homeomorphism $g : X \to Y$, there exists a Teichmüller mapping $f : X \to Y$ which is homotopic to g (Theorem 1 in §1.5 of Chapter II of [Abi80]).

Let $f : X \to Y$ be a Teichmüller mapping and let q be the associated unit norm holomorphic quadratic differential on X. In this situation, there exists a unit norm holomorphic quadratic differential φ on Y such that f maps each zero of order n of q to a zero of order n of φ, and is represented by $w \circ f \circ z^{-1}(z) = K(f)^{-\frac{1}{2}}x + iK(f)^{\frac{1}{2}}y$ where $z = x + iy$ and w are q and φ-coordinates respectively. Such φ is uniquely determined. For more details of this discussion, we refer the reader to [IT92].

Let $p = [Y, f]$, let $q \neq 0$ be a unit norm holomorphic quadratic differential on Y, and z any q-coordinate. The mapping $r : \mathbb{R}_{\geq 0} \to T(X)$ is called a *Teichmüller geodesic ray from p determined by q* if for any $t \geq 0$, we assign a point $[Y_t, g_t \circ f]$ in $T(X)$ to $r(t)$ where g_t is a Teichmüller mapping on Y which is of the form $z = x + iy \mapsto z_t = e^{-t}x + ie^{t}y$, and Y_t is a Riemann surface which is determined by the coordinates z_t. We assume that $g_0 = id_Y$ and $Y_0 = Y$. By properties of Teichmüller mappings, we have $d_{T(X)}(r(s), r(t)) = |s - t|$ for any $s, t \geq 0$. If q is Jenkins-Strebel, we call r a *Jenkins-Strebel ray.*

Let r, r' be any two Jenkins-Strebel rays on $T(X)$ from $r(0) = [Y, f]$, $r'(0) = [Y', f']$ determined by Jenkins-Strebel differentials q, q' with unit norm on Y, Y' respectively. The rays r, r' are *similar* if there exist mutually disjoint simple closed curves $\gamma_1, \cdots, \gamma_k$ on X such that the set of homotopy classes of core curves of the annuli corresponding to q, q' are represented by $f(\gamma_1), \cdots, f(\gamma_k)$ on Y and $f'(\gamma_1), \cdots, f'(\gamma_k)$ on Y' respectively.

2.4. **Augmented Teichmüller spaces.** We refer to [Abi77] for augmented Teichmüller spaces; see also [IT92] and [HS07]. Let R be a connected Hausdorff space which satisfies the following conditions:

- Any $p \in R$ has a neighborhood which is homeomorphic to the unit disk $\mathbb{D} = \{|z| < 1\}$ or the set $\{(z_1, z_2) \in \mathbb{C}^2 | \ |z_1| < 1, |z_2| < 1, z_1 \cdot z_2 = 0\}$. (In the latter case, p is called a *node* of R.)
- Let $p_1, \cdots, p_k$ be nodes of R. We denote by $R_1, \cdots, R_r$ the connected components of $R - \{p_1, \cdots, p_k\}$. For any $i = 1, \cdots, r$, each R_i is a Riemann surface of type (g_i, n_i) which satisfies $2g_i - 2 + n_i > 0$, $n = \sum_{i=1}^{r} n_i - 2k$, and $g = \sum_{i=1}^{r} g_i - r + k + 1$.

We call R the *Riemann surface of type (g, n) with nodes.*

The *augmented Teichmüller space* $\hat{T}(X)$ is the set of equivalence classes of pairs of a Riemann surface of type (g, n) with or without nodes R and a deformation $f : X \to R$. The deformation f is a continuous mapping such that some disjoint loops on X are contracted to nodes of R, and is homeomorphic except to these loops. Two pairs (R, f) and (R', f') are equivalent if there is a conformal mapping $h : R \to R'$ such that $h \circ f$ is homotopic to f', where the conformal mapping means that each restricted mapping of a component of $R - \{\text{nodes of } R\}$ onto a component of $R' - \{\text{nodes of } R'\}$ is conformal. Obviously, $T(X)$ is included in $\hat{T}(X)$. A topology of $\hat{T}(X)$ is induced by the following. Let $[R, f]$ in $\hat{T}(X)$. For any compact neighborhood V of the set of nodes of R and any $\varepsilon > 0$, a neighborhood $U_{V,\varepsilon}$ of $[R, f]$ is defined by the set of $[S, g]$ in $\hat{T}(X)$ such that there is a deformation $h : S \to R$ which is $(1+\varepsilon)$-quasiconformal on $h^{-1}(R-V)$ such that f is homotopic to $h \circ g$.

