

MONODROMY, LIFTINGS OF HOLOMORPHIC MAPS, AND EXTENSIONS OF HOLOMORPHIC MOTIONS

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In memory of Professor Clifford J. Earle

ABSTRACT. We study monodromy of holomorphic motions and show the equivalence of triviality of monodromy of holomorphic motions and extensions of holomorphic motions to continuous motions of the Riemann sphere. We also study liftings of holomorphic maps into certain Teichmüller spaces. We use this “lifting property” to prove that, under the condition of trivial monodromy, any holomorphic motion of a closed set in the Riemann sphere, over a hyperbolic Riemann surface, can be extended to a holomorphic motion of the sphere, over the same parameter space. We conclude that this extension can be done in a conformally natural way.

INTRODUCTION

Throughout this paper, we shall use the following notation: \mathbb{C} for the complex plane, $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ for the Riemann sphere, and Δ for the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$.

The subject of holomorphic motions was introduced in the study of the dynamics of rational maps; see [19]. Since its inception, an important topic has been the question of extending holomorphic motions. The papers [5] and [26] contained partial results. Subsequently, Slodkowski showed that any holomorphic motion of a set in $\widehat{\mathbb{C}}$, over Δ as the parameter space, can be extended to a holomorphic motion of $\widehat{\mathbb{C}}$ over Δ ; see [25]; also refer to the papers [6] and [12]. The paper [11] used a group-equivariant version of Slodkowski’s theorem to prove results in Teichmüller theory.

A natural question is to investigate holomorphic motions over more general parameter spaces. In Theorem 8.1 of [11], an explicit counterexample was given to show that Slodkowski’s theorem does not, in general, hold when the parameter space is a domain in \mathbb{C}^2 . In [15], it was shown that Slodkowski’s theorem does not hold for many higher-dimensional, simply connected parameter spaces; see also Proposition 5.4 in [16]. This naturally leads us to consider the case when the parameter space is a hyperbolic Riemann surface. Theorem D of our paper answers this question.

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A central topic is to study the question of obstructions to extending a holomorphic motion. More precisely, what are the analytic and topological obstructions to the extension of holomorphic motions? These two questions are intimately linked with Teichmüller spaces. The example in Proposition 5.4 of [16] studied the analytic obstruction. It is related to the non-existence of holomorphic sections for Teichmüller spaces; see the papers [8] and [9], and also [13]. The example (Theorem 1) in [15] linked the question of extending holomorphic motions with Earle's result in [8]. In our present paper, we study the question of topological obstructions. This is the essence of Theorem D.

The main purpose of our paper is to study necessary and sufficient topological conditions for extending holomorphic motions. We study monodromy of a holomorphic motion ϕ of a finite set E in $\widehat{\mathbb{C}}$, defined over a connected complex Banach manifold V . We show the equivalence of the triviality of this monodromy and the extendability of ϕ to a continuous motion of $\widehat{\mathbb{C}}$ over V . We then show that, if V is a hyperbolic Riemann surface, and E is any set in $\widehat{\mathbb{C}}$, then, under the condition of trivial monodromy, any holomorphic motion of E extends to a holomorphic motion of $\widehat{\mathbb{C}}$ over V .

The relationship between extending holomorphic motions and liftings of holomorphic maps into Teichmüller spaces of $\widehat{\mathbb{C}}$ with punctures, was first observed in Section 7 of the paper [5]. In that paper, the parameter space is the open unit disk. The main technique in our paper is to study the liftings of holomorphic maps from a hyperbolic Riemann surface into such Teichmüller spaces. Theorem B gives the general setting, and Theorem C is the key step in proving Theorem D, which is the main result of this paper.

Our paper is organized as follows. In Section 1, we give all precise definitions and discuss the useful facts that will be necessary in the proofs of the main theorems of this paper. In Section 2, we give precise statements of the main theorems. In Sections 3, 4, 5, and 6, we prove the main theorems.

1. DEFINITIONS AND SOME FACTS

Definition 1.1. Let V be a connected complex manifold with a basepoint x_0 and let E be any subset of $\widehat{\mathbb{C}}$. A *holomorphic motion* of E over V is a map $\phi : V \times E \rightarrow \widehat{\mathbb{C}}$ that has the following three properties:

- (1) $\phi(x_0, z) = z$ for all z in E ,
- (2) the map $\phi(x, \cdot) : E \rightarrow \widehat{\mathbb{C}}$ is injective for each x in V , and
- (3) the map $\phi(\cdot, z) : V \rightarrow \widehat{\mathbb{C}}$ is holomorphic for each z in E .

We say that V is a *parameter space* of the holomorphic motion ϕ . We will always assume that ϕ is a *normalized* holomorphic motion; i.e. $0, 1$, and ∞ belong to E and are fixed points of the map $\phi(x, \cdot)$ for every x in V . It is sometimes useful to write $\phi(x, z)$ as $\phi_x(z)$.

