CONFORMAL GEOMETRY AND DYNAMICS
An Electronic Journal of the American Mathematical Society
Volume 23, Pages 262–282 (December 3, 2019)
https://doi.org/10.1090/ecgd/341

STRUCTURAL PROPERTIES OF QUOTIENT SURFACES OF A HECKE GROUP

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Abstract. We study the properties of the surface Σ_q, which is a $2q$-fold cover of $\mathbb{H}/G_q$, where G_q is a Hecke group and q is an integer greater than 3. We have slightly different situations for the even and odd values of q. For odd values of q the surface Σ_q is a $\frac{q-1}{2}$ genus surface with a cusp, whereas, for even values it is a $\frac{q-2}{2}$ genus surface with two cusps. We prove that there exist g embedded tori with a hole on Σ_q, where $g = \frac{q-1}{2}$ when q is an odd integer and $g = \frac{q-2}{2}$ when q is even, with g boundary geodesics at different heights. These boundary geodesics are the separating geodesics intersecting each other transversally. We also prove that the surface Σ_q is a hyper-elliptic surface for every integer $q > 3$.

1. Introduction

In [1], Haas and Series described the relation between the height spectrum of geodesics of the quotient surface $\mathbb{H}/G_q$ and the generalized Diophantine approximation of G_q. They were able to give a formula for the Hurwitz constant of a Hecke group G_q, where $q \geq 3$. Our aim is to further study the properties of the quotient surface of a Hecke group G_q for $q > 3$. More precisely, we will be studying the $2q$-fold cover, Σ_q, of the surface $\mathbb{H}/G_q$. In [1] and [4], Series introduced some detailed coding systems for the geodesics lying on Σ_q. We use these techniques to further explore the structural properties of these surfaces.

This article is arranged as follows: Section 2 presents some preliminary definitions. In Section 3 we discuss some of the important generators and fundamental domains of the fundamental group of Σ_q. In Section 4 we define the symbolic systems, introduced by Series, and the relation between them. With the help of these symbolic systems we will be able to approximate the endpoints of the lifts of geodesics from the surface Σ_q to the hyperbolic plane $\mathbb{H}$. Section 5 is dedicated to proving that the surface Σ_q is hyper-elliptic; the surface has an order two symmetry. In this section we will prove that there are v embedded tori with a hole in Σ_q, where $v = \frac{q-2}{2}$ when q is an even integer greater than 3, whereas, $v = \frac{q-1}{2}$ when q is an odd integer. In Section 6 we will discuss how to numerically calculate heights of closed geodesics lying on the surface Σ_q for a fixed value of $q \geq 3$.

Received by the editors September 4, 2014, and, in revised form, July 26, 2017, and July 13, 2019.

2010 *Mathematics Subject Classification.* Primary 20H10, 30B70, 57M50; Secondary 11K60.

Key words and phrases. Quotient surfaces, Hecke groups, symbolic sequences.

The author was supported by WPRS grant from the University of Warwick, and HEC Partial Support by the Government of Pakistan.

2. Preliminary

The classical Diophantine approximation is the problem of approximating any real number x by a rational number p/q. It seeks the smallest possible value of k where $|x - p/q| < k/q^2$ for infinitely many q. Depending on x we can define a function $\mu(x)$ such that

$$\mu(x) = \inf\{k|\, |x - p/q| < k/q^2 \text{ for infinitely many } q\}.$$

Consider the action of $\mathrm{SL}(2,\mathbb{Z})$ on the hyperbolic plane $\mathbb{H}$. It is known that the orbit of ∞ under $\mathrm{SL}(2,\mathbb{Z})$ is the set of all rational numbers p/q and that for every $x \in \mathbb{R}$, $\mu(x)$ remains invariant throughout the $\mathrm{SL}(2,\mathbb{Z})$-orbit of x. The set $\{1/\mu(x)|x \in \mathbb{R}\}$ is called the Markoff spectrum. There is one-to-one correspondence between the Markoff spectrum and the height spectrum of geodesics on certain covers of $\mathbb{H}/\mathrm{SL}(2,\mathbb{Z})$. One such cover is a punctured torus, which is a six-fold cover, and its fundamental group is the commutator subgroup of $\mathrm{SL}(2,\mathbb{Z})$. The infimum of the Markoff spectrum, defined as the *Hurwitz constant*, is $\sqrt{5}$.

