

VEECH SURFACES AND THEIR PERIODIC POINTS

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ABSTRACT. We give inequalities comparing widths or heights of cylinder decompositions of Veech surfaces with the signatures of their Veech groups. As an application of these inequalities, we estimate the numbers of periodic points of non-arithmetic Veech surfaces. The upper bounds depend only on the topological types of Veech surfaces and the signatures of Veech groups as Fuchsian groups. The upper bounds also estimate the numbers of holomorphic sections of holomorphic families of Riemann surfaces constructed from Veech groups of non-arithmetic Veech surfaces.

1. INTRODUCTION

A flat structure u on a surface X is an atlas on X without a finite subset C such that every transition function is of the form $w = \pm z + c$. The pair (X, u) of the surface X and the flat structure u is called a flat surface. The points of C are called critical points of (X, u) . On (X, u) , we may use terminologies of Euclidean geometry. We focus on the case where (X, u) is of finite Euclidean area. An affine map $h : (X, u) \rightarrow (X, u)$ is a quasiconformal map of (X, u) such that h preserves C and is an affine map on the Euclidean plane with respect to u . The affine group $\text{Aff}^+(X, u)$ of (X, u) is the group of all affine maps of (X, u) . For an affine map h , the derivative $\pm A \in \text{SL}(2, \mathbb{R})$ of its descriptions as affine maps on the Euclidean plane by u is uniquely determined up to the sign. Thus, we have the homomorphism $D : \text{Aff}^+(X, u) \rightarrow \text{PSL}(2, \mathbb{R})$ which maps each affine map h to its derivative $\pm A$. The image $\Gamma(X, u) = \text{Im}(D)$ of this homomorphism D is called the Veech group of (X, u) . It is proved by Veech ([Vee89], Proposition 2.7) that Veech groups are Fuchsian groups. Conversely, there is the problem of which kinds of Fuchsian groups are realized as Veech groups (see [HMSZ06], Problem 5). For this problem, we have a result comparing the Veech group $\Gamma(X, u)$ with geometrical values for cylinder decompositions of (X, u) . If the Veech group $\Gamma(X, u)$ is a lattice in $\text{PSL}(2, \mathbb{R})$, a flat surface (X, u) is called a Veech surface. The Veech dichotomy theorem (Theorem 3.2) claims that if one of the geodesics with direction θ is not dense in a Veech surface (X, u) , then every geodesic with direction θ is closed or a saddle connection, that is, a segment connecting critical points. Let θ be a direction such that the geodesics with direction θ are closed or saddle connections. Since there are only finitely many critical points, almost all points lie in closed geodesic with direction θ . All connected components of the union of such closed geodesics are Euclidean cylinders whose boundaries consist of saddle connections.

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The family of the cylinders are called a cylinder decomposition of (X, u) . For a Euclidean cylinder R , the distance H between two boundary components is called the height of R , the shortest Euclidean length W of core curves of R is called the width of R . We call the ratio W/H the modulus of the cylinder R and denote it by $\text{mod}(R)$. We showed the following theorem in [Shi14a], Theorem 3.2.

Theorem 1.1. *Let (X, u) be a Veech surface such that X is a surface of type (g, n) with $3g - 3 + n > 0$. Let $\{R_i\}_{i=1}^m$ be any cylinder decomposition of (X, u) . Then, the inequality*

$$\left(\frac{\text{mod}(R_i)}{\text{mod}(R_j)} \right)^{\frac{1}{2}} < 2 \exp \left(\frac{5}{e} (3g - 3 + n) \right) \text{Area}(\mathbb{H}/\Gamma(X, u))$$

holds for all $i, j \in \{1, \dots, m\}$.

In this paper, we show the same kinds of inequalities for heights and widths of cylinders.

Theorem 1.2. *Let (X, u) be a Veech surface such that X is a surface of type (g, n) with $3g - 3 + n > 0$. Let $G(\cdot)$ be the Landau function and k_0 the number of punctures of $\mathbb{H}/\Gamma(X, u)$. We set $d = 3g - 3 + n$, $\lambda = \lambda(g, n) = 2G(2d)$ and $\mu = \text{Area}(\mathbb{H}/\Gamma(X, u)) - k_0 + 1$. Let $\{R_i\}_{i=1}^m$ be any cylinder decomposition of (X, u) and W_i, H_i the width and height of the cylinder R_i , respectively. Then, we have*

$$(\lambda\mu)^{-2(d-1)} < \frac{H_i}{H_j} < (\lambda\mu)^{2(d-1)}$$

and

$$(\lambda\mu)^{-2d} < \frac{W_i}{W_j} < (\lambda\mu)^{2d}$$

for all $i, j \in \{1, \dots, m\}$.

In this paper, we also study periodic points of non-arithmetic Veech surfaces. In [GHS03], Theorem 1, Gutkin-Hubert-Schmidt showed the finiteness of the numbers of periodic points of non-arithmetic prelattice surfaces. Even though not explicitly stated, they obtained an upper bound of the number of periodic points of a non-arithmetic prelattice translation surface depending only on the parameters of two cylinder decompositions of the surface. Möller ([Möl06], Theorem 3.3) characterized periodic points by torsion on some quotient of Jacobian variety of the flat surface. This result also gives us an upper bound of the number of periodic points of a compact non-arithmetic Veech surface depending only on the genus of the Veech surface (see [Möl06], Corollary 3.6 and [Bui94], Theorem A). However, the orders of these upper bounds are too large. We give an upper bound of the number of periodic points of non-arithmetic Veech surface whose underlying surface is of finite type. The upper bounds depend only on the topological types of Veech surfaces and the signatures of the Veech groups as Fuchsian groups. The order of the upper bounds are better than the previous ones.

Theorem 1.3. *Let (X, u) be a non-arithmetic Veech surface such that X is of type (g, n) . Then, the number of periodic points of (X, u) is less than*

$$\frac{1}{2^{26}} d^{10} (\lambda \mu)^{106d-1}.$$

As an application of this theorem, we estimate the numbers of holomorphic sections of holomorphic families of Riemann surfaces constructed from Veech groups. A triple (M, π, B) of a two-dimensional complex manifold M , a Riemann surface B and a holomorphic map $\pi : M \rightarrow B$ such that the fiber $X_t = \pi^{-1}(t)$ is a Riemann surface of type (g, n) for each $t \in B$ and the complex structure of X_t depends holomorphically on the parameter t is called a holomorphic family of Riemann surface of type (g, n) over B . A holomorphic family (M, π, B) of Riemann surface is called locally non-trivial if the induced map $B \ni t \mapsto \pi^{-1}(t) \in M(g, n)$ is non-constant. Here, $M(g, n)$ is the moduli space of Riemann surfaces of type (g, n) . A holomorphic section $s : B \rightarrow M$ of a holomorphic family of Riemann surface (M, π, B) is a holomorphic map satisfying $\pi \circ s = id_B$. Manin ([Man63], Teopema on page 1395), Grauert ([Gra65], Satz on page 132) and Miwa ([Miw66], Theorem 1) independently proved that if the base space B is of finite type, then the numbers of holomorphic sections of locally non-trivial holomorphic families (M, π, B) of Riemann surfaces of finite type is finite. In Manin's proof, there was a gap. It was found and fixed by Coleman [Col90] (see also [Man89]). By using Teichmüller theory, Iwayoshi and Shiga ([IS88], Section 5) also proved the finiteness. Shiga ([Shi97], Theorem 2 and Corollary 3) gave upper bounds of the numbers of holomorphic families of Riemann surfaces whose fibers are of type $(g, n) = (0, n)$ ($n \geq 4$), $(1, 2)$ and $(2, 0)$. These upper bounds also give upper bounds of the numbers of holomorphic sections of holomorphic families of Riemann surfaces of the above types. In [Shi14a], Theorem 3.1 and [Shi14b], Theorem 7.1, we estimate the numbers of holomorphic sections of holomorphic families of Riemann surfaces constructed from Veech groups.