If E is a proper subset of $\widehat{\mathbb{C}}$ and $\phi : V \times E \rightarrow \widehat{\mathbb{C}}$, $\widehat{\phi} : V \times \widehat{E} \rightarrow \widehat{\mathbb{C}}$ are two holomorphic motions, we say that $\widehat{\phi}$ *extends* ϕ if $\widehat{\phi}(x, z) = \phi(x, z)$ for all (x, z) in $V \times E$.

Remark 1.2. Let V and W be connected complex manifolds with basepoints, and let f be a basepoint preserving holomorphic map of W into V . If ϕ is a holomorphic

motion of E over V its *pullback* by f is the holomorphic motion

$$(1.1) \quad f^*(\phi)(x, z) = \phi(f(x), z) \quad \forall (x, z) \in W \times E,$$

of E over W .

Definition 1.3. Let V be a connected complex manifold with a basepoint. Let G be a group of Möbius transformations, let $E \subset \widehat{\mathbb{C}}$ be G -invariant, which means, $g(E) = E$ for each g in G . A holomorphic motion $\phi : V \times E \rightarrow \widehat{\mathbb{C}}$ is G -**equivariant** if for any $x \in V, g \in G$ there is a Möbius transformation, denoted by $\theta_x(g)$, such that

$$\phi(x, g(z)) = (\theta_x(g))(\phi(x, z))$$

for all z in E .

It is well known that if $\phi : V \times E \rightarrow \widehat{\mathbb{C}}$ is a holomorphic motion, where V is a connected complex manifold with a basepoint x_0 , then ϕ extends to a holomorphic motion of the closure \overline{E} , over V ; see [19] and [3]. Hence, throughout this paper, we will assume that E is a closed set in $\widehat{\mathbb{C}}$ (that contains the points 0, 1, and ∞).

Recall that a homeomorphism of $\widehat{\mathbb{C}}$ is called *normalized* if it fixes the points 0, 1, and ∞ . The blanket assumption that E is a closed set in $\widehat{\mathbb{C}}$ containing the points 0, 1, and ∞ holds.

Definition 1.4. Two normalized quasiconformal self-mappings f and g of $\widehat{\mathbb{C}}$ are said to be E -equivalent if and only if $f^{-1} \circ g$ is isotopic to the identity rel E . The *Teichmüller space* $T(E)$ is the set of all E -equivalence classes of normalized quasiconformal self-mappings of $\widehat{\mathbb{C}}$.

Let $M(\mathbb{C})$ be the open unit ball of the complex Banach space $L^\infty(\mathbb{C})$. Each μ in $M(\mathbb{C})$ is the Beltrami coefficient of a unique normalized quasiconformal homeomorphism w^μ of $\widehat{\mathbb{C}}$ onto itself. The basepoint of $M(\mathbb{C})$ is the zero function.

We define the quotient map

$$P_E : M(\mathbb{C}) \rightarrow T(E)$$

by setting $P_E(\mu)$ equal to the E -equivalence class of w^μ , written as $[w^\mu]_E$. Clearly, P_E maps the basepoint of $M(\mathbb{C})$ to the basepoint of $T(E)$.

In his doctoral dissertation [18], G. Lieb proved that $T(E)$ is a complex Banach manifold such that the projection map $P_E : M(\mathbb{C}) \rightarrow T(E)$ is a holomorphic split submersion; see [11] for the details.

Remark 1.5. Let E be a finite set. Its complement $\Omega = \widehat{\mathbb{C}} \setminus E$ is the Riemann sphere with punctures at the points of E . Then, $T(E)$ is biholomorphic to the classical Teichmüller space $Teich(\Omega)$; see Example 3.1 in [20] for the proof. This canonical identification will be very important in our paper. (The reader is referred to [14] or [23] for standard facts in Teichmüller theory.)

Proposition 1.6. *There is a continuous basepoint preserving map s from $T(E)$ to $M(\mathbb{C})$ such that $P_E \circ s$ is the identity map on $T(E)$.*

See [11], [16] for all details.

Definition 1.7. The map s from $T(E)$ to $M(\mathbb{C})$ is called the *Douady-Earle section* of P_E for the Teichmüller space $T(E)$.

Remark 1.8. When E is finite, $T(E)$ is canonically identified with the classical Teichmüller space $Teich(\widehat{\mathbb{C}} \setminus E)$, and hence s is the continuous section studied in Lemma 5 in [7].

Definition 1.9. The *universal holomorphic motion* $\Psi_E : T(E) \times E \rightarrow \widehat{\mathbb{C}}$ is defined as follows:

$$\Psi_E(P_E(\mu), z) = w^\mu(z) \text{ for } \mu \in M(\mathbb{C}) \text{ and } z \in E.$$

It is clear from the definition of P_E that the map Ψ_E is well-defined. It is a holomorphic motion because P_E is a holomorphic split submersion and $\mu \mapsto w^\mu(z)$ is a holomorphic map from $M(\mathbb{C})$ to $\widehat{\mathbb{C}}$ for every fixed z in $\widehat{\mathbb{C}}$, by Theorem 11 in [2].

This holomorphic motion is “universal” in the following sense.