The generalized Diophantine approximation, in [1], for a Fuchsian group G containing a parabolic transformation P fixing ∞ is defined as follows. Let $g \in G$; then g is of the form $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, where $ad - bc = 1$ and $a, b, c, d \in \mathbb{R}$. For our convenience, we can write $a = a(g)$, $b = b(g)$ and so on. We define the set of *G-rational numbers*, denoted by $\mathbb{Q}(G)$, as the set of points in $\mathbb{R}$ which are fixed by conjugates of the parabolic transformation P by elements in G, and we call the complement of $\mathbb{Q}(G)$ in $\mathbb{R}$ the set of *G-irrational numbers*. We need to approximate the G-irrational number x by G-rational numbers. In other words given $x \in \mathbb{R}\backslash\mathbb{Q}(G)$ we want to find that $k > 0$, such that

$$\left| x - \frac{a(g)}{c(g)} \right| < \frac{k}{c(g)^2}, \tag{1}$$

is satisfied for infinitely many $c(g)$. We define

$$\mu_G(x) = \inf\left\{ k > 0 \middle| \left| x - \frac{a(g)}{c(g)} \right| < \frac{k}{q^2} \text{ for infinitely many } g \in G_\infty \right\}.$$

The inequality (1) is the generalized Diophantine approximation. The fraction $\frac{a(g)}{c(g)}$ is actually $g(\infty)$, that is, the G-rational numbers are the orbit points of infinity under the action of G. As in the classical case,

$$M(G) = \{1/\mu_G(x)|x \in \mathbb{R}\backslash\mathbb{Q}(G)\}$$

is defined to be the *Markoff spectrum* for G and $h(G) = \inf M(G)$ the *Hurwitz constant* of the group G. The formula for the Hurwitz constant for a Hecke group G_q was proved by Haas and Series in [1], for any $q \geq 3$.

There is a relation between between $\mu_G(x)$ and the essential height $\mathfrak{h}(\gamma)$ (defined later) of hyperbolic geodesic γ with one endpoint at x. Since the Markoff spectrum is the set of all $1/\mu_G(x)$, for G-irrational x, we can use geodesics in $\mathbb{H}$ to discuss the Markoff spectrum of the group G.

For a geodesic $\gamma(x, y)$ in $\mathbb{H}$, where x and y are the endpoints of γ, the height is defined as

$$\mathrm{ht}(\gamma) = \begin{cases} \frac{1}{2}|x - y| & x, y \in \mathbb{R} \\ \infty & \text{otherwise.} \end{cases}$$

As defined in [1], let G_∞ be the set of equivalence classes of elements in G, where the equivalence relation is defined as: two transformations g and h are equivalent if there is a transformation V in G fixing ∞ such that $g = Vh$. Here it can be noted that for every geodesic $\gamma \in \mathbb{H}$, the height of the images of γ under elements of an equivalence class remains the same. The *essential height* is then defined as

$$\mathfrak{h}(\gamma) = \sup\{k |\ \mathrm{ht}(g\gamma) \geq k \text{ for infinitely many } [g] \in G_\infty\}.$$

It is proved in [1] that if γ_x is a geodesic in $\mathbb{H}$ joining x to ∞ and $\lambda(x, y)$ is the axis of a hyperbolic transformation in G, then $\mathfrak{h}(\gamma_x) = \mathfrak{h}(\lambda) = \sup\{\mathrm{ht}(g\lambda) | \, g \in G\}$, as each geodesic in $\mathbb{H}$ together with its images under the action of the group G projects to a geodesic on the hyperbolic surface $\mathbb{H}/G$.

We can also define the essential height in terms of horocycles. If C is a horocycle on $\mathbb{H}$, tangent at ∞, then its projection bounds a cusp region. The height of such a horocycle is defined to be the Euclidean distance from the line $\Im z = 0$ to the boundary of C. Let C be the horocycle with smallest possible height such that no image of a geodesic $\gamma \subset \mathbb{H}$ intersects with C. Then the projection γ' of γ will not intersect the cusp region bounded by the projection of C. The essential height of a geodesic tells us how far the projected geodesic goes towards the cusp on the surface. The height of a geodesic γ' on the surface is greater if the area of the cusp region bounded by such a horocycle is less.