2.5. **The end points of Jenkins-Strebel rays.** We consider the end points of Jenkins-Strebel rays. In the following discussion, we use the detailed description in §4.1 of [HS07]. Let r be a Jenkins-Strebel ray on $T(X)$ from $r(0) = [Y, f]$ determined by a Jenkins-Strebel differential q with unit norm on Y. All components of $Y - \Gamma_q$ are represented by rectangles $C_1, \cdots, C_k$ with identifications of vertical edges of them in q-coordinates. Let $m_1, \cdots, m_k$ be the moduli of $C_1, \cdots, C_k$ respectively. We cut off each rectangle in the half height, and the resulting half rectangle C_j^l is mapped conformally to the annulus $A_j^l(0) = \{e^{-m_j\pi} \leq |z| < 1\}$ for any $j = 1, \cdots, k$ and $l = 1, 2$. Then, we can assume that the original surface Y is constructed by $\{\overline{A_j^l(0)}\}_{j=1,\cdots,k}^{l=1,2}$ with gluing mappings which are determined naturally. Let $r(t) = [Y_t, g_t \circ f]$ be the representation of r for any $t \geq 0$. The Teichmüller mapping g_t is represented by $z = re^{i\theta} \mapsto r^{e^{2t}}e^{i\theta}$ on each $A_j^l(0)$. We set $A_j^l(t) = \{e^{-e^{2t}m_j\pi} \leq |z| < 1\}$ for any $j = 1, \cdots, k$, $l = 1, 2$, and $t \geq 0$, then Y_t is constructed by them as in the case of $t = 0$. In this representation, we can set $A_j^l(\infty)$ as the unit disk $\mathbb{D} = \{|z| < 1\}$ for any $j = 1, \cdots, k$ and $l = 1, 2$. We obtain the Riemann surface with nodes Y_∞ by $\{\overline{A_j^l(\infty)}\}_{j=1,\cdots,k}^{l=1,2}$ with the similar gluing mappings as in the case of $t \geq 0$. The deformation $g_\infty : Y \to Y_\infty$ is obtained by $z = re^{i\theta} \mapsto h_j(r)e^{i\theta}$ on $A_j^l(\infty)$ where $h_j : [e^{-m_j\pi}, 1) \to [0, 1)$ is an arbitrary monotonously increasing diffeomorphism for any $j = 1, \cdots, k$ and $l = 1, 2$. The homotopy class of g_∞ is independent of the choices of h_j for any $j = 1, \cdots, k$.

Proposition 2.1 (cf. [HS07]). *The Jenkins-Strebel ray $r(t) = [Y_t, g_t \circ f]$ on $T(X)$ converges to a point $r(\infty) = [Y_\infty, g_\infty \circ f]$ in $\hat{T}(X)$ as $t \to \infty$.*

Suppose that r, r' are similar Jenkins-Strebel rays on $T(X)$ from $r(0) = [Y, f]$, $r'(0) = [Y', f']$ determined by Jenkins-Strebel differentials q, q' with unit norm on Y, Y' respectively. Let $\gamma_1, \cdots, \gamma_k$ be as in the definition of "similar" in §2.3. There is a homeomorphism $\alpha : X - f^{-1}(\Gamma_q) \to X - f'^{-1}(\Gamma_{q'})$ which is homotopic to the identity such that the mapping $f' \circ \alpha \circ f^{-1}$ maps the core curves of the annuli corresponding to $f(\gamma_j)$ to the core curves of the annuli corresponding to $f'(\gamma_j)$ for any $j = 1, \cdots, k$. We set $r(\infty) = [Y_\infty, g_\infty \circ f]$, $r'(\infty) = [Y'_\infty, g'_\infty \circ f']$ and let $\{Y_{\infty,\lambda}\}_{\lambda=1,\cdots,\Lambda}$, $\{Y'_{\infty,\lambda}\}_{\lambda=1,\cdots,\Lambda}$ be the components of $Y_\infty - \{\text{nodes of } Y_\infty\}$, $Y'_\infty - \{\text{nodes of } Y'_\infty\}$ respectively, such that $(g'_\infty \circ f') \circ \alpha \circ (g_\infty \circ f)^{-1}(Y_{\infty,\lambda}) = Y'_{\infty,\lambda}$ for any $\lambda = 1, \cdots, \Lambda$. We define the Teichmüller distance between $r(\infty)$, $r'(\infty)$ by

$$d_{\hat{T}(X)}(r(\infty), r'(\infty)) = \max_{\lambda=1,\cdots,\Lambda} \frac{1}{2} \log \inf K(h_\lambda),$$

where the infimum ranges over all quasiconformal mappings $h_\lambda : Y_{\infty,\lambda} \to Y'_{\infty,\lambda}$ such that h_λ is homotopic to $(g'_\infty \circ f') \circ \alpha \circ (g_\infty \circ f)^{-1}$. This definition means that the distance between end points of Jenkins-Strebel rays is the maximum of the distance between the corresponding points of the Teichmüller space of each component of the end points of the rays.

3. Proof of Theorem 1.1

We recall our main theorem.

Theorem 1.1. *For any two Jenkins-Strebel rays r, r',*

$$(1) \lim_{t\to\infty} d_{T(X)}(r(t), r'(t))$$
$$= \begin{cases} \max\left\{\dfrac{1}{2}\log \max\limits_{j=1,\cdots,k}\left\{\dfrac{m'_j}{m_j}, \dfrac{m_j}{m'_j}\right\}, d_{\hat{T}(X)}(r(\infty), r'(\infty))\right\} \\ \hfill (\text{if } r,\, r' \text{ are similar}) \\ +\infty \ (\text{otherwise}). \end{cases}$$

We use the following lemma.

Lemma 3.1. *Let R, R' be Riemann surfaces with nodes and let $f : R \to R'$ be a K-quasiconformal mapping. This means that f is a homeomorphism, each restricted mapping of f which maps a component of $R - \{$nodes of $R\}$ onto a component of $R' - \{$nodes of $R'\}$ is a quasiconformal mapping, and the maximum of maximal dilatations of such mappings is K. Then, for any sufficiently small $\varepsilon > 0$, there exists a $(K + o(1))$-quasiconformal mapping $g_\varepsilon : R \to R'$ such that g_ε is conformal on a neighborhood of the set of nodes of R, and is homotopic to f.*

The lemma is proved in the paper of [FM10] (Lemma 3.12, p. 206), however, we give a new proof of the latter part of their proof.