Let (X, u) be a flat surface such that X is a surface of type (g, n) . We may also regard (X, u) as a Riemann surface. For each $A \in \mathrm{SL}(2, \mathbb{R})$, let $A \circ u$ be a flat structure given by composing A as a linear map with charts of u . The pair $(X, A \circ u)$ is also a flat surface. Thus, we have the $\mathrm{SL}(2, \mathbb{R})$ -orbit of (X, u) in the moduli space $M(g, n)$. It is proved that the $\mathrm{SL}(2, \mathbb{R})$ -orbit of (X, u) in $M(g, n)$ coincides with the orbifold $\mathbb{H}/\bar{\Gamma}(X, u)$. Here, $R = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\bar{\Gamma}(X, u) = R \cdot \Gamma(X, u) \cdot R^{-1}$ (see [EG97], Theorem 1 and Remarks on page 173 and [HS07], Corollary 2.21). Now, we have a holomorphic local isometry $f_0 : \mathbb{H}/\bar{\Gamma}(X, u) \rightarrow M(g, n)$. Composing a finite covering ϕ from a Riemann surface B of finite type onto $\mathbb{H}/\bar{\Gamma}(X, u)$, we obtain a holomorphic local isometry $f = \phi \circ f_0 : B \rightarrow M(g, n)$ that is called a Teichmüller curve. Through this map f , each point t of B corresponds to a Riemann surface $X_t = f(t)$. The correspondence gives us a holomorphic family of Riemann surfaces over B whose fibers are X_t . Our upper bounds of the numbers of holomorphic sections of such holomorphic families given in [Shi14a], Theorem 3.1 and [Shi14b], Theorem 7.1 depend only on g, n and the types of the base spaces B . However, if the degree of the finite covering $\phi : B \rightarrow \mathbb{H}/\bar{\Gamma}(X, u)$ tends to infinity, our upper bound also tends to infinity. In this paper, we give a uniform upper bound of the numbers of holomorphic sections of holomorphic families of Riemann surfaces constructed from the Veech group $\Gamma(X, u)$ of a fixed Veech surface (X, u) . The upper bound does not depend on the degrees of finite coverings $\phi : B \rightarrow \mathbb{H}/\bar{\Gamma}(X, u)$.

Theorem 1.4. *Let (X, u) be an non-arithmetic Veech surface such that X is of type (g, n) . For every holomorphic family of Riemann surfaces constructed from the Veech group $\Gamma(X, u)$, the number of holomorphic sections is less than*

$$\frac{1}{2^{26}} d^{10} (\lambda \mu)^{106d-1}.$$

2. PRELIMINARIES

In this section, we give definitions of flat surfaces, Veech groups, and periodic points.

Let \overline{X} be a (connected) compact Riemann surface of genus g . Let X be a Riemann surface of type (g, n) with $3g - 3 + n > 0$ that is \overline{X} with n points removed.

Definition 2.1 (Flat structures and flat surfaces). A flat structure u on X is an atlas of X minus a finite subset satisfying the following conditions:

- (1) the local coordinates of u are compatible with the orientation of X ,
- (2) the transition functions are of the form

$$(2.1) \quad w = \pm z + c$$

in $z(U \cap V)$ for $(U, z), (V, w) \in u$ with $U \cap V \neq \emptyset$,

- (3) the atlas u is the maximal atlas with respect to (1) and (2).

A pair (X, u) of a Riemann surface X and a flat structure u on X is called a flat surface. The punctures of X and the points which do not have neighborhoods in u are called critical points of (X, u) . We denote the set of all critical points of (X, u) by $C(X, u)$.

Remark 2.2. In Definition 2.1, if we replace the form of (2.1) to translations

$$w = z + c,$$

the pair (X, u) is called a translation surface.

We may consider terminologies of Euclidean geometry on (X, u) such as area, segments, lengths or directions of the segments. Hereafter, we assume that the Euclidean area of (X, u) is finite. By definition, we may regard u as a conformal structure on X . On the Riemann surface (X, u) , we have the integrable meromorphic quadratic differential q which is dz^2 with respect to each of the charts z of u . In this paper, we assume that q is holomorphic on (X, u) . These assumptions imply that all points of $C(X, u) \cap X$ are zeros of q and punctures may be poles of order at most 1. By the Riemann-Roch theorem, the sum of the orders of q at each point of $C(X, u)$ equals $4g - 4$. Therefore, we have the following.

Lemma 2.3. *Let X be a surface of type (g, n) and u a flat structure on X . The set $C(X, u)$ contains at most $4g - 4 + 2n$ points. The set $C(X, u)$ contains just $4g - 4 + 2n$ points if and only if all punctures are poles of order 1 and all zeros are of order 1.*

Definition 2.4 (θ -geodesics, closed θ -geodesics, θ -saddle connections). Let $\theta \in [0, \pi)$ be a direction. A (closed) θ -geodesic on (X, u) is a (closed) geodesic of direction $\theta \in [0, \pi)$ with respect to u which does not pass through critical points. A θ -saddle connection is a segment of direction θ whose end points are critical points and which does not contain critical points except for the end points. If $\theta = 0, \frac{\pi}{2}$, we say that θ -geodesics and θ -saddle connections are horizontal, vertical, respectively.

Definition 2.5 (Jenkins-Strebel directions). A direction $\theta \in [0, \pi)$ is a Jenkins-Strebel direction of (X, u) if every point of (X, u) lies in a closed θ -geodesic or a θ -saddle connection.

Let $\theta \in [0, \pi)$ be a Jenkins-Strebel direction. Since there exist only finitely many critical points of (X, u) , almost all points on (X, u) lie in closed θ -geodesics. Removing all θ -saddle connections from X , the resulting surface has finitely many connected components and they are cylinders foliated by closed θ -geodesics. Thus, the connected components are Euclidean cylinders. It is known that core curves of the cylinders are not homotopic to each other and not homotopic to a point or a puncture (See [Str84], Section 9).

Definition 2.6 (Cylinder decompositions). Let θ be a Jenkins-Strebel direction. If X is decomposed into m cylinders R_1, \dots, R_m by the direction θ , we call θ an m -Jenkins-Strebel direction. The family of cylinders $\{R_i\}_{i=1}^m$ is called the cylinder decomposition of (X, u) by the direction θ .

Remark 2.7. As core curves of the cylinders R_1, \dots, R_m are not homotopic to each other, the number of cylinders m is not greater than $3g - 3 + n$.

Definition 2.8 (Affine groups). An affine map h of (X, u) is a quasiconformal self-map of X such that h preserves critical points $C(X, u)$ and, for (U, z) and $(V, w) \in u$ with $h(U) \subset V$, the composition $w \circ h \circ z^{-1}$ is of the form $w = Az + c$ for some $A \in \text{GL}(2, \mathbb{R})$ and $c \in \mathbb{C}$. The affine group $\text{Aff}^+(X, u)$ of (X, u) is the group of all affine maps of (X, u) .

By the definition of flat structures, the matrix A which is the derivative of $w \circ h \circ z^{-1}$ is uniquely determined up to the sign. Moreover, A is in $\text{SL}(2, \mathbb{R})$ since we assume that the Euclidean area of (X, u) is finite. Therefore, we have the homomorphism $D : \text{Aff}^+(X, u) \rightarrow \text{PSL}(2, \mathbb{R})$ called derivative map which maps each affine map h to its derivative $\pm A$.

Definition 2.9 (Veech groups). The image of the derivative map D is called the Veech group of (X, u) and we denote it by $\Gamma(X, u)$.

It is proved by Veech ([Vee89], Proposition 2.7) that Veech groups are Fuchsian groups (see also [EG97], Theorem 1 and Remarks on page 173 and [HS07], Remarks 2.18 and 2.20).

Definition 2.10 (Veech surfaces and their arithmeticity). A flat surface (X, u) is called a Veech surface if $\Gamma(X, u)$ is a lattice in $\text{PSL}(2, \mathbb{R})$, that is, the corresponding orbifold $\mathbb{H}/\Gamma(X, u)$ is of a finite hyperbolic area. A flat surface is arithmetic if the Veech group is commensurable with $\text{PSL}(2, \mathbb{Z})$ and is non-arithmetic if it is not commensurable with $\text{PSL}(2, \mathbb{Z})$.

One of our interests is periodic points on Veech surfaces.

Definition 2.11 (Periodic points). Let (X, u) be a flat surface. A point $z \in (X, u)$ is a periodic point if its $\text{Aff}^+(X, u)$ -orbit $\text{Aff}^+(X, u)\{z\}$ is finite.

Example 2.12. Since the set $C(X, u)$ of all critical points of (X, u) is finite and affine maps preserve $C(X, u)$, critical points are periodic points.

Gutkin-Hubert-Schmidt proved that the number of periodic points on a prelattice flat surface is related to the arithmeticity of the flat surface. A flat surface is called prelattice if it has at least two Jenkins-Strebel directions.

Theorem 2.13 ([GHS03], Theorem 1). *If a prelattice flat surface is arithmetic, then its periodic points are dense. If a prelattice flat surface is non-arithmetic, then it has only finitely many periodic points.*

Finally, we see an important theorem about Fuchsian groups proved in [Shi14b], Theorem 7.15.

Theorem 2.14. *Let Γ be a lattice Fuchsian subgroup which is a lattice in $\mathrm{PSL}(2, \mathbb{R})$. Assume that Γ contains $\begin{bmatrix} 1 & b_0 \\ 0 & 1 \end{bmatrix}$ ($b_0 > 0$) as a primitive element. Then, Γ contains $\begin{bmatrix} * & * \\ c_1 & * \end{bmatrix} \in \Gamma$ satisfying the inequality*

$$1 \leq |b_0 c_1| < \mathrm{Area}(\mathbb{H}/\Gamma) - k_0 + 1.$$

Here, $\mathrm{Area}(\mathbb{H}/\Gamma)$ is the hyperbolic area of the orbifold \mathbb{H}/Γ and k_0 is the number of punctures of \mathbb{H}/Γ .

