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MATING, PAPER FOLDING, AND AN ENDOMORPHISM OF $\mathbb{PC}^2$

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In memory of Vitaly Sushchansky

Abstract. We are studying topological properties of the Julia set of the map $F(z,p) = \left(\left(\frac{2z}{p+1} - 1\right)^2, \left(\frac{p-1}{p+1}\right)^2\right)$ of the complex projective plane $\mathbb{PC}^2$ to itself. We show a relation between this rational function and an uncountable family of "paper folding" plane filling curves.

Contents

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1. Introduction

We study the topology of the Julia set of the map

$$F(z,p)=\left(\left(\frac{2z}{p+1}-1\right)^2,\left(\frac{p-1}{p+1}\right)^2\right)$$

of the projective plane $\mathbb{PC}^2$ to itself. In particular, we show an interesting connection between this map and an uncountable class of plane-filling curves coming from folding a strip of paper. One of the goals of the paper is to show how techniques of iterated monodromy groups can be used to obtain interesting topological properties of dynamical systems. All our results are proved using algebraic computations with self-similar groups.

Let us start from the end of the article, and describe the plane-filling curves. Take a long narrow strip of paper and fold it in two. Then repeat the procedure several times. Note that each time you have a choice of two directions to fold. Then unfold it so that you get right angles at the creases. You will get something as shown in Figure 1.

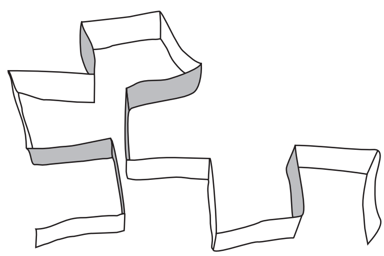

Figure 1. Folded paper

Let us record the way we folded the paper as a sequences of letters `L`, `R`, standing for "left" and "right", respectively.

Now fold two equal strips of paper in the same way (described by the same sequence of letters `L`, `R`), and rotate one with respect to the other by 180°. Put them down so that their endpoints touch; see Figure 2.

You will get a closed curve γ, bounding a connected maze of square rooms. See two examples of such mazes in Figure 3. The rooms are shaded black. Two red dots mark the endpoints of the strips of paper, and the green dots mark their midpoints (i.e., the creases of the first folding).

The marked points are vertices of a square (this easily follows from the construction). Let us choose an infinite sequence $w=X_1X_2\dots$ of letters `L` and `R`. Let us draw rescaled closed curves γ_{w_n} corresponding to finite sequences $w_n=X_1X_2\dots X_n$ of instructions in such a way that the marked points stay at the vertices of a fixed square Q. Let us parametrize the curves γ_{w_n} uniformly (proportionally to the arclength) by $t\in[0,1]$, so that $\gamma_{w_n}|_{[0,1/2]}$ and $\gamma_{w_n}|_{[1/2,1]}$ are the two folded paper strips. Then the vertices of Q are $\gamma_{w_n}(0)$, $\gamma_{w_n}(1/2)$ (the endpoints of the strips), and $\gamma_{w_n}(1/4)$, $\gamma_{w_n}(3/4)$ (their midpoints).

It follows from the description of the folding procedure that the maze $\gamma_{w_{n+1}}$ is obtained from the maze γ_{w_n} by replacing each wall by a corner (so that the old wall is the hypotenuse and the new walls are legs of an isosceles right triangle). Note

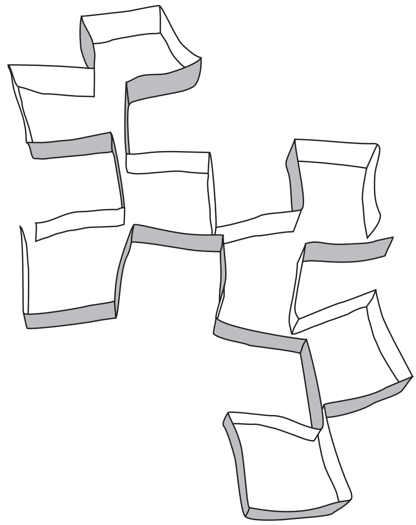

FIGURE 2. Two strips

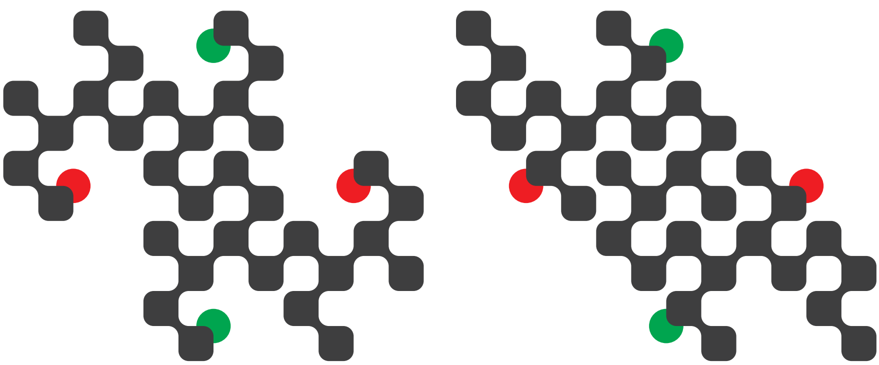

FIGURE 3. Mazes

that there are two ways of doing it (depending on the last letter of w_{n+1}). We then get a sequence of curves converging uniformly to some limit curve $\gamma_{X_1X_2\ldots}$.

The best known and studied example is the *Heighway dragon curve*, which corresponds to a constant sequence $w = \mathtt{LLL}\cdots$ (or $\mathtt{RRR}\cdots$). It was defined for the first time by NASA physicists J. Heighway, B. Banks, and W. Harter, and popularized by M. Gardner in "Scientific American." It is also called sometimes the "Jurassic Park Fractal", as the curves $\gamma_{\mathtt{L}^n}|_{[0,1/2]}$ appear at the beginning of each chapter of *Jurassic Park* by M. Crichton. The closed version γ_w is called sometimes the *twin-dragon curve*. See the images of these curves in Figure 4. In [8, p. 190] a relation of the twin-dragon curve to numeration systems on complex numbers is discussed.

Consider the group H of transformations of the plane generated by rotations by 180° around the vertices of the square Q (recall that it is the square whose vertices are the endpoints and the midpoints of the strips of paper). A fundamental domain of the group H is the square Q' of a twice bigger area such that the vertices of Q are the midpoints of the sides of Q'. The quotient $\mathbb{R}^2/H$ of the plane by the action

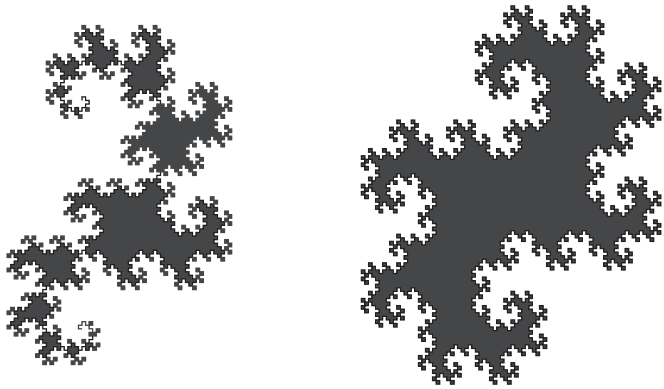

FIGURE 4. Dragon and twin-dragon curves

of H is homeomorphic to the sphere and can be realized as the pillowcase obtained from the square Q' by folding its corners over the sides of the square Q.

If we take the curve γ_{w_n}, then its image γ_{w_n}/H on the pillowcase $\mathbb{R}^2/H$ is a nice Eulerian path tracing a square grid on the pillow; see Figure 5. The figure shows how pieces of the curve γ_{w_n} that are outside the fundamental domain Q' are moved inside Q by elements of H (actually, just by the generators).

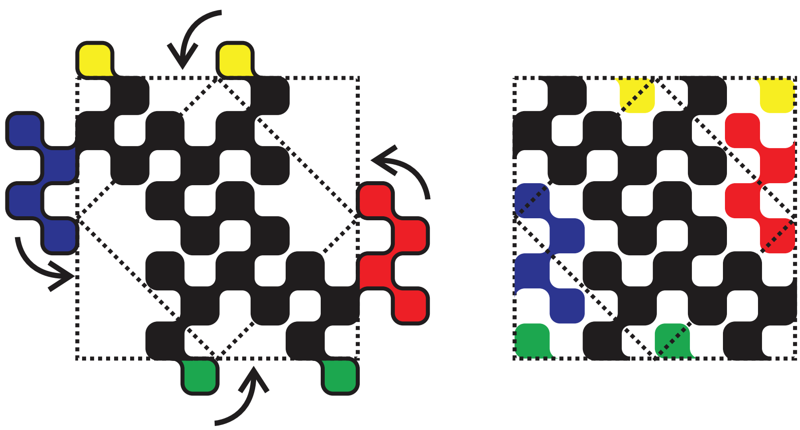

FIGURE 5. Curve on the pillowcase $\mathbb{R}^2/H$

For every infinite sequence $w = X_1X_2\ldots$ the image γ_w/H of the curve γ_w is a curve passing through every point of the sphere $\mathbb{R}^2/H$.

It follows directly from the construction that the curves $\gamma_{X_1X_2\ldots X_n}|_I$ for $I = [0,1/4], [1/4,1/2], [1/2,3/4], [3/4,1]$ are similar (with the similarity coefficient $1/\sqrt{2}$) to the curves $\gamma_{X_2X_3\ldots X_n}|_{[0,1/2]}$ and $\gamma_{X_2X_3\ldots X_n}|_{[1/2,1]}$. The partition of $\gamma_{X_1X_2\ldots X_n}$ into the above-defined sub-curves corresponds to splitting the original strip of paper in two (and "forgetting" about the first folding). Moreover, the similarities $\gamma_{X_1X_2\ldots X_n}|_{[0,1/4]} \longrightarrow \gamma_{X_2X_3\ldots X_n}|_{[0,1/2]}$, $\gamma_{X_1X_2\ldots X_n}|_{[1/4,1/2]} \longrightarrow \gamma_{X_2X_3\ldots X_n}|_{[1/2,1]}$, $\gamma_{X_1X_2\ldots X_n}|_{[1/2,3/4]} \longrightarrow \gamma_{X_2X_3\ldots X_n}|_{[0,1/2]}$, $\gamma_{X_1X_2\ldots X_n}|_{[3/4,1]} \longrightarrow \gamma_{X_2X_3\ldots X_n}|_{[1/2,1]}$ are restrictions of a branched self-covering $S_{X_1} : \mathbb{R}^2/H \longrightarrow \mathbb{R}^2/H$. See Figure 6 for a description of S_{X_1}. In the limit S_{X_1} induces a piecewise similarity map from $\gamma_{X_1X_2\ldots}$ to $\gamma_{X_2X_3\ldots}$.

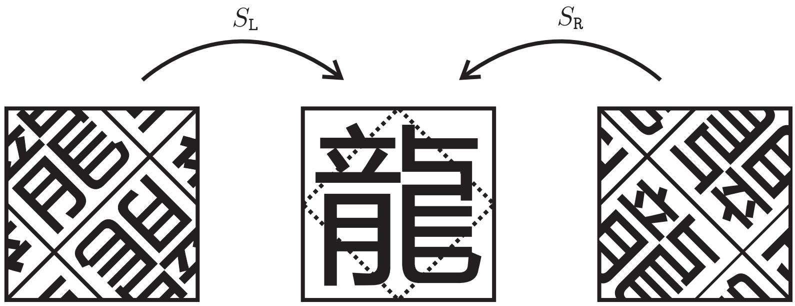

FIGURE 6. Pillowcase folding

Each of the curves $\gamma_{X_1X_2\ldots X_n}$ divides the sphere into two parts: it separates squares of different colors of a checkerboard coloring of the pillow $\mathbb{R}^2/\mathbb{Z}^2$. The squares of one color are connected to each other by the corridors of the maze in a tree-like fashion. It is reasonable to assume that in the limit the curve $\gamma_{X_1X_2\ldots}$ goes around a dendrite that is a limit of the sequence of the trees bounded by $\gamma_{X_1X_2\ldots X_n}$. See, for example [11, II.7], where the relation between plane-filling curves and the "rivers" they bound is explored.

The map S_{X_1} induces a degree two branched covering of the dendrite bounded by $\gamma_{X_2X_3\ldots}$ by the dendrite bounded by $\gamma_{X_1X_2\ldots}$.

The case of the twin-dragon curve was analyzed in detail by J. Milnor in [12]. He showed that the dendrites into which the curve $\gamma_{\mathtt{RRR}\ldots}/H$ separates the sphere can be naturally identified with the Julia set of the polynomial $f(z) = z^2+c$, where $c \approx -0.228 + 1.115i$ is a root of the polynomial $c^3 + 2c^2 + 2c + 2$. Moreover, the curve $\gamma_{\mathtt{RRR}\ldots}/H$ goes around the dendrite in the same way as the *Caratheodory loop* goes around the Julia set, so that the sphere $\mathbb{R}^2/H$ is the *mating* of the polynomial $z^2 + c$ with itself: it is obtained by gluing two Julia sets along the Caratheodory loops so that one loop is a complex conjugate of the other.

The general paper folding curves γ_w were studied in [6], where a connection with numeration systems on complex numbers was described (it is closely related to the "pillowcase folding" maps $S_{\mathtt{L}}, S_{\mathtt{R}} : \mathbb{R}^2/H \longrightarrow \mathbb{R}^2/H$).

We show in our paper a relation of the sphere-filling curves γ_w with the dynamics of the map

$$F(z,p) = \left(\left(\frac{2z}{p+1} - 1\right)^2, \left(\frac{p-1}{p+1}\right)^2\right)$$

of the projective plane $\mathbb{PC}^2$ to itself. Note that F on the second coordinate is the rational function $f : p \mapsto \left(\frac{p-1}{p+1}\right)^2$, whereas on the first coordinate the iterations of F are compositions of the polynomials $h_p : z \mapsto \left(\frac{2z}{p+1} - 1\right)^2$, where p runs through a forward orbit of iterations of f. Intersections of the Julia set of F with the lines $p = p_0$ are then the Julia sets of the non-autonomous iterations $\mathbb{C} \xrightarrow{h_{p_0}} \mathbb{C} \xrightarrow{h_{p_1}} \mathbb{C} \xrightarrow{h_{p_2}} \cdots$, where $p_{n+1} = \left(\frac{p_n-1}{p_n+1}\right)^2$. These Julia sets, which we will denote $J(p_0)$, are dendrites (see Figure 8) if p_0 belongs to the Julia set of f. Denote by J the union of the Julia sets $J(p_0)$ for p_0 running through the set of points of the Julia set of $f : p \mapsto \left(\frac{p-1}{p+1}\right)^2$. It coincides with the support of the measure of maximal entropy for the map F.

We show that the dendrites $J(p_0)$ are precisely the dendrites around which the loops γ_w go. Moreover, the pillow $\mathbb{R}^2/\mathbb{Z}^2$ can be obtained by gluing two copies of $J(p_0)$ along the Caratheodory loop going around $J(p_0)$ (one loop is glued to the other using reflection with respect to a diameter). The curve γ_w is the image of the Caratheodory loop. Moreover, the construction is dynamical: the map h_{p_0} agrees with the double coverings $\gamma_{X_1 X_2 \ldots} \longrightarrow \gamma_{X_2 X_3 \ldots}$.

The paper is organized as follows. We start with a short reminder of the main notions and techniques of self-similar groups. Section 3 is devoted to the study of dynamics of the endomorphism $F : \mathbb{PC}^2 \longrightarrow \mathbb{PC}^2$. We start with computation of the iterated monodromy group of F in Theorem 3.3. It is a self-similar group acting on a degree 4 rooted tree. A computationally more convenient group is an index two extension of $\mathrm{IMG}(F)$, which is defined in Subsection 3.2. We denote it $\overline{\mathcal{G}}$, and it is the iterated monodromy group of the quotient of the dynamical system $F : \mathbb{PC}^2 \longrightarrow \mathbb{PC}^2$ by complex conjugation automorphism.

The group $\mathrm{IMG}(F)$ contains a natural subgroup $\mathcal{G}$ corresponding to the non-autonomous iterations h_p of polynomials on the first coordinate. It is not transitive on the levels of the rooted tree, and we get an uncountable family of quotients of $\mathcal{G}$ coming from restricting of $\mathcal{G}$ to invariant binary subtrees. A similar family of groups was studied in [15, 17, 19, 21].

The group $\mathcal{G}$ corresponds to a dynamical system $\widetilde{F} : \mathcal{J}_{\mathcal{G}} \longrightarrow \mathcal{J}_{\mathcal{G}}$, where $\mathcal{J}_{\mathcal{G}}$ is a bundle of the Julia sets $J(p_0)$ over the Cantor set $\{0,1\}^\omega$ of one-sided binary sequences. It is obtained from J by "exploding" the Julia set of $f(p) = \left(\frac{p-1}{p+1}\right)^2$ into the Cantor set $\{0,1\}^\omega$, while keeping the Julia sets $J(p_0)$ intact. The inclusion $\mathcal{G} \hookrightarrow \mathrm{IMG}(F)$ induces a semiconjugacy $(\widetilde{F}, \mathcal{J}_{\mathcal{G}}) \longrightarrow (F, J)$. The semiconjugacy restricted to any connected component of $\mathcal{J}_{\mathcal{G}}$ is a homeomorphism onto $J(p_0)$ for some p_0. The projection of the semiconjugacy onto $\{0,1\}^\omega$ and onto the variable p of F is a semiconjugacy of the one-sided shift on $\{0,1\}^\omega$ with the action of the rational function $f(p)$ on its Julia set.

The graphs of the action of $\mathcal{G}$ on the levels of binary sub-trees are approximations of the Julia sets $J(p)$ of the non-autonomous iterations of h_p (i.e., the corresponding slices of the Julia set of F). They are trees, in accordance with the fact that the $J(p)$ are dendrites. We study these trees in Subsection 3.4 using a recursive description of the generators of the self-similar group $\mathcal{G}$. In particular, we describe inductive algorithms for constructing these graphs; see Corollaries 3.11 and 3.12.

In the next subsection we study the external angles to the Julia sets $J(p)$, i.e., the Caratheodory loops around them. This is also done using the theory of self-similar groups. Let $\mathcal{O}$ denote the bundle over the Julia set J_f of f of the Caratheodory loops around $J(p_0)$. The corresponding dynamical system $D : \mathcal{O} \longrightarrow \mathcal{O}$ is a skew product acting on the base J_f of the bundle as f, and as double covering maps on the fibers (circles). We have a semiconjugacy $\mathcal{O} \longrightarrow J$ acting identically on the base J_f and mapping the corresponding circles onto the slices $J(p)$ according to the Caratheodory loops.

The iterated monodromy group of the dynamical system $D : \mathcal{O} \longrightarrow \mathcal{O}$ is a subgroup $\mathcal{R}$ of $\mathrm{IMG}(F)$, and $\mathcal{O}$ is naturally homeomorphic to the limit space $\mathcal{J}_{\mathcal{R}}$.

The circle bundle $\mathcal{O} = \mathcal{J}_{\mathcal{R}}$ is a continuous image (homeomorphic on the fibers) of the trivial circle bundle $\mathbb{R}/\mathbb{Z} \times \{0,1\}^\omega$ over the Cantor set $\{0,1\}^\omega$, corresponding to a cyclic subgroup $\langle \tau \rangle$ of $\mathcal{G}$.

We get the following diagram of semiconjugacies of skew-product dynamical systems:

$$\begin{array}{ccc} \mathbb{R}/\mathbb{Z} \times \{0,1\}^\omega & \longrightarrow & \mathcal{O} \\ \big\downarrow & & \big\downarrow \\ \mathcal{J}_{\mathcal{G}} & \longrightarrow & J \end{array},$$

where the vertical arrows correspond to the Caratheodory loops in the fibers of the bundles (and homeomorphic on the bases of the bundles), and the horizontal arrows are homeomorphic on the fibers and are gluing the Cantor set $\{0,1\}^\omega$ together into the Julia set of f. The semiconjugacies are associated with the diagram of group monomorphisms

$$\begin{array}{ccc} \langle \tau \rangle & \longrightarrow & \mathcal{R} \\ \big\downarrow & & \big\downarrow \\ \mathcal{G} & \longrightarrow & \mathrm{IMG}\,(F) \end{array},$$

which is our main tool in the investigation of the semiconjugacies.

We use the group-theoretic information to understand which external angles land on the points of the line $z = p$. In particular, we show that for a countable set of parameters p (equal to the backward orbit of the unique real fixed point of $f(p) = \left(\frac{1-p}{1+p}\right)^2$) there are two external rays to $J(p)$ landing on (p,p), and that in all the other cases such a ray is unique (see Proposition 3.15).

In Section 4 we define matings of the non-autonomous iterations h_p. Namely, for every p in the Julia set of $f(p) = \left(\frac{1-p}{1+p}\right)^2$ consider the corresponding slice $J(p)$ of the Julia set of F and the Caratheodory loop around it. Take then another copy of $J(p)$ and glue them together along the Caratheodory loops so that one loop is a mirror image of the other with respect to the diameter containing an external angle landing on (p,p). Note that in the case when p belongs to the backward orbit of the real fixed point of f, there are two such rays, and we have therefore two possible choices for the mating.

We define the mating and study it in purely algebraic terms. We construct an "amalgam" of the group $\mathcal{G}$ with itself, generating a group $\widehat{\mathcal{G}} = \langle \mathcal{G}_1, \mathcal{G}_2 \rangle$ by two copies of $\mathcal{G}$. Then the inclusion of the two copies of $\mathcal{G}$ induce two semiconjugacies $\mathcal{J}_{\mathcal{G}} \longrightarrow \mathcal{J}_{\widehat{\mathcal{G}}}$ of the limit dynamical systems. We show that these semiconjugacies realize the matings as described in the previous paragraph.

We then study the limit dynamical system of $\widehat{\mathcal{G}}$, i.e., the obtained bundle of the matings. We show that the group $\widehat{\mathcal{G}}$ contains a virtually abelian subgroup $\mathcal{H}$ such that the inclusion $\mathcal{H} \hookrightarrow \widehat{\mathcal{G}}$ induces a conjugacy of the limit dynamical systems. We then show that the limit space $\mathcal{J}_{\widehat{\mathcal{G}}} = \mathcal{J}_{\mathcal{H}}$ is the direct product of the Cantor set $\{\mathtt{L}, \mathtt{R}\}^\infty$ with the sphere $\mathbb{C}/H$, where H is the group of affine transformations of the form $z \mapsto \pm z + a + ib$ for $a, b \in \mathbb{Z}$, i.e., the pillowcase described above. The limit dynamical system $\mathcal{J}_{\widehat{\mathcal{G}}} \longrightarrow \mathcal{J}_{\widehat{\mathcal{G}}}$ acts as the one-sided shift on the Cantor set and as multiplication by $1+i$ or $1-i$ on the pillowcase $\mathbb{C}/H$ (the choice of the coefficient $1 \pm i$ depends on the first letter of the corresponding element of the one-sided shift).

It follows that the constructed matings are *Lattès examples*, though non-autonomous, as we can choose one of the two multiplications.

Since $\mathcal{G} < \widehat{\mathcal{G}}$ the graphs of the action of $\mathcal{G}$ on the levels of the tree are sub-graphs of the graphs of action of $\widehat{\mathcal{G}}$. The graphs of the action of $\widehat{\mathcal{G}}$ are square grids on the pillowcases. We show that the graphs of the action of the two copies of $\mathcal{G}$ partition the edges of the grid into two disjoint sub-trees; see Figure 23. These partitions of a square grid into two subtrees converge to the decomposition of the pillowcase $\mathbb{C}/H$ into two dendrites.

We will relate, in Section 5, the obtained results about the mating with the paper folding curves. Namely, we show that the curve separating the two subtrees of the square grid on $\mathbb{C}/H$ coincide with the paper-folding curves γ_v described in this introduction and that in the limit the curves γ_v converge to the image of the Caratheodory loop in the mating.

Section 6 describes the structure of the slices of the Julia set of F that correspond to the values of p belonging to the boundaries of the Fatou components of $f(p) = \left(\frac{p-1}{p+1}\right)^2$. We show that they are obtained by "flattening" the boundaries of the Fatou components of the polynomials $16z^2(1-z)^2$ and $(2z^2-4z+1)^2$. In other terminology they are obtained by the (rotated) tuning of these polynomials by the polynomial z^2-2. As before, the proof is carried out using just algebraic computations with the iterated monodromy groups.

2. Self-similar groups

We present here, in a very condensed form, the main definitions and results of the theory of self-similar and iterated monodromy groups. For a more detailed account, see [14, 16, 18, 20].

2.1. Covering bisets.

Definition 2.1. Let G be a group. A *G-biset* is a set $\mathfrak{M}$ together with commuting left and right actions of G on $\mathfrak{M}$. In other words, we have two maps $G \times \mathfrak{M} \longrightarrow \mathfrak{M} : (g, x) \mapsto g \cdot x$ and $\mathfrak{M} \times G \longrightarrow \mathfrak{M} : (x, g) \mapsto x \cdot g$ satisfying $1 \cdot x = x \cdot 1 = x$ for all $x \in \mathfrak{M}$ and

$$g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x, \quad (x \cdot g_1) \cdot g_2 = x \cdot (g_1 g_2) \quad (g_1 \cdot x) \cdot g_2 = g_1 \cdot (x \cdot g_2)$$

for all $g_1, g_2 \in G$, $x \in \mathfrak{M}$.

We say that $\mathfrak{M}$ is a *covering biset* if the right action of G on $\mathfrak{M}$ is free, i.e., if $x \cdot g = x$ implies $g = 1$. We also assume then that the number of right orbits is finite.

We also consider $G_1 - G_2$ bisets $\mathfrak{M}$, which are sets with commuting left G_1-action and right G_2-action.

Definition 2.2. The isomorphism class of a pair $(G, \mathfrak{M})$, where G is a group and $\mathfrak{M}$ is a covering G-biset, is called a *self-similar group*. Here two pairs $(G_1, \mathfrak{M}_1)$ and $(G_2, \mathfrak{M}_2)$ are *isomorphic* (the corresponding self-similar groups are called *equivalent*) if there exists an isomorphism $\phi : G_1 \longrightarrow G_2$ and a bijection $f : \mathfrak{M}_1 \longrightarrow \mathfrak{M}_2$ such that $f(g_1 \cdot x \cdot g_2) = \phi(g_1) \cdot f(x) \cdot \phi(g_2)$ for all $g_1, g_2 \in G_1$ and $x \in \mathfrak{M}_1$.

Let $\mathfrak{M}_1, \mathfrak{M}_2$ be G-bisets. Then their *tensor product*, denoted $\mathfrak{M}_1 \otimes \mathfrak{M}_2$, is the quotient of $\mathfrak{M}_1 \times \mathfrak{M}_2$ by the equivalence relation $x_1 \otimes g \cdot x_2 = x_1 \cdot g \otimes x_2$, with the actions $g_1 \cdot (x_1 \otimes x_2) \cdot g_2 = (g_1 \cdot x_1) \otimes (x_2 \cdot g_2)$. It is easy to show that the tensor product of two covering bisets is a covering biset. It is also easy to see that $(\mathfrak{M}_1 \otimes \mathfrak{M}_2) \otimes \mathfrak{M}_3$ is naturally isomorphic to $\mathfrak{M}_1 \otimes (\mathfrak{M}_2 \otimes \mathfrak{M}_3)$.

Let $\mathfrak{M}$ be a covering G-biset. Consider the disjoint union $\mathfrak{M}^* = \bigcup_{n\geq 0} \mathfrak{M}^{\otimes n}$ of the tensor powers $\mathfrak{M}^{\otimes n}$, $n \geq 0$, where $\mathfrak{M}^{\otimes 0}$ is the group G with the natural left and right actions on itself. Denote by $T_{\mathfrak{M}}$ the set of orbits of the right action of G on $\mathfrak{M}^*$. Then G acts on $T_{\mathfrak{M}}$ from the left. The set $T_{\mathfrak{M}}$ has a natural structure of a rooted tree, where the right orbit of $x_1 \otimes x_2 \otimes \cdots \otimes x_n$ is connected by an edge to the right orbit of $x_1 \otimes x_2 \otimes \cdots \otimes x_n \otimes x_{n+1}$ for $x_i \in \mathfrak{M}$. The left action of G on $T_{\mathfrak{M}}$ is an action by automorphisms of the rooted tree (the root is the unique right orbit of the action of G onto itself).

