ON THE CLASSIFICATION OF CRITICALLY FIXED RATIONAL MAPS

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ABSTRACT. We discuss the dynamical, topological, and algebraic classification of rational maps $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$, each of whose critical points c is also a fixed-point of f , i.e., $f(c) = c$.

1. INTRODUCTION

Fix an integer $d \geq 2$, and let $f(z) = \frac{p(z)}{q(z)} \in \mathbb{C}(z)$ be a rational function of degree d . Regarded as a self-map of the complex projective line $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$, its iterates yield a holomorphic dynamical system. Let

$$C(f) := \{c_1, \dots, c_n\}$$

denote the set of critical points of f , i.e., points at which f is not locally injective. The orbits of these few critical points under iteration of f significantly influence the global dynamical behavior of f ; see e.g. [Mil]. Counted with multiplicity, there are $2d - 2$ critical points. The set $\text{Fix}(f)$ of fixed-points, counted similarly with multiplicity, has $d + 1$ elements.

In this article, we consider the classification of those rational maps f for which $C(f) \subset \text{Fix}(f)$, so that each critical point of f is also a fixed-point of f . Our investigations generalize those of Tischler [Tis], who considered the special case of polynomials.

Each critically fixed rational map f is also a member of the larger set of so-called *Thurston maps* [DH]. Thurston maps, by definition, are orientation-preserving branched coverings $f : S^2 \rightarrow S^2$ of degree at least two for which the set $Z(f) := \bigcup_{n \geq 0} f^n(C(f))$ is finite. A crude, set-theoretic invariant of a Thurston map is its *critical orbit portrait*

$$(Z(f) \xrightarrow{f} Z(f), Z(f) \xrightarrow{\deg(f, \cdot)} \mathbb{N}).$$

This invariant records the dynamics and local degree of f on the set $Z(f)$. One may also consider abstract critical orbit portraits $(Z \rightarrow Z, Z \xrightarrow{\deg(\cdot)} \mathbb{N})$, where points $z \in Z$ with $\deg(z) > 1$ are regarded as “critical”, and where suitable admissibility conditions are satisfied; see §2.3 for the precise formulation.

Recall that the multiplicity m of a critical point c is given by $m = \deg(f, c) - 1$. Our first main result is

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Theorem 1.1. *Suppose $(Z \rightarrow Z, Z \rightarrow \mathbb{N})$ is a critical orbit portrait of degree $d \geq 2$ consisting of n fixed critical points $c_1, \dots, c_n$, each of multiplicity $m_i \leq d - 1$, and satisfying the Riemann-Hurwitz relation $\sum_{i=1}^n m_i = 2d - 2$.*

Then there is a rational function realizing this critical orbit portrait if and only if $n \leq d$.

The necessity of Theorem 1.1 follows easily from the Holomorphic Fixed-Point Formula. We prove sufficiency by applying Thurston's topological characterization of rational functions among Thurston maps [DH] to the set of Thurston maps obtained by a certain surgery procedure on a planar multigraph. For our purposes, a *planar multigraph* $\mathcal{G}$ is a simple graph $S\mathcal{G}$ embedded in the plane, without loops or multiple edges, and with edges given a positive integer weight indicating the number of edges joining the two incident vertices. The “blowing up” surgery of [PT] assigns, to each such multigraph $\mathcal{G}$, a Thurston map $F_{\mathcal{G}}$. Moreover, $F_{\mathcal{G}}$ is combinatorially equivalent to a rational map $f_{\mathcal{G}}$ if and only if $\mathcal{G}$ is connected ([PT, §5.2]). By construction, the set of vertices $V(\mathcal{G})$ is in bijective correspondence with the set of critical points $\text{Crit}(f_{\mathcal{G}})$; the valence of a vertex is the multiplicity of the corresponding critical point.

Sufficiency is then deduced from the following purely graph-theoretic result:

Theorem 1.2. *Fix $d \geq 3$. Any partition*

$$2d - 2 = m_1 + \dots + m_n$$

with $n \leq d$ and $1 \leq m_i \leq d - 1$ for each $i = 1, \dots, n$ arises as the set of valences of a connected planar multigraph.

The method of proof yields a new proof of the topological realizability of branch data satisfying the hypotheses in the first paragraph of Theorem 1.1; see §2.7 and Theorem 8.4 at the end of §8.

We now consider the assignment $\mathcal{G} \mapsto F_{\mathcal{G}}$, regarded as a map from planar multigraphs to Thurston maps.

Theorem 1.3. *Multigraphs $\mathcal{G}_1, \mathcal{G}_2$ are planar isomorphic if and only if the corresponding Thurston maps F_1, F_2 are combinatorially equivalent.*

Theorem 1.3 implies that the planar isomorphism type of $\mathcal{G}$ is an invariant of the Thurston class of $F_{\mathcal{G}}$. Unfortunately we do not know if the function $\mathcal{G} \mapsto f_{\mathcal{G}}$ from the set of connected planar multigraphs to the set of Möbius conjugacy classes of critically fixed rational maps is surjective. We establish some partial results. First, if p is a critically fixed complex polynomial, then $p = P_{\mathcal{G}}$ for some graph $\mathcal{G}$ (see §9.3); indeed, in this case $\mathcal{G}$ is a star. Next, for nonpolynomial maps up to degree five, we find all critically fixed rational maps by an exhaustive algebraic search; comparing the resulting set of solutions with the set of maps obtained by blowing up graphs, we deduce that these two classes are the same; see §11. Making use of a recent topological result of Liu and Osserman [LO], we will establish the following weaker result.

Theorem 1.4. *Given two critically fixed Thurston maps F_1, F_2 , if their number and multiplicities of critical points coincide, then up to combinatorial equivalence, we have $F_2 = h \circ F_1$ where $h : (S^2, C(F_1)) \rightarrow (S^2, C(F_1))$ is an orientation-preserving homeomorphism fixing $C(F_1)$ pointwise.*

Put another way, given a critically fixed Thurston map R , its *Hurwitz class* in the sense of S. Koch [Koc] depends only on its *branch data*. Moreover, there is a connected planar multigraph $\mathcal{G}$ such that up to combinatorial equivalence, $R = h \circ f_{\mathcal{G}}$, where h is a homeomorphism fixing $C(f_{\mathcal{G}})$ pointwise; we say R is a *twist* of $f_{\mathcal{G}}$. We remark here that there are explicit examples of distinct graphs with the same branch data, so that the planar isomorphism type of a graph is not invariant under twisting. Perhaps this is not surprising: the rabbit and airplane polynomials have different Hubbard trees, but are related by twisting; cf. [BN].

Suppose now R is an arbitrary critically fixed rational map. Associated to R is another graph-theoretic invariant, which we call its *Tischler graph* $\mathcal{T}$, defined as the union of the closures of its finite many fixed internal rays; see [Tis] and §9 below. If $R = f_{\mathcal{G}}$ for some $\mathcal{G}$ then we show that one can compute $\mathcal{T}$ from $\mathcal{G}$ and vice-versa. Our lack of understanding regarding whether $R = f_{\mathcal{G}}$ in general stems in part from a lack of knowledge regarding the structure of $\mathcal{T}$; in particular, we do not know whether $\mathcal{T}$ is always connected.

Our investigations concern a very special family of dynamical systems. Within this family, a number of natural invariants coincide. To highlight the central issues, we take here the opportunity to explicitly record the various relationships that exist between these invariants in as general a context as possible. One reason is that a number of these invariants have close analogs in nondynamical settings. The distinctions, however, are subtle and important for applications in dynamics. The finest such invariant, and a complete invariant of combinatorial equivalence, is a group-theoretic one called a *wreath recursion* on the fundamental group $\pi_1(S^2 - P(f))$. Laurent Bartholdi [Bar] has implemented the iterative algorithm on Teichmüller space used in the proof of Thurston's characterization of rational functions. Taking as input a wreath recursion, it returns as output a numerical approximation of the corresponding rational map, if it exists, or an algebraic description of a topological obstruction.

Our investigations conclude in §11 with a discussion of examples. We first present several natural families of examples arising from, e.g., Newton's methods and Belyi functions. We then present a systematic enumeration of nonpolynomial examples for degrees up to 5, for which naive algebraic methods are sufficient, if a bit unwieldy in some cases. For maps of degree 6, we use a different strategy. We enumerate the set of corresponding multigraphs, compute their wreath recursions, and use Bartholdi's program to obtain numerical approximations. We present some representative data and discuss the computations of the wreath recursions in detail.

Conceivably, there are critically fixed maps R which do not arise as maps of the form $f_{\mathcal{G}}$ for some planar multigraph. In light of Theorem 1.4, R is a twist of some $f_{\mathcal{G}}$. Future work will address the classification of twists of critically fixed rational maps.

2. THURSTON MAPS AND THEIR INVARIANTS

2.1. Summary of relationships. A *Thurston map* is an orientation-preserving branched covering $f : S^2 \rightarrow S^2$ for which the postcritical set $P(f) := \bigcup_{n \geq 0} f^n(C(f))$ is finite, where $C(f)$ denotes the set of points x at which the local degree $\deg(f, x)$ is larger than 1. Two Thurston maps f, g are called *Thurston*, or *combinatorially* equivalent if there exist orientation-preserving homeomorphisms $h_0, h_1 :$

$(S^2, P(f)) \rightarrow (S^2, P(g))$ such that $h_0 \circ f = g \circ h_1$ and h_0 is isotopic to h_1 through homeomorphisms $h_t, 0 \leq t \leq 1$, for which $h_t|_{P(f)} = h_0|_{P(f)}$ for all $t \in [0, 1]$.

It is useful to think of a Thurston map as a noninvertible analog of a homeomorphism of a surface to itself, and of a combinatorial equivalence class as the analog of a conjugacy class of the corresponding mapping class.

Let $f : S^2 \rightarrow S^2$ be a Thurston map. Associated to f will be five other objects, organized as follows:

The top row consists of dynamical objects. The bottom row consists of classical, nondynamical objects. The middle row is a mild enhancement of the nondynamical objects. The right-hand column consists of set-theoretic objects, while the left-hand column consists of objects belonging to the realm of the topology of surfaces. The horizontal arrows arise by restricting to finite sets. The upper vertical arrows forget the identification of domain and range. The lower vertical arrows arise by forgetting the distinguished role played by a finite set.

For each type of object, we will have a corresponding notion of isomorphism and of automorphism, and this correspondence will be natural with respect to the implied maps above. We will thereby obtain six categories (whose morphisms are all isomorphisms) and corresponding functors

Now suppose f is a Thurston map.

2.2. Distinguished sets. Naturally associated to a Thurston map f are the following distinguished subsets:

- (1) the set $C(f) = \{x : \deg(f, x) > 1\}$ of critical (also called branch) points; here $\deg(f, x)$ is the local degree of f at x ;
- (2) the set $V(f) := f(C(f))$, the set of critical (also called branch) values;
- (3) the postcritical set $P(f) := \bigcup_{n \geq 0} f^n(C(f))$; by definition, this set is finite for a Thurston map. We also denote this by $B(f) := P(f)$ when forgetting dynamics (see below).
- (4) $Z(f) := P(f) \cup C(f)$;
- (5) $A(f) := P(f) \cap f^{-1}(B(f))$.

2.3. Critical orbit portrait. The *critical orbit portrait*¹ associated to f is the pair

$$(Z(f) \xrightarrow{f} Z(f), Z(f) \xrightarrow{\deg(f, \cdot)} \mathbb{N})$$

consisting of the restrictions of f and of the local degree function to the set $Z(f)$; it is therefore a dynamical, set-theoretic invariant. We say critical orbit portraits of f and g are *isomorphic* if there is a conjugacy $Z(f) \rightarrow Z(g)$ respecting the local degree functions. We obtain a corresponding category **CP** whose objects are abstract critical orbit portraits $(Z \rightarrow Z, Z \rightarrow \mathbb{N})$ and whose isomorphisms are defined similarly.

Note that if f, g are Thurston equivalent via (h_0, h_1) then the restriction $Z(f) \xrightarrow{h_1} Z(g)$ induces an isomorphism between the corresponding critical orbit portraits. So we obtain a functor **TM** $\rightarrow$ **CP**.

2.4. Augmented branched cover. The *augmented branched cover* associated to f is the pair $f : (S^2, A(f)) \rightarrow (S^2, B(f))$, where now domain and range are no longer identified.

The augmented branched coverings associated to Thurston maps f and g are isomorphic (or augmented-Hurwitz-equivalent)² if there are orientation-preserving homeomorphisms $h_0 : (S^2, B(f)) \rightarrow (S^2, B(g))$ and $h_1 : (S^2, A(f)) \rightarrow (S^2, A(g))$ such that $h_0 \circ f = g \circ h_1$. We obtain a corresponding category **ABC** of abstract augmented branched coverings $(S^2, A) \rightarrow (S^2, B)$.

Clearly, a Thurston equivalence yields an augmented Hurwitz-equivalence. We obtain in this way a forgetful functor **TM** $\rightarrow$ **ABC**.

Note that if f, g are (A, B) -Hurwitz equivalent via a pair (h_0, h_1) , then necessarily $A(f) \cup C(f) \xrightarrow{h_1} A(g) \cup C(g)$.

2.5. Augmented branch data. The *augmented branch data* associated to f is the pair

$$\left(A(f) \cup C(f) \xrightarrow{f} B(f), A(f) \cup C(f) \xrightarrow{\deg(f, \cdot)} \mathbb{N} \right)$$

obtained by restricting f and the local degree function to the set $A(f) \cup C(f)$. Here, domain and range of f are no longer identified, so that $A(f) \cup C(f)$ and $B(f)$ are disjoint.