Theorem 2.1 ([1])**.** *Let $x \in \partial\mathbb{H}$ such that x is a G_q-irrational number, and let γ_x be the geodesic joining x to ∞. Then $\mu(x) = \frac{1}{2}\mathfrak{h}^{-1}(\gamma_x)$.*

We are interested in studying closed geodesics such that both x and y are G_q-irrational numbers that are roots of the same quadratic equation.

3. The fundamental regions of the Hecke group and its subgroups

By definition, a Hecke group G_q is generated by

$$J_q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } S_q = \begin{pmatrix} 1 & 2\cos(\pi/q) \\ 0 & 1 \end{pmatrix},$$

where J_q is a $180°$ rotation about the point $\iota = \sqrt{-1}$, and S_q is a parabolic element fixing ∞. We denote $2\cos(\pi/q)$ by w_q. We also denote $E_q = S_q J_q$, which is an elliptic element of order q rotation about the point $\cos\frac{\pi}{q} + \iota \sin\frac{\pi}{q}$. An index q subgroup of G_q, denoted Γ_q, is generated by the q elliptic transformations $J_{q,n} = E_q^{-n} J_q E_q^n$ and $J_{q,\bar{n}} = E_q^{-\bar{n}} J_q E_q^{\bar{n}}$, where $n = 0, 1, \ldots, v$ such that $v = (q-1)/2$ when q is an odd element, whereas $v = q/2$ when q is an even integer, and $\bar{n}$ denotes inverse of n in $\mathbb{Z}_q$. All these elliptic transformations have order 2. Note that each of the transformations $J_{q,n}$ fixes the point $E_q^n(\iota)$. From these elliptic transformations we define hyperbolic transformations in Γ_q by $J_{q,n}J_{q,0}$ for $n = 1, \ldots, v$ where v is as defined above. Each of these transformations can be seen geometrically as a product of $180°$ rotations, first about ι and then about the point $E_q^n(\iota)$; see Figure 1. Similarly, we define $J_{q,0}J_{q,\bar{n}}$ to be hyperbolic transformation which first rotates any point on $\mathbb{H}$ about the point $E_q^{\bar{n}}(\iota)$ through an angle of $180°$ and then rotates about the point ι also through an angle of $180°$. We denote these transformations by $A_i = -J_{q,n}J_{q,0}$ and $B_i = -J_{q,0}J_{q,\bar{n}}$, where $n + i = v + 1$, for $i, n \in \{1, 2, \ldots, v\}$ where v is as defined above. These hyperbolic transformations generate an index 2 subgroup, H_q, of Γ_q. This is also a subgroup of G_q of index $2q$. Clearly each $\mathrm{Ax}(A_i)$ and $\mathrm{Ax}(B_i)$ have the same heights, which can easily be seen from Figure 1. Note that when q is even, for $i = 1$ the value of $n = \bar{n} = \frac{q}{2}$ gives us $A_1 = \bar{B}_1$. The matrices A_i and B_i depend only on q and n, and we can calculate each of these matrices. So we have

$$E_q^n = \begin{pmatrix} \sin\frac{(n+1)\pi}{q} & -\sin\frac{n\pi}{q} \\ \sin\frac{n\pi}{q} & -\sin\frac{(n-1)\pi}{q} \end{pmatrix}. \tag{2}$$

Then we have

$$A_i = \begin{pmatrix} \sin^2\frac{n\pi}{q} + \sin^2\frac{(n-1)\pi}{q} & 2\sin^2\frac{n\pi}{q}\cos\frac{\pi}{q} \\ 2\sin^2\frac{n\pi}{q}\cos\frac{\pi}{q} & \sin^2\frac{n\pi}{q} + \sin^2\frac{(n+1)\pi}{q} \end{pmatrix}, \tag{3}$$

whereas

$$B_i = \begin{pmatrix} \sin^2\frac{n\pi}{q} + \sin^2\frac{(n-1)\pi}{q} & -2\sin^2\frac{n\pi}{q}\cos\frac{\pi}{q} \\ -2\sin^2\frac{n\pi}{q}\cos\frac{\pi}{q} & \sin^2\frac{n\pi}{q} + \sin^2\frac{(n+1)\pi}{q} \end{pmatrix}. \tag{4}$$

All the entries in the matrices have been simplified such that $\det(A_i) = \det(B_i) = (\det(E_q^n))^2 = \left(\sin^2\frac{n\pi}{q} - \sin\frac{(n-1)\pi}{q}\sin\frac{(n+1)\pi}{q}\right)^2 = \sin^4\frac{\pi}{q} \neq 1$; also we can see that $\det(A_i) > 0$.