Theorem 1.10. *Let $\phi : V \times E \rightarrow \widehat{\mathbb{C}}$ be a holomorphic motion. If V is a simply connected complex Banach manifold with a basepoint, there is a unique basepoint preserving holomorphic map $f : V \rightarrow T(E)$ such that $f^*(\Psi_E) = \phi$.*

For a proof see Section 14 in [20].

Remark 1.11. Let $\phi : V \times E \rightarrow \widehat{\mathbb{C}}$ be a holomorphic motion, where V is a connected complex Banach manifold with a basepoint x_0 . Suppose there exists a basepoint preserving holomorphic map $f : V \rightarrow T(E)$ such that $f^*(\Psi_E) = \phi$. Let $\tilde{f} : V \rightarrow M(\mathbb{C})$, where $\tilde{f} = s \circ f$. By Proposition 1.6, \tilde{f} is a basepoint preserving continuous map. Then, for all $(x, z) \in V \times E$, we have

$$\phi(x, z) = \Psi_E(f(x), z) = \Psi_E(P_E(s(f(x))), z) = w^{s(f(x))}(z) = w^{\tilde{f}(x)}(z).$$

Definition 1.12. Let W be a path-connected Hausdorff space with a basepoint x_0 . A (normalized) continuous motion of $\widehat{\mathbb{C}}$ over W is a continuous map $\phi : W \times \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ such that:

- (a) $\phi(x_0, z) = z$ for all $z \in \widehat{\mathbb{C}}$ and
- (b) for each x in W , the map $\phi(x, \cdot) := \phi_x(\cdot)$ is a homeomorphism of $\widehat{\mathbb{C}}$ onto itself that fixes the points 0, 1, and ∞ .

In [21] the following was shown.

Theorem 1.13. *Let $\phi : V \times E \rightarrow \widehat{\mathbb{C}}$ be a holomorphic motion, where V is a connected complex Banach manifold with a basepoint x_0 . Then the following are equivalent:*

- (i) *There is a continuous motion $\tilde{\phi} : V \times \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ that extends ϕ .*
- (ii) *There exists a basepoint preserving holomorphic map $F : V \rightarrow T(E)$ such that $F^*(\Psi_E) = \phi$.*

The following corollary will be useful in this paper; see [21].

Corollary 1.14. *If the holomorphic motion ϕ can be extended to a continuous motion $\tilde{\phi}$, then $\tilde{\phi}$ can be chosen so that:*

- (i) *the map $\tilde{\phi}_x : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ is quasiconformal for each x in V and*
- (ii) *its Beltrami coefficient μ_x is a continuous function of x .*

Let w be a normalized quasiconformal self-mapping of $\widehat{\mathbb{C}}$, and let $\tilde{E} = w(E)$. By definition, the allowable map g from $T(\tilde{E})$ to $T(E)$ maps the \tilde{E} -equivalence class of f to the E -equivalence class of $f \circ w$ for every normalized quasiconformal self-mapping f of $\widehat{\mathbb{C}}$.

Lemma 1.15. *The allowable map $g : T(\tilde{E}) \rightarrow T(E)$ is biholomorphic. If μ is the Beltrami coefficient of w , then g maps the basepoint of $T(\tilde{E})$ to the point $P_E(\mu)$ in $T(E)$.*

See §7.12 in [11] or §6.4 in [20] for a complete proof. The following lemma will be useful in our paper.

Lemma 1.16. *Let B be a path-connected topological space. Let f and g be two continuous maps from B to $T(E)$, satisfying:*

- (i) $\Psi_E(f(t), z) = \Psi_E(g(t), z)$ for all z in E and
- (ii) $f(t_0) = g(t_0)$ for some t_0 .

Then, $f(t) = g(t)$ for all t in B .

See §12 in [20] for the proof.

If $f(t) = [w^\mu]_E$ and $g(t) = [w^\nu]_E$, condition (i) of the lemma means that $w^\mu(z) = w^\nu(z)$ for all z in E .

If E is a subset of the closed set \hat{E} and μ is in $M(\mathbb{C})$, then the \hat{E} -equivalence class of w^μ is contained in the E -equivalence class of w^μ . Therefore, there is a well-defined “forgetful map”

$$(1.2) \quad p_{\hat{E},E} : T(\hat{E}) \mapsto T(E)$$

such that $P_E = p_{\hat{E},E} \circ P_{\hat{E}}$. It is easy to see that this is a basepoint preserving holomorphic split submersion.

The following is a consequence of Lemma 1.16. Here, Ψ_E is the universal holomorphic motion of E and $\Psi_{\hat{E}}$ is the universal holomorphic motion of \hat{E} .

Proposition 1.17. *Let V be a connected complex Banach manifold with basepoint, and let f and g be basepoint preserving holomorphic maps from V into $T(E)$ and $T(\hat{E})$, respectively. Then $p_{\hat{E},E} \circ g = f$ if and only if $g^*(\Psi_{\hat{E}})$ extends $f^*(\Psi_E)$.*

See §13 in [20] for the proof. We say that the holomorphic map g lifts the holomorphic map f .