The augmented branch data of f and of g are called isomorphic if there are bijections $A(f) \cup C(f) \rightarrow A(g) \cup C(g)$ and $B(f) \rightarrow B(g)$ such that $A(f) \cup C(f) \rightarrow A(g) \cup C(g)$ respects the local degree functions and the diagram

$$\begin{array}{ccc} A(f) \cup C(f) & \longrightarrow & A(g) \cup C(g) \\ f \downarrow & & \downarrow g \\ B(f) & \longrightarrow & B(g) \end{array}$$

commutes. We obtain similarly a corresponding category **ABD** of abstract augmented branch data $(A \cup C \rightarrow B, A \cup C \rightarrow \mathbb{N})$. Sending an abstract augmented branched cover to the corresponding abstract augmented branch data is a functor **ABC** $\rightarrow$ **ABD**.

¹called a *mapping scheme* in [BBL⁺]

²In [Koc], this notion is formulated only for maps f and g with $A(f) = A(g), B(f) = B(g)$, and is there called (A, B) -Hurwitz-equivalent

2.6. Branched cover. The *branched cover* associated to f is the map $f : S^2 \rightarrow S^2$, where now domain and range are no longer identified.

The branched covers f, g are *Hurwitz equivalent* if there are orientation-preserving homeomorphisms h_0, h_1 satisfying $h_0 \circ f = g \circ h_1$. Note that $h_1 : C(f) \rightarrow C(g)$ and $h_0 : V(f) \rightarrow V(g)$. Thus (abstract, as opposed to arising from Thurston maps—but see below) branched covers and Hurwitz equivalences form a category **BC**.

An isomorphism of augmented branched covers gives a Hurwitz equivalence, so we again obtain a forgetful functor **ABC** $\rightarrow$ **BC**.

2.7. Branch data, or passport. The *branch data* associated to f is the pair

$$(C(f) \xrightarrow{f} V(f), C(f) \xrightarrow{\deg(f, \cdot)} \mathbb{N}).$$

More conventionally, branch data can be regarded as a collection of partitions of the degree d of f , one for each element v of $V(f)$, with each partition given by the unordered set of local degrees at points in $f^{-1}(v)$. The branch data of f and g are isomorphic if there are bijections $C(f) \rightarrow C(g), V(f) \rightarrow V(g)$ respecting the local degree functions and satisfying

$$\begin{array}{ccc} C(f) & \longrightarrow & C(g) \\ f \downarrow & & \downarrow g \\ V(f) & \longrightarrow & V(g) \end{array}$$

Since in this work we are restricting our attention to self-maps of spheres, it is convenient to now introduce three additional admissibility requirements that must be satisfied by abstract branch data $(C \rightarrow V, C \xrightarrow{\deg} \mathbb{N})$ which are necessary in order to arise from a branched covering $S^2 \rightarrow S^2$.

- $\deg(c) \geq 2$ for each $c \in C$, and $C \rightarrow V$ is surjective (else we are dealing with augmented branch data).
- The *Riemann-Hurwitz formula*

$$(2.1) \quad \sum_{c \in C} (\deg(c) - 1) = 2(d - 1)$$

is satisfied for some integer $d \geq 2$, called the *degree*.

- $\deg(c) \leq d$ for all $c \in C$.

Thus we have a category **BD** whose objects are abstract admissible pairs $(C \rightarrow V, C \rightarrow \mathbb{N})$. We obtain similarly a functor **BC** $\rightarrow$ **BD**. By forgetting values off the critical locus (that is, by restricting) we obtain a functor **ABD** $\rightarrow$ **BD**.

2.8. Hurwitz factorizations. There are necessary and sufficient algebraic conditions, algorithmically implementable, for determining Hurwitz equivalence and for determining the realizability of a given abstract admissible branch data by a branched covering; see also §3.2 below. Though classical, there are some subtleties regarding identifications, and this seems not to be widely known among researchers in dynamics. So we briefly review the theory, due to Hurwitz; see e.g. [BE].

Fix $d \geq 2$ and fix an abstract admissible branch data $(C \xrightarrow{f} V, C \xrightarrow{\deg} \mathbb{N})$ of degree d ; set $n := \#V$. Consider the (possibly empty) set of all branched covers $F : S^2 \rightarrow S^2$ realizing this branch data. We will assign to F a group-theoretic invariant, called a *Hurwitz factorization*. We will characterize the set of invariants

that arise, and we will give necessary and sufficient conditions in terms of Hurwitz factorizations for two branched covers F, G realizing this branch data to be Hurwitz equivalent. Loosely speaking, this is a coordinate-free classification of isomorphism classes of covering spaces; the base is allowed to vary.

The invariants we will consider, called *Hurwitz factorizations*, take the following form. Fix an enumeration $V = \{v_1, \dots, v_n\}$. For each $i = 1, \dots, n$ let

$$d_{i,1} + \dots + d_{i,m_i} = d$$

be the partition of d determined from the given branch data by the local degrees $d_{ij} := \deg(c_{ij})$ above v_i . A *Hurwitz factorization of degree d (and genus 0)* is an ordered n -tuple of permutations

$$(\sigma_1, \dots, \sigma_n)$$

satisfying the following conditions:

- (1) (transitivity) the subgroup of S_d generated by $\sigma_1, \dots, \sigma_n$ is transitive;
- (2) (local degrees) for each $i = 1, \dots, n$, the canonical decomposition into disjoint cycles

$$\sigma_i = \sigma_{i1} \cdot \dots \cdot \sigma_{im_i}$$

has the property that σ_{ij} is a cycle of length d_{ij} ;

- (3) (factorization) $\sigma_1 \cdot \dots \cdot \sigma_n = \text{id}$; here σ_1 is performed first, since monodromy is usually a right action.

The *Hurwitz Existence Theorem* asserts that the given branch data is realizable precisely when there exists a Hurwitz factorization satisfying the above properties; cf. [BE].

We now turn to the extent to which the set of such Hurwitz factorizations may be regarded as invariants of branched coverings realizing the given branch data.

For convenience, fix an identification $S^2 = \widehat{\mathbb{C}}$. Fix an identification $V = \{v_k := \exp(2\pi i k/n), k = 1, \dots, n\} \subset S^2$. Let $D := \{|z| \leq 2\} \subset \widehat{\mathbb{C}}$. Let $\text{Homeo}^+(S_n^2)$ and $\text{Homeo}^+(D_n, \partial D)$ denote respectively the group of orientation-preserving homeomorphisms of the sphere and of the disk D which preserve (not necessarily pointwise) the set V and which in the latter case are the identity on ∂D ; each is equipped with the compact-open topology. Let $\text{Mod}(S_n^2), \text{Mod}(D_n)$ denote the corresponding mapping class groups, that is, the group of path-components. By “capping” via an extension fixing the point at infinity, there is a natural inclusion $\text{Homeo}^+(D_n, \partial D) \hookrightarrow \text{Homeo}^+(S_n^2)$. By the Alexander Trick, this is surjective on the level of path-components, so the induced homomorphism $\text{Mod}(D_n) \rightarrow \text{Mod}(S_n^2)$ is surjective. The kernel is infinitely cyclic if $n \geq 3$; a generator is represented by a Dehn twist about the curve $|z| = 3/2$. Note that $\text{Mod}(D_n)$ is isomorphic to the Artin braid group B_n on n strands.

For $i = 1, \dots, n$ let r_i be the Euclidean segment joining the origin to v_i , and let γ_i be the loop in $\widehat{\mathbb{C}} \setminus V$ based at the origin which runs along r_i , loops around v_i in the counterclockwise direction, and returns along r_i . Let g_i be the corresponding element of $\pi_1(\widehat{\mathbb{C}} \setminus V, 0)$.

Now suppose F is a branched covering realizing the above branch data. By postcomposing with a (nonunique) homeomorphism ϕ , we may assume (i) $V(F) = V$, and (ii) above each $v_i \in V$, the partition of the integer d given by the ramification in the given abstract branch data coincides with that given by the ramification

of F . Fix an identification $\Lambda : F^{-1}(0) = \{1, \dots, d\}$. Then there is a monodromy representation

$$\rho_F : \pi_1(\widehat{\mathbb{C}} \setminus V, 0) \rightarrow S_d.$$

The *Hurwitz factorization* $\sigma(F)$ associated to F is the ordered n -tuple of permutations

$$\sigma(F) := (\sigma_1(F), \dots, \sigma_n(F)) \in S_d \times \dots \times S_d =: (S_d)^n$$

where $\sigma_i(F) = \rho_F(g_i)$.

The Hurwitz factorization associated to F constructed in the previous paragraph depends on (i) the choice of homeomorphism ϕ , and (ii) the choice of identification Λ . Clearly, a different choice of Λ changes ρ_F by post-composition with an inner automorphism of S_d . We see that we must consider two Hurwitz factorizations $(\alpha_1, \dots, \alpha_n), (\beta_1, \dots, \beta_n)$, to be equivalent if they are simultaneously conjugate, i.e., if there exists $x \in S_d$ such that

$$(\beta_1, \dots, \beta_n) = (\alpha_1^x, \dots, \alpha_n^x)$$

where we denote $g^x := x^{-1}gx$ and where x^{-1} is performed first (since monodromy is usually a right action).

If ϕ_1, ϕ_2 are two different homeomorphisms, the resulting coverings branched over V differ by postcomposition by an element h of $\text{Homeo}^+(S_n^2)$. If F, G are two coverings with $V(F) = V(G) = V$ and $G = h \circ F$ where $h \in \text{Homeo}^+(S_n^2)$ then $G^{-1} = F^{-1} \circ h^{-1}$. It follows that the monodromy representations of F and G are related:

$$G = h \circ F \implies \rho_G \doteq \rho_F \circ h_* \iff \rho_G \circ h_*^{-1} \doteq \rho_F$$

where $\doteq$ denotes equality up to postcomposition with inner automorphisms. Note that if the natural map

$$\text{Aut}(C \xrightarrow{f} V, C \xrightarrow{\deg} \mathbb{N}) \longrightarrow \text{Aut}(V)$$

(obtained by restricting an automorphism of branch data to the set of critical values) has trivial image, then the corresponding homeomorphism h fixes V pointwise, and so the corresponding mapping class (or any corresponding braid) element is pure. The effect of this ambiguity on the Hurwitz factorizations can be calculated as follows. Since we consider factorizations differing by simultaneous conjugation to be equivalent, we may assume $h(0) = 0$, so that h_* is well-defined. Then for each $i = 1, \dots, n$,

$$\sigma_i(G) = \rho_F(h_*^{-1}(g_i)).$$

More concretely, to find $\sigma_i(G)$, express $h_*^{-1}(g_i)$ as a word in the generators $g_1, \dots, g_n$, and replace the letters in this word by the corresponding permutations $\sigma_1(F), \dots, \sigma_n(F)$.

In this manner the groups $B_n \simeq \text{Mod}(D_n), \text{Mod}(S_n^2)$ act on the set of Hurwitz factorizations, with the action of the former factoring through the latter since the kernel acts trivially. Moreover, given F , the discussion in the previous paragraph shows that the orbit of $\sigma(F)$, under the action of both the braid group B_n and the diagonal action of simultaneous conjugation by elements of S_d , is a well-defined invariant of the branched cover F .

The formalism done, the proof of the following result is then very short:

Theorem 2.1. *Fix a given realizable admissible abstract branch data. Two branched coverings F, G realizing this data are Hurwitz equivalent if and only if $\sigma(F), \sigma(G)$ lie in the same B_n -orbit, up to simultaneous conjugation.*

Proof. By varying F and G within their Hurwitz classes as above, we may assume $V(F) = V(G) = V$ and that the ramification of F and of G above each point $v_i \in V$ is given by the common branch data. Consider the corresponding unramified coverings $F : S^2 \setminus F^{-1}(V) \rightarrow S^2 \setminus V$ and $G : S^2 \setminus G^{-1}(V) \rightarrow S^2 \setminus V$. If F, G are Hurwitz equivalent via a pair h, h_1 with $h : (S^2, V) \rightarrow (S^2, V)$, then the unramified covering spaces given by $h \circ F$ and G are isomorphic, and from the above discussion, the Hurwitz factorizations of F and G lie in the same orbit. Conversely, if the Hurwitz factorizations of F and G differ by the action of a homeomorphism h , then the corresponding monodromy representations of the unramified coverings $h \circ F$ and G are isomorphic, and so h lifts to a homeomorphism $h_1 : S^2 \setminus F^{-1}(V) \rightarrow S^2 \setminus G^{-1}(V)$ which extends over the punctures to a homeomorphism $h_1 : S^2 \rightarrow S^2$ so that the pair (h, h_1) gives a Hurwitz equivalence from F to G . $\square$

Practically speaking, then, to determine whether two given branched coverings F, G realizing the same branch data are Hurwitz equivalent, we do the following. Choose basepoints b_F, b_G in $S^2 \setminus V(F), S^2 \setminus V(G)$, respectively. Choose an indexing of $V(F), V(G)$ so that the ramification of F and G above the i th point are isomorphic. Choose a set of rays in the codomains of F and of G running from the basepoint to the critical values. This data determines, up to homeomorphism respecting the labelling of critical values, an identification of each of the codomains of F and G with the standard codomain and associated set of rays and critical values constructed above. One then computes the associated Hurwitz factorizations $\sigma(F), \sigma(G)$, which may be read off from a combinatorial presentation of F and of G . The problem of Hurwitz equivalence is then reduced to an (albeit complicated) problem in algebra.

2.8.1. Extension to augmented branched coverings. Assume that a given abstract augmented branch data $(A \cup C \rightarrow B, A \cup C \rightarrow \mathbb{N})$ is realizable by an augmented branched cover, f . Consider the problem of deciding the augmented Hurwitz equivalence of two such realizations f, g . One may attempt to adapt the classical framework to this new setting. However, if A does not consist of all inverse images (counted with multiplicity) of elements of B , then a homeomorphism h for which $h \circ f$ and g are covering equivalent need not have a lift h_1 which sends $A(f)$ to $A(g)$ as is required in the definition of augmented Hurwitz equivalence. One may attempt to demonstrate that some subgroup of the braid group will determine an equivalence relation wherein such lifts h_1 must send $A(f) \rightarrow A(g)$. The difficulty is that such lifts need not be unique, and formulating an appropriate group action is tricky. In the “pure” category this has been accomplished in [KPS, §3.1].