Remark 2.15. The lower bound of the inequality of Theorem 2.14 is a consequence of Shimizu's lemma ([Shi63], Lemma 4, see also [IT92], Lemma 2.21).

Lemma 2.16 (Shimizu's lemma). *Let Γ be a Fuchsian group in $\mathrm{PSL}(2, \mathbb{R})$. Suppose that Γ contains $\begin{bmatrix} 1 & b_0 \\ 0 & 1 \end{bmatrix}$ ($b_0 > 0$) as a primitive element. Then, the inequality $1 \leq |b_0 c_1|$ holds for every $\begin{bmatrix} * & * \\ c_1 & * \end{bmatrix} \in \Gamma$ with $c_1 \neq 0$.*

Remark 2.17. A Fuchsian group Γ of type $(p, k; \nu_1, \dots, \nu_k)$ is one whose corresponding orbifold \mathbb{H}/Γ has genus p and k cone points of order $\nu_1, \dots, \nu_k \in \{2, 3, \dots, \infty\}$. A cone point of order ∞ is a puncture of \mathbb{H}/Γ . If a Fuchsian group Γ is of type $(p, k; \nu_1, \dots, \nu_k)$, then we have

$$\mathrm{Area}(\mathbb{H}/\Gamma) = 2\pi \left(2p - 2 + \sum_{i=1}^k \left(1 - \frac{1}{\nu_i} \right) \right).$$

3. VEECH GROUPS VS. CYLINDER DECOMPOSITIONS

In this section, we show Theorem 1.2 which gives relations between Veech groups and cylinder decompositions of Veech surfaces.

Let (X, u) be a Veech surface such that X is a surface of type (g, n) with $3g - 3 + n > 0$.

Definition 3.1 (Heights, widths and moduli of cylinders). Let R be a Euclidean cylinder. The height of R is the shortest length of curves connecting two boundary components of R . The width of R is the shortest length of core curves of R . For a Euclidean cylinder R with height H and width W , the ratio $\mathrm{mod}(R) = W/H$ is called the modulus of R .

Given the cylinder decomposition $\{R_i\}_{i=1}^m$ by a Jenkins-Strebel direction θ . Take the orthogonal matrix $R_{-\theta} \in \mathrm{SO}(2)$. Let us describe u as $u = \{(U, z)\}$. Considering $A_{-\theta}$ as a linear map, $A_{-\theta} \circ u = \{(U, A_{-\theta} \circ z)\}$ is also a flat structure. The direction θ of (X, u) is regarded as a horizontal direction of $(X, A_{-\theta} \circ u)$. Moreover, $\Gamma(X, A_{-\theta} \circ u) = A_{-\theta} \cdot \Gamma(X, u) \cdot A_{-\theta}^{-1}$ and hence, $\mathrm{Area}(\mathbb{H}/\Gamma(X, A_{-\theta} \circ u)) = \mathrm{Area}(\mathbb{H}/\Gamma(X, u))$. Thus, we may assume that $\theta = 0$ is the given Jenkins-Strebel direction of (X, u) .

The following theorem ([Vee89], Proposition 2.10 and 2.11) gives us an affine map which preserves the horizontal directions.

Theorem 3.2 (The Veech dichotomy theorem). *Let (X, u) be a Veech surface. Every direction $\theta \in [0, \pi)$ satisfies one of the following properties:*

- *The direction θ is a Jenkins-Strebel direction. Let $\{R_i\}_{i=1}^m$ be the cylinder decomposition of (X, u) by the Jenkins-Strebel direction θ . Then, the ratio $\text{mod}(R_i)/\text{mod}(R_j)$ is a rational number for all $i, j \in \{1, \dots, m\}$. Moreover, there exists an affine map $h \in \text{Aff}^+(X, u)$ which preserves the direction θ and the derivative $D(h)$ is a parabolic element of $\text{PSL}(2, \mathbb{R})$.*
- *Every θ -geodesic is dense in X and uniquely ergodic. That is, the θ -geodesic flow has only one transverse measure μ up to scalar multiples such that the flow is ergodic with respect to μ .*

Assume that $\theta = 0$ is a Jenkins-Strebel direction of a Veech surface (X, u) . Let $\{R_i\}_{i=1}^m$ be any cylinder decomposition of (X, u) . Denote by W_i and H_i the width and height of the cylinder R_i , respectively. By Theorem 3.2, the Veech group $\Gamma(X, u)$ contains a parabolic element of the form $\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$. Let $B_0 = \begin{bmatrix} 1 & b_0 \\ 0 & 1 \end{bmatrix}$ ($b_0 > 0$) be the primitive element which is of this form. Fix $h_{B_0} \in D^{-1}(B_0)$. Veech ([Vee89], Proposition 2.4) also proved that there exists $\alpha \in \mathbb{N}$ such that $h_{B_0}^\alpha(R_i) = R_i$ and $h_{B_0}^\alpha|_{\partial R_i} = \text{id}$. Moreover, $h_{B_0}^\alpha$ is a power of Dehn twists along core curves of the cylinders R_1, \dots, R_m .

Definition 3.3 (The Landau function). The Landau function $G : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$ is one which maps each positive integer k to the greatest order of an element of the symmetric group S_k of degree k .

Remark 3.4. Landau ([Lan03], page 94) showed that

$$\lim_{k \rightarrow \infty} \frac{\log(G(k))}{\sqrt{k \log k}} = 1.$$

Massias [Mas84] showed that

$$\log G(k) \leq 1.05313 \dots \sqrt{k \log k}$$

with equality at $k = 1319766$.

Recall that (X, u) is a Veech surface such that X is a surface of type (g, n) with $3g - 3 + n > 0$ and the Veech group $\Gamma(X, u)$ is a Fuchsian group. Let k_0 be the number of punctures of $\mathbb{H}/\Gamma(X, u)$. Hereafter, we set

$$(3.1) \quad d = 3g - 3 + n,$$

$$(3.2) \quad \lambda = \lambda(g, n) = 2G(2d),$$

$$(3.3) \quad \mu = \text{Area}(\mathbb{H}/\Gamma(X, u)) - k_0 + 1.$$

Then, we have the following lemma.

Lemma 3.5. *There exists $\alpha < \lambda$ such that $h_B(R_i) = R_i$, $h_B|_{\partial R_i} = \text{id}$ and h_B is a power of Dehn twists along core curves of the cylinders R_1, \dots, R_m .*

Proof. Let s be the number of horizontal saddle connections. Recall that m is the number of cylinders. By the Euler characteristic, we have

$$2 - 2g = \sharp C(X, u) - (m + s) + m = \sharp C(X, u) - s.$$

By Lemma 2.3, the number s is at most $2(3g-3+n) = 2d$. Take $\alpha_0 < G(s) \leq G(2d)$ such that $h_{B_0}^{\alpha_0}$ preserves every saddle connection. Then, $h_{B_0}^{2\alpha_0}|_{\partial R_i} = id$, $h_{B_0}^{2\alpha_0}(R_i) = R_i$ for all i and $h_{B_0}^{2\alpha_0}$ is a power of Dehn twists along core curves of the cylinders R_1, \dots, R_m . Setting $\alpha = 2\alpha_0$, we obtain the claim. \square

We set $h_B = h_{B_0}^\alpha$ and $B = D(h_B) = \begin{bmatrix} 1 & \alpha b_0 \\ 0 & 1 \end{bmatrix}$ is the derivative of h_B . Let $i(\cdot, \cdot)$ be the geometric intersection number and C_i a core curve of R_i .

Lemma 3.6. *Let $h \in \text{Aff}^+(X, u)$ and $D(h) = \begin{bmatrix} * & * \\ c & * \end{bmatrix}$. We have*

$$|c|W_i = \sum_{k=1}^m i(h(C_i), C_k)H_k$$

and

$$|c|W_i = \sum_{k=1}^m i(C_i, h(C_k))H_k$$

for all i .

Proof. We take C_i to be a horizontal closed geodesic in R_i . Considering C_i as a vector $\begin{bmatrix} W_i \\ 0 \end{bmatrix}$, the image $h(C_i)$ is regarded as a vector $D(h) \begin{bmatrix} W_i \\ 0 \end{bmatrix} = \begin{bmatrix} * \\ cW_i \end{bmatrix}$. We may also consider every connected component of $h(C_i) \cap R_k$ as a vector $\begin{bmatrix} * \\ H_i \end{bmatrix}$. Since $h(C_i) \cap R_k$ has $i(h(C_i), C_k)$ connected components, we have

$$|c|W_i = \sum_{k=1}^m i(h(C_i), C_k)H_k.$$

Applying the same argument to the affine map h^{-1} , we have

$$|c|W_i = \sum_{k=1}^m i(h^{-1}(C_i), C_k)H_k = \sum_{k=1}^m i(C_i, h(C_k))H_k$$

as claimed. \square

The following lemma ([Shi14a], Lemma 3.11) is the key of our results. Theorem 1.1 is easily proved by Lemma 3.7.