The action of G on $T_{\mathfrak{M}}$ is not faithful in general. Let N be the kernel of the action. Denote then by $\mathfrak{M}/N$ the set of right N-orbits. It is easy to see that $h \cdot x$ and x belong to one right N-orbit for every $x \in \mathfrak{M}$ and $h \in N$, and that the left and right actions of G on $\mathfrak{M}$ descend to left and right actions of G/N on $\mathfrak{M}/N$. We call the self-similar group $(G/N, \mathfrak{M}/N)$ the *faithful quotient* of the self-similar group $(G, \mathfrak{M})$.

2.2. **Wreath recursions and virtual endomorphisms.** Let $(G, \mathfrak{M})$ be a self-similar group. We say that $X = \{x_1, x_2, \ldots, x_d\} \subset \mathfrak{M}$ is a *basis* if it is a transversal of the right G-orbits. In other words, it is such a subset of $\mathfrak{M}$ that for every $x \in \mathfrak{M}$ there exists a unique $x_i \in X$ such that $x = x_i \cdot g$ for some $g \in G$. Note that g is also uniquely determined by x, since the right action is free.

For every n the set $X^n = \{a_1 \otimes a_2 \otimes \cdots \otimes a_n \ : \ a_i \in X\}$ is a basis of $\mathfrak{M}^{\otimes n}$. Consequently, $X^* = \bigcup_{n\geq 0} X^n$ is a basis of $\mathfrak{M}^*$. Here X^0 consists of a single *empty word* identified with the identity element of $G = \mathfrak{M}^{\otimes 0}$. We will usually write $a_1 \otimes a_2 \otimes \cdots \otimes a_n$ just as the word $a_1 a_2 \ldots a_n$. We get a natural bijection between the set $T_{\mathfrak{M}}$ of the right orbits of $\mathfrak{M}^*$ and the set X^* of finite words over X. The vertex adjacency on the tree $T_{\mathfrak{M}}$ corresponds to a similar adjacency of elements of X^*: a word $v \in X^*$ is adjacent to the words of the form vx for $x \in X$. In other words, we consider X^* as the right Cayley graph of the free monoid generated by X. The action of G on $T_{\mathfrak{M}}$ is hence transformed to an action of G on X^*. We call this action the *self-similar action* associated with the basis X of the biset $\mathfrak{M}$.

Let $g \in G$ and $x \in X$; then there exist unique $y \in X$ and $h \in G$ such that $g \cdot x = y \cdot h$. The induced map $x \mapsto y$ is a permutation coinciding with the restriction of the action of g to the first level $X \subset X^*$ of the rooted tree X^*. Let us denote this permutation by $\sigma_g \in Symm(X)$. Denote also $h = g|_x$. We get a map

$$\Phi : g \mapsto \sigma_g (g|_x)_{x \in X}$$

from G to the semidirect product $Symm(X) \ltimes G^X$. The semidirect product is called the *(permutational) wreath product* of the symmetric group $Symm(X)$ with G.

It is easy to check that $\Phi : G \longrightarrow Symm(X) \ltimes G^X$ is a homomorphism. We call it the *wreath recursion* associated with the self-similar group $(G, \mathfrak{M})$ (and the basis X). If we change the basis X to another basis Y and identify G^X and G^Y with $G^{|X|} = G^{|Y|}$ using some bijections $X \longleftarrow \{1, 2, \ldots, d\} \longrightarrow Y$, then the wreath recursions associated with X and Y differ from each other by post-composition with an inner automorphism of the wreath product $Symm(d) \ltimes G^d$.

If the left action of G on the set of right orbits of $\mathfrak{M}$ (i.e., on the first level of the tree $T_{\mathfrak{M}}$) is transitive, then the self-similar group is uniquely determined by the *associated virtual endomorphism*. Let $x \in X$, and denote by G_x the stabilizer of the vertex x of the tree X^*. Then the *associated virtual endomorphism* ϕ_x is the homomorphism $\phi_x : G_x \longrightarrow G$ defined by the condition

$$g \cdot x = x \cdot \phi_x(g).$$

The biset $\mathfrak{M}$ is reconstructed (up to an isomorphism) from the associated virtual endomorphism in the following way. Let $\phi : G_1 \longrightarrow G$ be a virtual endomorphism (where G_1 is a subgroup of finite index in G). Let $\mathfrak{M}_\phi$ be the set of formal expressions $[\phi(g_1)g_2]$, for $g_1, g_2 \in G$, where $[\phi(g_1)g_2] = [\phi(h_1)h_2]$ if and only if $h_1^{-1}g_1 \in G_1$ and $\phi(h_1^{-1}g_1) = h_2g_2^{-1}$. This convention agrees with the identification of an expression $[\phi(g_1)g_2]$ with the partial transformation

$$[\phi(g_1)g_2] : x \mapsto \phi(xg_1)g_2$$

of G. Note that $\phi(xg_1)g_2 = \phi(xh_1)h_2$ is equivalent to $\phi(xg_1) = \phi(xg_1)\phi(g_1^{-1}h_1)h_2g_2^{-1}$, and hence to $\phi(h_1^{-1}g_1) = h_2g_2^{-1}$.

The set $\mathfrak{M}_\phi$ is invariant with respect to pre- and post-composition with the right translations $x \mapsto xg$. We hence get a natural G-biset structure on $\mathfrak{M}_\phi$. It is given by the formulas

$$[\phi(g_1)g_2] \cdot g = [\phi(g_1)(g_2g)], \qquad g \cdot [\phi(g_1)g_2)] = [\phi(gg_1)g_2].$$

It is easy to see that if $\phi = \phi_x$ is the virtual endomorphism associated with a covering biset $\mathfrak{M}$, then $\mathfrak{M}$ is isomorphic to $\mathfrak{M}_\phi$. Since the action of G on the set of right orbits is transitive, every element $y \in \mathfrak{M}$ can be written as $y = g_1 \cdot x \cdot g_2$, and then the isomorphism maps y to $[\phi(g_1)g_2]$.

2.3. **Iterated monodromy groups.** Let M_1, M_0 be path connected and locally path connected topological spaces or orbispaces. A *topological correspondence* is a pair of maps $f, \iota : M_1 \longrightarrow M_0$, where f is a finite degree covering map and ι is a continuous map.

Choose a basepoint $t \in M_0$. Let $\mathfrak{M}$ be the set of pairs (z, ℓ), where $z \in f^{-1}(t)$ and ℓ is the homotopy class of a path in M_0 from t to $\iota(z)$. The set $\mathfrak{M}$ has a natural structure of a covering $\pi_1(M_0, t)$-biset. Namely, for a loop $\gamma \in \pi_1(M_0, t)$, denote by $(z, \ell) \cdot \gamma$ the element $(z, \ell\gamma)$. Here and in the sequel we compose paths as functions: in a product $\ell\gamma$ the path γ is passed first, and then ℓ. Denote by $\gamma \cdot (z, \ell)$ the element $(y, \iota(\gamma_z)\ell)$, where γ_z is the unique lift of γ by f that starts at z and y is the end of γ_z.

A basis of $\mathfrak{M}$ is any set $\{(z_1, \ell_1), (z_2, \ell_2), \ldots, (z_d, \ell_d)\}$, where $d = \deg f$ and $f^{-1}(t) = \{z_1, z_2, \ldots, z_d\}$.

The faithful quotient of the self-similar group $(\pi_1(M_0, t), \mathfrak{M})$ is called the *iterated monodromy group* of the correspondence $f, \iota : M_1 \longrightarrow M_0$.

The virtual endomorphism associated with the biset $\mathfrak{M}$ is equal to $\iota_* : \pi_1(M_1) \longrightarrow \pi_1(M_0)$, where $\pi_1(M_1)$ is identified with a subgroup finite index in $\pi_1(M_0)$ by the isomorphism f_*. It is well defined up to compositions with inner automorphisms of $\pi_1(M_0)$, as is any virtual endomorphism associated with a self-similar group.

If $f : M \longrightarrow M$ is a branched self-covering, then we can transform it into a topological correspondence by removing from M the closure P of the union of the forward orbits of branch points of M. If P is not too big, in particular, if it does not disconnect M, then we can consider the topological correspondence $f, \iota : M_1 \longrightarrow M_0$, where $M_0 = M \setminus P$, $M_1 = f^{-1}(M_0)$, and $\iota : M_1 \longrightarrow M_0$ is the identical embedding. This is done, for example, if f is a post-critically finite rational function or a post-critically finite endomorphism of $\mathbb{PC}^2$ (which means that P is the union of a finite number of varieties). In these cases we represent the elements of $\mathfrak{M}$ just as paths $\ell = \iota(\ell)$, since their endpoints are uniquely determined by ℓ.

2.4. **Contracting self-similar groups.** Let $\mathfrak{M}$ be a covering G-biset, and let X be a basis of $\mathfrak{M}$. For every $v \in X^*$ and $g \in G$ denote by $g|_v$ the unique element of G such that $g \cdot v = u \cdot g|_v$ for some $u \in X^*$. We call it the *section* of g in v.

Definition 2.3. The self-similar group $(G, \mathfrak{M})$ (with a chosen basis X of $\mathfrak{M}$) is said to be *contracting* if there exists a finite set $\mathcal{N} \subset G$ such that for every $g \in G$ there exists n such that $g|_v \in \mathcal{N}$ for every $v \in X^k$ such that $k \geq n$. The smallest set $\mathcal{N}$ satisfying the above condition is called the *nucleus* of the group.

It is proved in [14, Corollary 2.11.7] that the property of a biset to be contracting does not depend on the choice of a basis X. The nucleus, however, depends on X.

Since a biset $\mathfrak{M}$ and a basis are uniquely determined by the associated wreath recursion, we call a wreath recursion $G \longrightarrow Symm(X) \ltimes G^X$ contracting if the corresponding self-similar group is contracting. Sometimes we say that a G-biset $\mathfrak{M}$ is *hyperbolic* if the self-similar group $(G, \mathfrak{M})$ is contracting. It is easy to see that if $(G, \mathfrak{M})$ is contracting, then its faithful quotient is also contracting.

The nucleus $\mathcal{N}$ satisfies the property that $g|_x \in \mathcal{N}$ for all $g \in \mathcal{N}$ and $x \in X$. We will often represent $\mathcal{N}$ as an *automaton* using its *Moore diagram*. It is the oriented graph with the set of vertices $\mathcal{N}$ in which for every $g \in \mathcal{N}$ and $x \in X$ we have an arrow from g to $g|_x$ labeled by $x|y$, where y is the image of x under the action of g on the first level of the tree X^*, i.e., we have $g \cdot x = y \cdot g|_x$.

Let $(G, \mathfrak{M})$ be a contracting self-similar group, and let $X \subset \mathfrak{M}$ be a basis. Consider the space $X^{-\omega}$ of left-infinite sequences $\ldots x_2 x_1$ of elements of X. We say that $\ldots x_2 x_1, \ldots y_2 y_1 \in X^{-\omega}$ are *asymptotically equivalent* if there exists a sequence $g_n \in G$ taking values in a finite subset of G such that $g_n(x_n \ldots x_1) = y_n \ldots y_1$ (with respect to the action of G on X^*). The quotient of the topological space $X^{-\omega}$ by the asymptotic equivalence relation is called the *limit space* of the group $(G, \mathfrak{M})$ and is denoted $\mathcal{J}_G$. The shift $\ldots x_2 x_1 \mapsto \ldots x_3 x_2$ agrees with the asymptotic equivalence relation, so that it induces a continuous map $\mathsf{s} : \mathcal{J}_G \longrightarrow \mathcal{J}_G$. We call the pair $(\mathcal{J}_G, \mathsf{s})$ the *limit dynamical system* of the self-similar group.

One can show (see [14, Theorem 3.6.3]) that two sequences $\ldots x_2 x_1$ and $\ldots y_2 y_1$ are asymptotically equivalent if and only if there exists an oriented path $\ldots e_2 e_1$ of arrows in the Moore diagram of the nucleus such that e_n is labeled by $x_n | y_n$.

Consider now $X^{-\omega} \times G$, where G is discrete. We write elements of the space $X^{-\omega} \times G$ as $\ldots x_2 x_1 \cdot g$ for $x_i \in X$ and $g \in G$. Two sequences $\ldots x_2 x_1 \cdot g$ and $\ldots y_2 y_1 \cdot h$ are said to be asymptotically equivalent if there exists a sequence $g_k \in G$ taking values in a finite subset of G such that $g_n \cdot x_n \ldots x_2 x_1 \cdot g = y_n \ldots y_2 y_1 \cdot h$ in $\mathfrak{M}^{\otimes n}$ for all n. One can show that $\ldots x_2 x_1 \cdot g$ and $\ldots y_2 y_1 \cdot h$ are asymptotically equivalent if and only if there exists an oriented path $\ldots e_2 e_1$ in the Moore diagram of the nucleus such that e_n is labeled by $x_n | y_n$ for every n and the last vertex of the path is hg^{-1}.

The quotient of $X^{-\omega} \times G$ by the asymptotic equivalence relation is called the *limit G-space* and is denoted $\mathcal{X}_G$. The group G acts naturally on $X^{-\omega} \times G$ by the right multiplication. This action agrees with the asymptotic equivalence relation, so that it induces a right action of G on $\mathcal{X}_G$ by homeomorphisms.

For every $x \in \mathfrak{M}$ and $\ldots x_2 x_1 \cdot g \in X^{-\omega} \times G$ the asymptotic equivalence class of $\ldots x_2 x_1 \cdot g \otimes x = \ldots x_2 x_1 y \cdot h$, where $h \in G$ and $x \in X$ are such that $g \cdot x = y \cdot h$, is uniquely determined by x and the asymptotic equivalence class of $\dots x_2 x_1 \cdot g$. It follows that we get a well-defined continuous map $\xi \mapsto \xi \otimes x$ of $\mathcal{X}_G$ to itself.

The biset structure of $\mathfrak{M}$ agrees with the maps $\xi \mapsto \xi \otimes x$ on $\mathcal{X}_G$, so that $(\xi \cdot g_1 \otimes x)\, g_2 = \xi \otimes (g_1 \cdot x \cdot g_2)$ for all $\xi \in \mathcal{X}_G$, $g_1, g_2 \in G$, $x \in \mathfrak{M}$. Moreover, we have the following rigidity theorem; see [14, Theorem 3.4.13].

Theorem 2.1. *Let $\mathfrak{M}$ be a hyperbolic G-biset. Let $\mathcal{X}$ be a metric space such that G acts on $\mathcal{X}$ co-compactly and properly by isometries from the right. Suppose that for every $x \in \mathfrak{M}$ we have a continuous strictly contracting map $\xi \mapsto \xi \otimes x$ such that $(\xi \cdot g_1 \otimes x) \cdot g_2 = \xi \otimes (g_1 \cdot x \cdot g_2)$ for all $\xi \in \mathcal{X}$, $g_1, g_2 \in G$, and $x \in \mathfrak{M}$. Then there exists a homeomorphism $\Phi : \mathcal{X} \longrightarrow \mathcal{X}_G$ such that $\Phi(\xi \cdot g) = \Phi(\xi) \cdot g$ and $\Phi(\xi \otimes x) = \Phi(\xi) \otimes x$ for all $\xi \in \mathcal{X}$, $g \in G$, and $x \in \mathfrak{M}$.*

2.5. **Contracting correspondences.** Let $f, \iota : M_1 \longrightarrow M_0$ be a topological correspondence. Its *limit space* M_∞ is a subspace of all sequences $(x_1, x_2, \dots) \in M_1^\infty$ such that $f(x_n) = \iota(x_{n+1})$. For example, if ι is an identical embedding, then M_∞ is naturally identified (using the first coordinate) with the intersection of the domains of all iterations of the partial map f.

The shift $(x_1, x_2, \dots) \mapsto (x_2, x_3, \dots)$ is a continuous self-map on M_∞, which we will denote f_∞. We call (f_∞, M_∞) the *limit dynamical system* of the topological correspondence.

Definition 2.4. Let $f, \iota : M_1 \longrightarrow M_0$ be a topological correspondence. We say that it is *contracting* if M_0 is a compact length metric space (i.e., there is a notion of length of arcs such that distance between two points is the infimum of lengths of arcs connecting them) and ι is contracting with respect to the length metric on M_0 and the lift of the length metric from M_0 to M_1 by f.

In particular, if f is expanding and ι is an identical embedding, then the correspondence $f, \iota : \mathcal{M}_1 \longrightarrow M_0$ is contracting.

One can show that the iterated monodromy group of a contracting topological correspondence is a contracting self-similar group; see [20].

Theorem 2.2. *The limit dynamical system (f_∞, M_∞) of a contracting topological correspondence $\mathcal{F}$ is topologically conjugate to the limit dynamical system of the iterated monodromy group of $\mathcal{F}$.*

The correspondence between contracting self-similar groups and expanding self-coverings is functorial in a precise way; see [16]. For example, any embedding of self-similar contracting groups (preserving self-similarity) induces a semiconjugacy of their limit dynamical systems.

3. The Julia set of an endomorphism of $\mathbb{PC}^2$

3.1. The endomorphism and its iterated monodromy group.

Consider the following map on $\mathbb{C}^2$:

$$F(z,p)=\left(\left(\frac{2z}{p+1}-1\right)^2,\left(\frac{p-1}{p+1}\right)^2\right).$$

It can be extended to an endomorphism of $\mathbb{PC}^2$ given in homogeneous coordinates by the formula

$$F[z:p:u]=[(2z-p-u)^2:(p-u)^2:(p+u)^2].$$

Note that this map has no points of indeterminacy, since $(2z-p-u)^2=(p-u)^2=(p+u)^2=0$ implies $p=u=z=0$. The Jacobian of the map F is

$$\begin{vmatrix} 4(2z-p-u) & 0 & 0\\ -2(2z-p-u) & 2(p-u) & 2(p+u)\\ -2(2z-p-u) & -2(p-u) & 2(p+u)\end{vmatrix}=32(2z-p-u)(p-u)(p+u),$$

hence the critical locus consists of three lines, $2z-p-u=0$, $p=u$, and $p+u=0$. Their orbits under the action of F are

$$\{2z-p-u=0\}\mapsto\{z=0\}\mapsto\{z=u\}\mapsto\{z=p\}\mapsto\{z=p\},$$
$$\{p=-u\}\mapsto\{u=0\}\mapsto\{p=u\}\mapsto\{p=0\}\mapsto\{p=u\}.$$

We see that the post-critical set of F is the union of the six lines $z=0$, $z=u$, $z=p$, $p=0$, $p=u$, and $u=0$. (Or, in affine coordinates, $z=0$, $z=1$, $z=p$, $p=0$, $p=1$, and the line at infinity.)

The map F is a particular case of a general class of post-critically finite skew-product maps related to the Teichmüller theory of post-critically finite branched self-coverings of the sphere (*Thurston maps*). See [2], where the map F was (somewhat implicitly) constructed, and [4, 10], where other different classes of similar examples are studied.

Denote by J_2 the Julia set of F, i.e., the set of points without neighborhoods on which the sequence $F^{\circ n}$ is normal. Denote by J_1 the support of the measure of maximal entropy of F, which coincides with the attractor of backward iterations of F. Both sets are completely F-invariant and we have $J_1\subset J_2$; see more in [7].

Proposition 3.1. *The limit dynamical system of the iterated monodromy group of F is topologically conjugate with the action of F on J_1.*

Proof. By [14, Theorem 5.5.3] (see also Theorem 2.2 in our paper), it is enough to construct an orbifold metric on a neighborhood of J_1 with respect to which F is expanding. Let U be an open relatively compact subset of $\mathbb{C}\setminus\{0,1\}$ containing the Julia set of the rational function $f(p)=\left(\frac{p-1}{p+1}\right)^2$ and such that $f^{-1}(U)\subset U$. Consider the inverse image W of U in $\mathbb{PC}^2$ under the projection map $(z,p)\mapsto p$ of $\mathbb{C}^2\subset\mathbb{PC}^2$ onto $\mathbb{C}$. Note that the lines $p=0$ and $p=1$ are disjoint from W, hence the intersection points of the lines $z=p$, $z=0$, $z=1$, $p=0$ and $p=1$ do not belong to W. Consider the orbifold with the underlying space W, where the lines $z=0$, $z=1$ and $z=p$ are singular with the isotropy groups of order two uniformized in an atlas of the orbifold as rotations by $180°$ in the z-planes (and projected to the identity map on the p-plane). The function F can be realized as a covering $F:W_1\longrightarrow W$ of a sub-orbispace W_1 of W, where W_1 is the orbifold with the underlying space $F^{-1}(W)$ and singular lines $z = 0$, $z = 1$, $z = p$, $z = p + 1$ of order two (also uniformized by a rotation in the z-planes).

The orbifold W is a locally trivial bundle over the set U with hyperbolic fibers (as the fundamental group of every fiber is the free product of three copies of the group of order two). It follows from Proposition 3.2.2 and Theorem 3.2.15 of [9] that the orbifold W is Kobayashi hyperbolic (i.e., that its universal covering is Kobayashi hyperbolic). Consequently, the embedding of orbifolds $\iota : W_1 \longrightarrow W$ is contracting with respect to the Kobayashi metrics on W_1 and W, while the map $F : W_1 \longrightarrow W$ is a local isometry. Theorem 2.2 finishes the proof. □

An important property of the map F, greatly facilitating its study, is a skew-product structure: the second coordinate $\left(\frac{p-1}{p+1}\right)^2$ of $F(z, p)$ depends only on p. See Figure 7 for the Julia set of $f(p) = \left(\frac{p-1}{p+1}\right)^2$ together with marked post-critical points 0 and 1.

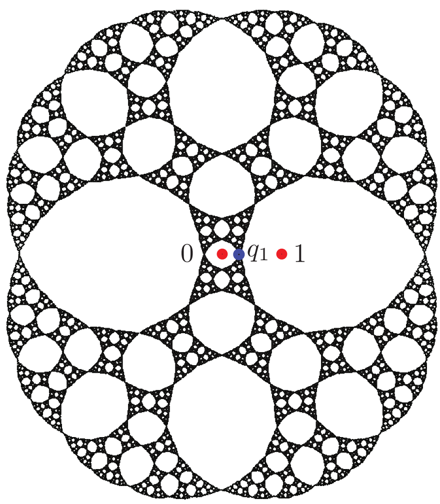

FIGURE 7. Julia set of $\left(\frac{p-1}{p+1}\right)^2$

We have on the first coordinate a quadratic polynomial $h_p(z) = (2z/(p+1)-1)^2$ in z, depending on the parameter p. This makes it possible, in particular, to draw the intersections of the Julia set of F with the z-lines $p = p_0$. See Figures 8 and 9, where some slices of the Julia set of F are shown. We will denote by $J(q)$ the intersection of the Julia set J_1 of F with the line $p = q$.

The rational function $\left(\frac{p-1}{p+1}\right)^2$ has three fixed points:

$$q_0 \approx -0.6478 + 1.7214i, \quad \overline{q_0} \approx -0.6478 - 1.7214i, \quad q_1 \approx 0.2956.$$

The corresponding polynomials $h_q(z) = \left(\frac{2z}{q+1} - 1\right)^2$, for $q \in \{q_0, \overline{q_0}, q_1\}$, are post-critically finite, with the dynamics

$$0 \mapsto 1 \mapsto q \mapsto q$$

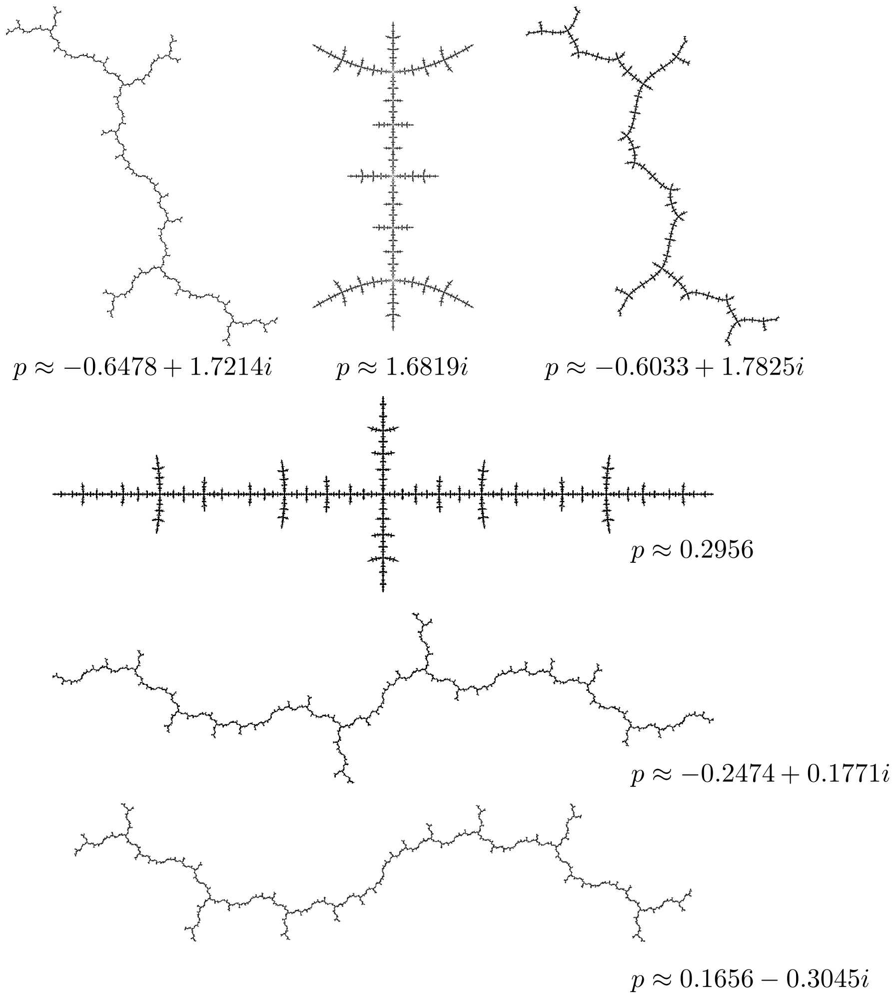

FIGURE 8. Slices of the Julia set J_1 of F

on the post-critical set. These polynomials and their iterated monodromy groups were studied in [2].

Consider the polynomial for $q_0 \approx -0.6478 + 1.7214i$. Its iterated monodromy group is generated by

$$\alpha = \sigma, \tag{1}$$

$$\beta = (1, \alpha), \tag{2}$$

$$\gamma = (\gamma, \beta), \tag{3}$$

where α, β and γ are loops around $0, 1$ and q_0, respectively (see a general formula for iterated monodromy groups of quadratic polynomials in [3]).

We can interpret the complement $\mathcal{M}$ of the post-critical set of F as the configuration space of pairs of complex numbers (z, p) which are different from $\infty, 0, 1,$

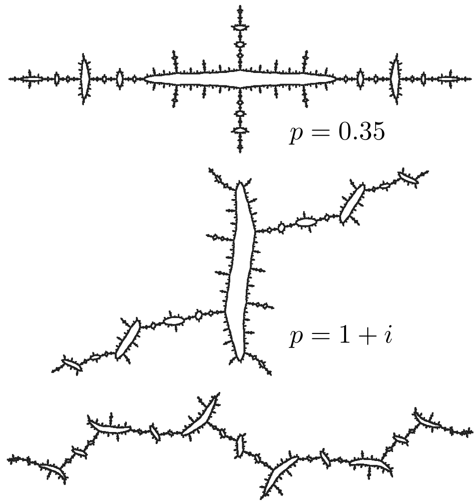

FIGURE 9. Slices of the Julia set J_2 of F

and from each other. We can identify the loops α, β, γ with the loops in the configuration space $\mathcal{M}$ coming from p staying fixed at q_0 and z traveling along the loops α, β, γ inside the line $p = q_0$.

Let S and P be the elements of the fundamental group of $\mathcal{M}$ uniquely determined by the following relations (see [19, Section 4]):

$$P\alpha P^{-1} = \beta\alpha\beta^{-1}, \qquad S\alpha S^{-1} = \alpha\gamma\alpha\gamma^{-1}\alpha^{-1}, \tag{4}$$

$$P\beta P^{-1} = \beta\alpha\beta\alpha^{-1}\beta^{-1}, \qquad S\beta S^{-1} = \beta, \tag{5}$$

$$P\gamma P^{-1} = \gamma, \qquad S\gamma S^{-1} = \alpha\gamma\alpha^{-1}. \tag{6}$$

Denote also $T = \gamma S^{-1} P^{-1} \beta\alpha$. We then have:

$$T\alpha T^{-1} = \alpha, \tag{7}$$

$$T\beta T^{-1} = \gamma\beta\gamma^{-1}, \tag{8}$$

$$T\gamma T^{-1} = \gamma\beta\gamma\beta^{-1}\gamma^{-1}. \tag{9}$$

In the same way, as in [19, Section 4], the loops S, T, and P are obtained by moving p around 0, 1, and z, respectively, in the configuration space $\mathcal{M}$. See also [19, Figure 2].