Proof of Lemma 3.1. Let μ be the Beltrami coefficient of f. For any $\varepsilon > 0$, we consider a new Beltrami coefficient

$$\mu_\varepsilon = \begin{cases} 0 \ (0 < |z| < \varepsilon), \\ \mu \ (\text{otherwise}) \end{cases}$$

on R, where z is each local coordinate near nodes of R and the domain $\{|z| < \varepsilon\}$ represents a neighborhood of nodes $z = 0$. Then, there exist a Riemann surface with nodes R_ε and a K-quasiconformal mapping $f_\varepsilon : R \to R_\varepsilon$ such that f_ε is conformal on the neighborhood of nodes of R. For sufficiently small ε, we confirm that there exists a $(1 + o(1))$-quasiconformal mapping between R_ε and R'. The mapping $f \circ f_\varepsilon^{-1} : R_\varepsilon \to R'$ is K-quasiconformal in a small neighborhood of nodes of R_ε and is conformal on the outside of the neighborhood. We use local coordinates such that nodes of R_ε and R' correspond to 0, and $f \circ f_\varepsilon^{-1}(0) = 0$. Let $p_1, \cdots, p_k$ be all nodes of R_ε. For any $j = 1, \cdots, k$, we take the pair of small disks N_j^1, N_j^2 about p_j in R_ε where $f \circ f_\varepsilon^{-1}$ is K-quasiconformal in $\overline{N_j^1} \cup \overline{N_j^2}$. We regard each of the disks N_j^1, N_j^2 as $\{|z| < \delta\}$. For any $j = 1, \cdots, k$ and $l = 1, 2$, the image $f \circ f_\varepsilon^{-1}(N_j^l)$ in R' is mapped to the disk $\{|z| < \delta\}$ by a conformal mapping $f_{j,\varepsilon}^l$ such that $f_{j,\varepsilon}^l(0) = 0$ and $f_{j,\varepsilon}^l(\delta) = \delta$. We denote simply by $F_\varepsilon := f_{j,\varepsilon}^l \circ f \circ f_\varepsilon^{-1}$. The family of K-quasiconformal mappings $\{F_\varepsilon\}$ is normal, then we can assume that F_ε converges to a K-quasiconformal mapping F_0 uniformly on any compact set of $\{|z| < \delta\}$ as $\varepsilon \to 0$. However, any point of $\{0 < |z| < \delta\}$ is a holomorphic point of F_ε for sufficiently small ε, then F_0 is holomorphic in $\{0 < |z| < \delta\}$. Since F_0 fixes 0 and δ, we can see that F_0 is an automorphism on $\{|z| < \delta\}$ and then it is the identity.

Now, we rescale $\{|z| < \delta\}$ to $\mathbb{D}$, and assume that $\{F_\varepsilon\}$ as mappings of $\mathbb{D}$ onto itself. We set a conformal mapping $\phi(z) := (z-i)/(z+i)$ of $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im} z > 0\}$ onto $\mathbb{D}$. We consider mappings $f_\varepsilon := \phi^{-1} \circ F_\varepsilon \circ \phi$ of $\mathbb{H}$ onto itself.

Lemma 3.2. *We have*

$$\lim_{\varepsilon\to 0}\sup_{x,t}\frac{f_\varepsilon(x+t)-f_\varepsilon(x)}{f_\varepsilon(x)-f_\varepsilon(x-t)}=1,$$

where the supremum ranges over all $x, t \in \mathbb{R}$ such that $t \neq 0$.

Proof of Lemma 3.2. We notice that $\frac{f_\varepsilon(x+t)-f_\varepsilon(x)}{f_\varepsilon(x)-f_\varepsilon(x-t)}$ is positive, and its reciprocal is the case of $-t$. Then, it suffices to show that $t > 0$. Let

$$(z_1, z_2, z_3, z_4) = \frac{z_1 - z_2}{z_1 - z_3}\cdot\frac{z_3 - z_4}{z_2 - z_4}$$

be a cross ratio for any $z_1, z_2, z_3, z_4 \in \hat{\mathbb{C}}$. We write $\phi(x) = e^{i\theta}$, $\phi(x+t) = e^{i(\theta+\varphi)}$, and $\phi(x-t) = e^{i(\theta-\psi)}$ where $0 \le \theta < 2\pi$ and $\varphi, \psi > 0$. Since all Möbius transformations preserve cross ratios,

$$\begin{aligned}\frac{f_\varepsilon(x+t)-f_\varepsilon(x)}{f_\varepsilon(x)-f_\varepsilon(x-t)} &= -(f_\varepsilon(x), f_\varepsilon(x+t), f_\varepsilon(x-t), \infty)\\ &= -(F_\varepsilon\circ\phi(x), F_\varepsilon\circ\phi(x+t), F_\varepsilon\circ\phi(x-t), 1)\\ &= \left|\frac{F_\varepsilon(e^{i(\theta+\varphi)})-F_\varepsilon(e^{i\theta})}{F_\varepsilon(e^{i\theta})-F_\varepsilon(e^{i(\theta-\psi)})}\cdot\frac{F_\varepsilon(e^{i(\theta-\psi)})-1}{F_\varepsilon(e^{i(\theta+\varphi)})-1}\right|. \qquad (2)\end{aligned}$$