2.9. Functional compositions and decompositions. A Thurston map

$$f : (S^2, P) \rightarrow (S^2, P)$$

may factor as a composition of augmented branched coverings.

Example ([BEKP, §5.1]). The quartic polynomial

$$f(z) = 2i \left(z^2 - \frac{1+i}{2} \right)^2$$

has postcritical set $P := \{0, 1, -1, \infty\}$. It factors as $f = g \circ s$ with $s(z) = z^2$ and $g(z) = 2i(z - \frac{1+i}{2})^2$. Put $A := \{0, 1, \infty\}$. Then f factors as a composition of

augmented branched covers

$$(\mathbb{P}^1, P) \xrightarrow{s} (\mathbb{P}^1, A) \xrightarrow{g} (\mathbb{P}^1, P)$$

as is easily verified. The dynamical properties of f are significantly influenced by the nondynamical properties of its compositional factors. For example, here $\#A = 3$, so the pullback map on Teichmüller space induced by f is constant.

If a Thurston map f with fixed critical points factors nontrivially as above, it is equivalent to $z \mapsto z^n$ for some n with $|n| \geq 2$. For if $f = g \circ s$ then each critical point of g must be a critical value of f above which f is totally ramified (else there are critical points with the same image). Thus s and g each have two critical points and two critical values, with $C(g) = V(s)$, so after conjugation the claim follows. Thus critically fixed Thurston maps are, with the above exception, all indecomposable from a function-theoretic point of view.

The preceding example demonstrates the importance of considering nondynamical objects—augmented branched coverings and augmented branch data—when considering the classification theory for Thurston maps. The next example, which arose in conversations with S. Koch and A. Saenz, shows that functional factors of Thurston maps can be augmented branched covers which do not themselves arise from Thurston maps via forgetting identification of domain and range.

Example. We identify $\widehat{\mathbb{C}}$ with S^2 . Let $g : (\widehat{\mathbb{C}}, P(g)) \rightarrow (\widehat{\mathbb{C}}, P(g))$ be an integral Lattès example induced by $z + \Lambda \mapsto 3z + \Lambda$ on a complex torus $\mathbb{C}/\Lambda$. Thus $\deg(g) = 9$, and each of the 16 critical points of g has local degree two. These critical points map surjectively to the postcritical set $P(g)$ of g , and each element of $P(g)$ is a repelling fixed-point of g . Thus each $p \in P(g)$ has five preimages: one equal to itself, and four that are critical points. Pick $p \in P(g)$, and let $h : S^2 \rightarrow S^2$ be an orientation-preserving homeomorphism which sends P bijectively onto $g^{-1}(P(g)) \setminus \{p\}$. Finally, put $f = h \circ g$. The map f is a *nearly Euclidean Thurston map* in the sense of [CFPP].

Here is a description of the critical orbit portrait of f . We may identify

$$C(f) = \{a_0, \dots, a_{11}, b_0, b_1, b_2, c\}$$

so that

$$a_i \mapsto b_{i \bmod 3} \mapsto c \mapsto c$$

and thus $P(f) = \{b_0, b_1, b_2, c\}$.

We now find a factorization of the augmented branched cover $f : (S^2, P(f)) \rightarrow (S^2, P(f))$. Separating domain and range, set $A := P(f) = \{a_1, \dots, a_4\}$, $C := P(f) = \{c_1, \dots, c_4\}$. It is easy to see that f factors as a composition of degree three maps

$$(S^2, A) \xrightarrow{g} (S^2, B) \xrightarrow{s} (S^2, C)$$

where

$$a_1, a_2, a_3 \xrightarrow{g, 1:1} b_4, \quad a_4 \xrightarrow{g, 2:1} b_5$$

and

$$b_i \xrightarrow{s, 1:1} c_i, i = 1, \dots, 3, \quad b_4 \xrightarrow{s, 2:1} c_4, \quad b_5 \xrightarrow{s, 1:1} c_4.$$

The symbol $2 : 1$ above an arrow indicates that the corresponding local degree is 2. Since $\#B = 5 > 4 = \#A$, we conclude that the augmented branched covering $g : (S^2, A) \rightarrow (S^2, B)$, which is a compositional factor of the Thurston map f ,

cannot itself arise from a Thurston map by forgetting identification of domain and range.

2.10. Dynamical compositions and decompositions. Suppose $P \subset S^2$ is a finite set. A *multicurve* is a collection $\Gamma = \{\gamma_1, \dots, \gamma_k\}$ of disjoint, pairwise nonhomotopic, simple, closed, essential, unoriented, nonperipheral curves up to homotopy in $S^2 \setminus P$. If $f : (S^2, P) \rightarrow (S^2, P)$ is a Thurston map, a multicurve Γ is *invariant* if each preimage of an element of Γ is either inessential, peripheral, or homotopic in $S^2 \setminus P$ to an element of Γ .

If $f^{-1}(\Gamma) = \Gamma$ (in the sense that for each $\gamma_i \in \Gamma$, there exists $\gamma_j \in \Gamma$ such that γ_i is homotopic in $S^2 - P$ to a component of $f^{-1}(\gamma_j)$), i.e., Γ is *fully invariant*, then by cutting along Γ one can decompose, in a dynamically natural way, a Thurston map into simpler pieces; see e.g. [Pil2]. In the case $f = f_{\mathcal{G}}$ for some multigraph $\mathcal{G}$, the boundary of any sufficiently small regular neighborhood of an edge will be a fully invariant multicurve; see Lemma 7.2 below.

Of course, one could formulate a procedure inverse to that of blowing up an arc, and regard such surgery as a decomposition or reduction of the dynamics.

2.11. Wreath recursions. Suppose $f : (S^2, P) \rightarrow (S^2, P)$ is a Thurston map of degree d . Fix a basepoint $b \in S^2 - P$ and, for convenience, set now $G := \pi_1(S^2 - P, b)$ and $X := \{1, \dots, d\}$. Fix a bijection $\Lambda : X \rightarrow f^{-1}(b)$ and use this to identify X with $f^{-1}(b)$. For each $x \in X$, choose a path $\lambda_x : [0, 1] \rightarrow S^2 - P$ joining b to x .

This data determines a *wreath recursion* of G : a homomorphism

$$\Phi_f : G \rightarrow G^d \rtimes S_d$$

$$g \mapsto (g_1, \dots, g_d)\sigma(g).$$

See [Nek1]. Here, $\sigma(g)$ is the monodromy permutation of X induced by g , with the right action written $i \mapsto i^\sigma$ and

$$g_i = \lambda_i * f^{-1}(g)[i] * \bar{\lambda}_i^\sigma.$$

(The asterisk stands for concatenation of paths; λ_i is traversed first; $f^{-1}(g)[i]$ is the lift of g under f based at i ; the notation $\bar{\lambda}$ means the reversal of the path λ .)

Multiplication in $G^d \rtimes S_d$ is computed like this. First, some conventions: permutations are *right* actions. So $\sigma = (1234)$ means

$$(1234) = (1 \mapsto 2 \mapsto 3 \mapsto 4 \mapsto 1).$$

We often write $2 = 1^\sigma$; equivalently, $2 = \sigma(1)$. Notice that $(123)(35) = (1253) \neq (5123)$ according to our convention; permutations are composed left to right. With this convention, here is how we multiply in $G^d \rtimes S_d$:

$$\langle (g_1, \dots, g_d)\sigma \rangle * \langle (h_1, \dots, h_d)\tau \rangle = \langle g_1 h_{1^\sigma}, \dots, g_d h_{d^\sigma} \rangle \sigma\tau.$$

For example,

$$\langle 1, ab, a, b^{-1} \rangle (142) * \langle ab, a^{-1}, 1, b \rangle (134) = \langle 1.b, ab.ab, a.1, b^{-1}a^{-1} \rangle (234).$$

This is nicely represented by regarding the elements as decorated permutations, and then concatenating them like braids:

The wreath recursion may be conveniently encoded as follows. For convenience we identify S^2 with $\widehat{\mathbb{C}}$. If $\#P = n$ we may identify $P = \{p_1, \dots, p_n\}$ with $\{\exp(2\pi i k/n), k = 1, \dots, n\}$, where $p_k = \exp(2\pi i k/n)$. For each $k = 1, \dots, n$ we choose as in §2.8 a generator g_k which runs radially from the origin to p_k , loops around p_k counterclockwise, and returns to the origin radially. Then the wreath recursion of f is encoded by the list of n expressions

$$\Phi(g_k) = (w_{1k}, \dots, w_{dk})\sigma(g_k), \quad k = 1, \dots, n$$

where w_{ik} are words in the generators g_k . If a topological model for f is given, the monodromy induced by g_k and the words w_{ik} may be calculated; see §2.11.

Up to a suitable notion of equivalence, the wreath recursion is a complete invariant of the Thurston combinatorial class of f . The following is a restatement of [Nek1, Theorem 6.5.2] into equivalent language.

Theorem 2.2. *Suppose $f : (S^2, P(f)) \rightarrow (S^2, P(f))$ and $g : (S^2, P(g)) \rightarrow (S^2, P(g))$ are Thurston maps. Choose basepoints $b_f \in S^2 - P(f)$, $b_g \in S^2 - P(g)$, and let G_f, G_g be the corresponding fundamental groups of $S^2 - P(f), S^2 - P(g)$, respectively. Then f and g are combinatorially equivalent via a homeomorphism $h : (S^2, P(f)) \rightarrow (S^2, P(g))$ if and only if the following diagram commutes up to inner automorphisms of $G_g^d \rtimes S_d$ induced by conjugation by elements of S_d :*

$$\begin{array}{ccc} G_f & \xrightarrow{\Phi_f} & G_f^d \rtimes S_d \\ h_* \downarrow & & \downarrow h_*^d \times \text{id} \\ G_g & \xrightarrow{\Phi_g} & G_g^d \rtimes S_d \end{array}$$

L. Bartholdi [Bar] has implemented a robust general version of the iterative algorithm on Teichmüller space used in the proof of Thurston's characterization theorem. Given a wreath recursion for a Thurston map, it is designed to return either a numerical approximation for the corresponding rational map, or an algebraic description of an obstruction.

2.12. Topological polynomials. A Thurston map f is a *topological polynomial* if it has a branch point c with $f(c) = c$ and $\deg(f, c) = \deg(f)$. Analogously, an abstract critical portrait is of polynomial type if it has a fixed-point of weight equal to its degree; a branch datum is of polynomial type if it has a point in its domain of weight equal to its degree. The combinatorial theory of polynomials is much simpler than that for rational functions, due mainly to the existence of good normal forms (Hubbard trees; critical (external ray) portraits; invariant spiders; kneading automata; etc.) for various invariants; see [Nek2].

3. TOPOLOGICAL REALIZABILITY CONDITIONS

It is natural to consider branched coverings $S^2 \rightarrow S^2$ without regard to dynamics, as well as to consider abstract directed graphs representing potential branch data, augmented branch data, and critical portraits. In this section, we consider the problem of characterizing the images of the natural functors between the categories given in §2.1.

3.1. $\mathbf{TM} \rightarrow \mathbf{BC}$. A branched covering $F : S^2 \rightarrow S^2$ determines a map of pairs $F : (S^2, C(F)) \rightarrow (S^2, V(F))$ where $\#V(F) \leq \#C(F)$, so there exists a (many, really) homeomorphism H yielding a map of pairs $H : (S^2, V(F)) \rightarrow (S^2, C(F))$. The composition $f := H \circ F$ will be a Thurston map and so every branched covering arises from some Thurston map. So $\mathbf{TM} \rightarrow \mathbf{BC}$ is (on the level of objects) surjective. A Hurwitz equivalence between the branched covers corresponding to two Thurston maps cannot always be lifted to a Thurston equivalence, however.

3.2. $\mathbf{BC} \rightarrow \mathbf{BD}$. Fix an integer $d \geq 2$. While necessary, admissibility is not sufficient for realizability of branch data by a branched covering.

Example. The branch data of degree 4 given by the partitions

$$\{[3, 1], [2, 2], [2, 2]\}$$

is not realized by any branched covering. To see this, assume the contrary, and consider an associated Hurwitz factorization. In this case, this is an ordered triple of permutations (α, β, γ) in S_4 generating a transitive subgroup and with $\alpha\beta\gamma = 1$. We may assume α is the three-cycle (123) and β is a product of two disjoint transpositions. By simultaneously conjugating α and β by a power of α we may assume $\beta = (14)(23)$. But then $\gamma = \beta^{-1} \cdot \alpha^{-1} = (143)$ is not a product of transpositions.

Indeed, a complete characterization is still unknown; for recent work see [PP] and the references therein. To give a flavor of the known results, we mention the following.

- Any admissible branch datum in degrees 2, 3, 5, 7 is realizable [EKS].
- In each nonprime degree there exist nonrealizable admissible branch data [EKS].
- Any polynomial branch datum is realizable [KZ].

In what follows, we shall always assume tacitly that branch data is required to be admissible in the sense of §2.7.

3.3. $\mathbf{CP} \rightarrow \mathbf{BD}, \mathbf{TM} \rightarrow \mathbf{CP}$. The preceding two paragraphs show that if some admissible abstract branch data $(C \rightarrow V, C \xrightarrow{\deg} \mathbb{N})$ is realizable, then it arises via restriction and forgetting identification of domain and range from some critical orbit portrait of a Thurston map; this is [BBL⁺, Prop. 2.9]. They also show that if the abstract admissible branch data associated to some portrait is realizable, then this portrait arises from a Thurston map.