Proposition 3.2. *The virtual endomorphism ϕ associated with F is given by*

$$\phi(\alpha^2) = 1, \quad \phi(\beta) = 1, \quad \phi(\gamma) = \gamma,$$
$$\phi(\alpha^{-1}\beta\alpha) = \alpha, \quad \phi(\alpha^{-1}\gamma\alpha) = \beta,$$
$$\phi(S^2) = \beta\alpha\gamma S^{-1} P^{-1}, \quad \phi(T) = P, \quad \phi(P^2) = 1.$$

Proof. The elements of the fundamental group of $\mathcal{M}$ are uniquely determined by their action by conjugation on the normal subgroup generated by α, β, and γ. Let us use this fact (see, for example, [2, Proposition 4.1] and [19, Proposition 4.2]) to compute the virtual endomorphism of $\pi_1(\mathcal{M})$ associated with the partial self-covering F.

The domain of the restriction ϕ_0 of the virtual endomorphism ϕ onto $\langle\alpha, \beta, \gamma\rangle$ (associated with the first coordinate of the recursion (1)–(3)) is generated by α^2, β, γ, $\alpha^{-1}\beta\alpha$, and $\alpha^{-1}\gamma\alpha$. The action of the virtual endomorphism is given by

$$\phi_0(\alpha^2) = 1, \quad \phi_0(\beta) = 1, \quad \phi_0(\gamma) = \gamma,$$
$$\phi_0(\alpha^{-1}\beta\alpha) = \alpha, \quad \phi_0(\alpha^{-1}\gamma\alpha) = \beta.$$

The domain of the virtual endomorphism ϕ associated with F is generated by the above generators of the domain of ϕ_0 and the automorphisms S^2, P^2 and $T = \gamma S^{-1}P^{-1}\beta\alpha$.

Denote $\tau = \gamma^{-1}\alpha^{-1}\beta^{-1}$ (which is the element given by z traveling around infinity for fixed p). A direct computation shows that τ commutes with S and with P. We also have

$$\tau^{-1}S^{-1}P^{-1}\alpha PS\tau = \beta\gamma\beta^{-1}\gamma^{-1}\alpha\gamma\beta\gamma^{-1}\beta, \tag{10}$$

$$\tau^{-1}S^{-1}P^{-1}\beta PS\tau = \beta\gamma\beta\gamma^{-1}\beta^{-1}, \tag{11}$$

and

$$\tau^{-1}S^{-1}P^{-1}\gamma PS\tau = \beta\gamma\beta^{-1}. \tag{12}$$

Let us find $\phi(S^2)$ by computing its action by conjugation on the subgroup $\langle\alpha, \beta, \gamma\rangle$. It follows from (1)–(2) that $\alpha = \phi_0(\alpha^{-1}\beta\alpha)$, and hence

$$\phi(S^2)\alpha\phi(S^{-2}) = \phi(S^2\alpha^{-1}\beta\alpha S^{-2}).$$

Using (4)–(6), we compute

$$S^2\alpha^{-1}\beta\alpha S^{-2} = (\alpha\gamma)^2\alpha^{-1}(\alpha\gamma)^{-2}\beta(\alpha\gamma)^2\alpha(\alpha\gamma)^{-2}.$$

Applying the wreath recursion (1)–(3) to the righthand side of the last equality, we see that the value of ϕ_0 (i.e., the first coordinate of the wreath recursion) is $\beta\gamma\beta^{-1}\gamma^{-1}\alpha\gamma\beta\gamma^{-1}\beta^{-1}$. It follows that

$$\phi(S^2)\alpha\phi(S^{-2}) = \phi_0((\alpha\gamma)^2\alpha^{-1}(\alpha\gamma)^{-2}\beta(\alpha\gamma)^2\alpha(\alpha\gamma)^{-2}) = \beta\gamma\beta^{-1}\gamma^{-1}\alpha\gamma\beta\gamma^{-1}\beta^{-1}.$$

We compute the action of $\phi(S^2)$ on β and γ using the same methods:

$$\phi(S^2)\beta\phi(S^{-2}) = \phi(S^2\alpha^{-1}\gamma\alpha S^{-2}) = \phi_0((\alpha\gamma)^2\alpha^{-1}\gamma\alpha(\alpha\gamma)^{-2}) = \beta\gamma\beta\gamma^{-1}\beta^{-1},$$
$$\phi(S^2)\gamma\phi(S^{-2}) = \phi(S^2\gamma S^{-2}) = \phi_0(\alpha\gamma\alpha\gamma\alpha^{-1}\gamma^{-1}\alpha^{-1}) = \beta\gamma\beta^{-1}.$$

We see that the action of $\phi(S^2)$ by conjugation on $\langle\alpha, \beta, \gamma\rangle$ is the same as the action of $\tau^{-1}S^{-1}P^{-1}$; see (10)–(12). We conclude that

$$\phi(S^2) = \tau^{-1}S^{-1}P^{-1}.$$

In order to compute $\phi(T)$, we find

$$\phi(T)\alpha\phi(T^{-1}) = \phi(T\alpha^{-1}\beta\alpha T^{-1}) = \phi_0(\alpha^{-1}\gamma\beta\gamma^{-1}\alpha^{-1}) = \beta\alpha\beta^{-1},$$
$$\phi(T)\beta\phi(T^{-1}) = \phi(T\alpha^{-1}\gamma\alpha T^{-1}) = \phi_0(\alpha^{-1}\gamma\beta\gamma\beta^{-1}\gamma^{-1}\alpha) = \beta\alpha\beta\alpha^{-1}\beta^{-1},$$

and

$$\phi(T)\gamma\phi(T^{-1}) = \phi(T\gamma T^{-1}) = \phi_0(\gamma\beta\gamma\beta^{-1}\gamma^{-1}) = \gamma,$$

which implies that $\phi(T) = P$.

It remains to compute $\phi(P^2)$. We have

$$\phi(P^2)\alpha\phi(P^{-2}) = \phi(P^2\alpha^{-1}\beta\alpha P^{-2}) = \phi_0(\beta\alpha\beta\alpha^{-1}\beta^{-1}) = \alpha,$$

$$\phi(P^2)\beta\phi(P^{-2}) = \phi(P^2\alpha^{-1}\gamma\alpha P^{-2}) \\ = \phi_0(\beta\alpha\beta\alpha^{-1}\beta^{-1}\alpha^{-1}\beta^{-1}\gamma\beta\alpha\beta\alpha\beta^{-1}\alpha^{-1}\beta^{-1}) = \beta,$$

$$\phi(P^2)\gamma\phi(P^{-2}) = \phi(P^2\gamma P^{-2}) = \phi_0(\gamma) = \gamma,$$

which implies that $\phi(P^2) = 1$. □

Theorem 3.3. *The iterated monodromy group* IMG (F) *is generated by the wreath recursion*

$$\begin{array}{rcl} \alpha & = & \sigma(\beta, \beta^{-1}, \beta\alpha, \alpha^{-1}\beta^{-1}), \\ \beta & = & (1, \beta\alpha\beta^{-1}, \alpha, 1), \\ \gamma & = & (\gamma, \beta, \gamma, \beta), \\ P & = & \pi, \\ S & = & \sigma\pi(P^{-1}\tau^{-1}, P^{-1}, S^{-1}\tau^{-1}, S^{-1}), \end{array}$$

where $\sigma = (12)(34)$*,* $\pi = (13)(24)$*, and* $\tau = \gamma^{-1}\alpha^{-1}\beta^{-1}$*.*

Note that it follows from the recursions that the elements α, β, γ, P of the iterated monodromy group are involutions, hence the wreath recursion can be written as

$$\begin{array}{rcl} \alpha & = & \sigma(\beta, \beta, \beta\alpha, \alpha\beta), \\ \beta & = & (1, \beta\alpha\beta, \alpha, 1), \\ \gamma & = & (\gamma, \beta, \gamma, \beta), \\ P & = & \pi, \\ S & = & \sigma\pi(P\tau^{-1}, P, S^{-1}\tau^{-1}, S^{-1}), \end{array}$$

where $\tau = \gamma\alpha\beta$.

Proof. Denote by $\mathtt{L}_0$ the element $[\phi(1)1]$ of the biset $\mathfrak{M}_\phi$. Then denote

$$\mathtt{L}_1 = \tau\cdot\mathtt{L}_0, \quad \mathtt{R}_0 = P\cdot\mathtt{L}_0, \quad \mathtt{R}_1 = P\tau\cdot\mathtt{L}_0,$$

and order the basis of the biset associated with ϕ in the sequence $(\mathtt{L}_0, \mathtt{L}_1, \mathtt{R}_0, \mathtt{R}_1)$.

Using the definitions in Subsection 2.2, we get

$$\alpha\cdot\mathtt{L}_0 = \tau\cdot\mathtt{L}_0\cdot\phi(\tau^{-1}\alpha) = \mathtt{L}_1\cdot\phi(\beta\alpha\gamma\alpha) = \mathtt{L}_1\cdot\beta,$$

$$\alpha\cdot\mathtt{L}_1 = \mathtt{L}_0\cdot\phi(\alpha\tau) = \mathtt{L}_0\cdot\phi(\alpha\gamma^{-1}\alpha^{-1}\beta^{-1}) = \mathtt{L}_0\cdot\beta^{-1},$$

$$\alpha\cdot\mathtt{R}_0 = P\tau\cdot\mathtt{L}_0\cdot\phi(\tau^{-1}P^{-1}\alpha P) = \mathtt{L}_1\cdot\phi(\beta\alpha\gamma\alpha^{-1}\beta^{-1}\alpha\beta\alpha) = \mathtt{L}_1\cdot\beta\alpha,$$

and

$$\alpha\cdot\mathtt{R}_1 = P\cdot\mathtt{L}_0\cdot\phi(P^{-1}\alpha P\tau) = \mathtt{R}_0\cdot\phi(\alpha^{-1}\beta^{-1}\alpha\beta\alpha\gamma^{-1}\alpha^{-1}\beta^{-1}) = \mathtt{R}_0\cdot\alpha^{-1}\beta^{-1}.$$

Consequently,

$$\alpha = \sigma(\beta, \beta^{-1}, \beta\alpha, \alpha^{-1}\beta^{-1}).$$

We have

$$\beta\cdot\mathtt{L}_0 = \mathtt{L}_0\cdot\phi(\beta) = \mathtt{L}_0\cdot 1,$$

$$\beta\cdot\mathtt{L}_1 = \tau\cdot\mathtt{L}_0\cdot\phi(\tau^{-1}\beta\tau) = \mathtt{L}_1\cdot\phi(\beta\alpha\gamma\beta\gamma^{-1}\alpha^{-1}\beta^{-1}) = \mathtt{L}_1\cdot\beta\alpha\beta^{-1},$$

$$\beta\cdot\mathtt{R}_0 = P\cdot\mathtt{L}_0\cdot\phi(P^{-1}\beta P) = \mathtt{R}_0\cdot\phi(\alpha^{-1}\beta\alpha) = \mathtt{R}_0\cdot\alpha,$$

and

$$\beta\cdot\mathtt{R}_1 = P\tau\cdot\mathtt{L}_0\cdot\phi(\tau^{-1}P^{-1}\beta P\tau) = \mathtt{R}_1\cdot\phi(\beta\alpha\gamma\alpha^{-1}\beta\alpha\gamma^{-1}\alpha^{-1}\beta^{-1}) = \mathtt{R}_1\cdot 1,$$

hence

$$\beta = (1, \beta\alpha\beta^{-1}, \alpha, 1).$$

We have

$$\gamma \cdot \mathtt{L}_0 = \mathtt{L}_0 \cdot \phi(\gamma) = \mathtt{L}_0 \cdot \gamma,$$

$$\gamma \cdot \mathtt{L}_1 = \tau \cdot \mathtt{L}_0 \cdot \phi(\tau^{-1}\gamma\tau) = \mathtt{L}_1 \cdot \phi(\beta\alpha\gamma\alpha^{-1}\beta^{-1}) = \mathtt{L}_1 \cdot \beta,$$

and also

$$\gamma \cdot \mathtt{R}_0 = \mathtt{R}_0 \cdot \gamma, \quad \gamma \cdot \mathtt{R}_1 = \mathtt{R}_1 \cdot \beta,$$

since P commutes with γ.

Since $\phi(P^2) = 1$ and P commutes with τ, we have

$$P = \pi.$$

Finally,

$$\begin{aligned} S \cdot \mathtt{L}_0 = P\tau \cdot \mathtt{L}_0 \cdot \phi(\tau^{-1}P^{-1}S) &= \mathtt{R}_1 \cdot \phi(\beta\alpha\gamma \cdot P^{-2}\beta\alpha \cdot \alpha^{-1}\beta^{-1}PS\gamma^{-1} \cdot \gamma) \\ &= \mathtt{R}_1 \cdot \phi(P^{-2} \cdot \beta\alpha\gamma\beta\alpha \cdot T^{-1} \cdot \gamma) = \mathtt{R}_1 \cdot \beta\alpha P^{-1}\gamma = \mathtt{R}_1 \cdot \tau^{-1}P^{-1}. \end{aligned}$$

Since T commutes with α, we have:

$$\begin{aligned} S \cdot \mathtt{L}_1 &= P \cdot \mathtt{L}_0 \cdot \phi(P^{-1}S\tau) \\ &= \mathtt{R}_0 \cdot \phi(P^{-2}\beta\alpha \cdot \alpha^{-1}\beta^{-1}PS\gamma^{-1} \cdot \alpha^{-1}\beta^{-1}) = \mathtt{R}_0 \cdot \phi(P^{-2}\beta\alpha T^{-1}\alpha^{-1}\beta^{-1}) \\ &= \mathtt{R}_0 \cdot \phi(\beta T^{-1}\beta^{-1}) = \mathtt{R}_0 \cdot P^{-1}. \end{aligned}$$

Since S commutes with $\alpha\gamma$, and T commutes with α, we have:

$$\begin{aligned} S \cdot \mathtt{R}_0 &= \tau \cdot \mathtt{L}_0 \cdot \phi(\tau^{-1}SP) \\ &= \mathtt{L}_1 \cdot \phi(\beta\alpha\gamma S^2\gamma^{-1} \cdot \gamma S^{-1}P^{-1}\beta\alpha \cdot \alpha^{-1}\beta^{-1}P^2) = \mathtt{L}_1 \cdot \phi(\beta S^2\alpha\gamma\alpha^{-1}T\beta^{-1}) \\ &= \mathtt{L}_1 \cdot \tau^{-1}S^{-1}P^{-1}P = \mathtt{L}_1 \cdot \tau^{-1}S^{-1}, \end{aligned}$$

and, since P and S commute with τ, and γ commutes with P:

$$\begin{aligned} S \cdot \mathtt{R}_1 = \mathtt{L}_0 \cdot \phi(SP\tau) &= \mathtt{L}_0 \cdot \phi(S^2\gamma^{-1} \cdot \gamma S^{-1}P^{-1}\beta\alpha \cdot \alpha^{-1}\beta^{-1}P^2\gamma^{-1}\alpha^{-1}\beta^{-1}) \\ &= \mathtt{L}_0 \cdot \phi(S^2\gamma^{-1}T\alpha^{-1}\beta^{-1}\gamma^{-1}\alpha^{-1}\beta^{-1}P^2) = \mathtt{L}_0 \cdot \tau^{-1}S^{-1}P^{-1}\gamma^{-1}P\alpha^{-1}\beta^{-1} \\ &= \mathtt{L}_0 \cdot \tau^{-1}S^{-1}\tau = \mathtt{L}_0 \cdot S^{-1}, \end{aligned}$$

which implies

$$S = \sigma\pi(P^{-1}\tau^{-1}, P^{-1}, S^{-1}\tau^{-1}, S^{-1}),$$

which finishes the proof. □

Computation (for example using the GAP packages [1] or [13]) gives the following nucleus of IMG (F):

$$\begin{aligned} (13)\quad &\{1, \alpha, \beta, \gamma, \alpha^\beta, \beta^\alpha, \gamma^\alpha, \gamma^\beta, \gamma^{\alpha\beta}, P, \gamma P, \gamma^{\alpha\beta}P\} \\ &\cup \{\alpha\beta, \alpha\gamma, \beta\gamma, \tau, \alpha\tau, \beta\tau, \\ &S, \alpha S, S\beta, S\gamma, \alpha\beta S, S\alpha\beta, S\gamma\beta, S\beta\gamma, S\gamma\alpha, S\alpha^\beta, \gamma^{\alpha\beta}S, \tau S, \beta\tau S, \alpha S\alpha\beta, \\ &P\alpha, P\beta, P\alpha\beta, P\tau\}^{\pm 1} \end{aligned}$$

consisting of 60 elements.

3.2. **An index two extension of** $\mathrm{IMG}(F)$**.** The wreath recursion defining $\mathrm{IMG}(F)$ (see Theorem 3.3) can be simplified by embedding $\mathrm{IMG}(F)$ into a bigger group $\overline{\mathcal{G}}$. We will do this just to simplify computations (in particular to get a smaller nucleus), though $\overline{\mathcal{G}}$ also has a dynamical interpretation. (It is the iterated monodromy group of the dynamical system obtained by taking the quotient of F by the action of the complex conjugation $(z, p) \mapsto (\overline{z}, \overline{p})$.)

Definition 3.1. Denote by $\overline{\mathcal{G}}$ the group generated by the elements

$$\begin{aligned} \alpha &= \sigma, \quad \beta = (1, \alpha, \alpha, 1), \qquad \gamma = (\gamma, \beta, \gamma, \beta), \\ a &= \pi, \quad b = (a, a, \alpha a, \alpha a), \quad c = (\beta b, \beta b, \gamma c, \gamma c), \end{aligned}$$

where $\sigma = (12)(34)$ and $\pi = (13)(24)$.

We will see later that the subgroup of $\overline{\mathcal{G}}$ generated by α, β, γ is equivalent as a self-similar group to the subgroup $\langle \alpha, \beta, \gamma \rangle < \mathrm{IMG}(F)$, so we use the same letters for the corresponding generators of these groups.

It is easy to check that $a^2 = b^2 = c^2 = 1$,

$$a\beta a = \alpha\beta\alpha, \quad b\gamma b = \beta\gamma\beta, \quad c\beta c = \gamma\beta\gamma, \tag{14}$$

and that a, b, c commute with the remaining generators α, β, γ.

Denote by $(\mathtt{e}_{00}, \mathtt{e}_{01}, \mathtt{e}_{10}, \mathtt{e}_{11})$ the ordered basis of the biset in the definition of the group $\overline{\mathcal{G}}$. Then $\sigma(\mathtt{e}_{i,j}) = \mathtt{e}_{i,(1-j)}$ and $\pi(\mathtt{e}_{i,j}) = \mathtt{e}_{(1-i),j}$.

Proposition 3.4. *The group* $\mathrm{IMG}(F)$ *is equivalent as as a self-similar group to an index two subgroup of the group* $\overline{\mathcal{G}}$*. The isomorphism maps* α, β, γ *to the corresponding generators of* $\mathrm{IMG}(F)$ *and maps* S *and* P *to* $ac\alpha\gamma$ *and* βba*, respectively. The bisets of the wreath recursions for* $\mathrm{IMG}(F)$ *(as in Theorem* 3.3*) and* $\overline{\mathcal{G}}$ *(as in Definition* 3.1*) are identified with each other by the equalities*

$$\{\mathtt{L}_0 = \mathtt{e}_{00}, \quad \mathtt{L}_1 = \mathtt{e}_{01} \cdot \beta, \quad \mathtt{R}_0 = \mathtt{e}_{10} \cdot a, \quad \mathtt{R}_1 = \mathtt{e}_{11} \cdot \beta\alpha a\}.$$

Proof. Conjugating the righthand side of the recursion from Definition 3.1 by $(1, \beta, a, \beta\alpha a)$, we get

$$\begin{gathered} \alpha = \sigma(\beta, \beta, a\alpha\beta a, a\beta\alpha a) = \sigma(\beta, \beta, \beta\alpha, \alpha\beta), \\ \beta = (1, \beta\alpha\beta, a\alpha a, 1) = (1, \beta\alpha\beta, \alpha, 1), \\ \gamma = (\gamma, \beta, a\gamma a, a\alpha\beta\alpha a) = (\gamma, \beta, \gamma, \beta), \\ a = \pi(a, \alpha a, a, \alpha a), \\ b = (a, \beta a\beta, a\alpha a a, a\alpha\beta\alpha a\beta\alpha a) = (a, \beta\alpha\beta\alpha a, \alpha a, \alpha a), \\ c = (\beta b, \beta\beta b\beta, a\gamma c a, a\alpha\beta\gamma c\beta\alpha a) = (\beta b, \beta b, \gamma a c a, \alpha\gamma\alpha a c a). \end{gathered}$$

Let us show that $S = ac\alpha\gamma$ and $P = \beta ba$ satisfy the wreath recursion of Proposition 3.4.

We have, using commutation of the involutions a and α,

$$\beta ba = \pi(a\alpha a\alpha a a, \alpha a\alpha a, aa, \beta\alpha\beta\beta\alpha\beta\alpha a\alpha a) = \pi,$$

which agrees with the condition $P = \pi$.

We have

$$\begin{aligned} ac\alpha\gamma &= \pi(a, \alpha a, a, \alpha a)(\beta b, \beta b, \gamma a c a, \alpha\gamma\alpha a c a)\sigma(\beta, \beta, \beta\alpha, \alpha\beta)(\gamma, \beta, \gamma, \beta) \\ &= \pi\sigma(\alpha a b\gamma, ab\beta, \gamma\alpha c a\beta\alpha\gamma, \gamma c a\alpha) = \pi\sigma(ab\beta \cdot \beta\alpha\gamma, ab\beta, \gamma\alpha c a\beta\alpha\gamma, \gamma\alpha c a), \end{aligned}$$

which also agrees with

$$S = \pi\sigma(P^{-1}\tau^{-1}, P^{-1}, S^{-1}\tau^{-1}, S^{-1}),$$

and finishes the proof. □

3.3. **Properties of the groups $\mathcal{G}, \overline{\mathcal{G}}$, and** IMG (F)**.** Denote by $\mathcal{G}$ the subgroup of IMG (F) generated by α, β, γ. It is also equivalent to the subgroup of $\overline{\mathcal{G}}$ generated by the same elements.

Note that for every element $g \in \mathcal{G}$ and for every e_{ij} we have $g \cdot \mathsf{e}_{ij} = \mathsf{e}_{ik} \cdot h$ for some $h \in \mathcal{G}$ and $k \in \{0, 1\}$. In other words, the self-similarity biset of $\mathcal{G}$ is a disjoint union ("direct sum") of the bisets $\mathfrak{M}_0 = \{\mathsf{e}_{00}, \mathsf{e}_{01}\} \cdot \mathcal{G}$ and $\mathfrak{M}_1 = \{\mathsf{e}_{10}, \mathsf{e}_{11}\} \cdot \mathcal{G}$.

Let us denote $E_i = \{\mathsf{e}_{i0}, \mathsf{e}_{i1}\}$ for $i \in \{0, 1\}$. We will also denote $E_\emptyset = \{\emptyset\}$ and

$$E_{i_1 i_2 \ldots i_n} = E_{i_1} E_{i_2} \ldots E_{i_n} \subset \{\mathsf{e}_{00}, \mathsf{e}_{01}, \mathsf{e}_{10}, \mathsf{e}_{11}\}^n.$$

Then for every sequence $w = i_1 i_2 \ldots \{0, 1\}^\omega$ the subtree

$$T_w = \bigcup_{n \geq 0} E_{i_1 i_2 \ldots i_n}$$

of the tree $T = \{\mathsf{e}_{00}, \mathsf{e}_{01}, \mathsf{e}_{10}, \mathsf{e}_{11}\}^*$ is invariant under the action of the group $\mathcal{G}$.

Let us identify T_w, for $w = x_1 x_2 \ldots$, with the binary tree $\{0, 1\}^*$ by the isomorphism

$$\mathsf{e}_{x_1 i_1} \mathsf{e}_{x_2 i_2} \ldots \mathsf{e}_{x_n i_n} \mapsto i_1 i_2 \ldots i_n.$$

Denote by $\mathcal{G}_w$ the restriction of the action of $\mathcal{G}$ onto the subtree T_w, seen as an automorphism group of the binary tree.

Then it follows directly from the wreath recursion for $\mathcal{G}$ that the group $\mathcal{G}_w$ is generated by automorphisms $\alpha_w, \beta_w, \gamma_w$ (the images of α, β, γ) which are defined by the following recursions:

$$\alpha_w = \sigma, \quad \gamma_w = (\gamma_{\overline{w}}, \beta_{\overline{w}})$$

and

$$\beta_w = \begin{cases} (1, \alpha_{\overline{w}}) & \text{if } x_1 = 0, \\ (\alpha_{\overline{w}}, 1) & \text{if } x_1 = 1. \end{cases}$$

Here $\overline{w} = x_2 x_3 \ldots$ is the shift of w.

The group $\mathcal{G}$ is the universal group of the family $\{\mathcal{G}_w \ : \ w \in \{0, 1\}^\omega\}$, i.e., $\mathcal{G}$ is the quotient of the free group $\langle \alpha, \beta, \gamma \mid \emptyset \rangle$ by the normal subgroup $R = \bigcap_{w \in \{0,1\}^\omega} R_w$, where R_w is the kernel of the natural epimorphism $\alpha \mapsto \alpha_w$, $\beta \mapsto \beta_w$, $\gamma \mapsto \gamma_w$ of the free group $\langle \alpha, \beta, \gamma \mid \emptyset \rangle$ onto the group $\mathcal{G}_w$. This follows from the fact that the subtrees T_w cover the tree T.

Proposition 3.5. *The group $\mathcal{G}$ is contracting with the nucleus (for the recursion from Definition* 3.1*) $\mathcal{N} = \{1, \alpha, \beta, \gamma\}$.*

Proof. By [14, Lemma 2.11.2] we have to show that sections of $\mathcal{N} \cdot \{\alpha, \beta, \gamma\}$ eventually belong to $\mathcal{N}$. But sections of the elements α, β in words of length more than one are trivial, while γ is of order two. Hence, sections of the elements of $\mathcal{N} \cdot \{\alpha, \beta, \gamma\}$ in words of length two belong to $\mathcal{N}$. □

Consider the map $\{\mathsf{e}_{00}, \mathsf{e}_{01}, \mathsf{e}_{10}, \mathsf{e}_{11}\}^* \longrightarrow \{0, 1\}^*$ generated by the map $\mathsf{e}_{ij} \mapsto j$ (i.e., applying $\mathsf{e}_{ij} \mapsto j$ to every letter of a word). It is a surjective morphism of the trees. It follows directly from the recursion defining the group $\overline{\mathcal{G}}$ that the action of $\overline{\mathcal{G}}$ on $\{\mathsf{e}_{ij}\}^*$ projects to an action on $\{0, 1\}^*$. Note also that it follows from the discussion above that the action of $\mathcal{G}$ on $\{\mathtt{e}_{ij}\}^*$ is projected to the trivial action on $\{0,1\}^*$. The action of $\overline{\mathcal{G}}$ on $\{0,1\}^*$ is not faithful. Let us denote by $\mathcal{K}$ the quotient of $\overline{\mathcal{G}}$ by the kernel of this action.

The following proposition then follows directly from the wreath recursion defining $\overline{\mathcal{G}}$.

Proposition 3.6. *Denote by $\mathcal{K}$ the self-similar group generated by*

$$\widetilde{a} = \sigma, \quad \widetilde{b} = (\widetilde{a}, \widetilde{a}), \quad \widetilde{c} = (\widetilde{b}, \widetilde{c}),$$

where σ is the transposition. The map $g \mapsto \widetilde{g}$ defined on the generators by

$$a \mapsto \widetilde{a}, \quad b \mapsto \widetilde{b}, \quad c \mapsto \widetilde{c}$$

and $g \mapsto 1$ for $g \in \mathcal{G}$ defines the epimorphism $\overline{\mathcal{G}} \longrightarrow \mathcal{K}$. Together with the map $\mathtt{e}_{ij} \mapsto j$ it generates an epimorphism of bisets.

The image of the subtree T_w under the action of an element $h \in \overline{\mathcal{G}}$ is the subtree $T_{\widetilde{h}(w)}$.