Lemma 3.7. *Let λ, μ be the same as (3.2) and (3.3). Recall that $b_0 > 0$ is the $(1, 2)$ -entry of the primitive element of $\Gamma(X, u)$ which is of this form $B_0 = \begin{bmatrix} 1 & b_0 \\ 0 & 1 \end{bmatrix}$. Then, the inequality*

$$1 < \frac{\lambda b_0}{\text{mod}(R_i)} < (\lambda \mu)^2$$

holds for all $i \in \{1, \dots, m\}$.

Proof. Recall that h_B is a power of Dehn twists along core curves of the cylinders R_1, \dots, R_m with $B = D(h_B) = \begin{bmatrix} 1 & \alpha b_0 \\ 0 & 1 \end{bmatrix}$. Assume that $h_B|_{R_i}$ is the N_i -th power

of the Dehn twist along a core curve of R_i . Then the derivative of $h_B|_{R_i}$ is of the form $\begin{bmatrix} 1 & \text{mod}(R_i) \\ 0 & 1 \end{bmatrix}^{N_i}$. Thus, we have

$$\begin{bmatrix} 1 & \alpha b_0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \text{mod}(R_i) \\ 0 & 1 \end{bmatrix}^{N_i}.$$

This implies the inequality

$$\frac{\text{mod}(R_i)}{b_0} = \frac{\alpha}{N_i} < 2G(2d) = \lambda$$

which is equivalent to the lower bound of our claim.

Next, we show the second inequality. By Theorem 2.14, we can also take $A_0 = \begin{bmatrix} * & * \\ c_1 & * \end{bmatrix} \in \Gamma(X, u)$ satisfying $1 \leq b_0 c_1 < \mu$. Fix $h_{A_0} \in D^{-1}(A_0)$ and put $h_A = h_{A_0}^{-1} \circ h_B \circ h_{A_0}$. The derivative A of h_A is

$$A = D(h) = A_0^{-1} B A_0 = \begin{bmatrix} * & * \\ -\alpha b_0 c_1^2 & * \end{bmatrix}.$$

Applying Lemma 3.6 to h_A , we have

$$\begin{aligned} (3.4) \quad \alpha b_0 c_1^2 W_i &= \sum_{k=1}^m i(h_A(C_i), C_k) H_k \\ &\geq i(h_A(C_i), C_i) H_i \\ &= i(h_B(h_{A_0}(C_i)), h_{A_0}(C_i)) H_i \end{aligned}$$

for all i . By the assumption, h_B is a composition of Dehn twists along C_i 's. Since the curve $h_{A_0}(C_i)$ intersects some C_j , we have $i(h_B(h_{A_0}(C_i)), h_{A_0}(C_i)) \geq 1$. Thus, the inequality

$$\frac{\lambda b_0}{\text{mod}(R_i)} = \frac{\lambda(b_0 c_1)^2 H_i}{b_0 c_1^2 W_i} \leq \alpha \lambda (b_0 c_1)^2 < (\lambda \mu)^2$$

holds for each i . □

In the proof of Lemma 3.7, we show that $N_i = \alpha b_0 / \text{mod}(R_i)$ if N_i is the number of twists of $h_B|_{R_i}$. Thus, we obtain the following lemma which is used in section 4.

Lemma 3.8. *The numbers of twists of h_B along core curves of R_1, \dots, R_m are less than $(\lambda \mu)^2$.*

Hereafter, let $h_A = h_{A_0}^{-1} \circ h_B \circ h_{A_0} \in \text{Aff}^+(X, u)$ be the same as in the proof of Lemma 3.7.

Lemma 3.9. *If $i(h_A(C_i), C_j) \neq 0$ or $i(C_i, h_A(C_j)) \neq 0$, we have*

$$(\lambda \mu)^{-2} < \frac{H_j}{H_i} < (\lambda \mu)^2$$

and

$$(\lambda \mu)^{-4} < \frac{W_j}{W_i} < (\lambda \mu)^4.$$

Proof. Suppose that $i(h_A(C_i), C_j) \neq 0$. Then, we have

$$\frac{H_j}{H_i} \leq \frac{1}{H_i} \sum_{k=1}^m i(h_A(C_i), C_k) H_k \leq \frac{\alpha b_0 c_1^2 W_i}{H_i} = \alpha (b_0 c_1)^2 \frac{\text{mod}(R_i)}{b_0} < (\lambda\mu)^2$$

and

$$\frac{H_i}{H_j} \leq \frac{1}{H_j} \sum_{k=1}^m i(C_j, h_A(C_k)) H_k \leq \frac{\alpha b_0 c_1^2 W_j}{H_j} = \alpha (b_0 c_1)^2 \frac{\text{mod}(R_j)}{b_0} < (\lambda\mu)^2$$

by Lemma 3.6 and Lemma 3.7. Moreover, the inequalities

$$\frac{W_j}{W_i} = \frac{W_j}{H_j} \cdot \frac{H_j}{H_i} \cdot \frac{H_i}{W_i} = \frac{\text{mod}(R_j)}{\text{mod}(R_i)} \cdot \frac{H_j}{H_i} < (\lambda\mu)^4$$

and

$$\frac{W_i}{W_j} = \frac{W_i}{H_i} \cdot \frac{H_i}{H_j} \cdot \frac{H_j}{W_j} = \frac{\text{mod}(R_i)}{\text{mod}(R_j)} \cdot \frac{H_i}{H_j} < (\lambda\mu)^4$$

hold for any i, j with $i(h_A(C_i), C_j) \neq 0$.

If $i(C_i, h_A(C_j)) \neq 0$, by the same argument as above, we obtain the claim. \square

Proof of Theorem 1.2. Fix any distinct $i, j \in \{1, \dots, m\}$. Since the surface X is connected, the set $\bigcup_{k=1}^m (C_k \cup h_A(C_k))$ is also connected. Thus, there exists a sequence $i = i_1, \dots, i_{k+1} = j$ such that $k < 2(m-1)$ and $i(h_A(C_{i_p}), C_{i_{p+1}}) \neq 0$ for $p = 1, 2, \dots, k$. Then, we have

$$\frac{H_i}{H_j} = \frac{H_{i_1}}{H_{i_2}} \cdot \frac{H_{i_2}}{H_{i_3}} \cdot \dots \cdot \frac{H_{i_k}}{H_{i_{k+1}}} < (\lambda\mu)^k < (\lambda\mu)^{2(m-1)} < (\lambda\mu)^{2(d-1)}.$$

Moreover, we have

$$\frac{W_i}{W_j} = \frac{W_i}{b_0 H_i} \cdot \frac{H_i}{H_j} \cdot \frac{b_0 H_j}{W_j} = \frac{\text{mod}(R_i)}{b_0} \cdot \frac{H_i}{H_j} \cdot \frac{b_0}{\text{mod}(R_j)} < (\lambda\mu)^{2d}$$

by Lemma 3.7. \square

4. UPPER BOUNDS OF THE NUMBERS OF PERIODIC POINTS

In this section, we show Theorem 1.3. We give upper bounds of the number of periodic points of non-arithmetic Veech surfaces. The basic idea is due to [GHS03]. They study periodic points on prelattice translation surface which is a translation surface (see Remark 2.2) with at least two Jenkins-Strebel directions. Given two distinct Jenkins-Strebel directions of a prelattice translation surface (X, u) . Let $\{R_i\}$ and $\{R'_j\}$ be the cylinder decompositions given by the directions, respectively. The following is the outline of their ideas.

Proposition 4.1 ([GHS03], Theorem 7, Lemma 6 and Corollary 5). *There exists a constant N depending only on some parameters of the decompositions $\{R_i\}$ and $\{R'_j\}$ such that if every cylinder of $\{R_i\}$ and $\{R'_j\}$ contains a periodic point whose period is greater than N , then (X, u) is arithmetic.*

They also proved that one periodic point with large period implies other periodic points with large periods.

Proposition 4.2 ([GHS03], Lemma 5 (ii)). *There exists a constant $c_3 > 0$ and $n_0 \in \mathbb{N}$ depending only on some parameters of the decompositions $\{R_i\}$ and $\{R'_j\}$ such that if one of the cylinders of $\{R_i\}$ (resp. $\{R'_j\}$) contains a periodic point whose period is $c_1 n^2$ with $n \geq n_0$, then there exists a periodic point whose period is n in each cylinder R'_j (resp. R_i) with $R_i \cap R'_j \neq \emptyset$.*

We assume that (X, u) has a periodic point whose period is sufficiently large. By iterating Proposition 4.2, we can conclude that every cylinder of $\{R_i\}$ and $\{R'_j\}$ contains a periodic point whose period is greater than the number N as in Proposition 4.1. Then, (X, u) is arithmetic by Proposition 4.1. This consideration gives the upper bound of the periods of periodic points of non-arithmetic translation surface depending only on some parameters of the two cylinder decompositions.