We set $z = e^{iy}$. By $\log F_\varepsilon(e^{iy}) = i(\arg F_\varepsilon(e^{iy}) + 2n\pi)$, we have

$$\frac{d\arg F_\varepsilon(e^{iy})}{dy} = \frac{z\frac{dF_\varepsilon(z)}{dz}}{F_\varepsilon(z)}.$$

Since $F_\varepsilon(z)$ and $\frac{dF_\varepsilon(z)}{dz}$ converge to z and 1 uniformly on $\partial\mathbb{D}$ respectively, then

$$\begin{aligned}\sup_{0\le y<2\pi}\left|\frac{d\arg F_\varepsilon(e^{iy})}{dy}-1\right| &= \sup\left|z\frac{dF_\varepsilon(z)}{dz}-F_\varepsilon(z)\right|\\ &\le \sup\left(\left|\frac{dF_\varepsilon(z)}{dz}-1\right|+|z-F_\varepsilon(z)|\right)\to 0\end{aligned}$$

as $\varepsilon \to 0$. For any $0 < E < 1$, we take sufficiently small ε such that

$$\left|\frac{d\arg F_\varepsilon(e^{iy})}{dy}-1\right| < E$$

holds for any $0 \le y < 2\pi$. Now, we calculate and estimate each term of (2).

$$\begin{aligned}
|F_\varepsilon(e^{i(\theta+\varphi)}) - F_\varepsilon(e^{i\theta})| &= 2\sin\frac{\arg F_\varepsilon(e^{i(\theta+\varphi)}) - \arg F_\varepsilon(e^{i\theta})}{2} \\
&= 2\sin\frac{\int_\theta^{\theta+\varphi} \frac{d\arg F_\varepsilon(e^{iy})}{dy}dy}{2} \\
&< 2\sin\frac{(1+E)\varphi}{2} \\
&= 2\sin\frac{(1+E)(\arg\phi(x+t) - \arg\phi(x))}{2} \\
&= 2\sin\frac{(1+E)\int_x^{x+t}\frac{d\arg\phi(y)}{dy}dy}{2} \\
&= 2\sin\frac{(1+E)\int_x^{x+t}\frac{\frac{d\phi(y)}{dy}}{i\phi(y)}dy}{2} \\
&= 2\sin\frac{(1+E)\int_x^{x+t}\frac{2}{1+y^2}dy}{2} \\
&= 2\sin\{(1+E)(\arctan(x+t) - \arctan x)\},
\end{aligned}$$

and similarly,

$$|F_\varepsilon(e^{i(\theta+\varphi)}) - F_\varepsilon(e^{i\theta})| > 2\sin\{(1-E)(\arctan(x+t) - \arctan x)\}.$$

For any $0 < \alpha \le \pi/2$,

$$\begin{aligned}
\left|\frac{\sin\{(1\pm E)\alpha\}}{\sin\alpha} - 1\right| &= \left|\frac{\sin\alpha\cos(E\alpha) \pm \cos\alpha\sin(E\alpha)}{\sin\alpha} - 1\right| \\
&= \left|\cos(E\alpha) \pm \cos\alpha\frac{\sin(E\alpha)}{\sin\alpha} - 1\right| \\
&\le |\cos(E\alpha) - 1| + \left|\frac{\sin(E\alpha)}{\sin\alpha}\right| \\
&= |\cos(E\alpha) - 1| + \left|\frac{\sin(E\alpha)}{E\alpha}\right|\left|\frac{\alpha}{\sin\alpha}\right|E \\
&\le E\alpha + \frac{\pi}{2}E \le \pi E \to 0
\end{aligned}$$

as $E \to 0$. This means that we can write $\sin\{(1\pm E)\alpha\} = (1+O(E))\sin\alpha$. Therefore,

$$\begin{aligned}
&2\sin\{(1\pm E)(\arctan(x+t) - \arctan x)\} \\
&\qquad = 2(1+O(E))\sin(\arctan(x+t) - \arctan x).
\end{aligned}$$

We conclude that

$$|F_\varepsilon(e^{i(\theta+\varphi)}) - F_\varepsilon(e^{i\theta})| < 2(1+O(E))\sin(\arctan(x+t) - \arctan x)$$

and

$$|F_\varepsilon(e^{i(\theta+\varphi)}) - F_\varepsilon(e^{i\theta})| > 2(1+O(E))\sin(\arctan(x+t) - \arctan x).$$

Also, we have similar estimates for

$$|F_\varepsilon(e^{i\theta}) - F_\varepsilon(e^{i(\theta-\psi)})|,\ |F_\varepsilon(e^{i(\theta-\psi)}) - 1|,\text{ and } |F_\varepsilon(e^{i(\theta+\varphi)}) - 1|.$$

Finally, we can see that

$$\begin{aligned}
&\frac{f_\varepsilon(x+t)-f_\varepsilon(x)}{f_\varepsilon(x)-f_\varepsilon(x-t)}\\
&\quad<(1+O(E))\frac{\sin(\arctan(x+t)-\arctan x)\sin(\frac{\pi}{2}-\arctan(x-t))}{\sin(\arctan x-\arctan(x-t))\sin(\frac{\pi}{2}-\arctan(x+t))}\\
&\quad=(1+O(E))\frac{\frac{t}{\sqrt{1+(x+t)^2}\sqrt{1+x^2}}\frac{1}{\sqrt{1+(x-t)^2}}}{\frac{t}{\sqrt{1+(x-t)^2}\sqrt{1+x^2}}\frac{1}{\sqrt{1+(x+t)^2}}}\\
&\quad=1+O(E).
\end{aligned}$$