Proposition 3.7. *For any element g of the kernel of the epimorphism $\overline{\mathcal{G}} \longrightarrow \mathcal{K}$ there exists n such that $g|_v \in \mathcal{G}$ for all words v of length greater than n.*

Proof. It is easy to check that the wreath recursion

$$\widetilde{a} = \sigma, \quad \widetilde{b} = (\widetilde{a}, \widetilde{a}), \quad \widetilde{c} = (\widetilde{b}, \widetilde{c})$$

is contracting on the abstract group $\widetilde{\mathcal{K}}$ given by the presentation $\widetilde{\mathcal{K}} = \langle \widetilde{a}, \widetilde{b}, \widetilde{c} \mid (\widetilde{a})^2 = (\widetilde{b})^2 = (\widetilde{c})^2 = 1\rangle$. The nucleus of $\widetilde{\mathcal{K}}$ is $\{1, \widetilde{a}, \widetilde{b}, \widetilde{c}\}$. The group $\mathcal{K}$ is the faithful quotient of the self-similar group $\widetilde{\mathcal{K}}$.

It follows from [14, Proposition 2.13.2] that a product $\widetilde{g}$ of the generators $\widetilde{a}, \widetilde{b}, \widetilde{c}$ is trivial in $\mathcal{K}$ if and only if there exists n such that $\widetilde{g}$ belongs to the kernel of the nth iterate of the wreath recursion on $\widetilde{\mathcal{K}}$. Let g be a product of the generators of $\overline{\mathcal{G}}$ equal to an element of the kernel of $\overline{\mathcal{G}} \longrightarrow \mathcal{K}$, and let $\widetilde{g}$ be the word obtained from g by removing all generators α, β, γ and applying the homomorphism $h \mapsto \widetilde{h}$ to every letter a, b, c. Then the word $\widetilde{g}$ represents a trivial element of $\mathcal{K}$. Let n be such that $\widetilde{g}$ belongs to the kernel of the nth iterate of the wreath recursion on $\widetilde{\mathcal{K}}$. Then it follows from the wreath recursion defining $\overline{\mathcal{G}}$ and normality of $\mathcal{G}$ in $\overline{\mathcal{G}}$ that the sections of g in all words of length n belong to $\mathcal{G}$. □

The epimorphism of self-similar groups (i.e., of groups and bisets) described in Proposition 3.7 induces a semiconjugacy $\mathcal{J}_{\overline{\mathcal{G}}} \longrightarrow \mathcal{J}_{\mathcal{K}}$. This semiconjugacy is induced by the same map $\mathtt{e}_{ij} \mapsto j$ as the epimorphism of self-similar groups. The image of $\mathrm{IMG}(F) < \overline{\mathcal{G}}$ under this epimorphism is the self-similar group generated by the wreath recursion

$$S = \sigma(P, S^{-1}), \qquad P = \sigma. \tag{15}$$

The corresponding epimorphism of the self-similarity bisets acts by the rule $\mathtt{L}_i \mapsto \mathtt{L}$ and $\mathtt{R}_i \mapsto \mathtt{R}$, where $(\mathtt{L}, \mathtt{R})$ is the ordered basis associated with the above recursion for $\mathrm{IMG}(f)$. This is the iterated monodromy group of $f(p) = \left(\frac{p-1}{p+1}\right)^2$, where the generators S and P of $\mathrm{IMG}(f)$ are identified with the images of the corresponding generators of $\mathrm{IMG}(F)$ under the projection $(z, p) \mapsto p$. See also an explicit computation of $\mathrm{IMG}(f)$ in the proof of Proposition 3.8 below.

It follows that the restriction of the epimorphism from Proposition 3.7 to the subgroup $\mathrm{IMG}\,(F) < \overline{\mathcal{G}}$ is the natural epimorphism $\mathrm{IMG}\,(F) \longrightarrow \mathrm{IMG}\,(f)$ induced by the projection $(z,p) \mapsto p$. Consequently, the induced map of the limit spaces $\mathcal{J}_{\mathrm{IMG}(F)} \longrightarrow \mathcal{J}_{\mathrm{IMG}(f)}$ coincides with the restriction of the projection $(z,p) \mapsto p$ to the Julia sets of F and f.

Proposition 3.8. *The transformation κ of the space $\{\mathtt{L},\mathtt{R}\}^{-\omega}$ changing in every sequence $w \in \{\mathtt{L},\mathtt{R}\}^{-\omega}$ each letter $\mathtt{L}$ to $\mathtt{R}$ and vice versa induces a homeomorphism of the limit space of* $\mathrm{IMG}\,(f)$*, corresponding to the complex conjugation on the Julia set of f.*

Proof. Let us compute the iterated monodromy group $\mathrm{IMG}\,(f)$ directly, in order to understand the geometric meaning of the elements $\mathtt{L}$ and $\mathtt{R}$.

The post-critical set of f is $\{\infty, 0, 1\}$. Take -1 as the basepoint. Let S and P be the loops going in the positive direction around 0 and around both 0 and 1, respectively, as it is shown in the top part of Figure 10. (This agrees with the interpretation of the generators S and P of $\mathrm{IMG}\,(F)$; see 3.1.) Connect the basepoint -1 with its preimages $\pm i$ by straight segments.

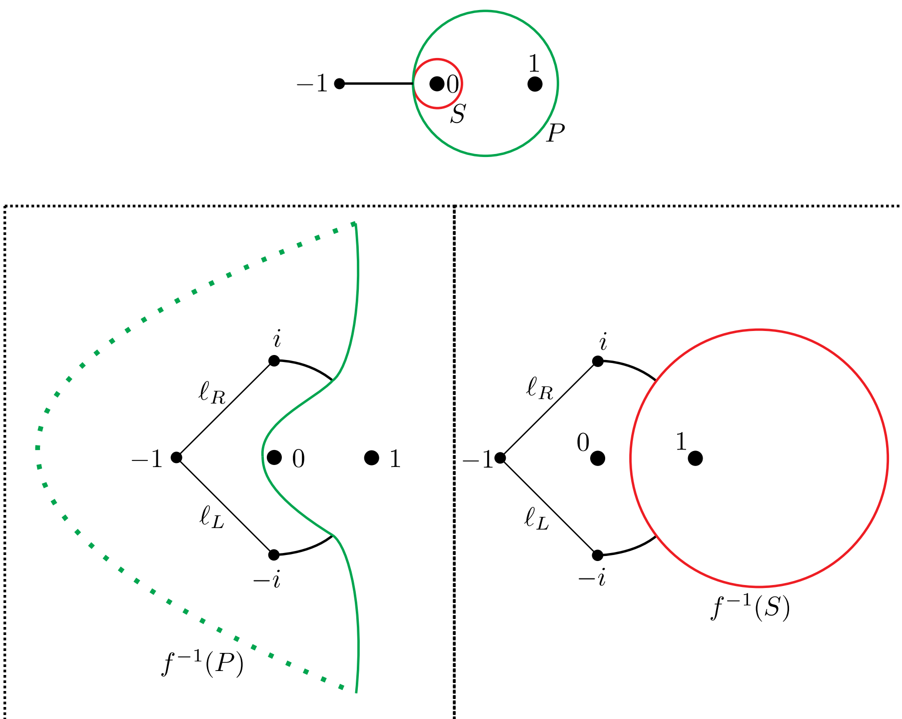

FIGURE 10. Computation of $\mathrm{IMG}\,(f)$

The bottom part of Figure 10 shows the inverse images of the generators under the action of the rational function. We see that if we label the path connecting the basepoint -1 to i by $\ell_{\mathtt{R}}$ and the path connecting -1 to $-i$ by $\ell_{\mathtt{L}}$, then the associated biset is defined by

$$P \cdot \mathtt{L} = \mathtt{R}, \quad P \cdot \mathtt{R} = \mathtt{L},$$

and

$$S \cdot \mathtt{R} = \mathtt{L} \cdot P, \quad S \cdot \mathtt{L} = \mathtt{R} \cdot S^{-1},$$

which agrees with the wreath recursion (15).

A sequence $\ldots X^{(1)}X^{(0)} \in \{\mathtt{L}, \mathtt{R}\}^{-\omega}$ represents the point of the Julia set equal to the limit of the path $\ell_0\ell_1\cdots$, where ℓ_k is a continuation of the path ℓ_{k-1} and is a lift of the path $\ell_{X^{(k)}}$ by the kth iteration of the rational function. Since the rational function $\left(\frac{p-1}{p+1}\right)^2$ has real coefficients, the basepoint -1 is real, and complex conjugation permutes the paths $\ell_{\mathtt{R}}$ and $\ell_{\mathtt{L}}$, then the transformation κ maps a sequence corresponding to a point z to the sequence corresponding to the conjugate point $\overline{z}$. $\square$

Recall that $J(q)$ denotes the intersection of the Julia set J_1 with the z-line $p = q$.

Proposition 3.9. *Each connected component of the limit space $\mathcal{J}_{\mathcal{G}}$ of the group $\mathcal{G}$ consists of points represented by the sequences of the form $\ldots X_{i_2}^{(2)}X_{i_1}^{(1)}$, where $w = \ldots X^{(2)}X^{(1)} \in \{\mathtt{R}, \mathtt{L}\}^{-\omega}$ is fixed and $i_k \in \{0, 1\}$ are arbitrary. The connected component corresponding to $w \in \{\mathtt{R}, \mathtt{L}\}^{-\omega}$ is homeomorphic $J(p_0)$, where p_0 is the point of the Julia set of f encoded by the sequence w.*

Proof. Since $\mathcal{G}$ is a self-similar subgroup of $\mathrm{IMG}(F)$, the equivalence relation associated with $\mathcal{G}$ is a sub-relation of the asymptotic equivalence relation associated with $\mathrm{IMG}(F)$. The group $\mathcal{G}$ changes only the indices of the symbols $X_{i_k}^{(k)}$, hence $\mathcal{G}$-equivalent sequences are of the form $\ldots X_{i_2}^{(2)}X_{i_1}^{(1)}$, $\ldots X_{j_2}^{(2)}X_{j_1}^{(1)}$ for some $i_k, j_k \in \{0, 1\}$ and $X^{(k)} \in \{\mathtt{L}, \mathtt{R}\}$. Since $\mathcal{G}$ is level-transitive on each of the subtrees, the image of the set $\{\ldots X_{i_2}^{(2)}X_{i_1}^{(1)} \ : \ \ldots i_2i_1 \in \{0, 1\}^{-\omega}\}$ in the limit space of $\mathcal{G}$ is connected (by the argument similar to that of [14, Section 3.5]); hence it is a connected component.

It remains to show that the equivalence relation associated with $\mathrm{IMG}(F)$ restricted to the set $\{\ldots X_{i_2}^{(2)}X_{i_1}^{(1)} \ : \ \ldots i_2i_1 \in \{0, 1\}^{-\omega}\}$ coincides with the restriction of the equivalence associated with $\mathcal{G}$, i.e., that the group $\mathrm{IMG}(F)$ does not introduce new identifications inside the connected components of the limit space of $\mathcal{G}$. Suppose that the sequences $\ldots X_{i_2}^{(2)}X_{i_1}^{(1)}$, $\ldots X_{j_2}^{(2)}X_{j_1}^{(1)}$ are equivalent with respect to the action of $\mathrm{IMG}(F)$. It means that there exists a sequence g_k of elements of $\mathrm{IMG}(F)$ assuming a finite set of values and such that $g_k \cdot X_{i_k}^{(k)} = X_{j_k}^{(k)} \cdot g_{k-1}$ for all $k \geq 1$. The limit space of $\mathrm{IMG}(f)$ has no singular points, since the rational function $f(p) = \left(\frac{p-1}{p+1}\right)^2$ is hyperbolic. It follows that the images of g_k in $\mathcal{K}$ are trivial. But this implies by Proposition 3.7 that the elements g_k belong to $\mathcal{G}$, i.e., that the sequences are equivalent with respect to the action of the group $\mathcal{G}$. $\square$

Let us summarize different groups and dynamical systems appearing in our analysis. We have self-similar groups and homomorphisms

$$\begin{array}{ccccc} \mathcal{G} & \hookrightarrow & \mathrm{IMG}(F) & \hookrightarrow & \overline{\mathcal{G}} \\ \downarrow & & \downarrow & & \downarrow \\ \{1\} & \hookrightarrow & \mathrm{IMG}(f) & \hookrightarrow & \mathcal{K}. \end{array} \tag{16}$$

They induce semiconjugacies

$$\begin{array}{ccccc} \mathcal{J}_{\mathcal{G}} & \longrightarrow & J_1 & \longrightarrow & J_1/((z,p)\sim(\overline{z},\overline{p})) \\ \downarrow & & \downarrow & & \downarrow \\ \{\mathtt{L},\mathtt{R}\}^{-\omega} & \longrightarrow & J_f & \longrightarrow & J_f/(z\sim\overline{z}) \end{array} \tag{17}$$

of the corresponding limit dynamical systems. The limit dynamical systems of $\mathrm{IMG}\,(F)$ and $\mathrm{IMG}\,(f)$ are $F : J_1 \longrightarrow J_1$ and the restriction of f onto its Julia set J_f, respectively. The limit dynamical system of $\mathcal{G}$, by Proposition 3.9, is a bundle of the Julia sets $J(p_0)$ over the Cantor set $\{\mathtt{R},\mathtt{L}\}^{-\omega}$, where $J(p_0)$ is the fiber above a sequence $w \in \{\mathtt{R},\mathtt{L}\}^{-\omega}$ if the sequence w is mapped to p_0 by the natural map from $\{\mathtt{R},\mathtt{L}\}^{-\omega}$ to J_f. The limit dynamical system of $\mathcal{G}$ acts on each fiber in the same way as the map F acts on the fibers $J(p_0)$ of J_1, and by the shift on $\{\mathtt{R},\mathtt{L}\}^{-\omega}$. The map $\mathcal{J}_{\mathcal{G}} \longrightarrow J_1$ identifies fibers corresponding to the same points of J_f. The epimorphisms $\mathrm{IMG}\,(F) \longrightarrow \mathrm{IMG}\,(f)$ and $\mathcal{G} \longrightarrow \{1\}$ are induced by the projection $(z,p) \mapsto p$, i.e., by collapsing the fibers to points.

The limit dynamical systems of $\overline{\mathcal{G}}$ and $\mathcal{K}$ are quotients of $F : J_1 \longrightarrow J_1$ and $f : J_f \longrightarrow J_f$ by complex conjugation (compare with Proposition 3.8). We will not give a full proof of this fact, since we do not need it here.

3.4. **The Schreier graphs of the groups $\mathcal{G}_w$.** Recall that for $v = i_1 i_2 \ldots i_n \in \{0,1\}^n$, we denoted by E_v the set of words of the form $\mathtt{e}_{i_1 j_1}\mathtt{e}_{i_2 j_2}\ldots\mathtt{e}_{i_n j_n}$, where $j_1 j_2 \ldots j_n \in \{0,1\}^n$. We will denote the word $\mathtt{e}_{i_1 j_1}\mathtt{e}_{i_2 j_2}\ldots\mathtt{e}_{i_n j_n}$ just by $j_1 j_2 \ldots j_n$, for simplicity of notation. This notation agrees with the interpretation of E_v as the nth level of the tree on which the group $\mathcal{G}_w$ acts.

Let $v \in \{0,1\}^*$. Denote by Γ_v the Schreier graph of the action of $\mathcal{G}$ on the set E_v, i.e., the graph with the set of vertices E_v in which two elements $w_1, w_2 \in X_v$ are adjacent if and only if $g(w_1) = w_2$ for some $g \in \{\alpha,\beta,\gamma\}$. We label the corresponding edge of Γ_v by g.

Note that the graph Γ_v is the Schreier graph of the action of the group $\mathcal{G}_w$ on the nth level of the tree, where w is any infinite word starting with v and n is the length of v.

It follows from the wreath recursion defining α,β,γ in Definition 3.1 that for every generator $g \in \{\alpha,\beta,\gamma\}$ of $\mathcal{G}$ there exists a unique word $z_{g,v} \in E_v$ of length n and a generator $h \in \{\alpha,\beta,\gamma\}$ such that $h|_{z_{g,v}} = g$. For the remaining pairs $h \in \{\alpha,\beta,\gamma\}$ and $u \in E_v$ we have $h|_u = 1$. Namely, we have, for $v \in \{0,1\}^n$:

$$z_{\alpha,v} = \underbrace{00\ldots0}_{n-2 \text{ times}}\ 1x', \quad z_{\beta,v} = \underbrace{00\ldots0}_{n-1 \text{ times}}\ 1, \quad z_{\gamma,v} = \underbrace{00\ldots0}_{n \text{ times}},$$

where $x' = 1 - x$ is the letter different from the last letter x of v. If the word v has length less than 2, one has to take the endings of length $|v|$ in the righthand sides of the equalities.

Proposition 3.10. *Let $v, u \in \{0,1\}^*$ be arbitrary finite words. Consider for each word $w \in \{0,1\}^{|v|}$ a copy $\Gamma_{u,w}$ of the edge-labeled graph Γ_u. Connect, for each $g \in \{\alpha,\beta,\gamma\}$ and $w \in \{0,1\}^{|v|}$, the copy of $z_{g,u}$ in $\Gamma_{u,w}$ with the copy of $z_{g,u}$ in $\Gamma_{u,g(w)}$ by an edge labeled by the element $h \in \{\alpha,\beta,\gamma\}$ such that $h|_{z_{g,u}} = g$. The obtained graph is isomorphic to Γ_{uv}.*

In this graph the vertex $z_{g,uv}$ is the copy of $z_{h,u}$ in $\Gamma_{u,w}$ for $h \in \{\alpha,\beta,\gamma\}$ and $w \in \{0,1\}^{|v|}$ such that $h|_w = g$.

Note that the copies $\Gamma_{u,w}$ of Γ_u are connected in Γ_{uv} in the same way as the vertices w are connected in the graph Γ_v.

Proof. Let $w_1w_2 \in X_{uv}$ and $|w_1| = |u|, |w_2| = |v|$. It follows from the definition of the words $z_{\alpha,u}, z_{\beta,u}, z_{\gamma,u}$ that a generator $g \in \{\alpha, \beta, \gamma\}$ changes the end of length $|w_2|$ in the word w_1w_2 only when $w_1 = z_{h,u}$ for some $h \in \{\alpha, \beta, \gamma\}$, and then we have $g(w_1w_2) = w_1h(w_2)$. In all the other cases $g(w_1w_2) = g(w_1)w_2$, since $g|_{w_1} = 1$. $\square$

In the case $|v| = 1$ we get the following inductive rule for constructing the graphs Γ_u.

Corollary 3.11. *In order to get Γ_{ux} one has to take two copies $\Gamma_u^{(0)}$ and $\Gamma_u^{(1)}$ of Γ_u and connect by an edge the copies of the vertices $z_{\alpha,u}$. The obtained graph is Γ_{ux}. The vertex $z_{\alpha,ux}$ is the copy of $z_{\beta,u}$ in $\Gamma_u^{(1-x)}$, the vertex $z_{\beta,ux}$ is the copy of $z_{\gamma,u}$ in $\Gamma_u^{(1)}$, and the vertex $z_{\gamma,ux}$ is the copy of $z_{\gamma,u}$ in $\Gamma_u^{(0)}$.*

In the opposite case (when $|u| = 1$) we get the following rule.

Corollary 3.12. *In order to get Γ_{xv} one has to replace in Γ_v each vertex w by a pair of vertices $0w$ and $1w$, connected by an edge (labeled by α), connect $(1-x)w$ to $(1-x)\alpha(w)$ by an edge (labeled by β), and connect $0w$ to $0\gamma(w)$ and $1w$ to $1\beta(w)$ by edges labeled by γ.*

We get nice pictures of the graphs Γ_v when we draw the edges labeled by α, β, and γ in such a way that they have equal length, and for every vertex w the edge labeled by β incident with w (if it exists) is obtained from the edge labeled by α by rotation by $\pi/2$ around w, while the edge labeled by γ is obtained from the edge labeled by α by rotation by $-\pi/2$. We can also use the opposite agreement (changing the signs of $\pi/2$ and $-\pi/2$). Note that since the generators are involutions, we draw them as single non-oriented edges (instead of drawing pairs of oriented edges) and we ignore the loops. See, for instance, Figure 11, where some graphs Γ_v are constructed this way. Figure 12 shows all graphs Γ_v for $|v| = 6$ (there are only 16 of them since every isomorphism class appears four times).

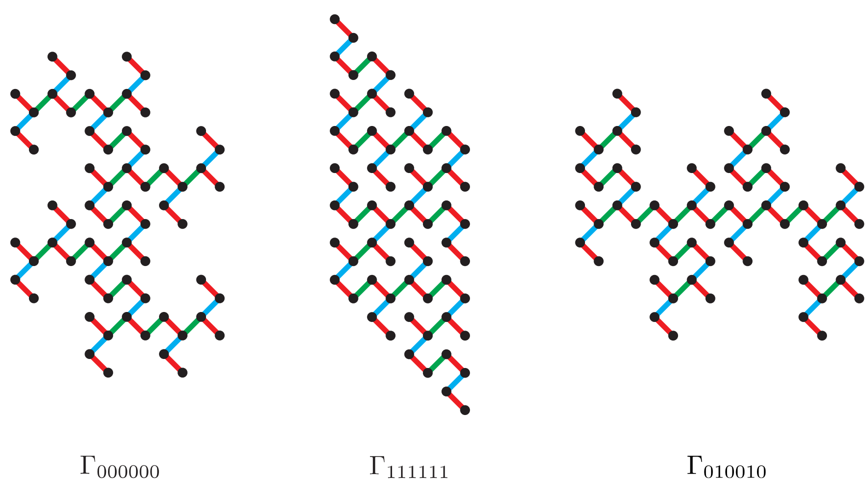

FIGURE 11. Graphs Γ_v

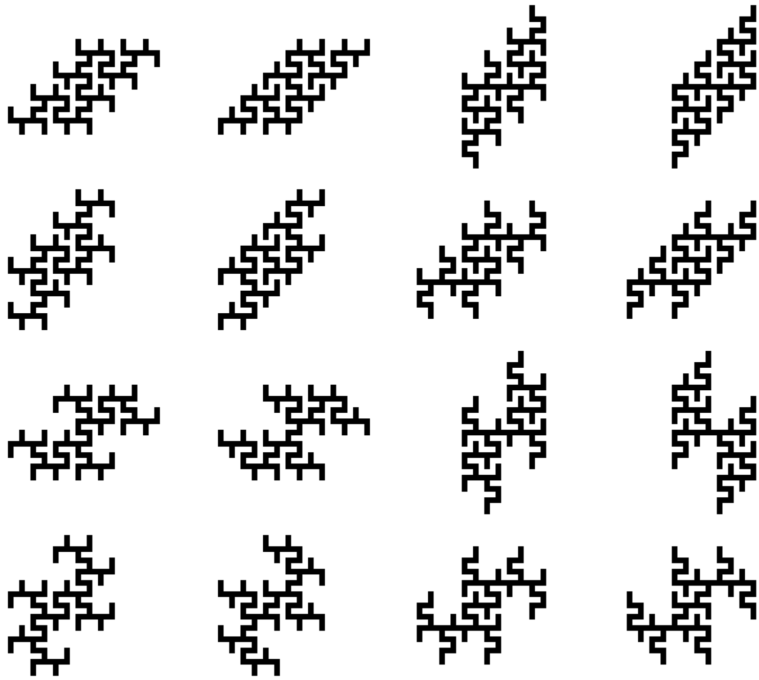

FIGURE 12. Graphs Γ_v for $|v| = 6$

The rule from Corollary 3.12 is shown in Figure 13. Note that in the transition from Γ_v to Γ_{1v} the relative position of the edges labeled by α (connecting $0w$ with $1w$), β (connecting $0w$ with $0\alpha(w)$) and γ (connecting $0w$ with $0\gamma(w)$ and $1w$ with $1\beta(w)$) is inverted. This can be corrected by taking the mirror image of Γ_{1v}.

The inductive rule shown in Figure 13 can be used to prove many properties of the graphs Γ_v, but we will use a more unified approach later.

3.5. **External angles.** The group generated by the binary adding machine (odometer) $\tau = \sigma(1, \tau)$ is the iterated monodromy group of the polynomial z^2. For every quadratic polynomial $h(z)$ the loop around infinity generates a self-similar cyclic subgroup of IMG (h) equivalent as a self-similar group to the group generated by the adding machine. If $h(z)$ is sub-hyperbolic, then the obtained embedding IMG $\left(z^2\right) \hookrightarrow$ IMG (h) induces a surjection from the circle (the Julia set of z^2) onto the Julia set of h, which agrees with the dynamics (i.e., is a semiconjugacy). This semiconjugacy coincides with the classical *Caratheodory loop* (see [12]) i.e., with the extension to the boundary of the biholomorphic conjugacy from the action of z^2 on the complement of the unit disc to the action of h on the complement of its filled Julia set (i.e., of the set of points that have bounded h-orbits).

Consider now the non-autonomous iterations of the polynomials $h_p(z)$, which are obtained by restricting the rational function $F(z, p)$ to the first coordinate z. Then there also exists a bi-holomorphic isomorphism Φ_q from the complement of the dendroid Julia set $J(q)$ in the z-plane $p = q$ to the complement of the unit disc. The isomorphism Φ_q is unique up to post-composition with a rotation. It follows that $\Phi_{f(p)} \circ h_p \circ \Phi_p^{-1}(z) = uz^2$, where u is a complex number of absolute value 1,

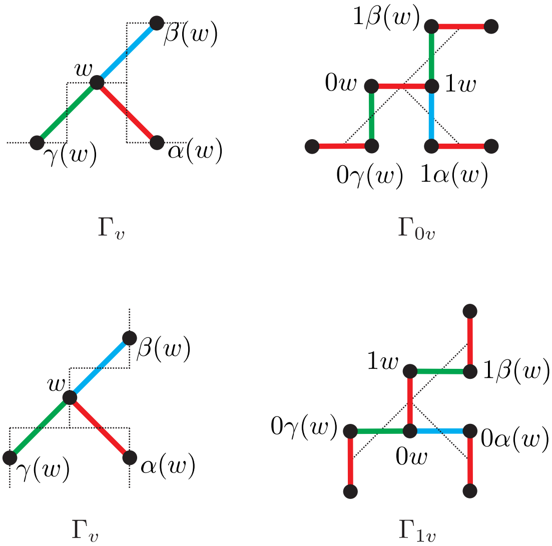

FIGURE 13. Inductive construction of Γ_v

depending on the particular choice of the isomorphisms Φ_p and $\Phi_{f(p)}$. Retracting the complement of the unit disc to the unit circle (using the map $z \mapsto z/|z|$) we get a circle bundle $\mathcal{O}$ over the Julia set J_f of f and the map $D = \Phi_{f(p)} \circ h_p \circ \Phi_p^{-1} : \mathcal{O} \longrightarrow \mathcal{O}$ acting as the angle doubling maps (composed with rotations) on the fibers of $\mathcal{O}$ and as $f : J_f \longrightarrow J_f$ on the base of the bundle.

The iterated monodromy of $D : \mathcal{O} \longrightarrow \mathcal{O}$ is the subgroup $\mathcal{R} = \langle P, S, \tau \rangle < \mathrm{IMG}\,(F)$, where $\tau = \gamma\alpha\beta$ is the loop around infinity in a z-plane. Recall that τ commutes with S and P, so that $\langle \tau \rangle$ is a normal subgroup of $\mathcal{R}$.

The generators of the subgroup $\mathcal{R}$ are given by the wreath recursion

$$\begin{aligned} P &= \pi, \\ S &= \pi\sigma(P\tau^{-1}, P, S^{-1}\tau^{-1}, S^{-1}), \\ \tau &= \sigma(1, \tau, 1, \tau). \end{aligned}$$

Recall that P is an involution, so $P = P^{-1}$.

It follows from the wreath recursion that $\langle \tau \rangle$ is a self-similar group whose limit space is $\mathbb{R}/\mathbb{Z} \times \{\mathtt{L}, \mathtt{R}\}^{-\omega}$. The limit dynamical system acts on it by the angle doubling map on $\mathbb{R}/\mathbb{Z}$ and by the shift on $\{\mathtt{L}, \mathtt{R}\}^{-\omega}$. The embedding $\langle \tau \rangle \hookrightarrow \mathcal{G}$ will induce a surjection from $\mathbb{R}/\mathbb{Z} \times \{\mathtt{L}, \mathtt{R}\}^{-\omega}$ to $\mathcal{J}_{\mathcal{G}}$. The surjection will map a circle $\mathbb{R}/\mathbb{Z} \times \{w\}$ for $w \in \{\mathtt{L}, \mathtt{R}\}^{-\omega}$ to the connected component of $\mathcal{J}_{\mathcal{G}}$ corresponding to the sequence w.