3.4. $\mathbf{TM} \rightarrow \mathbf{ABC}, \mathbf{CP} \rightarrow \mathbf{ABD}$. While natural in some nondynamical contexts, we do not pursue here the formulation of abstract augmented branched covers, abstract augmented branch data, and the characterization of such objects that arise from Thurston maps.

4. HOLOMORPHIC REALIZABILITY

Fix an identification $S^2 = \mathbb{P}^1 = \widehat{\mathbb{C}}$.

Letting the prefix “**H**” stand for holomorphic, we get three corresponding subcategories, **HTM**, **HCP**, **HBC**, where both the maps defining objects and those defining isomorphisms are, in addition, holomorphic. The objects of **HTM** are thus postcritically finite rational maps and the morphisms elements of $\text{Aut}(\mathbb{P}^1)$ giving conjugacies. The objects of **HBC** are rational maps; there is a morphism $f \rightarrow g$ if f and g differ by pre- and post-composition by (typically) distinct elements of $\text{Aut}(\mathbb{P}^1)$.

The lifting of complex structures under coverings, the removable singularities theorem, and the uniformization theorem imply that any branched covering is, up to pre-composition with homeomorphisms, a holomorphic map; this is sometimes attributed to R. Thom. It follows that a topologically realizable branch datum is isomorphic to the branch datum of an element of **HBC**.

Thurston’s fundamental characterization and rigidity theorem [DH] characterizes elements of **HTM** among elements of **TM**, and asserts that apart from the ubiquitous family of flexible Lattès counterexamples, if $f, g \in \mathbf{HTM}$ are isomorphic as elements of **TM**, then they are isomorphic as elements of **HTM**.

An unpublished theorem of Berstein-Levy [Hub] gives a sufficient condition for a topological polynomial to be combinatorially equivalent to a postcritically finite complex polynomial.

Theorem 4.1 (Berstein-Levy). *Suppose f is a topological polynomial. If each cycle of its critical orbit portrait contains a critical point, then f is combinatorially equivalent to a complex polynomial.*

Corollary 4.2. *Suppose f is a topological polynomial. If each cycle of its critical orbit portrait contains a critical point, then for each homeomorphism $h : (S^2, P(f)) \rightarrow (S^2, P(f))$ fixing the point at infinity, the map $h \circ f$ is combinatorially equivalent to a complex polynomial.*

Kelsey [Kel] provides a partial converse. If the critical orbit portrait of a topological polynomial has a cycle not containing a critical point, then under very general further conditions on the portrait, there exists an obstructed topological polynomial with this portrait; there may (and often do) exist complex polynomials realizing this portrait as well.

Not every critical portrait arises from a rational map.

Example. Suppose the cubic critical portrait consists of four fixed-points each of local degree two. Then this portrait does not arise from any cubic rational map: the Holomorphic Fixed-Point Formula [Mil, Theorem 12.4] asserts that if all fixed-points of a rational map are simple, then

$$\sum_{z \in \text{Fix}(f)} \frac{1}{1 - f'(z)} = 1.$$

A postcritically finite rational map has all fixed-points simple (multiple fixed-points are parabolic fixed-points with multiplier equal to 1; their immediate basins necessarily contain critical points with infinite forward orbit; see *op. cit.*, so a cubic rational map cannot have four fixed critical points.

A general characterization of holomorphically realizable critical orbit portraits seems at present far out of reach. Even the enumeration of critical orbit portraits is challenging; cf. [BBL⁺] where it is shown by brute force exhaustion that there are 134 admissible distinct cubic critical orbit portraits which have all cycles attractors, are nonpolynomial type, and have four or fewer postcritical points.

5. UNIQUENESS OF REALIZATIONS

Here, we consider the fibers of the natural functors in §2.1. That is, we discuss when two topological invariants have the same set-theoretic invariants.

5.1. $\mathbf{TM} \rightarrow \mathbf{CP}$. In [BBL⁺, Prop. 2.12] there is an explicit example given of a critical orbit portrait which arises from infinitely many distinct combinatorial classes. The idea is simple: one finds first a Thurston map f_1 and a surface Σ with $\Sigma \hookrightarrow S^2 \setminus P(f)$ injective on fundamental group and satisfying $f_1|_\Sigma = \text{id}_\Sigma$. Under suitable conditions, postcomposing f_1 with homeomorphisms h supported on Σ and representing distinct conjugacy classes of mapping classes in $\text{PMod}(S^2, P(f_1))$ yields the desired collection of Thurston maps f .

Another explicit class of such examples is given in [BN, Prop. 6.10].

Unsurprisingly, there are other families of examples coming from the exceptional class of Thurston maps with Euclidean orbifold.

In contrast, Thurston's fundamental rigidity theorem implies that apart from the flexible Lattès examples, there are up to holomorphic conjugacy only finitely many postcritically finite rational maps realizing a given critical orbit portrait; see [BBL⁺, Cor. 3.7].

5.2. $\mathbf{BC} \rightarrow \mathbf{BD}$. For “generic” branch data (all critical points simple, i.e., of local degree two, and mapping to distinct critical values) a classical result of Clebsch and Lüroth asserts that there is a unique Hurwitz class realizing the data; see [BE, Theorem 3.4] for a detailed proof in modern language. Indeed, a concrete model for a representative can be given by starting with the identity map and “blowing up” $d - 1$ disjoint arcs; see §7.3.

In principle, Hurwitz equivalence is algorithmically checkable. Indeed, given Hurwitz data for branched covers f and g , starting with the data for f one may randomly apply the elementary moves corresponding to generators of the mapping class group and check to see if the result is equivalent to the data for g . This can be done very quickly, and if f and g are equivalent it seems likely that one may find such an equivalence in a reasonable amount of time if the degree is not too large.

It is relatively easy to find examples of Hurwitz inequivalent covers with the same branch data.

Example 5.1. Consider the set of so-called *clean Belyi polynomials* $f : \mathbb{C} \rightarrow \mathbb{C}$ defined by the conditions that $V(f) = \{0, 1\}$ and $\deg(f, c) = 2$ for each $c \in f^{-1}(1)$. Such a polynomial has a *dessin*, the bipartite planar tree whose underlying set is $f^{-1}[0, 1]$ and whose vertices are $f^{-1}(\{0, 1\})$, with preimages of 0 being black and preimages of 1 being white. It is easy to prove, using triviality of the pure mapping class group of the thrice-marked sphere, that if f, g are two such polynomials each having at least one finite critical point of local degree at least 3, then f, g are Hurwitz equivalent if and only if the covering spaces of $\mathbb{C} \setminus \{0, 1\}$ they induce are isomorphic. This, in turn, holds if and only if their dessins are isomorphic as planar embedded bipartite trees. So to find examples it suffices to draw two such dessins

which are isomorphic as abstract, but not as planar, trees. An explicit example may be found in [Pil1, §4]. The point is that the branch data records only the cycle structure of the monodromy of peripheral generators, while the conjugacy class of the subgroup of the symmetric group determined by monodromy is an invariant of the Hurwitz class.

It is more difficult to give, say, primitive such pairs of examples with more branch values.

Example 5.2. The following is due to Jones and Zvonkin [JZ]. The polynomial branch data of degree 7 corresponding to the partitions

$$\{[7], [2, 2, 1, 1, 1], [2, 2, 1, 1, 1]\}$$

is topologically realizable by precisely 4 distinct Hurwitz classes.

In addition to the case of generic branch data, there are several partial results giving conditions on the branch data for there to exist a unique Hurwitz class. Khovanskii and Zdravkovska [KZ, §4] use analytical methods to show this is the case for topological polynomials if the sum of the local degrees at critical points is not too large compared to the degree. Relevant for the case of Thurston maps with fixed critical points is the following recent result of Liu-Osserman; they give a proof but assert that this result is well known.

Theorem 5.3. *Suppose a given admissible branch datum has the property that each associated partition has the form $[k, 1, 1, \dots, 1]$. Then any two branched coverings realizing this datum are Hurwitz equivalent.*

That such branch data is realizable seems to be well known; our Theorem 8.1 gives a combinatorial proof.

5.3. $\mathbf{ABC} \rightarrow \mathbf{ABD}$. It is easy to find examples of augmented branched covers which are Hurwitz equivalent, but not augmented Hurwitz equivalent; the following is taken verbatim from [Koc, §2.9]. We warn the reader that the term “Hurwitz equivalent” used in [Koc] is not the same as that used here; Koch’s notion is here (see below) called “purely augmented Hurwitz equivalent”.

Example. Let $F_\alpha(t) = (1 - (1 - \alpha)t)^d$ where $\alpha \neq 1$ is a d th root of unity. Each F_α is a unicritical polynomial of degree d , with critical point $1/(1 - \alpha)$, such that

$$F_\alpha\left(\frac{1}{1 - \alpha}\right) = 0, \quad F_\alpha(0) = 1, \quad \text{and} \quad F_\alpha(1) = 1.$$

Define the sets $A := \{0, 1, \infty\}$ and $B := \{0, 1, \infty\}$. Now suppose α, β are distinct d th roots of unity. Suppressing mention of local degrees when they are equal to 1, the corresponding augmented branch data for F_α and F_β are given respectively by

$$0 \mapsto 1, \quad 1 \mapsto 1, \quad \frac{1}{1 - \alpha} \xrightarrow{d} 0, \quad \infty \xrightarrow{d} \infty$$

and

$$0 \mapsto 1, \quad 1 \mapsto 1, \quad \frac{1}{1 - \beta} \xrightarrow{d} 0, \quad \infty \xrightarrow{d} \infty.$$

Clearly, the isomorphism type of this augmented branch data depends only on d , and the automorphism group of this augmented branch data (defined in the obvious way) is generated by two commuting involutions: one which swaps $0 \leftrightarrow 1$ in A leaving B fixed pointwise, while the other interchanges $\frac{1}{1 - \alpha} \rightarrow 0$ with $\infty \rightarrow \infty$.

For varying choices of α , the maps

$$F_\alpha : (\mathbb{P}^1, A) \rightarrow (\mathbb{P}^1, B)$$

are all ordinary Hurwitz equivalent (in the sense of this paper) to each other, since the underlying branched coverings have the same degree and are ramified only over two points. However, if $\alpha \neq \beta$ then F_α and F_β will not in general be augmented Hurwitz equivalent in the sense of §2.4.

To see this, suppose the pair (h_0, h_1) gives such an equivalence. We must have $h_0(1) = 1$. The (nonpure) mapping class group $\text{Mod}(\mathbb{P}^1, B)$ is represented by the six elements of $\text{Aut}(\mathbb{P}^1)$ preserving the set $\{0, 1, \infty\}$, which includes $z \mapsto 1/z$. This element lifts under any map F_β , so without loss of generality we may assume further that $h_0(0) = 0$ and $h_0(\infty) = \infty$, so that $h_0(z) = \text{id}$. Since h_0, F_α, F_β are holomorphic, so is h_1 . From the definition of augmented Hurwitz equivalent, we must have $h_1(\infty) = \infty$, so h_1 is affine, and we must also have $h_1(\{0, 1\}) = \{0, 1\}$, so either $h_1(z) = z$ or $h_1(z) = 1 - z$. But we must also have $h_1(\frac{1}{1-\alpha}) = \frac{1}{1-\beta}$, so $\alpha \neq \beta$ implies $\frac{1}{1-\alpha} + \frac{1}{1-\beta} = 1$. We conclude that F_α, F_β are augmented Hurwitz equivalent only if $\beta = \bar{\alpha}$. A straightforward calculation shows that in this latter case the maps F_α, F_β are in fact augmented Hurwitz equivalent.

We remark here that the arguments of [Koc, §2.9] show that F_α, F_β are never purely augmented Hurwitz equivalent if $\alpha \neq \beta$, and so these examples show that there is indeed a distinction between the two notions.

Remark. In many applications, including dynamical ones, in order to avoid singularities caused by configurations with symmetries, a stronger notion of Hurwitz equivalence is used which requires in essence that equivalences preserve a labelling of points. This is the point of view taken in [Koc].

6. PROOF OF THEOREM 1.4

Suppose F, G are two Thurston maps whose critical points are all fixed, and suppose the number n and multiplicities of their critical points coincide, so that both have isomorphic branch data. Let $V = \{\exp(2\pi i k/n), k = 1, \dots, n\}$ be as in §2.8.

By varying F, G within their respective combinatorial classes, we may assume $C(F) = C(G) = V(F) = V(G) = P(F) = P(G) = C = V$, and that the local degrees of F and G at each $c \in C$ coincide. By [LO, Proposition 1.1(a)], the Hurwitz factorizations associated to F and G lie in the same orbit of the pure braid group. It follows that there exist orientation-preserving homeomorphisms $h_0, h_1 : (S^2, P) \rightarrow (S^2, P)$ such that $h_0 \circ F = G \circ h_1$; since the braid is pure, we have $h_0|_P = \text{id}_P$. The functional equation and the fact that over each $v \in V$ the fiber of G contains a unique critical point then implies that $h_1|_P = \text{id}_P$ as well and so both h_0, h_1 lie in $\text{PMod}(S^2, P)$.

Set $h := h_1^{-1} \circ h_0$. Then

$$h \circ F = h_1^{-1} \circ G \circ h_1$$

which shows that G is combinatorially equivalent to a twist of F . $\square$

7. FROM PLANAR MULTIGRAPHS TO THURSTON MAPS

7.1. Planar multigraphs. Here, a *multigraph* $\mathcal{G}$ is a finite graph (V, E) with a nonempty set of vertices, without loops (edges from a vertex to itself), without

isolated vertices (i.e., each vertex is incident to some edge), and with multiple (“parallel”) edges joining distinct vertices permitted. The underlying *simple graph* SG is the graph obtained by removing any redundant edges. We encode $\mathcal{G}$ by weighting each edge of SG according to its multiplicity. $\mathcal{G}$ is *connected* if SG is connected.