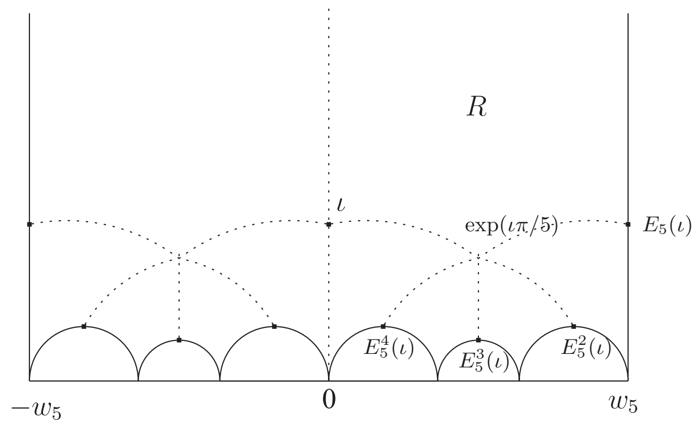

Figure 1. The fundamental domain of a genus 2 surface with a cusp.

The quotient surface $\mathbb{H}/G_q$ is a sphere with a cusp and two cone points of orders q and 2. We fix a fundamental region of G_q in $\mathbb{H}$ which is a quadrilateral with vertices ι, $\exp(\iota\pi/q)$, $E_q(\iota)$ and a vertex at ∞ denoted by R. Any quotient surface obtained by the action of any subgroup of G_q on $\mathbb{H}$ is a covering surface of $\mathbb{H}/G_q$. So the surface $\mathbb{H}/\Gamma_q$ is a q-fold covering surface of $\mathbb{H}/G_q$. This surface is a sphere with a cusp and q cone points all of order 2. There is a fundamental region, denoted by P, of Γ_q containing R, which is a q-gon with vertices $E_q^m(\infty)$ where $1 \leq m \leq q$. Finally, the quotient surface $\mathbb{H}/H_q$ is a two-fold and $2q$-fold cover of $\mathbb{H}/\Gamma_q$ and $\mathbb{H}/G_q$, respectively. The region Q refers to the fundamental region of H_q on $\mathbb{H}$ containing R and P. This region Q is an ideal $2(q-1)$-gon with vertices $E_q^m(\infty)$, $J(E_q^m(\infty))$ where $1 \leq m \leq q$. The quotient surface $\Sigma_q := \mathbb{H}/H_q$ is a genus $g = \frac{q-1}{2}$ surface with a cusp when q is an odd integer, whereas it is a genus $g = \frac{q-2}{2}$ with two cusps when q is an even integer. The fundamental domain of $\mathbb{H}/H_q$ is shown

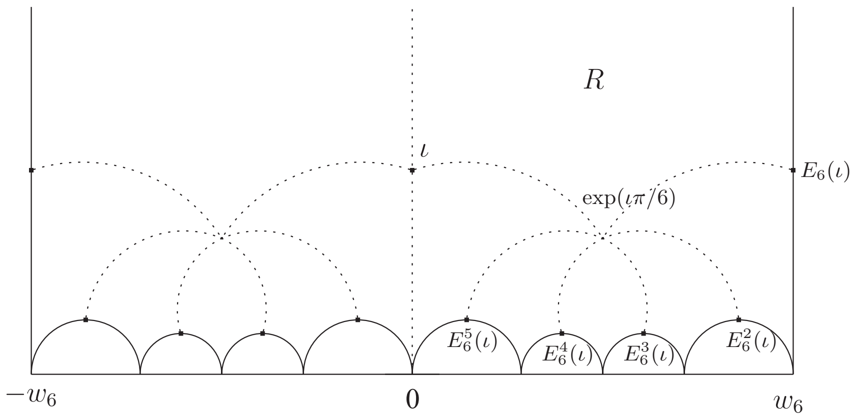

FIGURE 2. The fundamental domain of a genus 2 surface with two cusps.

in Figure 1 and Figure 2 for $q = 5$ and 6. The $2q$ copies of the fundamental domain R of $\mathbb{H}/G_q$ are shown by dotted lines on both figures; their union is the region Q.

4. Symbolic sequences

Series introduced two coding systems for geodesics lying on the quotient surface $\mathbb{H}/G_q$ and its covers. She also defined a way to relate these types of bi-infinite symbolic sequences for a geodesic on the surface $\mathbb{H}/H_q$.