We now discuss the concept of *monodromy* of a holomorphic motion. We closely follow the discussion in §2 in [4]. Let $\phi : V \times E \rightarrow \hat{\mathbb{C}}$ be a holomorphic motion, where V is a connected complex Banach manifold with a basepoint x_0 . Let $\pi : \tilde{V} \rightarrow V$ be a holomorphic universal covering, with the group of deck transformations Γ . We choose a point \tilde{x}_0 in \tilde{V} such that $\pi(\tilde{x}_0) = x_0$. Let $\pi_1(V, x_0)$ denote the fundamental group of V with basepoint x_0 .

Let $\Phi = \pi^*(\phi)$. Then, $\Phi : \tilde{V} \times E \rightarrow \hat{\mathbb{C}}$ is a holomorphic motion of E over \tilde{V} with \tilde{x}_0 as the basepoint. By Remark 1.11, there exists a basepoint preserving continuous map $\tilde{f} : \tilde{V} \rightarrow M(\mathbb{C})$ such that

$$\Phi(x, z) = w^{\tilde{f}(x)}(z)$$

for each $x \in \tilde{V}$ and each $z \in E$.

For each $z \in E$ and for each $\gamma \in \Gamma$, we have

$$w^{\tilde{f} \circ \gamma(\tilde{x}_0)}(z) = \Phi(\gamma(\tilde{x}_0), z) = \phi(\pi \circ \gamma(\tilde{x}_0), z) = \phi(x_0, z) = z.$$

Therefore, $w^{\tilde{f} \circ \gamma(\tilde{x}_0)}$ keeps every point of E fixed.

Lemma 1.18. *The homotopy class of $w^{\tilde{f} \circ \gamma(\tilde{x}_0)}$ relative to E does not depend on the choice of the continuous map \tilde{f} .*

See Lemma 2.12 in [4].

We now assume that E is a finite set containing n points, where $n \geq 4$; as usual, $0, 1,$ and ∞ are in E . Let $\phi : V \times E \rightarrow \widehat{\mathbb{C}}$ be a holomorphic motion. The map $w^{\tilde{f} \circ \gamma(\tilde{x}_0)}$ is a quasiconformal self-map of the hyperbolic Riemann surface $X_E := \widehat{\mathbb{C}} \setminus E$. Therefore, it represents a mapping class of X_E , and by Lemma 1.18, we have a homomorphism $\rho_\phi : \pi_1(V, x_0) \rightarrow \text{Mod}(0, n)$ given by

$$\rho_\phi(c) = [w^{\tilde{f} \circ \gamma(\tilde{x}_0)}],$$

where $\text{Mod}(0, n)$ is the mapping class group of the n -times punctured sphere, $\gamma \in \Gamma$ is the element corresponding to $c \in \pi_1(V, x_0)$, and $[w]$ denotes the mapping class group of X_E for w .

Definition 1.19. We call the homomorphism ρ_ϕ the *monodromy* of the holomorphic motion ϕ of the finite set E . The monodromy is called *trivial* if it maps every element of $\pi_1(V, x_0)$ to the identity of $\text{Mod}(0, n)$.

2. STATEMENTS OF THE MAIN THEOREMS

In this section, we give the precise statements of the main theorems of our paper.

Theorem A. *Let $\phi : V \times E \rightarrow \widehat{\mathbb{C}}$ be a holomorphic motion of a finite set E , containing the points $0, 1,$ and ∞ , where V is a connected complex Banach manifold with basepoint x_0 . The following are equivalent:*

- (i) *There exists a continuous motion $\tilde{\phi} : V \times \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$, such that $\tilde{\phi}$ extends ϕ .*
- (ii) *The monodromy ρ_ϕ is trivial.*

In the next theorem, let $\widehat{E} = E \cup \{\zeta\}$, where, E is a finite set containing $0, 1,$ and ∞ , and $\zeta \in \mathbb{C} \setminus E$. Let V be a connected complex Banach manifold with basepoint x_0 .

Theorem B. *Suppose every holomorphic map from V into $T(E)$ lifts to a holomorphic map from V into $T(\widehat{E})$. Then, if $\phi : V \times E \rightarrow \widehat{\mathbb{C}}$ is a holomorphic motion that has trivial monodromy, there exists a holomorphic motion $\widehat{\phi} : V \times \widehat{E} \rightarrow \widehat{\mathbb{C}}$ such that $\widehat{\phi}$ extends ϕ and also has trivial monodromy.*

In the next two theorems, X is a hyperbolic Riemann surface with a basepoint x_0 , and E is a closed set in $\widehat{\mathbb{C}}$ containing the points $0, 1,$ and ∞ .