A multigraph is $\mathcal{G}$ *planar* if its underlying simple graph SG can be drawn in the plane without crossing of edges. We will always consider planar multigraphs to be so embedded. A *face* of $\mathcal{G}$ is a component of the complement of SG . Two planar multigraphs are here termed *isomorphic* if there is an orientation-preserving homeomorphism sending one to the other and respecting the multiplicity of edges.

The case of $\#V = 2$ will be elementary and largely uninteresting; we assume throughout that $\#V \geq 3$ to avoid repeated mention of cumbersome special cases. In this case, each face of SG has at least three vertices on its boundary. In contrast, $\mathcal{G}$ may have bigons: faces bounded by a pair of parallel edges.

7.2. Arcs. Suppose $P \subset S^2$ is a finite set. An *arc* in (S^2, P) is an embedded arc in S^2 with endpoints in P and which otherwise avoids P ; two such arcs are *homotopic* if they can be continuously deformed one to the other through such arcs. If $F : (S^2, A) \rightarrow (S^2, B)$ is an augmented branched covering and β is an arc in (S^2, B) then (abusing terminology) by a *preimage* $\tilde{\beta}$ of $F^{-1}(\beta)$ we will mean the closure of a component of $F^{-1}(\beta \setminus \text{endpoints}(\beta))$; we say $\tilde{\beta}$ is *essential* if it is an arc in (S^2, A) . If $A = B = P$ then an arc α is *invariant* under pullback by F if it has a preimage homotopic to itself in (S^2, P) .

7.3. Blowing up an arc and proof of Theorem 3. For details, see [PT]. The process of *blowing up an arc* is a surgery procedure for modifying a given Thurston map to produce another Thurston map of higher degree. It is based on the following idea. See Figures 4 and 5 in §12 below for an example.

Suppose $F : (S^2, P) \rightarrow (S^2, P)$ is an augmented branched covering; we allow the case $\deg(F) = 1$. Suppose α is an arc in $(S^2, F^{-1}(P))$, and suppose $F|_{\alpha}$ is a homeomorphism onto its image. To modify F by blowing up along α to obtain a new map G , we cut the domain along α , open up the slit, map the complement via F , and map the disk homeomorphically to the complement of $F(\alpha)$ in the obvious way; see [PT, Figure 3] for details. The combinatorial class of G depends only on the pair (F, α) in the obvious way; an equivalence $F \rightarrow F'$ sending $\alpha \rightarrow \alpha'$ yields an equivalence $G \rightarrow G'$ between the corresponding modified maps, so the process is natural.

One can blow up the same arc multiple times as well, by mapping the complement in the obvious many-to-one fashion. Up to homotopy, the result is equivalent to blowing up each member in a family of parallel homotopic arcs joining the same endpoints; this point of view is sometimes convenient. Thus, if $\mathcal{G}$ is a planar multigraph, there is an associated Thurston map $F_{\mathcal{G}}$ of degree equal to one plus the number (counted with multiplicity) of edges of $\mathcal{G}$. Clearly, a planar isomorphism $\mathcal{G}_1 \rightarrow \mathcal{G}_2$ induces a combinatorial equivalence $F_{\mathcal{G}_1} \rightarrow F_{\mathcal{G}_2}$. So we have a procedure that starts with a planar multigraph $\mathcal{G}$ and builds in a natural way a Thurston map $F = F_{\mathcal{G}}$ for which $\text{Crit}(F) \subset \text{Fix}(F)$.

Suppose $F = F_{\mathcal{G}}$ is obtained by blowing up edges in a graph $\mathcal{G}$; let $P = P(F)$. Suppose vertices v, w are joined by an edge e of SG which is blown-up $m(e)$ times; we regard e as an arc in (S^2, P) . By construction, the arc e has $1 + m(e)$ components

of its preimage that are arcs homotopic to e in (S^2, P) . Since $m(e) \geq 1$, from [PT, Cor. 3] we have that up to homotopy no Thurston obstructions can intersect e ; from [PT, Cor. 4] we have that likewise no invariant arc meets e . The former observation implies the following, which is a special case of [PT, §5.2].

Theorem 7.1. *$F_{\mathcal{G}}$ is combinatorially equivalent to a rational map if and only if $\mathcal{G}$ is connected.*

The latter of the above observations implies Theorem 1.3 from the introduction, asserting that distinct graphs produce maps which are distinct, up to equivalence.

Proof of Theorem 1.3. Suppose $h_0, h_1 : (S^2, P_1) \rightarrow (S^2, P_2)$ give a combinatorial equivalence between $F_1 = F_{\mathcal{G}_1}$ and $F_2 = F_{\mathcal{G}_2}$. We claim $h := h_0$ induces a planar isomorphism $h : \mathcal{G}_1 \rightarrow \mathcal{G}_2$. Consider an edge e_1 of $\mathcal{G}_1$ having $k \geq 2$ preimages homotopic to itself under F_1 . Then its image $h(e_1)$ has $k \geq 2$ preimages homotopic to itself under F_2 . It follows from [PT, Cor. 4] that up to homotopy $h(e_1)$ cannot meet an edge of $\mathcal{G}_2$. So either $h(e_1) = e_2$ for some edge e_2 of $\mathcal{G}_2$ or $h(e_1)$ is an arc in a face D of $\mathcal{G}_2$ containing at least four vertices on its boundary. The latter case cannot occur: let δ be an arc in D which joins two nonincident vertices on ∂D and which intersects $h(e_1)$. Then δ is invariant under F_2 and meets $h(e_1)$ which has more than one preimage under F_2 homotopic to itself, which is impossible again by [PT, Cor. 4]. Thus h sends edges of $\mathcal{G}_1$ to edges of $\mathcal{G}_2$; clearly, if e is blown up m times for F_1 then $h(e)$ is blown up m times for F_2 , and the theorem is proved. $\square$

Invariant multicurves are natural invariant objects associated to a Thurston map. The next lemma shows that for maps $f_{\mathcal{G}}$ as above, there are many invariant multicurves with special properties.

Lemma 7.2. *Suppose $\mathcal{G}$ is a multigraph and $f = f_{\mathcal{G}}$ is obtained by blowing up each edge e of $\mathcal{G}$ some number $m(e)$ times. Suppose e is an edge of $\mathcal{G}$. Let γ be the boundary of a small regular neighborhood of e , so that γ separates the endpoints v, w of e from all other vertices of $\mathcal{G}$. Then the curve γ has a single essential nonperipheral preimage homotopic to itself, mapping by degree*

$$\sum_{\substack{e' \neq e, \\ \{v, w\} \cap e' \neq \emptyset}} (1 + m(e')),$$

and all other preimages are inessential. In particular, $\Gamma := \{\gamma\}$ is a completely invariant multi curve.

8. PROOF OF THEOREM 1.2

Suppose $\mathcal{G}$ is a multigraph. We will refer to the underlying simple graph of $\mathcal{G}$ by $S\mathcal{G}$, so that $V(S\mathcal{G}) = V(\mathcal{G})$ and $v, w \in V(S\mathcal{G})$ are connected by an edge if and only if v, w are connected by at least one edge in $\mathcal{G}$. We denote the degree of $v \in V(\mathcal{G})$ in $\mathcal{G}$ by $\deg_{\mathcal{G}}(v)$, and the degree of v in $S\mathcal{G}$ by $\deg_{S\mathcal{G}}(v)$. Recall that a face of $\mathcal{G}$ is a component of the complement of $S\mathcal{G}$.

Suppose $d \geq 3$ is an integer. An *admissible partition* of the integer $2d - 2$ is a partition $k_1 + \dots + k_n = 2d - 2$ where $n \leq d$ and $1 \leq k_i \leq d - 1$ for each $i = 1, \dots, n$; the order of the terms is irrelevant, and by convention we arrange the terms so $k_1 \geq \dots \geq k_n$. For convenience, we say that the *type* of the partition is the pair (n, d) . Any partition may be encoded by its *Young diagram*, an array

of congruent squares where each row corresponds to a term in the partition, with the i th row containing k_i squares. This point of view will help to organize our arguments. In this language, Theorem 1.2 can be stated as follows.

Theorem 8.1. *Given any Young diagram associated with an admissible partition of type (n, d) , we may find a corresponding connected planar multigraph $\mathcal{G}$ such that each Young diagram row corresponds to a unique vertex of $\mathcal{G}$ and the number of boxes in a row is equal to the degree of the associated vertex.*

Note that the restriction $n \leq d$ is necessary. For instance, if $d = 3$, $n = 4$, then we have $2 \cdot (3 - 2) = 4$ boxes and 4 rows, so each row must have exactly one square. But clearly there is no way to obtain a connected planar multigraph with 4 vertices each of degree 1.

Proof. Case of type $(2, d)$. We take $\mathcal{G}$ to consist of two vertices v_1 and v_2 connected by $d - 1$ edges.

For the remainder of the proof, we assume $n \geq 3$.

Case $n = d$. In this case the type is (d, d) .

We remark that any such graph is a planar tree and corresponds to a Newton's method map; see §11.

We will argue by induction on d . If $n = d = 3$, we take $\mathcal{G}$ to be a path of length 2. Now assume that we may find a connected planar multigraph $\mathcal{G}$ for any Young diagram associated with an admissible partition of type (d, d) . Since $\mathcal{G}$ is connected and without loops, we may find v_1 and $v_2 \in V(\mathcal{G})$ two adjacent vertices connected by edge $e \in E(\mathcal{G})$. We note that we may construct any of the Young diagrams associated with an admissible partition of $(d + 1, d + 1)$ (with $2d$ boxes) from one of the Young diagrams associated with an admissible partition of (d, d) (with $2d - 2$ boxes) by adding one row and two new boxes. Let the vertex corresponding to the added row be $v' \in V(\mathcal{G}')$. We consider two subcases.

Subcase I: *We add one new row with two boxes.*

In this case, we construct $\mathcal{G}'$ from $\mathcal{G}$ by bisecting the edge e with the vertex v' .

Subcase II: *We add one new row consisting of one box and also add one other box to some existing row (say the row corresponding to v_1).*

In this case, we may construct $\mathcal{G}'$ from $\mathcal{G}$ by adding v' in a face of $\mathcal{G}$ whose boundary contains v_1 , say, and then joining v' and v_1 by an edge in this face, forming a “sticker”.

Obviously the new graph $\mathcal{G}'$ is connected and planar.

Case $n < d$. We now fix the number n of vertices (equivalently, parts of the partition), and we argue by induction on d ; equivalently, on the number of edges in $\mathcal{G}$.

Again there are two subcases.

Subcase I: *We add one new box to each of two distinct rows of a Young diagram associated with an admissible partition of type (n, d) , $d \geq n$.*

Recall that, by assumption, $n \geq 3$. This corresponds to increasing the degree of two distinct vertices of $\mathcal{G}$ by 1.

Lemma 8.2. *Let $\mathcal{G}$ be a connected planar multigraph with $|V(\mathcal{G})| \geq 3$. Given $v_1, v_2 \in V(\mathcal{G})$ adjacent and $\deg_{\mathcal{G}}(v_1) \geq 2$, we may find a connected planar multigraph $\mathcal{G}'$ with $V(\mathcal{G}) = V(\mathcal{G}')$ such that*

1. $\deg_{\mathcal{G}'}(v) = \deg_{\mathcal{G}}(v)$ for all $v \in V(\mathcal{G})$, $v \neq v_1, v_2$,
2. $\deg_{\mathcal{G}'}(v_1) = -1 + \deg_{\mathcal{G}}(v_1)$,
3. $\deg_{\mathcal{G}'}(v_2) = 1 + \deg_{\mathcal{G}}(v_2)$.

Proof. $\mathcal{G}$ is a planar multigraph with $|V(\mathcal{G})| \geq 3$, so since $S\mathcal{G}$ contains no multiple edges or loops, the boundary of each face of $\mathcal{G}$ contains at least three distinct vertices. Since v_1 and v_2 are adjacent, they lie on the boundary of some face F of $S\mathcal{G}$. Furthermore, since $\deg_{S\mathcal{G}}(v_1) \geq 2$, there exists $w \in V(\mathcal{G})$ adjacent to v_1 , $w \neq v_2$, such that w also lies on ∂F . Let e_{w1}, e_{12} be edges in ∂F joining w to v_1 and v_1 to v_2 , respectively.

Let $\mathcal{G}'$ be the graph obtained from $\mathcal{G}$ by deleting edge e_{w1} and adding an edge e_{w2} in F . By construction, conditions 1–3 hold. Since w and v_2 are both on ∂F , edge e_{w2} does not cross other edges, so $\mathcal{G}'$ is planar. Since $\mathcal{G}$ is connected, $\mathcal{G}'$ is connected by construction. $\square$

The proof of Subcase I then follows from the following proposition:

Proposition 8.3. *Let $\mathcal{G}$ be a connected planar multigraph with $|V(\mathcal{G})| \geq 3$. Given arbitrary $s, t \in V(\mathcal{G})$, we may find a connected planar multigraph $\mathcal{G}'$ with $V(\mathcal{G}) = V(\mathcal{G}')$ such that*

1. $\deg_{\mathcal{G}'}(v) = \deg_{\mathcal{G}}(v)$ for all $v \in V(\mathcal{G})$, $v \neq s, t$,
2. $\deg_{\mathcal{G}'}(s) = 1 + \deg_{\mathcal{G}}(s)$,
3. $\deg_{\mathcal{G}'}(t) = 1 + \deg_{\mathcal{G}}(t)$.

Proof of Proposition. If s and t are adjacent in $\mathcal{G}$, we construct $\mathcal{G}'$ by simply adding another edge between them to obtain $\mathcal{G}'$.