We get the following commutative diagram of embeddings of self-similar groups:

$$\begin{array}{ccc} \langle \tau \rangle & \longrightarrow & \mathcal{R} \\ \downarrow & & \downarrow \\ \mathcal{G} & \longrightarrow & \mathrm{IMG}\,(F) \end{array} \tag{18}$$

inducing semiconjugacies of the corresponding dynamical systems

$$\begin{array}{ccc} \mathbb{R}/\mathbb{Z}\times\{\mathtt{L},\mathtt{R}\}^{-\omega} & \longrightarrow & \mathcal{O} \\ \Big\downarrow & & \Big\downarrow \\ \mathcal{J}_{\mathcal{G}} & \longrightarrow & J_1 \end{array} \tag{19}$$

The vertical arrows of (19) are maps that are identical on the bases of the corresponding fiber bundles and are Caratheodory loops around the Julia sets $J(p)$ on the fibers. The horizontal semiconjugacies are homeomorphisms on the fibers, and they make in the bases the necessary identifications in the Cantor set $\{\mathtt{L},\mathtt{R}\}^{-\omega}$ producing the Julia set of f.

We will now study the embeddings of the self-similar groups from the commutative diagram (18) to understand the identifications in the semiconjugacies from (19).

Proposition 3.13. *The nucleus of the group $\mathcal{R}=\langle P,S,\tau\rangle$ is the set*

$$\mathcal{N}=\{1,S,S^{-1},P,\tau,\tau^{-1},S\tau,S^{-1}\tau^{-1},P\tau,P\tau^{-1}\}.$$

Proof. It follows directly from the recursion (and the fact that τ commutes with P and S) that $\mathcal{N}$ is a symmetric state-closed set (a subset A of a self-similar group is called *state-closed* if for every $g\in A$ and $x\in X$ we have $g|_x\in A$). We have to prove that the sections of the elements

$$\{S,S^{-1},\tau,\tau^{-1},S\tau,S^{-1}\tau^{-1}\}\cdot\{S,\tau\}$$

eventually belong to $\mathcal{N}$ (sections of P are trivial in non-empty words, so we do not have to consider it). But this follows from the equalities

$$\begin{aligned} S^2&=(PS^{-1}\tau^{-1},PS^{-1}\tau^{-1},S^{-1}P\tau,S^{-1}P\tau),\\ S\tau^{-1}&=\pi(P\tau^{-1},P\tau^{-1},S^{-1}\tau^{-1},S^{-1}\tau^{-1}),\\ S^2\tau&=\sigma(PS^{-1}\tau^{-1},PS^{-1},S^{-1}P\tau,S^{-1}P),\\ \tau^2&=(\tau,\tau,\tau,\tau),\\ S\tau^2&=\pi\sigma(P,P\tau,S^{-1},S^{-1}\tau). \end{aligned}$$

See the Moore diagram of the nucleus in Figure 14. ☐

Let $w=\ldots X^{(3)}_{i_3}X^{(2)}_{i_2}X^{(1)}_{i_1}$ be an element of $\{\mathtt{L}_0,\mathtt{L}_1,\mathtt{R}_0,\mathtt{R}_1\}^{-\omega}$, where $X^{(k)}\in\{\mathtt{L},\mathtt{R}\}$ and $i_k\in\{0,1\}$. Then denote

$$p(w)=\ldots X^{(3)}X^{(2)}X^{(1)}\in\{\mathtt{L},\mathtt{R}\}^{-\omega}$$

and

$$\theta(w)=\sum_{k=1}^{\infty}\frac{i_k}{2^k}\in\mathbb{R}/\mathbb{Z}.$$

The following proposition describes which points $(\theta,p(w))\in\mathbb{R}/\mathbb{Z}\times\{\mathtt{L},\mathtt{R}\}^{\infty}$ are mapped to the same point of the bundle of Caratheodori loops $\mathcal{O}$ by the semiconjugacy $\mathbb{R}/\mathbb{Z}\times\{\mathtt{L},\mathtt{R}\}^{-\omega}\longrightarrow\mathcal{O}$ from the commutative diagram (19).

Proposition 3.14. *Two sequences are asymptotically equivalent with respect to $\mathcal{R}$ if and only if they are equal to sequences $w_1,w_2\in\{\mathtt{L}_0,\mathtt{L}_1,\mathtt{R}_0,\mathtt{R}_1\}^{-\omega}$ such that one of the following conditions is satisfied:*

(1) $p(w_1)=p(w_2),\quad \theta(w_1)=\theta(w_2)$,
(2) $p(w_1)=(\mathtt{RL})^{-\omega},\quad p(w_2)=(\mathtt{LR})^{-\omega},\quad \theta(w_1)=\theta(w_2)+\frac{2}{3}$,

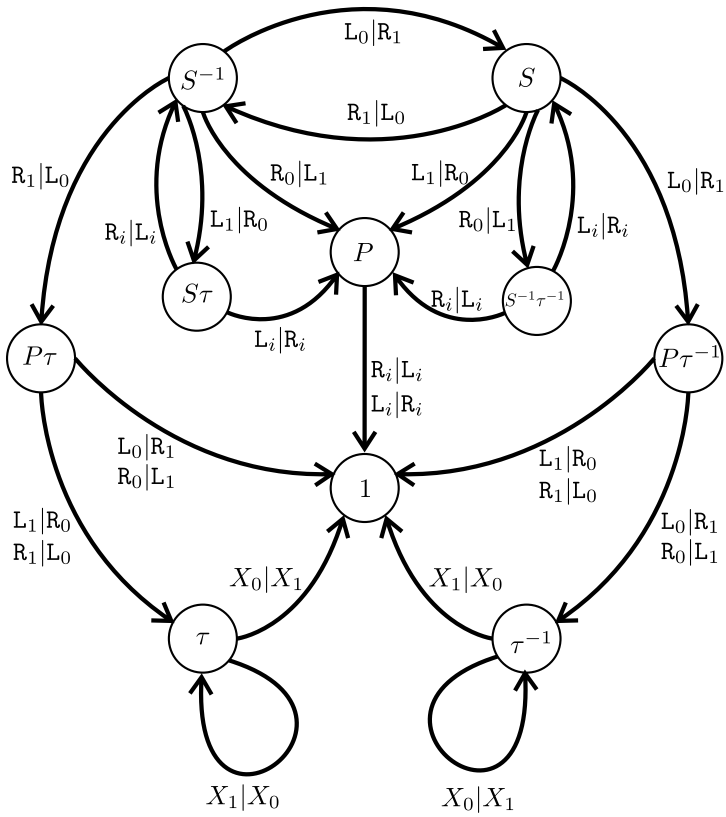

FIGURE 14. Moore diagram of the nucleus of $\mathcal{R}$

(3) $p(w_1) = (\mathtt{RL})^{-\omega}\mathtt{L}, \quad p(w_2) = (\mathtt{LR})^{-\omega}\mathtt{R}, \quad \theta(w_1) = \theta(w_2) + \frac{1}{3}$,

(4) *there exists a non-empty word* $v \in \{\mathtt{L}, \mathtt{R}\}^*$ *such that*

$$p(w_1) = (\mathtt{RL})^{-\omega}\mathtt{L}v, \quad p(w_2) = (\mathtt{LR})^{-\omega}\mathtt{R}v', \quad \theta(w_1) = \theta(w_2) + \frac{1}{2^{|v|}3},$$

where v' *is obtained from* v *by changing the first letter.*

Proof. Note that the wreath recursion for τ and P can be written in terms of the self-similarity biset as

$$\tau \cdot X_0 = X_1, \quad \tau \cdot X_1 = X_0 \cdot \tau,$$

$$P \cdot \mathtt{R}_i = \mathtt{L}_i, \quad P \cdot \mathtt{L}_i = \mathtt{R}_i,$$

where X is one of the symbols $\mathtt{L}, \mathtt{R}$ and i is one of the symbols $0, 1$.

Note also that $\tau S = \pi(P, P, S^{-1}, S^{-1})$; hence

$$S \cdot \mathtt{L}_i = \tau^{-1} \cdot \mathtt{R}_i \cdot P, \quad S \cdot \mathtt{R}_i = \tau^{-1} \cdot \mathtt{L}_i \cdot S^{-1},$$

which implies

$$S^{-1} \cdot \mathtt{L}_i = \tau \cdot \mathtt{R}_i \cdot S, \quad S^{-1} \cdot \mathtt{R}_i = \tau \cdot \mathtt{L}_i \cdot P,$$

since S and τ commute.

Examining the nucleus of the group in Figure 14, we see that left-infinite (infinite in the past) paths labeled by (w_1, w_2) in its Moore diagram belong to one of the following types:

(I) The path travels inside the set $\{1, \tau, \tau^{-1}\}$. In this case we have $p(w_1) = p(w_2)$ and $\theta(w_1) = \theta(w_2)$, and every pair (w_1, w_2) satisfying these two equalities can be obtained this way.

(II) Its vertices alternatively belong to $\{S, S\tau\}$ and to $\{S^{-1}, S^{-1}\tau^{-1}\}$.

If the last vertex of the path belongs to $\{S, S\tau\}$, then $p(w_1) = (\mathtt{RL})^{-\omega}$ and $p(w_2) = (\mathtt{LR})^{-\omega}$. If the last vertex belongs to $\{S^{-1}, S^{-1}\tau^{-1}\}$, then $p(w_1) = (\mathtt{LR})^{-\omega}$ and $p(w_2) = (\mathtt{RL})^{-\omega}$, which is symmetric with the first case.

(III) The last vertex of the path belongs to $\{P, P\tau, P\tau^{-1}\}$. Then either $p(w_1) = (\mathtt{RL})^{-\omega}\mathtt{L}$ and $p(w_2) = (\mathtt{LR})^{-\omega}\mathtt{R}$ (if the previous vertex belongs to $\{S, S\tau\}$), or $p(w_1) = (\mathtt{LR})^{-\omega}\mathtt{R}$ and $p(w_2) = (\mathtt{RL})^{-\omega}\mathtt{L}$ (otherwise).

(IV) One of the vertices of the path (but not the last) belongs to $\{P, P\tau, P\tau^{-1}\}$. Then $p(w_1) = (\mathtt{RL})^{-\omega}\mathtt{L}v$ and $p(w_2) = (\mathtt{LR})^{-\omega}\mathtt{R}v'$, or $p(w_1) = (\mathtt{LR})^{-\omega}\mathtt{R}v$ and $p(w_2) = (\mathtt{RL})^{-\omega}\mathtt{L}v'$, where v' is obtained from v by changing the first letter.

In the first case of **(II)**, if $w_1 = \dots \mathtt{R}_{i_4}\mathtt{L}_{i_3}\mathtt{R}_{i_2}\mathtt{L}_{i_1}$ and $w_2 = \dots \mathtt{L}_{j_4}\mathtt{R}_{j_3}\mathtt{L}_{j_2}\mathtt{R}_{j_1}$, then for any n either

$$S \cdot \mathtt{R}_{i_{2n}}\mathtt{L}_{i_{2n-1}} \dots \mathtt{R}_{i_2}\mathtt{L}_{i_1} = \mathtt{L}_{j_{2n}}\mathtt{R}_{j_{2n-1}} \dots \mathtt{L}_{j_2}\mathtt{R}_{j_1} \cdot S$$

or

$$S \cdot \mathtt{R}_{i_{2n}}\mathtt{L}_{i_{2n-1}} \dots \mathtt{R}_{i_2}\mathtt{L}_{i_1} = \mathtt{L}_{j_{2n}}\mathtt{R}_{j_{2n-1}} \dots \mathtt{L}_{j_2}\mathtt{R}_{j_1} \cdot S\tau$$

or

$$\tau S \cdot \mathtt{R}_{i_{2n}}\mathtt{L}_{i_{2n-1}} \dots \mathtt{R}_{i_2}\mathtt{L}_{i_1} = \mathtt{L}_{j_{2n}}\mathtt{R}_{j_{2n-1}} \dots \mathtt{L}_{j_2}\mathtt{R}_{j_1} \cdot S$$

or

$$\tau S \cdot \mathtt{R}_{i_{2n}}\mathtt{L}_{i_{2n-1}} \dots \mathtt{R}_{i_2}\mathtt{L}_{i_1} = \mathtt{L}_{j_{2n}}\mathtt{R}_{j_{2n-1}} \dots \mathtt{L}_{j_2}\mathtt{R}_{j_1} \cdot S\tau.$$

This implies that either

$$\tau^{-1} \cdot \mathtt{L}_{i_{2n}} \cdot \tau \cdot \mathtt{R}_{i_{2n-1}} \dots \tau^{-1} \cdot \mathtt{L}_{i_2} \cdot \tau \cdot \mathtt{R}_{i_1} \cdot S = \mathtt{L}_{j_{2n}}\mathtt{R}_{j_{2n-1}} \dots \mathtt{L}_{j_2}\mathtt{R}_{j_1} \cdot S$$

or

$$\tau^{-1} \cdot \mathtt{L}_{i_{2n}} \cdot \tau \cdot \mathtt{R}_{i_{2n-1}} \dots \tau^{-1} \cdot \mathtt{L}_{i_2} \cdot \tau \cdot \mathtt{R}_{i_1} \cdot S = \mathtt{L}_{j_{2n}}\mathtt{R}_{j_{2n-1}} \dots \mathtt{L}_{j_2}\mathtt{R}_{j_1} \cdot S\tau$$

or

$$\mathtt{L}_{i_{2n}} \cdot \tau \cdot \mathtt{R}_{i_{2n-1}} \dots \tau^{-1} \cdot \mathtt{L}_{i_2} \cdot \tau \cdot \mathtt{R}_{i_1} \cdot S = \mathtt{L}_{j_{2n}}\mathtt{R}_{j_{2n-1}} \dots \mathtt{L}_{j_2}\mathtt{R}_{j_1} \cdot S$$

or

$$\mathtt{L}_{i_{2n}} \cdot \tau \cdot \mathtt{R}_{i_{2n-1}} \dots \tau^{-1} \cdot \mathtt{L}_{i_2} \cdot \tau \cdot \mathtt{R}_{i_1} \cdot S = \mathtt{L}_{j_{2n}}\mathtt{R}_{j_{2n-1}} \dots \mathtt{L}_{j_2}\mathtt{R}_{j_1} \cdot S\tau.$$

In all cases, as $n \to \infty$, we get

$$\theta(w_2) = \theta(w_1) + 1/2 - 1/4 + 1/8 - 1/16 + \dots = \theta(w_1) + 1/3 \pmod 1,$$

i.e., $\theta(w_1) = \theta(w_2) + 2/3$.

In the first case of **(III)** we have

$$w_1 = \dots \mathtt{R}_{i_5}\mathtt{L}_{i_4}\mathtt{R}_{i_3}\mathtt{L}_{i_2}\mathtt{L}_{i_1}$$

and

$$w_2 = \dots \mathtt{L}_{j_5}\mathtt{R}_{j_4}\mathtt{L}_{j_3}\mathtt{R}_{j_2}\mathtt{R}_{j_1},$$

and we have

$$\tau^{k_1} S \cdot \mathtt{R}_{i_{2n+1}}\mathtt{L}_{i_{2n}} \dots \mathtt{R}_{i_3}\mathtt{L}_{i_2}\mathtt{L}_{i_1} = \mathtt{L}_{j_{2n+1}}\mathtt{R}_{j_{2n}} \dots \mathtt{L}_{j_3}\mathtt{R}_{j_2}\mathtt{R}_{j_1} \cdot P\tau^{k_2}$$

for $k_1 \in \{0,1\}$ and $k_2 \in \{0,-1\}$. Then

$$\tau^{k_1-1} \cdot \mathtt{L}_{i_{2n+1}} \cdot \tau \cdot \mathtt{R}_{i_{2n}} \ldots \tau^{-1} \cdot \mathtt{L}_{i_3} \cdot \tau \cdot \mathtt{R}_{i_2} \cdot \tau^{-1} \cdot \mathtt{R}_{i_1} \cdot P = \mathtt{L}_{j_{2n+1}} \mathtt{R}_{j_{2n}} \ldots \mathtt{L}_{j_3} \mathtt{R}_{j_2} \mathtt{R}_{j_1} \cdot P \tau^{k_2},$$

which implies

$$\theta(w_2) = \theta(w_1) - 1/2 + 1/4 - 1/8 + \cdots = \theta(w_1) - 1/3 \pmod 1,$$

which proves case **(III)** of the proposition.

Now consider case **(IV)**. We have

$$w_1 = \ldots \mathtt{R}_{i_5} \mathtt{L}_{i_4} \mathtt{R}_{i_3} \mathtt{L}_{i_2} \mathtt{L}_{i_1} u$$

and

$$w_2 = \ldots \mathtt{L}_{j_5} \mathtt{R}_{j_4} \mathtt{L}_{j_3} \mathtt{R}_{i_2} \mathtt{R}_{i_1} u'$$

for some $u, u' \in \{\mathtt{L}_0, \mathtt{L}_1, \mathtt{R}_0, \mathtt{R}_1\}^*$, and for every n we have

$$\tau^{k_1} S \cdot \mathtt{R}_{i_{2n+1}} \mathtt{L}_{i_{2n}} \ldots \mathtt{R}_{i_3} \mathtt{L}_{i_2} \mathtt{L}_{i_1} u = \mathtt{L}_{j_{2n+1}} \mathtt{R}_{j_{2n}} \ldots \mathtt{L}_{j_3} \mathtt{R}_{j_2} \mathtt{R}_{j_1} u',$$

for some $k_1 \in \{0,1\}$. Hence

$$\tau^{k_1-1} \cdot \mathtt{L}_{i_{2n+1}} \cdot \tau \cdot \mathtt{R}_{i_{2n}} \ldots \tau^{-1} \cdot \mathtt{L}_{i_3} \cdot \tau \cdot \mathtt{R}_{i_2} \cdot \tau^{-1} \cdot \mathtt{R}_{i_1} \cdot P \cdot u = \mathtt{L}_{j_{2n+1}} \mathtt{R}_{j_{2n}} \ldots \mathtt{L}_{j_3} \mathtt{R}_{j_2} \mathtt{R}_{j_1} u'.$$

This implies that

$$\theta(w_2) = \theta(w_1) + \frac{1}{2^{|u|}} \left(-\frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots \right) = \theta(w_1) - \frac{1}{2^{|u|} \cdot 3},$$

which finishes the proof. □

Recall that the iterated monodromy group $\mathrm{IMG}(f)$ of $f(p) = \left(\frac{p-1}{p+1}\right)^2$ is the image of the group $\mathcal{G} = \langle \alpha, \beta, \gamma, R, S \rangle$ under the natural epimorphism $\overline{\mathcal{G}} \longrightarrow \mathcal{K}$ of self-similar groups described in Proposition 3.6. See also the diagram (16). Restricting the diagram (16) to subgroups we get

$$\begin{array}{ccc} \langle \tau \rangle & \hookrightarrow & \mathcal{R} \\ \downarrow & & \downarrow \\ \{1\} & \hookrightarrow & \mathrm{IMG}(f) \end{array} \tag{20}$$

where the vertical arrows are still surjective.

The embedding $\{1\} \hookrightarrow \mathrm{IMG}(f)$ induces the semiconjugacy from $\{\mathtt{L}, \mathtt{R}\}^{-\omega}$ to the Julia set of f. The asymptotic equivalence relation defined by $\mathrm{IMG}(f)$ (i.e., the identifictions produced by this semiconjugacy) is generated by the identifications

$$(\mathtt{RL})^{-\omega} \sim (\mathtt{LR})^{-\omega}, \quad (\mathtt{RL})^{-\omega} \mathtt{L} v \sim (\mathtt{LR})^{-\omega} \mathtt{R} v',$$

where $v \in \{\mathtt{L}, \mathtt{R}\}^*$ is arbitrary and v' is obtained from v by changing the first letter. This also follows from Proposition 3.14 just by ignoring the indices, i.e., the map θ.

The morphisms (20) induce semiconjugacies

$$\begin{array}{ccc} \mathbb{R}/\mathbb{Z} \times \{\mathtt{L}, \mathtt{R}\}^{-\omega} & \longrightarrow & \mathcal{J}_{\mathcal{R}} \\ \downarrow & & \downarrow \\ \{\mathtt{L}, \mathtt{R}\}^{-\omega} & \longrightarrow & \mathcal{J}_{\mathrm{IMG}(f)} \end{array} \tag{21}$$

where $\mathcal{J}_{\mathrm{IMG}(f)}$ is naturally identified with the Julia set of f.

The map $\widetilde{p} : \mathcal{J}_{\mathcal{R}} \longrightarrow \mathcal{J}_{\mathrm{IMG}(f)}$ is induced by the natural projection

$$p : \{\mathtt{L}_0, \mathtt{L}_1, \mathtt{R}_0, \mathtt{R}_1\}^{-\omega} \longrightarrow \{\mathtt{L}, \mathtt{R}\}^{-\omega}.$$

The fibers of the map $\widetilde{p}$ are circles by Proposition 3.14. It follows that the limit space of $\mathcal{R}$ can be interpreted as the bundle $\mathcal{O}$ over the Julia set of f of the Caratheodory loops around the p-slices $J(p)$ of the Julia set of F, and the map $\widetilde{p}$ is the natural projection from $\mathcal{O}$ to the Julia set of f.

Let us describe the limit space of the group IMG(f) following [14, Section 3.10]. As the zero step approximation of the tile of the group take a rectangle. The vertices of the rectangle (which will correspond to the boundary points of the tile) are labeled by the sequences $(\mathtt{RL})^{-\omega}\mathtt{L}, (\mathtt{LR})^{-\omega}\mathtt{R}, (\mathtt{LR})^{-\omega}, (\mathtt{RL})^{-\omega}$ in the given cyclic order counterclockwise. Hence, the zero step approximation of the limit space will be the rectangle with two pairs of vertices identified. In order to get the next approximation of the tile one has to take two copies of the previous approximation, append $\mathtt{R}$ to the end of the label of one of them and append $\mathtt{L}$ to the label of the other. After that one has to identify the point labeled by $(\mathtt{LR})^{-\omega}\mathtt{RR}$ with the point labeled by $(\mathtt{RL})^{-\omega}\mathtt{LL}$ and the point labeled by $(\mathtt{LR})^{-\omega}\mathtt{RL}$ with the point labeled by $(\mathtt{RL})^{-\omega}\mathtt{LR}$.

See the sixth approximation of the tile in the middle picture of Figure 15. The two pairs of the boundary points of the tile, which are identified in the limit space, are drawn close to each other, so that we get a picture approximating the limit space. The lefthand side of Figure 15 shows the identifications of the vertices of 64 rectangles made in the process of constructing the approximation of the limit space. Compare the obtained pictures with the Julia set of the rational function $u \mapsto \frac{u^2+1}{u^2-1}$, shown on the righthand side of Figure 15. This rational function is conjugate to $f : p \mapsto \left(\frac{1-p}{1+p}\right)^2$ via the identification $p = \frac{u-1}{u+1}$.

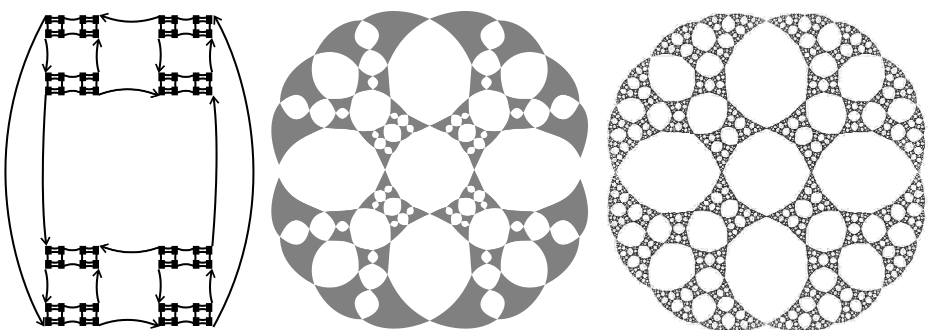

FIGURE 15. The Julia set of $(p-1)^2/(p+1)^2$ and its combinatorial model

If we apply just the identifications (1) of Proposition 3.14 to the space $\{\mathtt{L}_0, \mathtt{L}_1, \mathtt{R}_0, \mathtt{R}_1\}^{-\omega}$, i.e., if we consider the limit space of $\langle\tau\rangle$, then we will get the direct product of the Cantor set $\{\mathtt{L}, \mathtt{R}\}^{-\omega}$ with the circle $\mathbb{R}/\mathbb{Z}$.

Figure 16 shows the remaining identifications producing the limit space of $\mathcal{R}$. The arrows show which sequences $w \in \{\mathtt{L}, \mathtt{R}\}^{-\omega}$ are identified, while the labels are the rotations applied to the corresponding circles. Namely, if we have an arrow from w_1 to w_2 labeled by θ_0, then each point θ of the circle above w_1 is identified with the point $\theta + \theta_0$ of the circle above w_2. The limit dynamical system acts on the Julia set of f as f (equivalently, as the shift on $\{\mathtt{L}, \mathtt{R}\}^{-\omega}$), and on the circles as the map $\theta \mapsto 2\theta$. Note that the identifications described by Proposition 3.14 and Figure 16 are such that the resulting map on the limit space of $\mathcal{R}$ is well defined.

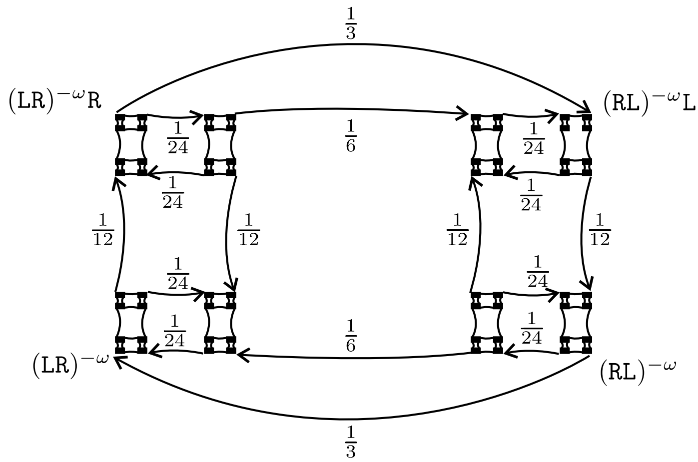

FIGURE 16. Building the space of external angles

The embedding $\mathcal{R} \hookrightarrow \mathrm{IMG}(F)$ induces a semiconjugacy of the limit spaces

$$\Phi : \mathcal{J}_{\mathcal{R}} \longrightarrow \mathcal{J}_{\mathrm{IMG}(F)} = J_1.$$

We call the points of $\mathcal{J}_{\mathcal{R}}$ *external rays.* We say that an external ray $\zeta \in \mathcal{J}_{\mathcal{R}}$ *lands* on $(z, p) \in J_1$ if $\Phi(\zeta) = (z, p)$. Points of $\mathcal{J}_{\mathcal{R}}$ are encoded by pairs (θ, w), where $w \in \{\mathtt{L}, \mathtt{R}\}^{-\omega}$ is a sequence representing a point $p \in J_1$ and $\theta \in \mathbb{R}/\mathbb{Z}$. The coordinate θ is called the *angle* of the external ray. Note that the angle of an external ray may not be uniquely defined, since a point of the Julia set of f may be represented by different sequences. On the other hand, the difference between angles of two external rays above the same point of the Julia set of f is well defined, since two circles are pasted to each other (in Proposition 3.14) using a rotation.

Proposition 3.15. *Denote by q_1 the fixed point ≈ 0.2956 of $f(p) = \left(\frac{1-p}{1+p}\right)^2$; see Figure 7. If p_0 belongs to the backward orbit $\bigcup_{n\geq 0} f^{-n}(q_1)$ of q_1, then there are two external rays landing on (p_0, p_0). The difference of the angles of these external rays is equal to $\frac{1}{2^{k-1}3}$, where k is the smallest integer such that $f^k(p_0) = q_1$. In all the other cases there is a unique ray landing on (p_0, p_0).*

Recall from Subsection 3.1 that the line $z = p$ is contained in J_1 and is an F-invariant subset of the post-critical locus of F.