The lower estimate is similar. □

Lemma 3.2 implies that the mapping $f^l_{j,\varepsilon}\circ f\circ f_\varepsilon^{-1}$ is $(1+o(1))$-quasisymmetric on the circle $\partial N^l_j=\{|z|=\delta\}$ for sufficiently small ε. We apply Lemma 4.1 in [Gup14]. Then, there exists a mapping $\eta^l_{j,\varepsilon}: N^l_j\to\{|z|<\delta\}$ which is $(1+o(1))$-quasiconformal, $\eta^l_{j,\varepsilon}|_{\partial N^l_j}=f^l_{j,\varepsilon}\circ f\circ f_\varepsilon^{-1}|_{\partial N^l_j}$, and $\eta^l_{j,\varepsilon}$ is the identity in a sufficiently small neighborhood of 0. The mapping $(f^l_{j,\varepsilon})^{-1}\circ\eta^l_{j,\varepsilon}: N^l_j\to f\circ f_\varepsilon^{-1}(N^l_j)$ is $(1+o(1))$-quasiconformal and is conformal in a sufficiently small neighborhood of 0 such that $(f^l_{j,\varepsilon})^{-1}\circ\eta^l_{j,\varepsilon}|_{\partial N^l_j}=f\circ f_\varepsilon^{-1}|_{\partial N^l_j}$ and $(f^l_{j,\varepsilon})^{-1}\circ\eta^l_{j,\varepsilon}(0)=0$. We consider the mapping h_ε of R_ε onto R' which is $(f^l_{j,\varepsilon})^{-1}\circ\eta^l_{j,\varepsilon}$ in N^l_j for any $j=1,\cdots,k$ and $l=1,2$, and is $f\circ f_\varepsilon^{-1}$ on $R_\varepsilon-\bigcup_{j=1,\cdots,k}^{l=1,2}N^l_j$. This mapping is $(1+o(1))$-quasiconformal and is clearly homotopic to $f\circ f_\varepsilon^{-1}$ by an appropriate homotopy on each disk N^l_j that is theidentity on the boundary. Therefore, we conclude that the composition $h_\varepsilon\circ f_\varepsilon$ is our desired mapping g_ε. □

Proof of Theorem 1.1. If r, r' are not similar, the result is already known; see [Iva01], [LM10], and also [Ama14].

Let r, r' be similar Jenkins-Strebel rays on $T(X)$ from $r(0)=[Y,f]$, $r'(0)=[Y',f']$ determined by Jenkins-Strebel differentials q, q' with unit norm on Y, Y' respectively. By definition, there exist mutually disjoint simple closed curves $\gamma_1,\cdots,\gamma_k$ on X such that the set of homotopy classes of core curves of the annuli corresponding to q, q' are represented by $f(\gamma_1),\cdots,f(\gamma_k)$ on Y and $f'(\gamma_1),\cdots,f'(\gamma_k)$ on Y' respectively. Moreover, there is a homeomorphism $\alpha: X-f^{-1}(\Gamma_q)\to X-f'^{-1}(\Gamma_{q'})$ which is homotopic to the identity such that the mapping $f'\circ\alpha\circ f^{-1}$ maps the core curves of the annuli corresponding to $f(\gamma_j)$ to the core curves of the annuli corresponding to $f'(\gamma_j)$ for any $j=1,\cdots,k$. We denote by m_j, m'_j the moduli of the annuli on Y, Y' with core curves homotopic to $f(\gamma_j)$, $f'(\gamma_j)$ respectively. For any $t\geq 0$, we set $r(t)=[Y_t,g_t\circ f]$, $r'(t)=[Y'_t,g'_t\circ f']$ where $g_t: Y\to Y_t$, $g'_t: Y'\to Y'_t$ are Teichmüller mappings. Let $r(\infty)=[Y_\infty,g_\infty\circ f]$, $r'(\infty)=[Y'_\infty,g'_\infty\circ f']$ be the end points of r, r' in the augmented Teichmüller space $\hat{T}(X)$ respectively. Let $\{Y_{\infty,\lambda}\}_{\lambda=1,\cdots,\Lambda}$, $\{Y'_{\infty,\lambda}\}_{\lambda=1,\cdots,\Lambda}$ be the components of $Y_\infty-\{\text{nodes of }Y_\infty\}$, $Y'_\infty-\{\text{nodes of }Y'_\infty\}$ respectively, such that $(g'_\infty\circ f')\circ\alpha\circ(g_\infty\circ f)^{-1}(Y_{\infty,\lambda})=Y'_{\infty,\lambda}$ for any $\lambda=1,\cdots,\Lambda$.