4.1. **Oriented geodesics.** In every quotient surface of the form $\mathbb{H}/G$ the Fuchsian group G is the fundamental group of the surface. We can relate every geodesic on the surface with a conjugacy class of a hyperbolic element in G. The axis of each hyperbolic element in a conjugacy class projects to a unique geodesic on the surface $\mathbb{H}/G$. We denote geodesics lines in $\mathbb{H}$ by s and γ, and oriented lines by $\mathbf{s}$ and $\boldsymbol{\gamma}$. Let $\boldsymbol{\gamma} \subset \mathbb{H}$ be a lift of an oriented geodesic on the surface. Then there exists a primitive hyperbolic element $g \in G$ such that $\boldsymbol{\gamma} = \mathrm{Ax}(g)$. We denote the attractive and repulsive endpoints of the geodesic $\boldsymbol{\gamma}$ by $\boldsymbol{\gamma}^+$ and $\boldsymbol{\gamma}^-$, respectively. If $\boldsymbol{\gamma}$ and $\boldsymbol{\delta}$ are two oriented lines in $\mathbb{H}$ intersecting at a point $z \in \mathbb{H}$, then we write $\boldsymbol{\gamma} \wedge \boldsymbol{\delta} > 0$ if the unit tangent vectors of these lines at z have the same orientation as that of the vectors $(1, 0)$ and $(0, 1)$ at the origin.

4.2. **$\mathfrak{T}$-sequences.** The $\mathfrak{T}$-sequence is defined with respect to a fundamental domain P of the group Γ_q, and the tessellation $\mathfrak{T}$ created by covering $\mathbb{H}$ by all the copies of P under the action of Γ_q, [1]. Let $\boldsymbol{\gamma}$ be an oriented geodesic in $\mathbb{H}$. We divide this geodesic into a sequence of segments by $\{\mathbb{H} \cap \boldsymbol{\gamma}\} \backslash \mathfrak{T}$, in the form

$$\cdots \boldsymbol{\gamma}_{-1}, \boldsymbol{\gamma}_0, \boldsymbol{\gamma}_1, \cdots .$$

Label the line I passing through 0, i to ∞ by s_0, and the rest of the sides of P by $s_1, s_2, \ldots, s_{q-1}$ in the clockwise direction. Since the geodesic γ is oriented, we denote γ_i^+ and γ_i^- as the final and initial endpoints of the segment γ_i, respectively. Now we can assign each segment a number as follows: for each $\boldsymbol{\gamma}_i$ there is a unique $g \in \Gamma_q$ such that $g(\boldsymbol{\gamma}_i)$ is in P with initial point on s_0. We assign the number j to this segment if the final endpoint of this image is on the side s_j, denoted as $\sigma(\gamma_i) = j$. Hence the *symbolic sequence* $\sigma(\boldsymbol{\gamma})$ of the geodesic $\boldsymbol{\gamma}$ is a sequence in

$\Sigma_{\mathfrak{T}} = \Pi_{n=-\infty}^{\infty}\{1, 2, \ldots, q-1\}$. There are some observations about the geodesics related with the symbolic sequences.

Theorem 4.1 (cf. [1], [3]). (1) *Two geodesics $\boldsymbol{\alpha}$, $\boldsymbol{\beta} \subset \mathbb{H}$ are equivalent under Γ_q, or in other words are images of each other under the action of Γ_q if and only if there is $m \in \mathbb{Z}$ such that $\sigma(\boldsymbol{\alpha}_n) = \sigma(\boldsymbol{\beta}_{n+m})$ for all $n \in \mathbb{Z}$* [1].

(2) *A geodesic $\boldsymbol{\gamma}$ in $\mathbb{H}$ projects to a closed geodesic on the surface $\mathbb{H}/\Gamma_q$ if and only if its symbol sequence is periodic* [1].