Let $E_n = \{0, 1, \infty, \xi_1, \dots, \xi_n\}$, where $n \geq 1$, and $E_{n+1} = E_n \cup \{\xi_{n+1}\}$, where $\xi_{n+1} \in \widehat{\mathbb{C}} \setminus E_n$. Let $p : T(E_{n+1}) \rightarrow T(E_n)$ denote the forgetful map in (1.2). Let $\phi_n : X \times E_n \rightarrow \widehat{\mathbb{C}}$ be a holomorphic motion that has trivial monodromy. We will see in the proof of Theorem A that there exists a unique basepoint preserving holomorphic map $f_n : X \rightarrow T(E_n)$ such that $f_n^*(\Psi_{E_n}) = \phi_n$. The following theorem is a key result in our paper.

Theorem C. *Let $\phi_n : X \times E_n \rightarrow \widehat{\mathbb{C}}$ be a holomorphic motion. If the monodromy of ϕ_n is trivial, there exists a basepoint preserving holomorphic map $f_{n+1} : X \rightarrow T(E_{n+1})$ such that $p \circ f_{n+1} = f_n$.*

The following corollary is an immediate consequence. Here $\Psi_{E_{n+1}} : T(E_{n+1}) \times E_{n+1} \rightarrow \widehat{\mathbb{C}}$ is the universal holomorphic motion of E_{n+1} .

Corollary 2.1. *Let $\phi_{n+1} := f_{n+1}^*(\Psi_{E_{n+1}})$. Then ϕ_{n+1} extends ϕ_n and has trivial monodromy.*

Theorem D. *Let $\phi : X \times E \rightarrow \widehat{\mathbb{C}}$ be a holomorphic motion such that ϕ restricted to $X \times E'$ has trivial monodromy, or extends to a continuous motion of $\widehat{\mathbb{C}}$ (over X), where E' is any finite subset of E , containing the points $0, 1$, and ∞ . Then, there exists a holomorphic motion $\widehat{\phi} : X \times \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ such that $\widehat{\phi}$ extends ϕ .*

Remark 2.2. H. Shiga [24] has recently announced a completely different approach to a part of this theorem. Our methods are totally independent and more direct. The crucial point in our approach is the lifting property as given in Theorem C.

3. PROOF OF THEOREM A

Let $\pi : \widetilde{V} \rightarrow V$ be a holomorphic universal covering with the group Γ of deck transformations, so that, $V = \widetilde{V}/\Gamma$, and $\pi(\widetilde{x}_0) = x_0$.

Suppose ϕ can be extended to a continuous motion $\widetilde{\phi}$ of $\widehat{\mathbb{C}}$ over V . Then, by Corollary 1.14, there exists a continuous map $f : V \rightarrow M(\mathbb{C})$ such that $\widetilde{\phi}(x, z) = w^{f(x)}(z)$ for all $(x, z) \in V \times \widehat{\mathbb{C}}$. Let $\widetilde{f} = f \circ \pi$. Then, for any $c \in \pi_1(V, x_0)$ with corresponding $\gamma \in \Gamma$, we have

$$\rho_\phi(c) = [w^{\widetilde{f} \circ \gamma(\widetilde{x}_0)}] = [w^{f \circ \pi \circ \gamma(\widetilde{x}_0)}] = [w^{f(x_0)}] = [Id].$$

This shows that the monodromy ρ_ϕ is trivial.

Let $\phi : V \times E \rightarrow \widehat{\mathbb{C}}$ be a holomorphic motion with trivial monodromy. Let $\phi_{\widetilde{V}} := \pi^*(\phi)$ be the holomorphic motion of E over \widetilde{V} . By Theorem 1.10, there exists a unique basepoint preserving holomorphic map $f_{\widetilde{V}} : \widetilde{V} \rightarrow T(E)$, such that $\phi_{\widetilde{V}} = f_{\widetilde{V}}^*(\Psi_E)$. For any element $\gamma \in \Gamma$, we also have $f_{\widetilde{V}} \circ \gamma : \widetilde{V} \rightarrow T(E)$. Note that

$$\begin{aligned} (f_{\widetilde{V}} \circ \gamma)^*(\Psi_E)(x, z) &= \Psi_E((f_{\widetilde{V}} \circ \gamma)x, z) = \phi_{\widetilde{V}}(\gamma(x), z) = \phi(\pi(\gamma(x)), z) \\ &= \phi(\pi(x), z) = \phi_{\widetilde{V}}(x, z) = (f_{\widetilde{V}})^*(\Psi_E)(x, z). \end{aligned}$$

By the triviality of the monodromy, we have $f_{\widetilde{V}} \circ \gamma(x_0) = f_{\widetilde{V}}(x_0)$ for all $\gamma \in \Gamma$. Lemma 1.16 implies that $f_{\widetilde{V}} \circ \gamma = f_{\widetilde{V}}$ for all $\gamma \in \Gamma$. Thus, $f_{\widetilde{V}}$ defines a unique basepoint preserving holomorphic map $f : V \rightarrow T(E)$ such that $\phi = f^*(\Psi_E)$. It then follows from Theorem 1.13 that there exists a continuous motion of $\widehat{\mathbb{C}}$ over V that extends ϕ . This completes the proof. \square