So assume s, t are not adjacent in $\mathcal{G}$. Let $s = a_0, a_1, \dots, a_m, a_{m+1} = t$ be the vertices contained in an edge-path of minimal length joining s and t . The minimality property implies that the valence at each of the vertices $a_1, \dots, a_m$ in $S\mathcal{G}$ is at least 2.

We begin by adding another edge between a_0 and a_1 along this edge-path to obtain a graph $\mathcal{G}^0$. Notice that $\deg_{\mathcal{G}^0}(a_0) = \deg_{\mathcal{G}}(a_0) + 1$, which is what we want. However, $\deg_{\mathcal{G}^0}(a_1) = \deg_{\mathcal{G}}(a_1) + 1$, which is something we do not want. Thinking of this “+1” as a “defect”, we are going to apply Lemma 1 to “push” this defect toward t along our chosen edge-path.

By construction, the valence of a_1 in $S\mathcal{G}$ is at least 2, hence its valence in $S\mathcal{G}^0$ is at least 2. We apply Lemma 8.2 to $\mathcal{G}^0$ with $v_1 := a_1, v_2 := a_2$ to obtain a graph $\mathcal{G}^1$. Then

$$\deg_{\mathcal{G}^1}(a_1) = \deg_{\mathcal{G}^0}(a_1) - 1 = \deg_{\mathcal{G}}(a_1) + 1 - 1 = \deg_{\mathcal{G}}(a_1)$$

and

$$\deg_{\mathcal{G}^1}(a_2) = \deg_{\mathcal{G}^0}(a_2) + 1 = \deg_{\mathcal{G}}(a_2) + 1.$$

If $a_2 = t$ we are done. Otherwise, we note that our modification via Lemma 8.2 did not decrease the valence at a_2 of the underlying simple graphs. We apply Lemma 8.2 again to the pair $v_1 := a_2, v_2 := a_3$. Continuing in this fashion proves the proposition. $\square$

Subcase II: *We add two new boxes to one row of a Young diagram associated with an admissible partition of type (n, d) .*

Following the given constraints, this case only pertains to rows of size $\leq d - 2$. Suppose that our row corresponds to $v_1 \in V(\mathcal{G})$. Since $n, d \geq 3$ and $\mathcal{G}$ is planar, we

may find $v_2, v_3 \in V(\mathcal{G})$ such that v_2 and v_1 are connected by edge e_{12} , v_2 and v_3 are connected by edge e_{23} , and v_1, v_2, v_3 lie along the boundary of a face F of $\mathcal{G}$. Delete e_{23} , add an edge e_{13} in F from v_1 to v_3 , and add another edge e'_{12} from v_1 to v_2 . The resulting graph $\mathcal{G}'$ is obviously planar and connected.

Theorem 1.2 (and Theorem 8.1) are proved. $\square$

The same ideas in fact show that if the assumption $n \leq d$ is dropped, one may still find that the branch data arises from the blowing up construction of a now not necessarily connected multigraph. This is the content of the following theorem.

Theorem 8.4. *Suppose $d \geq 2$ and branch data is given corresponding to the partition $k_1 + \dots + k_n = 2d - 2$, where $1 \leq k_i \leq d$ for $i = 1, \dots, n$. Then there exists a planar multigraph $\mathcal{G}$ such that blowing up the edges of $\mathcal{G}$ yields a critically fixed Thurston map, and hence a branched covering, realizing this branch data.*

Proof: If $d = 2$ we blow up a single edge. We now induct on the integer d and adapt the setup of the proof of Theorem 8.1. At the inductive step, we have a Young diagram whose corresponding partition is given by a planar multigraph $\mathcal{G}$. This time, there are three cases.

Suppose we add two new rows, each with one box. In this case we obtain $\mathcal{G}'$ from $\mathcal{G}$ by adding a disjoint copy of a segment (two vertices joined by a single edge).

Suppose we add one new row with two new boxes. Suppose this row corresponds to a vertex v of $\mathcal{G}$. We apply the preceding arguments (in the case $d > n$, Subcase II) to the connected component of $\mathcal{G}$ containing v to obtain $\mathcal{G}'$.

Finally, suppose we add one box to each of two existing rows. Suppose u, v are the vertices of $\mathcal{G}$ corresponding to these rows. If u, v belong to the same component of $\mathcal{G}$, we apply the preceding arguments (in the case $d > n$, Subcase I) to this component. Otherwise, suppose $\mathcal{G}_u, \mathcal{G}_v$ are the components of $\mathcal{G}$ containing u and v . By altering the planar embedding of $\mathcal{G}$ we may assume: (i) no component of $\mathcal{G}$ separates $\mathcal{G}_u$ from $\mathcal{G}_v$, and (ii) u, v are on the boundary of a common complementary component F (face) of $\mathcal{G}$. We then join u and v by an edge e in F to obtain $\mathcal{G}'$. $\square$

9. TISCHLER GRAPHS

Suppose f is a critically fixed rational map. Suppose $c \in C(f)$ has multiplicity m and let Ω_c be its immediate basin. Up to precomposition by multiplication by m th roots of unity, there is a unique holomorphic isomorphism $\Phi : (\mathbb{D}, 0) \rightarrow (\Omega_c, c)$ conjugating $w \mapsto w^{m+1}$ on $\mathbb{D}$ with f on Ω_c which extends continuously to a semiconjugacy on the closures. The image $\Phi(r \exp(2\pi it)), r \in [0, 1]$ is the *internal ray of angle $t \in \mathbb{R}/\mathbb{Z}$* in Ω_c ; when $c = \infty$ and f is a polynomial these are called *external rays*. Note that the ray of angle t is fixed if and only if $t = j/m \bmod 1$ for some $j \in \{0, 1, \dots, m-1\}$.

The *Tischler graph* is a planar embedded graph; as a subset of $\mathbb{P}^1$ it is the union of the fixed internal rays as in the previous paragraph. Below, $C := C(f)$ and R is the set of landing points, necessarily fixed by f , of the fixed internal rays constructed above.

Lemma 9.1. *Let $\mathcal{T} = (V, E)$ be the Tischler graph of a critically fixed rational map f of degree d . Then $f(\mathcal{T}) = \mathcal{T}$, $f|_{\mathcal{T}} \simeq \text{id}_{\mathcal{T}}$ relative to $C \sqcup R$, and the following hold:*

- (1) $\mathcal{T}$ is bipartite, with $V = C \sqcup R$ and edges joining elements of C to elements of R ,
- (2) $|E| = 2d - 2$,
- (3) $|C| + |R| \leq d + 1$,
- (4) $2 \leq |C|$,
- (5) $|C| \leq d$,
- (6) for any 2-cycle in $\mathcal{T}$, each of the complementary disks meets a point in C .

Proof. The invariance properties and the first five statements are clear. To prove the last, suppose distinct edges $e_{\pm}$ join a critical point c of multiplicity m to a repelling fixed-point r and $e_+ \cup e_-$ bounds a Jordan domain D whose interior does not meet C . Then $f : D \rightarrow D$ maps by degree 1. (To see this: consider the preimages of D ; the map f is injective near r , so D is invariant under f near r ; but if $D \cap C = \emptyset$ then all preimages of D map by degree 1.) But this is impossible, since the difference in angles between the rays comprising e_+ and e_- must be at least $1/m$ which is larger than $1/(m + 1)$. $\square$

9.1. From multigraphs to Tischler graphs. Suppose $\mathcal{G}$ is a connected planar multigraph. By blowing up the edges of $\mathcal{G}$, we obtain a Thurston map F ; it is equivalent to a rational map f (Theorem 7.1) that has a Tischler graph $\mathcal{T}_f$. We now show how to reconstruct $\mathcal{T}_f$ from $\mathcal{G}$.

Fix a topological realization of such a graph $\mathcal{G}$; we emphasize that multiple edges are drawn as disjoint. We therefore allow *bigons* as faces of the cell structure of S^2 determined by $\mathcal{G}$. For each face of $\mathcal{G}$, choose a point p in its interior, and denote by R_F the resulting collection of points (we will construct F momentarily). Put $C_F = V(\mathcal{G})$. Consider the identity map, as an augmented (“marked”) branched covering with marked set $C_F \cup R_F$.

Let F be the result of blowing up this marked branched covering along the edges of $\mathcal{G}$, as in [PT]. By *ibid.*, Proposition 2, the resulting combinatorial class is well defined. We claim that F is combinatorially equivalent, as a marked branched map, to a marked rational map f , with a set of marked repelling fixed-points R_f corresponding to R_F and C_f to C_F . In particular, f has a repelling fixed point for each face of $\mathcal{G}$. Since we have assumed throughout that $\#C \geq 2$, the orbifold of F is hyperbolic, so it is sufficient to rule out obstructions. But this is clear, since no obstruction can meet an edge of $\mathcal{G}$, and each face contains only one element of R_F .

In this paragraph, we construct a combinatorial candidate for the Tischler graph. Consider the sphere S^2 with its cell structure induced from $\mathcal{G}$. Let D^2 denote the closed unit disk, let D be a face of $\mathcal{G}$, and let $\psi_D : D^2 \rightarrow D$ be an attaching map. We may assume that $\psi_D(0) \in R_F$. Consider the subdivision of D obtained by joining the origin to the 0-cells on ∂D^2 via what we will suggestively call “internal” 1-cells and mapping the result forward by ψ_D . We obtain in this way a bipartite graph $\mathcal{T}_F$ whose edges are internal 1-cells, and whose vertices are the disjoint union of $R_F \cup C_F$.

Theorem 9.2. *Suppose $\phi : (S^2, C_F \cup R_F) \rightarrow (\widehat{\mathbb{C}}, C_f \cup R_f)$ is a combinatorial equivalence from F to f . Then $\phi(\mathcal{T}_F)$ is isotopic relative to $C_f \cup R_f$ to the Tischler graph of f .*

Proof. The ideas are by now standard; we merely sketch the main ingredients. By conjugating F with ϕ we may assume $C = C_F = C_f$, $R = R_F = R_f$, $\phi \simeq \text{id}$ relative to $C \cup R$, and $\mathcal{T}_F \subset \widehat{\mathbb{C}}$; we replace F with this conjugate, so that $F : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$. By

FIGURE 1. Detail of Example **33211a** of §11. Infinity is a fixed critical point. Shown in black is the Tischler graph $\mathcal{T}$ and in red the graph $\mathcal{G}$.

construction, each $c \in C_f$ is incident to $m(c)$ edges in $\mathcal{T}_F$ and in $\mathcal{T}_f$, where $m(c)$ is the multiplicity of c . It follows that F is homotopic relative to $C \cup R$ to a map F' which agrees with f near C ; we replace F with F' and denote the result by F again. Furthermore, we may assume $\mathcal{T}_F = \mathcal{T}_f$ near C . Since $F(\mathcal{T}_F) = \mathcal{T}_F$, there is a preferred subgraph of $f^{-1}(\mathcal{T}_F)$ isotopic to $\mathcal{T}_F$ via a family of homeomorphisms $h_t, t \in [0, 1]$.

Inductively, we may lift the isotopy $h_t, t \in [0, 1]$ under iterates of f to a real one-parameter family $h_t, t \in [0, \infty)$. Since f^{-1} is uniformly contracting away from C , the family h_t converges as $t \rightarrow \infty$, yielding an isotopy from $\mathcal{T}_F$ to $\mathcal{T}_f$. $\square$

9.2. From Tischler graphs to multigraphs. Suppose f is a critically fixed rational map which arises from blowing up a planar multigraph $\mathcal{G}$, and let $\mathcal{T}$ be its Tischler graph, as constructed in the preceding subsection. Thinking of edges of $\mathcal{G}$ as black, and edges of $\mathcal{T}$ as red, we observe the following. The union $\mathcal{G} \cup \mathcal{T}$ gives a cell structure to the sphere in which each cell is a triangle bounded by two red edges and one black edge, with the red edges incident to a common vertex in R_f , and the black edges incident to two distinct vertices in C_f . Moreover, each black edge e lies in the boundary of exactly two such triangles, which therefore forms a quadrilateral.

This black edge e , which is a diagonal of this quadrilateral, has two distinguished preimages: one each obtained by lifting the homotopy to either “side” formed by the union of two red edges. By construction, each such preimage $\tilde{e}$ of e is homotopic to e relative to $C(f)$.

This observation shows how $\mathcal{G}$ may be reconstructed from $\mathcal{T}$: the Tischler graph divides the sphere into quadrilaterals; the boundary of each quadrilateral has precisely two vertices in C_f ; the graph $\mathcal{G}$ is obtained from $\mathcal{T}$ by joining, for each

quadrilateral, this pair of vertices by a black edge. To see this, simply note that each black edge has two preimages, each one obtained by lifting a homotopy joining this edge to one side of the quadrilateral in which it lies. We conclude:

Theorem 9.3. *Suppose f is a critically fixed rational map with the property that its Tischler graph $\mathcal{T}_f$ is connected. Then f is equivalent to $f_{\mathcal{G}}$, where $\mathcal{G}$ is the planar multigraph obtained from $\mathcal{T}_f$.*

9.3. Every Tischler polynomial is a blown-up star. As a special case, we have

Corollary 9.4. *Suppose f is a polynomial of degree d each of whose critical points is also a fixed-point. Then f is equivalent to $F_{\mathcal{G}}$, where $\mathcal{G}$ is a “star” graph, i.e., one with at most one vertex of valence greater than 1.*

10. SUBDIVISION RULES

Cannon, Floyd, and Parry [CFP] have formulated the notion of a *finite subdivision rule*. In our setting, these correspond to Thurston maps $f : (S^2, P) \rightarrow (S^2, P)$ which preserve some additional structure: there exist cell structures X_0, X_1 on S^2 such that X_1 refines X_0 and the restriction of f to each cell of X_1 is a homeomorphism.