First, we consider the upper estimate. Let $h_\lambda: Y_{\infty,\lambda}\to Y'_{\infty,\lambda}$ be the Teichmüller mapping which is homotopic to $(g'_\infty\circ f')\circ\alpha\circ(g_\infty\circ f)^{-1}$, and we set the mapping $h: Y_\infty\to Y'_\infty$ constructed by $\{h_\lambda\}_{\lambda=1,\cdots,\Lambda}$. We set $K=\exp(2d_{\hat{T}(X)}(r(\infty),r'(\infty)))=\max_{\lambda=1,\cdots,\Lambda}K(h_\lambda)$. The Riemann surfaces with nodes Y_∞, Y'_∞ are represented by the unions of closed unit disks $\{\overline{A_j^l(\infty)}\}_{j=1,\cdots,k}^{l=1,2}$, $\{\overline{A_j'^l(\infty)}\}_{j=1,\cdots,k}^{l=1,2}$ respectively. Let $h_j^l := h|_{A_j^l(\infty)}$ be the restriction to $A_j^l(\infty)$ of h for any $j = 1,\cdots,k$ and $l = 1,2$, then there is λ such that h_j^l is equal to $h_\lambda|_{A_j^l(\infty)}$. We choose any sufficiently small $\varepsilon > 0$ and apply Lemma 3.1 to the mapping $h : Y_\infty \to Y'_\infty$. Hence, for any $j = 1,\cdots,k$ and $l = 1,2$, we assume that h_j^l is $(K + o(1))$-quasiconformal such that it is conformal in a small neighborhood of 0 in $A_j^l(\infty)$ and is homotopic to $(g'_\infty \circ f') \circ \alpha \circ (g_\infty \circ f)^{-1}$. In this small neighborhood in $A_j^l(\infty)$, h_j^l is represented by a power series, i.e., we can write $h_j^l(z) = c_j^l z + c_{j,2}^l z^2 + \cdots = c_j^l z + \psi_j^l(z)$ where $c_j^l \neq 0$, $-\pi < \arg c_j^1 \leq \pi$, $-\pi \leq \arg c_j^2 < \pi$, and $\psi_j^l(z)$ is holomorphic. Now, in the small neighborhood in each $A_j^l(\infty)$, we can use the idea of the proof of Theorem 1.1 in [Ama14]. One follows the proof of Theorem 1.1 in [Ama14] verbatim. For any $j = 1,\cdots,k$, we set $M_j = \frac{m_j'}{m_j}$, and for any $t \geq 0$, $\delta_j(t) = e^{-e^{2t}m_j\pi}$, $\delta_j'(t) = e^{-e^{2t}m_j'\pi}$, then $\delta_j'(t) = \delta_j(t)^{M_j}$. We only consider the case of $M_j > 1$, so we fix such j. Again, we use $\varepsilon' < 1$ instead of $o(1)$. We take X_j as

$$X_j < \frac{\log \frac{\varepsilon'}{M_j+\varepsilon'-1}}{\log M_j} < 0.$$

We take sufficiently large t such that an inequality $\delta_j(t)^{M_j} < |c_j^l|\delta_j(t)^{M_j^{X_j}}$ holds, and set $\Delta_j(t) = \delta_j(t)^{M_j^{X_j}}$. Also, we assume that a domain such that h_j^l can be represented by the power series contains $\{|z| \leq 2\Delta_j(t)\}$. We construct $F_{j,t}^l : A_j^l(t) \to h(A_j^l(t)) - \{|z| < \delta_j'(t)\}$ by the following:

$$F_{j,t}^l(z) = \begin{cases} P_{j,t}^l(z) & (\delta_j(t) \leq |z| \leq \Delta_j(t)) \quad \text{(i)}, \\ Q_{j,t}^l(z) & (\Delta_j(t) \leq |z| \leq 2\Delta_j(t)) \quad \text{(ii)}, \\ h_j^l(z) & (2\Delta_j(t) \leq |z| < 1) \quad \text{(iii)}. \end{cases}$$

(i) In $\delta_j(t) \leq |z| \leq \Delta_j(t)$, we set

$$P_{j,t}^l(z) = \Delta_j(t)^{\frac{1-M_j}{1-M_j^{X_j}}} \cdot {c_j^l}^{\frac{1}{1-M_j^{X_j}} + \frac{\log|z|}{\log\Delta_j(t) - \log\delta_j(t)}} \cdot |z|^{-\frac{1-M_j}{1-M_j^{X_j}}} \cdot z$$

which satisfies $P_{j,t}^l(z) = \delta_j(t)^{M_j-1} \cdot z$ on $|z| = \delta_j(t)$, $P_{j,t}^l(z) = c_j^l z$ on $|z| = \Delta_j(t)$. The mapping $P_{j,t}^l$ is a quasiconformal mapping because it is conjugate to a one-to-one affine mapping by $\log z$. The maximal dilatation of $P_{j,t}^l$ is

$$K(P_{j,t}^l) = \frac{\left|\frac{\log c_j^l}{2(M_j^{X_j}-1)\log\delta_j(t)} + \frac{\alpha_j}{2} + 1\right| + \left|\frac{\log c_j^l}{2(M_j^{X_j}-1)\log\delta_j(t)} + \frac{\alpha_j}{2}\right|}{\left|\frac{\log c_j^l}{2(M_j^{X_j}-1)\log\delta_j(t)} + \frac{\alpha_j}{2} + 1\right| - \left|\frac{\log c_j^l}{2(M_j^{X_j}-1)\log\delta_j(t)} + \frac{\alpha_j}{2}\right|},$$

where $\alpha_j = -\frac{1-M_j}{1-M_j^{X_j}}$. We see that

$$K(P_{j,t}^l) \to \frac{M_j - M_j^{X_j}}{1 - M_j^{X_j}} < M_j + \varepsilon'$$

as $t \to \infty$.