4. PROOF OF THEOREM B

Let $\phi : V \times E \rightarrow \widehat{\mathbb{C}}$ be a holomorphic motion such that it has trivial monodromy. By the proof of Theorem A, there exists a basepoint preserving holomorphic map $f : V \rightarrow T(E)$ such that $\phi = f^*(\Psi_E)$. Let $p : T(\widehat{E}) \rightarrow T(E)$ denote the forgetful map defined in (1.2). By hypothesis, there exists a basepoint preserving holomorphic map $\widehat{f} : V \rightarrow T(\widehat{E})$ such that $p \circ \widehat{f} = f$. Let $\widehat{\phi} := \widehat{f}^*(\Psi_{\widehat{E}})$. By Proposition 1.17, $\widehat{\phi}$ extends ϕ . Note that, for $x \in V$, and $z \in \widehat{E}$, we have $\widehat{\phi}(x, z) = w^{\alpha(x)}(z)$, where $\alpha = s_{\widehat{E}} \circ \widehat{f}$ and $s_{\widehat{E}}$ is the continuous section of the projection $P_{\widehat{E}} : M(\mathbb{C}) \rightarrow T(\widehat{E})$; see Remark 1.8.

Let $\pi : \widetilde{V} \rightarrow V$ be the holomorphic universal cover, and $\pi(\widetilde{x}_0) = x_0$. Then, $\pi^*(\widehat{\phi}) : \widetilde{V} \times \widehat{E} \rightarrow \widehat{\mathbb{C}}$ is a holomorphic motion. Since \widetilde{V} is simply connected, there exists a basepoint preserving continuous map $\beta : \widetilde{V} \rightarrow M(\mathbb{C})$ such that $\beta^*(\Psi_{\widehat{E}}) =$

$\pi^*(\widehat{\phi})$ (see Remark 1.11). That implies $\pi^*(\widehat{\phi})(x, z) = w^{\beta(x)}(z)$. Recall that the monodromy $\rho : \pi_1(V) \rightarrow \text{Mod}(0, n + 1)$ is defined by

$$\rho(c) = [w^{\beta \circ \gamma(x_0)}]$$

for any $c \in \pi_1(V, x_0)$ with the corresponding $\gamma \in \Gamma$. Furthermore, it is independent of the choice of β . In particular, if we choose $\beta = \alpha \circ \pi$, we see that

$$\rho(c) = [w^{\alpha \circ \pi \circ \gamma(x_0)}] = [w^{\alpha(x_0)}] = [Id].$$

Note that $\pi \circ \gamma(x_0) = x_0$. This implies that ρ is trivial. □

5. PROOFS OF THEOREM C AND COROLLARY 2.1

We recall the following result, due to S. Nag, that will be fundamental in our paper; see [22].

Theorem 5.1. *Given $n > 0$, choose a point $(\zeta_1, \dots, \zeta_n)$ in the domain*

$Y_n = \{(z_1, \dots, z_n) \in \mathbb{C}^n : z_i \neq z_j \text{ for } i \neq j \text{ and } z_i \neq 0, 1 \text{ for all } i = 1, \dots, n\}$
and let $E_n = \{0, 1, \infty, \zeta_1, \dots, \zeta_n\}$. Then, the map $p_n : T(E_n) \rightarrow Y_n$ defined by

$$p_n([w^\mu]_{E_n}) = (w^\mu(\zeta_1), \dots, w^\mu(\zeta_n)) \text{ for all } \mu \in M(\mathbb{C})$$

is a holomorphic universal covering.

Let $E_n = \{0, 1, \infty, \zeta_1, \dots, \zeta_n\}$ and $E_{n+1} = E_n \cup \{\zeta_{n+1}\}$, where $\zeta_{n+1} \in \widehat{\mathbb{C}} \setminus E_n$. We have also a holomorphic universal covering $p_{n+1} : T(E_{n+1}) \rightarrow Y_{n+1}$, where

$$Y_{n+1} = \{(z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1} : z_i \neq z_j \text{ for } i \neq j \text{ and } z_i \neq 0, 1 \text{ for all } i = 1, \dots, n+1\}.$$

Let $p : T(E_{n+1}) \rightarrow T(E_n)$ denote the forgetful map defined in (1.2).

We need some preliminaries. The reader is referred to Lemmas 3.1–3.5 in [17] for all details. Let $\mathcal{C}(\mathbb{C})$ denote the complex Banach space of bounded, continuous functions ϕ on \mathbb{C} with the norm

$$\|\phi\| = \sup_{z \in \mathbb{C}} |\phi(z)|.$$

In §3 of [17] we construct a continuous compact operator

$$\mathcal{K} : \mathcal{C}(\mathbb{C}) \rightarrow \mathcal{C}(\mathbb{C}).$$

By Lemma 3.2 in [17], there exists a constant $C_3 > 0$ such that

$$\|\mathcal{K}\| \leq C_3 \text{ for all } f \in \mathcal{C}(\mathbb{C}).$$

For ζ_{n+1} , let

$$\mathcal{B} = \{f \in \mathcal{C}(\mathbb{C}) : \|f\| \leq |\zeta_{n+1}| + C_3\}.$$