In this section, we estimate such upper bounds by the numbers d, λ and μ given in (3.1), (3.2) and (3.3). As a result, we also obtain Theorem 1.3. We use the same notations as in Section 3. Let (X, u) be a Veech surface such that the surface X is of type (g, n) with $d = 3g - 3 + n > 0$. We may assume that $\theta = 0$ is a Jenkins-Strebel direction of (X, u) . Then, we take $B_0 = \begin{bmatrix} 1 & b_0 \\ 0 & 1 \end{bmatrix}$ ($b_0 > 0$) to be the primitive element of $\Gamma(X, u)$ which is of the form $\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$ and $h_{B_0} \in D^{-1}(B_0)$. By Lemma 3.5, there exists $\alpha < 2G(2d) = \lambda$ such that $h_B = h_{B_0}^\alpha$ is a composition of the Dehn twists along core curves C_1, \dots, C_m of cylinders R_1, \dots, R_m given by the direction $\theta = 0$. We set $B = D(h_B) = B_0^\alpha$. Theorem 2.14 gives us an element $A_0 = \begin{bmatrix} * & * \\ c_1 & * \end{bmatrix}$ of the Veech group $\Gamma(X, u)$ satisfying $1 \leq |b_0 c_1| < \mu = \text{Area}(\mathbb{H}/\Gamma(X, u)) - k_0 + 1$. Here, k_0 is the number of punctures of $\mathbb{H}/\Gamma(X, u)$. Now, we fix $h_{A_0} \in D^{-1}(A_0)$ and set $h_A = h_{A_0}^{-1} \circ h_B \circ h_{A_0}$. We have

$$A = D(h_A) = A_0^{-1} B A_0 = \begin{bmatrix} * & * \\ -\alpha b_0 c_1^2 & * \end{bmatrix}.$$

In [GHS03], they iterate Proposition 4.2 many times so that every cylinder contains periodic points with large periods. This enlarges the upper bounds of the periods of the periodic points of non-arithmetic translation surfaces. To reduce this, we construct an affine map h_{M_0} of (X, u) such that all pairs of R_i and $h_{M_0}(R_j)$ intersect each other. Recall that m is the number of cylinders given by the horizontal direction.

Proposition 4.3. *Let $h_{M_0} = h_B^{-1} \circ (h_B \circ h_A)^m$. We have*

$$i(h_{M_0}(C_i), C_j) \neq 0$$

for any $i, j \in \{1, \dots, m\}$.

Proof. The map h_{M_0} is constructed by composing h_A and h_B alternately. Fix any $i_0 \in \{1, \dots, m\}$. We see that the number of $j \in \{1, \dots, m\}$ satisfying

$$i(h_B^{-1} \circ (h_B \circ h_A)^k(C_{i_0}), C_j) \neq 0$$

becomes larger as k becomes larger. When k tends to m , the number of such j equals m .

We construct subsets $I_0, \dots, I_m, J_1, \dots, J_m$ of $\{1, \dots, m\}$ inductively as follows:

$$\begin{aligned} I_0 &= \{i_0\}, \\ J_{k+1} &= \{j \in \{1, \dots, m\} \mid i(h_A(C_i), C_j) \neq 0 \text{ for some } i \in I_k\}, \\ I_{k+1} &= \{i \in \{1, \dots, m\} \mid i(h_A(C_i), C_j) \neq 0 \text{ for some } j \in J_{k+1}\}. \end{aligned}$$

By definition, we have $I_0 \subset I_1 \subset \dots \subset I_m$ and $J_1 \subset \dots \subset J_m$. The curve $h_A(C_{i_0})$ intersects C_j for every $j \in J_1$. Since h_B is a composition of the Dehn twists along C_1, \dots, C_m , the curve $h_B h_A(C_{i_0})$ intersects every curve $h_A(C_i)$ intersecting C_j for some $j \in J_1$. That is, the curve $h_B h_A(C_{i_0})$ intersects $h_A(C_i)$ for every $i \in I_1$. Again, since h_A is a composition of the Dehn twists along $h_{A_0}(C_1), \dots, h_{A_0}(C_m)$, the curve $h_A h_B h_A(C_{i_0})$ intersects every curve C_j intersecting $h_A(C_i)$ for some $j \in I_1$. That is, the curve $h_A h_B h_A(C_{i_0})$ intersects C_j for every $j \in J_2$. Continuing this process, we conclude that $h_{M_0}(C_{i_0})$ intersects C_j for every $j \in J_m$.

We show that $J_m = \{1, \dots, m\}$. If $J_k \neq J_{k+1}$ for all k , we have $J_m = \{1, \dots, m\}$. If there exists k such that $J_k = J_{k+1}$, then $J_k = J_{k+1} = \dots = J_m$ and $I_k = I_{k+1} = \dots = I_m$. As the surface X is connected, so is $\bigcup_{i=1}^m (h_{M_0}(C_i) \cup C_j)$. Thus, $J_k = J_{k+1} = \dots = J_m$ must be the set $\{1, \dots, m\}$. \square

We set $M_0 = D(h_{M_0})$ and $h_M = h_{M_0}^{-1} \circ h_B \circ h_{M_0}$. Replacing (X, u) with $(X, U \circ u)$ for some matrix $U = \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}$ we may assume that R_1, \dots, R_m are given by the horizontal direction and $h_{M_0}(R_1), \dots, h_{M_0}(R_m)$ are given by the vertical direction. Note that the number μ is invariant under this deformation since $\Gamma(X, U \circ u) = U\Gamma(X, u)U^{-1}$. Then, we have

$$(4.1) \quad M_0 = D(h_{M_0}) = \begin{bmatrix} 0 & * \\ c & * \end{bmatrix}$$

for some $c \in \mathbb{R}$. Note that the affine map h_M is a composition of Dehn twists along the core curves $h_{M_0}(C_1), \dots, h_{M_0}(C_m)$ of the cylinders $h_{M_0}(R_1), \dots, h_{M_0}(R_m)$. The width and height of the cylinder $h_{M_0}(R_i)$ are cW_i and H_i/c , respectively.

In the following lemma, we estimate the number $|b_0 c|$ by d, λ and μ .

Lemma 4.4. *The $(2, 1)$ -entry c of M_0 satisfies*

$$c = -\alpha b_0 c_1^2 \cdot \frac{\tau_+^m - \tau_-^m}{\tau_+ - \tau_-}.$$

Here, the numbers τ_+ and τ_- ($\tau_+ > \tau_-$) are solutions of the equation $x^2 - (2 - \alpha^2 b_0^2 c_1^2)x + 1 = 0$. In particular, we have

$$|b_0 c| < \mu (\lambda \mu)^{2d-1}.$$

Proof. Let $A_{2,2}$ be the $(2, 2)$ -entry of A . By computation, we see that

$$BA = \begin{bmatrix} 1 + \alpha b_0 c A_{2,2} - \alpha^2 b_0^2 c_1^2 & \alpha b_0 (1 + A_{2,2}^2 - \alpha b_0 c A_{2,2}) \\ -\alpha b_0 c_1^2 & 1 - \alpha b_0 c_1 A_{2,2} \end{bmatrix}$$

and the eigenvalue equation of BA is $x^2 - (2 - \alpha^2 b_0^2 c_1^2)x + 1 = 0$. Let τ_+, τ_- ($\tau_+ > \tau_-$) be the solutions of this equation. Setting

$$P = \begin{bmatrix} \alpha b_0 c_1 A_{2,2} + \tau_+ - 1 & \alpha b_0 c_1 A_{2,2} + \tau_- - 1 \\ -\alpha b_0 c_1^2 & -\alpha b_0 c_1^2 \end{bmatrix},$$

we have

$$BA = P \begin{bmatrix} \tau_+ & 0 \\ 0 & \tau_- \end{bmatrix} P^{-1}.$$

Thus, we also have

$$M_0 = B^{-1} (BA)^m = \begin{bmatrix} * & * \\ -\alpha b_0 c_1^2 \cdot \frac{\tau_+^m - \tau_-^m}{\tau_+ - \tau_-} & * \end{bmatrix}$$

and

$$|c| = \alpha b_0 c_1^2 \left| \frac{\tau_+^m - \tau_-^m}{\tau_+ - \tau_-} \right| \leq \alpha b_0 c_1^2 (|\tau_+| + |\tau_-|)^{m-1} \leq \alpha b_0 c_1^2 (\alpha^2 b_0^2 c_1^2 - 2)^{m-1}.$$

The constants α, b_0 and c_1 are taken so that $\alpha < \lambda$ and $1 \leq b_0 c_1 < \mu$. By Remark 2.7, we have

$$|b_0 c| < \mu (\lambda \mu)^{2d-1}$$

as claimed. \square

We can estimate the intersection number $i(h_{M_0}(C_i), C_j)$ by the numbers d, λ and μ .

Lemma 4.5. *We have*

$$0 < i(h_{M_0}(C_i), C_j) < (\lambda \mu)^{2d-1}$$

for all i, j .