Proof. It follows from the description of the asymptotic equivalence relation of the group $\mathrm{IMG}(f)$ that the fixed points of f are encoded in the limit space by the sequences $\mathtt{R}^{-\omega}, \mathtt{L}^{-\omega}$ and $(\mathtt{RL})^{-\omega} \sim (\mathtt{LR})^{-\omega}$. The transformation κ permutes the first two sequences and fixes the last one. Since κ corresponds to complex conjugation (see Proposition 3.8), we conclude that the real fixed point of f is encoded by the sequences $(\mathtt{RL})^{-\omega} \sim (\mathtt{LR})^{-\omega}$. Hence, the points of the backward orbit of q_1 are the points encoded by the sequences of the form $(\mathtt{RL})^{-\omega}v$, for $v \in \{\mathtt{R}, \mathtt{L}\}^*$.

It follows from the dynamics on the post-critical set of F that the points of the line $z = p$ are the only singular points with the isotropy group a conjugate of $\langle\gamma\rangle$. The wreath recursion in Theorem 3.3 implies that the points of the limit dynamical system encoded by the sequences $\ldots X_0^{(2)} X_0^{(1)}$ for $X^{(k)} \in \{\mathtt{L}, \mathtt{R}\}$ are the only singular points with isotropy group a conjugate of $\langle\gamma\rangle$. Hence these sequences encode the points $z = p_0$, where p_0 is encoded by $\ldots X^{(2)}X^{(1)}$ in the limit space of $\mathrm{IMG}\,(f)$. (For the orbispace structure of the limit dynamical systems, see Chapters 4 and 5 of [14].)

Let us see to which sequences of the form $\ldots X_{i_2}^{(2)}X_{i_1}^{(1)}$ the sequence $\ldots X_0^{(2)}X_0^{(1)}$ can be equivalent. By Proposition 3.9, two such sequences, if they are equivalent with respect to $\mathrm{IMG}\,(F)$, are equivalent with respect to $\mathcal{G}$.

The nucleus of $\mathcal{G}$ for the wreath recursion of Theorem 3.3 is equal to

$$\{\alpha, \beta, \gamma, (\alpha\beta)^{\pm1}, \alpha^\beta, (\alpha\gamma)^{\pm1}, \beta^\alpha, (\beta\gamma)^{\pm1}, \gamma^\alpha, \gamma^\beta, \gamma^{\alpha\beta}, \tau^{\pm1}, (\alpha\tau)^{\pm1}, (\beta\tau)^{\pm1}\};$$

see (13) on page 321.

We are interested in the left-infinite paths in the Moore diagram of the nucleus with the arrows labeled by pairs of the form (X_0, X_i) for $X \in \{\mathtt{L}, \mathtt{R}\}$ and $i \in \{0, 1\}$. Removing all the other arrows and removing all arrows which do not belong to any left-infinite path, we get the graph shown in Figure 17.

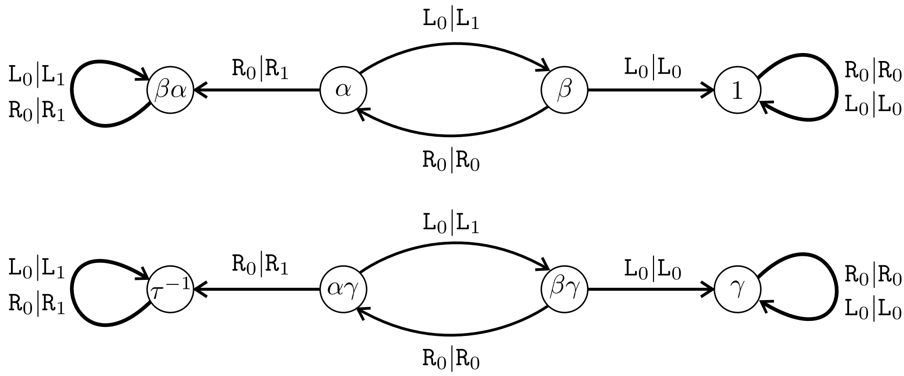

FIGURE 17. A part of the nucleus of $\mathcal{G}$.

It follows that we have only the following non-trivial identifications

$$(\mathtt{L}_0\mathtt{R}_0)^{-\omega}\mathtt{R}_0X_0^{(n)}\ldots X_0^{(2)}X_0^{(1)} \sim (\mathtt{L}_1\mathtt{R}_0)^{-\omega}\mathtt{R}_1X_1^{(n)}\ldots X_1^{(2)}X_1^{(1)},$$
$$(\mathtt{R}_0\mathtt{L}_0)^{-\omega}\mathtt{L}_0X_0^{(n)}\ldots X_0^{(2)}X_0^{(1)} \sim (\mathtt{R}_0\mathtt{L}_1)^{-\omega}\mathtt{L}_0X_0^{(n)}\ldots X_0^{(2)}X_0^{(1)},$$

their shifts and the identification

$$\ldots X_0^{(2)}X_0^{(1)} \sim \ldots X_1^{(2)}X_1^{(1)}.$$

The last identification is trivial in terms of the external angles.

It follows that the point $z = p_0$ is a landing point of one external ray to the slice $p = p_0$ of the Julia set of F except when p_0 is in the backward orbit of the fixed point q_1, when it is a landing point of exactly two external rays.

The remaining statements follow from Proposition 3.14. □

Note that the points of the backward orbit of q_1 are precisely the points where different Fatou components of f touch each other, i.e., the points belonging to boundaries of two Fatou components of f. This follows from the fact that the fixed point q_1 belongs to the boundaries of the Fatou components containing 0 and 1, and that every Fatou component of f is mapped by some iterations of f onto the Fatou components containing 0 and 1.

See Figure 18, where the external rays to the point (q_1, q_1) are shown.

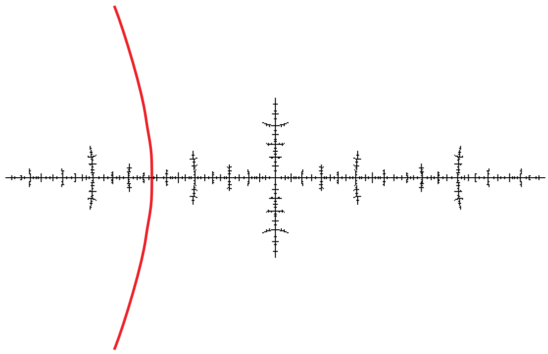

FIGURE 18. External rays landing at (q_1, q_1)

4. MATINGS

4.1. **An amalgam of $\mathcal{G}$ with itself.** Consider a copy $\mathcal{G}_1$ of the group $\mathcal{G}$ generated by

$$\alpha_1 = \sigma(\beta_1, \beta_1, \beta_1\alpha_1, \alpha_1\beta_1), \tag{22}$$
$$\beta_1 = (1, \beta_1\alpha_1\beta_1, \alpha_1, 1), \tag{23}$$
$$\gamma_1 = (\gamma_1, \beta_1, \gamma_1, \beta_1), \tag{24}$$

as in Theorem 3.3.

Let us conjugate the righthand side of the recursion defining $\alpha_1, \beta_1, \gamma_1$ by $\pi = (1,3)(2,4)$ (which corresponds to changing each $\mathtt{L}_i$ by $\mathtt{R}_i$ and vice versa). We then get an equivalent copy $\mathcal{G}_2$ of $\mathcal{G}$:

$$\alpha_2 = \sigma(\beta_2\alpha_2, \alpha_2\beta_2, \beta_2, \beta_2), \tag{25}$$
$$\beta_2 = (\alpha_2, 1, 1, \beta_2\alpha_2\beta_2), \tag{26}$$
$$\gamma_2 = (\gamma_2, \beta_2, \gamma_2, \beta_2). \tag{27}$$

Note that

$$\gamma_1\alpha_1\beta_1 = \sigma(1, \gamma_1\alpha_1\beta_1, 1, \gamma_1\alpha_1\beta_1).$$

Similarly,

$$\gamma_2\alpha_2\beta_2 = \sigma(1, \gamma_2\alpha_2\beta_2, 1, \gamma_2\alpha_2\beta_2),$$

which implies that $\gamma_1\alpha_1\beta_1 = \gamma_2\alpha_2\beta_2 = \tau$.

Denote by $\widehat{\mathcal{G}}$ the group generated by the set $\mathcal{G}_1 \cup \mathcal{G}_2$.

Lemma 4.1. *The elements β_1 and β_2 act non-trivially on disjoint sets of words and hence commute. The same is true for γ_1 and γ_2.*

Proof. The first statement follows directly from the wreath recursion. The second statement follows from the first. $\square$

Computer computation using GAP shows that $\widehat{\mathcal{G}}$ is contracting with the following nucleus of 122 elements:

$$\begin{aligned}\{1,\alpha_i,\beta_i,\gamma_i,\alpha_i^{\beta_i},\gamma_i^{\alpha_i},\gamma_i^{\beta_i},\beta_i^{\alpha_i},\\ \alpha_i^{\beta_j},\beta_i^{\alpha_j},\gamma_i^{\alpha_j},\gamma_i^{\beta_j},\alpha_i^{\alpha_j\beta_j},\gamma_i^{\alpha_i\beta_i},\beta_i^{\alpha_j\beta_j},\\ \alpha_i^B,\alpha_i^C,B,C,B^{\alpha_i},B^{\alpha_i\beta_i},C^{\alpha_i},C^{\beta_i},C^\tau,\alpha_i\beta_j\alpha_j\beta_j\}\\ \cup\{\alpha_i\beta_i,\alpha_i\gamma_i,\beta_i\gamma_i,\beta_i\alpha_j,\beta_i\gamma_j,\alpha_i\gamma_j,\\ \tau,\beta_i\alpha_i\beta_j,\beta_i\alpha_i\alpha_j,\alpha_i\beta_i\alpha_j\beta_j,\beta_i\gamma_i\alpha_j,\alpha_i\tau,\beta_i\tau,\alpha_iB,\beta_iC,\beta_i\alpha_iB,\\ \beta_i\tau\beta_j,C\tau,\alpha_iC\tau,\beta_iC\tau\}^{\pm1}.\end{aligned}$$

Here $\{i,j\}=\{0,1\}$ and $B=\beta_1\beta_2$, $C=\gamma_1\gamma_2$.

Proposition 4.2. *The connected components of the limit space of the group $\widehat{\mathcal{G}}$ are obtained by taking the slices $J(p)$ and $J(\overline{p})$ of the Julia set of F and gluing one to the other along the Caratheodory loop, where the external ray landing on (p,p) is identified with the external ray landing on $(\overline{p},\overline{p})$. If there are two external rays landing on (p,p) (i.e., if p belongs to the backward orbit of the real fixed point q_1 of f), then the Caratheodory loops are aligned in such a way that only one external ray landing on (p,p) is identified with the external ray landing on $(\overline{p},\overline{p})$.*

Equivalently, the connected components of the limit space of $\widehat{\mathcal{G}}$ are obtained by taking two copies of $J(p)$ and gluing the Caratheodory loop around one copy of $J(p)$ to its mirror reflection along the diameter containing a ray landing on (p,p).

Proof. It follows from Propositions 3.9 and 3.8 that the connected components of the limit space of $\widehat{\mathcal{G}}$ are obtained by gluing together the slice $J(p)$ of the Julia set of F with the slice $J(\overline{p})$. Since $\gamma_1\alpha_1\beta_1=\gamma_2\alpha_2\beta_2=\tau$, the Caratheodory loop around $J(p)$ is identified with the Caratheodory loop around $J(\overline{p})$ by the map induced by the map $\ldots X_{i_2}^{(2)}X_{i_1}^{(1)}\mapsto\ldots Y_{i_2}^{(2)}Y_{i_1}^{(1)}$ on the corresponding sets of sequences. Here $\ldots i_2i_1\in\{0,1\}^{-\omega}$ encodes the points of the circle $\mathcal{J}_{\langle\tau\rangle}$ and $\ldots X^{(2)}X^{(1)}=\kappa(\ldots Y^{(2)}Y^{(1)})$ is the sequence encoding the point p. The identification rule of the circles of external rays then follows from Proposition 3.14. $\square$

Classically (see [12]) the identifications described in Proposition 4.2 are called "matings". The only difference is that in the case of the classical mating the polynomials are monic, and the corresponding Caratheodory loops are reflected with respect to the real axis (which corresponds to the angle 0 external ray of a special fixed point of the polynomial). In our case we reflect the Caratheodory loop with respect to the diameter containing a ray landing on the points of the invariant line (p,p). Since there can be two rays landing on (p,p), there can be two "rotated matings".

As particular cases of the components described in Proposition 4.2 we get the mating of the polynomial $h_{q_0}(z)=\left(\frac{2z}{q_0+1}-1\right)^2$, for $q_0\approx-0.6478+1.7214i$, with itself (see a detailed analysis of this mating in [12]), and two rotated matings of the polynomial $h_{q_1}(z)=\left(\frac{2z}{q_1+1}-1\right)^2$, for $q_1\approx0.2956$, with itself.

4.2. **The self-similarity biset of $\widehat{\mathcal{G}}$.** Let $(\hat{\mathtt{L}}_0, \hat{\mathtt{L}}_1, \hat{\mathtt{R}}_0, \hat{\mathtt{R}}_1)$ be the ordered basis of the self-similarity $\widehat{\mathcal{G}}$-biset corresponding to the original wreath recursion (22)–(27).

Then the $\mathcal{G}_1$-biset $\{\hat{\mathtt{L}}_0, \hat{\mathtt{L}}_1, \hat{\mathtt{R}}_0, \hat{\mathtt{R}}_1\} \cdot \mathcal{G}_1$ is naturally isomorphic to the self-similarity biset of $\mathcal{G}$ (if we identify $\mathcal{G}_1$ with $\mathcal{G}$ in the natural way). The isomorphism is given by the map

$$\hat{\mathtt{L}}_0 \mapsto \mathtt{L}_0, \quad \hat{\mathtt{L}}_1 \mapsto \mathtt{L}_1, \quad \hat{\mathtt{R}}_0 \mapsto \mathtt{R}_0, \quad \hat{\mathtt{R}}_1 \mapsto \mathtt{R}_1,$$

where $\{\mathtt{L}_0, \mathtt{L}_1, \mathtt{R}_0, \mathtt{R}_1\}$ is the usual basis of the self-similarity biset of $\mathcal{G}$ (giving the wreath recursion from Theorem 3.3).

The $\mathcal{G}_2$-biset $\{\hat{\mathtt{L}}_0, \hat{\mathtt{L}}_1, \hat{\mathtt{R}}_0, \hat{\mathtt{R}}_1\} \cdot \mathcal{G}_2$ is also isomorphic to the self-similarity biset of $\mathcal{G}$ via the mapping

$$\hat{\mathtt{L}}_0 \mapsto \mathtt{R}_0, \quad \hat{\mathtt{L}}_1 \mapsto \mathtt{R}_1, \quad \hat{\mathtt{R}}_0 \mapsto \mathtt{L}_0, \quad \hat{\mathtt{R}}_1 \mapsto \mathtt{L}_1.$$

The self-similarity biset of $\widehat{\mathcal{G}}$ is a direct sum (i.e., disjoint union) of the biset $\mathfrak{L} = \{\hat{\mathtt{L}}_0, \hat{\mathtt{L}}_1\} \cdot \widehat{\mathcal{G}}$ and $\mathfrak{R} = \{\hat{\mathtt{R}}_0, \hat{\mathtt{R}}_1\} \cdot \widehat{\mathcal{G}}$. Also denote for $i = 1, 2$

$$\mathfrak{L}_i = \{\hat{\mathtt{L}}_0, \hat{\mathtt{L}}_1\} \cdot \mathcal{G}_i, \qquad \mathfrak{R}_i = \{\hat{\mathtt{R}}_0, \hat{\mathtt{R}}_1\} \cdot \mathcal{G}_i.$$

Let us identify $\mathcal{G}_1$ and $\mathcal{G}_2$ with $\mathcal{G}$ in the natural way, so that $\mathfrak{L}_i$ and $\mathfrak{R}_i$ become $\mathcal{G}$-bisets. Note that then $\mathfrak{L}_1 = \mathfrak{R}_2$ and $\mathfrak{L}_2 = \mathfrak{R}_1$.

Let $a = \pi(a, a\alpha, a, \alpha a)$, which is the element $a = \pi$ of $\overline{\mathcal{G}}$ written with respect to the basis $\mathtt{L}_0 = \mathtt{e}_{00}$, $\mathtt{L}_1 = \mathtt{e}_{01} \cdot \beta$, $\mathtt{R}_0 = \mathtt{e}_{10} \cdot a$, $\mathtt{R}_1 = \mathtt{e}_{11} \cdot \beta\alpha_1 a$; see Proposition 3.4. Then a induces an automorphism of $\mathcal{G}$ by conjugation:

$$\alpha^a = \alpha, \quad \beta^a = \beta^\alpha, \quad \gamma^a = \gamma.$$

Let $\mathfrak{M}_0 = \{\mathtt{e}_{00}, \mathtt{e}_{01}\} \cdot \mathcal{G}$ and $\mathfrak{M}_1 = \{\mathtt{e}_{10}, \mathtt{e}_{11}\} \cdot \mathcal{G}$ be the natural $\mathcal{G}$-bisets; see Subsection 3.3.

Proposition 4.3. *Let $v = X^{(1)}X^{(2)} \dots X^{(n)} \in \{\mathfrak{L}, \mathfrak{R}\}^n$ denote*

$$x_i = \begin{cases} 0 & \text{if } X^{(i)} = \mathfrak{L}, \\ 1 & \text{if } X^{(i)} = \mathfrak{R}. \end{cases}$$

Then the biset $X_1^{(1)} \otimes X_1^{(2)} \otimes \cdots \otimes X_1^{(n)}$ is isomorphic to the biset

$$\mathfrak{M}_{x_1} \otimes \mathfrak{M}_{x_1+x_2} \otimes \mathfrak{M}_{x_2+x_3} \otimes \cdots \otimes \mathfrak{M}_{x_{n-1}+x_n} \cdot a^{x_n},$$

where addition of indices is modulo two.

The $\mathcal{G}$-biset $X_2^{(1)} \otimes X_2^{(2)} \otimes \cdots \otimes X_2^{(n)}$ is isomorphic to the biset

$$\mathfrak{M}_{1+x_1} \otimes \mathfrak{M}_{x_1+x_2} \otimes \mathfrak{M}_{x_2+x_3} \otimes \cdots \otimes \mathfrak{M}_{x_{n-1}+x_n} \cdot a^{1+x_n}.$$

Proof. We have $\mathtt{L}_0 = \mathtt{e}_{00}$, $\mathtt{L}_1 = \mathtt{e}_{01} \cdot \beta_1$, so $\mathfrak{L}_1 = \mathfrak{M}_0$. We have $\mathtt{R}_0 = \mathtt{e}_{10} \cdot a$ and $\mathtt{L}_1 = \mathtt{e}_{11} \cdot \beta\alpha a$, hence $\mathfrak{R}_1$ is identified with $\mathfrak{M}_1 \cdot a$. Consequently, $\mathfrak{L}_2$ is isomorphic to $\mathfrak{M}_1 \cdot a$, and $\mathfrak{R}_2$ is isomorphic to $\mathfrak{M}_0$.

Note that it follows from the wreath recursions defining $\overline{\mathcal{G}}$ that $a \cdot \mathfrak{M}_0 \cong \mathfrak{M}_1$ and $a \cdot \mathfrak{M}_1 \cong \mathfrak{M}_0$.

Consequently, the biset $X_1^{(1)} \otimes X_1^{(2)} \otimes \cdots \otimes X_1^{(n)}$ is isomorphic to the biset $\mathfrak{M}_{x_1} \cdot a^{x_1} \otimes \mathfrak{M}_{x_2} \cdot a^{x_2} \otimes \cdots \otimes \mathfrak{M}_{x_n} \cdot a^{x_n}$, which is isomorphic to $\mathfrak{M}_{x_1} \otimes \mathfrak{M}_{x_1+x_2} \otimes \mathfrak{M}_{x_2+x_3} \otimes \cdots \otimes \mathfrak{M}_{x_{n-1}+x_n} \cdot a^{x_n}$.

Similarly, the $\mathcal{G}$-biset $X_2^{(1)} \otimes X_2^{(2)} \otimes \cdots \otimes X_2^{(n)}$ is isomorphic to $\mathfrak{M}_{1+x_1} \cdot a^{1+x_1} \otimes \mathfrak{M}_{1+x_2} \cdot a^{1+x_2} \otimes \cdots \otimes \mathfrak{M}_{x_n} \cdot a^{1+x_n}$, which is isomorphic to $\mathfrak{M}_{1+x_1} \otimes \mathfrak{M}_{x_1+x_2} \otimes \mathfrak{M}_{x_2+x_3} \otimes \cdots \otimes \mathfrak{M}_{x_{n-1}+x_n} \cdot a^{x_n}$. $\square$

4.3. **A virtually abelian subgroup of $\widehat{\mathcal{G}}$.** Denote $A = \alpha_2$, $B = \beta_1\beta_2$, and $C = \gamma_1\gamma_2$. We then have $\alpha_1 = CAB = BAC$.

Let us pass to the basis

$$\mathtt{x}_1 = \hat{\mathtt{L}}_0, \quad \mathtt{x}_2 = \hat{\mathtt{L}}_1 \cdot \beta_1, \quad \mathtt{x}_3 = \hat{\mathtt{R}}_0, \quad \mathtt{x}_4 = \hat{\mathtt{R}}_1 \cdot \beta_2,$$

i.e., conjugate the wreath recursion defining $\widehat{\mathcal{G}}$ by $(1, \beta_1, 1, \beta_2)$. We then get

$$\begin{aligned} CAB = \alpha_1 &= \sigma(1, 1, AC, CA), \\ \beta_1 &= (1, CAB, CAB, 1), \\ \gamma_1 &= (\gamma_1, \beta_1, \gamma_1, \beta_1), \\ A = \alpha_2 &= \sigma(BA, AB, 1, 1), \\ \beta_2 &= (A, 1, 1, A), \\ \gamma_2 &= (\gamma_2, \beta_2, \gamma_2, \beta_2). \end{aligned}$$

It follows that

$$\begin{aligned} A &= \sigma(BA, AB, 1, 1), \\ B &= (A, CAB, CAB, A), \\ C &= (C, B, C, B). \end{aligned}$$

Proposition 4.4. *The subgroup $\mathcal{H} = \langle A, B, C \rangle$ of $\widehat{\mathcal{G}}$ is equivalent as a self-similar group to the group of affine transformations of $\mathbb{C}$ of the form $z \mapsto \pm z + q$, where $q \in \mathbb{Z}[i]$. The isomorphism identifies A, B and C with the affine transformations*

$$z \cdot A = -z + 1, \quad z \cdot B = -z + 1 + i, \quad z \cdot C = -z.$$

The basis of the self-similarity biset is identified with the affine transformations

$$\begin{aligned} z \cdot \mathtt{x}_1 &= \frac{1}{1+i} z = \frac{1-i}{2} z, \\ z \cdot \mathtt{x}_2 &= \frac{1}{1+i}(-z + i) = -\frac{1-i}{2} z + \frac{1+i}{2}, \\ z \cdot \mathtt{x}_3 &= \frac{1}{1-i} z = \frac{1+i}{2} z, \\ z \cdot \mathtt{x}_4 &= \frac{1}{1-i}(-z + 1) = -\frac{1+i}{2} z + \frac{1+i}{2}. \end{aligned}$$

For identification of permutational bisets with sets of (partial) transformations, see Subsection 2.2. The biset structure comes from pre- and post-composition with the group action.

Proof. We have $\mathtt{x}_4 = A \cdot \mathtt{x}_3$ and $A \cdot \mathtt{x}_4 = \mathtt{x}_3$,

$$z \cdot A \cdot \mathtt{x}_1 = -\frac{1-i}{2} z + \frac{1-i}{2} = -\left(-\left(-\frac{1-i}{2} z + \frac{1+i}{2}\right) + 1 + i\right) + 1 = \mathtt{x}_2 \cdot BA;$$

hence $A \cdot \mathtt{x}_1 = \mathtt{x}_2 \cdot BA$ and $A \cdot \mathtt{x}_2 = \mathtt{x}_1 \cdot AB$, which agrees with the wreath recursion.

We have

$$z \cdot B \cdot \mathtt{x}_1 = \frac{1-i}{2}(-z + 1 + i) = -\frac{1-i}{2} z + 1 = z \cdot \mathtt{x}_1 \cdot A,$$

$$z \cdot B \cdot \mathtt{x}_2 = -\frac{1-i}{2}(-z + 1 + i) + \frac{1+i}{2} = \frac{1-i}{2} z - \frac{1-i}{2} = z \cdot \mathtt{x}_2 \cdot CAB,$$

since $z \cdot CAB = -z + i$,

$$z \cdot B \cdot \mathtt{x}_3 = \frac{1+i}{2}(-z+1+i) = -\frac{1+i}{2}z + i = z \cdot \mathtt{x}_3 \cdot CAB,$$

and

$$z \cdot B \cdot \mathtt{x}_4 = -\frac{1+i}{2}(-z+1+i) + \frac{1+i}{2} = \frac{1+i}{2}z + \frac{1-i}{2} = z \cdot \mathtt{x}_4 \cdot A,$$

which also agrees with the wreath recursion.

Finally, it is easy to check that $z \cdot C \cdot \mathtt{x}_1 = z \cdot \mathtt{x}_1 \cdot C$, $z \cdot C \cdot \mathtt{x}_3 = z \cdot \mathtt{x}_3 \cdot C$ and

$$z \cdot C \cdot \mathtt{x}_2 = \frac{1-i}{2}z + \frac{1+i}{2} = z \cdot \mathtt{x}_2 \cdot B,$$

and

$$z \cdot C \cdot \mathtt{x}_4 = \frac{1+i}{2}z + \frac{1+i}{2} = z \cdot \mathtt{x}_4 \cdot B.$$

□

4.4. **The limit dynamical system of $\mathcal{H}$.**

Proposition 4.5. *The limit $\mathcal{H}$-space $\mathcal{X}_{\mathcal{H}}$ is homeomorphic to the direct product of $\mathbb{C}$ with the Cantor set $\{\mathtt{L}, \mathtt{R}\}^{-\omega}$ with the natural (right) action of $\mathcal{H}$ on $\mathbb{C}$ and trivial action on $\{\mathtt{L}, \mathtt{R}\}^{-\omega}$. The self-similarity structure is given by*

$$\begin{aligned}
(z, \ldots y_2 y_1) \otimes \mathtt{x}_1 &= \left(\frac{1-i}{2}z, \ldots y_2 y_1 \mathtt{L}\right), \\
(z, \ldots y_2 y_1) \otimes \mathtt{x}_2 &= \left(-\frac{1-i}{2}z + \frac{1+i}{2}, \ldots y_2 y_1 \mathtt{L}\right), \\
(z, \ldots y_2 y_1) \otimes \mathtt{x}_3 &= \left(\frac{1+i}{2}z, \ldots y_2 y_1 \mathtt{R}\right), \\
(z, \ldots y_2 y_1) \otimes \mathtt{x}_4 &= \left(-\frac{1+i}{2}z + \frac{1+i}{2}, \ldots y_2 y_1 \mathtt{R}\right).
\end{aligned}$$

Proof. Direct corollary of Proposition 4.4 and Theorem 2.1. □

The orbispace $\mathbb{C}/\mathcal{H}$ is a flat surface homeomorphic to the sphere with four singular points, which are the images of the fixed points $1/2, (1+i)/2, 0$, and $i/2$ of the transformations A, B, C and CAB, respectively. Let us denote these singular points by Z_A, Z_B, Z_C and Z_{CAB}, respectively. A fundamental domain D of $\mathcal{H}$ is the rectangle with the vertices $i/2, 0, 1$ and $1 + i/2$.

The natural map $D \longrightarrow \mathbb{C}/\mathcal{H}$ folds this rectangle along the segment connecting $1/2$ and $(1+i)/2$ in two, so that we get a "pillowcase", whose vertices are the points Z_A, Z_B, Z_C and Z_{CAB}; see Figure 19.

The following is a direct corollary of the description of the limit $\mathcal{H}$-space given in Proposition 4.5.

Corollary 4.6. *The limit space $\mathcal{J}_{\mathcal{H}}$ is homeomorphic to the direct product $\mathbb{C}/\mathcal{H} \times \{\mathtt{L}, \mathtt{R}\}^{-\omega}$. The shift map $\mathsf{s} : \mathcal{J}_{\mathcal{H}} \longrightarrow \mathcal{J}_{\mathcal{H}}$ acts by the rule*

$$\mathsf{s}(z, \ldots y_2 y_1) = \begin{cases} ((1-i)z, \ldots y_3 y_2), & \text{if } y_1 = \mathtt{L}, \\ ((1+i)z, \ldots y_3 y_2), & \text{if } y_1 = \mathtt{R}. \end{cases}$$

Here $z, (1-i)z$, and $(1+i)z$ are complex numbers representing the corresponding points of the orbispace $\mathbb{C}/\mathcal{H}$.