It is a bounded convex subset in $\mathcal{C}(\mathbb{C})$. The continuous compact operator $\zeta_{n+1} + \mathcal{K}$ maps \mathcal{B} into itself. By Schauder fixed point theorem (see Theorem 2A on page 56 of [27]; also page 557 of [17]), $\zeta_{n+1} + \mathcal{K}$ has a fixed point in \mathcal{B} . This says that we can find a $g_{n+1} \in \mathcal{B}$ such that

$$g_{n+1}(z) = \zeta_{n+1} + \mathcal{K}g_{n+1}(z) \text{ for all } z \in \mathbb{C}.$$

The reader is referred to §3 of [17] for all details; especially page 557 of that paper. By Lemma 3.5 of [17], the solution $g_{n+1}(z)$ is the unique fixed point of the operator $\zeta_{n+1} + \mathcal{K}$.

Suppose $\phi_n : X \times E_n \rightarrow \widehat{\mathbb{C}}$ is a holomorphic motion that has trivial monodromy. By Theorem A, it can be extended to a continuous motion $\widetilde{\phi} : X \times \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$. By

Theorem 1.13, there exists a basepoint preserving holomorphic map $f_n : X \rightarrow T(E_n)$ such that $f_n^*(\Psi_E) = \phi_n$.

Proof of Theorem C. Let

$$p_n : T(E_n) \rightarrow Y_n \quad \text{and} \quad p_{n+1} : T(E_{n+1}) \rightarrow Y_{n+1}$$

be the two holomorphic coverings in Theorem 5.1.

Since X is a hyperbolic Riemann surface, its universal covering space is the open unit disk Δ . Let $\pi : \Delta \rightarrow X$ be the holomorphic universal cover. Let Γ be the group of deck transformations such that $X = \Delta/\Gamma$.

Define the holomorphic map

$$f_{\Delta,n} = f_n \circ \pi : \Delta \rightarrow T(E_n)$$

and for any given $\gamma \in \Gamma$, consider the holomorphic map

$$f_{\Delta,n} \circ \gamma : \Delta \rightarrow T(E_n).$$

So we have two holomorphic maps

$$F_n = p_n \circ f_{\Delta,n} : \Delta \rightarrow Y_n \quad \text{and} \quad F_n \circ \gamma = p_n \circ (f_{\Delta,n} \circ \gamma) : \Delta \rightarrow Y_n.$$

Let us write

$$F_n = (h_1, \dots, h_n) \quad \text{and} \quad F_n \circ \gamma = (h_1 \circ \gamma, \dots, h_n \circ \gamma).$$

Each h_i (as well as $h_i \circ \gamma$) is holomorphic in Δ . In §3 of [10], we constructed a map g_i (as well as a map $g_i \circ \gamma$) by using h_i (as well as $h_i \circ \gamma$) which is holomorphic outside Δ and continuous on \mathbb{C} . By using g_i for all $1 \leq i \leq n$, we constructed a continuous compact operator

$$\mathcal{K} = \mathcal{K}(F_n) : \mathcal{C}(\mathbb{C}) \rightarrow \mathcal{C}(\mathbb{C})$$

in §3 of [10]. Similarly, by using $g_i \circ \gamma$ for all $1 \leq i \leq n$, we have a continuous compact operator

$$\mathcal{K}_\gamma = \mathcal{K}(F_n \circ \gamma) : \mathcal{C}(\mathbb{C}) \rightarrow \mathcal{C}(\mathbb{C}).$$

The main point in §3 of [10] is that we can find the unique fixed point g_{n+1} for $\xi_{n+1} + \mathcal{K}$ and the unique fixed point $g_{n+1,\gamma}$ for $\xi_{n+1} + \mathcal{K}_\gamma$. That is,

$$(5.1) \quad g_{n+1}(z) = \xi_{n+1} + \mathcal{K}g_{n+1}(z)$$

and

$$(5.2) \quad g_{n+1,\gamma}(z) = \xi_{n+1} + \mathcal{K}_\gamma g_{n+1,\gamma}(z).$$

We also have

$$(5.3) \quad g_{n+1} \circ \gamma(z) = \xi_{n+1} + \mathcal{K}_\gamma(g_{n+1} \circ \gamma(z)).$$

From g_{n+1} (as well as $g_{n+1,\gamma}$), which is holomorphic outside Δ and continuous in \mathbb{C} , we get a holomorphic map h_i (as well as $h_{n+1,\gamma}$) in Δ for $i = 1, \dots, n$. Then we form two holomorphic maps

$$(5.4) \quad F_{n+1} = (h_1, \dots, h_n, h_{n+1}) : \Delta \rightarrow Y_{n+1}$$

and

$$(5.5) \quad F_{n+1,\gamma} = (h_1 \circ \gamma, \dots, h_n \circ \gamma, h_{n+1,\gamma}) : \Delta \rightarrow Y_{n+1}.$$

Since Δ is simply connected and since $p_{n+1} : T(E_{n+1}) \rightarrow Y_{n+1}$ is the universal cover, we can lift F_{n+1} and $F_{n+1,\gamma}$ to two holomorphic maps

$$f_{\Delta,n+1} : \Delta \rightarrow T(E_{n+1}) \quad \text{and} \quad f_{\Delta,n+1,\gamma} : \Delta \rightarrow T(E_{n+1})$$

such that $p_{n+1} \circ f_{\Delta,n+1} = F_{n+1}$ and $p_{n+1} \circ f_{\Delta,n+1,\gamma} = F_{n+1,\gamma}$.