Proof. By Theorem 1.2, Lemma 3.6, Proposition 4.3 and Lemma 4.4, we have

$$i(h_{M_0}(C_i), C_j) \leq \frac{1}{H_j} \sum_{k=1}^m i(h_{M_0}(C_i), C_k) H_k = |b_0 c| \cdot \frac{W_i}{b_0 H_i} \cdot \frac{H_i}{H_j} < (\lambda \mu)^{2d-1}$$

as claimed. \square

Definition 4.6 (G -periodic points and G -periods). Let G be a subgroup of the affine group $\text{Aff}^+(X, u)$. A point $z \in (X, u)$ is a G -periodic point if its G -orbit is finite. The cardinal of the G -orbit is called the G -period. We denote by P_n^G the set of all G -periodic points whose G -periods are less than or equal to n .

We set $G = \langle h_B, h_M \rangle$. If $z \in (X, u)$ is a periodic point, then its G -orbit is also finite. We estimate the number of G -periodic points of non-arithmetic Veech surfaces.

Let $\phi(n) : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$ be the Euler totient function. That is, $\phi(n)$ is the number of positive integers less than or equal to n that are relatively prime to n . We also set $\Phi(n) = \sum_{k=1}^n \phi(k)$. It is known that $\Phi(n) = (3/\pi^2) n^2 + O(n \log n)$ (see [HW08], Theorem 330). In this paper, we estimate it as $\Phi(n) < n^2/2$. We can understand the sets $P_n^{(h_B)}$ and $P_n^{(h_M)}$ by the next lemma.

Lemma 4.7. *For each cylinder R_i (resp. $h_{M_0}(R_i)$), the set $P_n^{(h_B)} \cap R_i$ (resp. $P_n^{(h_M)} \cap h_{M_0}(R_i)$) consists of $N_i \Phi(n)$ horizontal (resp. vertical) closed geodesics. Here, the number N_i is the number of twists of h_B along the core curve C_i of R_i .*

Proof. As $h_M = h_{M_0}^{-1} \circ h_B \circ h_{M_0}$, we only need to prove for the set $P_n^{\langle h_B \rangle} \cap R_i$. We identify the cylinder R_i as the rectangle $[0, W_i) \times [0, H_i]$. Recall that W_i and H_i are the width and height of R_i , respectively. Since $h_B|_{R_i}$ is the N_i -th power of the Dehn twist, we have

$$D(h_B) = \begin{bmatrix} 1 & N_i \bmod(R_i) \\ 0 & 1 \end{bmatrix}.$$

Assume that a point $p = (x, y) \in [0, W_i) \times [0, H_i]$ is one whose $\langle h_B \rangle$ -period is k . Since $h_B^k(p) = \begin{bmatrix} 1 & N_i \bmod(R_i) \\ 0 & 1 \end{bmatrix}^k \begin{bmatrix} x \\ y \end{bmatrix}$, there exists $l \in \mathbb{N}$ such that $(k, l) = 1$ and $kN_i \bmod(R_i)y = lW_i$. This implies that $l/(kN_i) = y/H_i \leq 1$. Thus, the integer l is of the form $l = qk + r$ for some $q \in \{0, \dots, N_i - 1\}$ and $r \in \{1, \dots, k - 1\}$ with $(k, r) = 1$. This implies that the set of all points whose $\langle h_B \rangle$ -periods are k consists of $N_i\phi(k)$ horizontal closed curves. Thus, we obtain the claim. \square

The next proposition estimates the cardinals of the sets P_n^G and $P_n^{\langle h_B \rangle} \cap P_n^{\langle h_M \rangle}$.

Proposition 4.8. *Set $F_1(n) = \frac{1}{4}d^2(\lambda\mu)^{2d+3}n^4$. Then, we have*

$$\#P_n^G \leq \#(P_n^{\langle h_B \rangle} \cap P_n^{\langle h_M \rangle}) < F_1(n).$$

Proof. The first inequality is clear since $P_n^G \subset P_n^{\langle h_B \rangle} \cap P_n^{\langle h_M \rangle}$. By Lemma 3.8, Lemma 4.5 and Lemma 4.7, we have

$$\begin{aligned} \#(P_n^{\langle h_B \rangle} \cap P_n^{\langle h_M \rangle}) &\leq \sum_{i,j=1}^m \# \left\{ (P_n^{\langle h_B \rangle} \cap R_i) \cap (P_n^{\langle h_M \rangle} \cap h_{M_0}(R_j)) \right\} \\ &= \sum_{i,j=1}^m i(C_i, h_{M_0}(C_j)) N_i \Phi(n) N_j \Phi(n) \\ &< \frac{1}{4} d^2 (\lambda\mu)^{2d+3} n^4 = F_1(n) \end{aligned}$$

as desired. \square

From the above proposition, we obtain the following lemma.

Lemma 4.9. *Let $z_0 \in (X, u)$ be a G -periodic point. If G -period of z_0 is greater than or equal to $F_1(n)$, then there exists $z \in G\{z_0\}$ whose $\langle h_B \rangle$ -period or $\langle h_M \rangle$ -period is greater than n .*

The following lemma claims that a large $\langle h_B \rangle$ -orbit (resp. $\langle h_M \rangle$ -orbit) of a G -periodic point contains $\langle h_M \rangle$ -periodic (resp. $\langle h_B \rangle$ -periodic) points with large periods. Note that the cylinders R_i and $h_{M_0}(R_j)$ intersect for all $i, j \in \{1, \dots, m\}$ by Proposition 4.3.

Lemma 4.10. *Set $F_2(n) = \frac{1}{2}(\lambda\mu)^{2d+1}n^2$. Let $z_0 \in (X, u)$ be a G -periodic point. Suppose that $G\{z_0\} \cap R_i$ (resp. $G\{z_0\} \cap h_{M_0}(R_j)$) contains a point z whose $\langle h_B \rangle$ -period (resp. $\langle h_M \rangle$ -period) is greater than $F_2(n)$. For all $j \in \{1, \dots, m\}$, there exists $w \in \langle h_B \rangle\{z\} \cap R_i \cap h_{M_0}(R_j)$ (resp. $w \in \langle h_M \rangle\{z\} \cap R_i \cap h_{M_0}(R_j)$) whose $\langle h_M \rangle$ -period (resp. $\langle h_B \rangle$ -period) is greater than n .*

Proof. Let L be a connected component of $\text{Int}(R_i) \cap \text{Int}(h_{M_0}(R_j))$. Let N_i be the number of twists of h_B along the core curve C_i of R_i . By Lemma 4.7, the set $L \cap P_{n-1}^{\langle h_M \rangle}$ consists of $N_j \Phi(n-1)$ vertical segments. Therefore, the set $\langle h_B \rangle \{z\} \cap L \cap P_{n-1}^{\langle h_M \rangle}$ contains at most $N_j \Phi(n-1)$ points. We show that $\langle h_B \rangle \{z\} \cap L$ contains more than $N_j \Phi(n-1)$ points. Then, we conclude that the $\langle h_M \rangle$ -period of one of the points in $\langle h_B \rangle \{z\} \cap L$ must be greater than n .

Assume that there exists $z \in G\{z_0\} \cap R_i$ whose $\langle h_B \rangle$ -period is greater than $F_2(n)$. The distance between two points of $\langle h_B \rangle \{z\}$ which are adjacent to each other is less than $W_i/F_2(n)$. Fix any $j \in \{1, \dots, m\}$. By Proposition 4.3, the cylinder $h_{M_0}(R_j)$ intersects R_i . Every connected component L of $\text{Int}(R_i) \cap \text{Int}(h_{M_0}(R_j))$ is a rectangle whose horizontal edge has length $H_j/|c|$. By Theorem 1.2, Proposition 3.7 and Lemma 3.8, we have

$$\begin{aligned}
 \sharp(\langle h_B \rangle \{z_0\} \cap L) &> \frac{H_j}{|c|} \cdot \frac{F_2(n)}{W_i} \\
 &= \frac{1}{|cb_0|} \cdot \frac{b_0}{\text{mod}(R_i)} \cdot \frac{H_j}{H_i} \cdot F_2(n) \\
 &> \mu^{-1} \lambda^{-1} (\lambda \mu)^{-2(d-1)} F_2(n) \\
 &= \frac{1}{2} (\lambda \mu)^2 n^2 \\
 &> N_j \Phi(n-1).
 \end{aligned}$$

Hence, we obtain the claim. \square

The next proposition is a consequence of Lemma 4.9 and Lemma 4.10. We see that a G -periodic point whose period is sufficiently large induces periodic points with large periods in all cylinders R_1, \dots, R_m or $h_{M_0}(R_1), \dots, h_{M_0}(R_m)$.