If f is a rational map equivalent to $F_{\mathcal{G}}$ for some graph $\mathcal{G}$, then its Tischler graph $\mathcal{T}$ forms the 1-skeleton of a cell structure X_0 on the sphere. This structure may be lifted under f to obtain a structure cell structure X_1 such that f becomes the subdivision map of a finite subdivision rule.

11. EXAMPLES

Critically fixed polynomials are already classified by their Tischler graphs; equivalently, by the corresponding star graph. We therefore focus exclusively on the non-polynomial cases. After classifying by degree, we classify first by the corresponding partition of $2d - 2$; the exclusion of polynomials implies that we need consider only those partitions of $2d - 2$ whose largest part is at most $d - 2$. Next we classify by abstract graphs, and then by planar multigraph.

In degrees 2 and 3, the only examples are conjugate to polynomials, as is easily verified. The results given below for degree 4 and 5 were obtained by using the software package Maple to find all rational maps realizing the given partition, computing the number of conjugacy classes, and then comparing this count with the count of planar multigraphs. Below, by a “sticker” we mean an edge of a graph incident to a vertex of valence 1. The assertions for degrees 4 and 5 below follow from solving the equations arising from the critical orbit portraits as in [BBL⁺].

Before turning to the data, we pause to make mention of a few special subfamilies.

Newton’s method. Suppose $d \geq 2$ and p is a polynomial of degree d with d distinct roots $r_1, \dots, r_d$. The *Newton method map* is defined as $N_p(z) = z - p(z)/p'(z)$. Each root r_i is a fixed-point and a critical point of N_p , so the holomorphic fixed-point index formula implies that the point at infinity is the remaining fixed-point of N_p and has multiplier $d/(d - 1)$. There may, however, be other critical points of N_p . Suppose now that N_p is critically fixed. Then each critical point of N_p is one of the roots of p . Such examples are very special among Newton methods.

Conversely, suppose a rational map f has degree d , is critically fixed, and has the maximum allowed number, d , of fixed critical points $r_1 \dots, r_d$. We may assume each fixed critical point lies in $\mathbb{C}$ and that the remaining fixed-point, necessarily repelling and of multiplier $d/(d-1)$, lies at infinity. Let $p(z) = \prod_{i=1}^d (z - r_i)$ and consider the Newton map $N_p(z)$. Let w be a generic projective coordinate, and consider the difference between f and N_p , computed in this coordinate. The difference is a rational function of degree at most $2d$ with $2d+1$ zeros, and therefore vanishes identically. Therefore f and N_p coincide.

We conclude that each critically fixed rational map f of degree d having the maximum number, d , of critical points is a Newton method map for some polynomial of degree d with distinct roots. It is known [HSS, Prop. 6] that each immediate basin of a critical point of f of multiplicity k contains k *accesses to infinity*. These accesses coincide with edges of the Tischler graph. We conclude that for such f , the Tischler graph is connected and has a single repelling vertex. From Theorem 9.3, we conclude that if a connected multigraph $\mathcal{G}$ is a tree with no multiple edges, then the rational map $f_{\mathcal{G}}$ is a critically fixed Newton method, and conversely, that each critically fixed Newton method arises from this construction.

We remark that Rückert [R] studies the classification of postcritically fixed Newton maps—a class of maps that is larger than those considered here. In the case of critically fixed Newton maps, the Tischler graph coincides with his *channel diagram*.

Examples. The discussion here was contributed by E. Crane. Starting from the observation that

$$N'_p(z) = \frac{p''(z)p(z)}{(p'(z))^2},$$

Crane [Cr] observes that if a polynomial p has distinct roots and if the roots of p'' are also roots of p , then N_p has all critical points fixed, and these critical points are zeros of p . We may normalize two roots so that they are located at two convenient points, e.g. ± 1 .

For example, if p satisfies the linear homogeneous ODE

$$(z-1)(z+1)p''(z) - d(d-1)p(z) \equiv 0,$$

then each root of p'' is also a root of p . Writing $p(z) = (z^2-1)g(z)$ we may write this ODE as

$$(1-z^2)g'' - 4zg' + (d-2)(d+1)g = 0.$$

It is straightforward to see that there exist, up to scale, unique polynomial solutions whose coefficients are given by products of rational numbers satisfying simple recurrence relations. Appropriately normalized, it turns out these are the so-called *Gegenbauer polynomials* with parameter $3/2$. They are orthogonal on $[-1, 1]$; their zeros are all real, simple, and contained in the interval $[-1, 1]$; they satisfy a recurrence relation, which yields such a relation for $p = p_d$:

$$p_2(z) = z^2 - 1, \quad p_3(z) = z^3 - z$$

and, for $d \geq 4$,

$$p_d(z) = zp_{d-1}(z) - \frac{(d-1)(d-3)}{(2d-3)(2d-5)}p_{d-2}(z).$$

For $d = 11$ the basins of the corresponding Newton method appear as below; the graph $\mathcal{G}$ is a segment of length 10, with each edge blown up once:

Another family arising from this construction is the following. Fix integers $m, n \geq 1$, set $d = m + n + 2$, and set

$$p''(z) = d(d-1)(z-1)^m(z+1)^n.$$

Then p is determined up to an additive term of the form $az + b$. Requiring that $p(1) = p(-1) = 0$ gives linear equations on a, b that can be easily solved. This yields a polynomial $p = p_{m,n}$ with rational coefficients whose corresponding Newton method now has possibly nonreal critical points, each of which is fixed.

Belyi maps. By considering differences and computing local degrees, one sees easily that any critically fixed rational map f with $\#C(f) = 3$ is determined, up to Möbius conjugacy, by its critical portrait and hence is a blown-up triangle; see [BBL⁺, Prop. 3.1] for details. Since $\#C(f) = 3$ implies $\#V(f) = 3$, the map f is a so-called *Belyi map*.

While we do not have a closed-form formula in terms of the critical orbit portrait, it is easy to find a system of linear equations with integer coefficients whose unique solution yields the coefficients for such a map, as we now explain.

Suppose we blow up the edges of a triangle a, b, c times, respectively, so that we obtain a map of degree $d := a + b + c + 1$ with three fixed critical points of local degrees

$$\begin{aligned} d_\infty &:= a + b + 1, \\ d_1 &:= b + c + 1, \\ d_0 &:= a + c + 1, \end{aligned}$$

so that $d_0 + d_1 + d_\infty - 3 = 2d - 2$. By reindexing we may assume

$$d_\infty \geq d_1 \geq d_0.$$

Put $\delta_i = d - d_i, i \in \{0, 1, \infty\}$. We search for relatively prime polynomials p and q , with p monic and of degree d and q of degree δ_∞ , so that the map $f(z) = p(z)/q(z)$ fixes $0, 1, \infty$ and has the desired ramification. We use a technique we learned from

Edray Goins. The condition at infinity is satisfied automatically by the assumption on the degrees of p, q . The local degree condition at $z = 0$ holds if and only if there exists a monic polynomial P such that

$$(11.1) \quad p(z) = z^{d_0} \cdot P(z), \quad \deg(P) = \delta_0, \quad P(0) \neq 0.$$

Similarly, the local degree condition at the point $z = 1$ is equivalent to the condition that for some monic polynomial R ,

$$(11.2) \quad p(z) - q(z) = (z - 1)^{d_1} \cdot R(z), \quad \deg(R) = \delta_1, \quad R(1) \neq 0.$$

Putting $Q(z) = q(z)$, noting that $\deg(Q) = \delta_\infty$, considering the expression

$$p(z) - q(z) - (p(z) - q(z)) \equiv 0$$

which vanishes identically, and substituting in the left-hand sides of (2) and (3) yields

$$(11.3) \quad z^{d_0} P(z) - Q(z) - (z - 1)^{d_1} R(z) \equiv 0.$$

Viewing the coefficients of P, Q, R as unknowns, (4) gives a system of linear equations. Thurston's rigidity theorem and the discussion in the preceding paragraphs implies the solution exists, and is unique.

With $d_0 = 5, d_1 = 18, d_\infty = 20$ we get easily using Maple

$$f(z) = (-4 + 21z)^{-1} \cdot z^5.$$

$$\begin{aligned} & (20349 - 217056z + 1162800z^2 - 4069800z^3 + 10287550z^4 - 19752096z^5 \\ & + 29628144z^6 - 35271600z^7 + 33575850z^8 - 25581600z^9 + 15519504z^{10} \\ & - 7407036z^{11} + 2723175z^{12} - 744800z^{13} + 142800z^{14} - 17136z^{15} + 969z^{16}). \end{aligned}$$

Its Julia set is shown below; the blue, green, and grey basins are those of $0, 1, \infty$, respectively.

Other special families. Speyer [Sp] gives the derivation of an explicit formula for the critically fixed rational map obtained by blowing up, once, the edges of a regular n -gon, $n \geq 3$:

$$f_n(z) = \frac{(n-2)z^n + nz}{nz^{n-1} + (n-2)}.$$

It is easily verified that $z = 1$ is a fixed critical point of multiplicity two, and that $\text{Aut}(f_n) = \langle z \mapsto 1/z, z \mapsto \omega_n z \rangle$ where $\omega_n = \exp(2\pi i/(n-1))$, so that these double critical points are located at the $(n-1)$ st roots of unity. Comparing this map with the equally symmetric map obtained by blowing up, once, the edges of a regular $n-1$ -gon, we conclude that these two maps are the same.

We next consider a systematic enumeration of examples in degrees 4 and 5.

Degree 4. [2, 2, 2] There is a unique such example, obtained by blowing up the edges of a graph whose underlying set is a triangle; it is conjugate to $z \mapsto z^3(z - 2)/(1 - 2z)$. McMullen [McM, Prop. 1.2] uses this to give a generally convergent iterative algorithm for approximating the (distinct) roots of the general cubic polynomial. Curiously, it also arises as one of the two maps comprising the correspondence on moduli space induced by the generic cubic rational function; see [BEKP] and [Lod]. Even more curiously, up to pre- and post-composition with Möbius transformations, it appears to be the unique nonbicritical rational map whose fibers are conformally rigid; this observation is due to A. Saenz. To see this, simply notice that as $w \rightarrow c \in C$, the fiber above w approaches a configuration which has nearly a symmetry of order 3, so that the j -invariant of this fiber is a bounded holomorphic function of w .

[2, 2, 1, 1] There is a unique such example, obtained by blowing up a planar segment of length 3; it is conjugate to a Newton method. Up to conjugacy it is given by

$$z \mapsto -\frac{z^3(2z + 3zc - 3 - 4c)}{z + c}$$

where $c = -3/8 \pm \sqrt{5}/8$.

Degree 5.

[3, 3, 2] There is a unique such example, obtained by blowing up a triangle with a single doubled edge. It is given by

$$(5.1) \quad z \mapsto -\frac{z^4(3z - 5)}{5z - 3}.$$

[3, 3, 1, 1] As an abstract graph, there is a unique such example: a segment of length three with its central edge doubled. However, as a planar graph, there are two possibilities: (i) a planar segment with a doubled central edge, and (ii) a bigon with a sticker on both the outside and inside.

They are given by

$$z \mapsto -\frac{z^4(3z + 4zc - 4 - 5c)}{z + c}$$

where in case (i) we have

$$(5.2) \quad c = -1/2 \pm \sqrt{5}/10$$

and where in case (ii) we have

$$(5.3) \quad c = -3/10 \pm \sqrt{21}/10.$$

Galois conjugate pairs turn out to yield Möbius conjugate maps.

[3, 2, 2, 1] There are two graphs realizing this portrait: (i) a segment of length 3 with an end segment doubled, yielding

$$(5.4) \quad z \mapsto \frac{z^3(16.39759477z^2 - 43.99398693z + 33.32932462)}{6z - .267067538},$$

and (ii) a triangle with a sticker, yielding

$$(5.5) \quad z \mapsto \frac{z^3(1.02035014Iz^2 - (3 - 2.55087534I)z + 6. + 1.70058356I)}{6z - 3. + .17005836I}.$$

[2, 2, 2, 2] There is a unique such example, obtained by blowing up the edges of a square. Up to conjugacy it is given by

$$(5.6) \quad z \mapsto -\frac{z^3(-\frac{3}{5}z^2 + 3z - 3)}{-3z^2 + 6z - \frac{12}{5}}.$$

[3, 2, 1, 1, 1] There seems to be a unique such example: a letter “Y” with vertical segment of length 2, and no doubled edges. It is conjugate to a Newton method. Up to conjugacy it is given by the formula

$$(5.7) \quad z \mapsto \frac{z^3(0.00391776z^2 - 0.06348335z + .41144821)}{z - .64811739}.$$

Finding this map algebraically is formidable: with one normalization, one is left with a system of equations which reduces to solving a sextic univariate polynomial; it can be shown that certain pairs of roots yield conjugate maps, while other pairs yield the same map.

[2, 2, 2, 1, 1] The graph is a planar segment of length 4, with no doubled edges. There is a unique such example, given by the formula

$$(5.8) \quad z \mapsto \frac{z^3(-\frac{1}{20}z^2 - \frac{1}{4}z + 1)}{z^2 - \frac{1}{4}z - \frac{1}{20}}.$$

Shown below are the corresponding Julia sets of the maps (5.1)–(5.8); the colors correspond to the basins of attraction.

Degree 6. Given the complexity encountered in the enumeration of quintics via naive algebraic methods, we take a different approach to the enumeration of sextics.

Our first task is to sensibly enumerate the corresponding partitions. Within maps of a fixed degree, one naive way to do this is to note that given any positive integer N , the set of partitions of N is totally ordered by regarding the parts as digits in a fixed base B larger than N and computing the resulting integer. We obtain using, e.g., $B = 10$:

421111
331111
322111
222211
43111
42211
33211
32221
22222
4411
4321
4222
3331
3322
442
433

FIGURE 2. From upper-left to lower-right: Julia sets of the maps (5.1)–(5.8). Pixels are colored according to their basin of attraction.