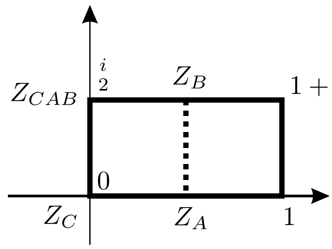

FIGURE 19. Fundamental domain of $\mathcal{H}$

4.5. **Schreier graphs of $\mathcal{H}$ and $\widehat{\mathcal{G}}$.** Consider the natural (right) action of $\mathcal{H}$ on $\mathbb{C}$. Take the basepoint $\xi = \frac{1+i}{4}$. It has trivial stabilizer in $\mathcal{H}$, hence we can consider the orbit of ξ as a vertex set of the left Cayley graph $\Gamma_{\mathcal{H}}$ of $\mathcal{H}$ with respect to the generating set A, B, C, CAB. See the Cayley graph on Figure 20. Here edges corresponding to the generators A, B, C, and CAB are orange, blue, green, and red, respectively.

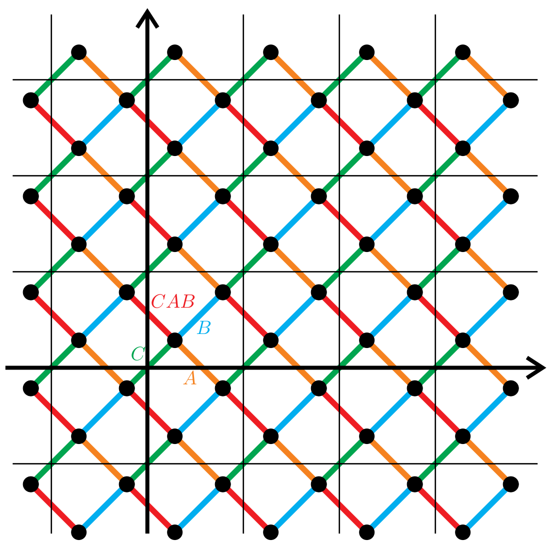

FIGURE 20. Cayley graph of $\mathcal{H}$

Denote by $\mathcal{H}_n$ the subgroup of $\mathcal{H}$ consisting of the affine transformations of the form $z \mapsto \pm z + q$, where $q \in \mathbb{Z}[i]$ is a Gaussian integer divisible by $(1+i)^n$.

It follows then from Proposition 4.4 that the Schreier graph of the action of $\mathcal{H}$ on the nth level of the tree consists of 2^n copies of the graph $\Gamma_n(\mathcal{H}) := \Gamma_{\mathcal{H}}/\mathcal{H}_n$. A fundamental domain of $\mathcal{H}_n$ is the rectangle with vertices 0, $(1+i)^n$, $i(1+i)^n/2$, and $(1+i/2)(1+i)^n$. Note that its sides are either parallel to the real and imaginary axis (for even n) or parallel to the diagonals $\Re(z) = \Im(z)$ and $\Re(z) = -\Im(z)$.

See Figure 21 for the Schreier graphs $\Gamma_5(\mathcal{H})$ and $\Gamma_4(\mathcal{H})$. Note that they have four loops, which we will usually omit in the sequel.

Figure 22 shows a more convenient way of drawing the Schreier graphs $\Gamma_n(\mathcal{H})$ and their subgraphs. Here the graphs $\Gamma_5(\mathcal{H})$ and $\Gamma_4(\mathcal{H})$ are drawn inside the

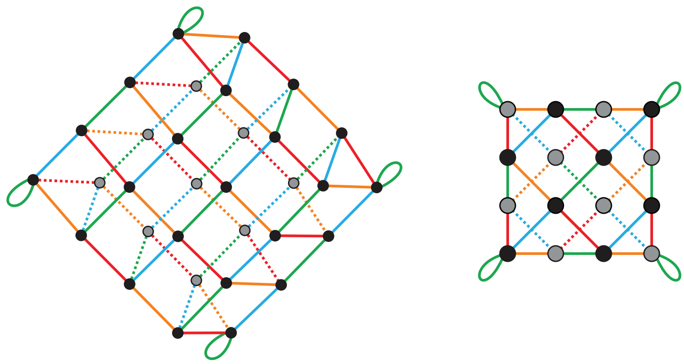

FIGURE 21. Schreier graphs $\Gamma_n(\mathcal{H})$

fundamental domains of the action of $\mathcal{H}_n$ on $\mathbb{C}$. In order to get the Schreier graphs one has to fold the rectangle into a square pillowcase (which corresponds to taking the quotient $\mathbb{C}/\mathcal{H}_n$).

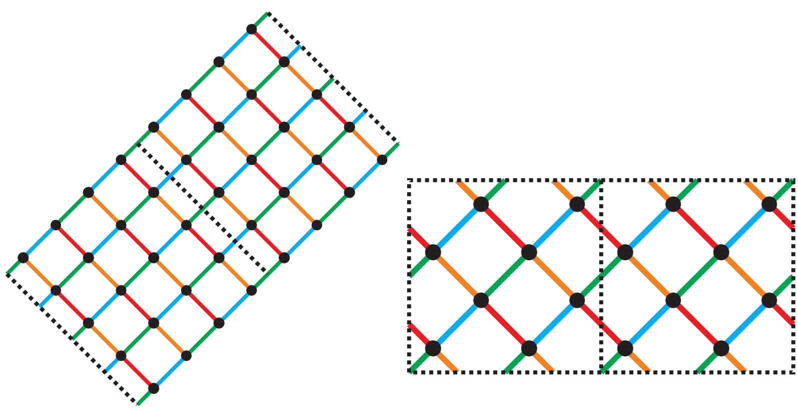

FIGURE 22. Unfolded Schreier graphs $\Gamma_{\mathcal{H}_n}$

Denote, for $v = X^{(1)}X^{(2)}\dots X^{(n)} \in \{\mathtt{L},\mathtt{R}\}^n$, by $\Gamma_{1,v}$ and $\Gamma_{2,v}$ the corresponding connected components of the Schreier graphs of the actions of $\mathcal{G}_1$ and $\mathcal{G}_2$ on the nth level of the tree. More precisely, they are the Schreier graphs of the left actions of $\mathcal{G}_i$ on the spaces of right orbits of the bisets

$$\{X_0^{(1)}, X_1^{(1)}\} \otimes \{X_0^{(2)}, X_1^{(2)}\} \otimes \cdots \otimes \{X_0^{(n)}, X_1^{(n)}\} \cdot \mathcal{G}_i.$$

The graphs $\Gamma_{1,v}$ and $\Gamma_{2,v}$ are isomorphic to the Schreier graphs Γ_{w_1} and Γ_{w_2} of the group $\mathcal{G}$, where $w_1, w_2 \in \{0,1\}^*$ are determined by the rules given in Proposition 4.3. Note that w_2 is obtained from w_1 by changing the first letter.

By Lemma 4.1, the graphs $\Gamma_{1,v}$ and $\Gamma_{2,v}$ have disjoint sets of edges such that their union is the set of edges of $\Gamma_n(\mathcal{H})$ (if we ignore the loops). Namely, the red edges of Figure 20 correspond to α_1, and the orange ones to α_2; each blue edge corresponds either to β_1 or to β_2, and each green edge either to γ_1 or to γ_2.

See Figure 23 for an example of the subgraphs $\Gamma_{1,v}$ and $\Gamma_{2,v}$ (colored red and black, respectively) of $\Gamma_6(\mathcal{H})$. We have removed the edges corresponding to the loops in $\Gamma_6(\mathcal{H})$.

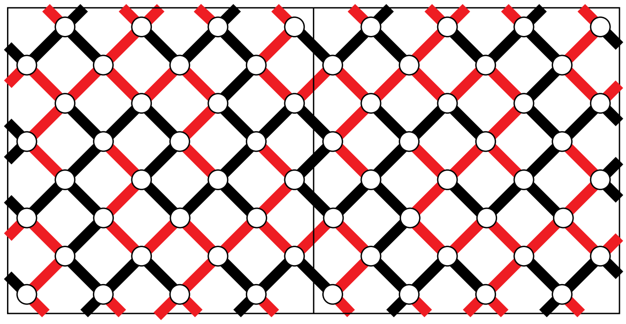

FIGURE 23. A component of a Schreier graph of $\widehat{\mathcal{G}}$

Note that each edge of $\Gamma_{\mathcal{H}}$ is a diagonal of a square with the sides of length $1/2$ parallel to the real and imaginary axes. These squares tile the plane and the pillowcases $\mathbb{C}/\mathcal{H}_n$, and each square of the tiling has precisely one diagonal belonging to the Cayley graph $\Gamma_{\mathcal{H}}$. By coloring the squares containing the edges of $\Gamma_{1,v}$ and $\Gamma_{2,v}$ in different colors (e.g., black and white), we get a nice visualization of the partition of $\Gamma_n(\mathcal{H})$ into the trees $\Gamma_{1,v}$ and $\Gamma_{2,v}$; see Figure 24. Here the squares whose diagonals are loops of $\Gamma_n(\mathcal{H})$ are colored blue. We will call them *singular*.

Let us denote by $K_{1,v}$ the union of the squares whose diagonals belong to $\Gamma_{1,v}$ and by $K_{2,v}$ the union of the squares whose diagonals belong to $\Gamma_{2,v}$ and of the singular squares.

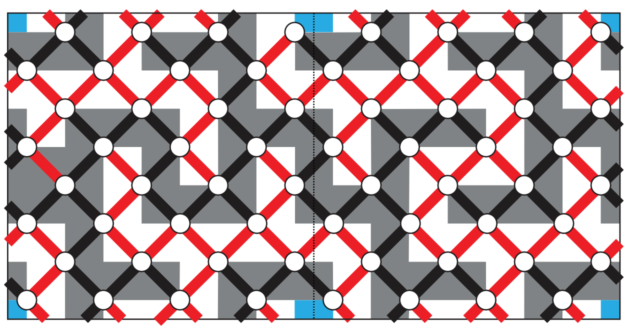

FIGURE 24. Partition into sets $K_{i,v}$

Proposition 4.7. *The boundary between $K_{1,v}$ and $K_{2,v}$ is a closed broken line λ_v describing the action of τ on the vertex set of $\Gamma_n(\mathcal{H})$. Namely, for every vertex u there are no vertices of $\Gamma_n(\mathcal{H})$ on λ_v between u and $\tau(u)$.*

Note that our choice to include the singular squares in $K_{2,v}$ is not very important. It will change only the side on which the path λ_v goes around the singular point of $\mathbb{C}/\mathcal{H}_n$.

Proof. We have $\tau = \gamma_1\alpha_1\beta_1 = \gamma_2\alpha_2\beta_2$. Consider the little squares of $\Gamma_n(\mathcal{H})$. Two of their sides (opposite to each other) correspond to $\alpha_1 = CAB$ and $\alpha_2 = A$, and the other two sides correspond to B and C. Each of the latter two edges may belong either to $\Gamma_{1,v}$ or to $\Gamma_{2,v}$. Figure 25 shows all four possible cases. Note that if an edge corresponding to B or C belongs to $\Gamma_{i,v}$, then its endpoints are fixed under the action of β_{1-i}, γ_{1-i}, respectively. This information makes it possible to determine for one of the pairs of vertices of the square that one is the image of the other under the action of τ, as it is shown by the black arrows in Figure 25. If one of the edges of the squares is a loop of $\Gamma_n(\mathcal{H})$, then we assume that it belongs to $\Gamma_{2,v}$ (according to our convention about the set $K_{2,v}$). Note that the other agreement does not change the order in which λ_v connects the vertices of $\Gamma_n(\mathcal{H})$.

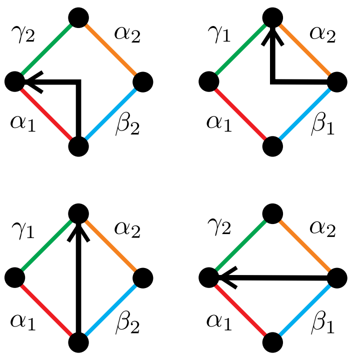

FIGURE 25. Action of τ

We see that the arrows describing the action of τ belong to the boundary λ_v between the sets $K_{1,v}$ and $K_{2,v}$. □

Consequently, the partition of the pillowcase $\mathbb{C}/\mathcal{H}_n$ into the sets $K_{1,v}$ and $K_{2,v}$ is an approximation of the mating described in Proposition 4.2. The boundary λ_v between the sets converges (as v converges to a left-infinite sequence $w \in \{\mathtt{L}, \mathtt{R}\}^{-\omega}$) to the map from the circle to a connected component of $\mathcal{J}_{\widehat{\mathcal{G}}}$ induced by the inclusion $\langle\tau\rangle < \widehat{\mathcal{G}}$.

See Figure 26, where two examples (approximations for $w = \mathtt{R}^{-\omega}$ and $(\mathtt{LR})^{-\omega}$) of the partition are given.

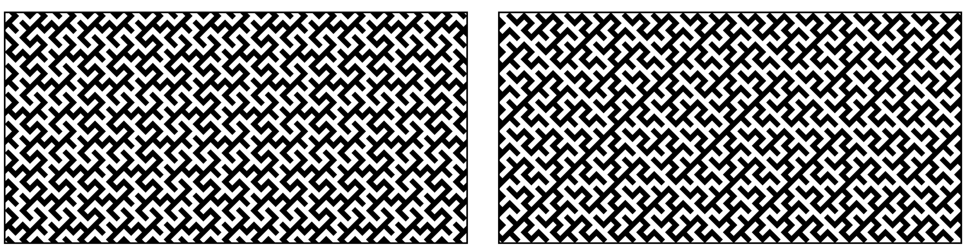

FIGURE 26. "Pillowcase ornaments"

We orient λ_v according to the action of τ, so that the oriented segments go from u to $\tau(u)$. Then λ_v goes around $\Gamma_{1,v}$ in the positive direction and around $\Gamma_{2,v}$ in the negative direction (if we orient $\mathbb{C}$ in the standard way); see Figure 25.

4.6. **Recursive rule of constructing $\Gamma_{1,v}$ and $\Gamma_{2,v}$.** It follows from Proposition 4.3 that the graphs $\Gamma_{1,v}$ and $\Gamma_{2,v}$ for $v = z_1z_2\ldots z_n \in \{\mathtt{L},\mathtt{R}\}^n$ are isomorphic to the Schreier graphs Γ_{w_1} and Γ_{w_2} of $\mathcal{G}$, where $w_1 = x_1x_2\ldots x_n$, $w_2 = y_1y_2\ldots y_n \in \{0,1\}^n$ are defined by the rule

$$x_k = y_k = \begin{cases} 0 & \text{if } z_{k-1} = z_k, \\ 1 & \text{otherwise,} \end{cases}$$

for $k \geq 2$, while x_1 and y_1 are defined by the same rule with the assumption $z_0 = \mathtt{L}$ and $z_0 = \mathtt{R}$, respectively.

Let us now translate the recursive rule from Corollary 3.11 of construction of the graphs Γ_v in terms of the sequences over the alphabet $\{\mathtt{L},\mathtt{R}\}$ and subgraphs of the graph $\Gamma_{\mathcal{H}}$.

Note that since the graphs $\Gamma_{i,v}$ are trees (as they are isomorphic to Γ_w for some w), they can be lifted by the natural quotient map $\Gamma_{\mathcal{H}} \longrightarrow \Gamma_{\mathcal{H}}/\mathcal{H}_n$ to a subgraph of the Cayley graph $\Gamma_{\mathcal{H}}$ of $\mathcal{H}$.

On the initial step (for the empty word $v = \emptyset$) the graphs $\Gamma_{i,\emptyset}$ consist of one vertex only, which is marked by $z_{\alpha,\emptyset}, z_{\beta,\emptyset}$, and $z_{\gamma,\emptyset}$ simultaneously. Choose a point in $\Gamma_{\mathcal{H}}$, which will be the lift of the graphs $\Gamma_{\emptyset,i}$. We will add, for convenience, halves of the incident edges of $\Gamma_{\mathcal{H}}$, corresponding to CAB, B, C for $\Gamma_{1,\emptyset}$ and A, B, C for $\Gamma_{2,\emptyset}$. Let us denote the obtained graphs by $\Delta_{1,\emptyset}$ and $\Delta_{2,\emptyset}$, respectively.

Suppose that we have constructed the graphs $\Delta_{1,v}$ and $\Delta_{2,v}$, which are lifts of the graphs $\Gamma_{1,v}$ and $\Gamma_{2,v}$, respectively, with three marked vertices $z_{\alpha,v}$, $z_{\beta,v}$, and $z_{\gamma,v}$ and halves of some edges of $\Gamma_{\mathcal{H}}$ attached to the marked vertices. Let $z'_{\alpha,v}$, $z'_{\beta,v}$, and $z'_{\gamma,v}$ be the other ("hanging") vertices of the half-edges.

Then the graphs $\Delta_{i,vx}$ for $i \in \{1,2\}$, $x \in \{\mathtt{L},\mathtt{R}\}$ together with the marking are obtained by the following rule.

Denote $\Delta_{i,v,0} = \Delta_{i,v}$ and let $\Delta_{i,v,1}$ be $\Delta_{i,v}$ rotated by 180° around $z'_{\alpha,v}$. Take the union of $\Delta_{i,v,0}$ with the $\Delta_{i,v,1}$, connecting in this way the respective copies of $z_{\alpha,v}$ by an edge.

The copy of $z_{\gamma,v}$ in $\Delta_{i,v,0}$ is the vertex $z_{\gamma,vx}$. The copy of $z_{\gamma,v}$ in $\Delta_{i,v,1}$ is the vertex $z_{\beta,vx}$. If the last letter of v coincides with x (or if $v = \emptyset$, $x = \mathtt{L}$, $i = 1$, or $v = \emptyset, x = \mathtt{R}, i = 2$), then the copy of $z_{\beta,v}$ in $\Delta_{i,v,1}$ is $z_{\alpha,vx}$; otherwise $z_{\alpha,vx}$ is the copy of $z_{\beta,v}$ in $\Delta_{i,v,0}$. Remove the half-edge attached to the other (unmarked) copy of $z_{\beta,v}$. The obtained graph is $\Delta_{i,vx}$. The graph $\Gamma_{i,vx}$ is obtained from it by removing the three half-edges attached to the marked vertices.

See the first three steps of this recursion (for $i = 1$) in Figure 27.

Note that it follows directly from the construction that the points $z'_{\alpha,v}$, $z'_{\beta,v}$ and $z'_{\gamma,v}$ are vertices of a right isosceles triangle. Orientation of the triangles $z'_{\alpha,v}z'_{\beta,v}z'_{\gamma,v}$ depends on the last letter of v: it is counterclockwise if it is $\mathtt{L}$ and clockwise if it is $\mathtt{R}$.

5. Paper-folding curves

5.1. **Mazes associated with graphs $\Gamma_{1,v}$.** Consider again the Cayley graph $\mathcal{H}$ drawn in $\mathbb{C}$, as in Figure 20. Consider the half-integral grid on $\mathbb{C}$, i.e., the tiling of the plane by the parallel translations by the elements of $\mathbb{Z}[i]/2$ of the square with the vertices $0, 1/2, i/2$, and $1/2 + i/2$. The group $\mathcal{H}$ acts freely on the set of these squares with two orbits (corresponding to the two colors of the checkerboard

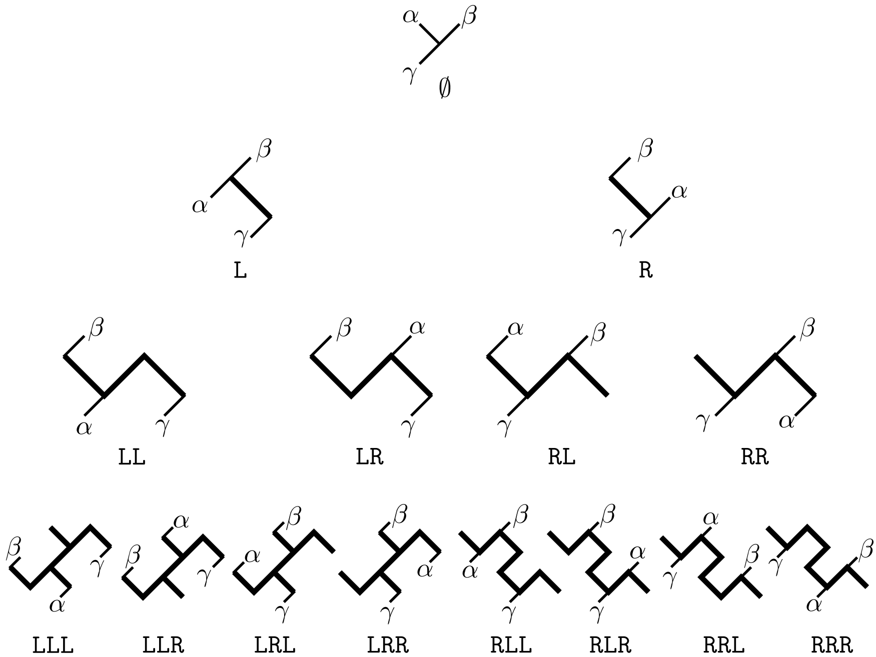

FIGURE 27. Graphs $\Gamma_{1,v}$

coloring). We get this way a checkerboard coloring of the pillowcases $\mathbb{C}/\mathcal{H}_n$. Let Q_n be the graph consisting of the sides of the half-integral grid on $\mathbb{C}/\mathcal{H}_n$.

The vertices of the graph $\Gamma_{1,v}$ are the centers of the squares of one color in the checkerboard coloring of $\mathbb{C}/\mathcal{H}_n$. Since $\Gamma_{1,v}$ is a tree, there is a closed Eulerian path ρ_v in Q_n without transversal self-intersections, which goes around $\Gamma_{1,v}$, i.e., does not intersect it transversally (see Figure 28) where the squares containing $\Gamma_{1,v}$ are colored red and the other squares are white. The path ρ_v is the boundary of the white region (after we glue the picture into a pillowcase).

Note that, unlike the path λ_v, there are no problems in the definition of ρ_v concerning the singular points.

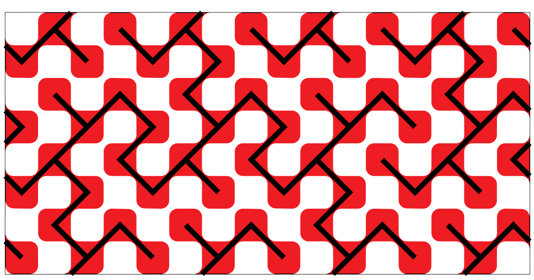

FIGURE 28. Path ρ_v

See more examples of the paths ρ_v in Figure 29, where their connected lifts to $\mathbb{C}$ are shown.

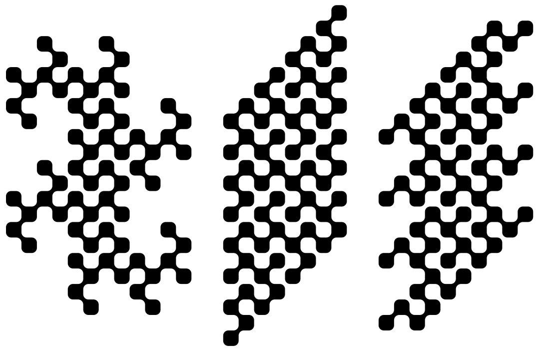

FIGURE 29. Paths ρ_v around Γ_v

Recall that the curve λ_v describes the action of τ on the vertices of $\Gamma_{1,v}$. It is easy to check that ρ_v is obtained from λ_v by the replacements of the segments of λ_v between the vertices of $\Gamma_{1,v}$ by the curves shown in Figure 30. In particular, the curves ρ_v also approximate the plane-filling curves coming from the matings described in Proposition 4.2.

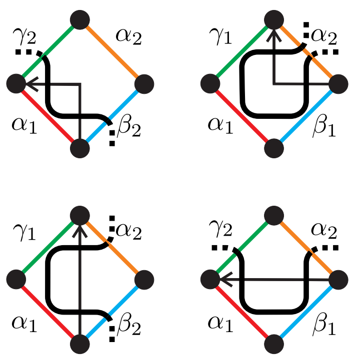

FIGURE 30. Changing λ_v to ρ_v

5.2. **Another pair of Schreier graphs.** The curve ρ_n in Figure 28 goes around $\Gamma_{1,v}$, bounding the red cells of the checkerboard tiling of the pillowcase $\mathbb{C}/\mathcal{H}_n$. It also goes around the white cells, and these cells are arranged into a tree around which ρ_v travels. Let us try to interpret this "white" tree in terms of the group $\mathcal{G}$.

Let $\zeta_1 = 1/4 + i/4$ and $\zeta_2 = 1/4 - i/4$. Let $\Sigma_1 = \Gamma_{\mathcal{H}}$ be the left Cayley graph of $\mathcal{H}$ with the set of vertices $\zeta_1 \cdot \mathcal{H}$ and the edges corresponding to the generators A, B, C, CAB. Let Σ_2 be the left Cayley graph of $\mathcal{H}$ with the set of vertices $\zeta_2 \cdot \mathcal{H}$ and the edges corresponding to the generators A, ABA, C, ABC.

See Figure 31 for the graphs Σ_1, Σ_2. Note that each edge of Σ_1 intersects exactly one edge of Σ_2 and vice versa. Namely, the edges of Σ_1 corresponding to A, B, C, and CAB intersect the edges of Σ_2 corresponding to A, ABA, C, and ABC, respectively.

For a given word $v = X^{(1)}X^{(2)} \dots X^{(n)} \in \{\mathtt{L}, \mathtt{R}\}^*$ of length n, denote by $\Sigma_{1,v}$ and $\Sigma_{2,v}$ the Schreier graphs of the groups $\mathcal{G}_1$ and $\mathcal{G}_2$ acting on the set $X^{(1)} \times X^{(2)} \times$

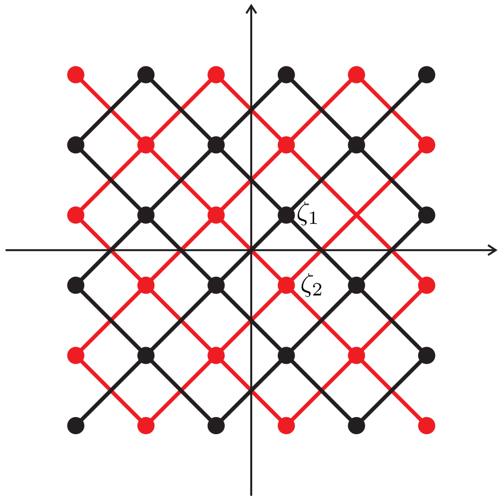

FIGURE 31. Graphs Σ_1 and Σ_2.

$\cdots \times X^{(n)}$ (i.e., on the set of the right orbits of the bisets $X_1^{(1)} \otimes X_1^{(2)} \otimes \cdots \otimes X_1^{(n)}$ and $X_2^{(1)} \otimes X_2^{(2)} \otimes \cdots \otimes X_2^{(n)}$, respectively), defined with respect to the generating sets $\{\alpha_1 = CAB, \beta_1, \gamma_1\}$ and $\{\alpha_2 = A, \alpha_2\beta_2\alpha_2, \gamma_2\}$. We identify them with the corresponding sub-graphs of the graphs $\Sigma_1/\mathcal{H}_n$ and $\Sigma_2/\mathcal{H}_n$, respectively.

Proposition 5.1. *The graphs $\Sigma_{1,v}$ and $\Sigma_{2,v}$ do not intersect (as subsets of $\mathbb{C}/\mathcal{H}_n$). In each pair of intersecting edges of $\Sigma_1/\mathcal{H}_n$ and $\Sigma_2/\mathcal{H}_n$ one edge belongs to one of the graphs $\Sigma_{1,v}$ and $\Sigma_{2,v}$ and the other edge does not belong to either graph.*

It follows that ρ_v separates the trees $\Sigma_{1,v} = \Gamma_{1,v}$ and $\Sigma_{2,v}$ and that the graphs $\Sigma_{1,v}$ and $\Sigma_{2,v}$ describe adjacency of the cells on the corresponding side of the curve ρ_v.