Proposition 4.11. *If the G -period of a point $z_0 \in (X, u)$ is greater than $F_1 \circ F_2(n)$, then one of the following holds:*

- (i) *there exists $z_i \in R_i \cap G\{z_0\}$ whose $\langle h_B \rangle$ -period is greater than n for all $i \in \{1, \dots, m\}$,*
- (ii) *there exists $w_j \in h_{M_0}(R_j) \cap G\{z_0\}$ whose $\langle h_M \rangle$ -period is greater than n for all $j \in \{1, \dots, m\}$.*

Proof. Suppose that the G -period of a point $z_0 \in (X, u)$ is greater than $F_1 \circ F_2(n)$. By Lemma 4.9, there exists $z'_0 \in G\{z_0\}$ whose $\langle h_B \rangle$ -period or $\langle h_M \rangle$ -period is greater than $F_2(n)$. To make the argument easy, we assume that $z'_0 \in R_1 \cap h_{M_0}(R_1)$. By Lemma 4.10, if the $\langle h_M \rangle$ -period of z'_0 is greater than $F_2(n)$, every R_i contains a point z_i whose $\langle h_B \rangle$ -period is greater than n . If the $\langle h_B \rangle$ -period of z'_0 is greater than $F_2(n)$, every $h_{M_0}(R_j)$ contains a point w_j whose $\langle h_M \rangle$ -period is greater than n . \square

Next, we observe the relation between periodic points with large periods and the arithmeticity of Veech surfaces. By the equation (4.1) and Lemma 4.4, we recall that h_{M_0} is an affine map whose derivative is $M_0 = D(h_{M_0}) = \begin{bmatrix} 0 & * \\ c & * \end{bmatrix}$ with $c < 0$. The width and height of the cylinder $h_{M_0}(R_i)$ are $|c|W_i$ and $H_i/|c|$, respectively.

Lemma 4.12. *We identify $h_{M_0}(R_i)$ with the rectangle $L = [0, H_i/|c|] \times [0, |c|W_i]$. Let $z = (x, y) \in L$ be a $\langle h_M \rangle$ -periodic point. Then, we have $cx/H_i \in \mathbb{Q}$.*

Proof. Since z is a $\langle h_M \rangle$ -periodic point, $k = \# \langle h_M \rangle \{z\}$ is finite. The affine map h_M is the N_i -th power of the Dehn twist on a cylinder $h_{M_0}(R_i)$ given by the vertical direction. Thus, we have

$$M = \begin{bmatrix} 1 & 0 \\ -N_i \bmod(h_{M_0}(R_i)) & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -c^2 N_i \bmod(R_i) & 1 \end{bmatrix}$$

and

$$M^k \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -kc^2 N_i \bmod(R_i)x + y \end{bmatrix}.$$

This implies that $-kc^2 N_i \bmod(R_i) = cW_i l$ for some $l \in \mathbb{Z}$. Now, we obtain our claim $cx/H_i \in \mathbb{Q}$. \square

Lemma 4.13. *Let L be a connected component of $\text{Int}(R_i) \cap \text{Int}(h_{M_0}(R_j))$. If L contains two G -periodic points z_1 and z_2 with $\langle h_B \rangle \{z_1\} = \langle h_B \rangle \{z_2\}$, then we have $cW_i/H_j \in \mathbb{Q}$.*

Proof. We identify R_i with the rectangle $[0, W_i] \times [0, H_i]$. Let us consider $z_1 = (x_1, y_1), z_2 = (x_2, y_2) \in [0, W_i] \times [0, H_i]$. By the assumption, $y_1 = y_2$. By Lemma 4.12, we have $c(x_1 - x_2)/H_j \in \mathbb{Q}$. Setting $k = \# \langle h_B \rangle \{z_1\}$, the distance between two points of $\langle h_B \rangle \{z_1\}$ which are adjacent to each other is W_i/k . Then, $x_1 - x_2 = tW_i/k$ for some $t \in \mathbb{Z}$. Hence,

$$c \frac{W_i}{H_j} = \frac{W_i}{x_1 - x_2} \cdot \frac{c(x_1 - x_2)}{H_j} = \frac{k}{t} \cdot \frac{c(x_1 - x_2)}{H_j} \in \mathbb{Q}$$

holds. \square

We see from the following lemma that a G -periodic point with sufficiently large period induces commensurability of the widths W_i and heights H_i of the cylinders R_1, \dots, R_m .

Proposition 4.14. *If there exists $z_0 \in (X, u)$ with $\#G\{z_0\} > F_1 \circ F_2((\lambda\mu)^{2d-1})$, then the numbers cW_i/H_j , W_i/W_j and H_i/H_j are rational numbers for all $i, j \in \{1, \dots, m\}$.*

Proof. Suppose that the G -period of $z_0 \in (X, u)$ is greater than $F_1 \circ F_2((\lambda\mu)^{2d-1})$. We see that the G -period of z_0 is sufficiently large such that $G\{z_0\}$ contains periodic points with large periods to which we can apply Lemma 4.13.

By Proposition 4.11, there are two possibilities (i) and (ii) as follows:

- (i) there exists $z_i \in R_i \cap G\{z_0\}$ whose $\langle h_B \rangle$ -period is greater than $(\lambda\mu)^{2d-1}$ for all $i \in \{1, \dots, m\}$,
- (ii) there exists $w_j \in h_{M_0}(R_j) \cap G\{z_0\}$ whose $\langle h_M \rangle$ -period is greater than $(\lambda\mu)^{2d-1}$ for all $j \in \{1, \dots, m\}$.

Fix any i and j . If case (i) holds, then the distance between two points of $\langle h_B \rangle \{z_i\}$ which are adjacent to each other is less than $W_i/(\lambda\mu)^{2d-1}$. By Theorem 1.2 and Proposition 3.7, we have

$$|c| \frac{W_i}{H_j} = |cb_0| \cdot \frac{\bmod(R_i)}{b_0} \cdot \frac{H_i}{H_j} < (\lambda\mu)^{2d-1}.$$

Now, we have $W_i/(\lambda\mu)^{2d-1} < H_j/|c|$. This means that every connected component of $\text{Int}(R_i) \cap \text{Int}(h_{M_0}(R_j))$ contains two points that have the same $\langle h_B \rangle$ -orbits. By

Lemma 4.13, we conclude that $cW_i/H_j \in \mathbb{Q}$. If case (ii) holds, then we again see that $cW_i/H_j \in \mathbb{Q}$ by the same argument as in case (i).

Moreover, we have

$$\frac{H_i}{H_j} = \frac{H_i}{cW_i} \cdot \frac{cW_i}{H_j} \in \mathbb{Q}$$

and

$$\frac{W_i}{W_j} = \frac{cW_i}{H_i} \cdot \frac{H_i}{cW_j} \in \mathbb{Q}$$

for any i, j . □

Finally, we prove Theorem 1.3. We show that a G -periodic point with a large period implies commensurability of the widths W_i and heights H_i of the cylinders R_1, \dots, R_m and then the Veech surface (X, u) must be arithmetic. Therefore, we conclude that non-arithmetic Veech surface cannot have periodic points whose periods are too large.

Proof of Theorem 1.3. Let (X, u) be an non-arithmetic Veech surface. Assume that there exists a G -periodic point $z_0 \in (X, u)$ such that $\sharp G\{z_0\} \geq F_1 \circ F_2((\lambda\mu)^{2d-1}) = \frac{1}{64}d^2(\lambda\mu)^{26d-1}$. By Proposition 4.14, H_i/H_j and W_i/W_j are rational numbers for any i, j . Considering $(X, K \circ u)$ for some $K = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ with $k > 0$, the flat surface $(X, K \circ u)$ is constructed by gluing finitely many unit squares. Moreover, there exists a translation surface (Y, u') which is a double covering of $(X, K \circ u)$ and is a finite covering of a torus with at most one branched point. (See Remark 2.2 for the definition of translation surfaces.) By [GJ96], Theorem 2, we conclude that $\Gamma(X, u)$ is commensurable with $\mathrm{PSL}(2, \mathbb{Z})$ and hence, (X, u) is arithmetic. This is a contradiction. Therefore, G -periods of G -periodic points of (X, u) are less than $d^2(\lambda\mu)^{26d-1}/64$. By Proposition 4.8, the number of periodic points of (X, u) is less than $d^{10}(\lambda\mu)^{106d-1}/2^{26}$. □

5. APPLICATION TO HOLOMORPHIC FAMILIES OF RIEMANN SURFACES

Veech groups give us holomorphic families of Riemann surfaces. The holomorphic families of Riemann surfaces are corresponding to Teichmüller curves that are holomorphic local isometry from Riemann surfaces into moduli spaces of Riemann surfaces. In this section, we see that the upper bound of the numbers of periodic points given in Theorem 1.3 is also an upper bound of the number of holomorphic sections of holomorphic families of Riemann surfaces constructed from Veech groups.

Definition 5.1 (Holomorphic families of Riemann surfaces). Let \bar{M} be a two-dimensional complex manifold and A be a one-dimensional analytic subset of \bar{M} or an empty set. Let B be a Riemann surface and $\bar{\pi} : \bar{M} \rightarrow B$ a proper holomorphic map satisfying the following two conditions:

- (1) setting $M = \bar{M} - A$ and $\pi = \bar{\pi}|_M$, the holomorphic map π is of maximal rank at every point of M ,
- (2) the fiber $X_t = \pi^{-1}(t)$ over each $t \in B$ is a Riemann surface of fixed finite type (g, n) with $3g - 3 + n > 0$.