Note that the first four will correspond to Newton methods, and the last two to Belyi maps.

Instead of systematically tabulating data, we discuss an example.

The partition **33211** corresponds to three abstract isomorphisms and four planar isomorphism classes of multigraph $\mathcal{G}$. The four possibilities for $\mathcal{G}$ are shown below, along with a corresponding wreath recursion Φ for the map $F_{\mathcal{G}}$ obtained by blowing up $\mathcal{G}$. In each, the i th generator surrounds the i th vertex, but further dependence of the wreath recursion on the precise choice of the generators and connecting paths has been suppressed. For the graph **33211a**, we also give the numerical approximation given by Bartholdi's program, the approximate locations of the attractors, and an image of the Julia set (Figure 3); detail is shown in Figure 1. The first two tables give the coefficients of the numerator and of the denominator.

FIGURE 3. The closures of the five immediate basins meet at a repelling fixed-point, located near the bottom of the image, and three (light blue, green, dark blue) also meet at another repelling fixed-point, a bit higher up in the image. Detail is given in Figure 1.

numerator			denominator		
degree	Re	Im	degree	Re	Im
0	−.2823382388	.0873666659	0	−1.0000000000	0
1	−.6106212003	−.2210949962	1	−3.6399623589	.5053341577
2	−1.5258285009	1.6871931450	2	1.8652782469	−2.2777114530
3	3.3728946411	−2.2256057710	3	−2.2515718241	7.7088079910
4	−.5227487239	3.1529307220	4	−9.2611252563	.6459068671
5	−8.5737682757	.3796365169	5	0	0
6	−.3915110282	−1.2571815540	6	0	0

vertex	Re	Im
p_1	0.26444	−0.046607
p_2	0.52144	−0.66036
p_3	−0.59233	0.023373
p_4	−0.47002	1.7073
p_5	∞	∞

33211a.

$$\Phi(a) = \langle abc, 1, b^{-1}, c^{-1}, 1, 1 \rangle (1432)$$
$$\Phi(b) = \langle 1, b, 1, 1, 1, 1 \rangle (12)$$
$$\Phi(c) = \langle 1, 1, c, 1, 1, 1 \rangle (135)$$
$$\Phi(d) = \langle 1, 1, 1, d, 1, 1 \rangle (1546)$$
$$\Phi(e) = \langle 1, 1, 1, 1, 1, e \rangle (16)$$

33211b.

$$\Phi(a) = \langle abc, 1, b^{-1}, c^{-1}, 1, 1 \rangle (1432)$$

$$\Phi(b) = \langle 1, b, 1, 1, 1, 1 \rangle (125)$$

$$\Phi(c) = \langle 1, 1, c, 1, 1, 1 \rangle (53)$$

$$\Phi(d) = \langle 1, 1, 1, d, 1, 1 \rangle (1546)$$

$$\Phi(e) = \langle 1, 1, 1, 1, 1, e \rangle (16)$$

33211c.

$$\Phi(a) = \langle ab, 1, 1, b^{-1}, 1, 1 \rangle (1423)$$

$$\Phi(b) = \langle 1, b, 1, 1, 1, 1 \rangle (132)$$

$$\Phi(c) = \langle 1, 1, 1, c, 1, 1 \rangle (145)$$

$$\Phi(d) = \langle 1, 1, 1, 1, d, 1 \rangle (156)$$

$$\Phi(e) = \langle 1, 1, 1, 1, 1, e \rangle (16)$$

33211d.

$$\Phi(a) = \langle abc, 1, b^{-1}, c^{-1}, 1, 1 \rangle (1432)$$

$$\Phi(b) = \langle 1, b, 1, 1, 1, 1 \rangle (12)$$

$$\Phi(c) = \langle 1, 1, c, 1, 1, 1 \rangle (13)$$

$$\Phi(d) = \langle 1, 1, 1, d, 1, 1 \rangle (1456)$$

$$\Phi(e) = \langle 1, 1, 1, 1, e, 1 \rangle (165)$$

Reflecting on the fact that wreath recursions depend on choices, it is perhaps remarkable that the preceding examples have such a simple form. Though perhaps hard to define “simple” precisely, indeed, this is not an accident. In [NP], an algorithm is given which takes, as input, a connected planar multigraph $\mathcal{G}$, together with certain additional data, and returns as output a wreath recursion for the

corresponding Thurston map $f_{\mathcal{G}}$. This algorithm was effectively implementable by hand, resulting in the efficient calculation of the wreath recursions for all quartic, quintic, and sextic examples.

In the following section, we mention a few key features of this algorithm.

12. COMPUTATION OF WREATH RECURSIONS

Suppose $\mathcal{G}$ is a planar multigraph with $n = \#V(G)$ vertices, and $f = f_{\mathcal{G}}$ is the corresponding Thurston map. In outline, here is the algorithm which was employed for the computation of a wreath recursion associated with the maps $f_{\mathcal{G}}$.

- (1) Begin with a topological realization of $\mathcal{G}$ (that is, drawing multiple edges as parallel), and choose a spanning tree T for the planar dual $\mathcal{G}^*$. Choose a basepoint vertex b for T .
- (2) Choose a set of arcs $r_i, i = 1, \dots, n$ from b to the elements of V as in §2.8 so that in addition the following hold: (i) no arc r_i crosses an edge of $\mathcal{G}$ more than once; (ii) no arc r_i crosses an edge incident to a vertex at which r_i terminates.

That this is possible follows easily by the greedy algorithm. Initialization: let F be the face of $\mathcal{G}$ containing b ; join b to each vertex on ∂F . Induction: suppose there exists a face F not all of whose vertices have been visited. There exists a closest such face to b , as measured by the edge-path length in T . Choose one such face F , run from b to F along edges of T , and then run to the unique vertex on ∂F which is not yet visited and which is first in the counterclockwise ordering of these vertices, starting from the vertex immediately to the right of the traversed edge-path in T as it enters F .

Label these arcs r_i in the usual counterclockwise cyclic fashion, and label their terminal points v_i .

- (3) Next, label the remaining preimages of the basepoint $b_2, \dots, b_d$ as follows.

The map f is obtained by “blowing up” the edges e_j to disks D_j , where $f : D_j \rightarrow S^2 - e_j$ is a homeomorphism. Thus each D_j contains a unique element $b_j \in f^{-1}(b)$. We may assume, for convenience of representation, that b_j is drawn as a midpoint of e_j . We may identify $f^{-1}(b) - \{b\}$ with the set of edges of $\mathcal{G}$. Set $b_1 = b$. For each $v_i, i = 1, \dots, n$, do the following. Examine the neighbors of v_i . Find the neighbor with smallest index, j , for which there exists a nonempty set E of unlabelled edges e joining v_i and v_j ; if no such neighbor exists, proceed to v_{i+1} . Otherwise, the set $E \cup \{r_i\}$ has a total order, given by the counterclockwise ordering in which they are incident to v_i , with r_i as the least element. Label the elements of E consecutively, with increasing indices on the labels b_k ’s.

The proof of the following theorem uses the way in which the arcs r_i were chosen, but not the way in which the labels of edges (preimages of b) were assigned. To describe the result, we imagine that the subset of the arcs r_i which join the basepoint b to vertices w on the boundary of the face containing b are “labelled” by the symbol 1, and for the purposes of describing the monodromy, we imagine that these arcs are new “edges” incident to such vertices w . See Figures 4 and 8 for a clarifying example.

Theorem 12.1. *The monodromy induced by g_i is the pure cycle consisting of the indices of the edge labels incident to v_i , with the counterclockwise cyclic order.*

- 4 Next, choose connecting arcs as follows. Set λ_1 to be the constant path at $b_1 = b$. Given $j > 1$, the edge e_j is incident to two vertices v_k, v_ℓ . Let $i = \max\{k, \ell\}$, and choose λ_j so that it follows r_i to v_i , loops around v_i counterclockwise in the direction of g_i until it reaches e_j , and then follows e_j to b_j .

FIGURE 4. Choosing the arcs $r_1, \dots, r_4$ and the preimage (edge) labels $e_2, \dots, e_6$.

FIGURE 5. Preimages of edges and of the basepoint b .

A worked example. Consider the multigraph, connecting arcs, and edge-labels as chosen in Figure 4. Figure 5 shows the preimages of the edges and basepoints under the corresponding branched covering, labelled by their images, and with the generators (but not their preimages) superimposed.

Consider the monodromy induced by g_4 . The local pictures, gleaned from the previous figure, are as follows. Near the branch point v_4 circled by g_4 , we see a cycle:

FIGURE 6. A cycle in the monodromy induced by g_4 .

For elements b_j in $f^{-1}(b)$ which correspond to edges e_j not adjacent to v_4 , we see that the interior of D_j contains a preimage copy of v_4 , and thus the lift $f^{-1}(g_4)[b_j]$ of g_4 based at b_j circles this copy of v_i and returns to b_j within D_j so that $j^{g_4} = j$:

FIGURE 7. Fixed-points in the monodromy induced by g_4 .

Thus the monodromy induced by g_4 is given by the pure cycle (165). The calculation of the remainder of the monodromy is similar, yielding:

FIGURE 8. The monodromy of f .

Figure 9 shows the result of our algorithm for constructing connecting paths.

Since the disks D_j are disjoint from the postcritical set, the connecting paths λ_j may take any route to b_j once it has entered D_j . So we may represent the connecting paths by drawing them on the same codomain plane as the graph $\mathcal{G}$. See Figure 10.

We now compute the corresponding wreath recursion Φ for this example, choice of generators, labels of preimages, and choice of connecting paths. For brevity, we give only the computation of the wreath position $g_2|_5$ and informally describe several helpful observations.

Let us compute $g_2|_5$. Note first that $5^{g_2} = 3$. We express the loop $[\lambda_5 * \tilde{g}_2[b_5] * \bar{\lambda}_3]$ as a word in the generators. Here, $\tilde{g}_1[b_5]$ is the lift under f of a loop at b representing g_1 starting at $b_5 \in f^{-1}(b)$.

By examining the enclosed region (shaded in Figure 11), we see that $\lambda_5 * \tilde{g}_{2_{b_5}} * \bar{\lambda}_{5^{\sigma(g_2)}}$ is homotopic to g_3^{-1} and thus conclude $g_2|_5 = g_3^{-1}$.

FIGURE 9. Choosing connecting paths.

FIGURE 10. Representation of the connecting paths via paths in the codomain.

FIGURE 11. Computation of $g_2|_5$.

Performing this computation for each wreath position $g_i|_j$, $1 \leq i \leq 4, 1 \leq j \leq 6$, we notice several striking patterns. First, if j is fixed by the monodromy $\sigma(g_i)$ then $g_i|_j$ is trivial. The intuition is latent in the procedure we have described: if v_k is the vertex of higher degree attached to the edge e_j , then λ_j follows g_k to e_j . Since j is fixed by the monodromy, the lift $\tilde{g}_i[b_j]$ stays within the disk obtained by blowing up e_j and thus $\bar{\lambda}_{j^{g_i}} = \bar{\lambda}_j$ traverses λ_j backwards, yielding a path homotopic to the constant map at b . Together with our computation $g_2|_5 = g_3^{-1}$, this gives us 10 of the 24 wreath positions:

$$\begin{aligned}\Phi(g_1) &= \langle \ , \ , \ 1, \ 1, \ 1, \ 1 \rangle (12), \\ \Phi(g_2) &= \langle \ , \ , \ , \ , \ g_3^{-1}, \ 1 \rangle (12534), \\ \Phi(g_3) &= \langle \ , \ 1, \ , \ , \ 1, \ \rangle (1436), \\ \Phi(g_4) &= \langle \ , \ 1, \ 1, \ 1, \ , \ \rangle (165).\end{aligned}$$

The second observation we make is that if $j' := j^{g_i} \neq j$, if edges $e_j, e_{j'}$ are not parallel, and if the index of v_i (at which both $e_j, e_{j'}$ are both incident) is largest among the three vertices comprised by the endpoints of e_j and $e_{j'}$, then the wreath

position $g_i|_j$ is trivial. In this case λ_i follows g_i to the edge e_j and runs up e_j to b_j . Then the lift $\tilde{g}_i[b_j]$ travels back down e_j to g_i , follows g_i to $e_{j^{g_i}}$ and runs up $e_{j^{g_i}}$ to $b_{j^{g_i}}$. Then since v_i is the vertex of higher index attached to $e_{j^{g_i}}$, $\bar{\lambda}_{j^{g_i}}$ goes back down $e_{j^{g_i}}$ to g_i and follows g_i out backwards to b . Thus $g_i|_j$ encircles no elements of V and is therefore homotopic to the constant map. This gives us five more wreath positions.

$$\Phi(g_1) = \langle \ , \ , \ 1, \ 1, \ 1, \ 1 \rangle (12),$$

$$\Phi(g_2) = \langle \ 1, \ , \ , \ , \ g_3^{-1}, \ 1 \rangle (12534),$$

$$\Phi(g_3) = \langle \ 1, \ 1, \ , \ , \ 1, \ 1, \ \rangle (1436),$$

$$\Phi(g_4) = \langle \ 1, \ 1, \ 1, \ 1, \ 1, \ , \ 1 \rangle (165).$$

Now we fill in the rest of the tabl1, encouraging the reader to check our work using the processes described above.

$$\Phi(g_1) = \langle \ g_1, \ 1, \ 1, \ 1, \ 1, \ 1 \rangle (12),$$

$$\Phi(g_2) = \langle \ 1, \ g_2g_3, \ 1, \ 1, \ g_3^{-1}, \ 1 \rangle (12534),$$

$$\Phi(g_3) = \langle \ 1, \ 1, \ g_3, \ 1, \ 1, \ 1 \rangle (1436),$$

$$\Phi(g_4) = \langle \ 1, \ 1, \ 1, \ 1, \ g_4, \ 1 \rangle (165).$$

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