Proof. Connect the basepoints ζ_1 and ζ_2 by a straight segment ℓ. The points ζ_1 and ζ_2 as vertices of the Cayley graphs Σ_1 and Σ_2 correspond to the identity in the group $\mathcal{H}$. Consequently, the images of ℓ under the action of $\mathcal{H}$ connect the vertices of Σ_1 and Σ_2 corresponding to the same elements of $\mathcal{H}$. It follows now from the construction of the graphs Σ_i (see Figure 32) that the edge connecting $h \in \mathcal{H}$ to Ch in Σ_1 intersects with the edge connecting the corresponding vertices in Σ_2. The same statement for the edges connecting h to Ah is true. The edge in Σ_1 connecting Ah to BAh intersects the edge in Σ_2 connecting h to $ABAh$.

If the graph $\Sigma_{1,v}$ contains an edge from w to $\gamma_1(w) = C(w)$, then the graph $\Sigma_{2,v}$ does not contain an edge connecting w to $\gamma_2(w)$, and vice versa. The edge w to $\gamma_1(w)$ coincides with the edge from some h to Ch in Σ_1, while the edge from w to $\gamma_2(w)$ coincides with the edge from the same element h to Ch in Σ_2. This settles the statement for the edges $(w, \gamma_i(w))$.

The edge from w to $\alpha_2(w)$ corresponds to the edge from h to Ah, which is not included into Σ_1. Similarly, the edge from w to $\alpha_1(w) = CAB(w)$ is not included into Σ_2.

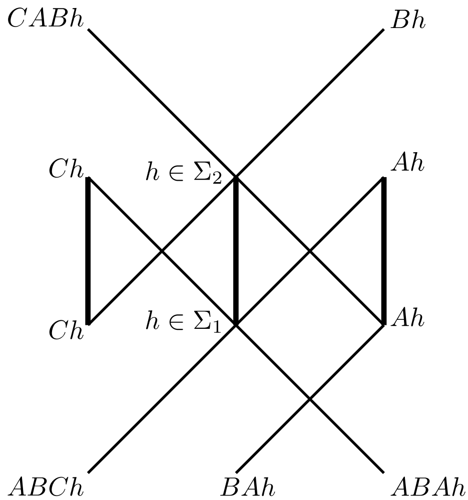

FIGURE 32. Proof of Proposition 5.1

If $\Sigma_2(w)$ contains an edge from w to $\alpha_2\beta_2\alpha_2(w) = ABA(w)$, then $w \neq \alpha_2\beta_2\alpha_2(w)$, i.e., $\alpha_2(w) \neq \beta_2\alpha_2(w)$, which is equivalent to the condition that the graph $\Sigma_1(w)$ does not have an edge from $A(w)$ to $BA(w)$. $\square$

Proposition 5.2. *The graphs $\Sigma_{1,v}$ and $\Sigma_{2,v}$ are isomorphic for every word v.*

Proof. We know (see Proposition 4.3) that for every finite sequence $v = X_1X_2 \dots X_n \in \{\mathtt{L}, \mathtt{R}\}^*$ the $\mathcal{G}_1 \cong \mathcal{G}$-biset $X_1 \otimes X_2 \otimes \cdots \otimes X_n \cdot \mathcal{G}_1$ is isomorphic to the $\mathcal{G}_2 \cong \mathcal{G}$-biset $a \cdot X_1 \otimes X_2 \otimes \cdots \otimes X_n \cdot \mathcal{G}_2 \cdot a$. We have $\alpha = a\alpha a, \alpha\beta\alpha = a\beta a$ and $\gamma = a\gamma a$, which implies that the map

$$w \mapsto a(w)$$

is an isomorphism of the Schreier graphs $\Sigma_{2,v} \longrightarrow \Sigma_{1,v}$. $\square$

5.3. **Paper-folding.** Consider a lift $\Sigma'_{1,v}$ of the graph $\Sigma_{1,v}$ to $\mathbb{C}$, and let ρ'_v be the corresponding lift of the path ρ_v (i.e., a closed path going around the lift of the tree $\Sigma_{1,v}$).

Since the singular points Z_A, Z_B, Z_C, and Z_{CAB} of the orbifold $\mathbb{C}/\mathcal{H}_n$ belong to the graph Q_n for every n, the path ρ_v also passes through these points. The lift ρ'_v will pass through preimages Z'_A, Z'_B, Z'_C and Z'_{CAB} of the singular points.

Proposition 5.3. *The path ρ'_v consists of four pieces: $\rho_v(A, B)$ from Z'_A to Z'_B, $\rho_v(B, CAB)$ from Z'_B to Z'_{CAB}, $\rho_v(CAB, C)$ from Z'_{CAB} to Z'_C, and $\rho_v(C, A)$ from Z'_C to Z'_A.*

If the last letter of v is $\mathtt{L}$, or if v is empty (resp., if the last letter of v is $\mathtt{R}$), then the path, going consecutively through $\rho_v(A, B), \rho_v(B, CAB)$, $\rho_v(CAB, C)$, and $\rho_v(C, A)$, goes in the positive (resp., negative) direction around $\Sigma'_{1,v}$. Each path $\rho_v(X_1, X_2)$ is equal to the image of the previous path $\rho_v(X_0, X_1)$ under rotation by $-\pi/2$ (resp. $\pi/2$) around Z'_{X_1}.

The path $\rho_{v\mathtt{L}}(C, A)$ (resp. $\rho_{v\mathtt{R}}(C, A)$) can be taken equal to the union of the path $\rho_v(C, A) \cup \rho_v(A, B)$ with its image under rotation by $-\pi/2$ (resp. $\pi/2$) around Z'_B.

Proof. The proof is obtained by straightforward induction, using the inductive rule of constructing the graphs $\Gamma_{1,v}$. See Figure 33 for the inductive step. The points $z'_{\alpha,v}$, $z'_{\beta,v}$ and $z'_{\gamma,v}$ will coincide with the points Z'_{CAB}, Z'_B and Z'_C (which are denoted CAB, B and C in Figure 33). □

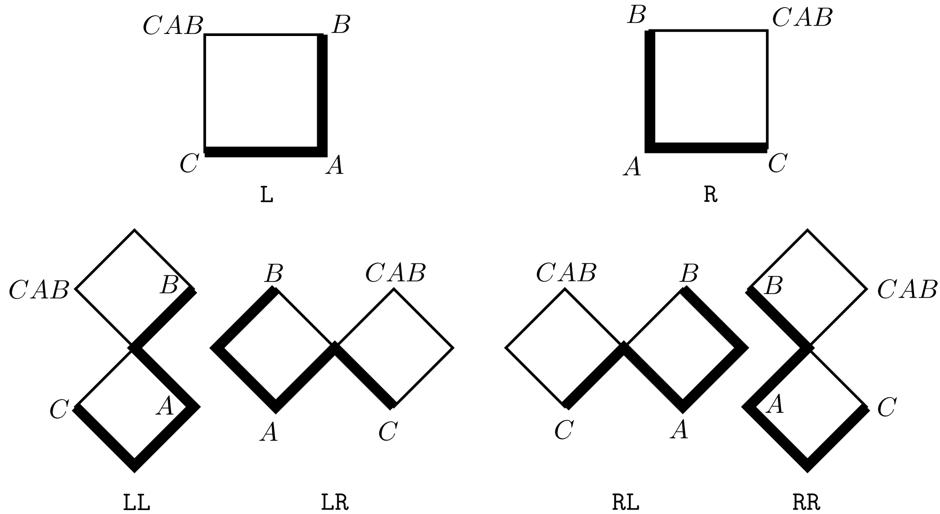

FIGURE 33. Inductive proof of Proposition 5.3

Consider a strip of paper of length 2^{n-1}. Let us denote one end of the strip by C. For a given word $v = X_1X_2\dots X_n$ of letters $X_i \in \{\mathtt{L}, \mathtt{R}\}$ fold the strip in two, fixing C and moving the other end of the strip to C on the left side, if $X_n = \mathtt{L}$, or on the right side, if $X_n = \mathtt{R}$ (see Figure 34). Now repeat the procedure for the word $X_1X_2\dots X_{n-1}$. After n steps unfold the strip so that all bends are at right angles. This way we get a broken line P_v. Take a copy of P_v, rotate it by 180° and connect its endpoints with the endpoints of P_v. We get a closed broken line $\overline{P}_v$.

The following statement is a direct corollary of Proposition 5.3.

Corollary 5.4. *The broken line $\overline{P}_v$ is isometric to the path ρ'_v going around the graph $\Sigma'_{1,v}$.*

6. Boundaries of Fatou components of f and rotated tunings

Consider the group $\mathcal{B} = \langle S, \gamma\alpha, \beta\rangle = \langle S\rangle \times \langle \gamma\alpha, \beta\rangle$. Let us write the images of the generators under the wreath recursion:

$$\begin{aligned} S &= \sigma\pi(P\beta\alpha\gamma, P, S^{-1}\beta\alpha\gamma, S^{-1}),\\ \gamma\alpha &= \sigma(1, \gamma\beta, \alpha, \gamma\alpha\beta),\\ \beta &= (1, \beta\alpha\beta, \alpha, 1). \end{aligned}$$

Conjugate the righthand side of the recursion by $(34)(\beta, \beta, P\beta\alpha\gamma\beta, P\beta)$:

$$\begin{aligned} S &= (13)(24)(1, 1, T, T),\\ \gamma\alpha &= \sigma(1, \beta\gamma, 1, \beta\gamma),\\ \beta &= (1, \alpha, 1, \alpha). \end{aligned}$$

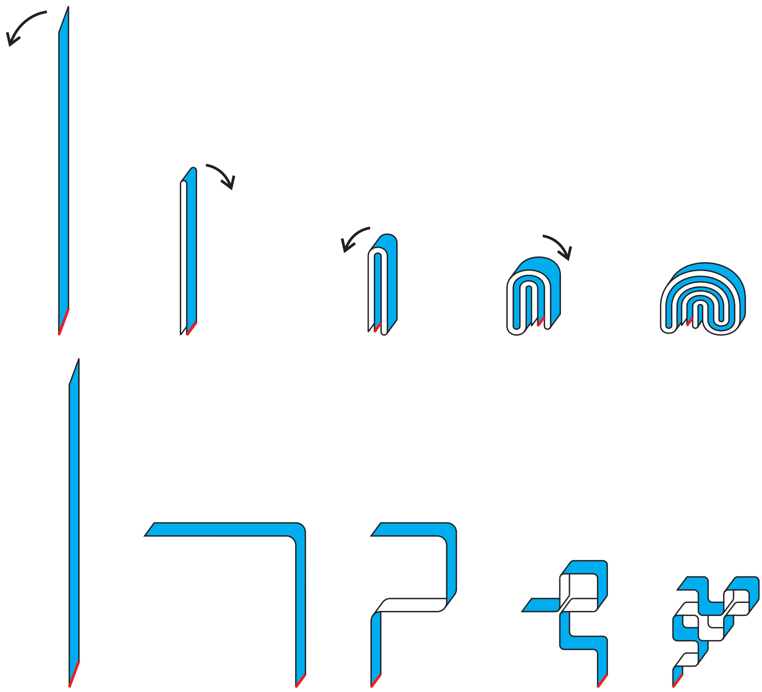

FIGURE 34. Folding the paper according to `RLRL`

We see that on the righthand side we get elements of the group $\mathcal{A} = \langle T, \beta\gamma, \alpha\rangle = \langle T\rangle \times \langle \beta\gamma, \alpha\rangle$. Moreover, the corresponding $(\mathcal{B}-\mathcal{A})$-biset $\mathfrak{M}_{\mathcal{B},\mathcal{A}}$ is the direct product of the $(\langle S\rangle - \langle T\rangle)$-biset given by the binary recursion

$$S = \sigma(1, T),$$

with the $(\langle \gamma\alpha, \beta\rangle - \langle \beta\gamma, \alpha\rangle)$-biset given by the recursion

$$\gamma\alpha = \sigma(1, \beta\gamma), \tag{28}$$

$$\beta = (1, \alpha). \tag{29}$$

The following proposition is proved by direct computation.

Proposition 6.1. *The biset over the free groups of rank two given by* (28)–(29) *is isomorphic to the biset associated with the partial covering*

$$\mathbb{C} \setminus \{0, 1\} \supset \mathbb{C} \setminus \{1, 0, 2\} \longrightarrow \mathbb{C} \setminus \{0, 1\}$$

defined by the polynomial $(1 - z)^2$, *where* $\gamma\alpha$ *and* β *correspond to the loops around the punctures* 0 *and* 1 *in the range of the covering, respectively, and the generators* $\beta\gamma$ *and* α *correspond to the loops around the punctures* 1 *and* 0 *in the domain of the covering, respectively.*

The generators of the group $\mathcal{A}$ are decomposed (in the original recursion) as follows:

$$\begin{aligned} T &= (P, \beta P\beta, \gamma S\gamma, S), \\ \beta\gamma &= (\gamma, \beta\alpha, \alpha\gamma, \beta), \\ \alpha &= \sigma(\beta, \beta, \beta\alpha, \alpha\beta). \end{aligned}$$

Conjugate the wreath recursion by $(1, \beta, \alpha, \beta)$:

$$\begin{aligned} T &= (P, P, S, S), \\ \beta\gamma &= (\gamma, \alpha\beta, \gamma\alpha, \beta), \\ \alpha &= \sigma. \end{aligned}$$

Restricting to the last two coordinates of the wreath product (i.e., to the biset $\mathfrak{N}$) we get a biset $\mathfrak{M}_{\mathcal{A},\mathcal{B}}$, which is a direct product of the biset defined by the isomorphism $T \mapsto S$ and the $(\langle\beta\gamma, \alpha\rangle - \langle\gamma\alpha, \beta\rangle)$-biset given by the recursion

$$\beta\gamma = (\gamma, \alpha\beta), \tag{30}$$

$$\alpha = \sigma. \tag{31}$$

The proof of the following proposition is also straightforward.

Proposition 6.2. *The biset over the free groups defined by the wreath recursion* (30)–(31) *is isomorphic to the biset associated with the partial covering*

$$\mathbb{C} \setminus \{0, 1\} \supset \mathbb{C} \setminus \{0, 1, 1/2\} \longrightarrow \mathbb{C} \setminus \{0, 1\},$$

given by the polynomial $(2z - 1)^2$, where the generators α and $\beta\gamma$ correspond to the loops around 0 *and* 1 *in the domain, while the generators γ and $\alpha\beta$ correspond to the loops around* 1 *and* 0 *in the range, respectively.*

Taking tensor products $\mathfrak{M}_{\mathcal{B},\mathcal{A}} \otimes_{\mathcal{A}} \mathfrak{M}_{\mathcal{A},\mathcal{B}}$ and $\mathfrak{M}_{\mathcal{A},\mathcal{B}} \otimes_{\mathcal{B}} \mathfrak{M}_{\mathcal{B},\mathcal{A}}$ we see that $\mathcal{A}$ and $\mathcal{B}$ are self-similar subgroups of IMG $\left(F^{\circ 2}\right)$.

Corollary 6.3. *The bisets $\mathfrak{M}_{\mathcal{B},\mathcal{A}} \otimes_{\mathcal{A}} \mathfrak{M}_{\mathcal{A},\mathcal{B}}$ and $\mathfrak{M}_{\mathcal{A},\mathcal{B}} \otimes_{\mathcal{B}} \mathfrak{M}_{\mathcal{B},\mathcal{A}}$ are isomorphic to the bisets associated with the post-critically finite polynomials $(2(1 - z)^2 - 1)^2 = (2z^2 - 4z + 1)^2$ and $(1 - (2z - 1)^2)^2 = 16z^2(1 - z)^2$, respectively.*

Proof. It follows from Propositions 6.1 and 6.2. ☐

Note that the polynomials $16z^2(1 - z)^2$ and $(2z^2 - 4z + 1)^2$ coincide with the restrictions of the second iteration of the endomorphism F of $\mathbb{PC}^2$ to the post-critical lines $p = 1$ and $p = 0$, respectively. The action of F is written in the homogeneous coordinates as

$$[z : p : u] \mapsto [(2z - p - u)^2 : (p - u)^2 : (p + u)^2],$$

hence its restriction to the line $p = u$ is

$$[z : p : p] \mapsto [(2z - 2p)^2 : 0 : 4p^2],$$

so that it acts on the first coordinate (in the non-homogeneous coordinates) as

$$z \mapsto (z - 1)^2.$$

The restriction of F to the line $p = 0$ is

$$[z : 0 : u] \mapsto [(2z - u)^2 : u^2 : u^2],$$

i.e.,

$$z \mapsto (2z - 1)^2.$$

Proposition 6.4. *The limit spaces of $(\mathcal{A}, \mathfrak{M}_{\mathcal{A},\mathcal{B}} \otimes \mathfrak{M}_{\mathcal{B},\mathcal{A}})$ and $(\mathcal{B}, \mathfrak{M}_{\mathcal{B},\mathcal{A}} \otimes \mathfrak{M}_{\mathcal{A},\mathcal{B}})$ are direct products of the circle with the Julia sets of the polynomials $16z^2(1 - z)^2$ and $(2z^2 - 4z + 1)^2$, respectively. The images of $\mathcal{J}_{\mathcal{A}}$ and $\mathcal{J}_{\mathcal{B}}$ in $\mathcal{J}_{\mathrm{IMG}(F)}$ are identified with the subsets of the Julia set projected by $(z, p) \mapsto p$ to the boundaries of the Fatou components of $f(p) = \left(\frac{p-1}{p+1}\right)^2$, containing* 1 *and* 0*, respectively.*

Proof. The description of the limit space follows directly from the structure of the wreath recursion and Corollary 6.3. The groups $\langle\alpha, \beta\gamma\rangle$ and $\langle\beta, \gamma\alpha\rangle$ are level-transitive, which implies that the map from the limit spaces of $\mathcal{A}$ and $\mathcal{B}$ to the limit space of IMG (F) is surjective on the fibers of the natural projection $\mathcal{J}_{\mathrm{IMG}(F)} \longrightarrow \mathcal{J}_{\mathrm{IMG}(f)}$.

It follows from the post-critical dynamics of f and the interpretation of the maps S and P as loops in the space $\mathbb{C} \setminus \{0, 1\}$ (see Proposition 3.8) that the images of the limit spaces $\mathcal{J}_{\mathcal{A}}$ and $\mathcal{J}_{\mathcal{B}}$ in the Julia set of f are the boundaries of the Fatou components of 1 and 0, respectively. □

See Figure 35, where the Julia sets of the polynomials $16z^2(1-z)^2$ and $(2z^2 - 4z + 1)^2$ are shown.

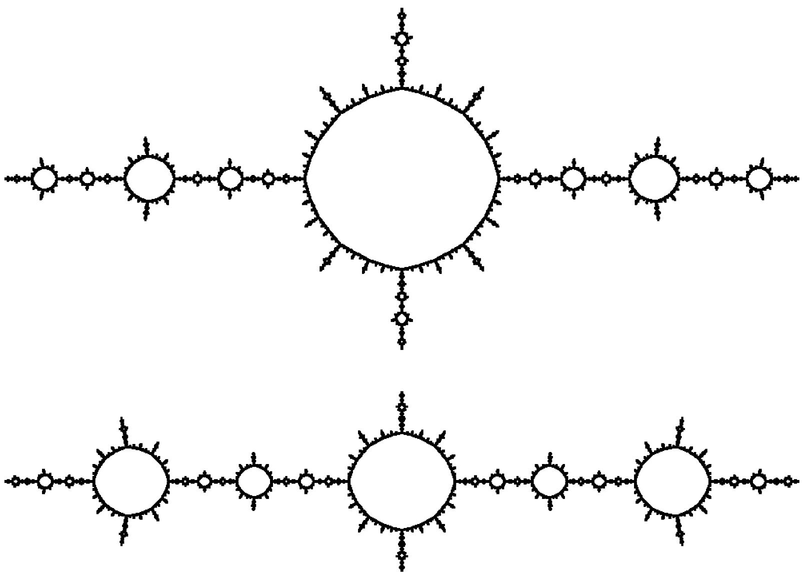

FIGURE 35. The Julia sets of $16z^2(1-z)^2$ and $(2z^2 - 4z + 1)^2$

The natural map from $\mathcal{J}_{\mathcal{A}}$ and $\mathcal{J}_{\mathcal{B}}$ to the Julia set of F are not injective. Let us see which points of the limit space are identified under these maps.

Consider the group $\mathcal{B}_1 = \langle\gamma, \alpha, S\rangle$. Its generators are written as

$$\begin{aligned} \alpha &= \sigma(\beta, \beta, \beta\alpha, \alpha\beta), \\ \gamma &= (\gamma, \beta, \gamma, \beta), \\ S &= \pi\sigma(P\beta\alpha\gamma, P, S^{-1}\beta\alpha\gamma, S^{-1}). \end{aligned}$$

Conjugating the righthand side by $(1, \beta, P, P\beta\alpha)$ we get

$$\begin{aligned} \alpha &= \sigma, \\ \gamma &= (\gamma, \beta, \gamma, \beta), \\ S\gamma\alpha &= \pi(1, 1, T\beta\gamma, T\beta\gamma). \end{aligned}$$

Consider now the group $\mathcal{A}_1 = \langle \gamma, \beta, T\rangle$. We have

$$\begin{aligned} \beta &= (1, \beta\alpha\beta, \alpha, 1), \\ \gamma &= (\gamma, \beta, \gamma, \beta), \\ T &= (P, \beta P\beta, \alpha S\alpha, S), \\ T\beta\gamma &= (P\gamma, \beta P\alpha, S\gamma\alpha, S\beta). \end{aligned}$$

The restriction to the third coordinate of the righthand side gives the homomorphism

$$\beta \mapsto \alpha,$$
$$\gamma \mapsto \gamma,$$
$$T\beta\gamma \mapsto S\gamma\alpha.$$

Note that the subgroups $\langle \gamma\alpha, S\rangle$ and $\langle \beta\gamma, T\rangle$ are also subgroups of $\mathcal{B}$ and $\mathcal{A}$, respectively. The corresponding wreath recursions for these groups are

$$\gamma\alpha = \sigma(1, \beta\gamma),$$
$$S = \sigma(1, T)$$
$$S\gamma\alpha = (T, \beta\gamma),$$

(here we use a conjugated version (28)–(29)), and

$$\beta\gamma \mapsto \gamma\alpha,$$
$$T \mapsto S,$$
$$T\beta\gamma \mapsto S\gamma\alpha.$$

It follows from the recursions and post-critical dynamics of the polynomials $16z^2(1-z)^2$ and $(2z^2-4z+1)^2$ that the limit space of $\langle \gamma\alpha, S\rangle$ with respect to these recursions is the natural direct product of the boundary of the Fatou component of f containing 0 (which is the limit space of the subgroup $\langle S\rangle$) and the boundary of the Fatou component of $(2z^2-4z+1)^2$ containing 0 (the limit space of $\langle\gamma\alpha\rangle$). Similarly, the limit space of $\langle \beta\gamma, T\rangle$ is the direct product of the the boundary of the Fatou component of f containing 1 (the limit space of $\langle T\rangle$) with the boundary of the Fatou component of $16z^2(1-z)^2$ containing 1 (the limit space of $\langle\beta\gamma\rangle$). Hence, both limit spaces are tori.

In particular, the natural maps from the limit spaces of the groups $\langle \gamma\alpha, S\rangle$ and $\langle \beta\gamma, T\rangle$ to $\mathcal{J}_{\mathcal{B}}$ and $\mathcal{J}_{\mathcal{A}}$, respectively, are injective.

Let us compare the limit spaces of these groups with the limit spaces of their extensions $\mathcal{B}_1$ and $\mathcal{A}_1$. We have $\mathcal{B}_1 = \langle \alpha, \gamma\rangle \times \langle S\gamma\alpha\rangle$, and $\mathcal{A}_1 = \langle \beta, \gamma\rangle \times \langle T\beta\gamma\rangle$. The direct factors $\langle \alpha, \gamma\rangle$ and $\langle \beta, \gamma\rangle$ are infinite dihedral with the wreath recursions

$$\alpha = \sigma, \quad \gamma = (\gamma, \beta)$$

and

$$\beta \mapsto \alpha, \quad \gamma \mapsto \gamma.$$

It follows that the limit spaces of the subgroups $\langle \alpha, \gamma\rangle$ and $\langle \beta, \gamma\rangle$ are segments with singular endpoints (one with the isotropy group $\langle\alpha\rangle$ or $\langle\beta\rangle$ and the other with the isotropy group $\langle\gamma\rangle$).

Consequently, the limit spaces of $\mathcal{B}_1$ and $\mathcal{A}_1$ are annuli (direct products of the circle and the segment) and the natural map from the limit spaces of $\langle \gamma\alpha, S\rangle$ and $\langle \beta\gamma, T\rangle$ to the limit spaces of $\mathcal{B}_1$ and $\mathcal{A}_1$ "flattens" the tori into annuli, by flattening the circles $\mathcal{J}_{\langle\gamma\alpha\rangle}$ and $\mathcal{J}_{\langle\beta\gamma\rangle}$ to the segments $\mathcal{J}_{\langle\gamma,\alpha\rangle}$ and $\mathcal{J}_{\langle\beta,\gamma\rangle}$. Note that the natural direct product decomposition of the spaces $\mathcal{J}_{\langle\gamma\alpha,S\rangle}$ and $\mathcal{J}_{\langle\beta\gamma,T\rangle}$ comes from the direct product decompositions $\langle\gamma\alpha\rangle\times\langle S\rangle$ and $\langle\beta\gamma\rangle\times\langle T\rangle$, while we have decompositions of the self-similar groups $\mathcal{B}_1 = \langle\gamma,\alpha\rangle\times\langle S\gamma\alpha\rangle$ and $\mathcal{A}_1 = \langle\beta,\gamma\rangle\times\langle T\beta\gamma\rangle$. Consequently, the preimages of the boundaries of the annuli in the tori are diagonals with respect to the natural decomposition of the torus into the product of the boundaries of the Fatou components. It means that the diameter with respect to which we flatten the boundary of the Fatou component of the polynomials $16z^2(1-z)^2$ and $(2z^2-4z+1)^2$ rotates as p travels along the boundary of the Fatou component of f.

This flattening of the circles into segments can be interpreted as a "rotated tuning" of the polynomials $16z^2(1-z)^2$ and $(2z^2-4z+1)^2$ by the polynomial z^2-2 (which is the quadratic polynomial with the dihedral iterated monodromy group and the Julia set a segment). It can be nicely illustrated by the slices of the Julia set of F as p travels close to the boundary of the Fatou component of f, but stays inside it. Then the slices are still homeomorphic to the Julia sets of $16z^2(1-z)^2$ and $(2z^2-4z+1)^2$, but are close to the dendrite slices of the Julia set of F along the boundary of the Fatou component of f.

See Figure 36 where slices of the Julia set of F are shown when p is traveling close to the boundary of the Fatou component of f containing 1 (the top half of the figure). Corresponding slices for p on the boundary of the Fatou component are shown in the bottom part of the figure.

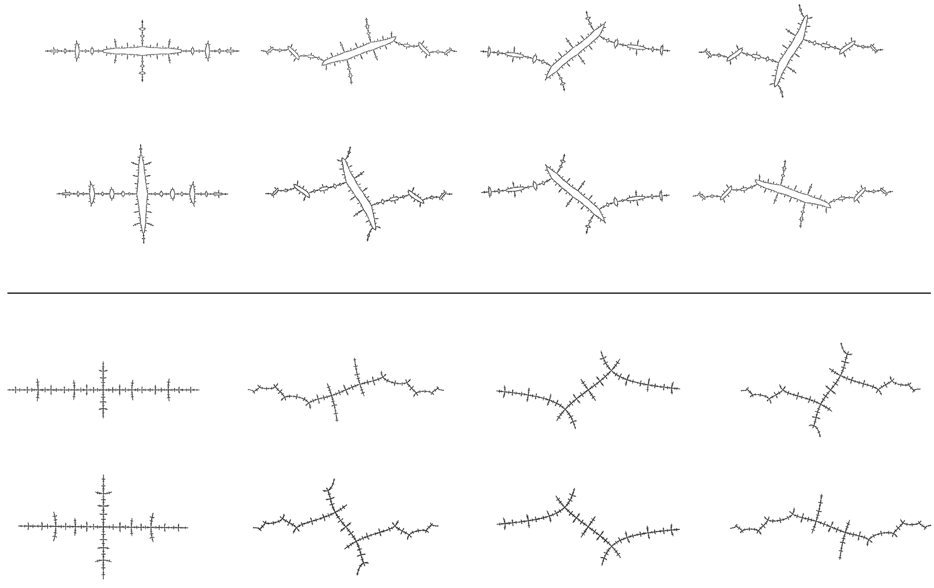

FIGURE 36. Rotated tuning

The described rotated tunings are analgous to the "rotated matings" studied in [5].

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