Under the assumption of the trivial monodromy, we know that $f_{\Delta,n} = f_{\Delta,n} \circ \gamma$ (see the proof of Theorem A). That is, $h_i = h_i \circ \gamma$ and $g_i = g_i \circ \gamma$ for all $1 \leq i \leq n$. Thus $\mathcal{K}_\gamma = \mathcal{K}$. Since the fixed point is unique, we get

$$g_{n+1} \circ \gamma = g_{n+1,\gamma} = g_{n+1}.$$

This implies that $h_{n+1} \circ \gamma = h_{n+1}$ and $F_{n+1} \circ \gamma = F_{n+1}$ and $f_{\Delta,n+1} \circ \gamma = f_{\Delta,n+1}$. Since this holds for all $\gamma \in \Gamma$, the map $f_{\Delta,n+1}$ defines a holomorphic map $f_{n+1} : X \rightarrow T(E_{n+1})$ such that $p \circ f_{n+1} = f_n$. Therefore, f_{n+1} is a lift of f_n . This completes the proof. \square

Proof of Corollary 2.1. This follows at once from Theorem C, Proposition 1.17, and Theorem B. \square

6. PROOF OF THEOREM D

Let E be an arbitrary closed set in $\widehat{\mathbb{C}}$ such that $0, 1,$ and ∞ are in E , and let X be a hyperbolic Riemann surface with a basepoint x_0 .

We will need the following result; see Theorem C in [4].

Theorem 6.1. *Let $\phi : X \times E \rightarrow \widehat{\mathbb{C}}$ be a holomorphic motion. Suppose the restriction of ϕ to $X \times E'$ extends to a holomorphic motion $\tilde{\phi} : X \times \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$, whenever $\{0, 1, \infty\} \subset E' \subset E$ and E' is finite. Then ϕ can be extended to a holomorphic motion of $\widehat{\mathbb{C}}$ over X .*

Proof of Theorem D. Let $\phi : X \times E \rightarrow \widehat{\mathbb{C}}$ be a holomorphic motion with the property that ϕ restricted to $X \times E'$ has trivial monodromy, where E' is any finite subset of E , containing $0, 1,$ and ∞ .

Fix some $E' \subset E$ such that E' contains the points $0, 1, \infty$ and ϕ restricted to $X \times E'$ has trivial monodromy. Let $E_0 = E'$. Consider $E_1 = E_0 \cup \{\zeta_1\}$ for $\zeta_1 \notin E_0$. Inductively, consider $E_{n+1} = E_n \cup \{\zeta_{n+1}\}$ for $\zeta_{n+1} \notin E_n$ for all $n \geq 0$. Then we eventually get a countable set $E_\infty = \bigcup_{n=0}^\infty E_n$ in $\widehat{\mathbb{C}}$. We can assume that E_∞ is dense in $\widehat{\mathbb{C}}$.

Define $\phi_0 := \phi$ restricted to $X \times E_0$. Using Theorem C and Corollary 2.1 inductively, we know that the holomorphic motion $\phi_n : X \times E_n \rightarrow \widehat{\mathbb{C}}$ can be extended to a holomorphic motion $\phi_{n+1} : X \times E_{n+1} \rightarrow \widehat{\mathbb{C}}$ with trivial monodromy for all $n \geq 0$. Thus we can extend ϕ_0 to a holomorphic motion $\phi_\infty : X \times E_\infty \rightarrow \widehat{\mathbb{C}}$. Since E_∞ is dense in $\widehat{\mathbb{C}}$, it can be further extended to a holomorphic motion $\tilde{\phi} : X \times \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$. The conclusion now follows from Theorem 6.1. \square

Remark 6.2. Let G be a group of Möbius transformations, such that the closed set E is G -invariant. Let $\phi : X \times E \rightarrow \widehat{\mathbb{C}}$ be a G -equivariant holomorphic motion; see Definition 1.3. If ϕ has the property that ϕ restricted to $X \times E'$ has trivial monodromy, where E' is any subset of E , containing the points $0, 1,$ and ∞ , then ϕ can be extended to a holomorphic motion $\tilde{\phi} : X \times \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ which is also G -equivariant. The proof is very similar to the proof of Theorem 1 in [10] or of Theorem B in [3].

Remark 6.3. The question of liftings of holomorphic maps into complex manifolds is a more general and fundamental question. In our paper, we have studied liftings of holomorphic maps into Teichmüller spaces and their intimate relationship with extending holomorphic motions. In future research, we plan to investigate whether the tools and techniques in our paper can be extended to other kinds of Teichmüller spaces.

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