(ii) In $\Delta_j(t) \le |z| \le 2\Delta_j(t)$, we set

$$Q^l_{j,t}(z) = c^l_j z + \phi_{\Delta_j(t)}(|z|)\psi^l_j(z),$$

where $\phi_{\Delta_j(t)} : [\Delta_j(t), 2\Delta_j(t)] \to [0,1]$ is defined by

$$\phi_{\Delta_j(t)}(|z|) = \frac{|z|}{\Delta_j(t)} - 1.$$

Then $Q^l_{j,t}(z) = c^l_j z$ on $|z| = \Delta_j(t)$, $Q^l_{j,t}(z) = h^l_j(z)$ on $|z| = 2\Delta_j(t)$. We consider the partial derivatives of $Q^l_{j,t}$,

$$\partial_{\bar z} Q^l_{j,t} = \frac{1}{2\Delta_j(t)} z^{\frac12}\bar z^{-\frac12}\psi^l_j(z),$$
$$\partial_z Q^l_{j,t} = c^l_j + \frac{1}{2\Delta_j(t)} z^{-\frac12}\bar z^{\frac12}\psi^l_j(z) + \phi_{\Delta(t)}(|z|)\frac{d\psi^l_j(z)}{dz}.$$

These are continuous in $\Delta_j(t) \le |z| \le 2\Delta_j(t)$. There is $C > 0$ such that $|\psi^l_j(z)| \le C\Delta_j(t)^2$ for sufficiently large t. We see that

$$\left|\frac{1}{2\Delta_j(t)} z^{\frac12}\bar z^{-\frac12}\psi^l_j(z)\right| = \left|\frac{1}{2\Delta_j(t)} z^{-\frac12}\bar z^{\frac12}\psi^l_j(z)\right| = \frac{|\psi^l_j(z)|}{2\Delta_j(t)} \le \frac{C\Delta_j(t)}{2} \to 0$$

and then $|\partial_{\bar z} Q^l_{j,t}| \to 0$, $|\partial_z Q^l_{j,t}| \to |c^l_j| \ne 0$ as $t \to \infty$. Hence, for sufficiently large t, $\mathrm{Jac}\, Q^l_{j,t} = |\partial_z Q^l_{j,t}|^2 - |\partial_{\bar z} Q^l_{j,t}|^2 > 0$, and we conclude that $Q^l_{j,t}$ is a local C^1-diffeomorphism. We denote by D the closed set whose fundamental group is $\pi_1(D) = \mathbb{Z}$ and its boundary components are $Q^l_{j,t}(\{|z| = \Delta_j(t)\}) = \{|w| = |c^l_j|\Delta_j(t)\}$ and $Q^l_{j,t}(\{|z| = 2\Delta_j(t)\}) = h^l_j(\{|z| = 2\Delta_j(t)\})$. Since $Q^l_{j,t}$ is a local C^1-diffeomorphism, we have $Q^l_{j,t}(\{\Delta_j(t) \le |z| \le 2\Delta_j(t)\}) = D$. Furthermore, by the compactness of $\{\Delta_j(t) \le |z| \le 2\Delta_j(t)\}$, $Q^l_{j,t}$ is proper. Then we can regard the mapping $Q^l_{j,t} : \{\Delta_j(t) \le |z| \le 2\Delta_j(t)\} \to D$ as a covering. Let $Q^l_{j,t*} : \pi_1(\{\Delta_j(t) \le |z| \le 2\Delta_j(t)\}) \to \pi_1(D)$ be the group homomorphism induced by $Q^l_{j,t}$. We see that $Q^l_{j,t*}(\pi_1(\{\Delta_j(t) \le |z| \le 2\Delta_j(t)\})) = \mathbb{Z} \lhd \pi_1(D)$ because $Q^l_{j,t}(z) = c^l_j z$ on $|z| = \Delta_j(t)$. Then, the covering $Q^l_{j,t}$ is regular, and its covering transformation group is $\mathbb{Z}/\mathbb{Z} = 1$. Therefore, we conclude that $Q^l_{j,t}$ is a C^1-diffeomorphism. By the partial derivatives of $Q^l_{j,t}$, for sufficiently large t, it is a quasiconformal mapping and satisfies $K(Q^l_{j,t}) \to 1$ as $t \to \infty$.

(iii) In $2\Delta_j(t) \le |z| < 1$, we see that $F^l_{j,t}(z) = h^l_j(z)$ and $K(h^l_j) \le K$.

By the above discussions, for sufficiently large t, we obtain a quasiconformal mapping $F^l_{j,t}$ such that

$$\begin{aligned} K(F^l_{j,t}) = \max\{K(P^l_{j,t}), K(Q^l_{j,t}), K(h^l_j)\} \quad &\to \quad \max\left\{\frac{M_j - M_j^{X_j}}{1 - M_j^{X_j}}, K(h^l_j)\right\} \\ &< \quad \max\{M_j, K\} + \varepsilon' \end{aligned}$$

as $t \to \infty$.

In the cases of $M_j < 1$, $M_j = 1$, we also have

$$\lim_{t\to\infty} K(F^l_{j,t}) < \max\left\{\frac{1}{M_j}, K\right\} + \varepsilon'$$

and

$$\lim_{t\to\infty} K(F^l_{j,t}) = K$$

by similar arguments.