We call such a triple (M, π, B) a holomorphic family of Riemann surfaces of type (g, n) over the base space B .

Let (M, π, B) be a holomorphic family of Riemann surfaces of type (g, n) over B . Let $M(g, n)$ be the moduli space of Riemann surfaces of type (g, n) . We have the holomorphic map $f : B \ni t \mapsto X_t = \pi^{-1}(t) \in M(g, n)$. Fix a base point $t_0 \in B$ and set $X = \pi^{-1}(t_0)$. The Teichmüller space $T(X)$ is the lift of the moduli space $M(g, n)$.

Definition 5.2 (Representations). A lift $\tilde{f} : \mathbb{H} \rightarrow T(X)$ of the holomorphic map $f : B \rightarrow M(g, n)$ to the universal coverings of B and $M(g, n)$ is called a representation of (M, π, B) in $T(X)$.

Definition 5.3 (Local triviality and local non-triviality). A holomorphic family of Riemann surfaces (M, π, B) is locally trivial if the induced map $f : B \rightarrow M(g, n)$ is constant and is locally non-trivial if the induced map $f : B \rightarrow M(g, n)$ is non-constant.

Definition 5.4 (Holomorphic sections). Let (M, π, B) be a holomorphic family of Riemann surfaces over B . A holomorphic map $s : B \rightarrow M$ is a holomorphic section of (M, π, B) if it satisfies $\pi \circ s = id_B$.

We study a way to construct holomorphic families of Riemann surfaces from Veech groups. Let (X, u) be a flat surface. For each $A \in \mathrm{SL}(2, \mathbb{R})$, we obtain a flat surface $(X, A \circ u)$ which is not conformal equivalent to (X, u) in general.

Lemma 5.5. *For $A \in \mathrm{SL}(2, \mathbb{R})$ and $U \in \mathrm{SO}(2)$, the flat surfaces $(X, A \circ u)$ and $(X, UA \circ u)$ are conformal equivalent.*

Proof. The identity map $id_X : (X, A \circ u) \rightarrow (X, UA \circ u)$ is a conformal map between them. \square

From the above lemma, we have the map \tilde{f} from $\mathrm{SO}(2) \backslash \mathrm{SL}(2, \mathbb{R})$ into the Teichmüller space $T(X)$ of X which maps each $\mathrm{SO}(2) \cdot A$ to the Teichmüller equivalence class $[(X, A \circ u), id_X]$. Let us consider $A \in \mathrm{SL}(2, \mathbb{R})$ as a Möbius transformation acting on \mathbb{H} . Identifying $\mathrm{SO}(2) \backslash \mathrm{SL}(2, \mathbb{R})$ with \mathbb{H} by the bijection $\mathrm{SO}(2) \cdot A \mapsto -\overline{A^{-1}(i)}$, we have the map $\tilde{f} : \mathbb{H} \rightarrow T(X)$.

Proposition 5.6. *The map $\tilde{f} : \mathbb{H} \rightarrow T(X)$ is a holomorphic isometry with respect to the hyperbolic and Teichmüller metric.*

For the proof, see [HS07], Proposition 2.11. Holomorphic isometry from the upper half-plane \mathbb{H} into a Teichmüller space $T(X)$ is called a Teichmüller disk. Proposition 5.6 claims that flat surfaces give Teichmüller disks. By Teichmüller's theorem, it is also true that every Teichmüller disk is given by a flat surface. See [EG97], Section 5 and [HS07], Section 2.

To construct holomorphic families of Riemann surfaces from Veech groups, we consider the images of Teichmüller disks into the moduli spaces of Riemann surfaces. Let X be a surface of type (g, n) , u a flat structure on X and $\tilde{f} : \mathbb{H} \rightarrow T(X)$ the Teichmüller disk constructed from the flat surface (X, u) . The moduli space $M(g, n)$ of Riemann surfaces of type (g, n) is the quotient of $T(X)$ by the mapping class group $\mathrm{Mod}(X)$. Hence, the image of the Teichmüller disk $\Delta = \tilde{f}(\mathbb{H})$ into the moduli space is represented by $\Delta/\mathrm{Stab}(\Delta)$. Here, the stabilizer $\mathrm{Stab}(\Delta)$ of Δ is the subgroup of $\mathrm{Mod}(X)$ consisting of all mapping classes preserving Δ .

Theorem 5.7. *Let $R = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\bar{\Gamma}(X, u) = R \cdot \Gamma(X, u) \cdot R^{-1}$. The stabilizer $\text{Stab}(\Delta)$ coincides with $\text{Aff}^+(X, u)$. The pull-back of the action on $h \in \text{Aff}^+(X, u)$ through \tilde{f} is the action of the matrix $R \cdot D(h) \cdot R^{-1}$ on \mathbb{H} as a Möbius transformation. In particular, $\Delta/\text{Stab}(\Delta)$ coincides with the orbifold $\mathbb{H}/\bar{\Gamma}(X, u)$.*

For the proof, see [EG97], Theorem 1 and Remarks on page 173 and [HS07], Corollary 2.21.

Remark 5.8. It is known that any two affine maps of a flat surface (X, u) are not homotopic to each other (see [Vee89], Proposition 2.5 and [EG97], Lemma 5.2). Thus, the affine group $\text{Aff}^+(X, u)$ is regarded as a subgroup of the mapping class group $\text{Mod}(X)$.

Remark 5.9. The discreteness of Veech groups is due to Theorem 5.7.

By Theorem 5.7, the Teichmüller disk $\tilde{f} : \mathbb{H} \rightarrow T(X)$ is projected to a map $f_0 : \mathbb{H}/\bar{\Gamma}(X, u) \rightarrow M(g, n)$ which is a holomorphic local isometry between the hyperbolic and Teichmüller metric. Assume that (X, u) is a Veech surface. Then, the orbifold $\mathbb{H}/\bar{\Gamma}(X, u)$ is of finite hyperbolic area. It is not true in general that $\mathbb{H}/\bar{\Gamma}(X, u)$ is a Riemann surface. However, composing a finite covering ϕ from a Riemann surface B of finite type onto $\mathbb{H}/\bar{\Gamma}(X, u)$, we have a holomorphic local isometry $f = \phi \circ f_0 : B \rightarrow M(g, n)$ which is called a Teichmüller curve. Then, each point $t \in B$ corresponds to a conformal equivalence class X_t of Riemann surfaces. Now, we can construct a holomorphic family (M, π, B) of Riemann surfaces over B such that $M = \{(t, p) : t \in B, p \in f(t) = X_t\}$ and $\pi : M \ni (t, p) \mapsto t \in B$. Such holomorphic families of Riemann surfaces are ones constructed from Veech groups. Teichmüller disks are representations of such holomorphic families of Riemann surfaces. Note that every fiber of such a holomorphic family of Riemann surfaces coincides with a flat surface $(X, A \circ u)$ for some $A \in \text{SL}(2, \mathbb{R})$. In particular, the equivalence class $[i] \in \mathbb{H}/\bar{\Gamma}(X, u)$ of the imaginary unit $i \in \mathbb{H}$ is the original flat surface (X, u) .

Let (X, u) be a Veech surface such that X is a surface of type (g, n) . Let (M, π, B) be a holomorphic family of Riemann surfaces constructed from the Veech group $\Gamma(X, u)$. Given a base point $t_0 = [i]$ of B . Then, the fiber $\pi^{-1}(t_0)$ coincides with (X, u) . Denote by \mathcal{S} the family of all holomorphic sections of (M, π, B) . Möller ([Möl06], Lemma 1.2) showed that a point of (X, u) is a periodic point if and only if there exists a holomorphic section of a holomorphic family of Riemann surfaces obtained from (X, u) which pass through the point on the fiber (X, u) . We also showed that a holomorphic section of a holomorphic family of Riemann surfaces obtained from (X, u) corresponds to a point satisfying certain condition ([Shi14b], Corollary 5.6). The condition implies that holomorphic sections correspond to periodic points of (X, u) . The section is realized as the image of the point by Teichmüller deformations around the base point t_0 of B . Hence, holomorphic sections of such holomorphic families of Riemann surfaces do not intersect each other. As a result, we obtain the following.

Theorem 5.10. *The map $\mathcal{S} \ni (s : B \rightarrow M) \mapsto s(t_0) \in (X, u)$ is injective. Moreover, the point $s(t_0)$ is a periodic point of (X, u) .*

From this theorem, our upper bound of the numbers of periodic points in Theorem 1.3 estimates the numbers of holomorphic sections of the holomorphic family of

Riemann surfaces constructed from Veech groups of non-arithmetic Veech surfaces. Thus, we obtain Theorem 1.4.

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