Thus, for sufficiently large t, we can construct the quasiconformal mapping $F_t : Y_t \to Y'_t$ by gluing $\{F^l_{j,t}\}^{l=1,2}_{j=1,\cdots,k}$. We obtain the inequality

$$\lim_{t\to\infty} K(F_t) < \max\left\{\max_{j=1,\cdots,k}\left\{\frac{m'_j}{m_j}, \frac{m_j}{m'_j}\right\}, K\right\} + \varepsilon'.$$

Next, we confirm that F_t is homotopic to $(g'_t \circ f') \circ (g_t \circ f)^{-1}$. In any case, each h^l_j is homotopic to $(g'_t \circ f') \circ \alpha \circ (g_t \circ f)^{-1}$ in $\{2\Delta_j(t) < |z| < 1\}$. Each $Q^l_{j,t}$ satisfies $K(Q^l_{j,t}) \to 1$ as $t \to \infty$ and the domain $\{\Delta_j(t) < |z| < 2\Delta_j(t)\}$ has the constant modulus for any t. Finally, each $P^l_{j,t}$ produces a twist of angle $\arg c^l_j$ in $\{\delta_j(t) < |z| < \Delta_j(t)\}$ and satisfies $|\arg c^1_j + \arg c^2_j| < 2\pi$. Therefore, for sufficiently large t, the mapping F_t is homotopic to $(g'_t \circ f') \circ \alpha \circ (g_t \circ f)^{-1}$. Since α is homotopic to the identity on X, we are done. We conclude that

$$\limsup_{t\to\infty} d_{T(X)}(r(t), r'(t)) \le \max\left\{\frac{1}{2}\log \max_{j=1,\cdots,k}\left\{\frac{m'_j}{m_j}, \frac{m_j}{m'_j}\right\}, d_{\hat{T}(X)}(r(\infty), r'(\infty))\right\}.$$

For the lower estimate, we can use the following inequality.

Proposition 3.3 ([Ama14]). *We have*

$$\liminf_{t\to\infty} d_{T(X)}(r(t), r'(t)) \ge \frac{1}{2}\log \max_{j=1,\cdots,k}\left\{\frac{m'_j}{m_j}, \frac{m_j}{m'_j}\right\}.$$

Furthermore, we use the following fact.

Proposition 3.4 ([Mas75]). *We have*

$$\liminf_{t\to\infty} d_{T(X)}(r(t), r'(t)) \ge d_{\hat{T}(X)}(r(\infty), r'(\infty)).$$

Combining the two inequalities above, we obtain the inequality

$$\liminf_{t\to\infty} d_{T(X)}(r(t), r'(t)) \ge \max\left\{\frac{1}{2}\log \max_{j=1,\cdots,k}\left\{\frac{m'_j}{m_j}, \frac{m_j}{m'_j}\right\}, d_{\hat{T}(X)}(r(\infty), r'(\infty))\right\}.$$

□

Corollary 1.2. *If r, r' are similar, the minimum value of equation* (1) *when we shift the base points of r, r' along the rays is given by*

$$\max\left\{\frac{1}{2}\delta, d_{\hat{T}(X)}(r(\infty), r'(\infty))\right\},$$

where $\delta = \frac{1}{2}\log \max_{j=1,\cdots,k} \frac{m'_j}{m_j} + \frac{1}{2}\log \max_{j=1,\cdots,k} \frac{m_j}{m'_j}$.

Proof of Corollary 1.2. We see that

$$\frac{1}{2}\log \max_{j=1,\cdots,k}\left\{\frac{m'_j}{m_j}, \frac{m_j}{m'_j}\right\} \ge \frac{1}{2}\delta.$$

The values $\frac{1}{2}\delta$ and $d_{\hat{T}(X)}(r(\infty), r'(\infty))$ are invariant when we shift the base points of the rays r, r'. Hence, by Theorem 1.1,

$$\lim_{t\to\infty} d_{T(X)}(r(t), r'(t+\alpha)) \geq \max\left\{\frac{1}{2}\delta, d_{\hat{T}(X)}(r(\infty), r'(\infty))\right\}$$

for any $\alpha \in \mathbb{R}$. The equality holds if

$$\alpha = \frac{1}{4}\log\frac{\displaystyle\max_{j=1,\cdots,k}\frac{m_j}{m'_j}}{\displaystyle\max_{j=1,\cdots,k}\frac{m'_j}{m_j}}.$$

Indeed, we calculate that

$$\begin{aligned}\max_{j=1,\cdots,k}\frac{e^{2\alpha}m'_j}{m_j} = \max_{j=1,\cdots,k}\left\{\frac{\sqrt{\displaystyle\max_{j=1,\cdots,k}\frac{m_j}{m'_j}}\cdot m'_j}{\sqrt{\displaystyle\max_{j=1,\cdots,k}\frac{m'_j}{m_j}}\cdot m_j}\right\} &= \sqrt{\max_{j=1,\cdots,k}\frac{m'_j}{m_j}}\cdot\sqrt{\max_{j=1,\cdots,k}\frac{m_j}{m'_j}} \\ &= \max_{j=1,\cdots,k}\frac{m_j}{e^{2\alpha}m'_j}.\end{aligned}$$

Therefore, we conclude that

$$\begin{aligned}&\lim_{t\to\infty} d_{T(X)}(r(t), r'(t+\alpha)) \\ &\quad= \max\left\{\frac{1}{2}\log\max_{j=1,\cdots,k}\left\{\frac{e^{2\alpha}m'_j}{m_j}, \frac{m_j}{e^{2\alpha}m'_j}\right\}, d_{\hat{T}(X)}(r(\infty), r'(\infty))\right\} \\ &\quad= \max\left\{\frac{1}{2}\left(\frac{1}{2}\log\max_{j=1,\cdots,k}\frac{m'_j}{m_j} + \frac{1}{2}\log\max_{j=1,\cdots,k}\frac{m_j}{m'_j}\right), d_{\hat{T}(X)}(r(\infty), r'(\infty))\right\} \\ &\quad= \max\left\{\frac{1}{2}\delta, d_{\hat{T}(X)}(r(\infty), r'(\infty))\right\}. \qquad \square\end{aligned}$$

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