(3) *For each $\sigma = (\sigma_n) \in \Sigma_{\mathfrak{T}}$ and each oriented side $\mathbf{s}$ of $\mathfrak{T}$ and $m \in \mathbb{Z}$, there is a unique oriented geodesic $\boldsymbol{\gamma} \subset \mathbb{H}$ with $\sigma(\boldsymbol{\gamma}_n) = \sigma_n$ and such that $\boldsymbol{\gamma}_m^- \in \mathbf{s}$ and $\boldsymbol{\gamma} \wedge \mathbf{s} > 0$. We denote such a geodesic by $\ldots\sigma_{m-1}\mathbf{s}\sigma_m\ldots$. And with varying $\mathbf{s}$ and m, we get varying geodesics with the same sequence. All such geodesics are called lifts of σ.*

(4) *Let $\mathbf{s}$ be an oriented side of $\mathfrak{T}$, and let $\sigma, \sigma' \in \Sigma_{\mathfrak{T}}$ with $\sigma_n = \sigma'_n$ for $n \geq N$. Let $\boldsymbol{\gamma} = \ldots\sigma_{N-1}\mathbf{s}\,\sigma_N\ldots$ and $\boldsymbol{\gamma}' = \ldots\sigma'_{N-1}\mathbf{s}\,\sigma'_N\ldots$. Then $\boldsymbol{\gamma}^+ = \boldsymbol{\gamma}'^+$. We denote $\boldsymbol{\gamma}^+$ by $\mathbf{s}\,\sigma_N\sigma_{N+1}\ldots$ and $\boldsymbol{\gamma}^-$ by $\bar{\mathbf{s}}\,\bar{\sigma}_{N-1}\bar{\sigma}_{N-2}\ldots$, where $\bar{i}$ is the inverse of i in $\mathbb{Z}_q$. Here the sequence $\mathbf{s}\sigma_N\sigma_{N+1}\ldots$ means starting from the side $\mathbf{s}$ with positive orientation and then following the sequence cutting the adjacent sides. Note that this will give us a unique point on $\partial\mathbb{H}$.*

4.3. **$\mathfrak{O}$-sequences.** The $\mathfrak{O}$-sequence for each geodesic is defined with reference to the $\mathfrak{O}$ tessellations; see [3]. The sequence of a geodesic represents the order in which it crosses the sides of tessellations created by the copies of the region Q. With each intersection or crossing we assign a value from the set

$$H_0 = \{A_i, B_i, \bar{A}_i, \bar{B}_i | 1 \leq i \leq v\},$$

where $v = \frac{q-1}{2}$ if q is odd and $v = \frac{q}{2}$ if q is even; see Figure 3. So corresponding to each geodesic we have a bi-infinite sequence from the set $\Pi_{i=-\infty}^{\infty} H_0$. Let $\Sigma_{\mathfrak{O}}$ be the set of all such reduced bi-infinite sequences, which means no element in the sequence is followed by its inverse. And let $\Sigma_{\mathfrak{O}}^n$ denote reduced $\mathfrak{O}$ sequence of length n, for $n \in \mathbb{Z}$. We define each oriented side of the tessellation as follows. Let $\mathbf{s}$ be a side joining two regions S and S' in $\mathfrak{O}$. Then the side of $\mathbf{s}$ interior to S is labelled $h \in H_0$ if and only if $S = hS'$. In this case the side of $\mathbf{s}$ interior to S' is labelled $\bar{h}$. Now if a geodesic $\boldsymbol{\gamma} \subset \mathbb{H}$ crosses the regions $S_0, S_1, \ldots, S_n$ of $\mathfrak{O}$ in order, and if $h_i \in H_0$ is the label for the side $S_{i-1} \cap S_i$ interior to S_i for $0 < i \leq n$ and $n \in \mathbb{Z}$, then from above, we have $S_n = h_1 \ldots h_n S_0$. The sequence corresponding to $\boldsymbol{\gamma}$ is denoted as $\chi(\boldsymbol{\gamma}) \in \Sigma_{\mathfrak{O}}$. We also denote $\partial_h S$ as the side interior to the region S in $\mathfrak{O}$ with label h.

Proposition 4.2 (cf. [3]). *Here are some important results related to the $\mathfrak{O}$-sequences of geodesics on the surface:*

(1) *Two oriented geodesics $\boldsymbol{\gamma}, \boldsymbol{\delta} \subset \mathbb{H}$ are equivalent under H if and only if there exists $m \in \mathbb{Z}$ such that $\chi(\boldsymbol{\gamma})_n = \chi(\boldsymbol{\delta})_{n+m}$ for all $n \in \mathbb{Z}$.*

(2) *For every $\chi = (h_n) \in \Sigma_{\mathfrak{O}}$, a region S of $\mathfrak{O}$, and $m \in \mathbb{Z}$ there exists a unique geodesic $\boldsymbol{\gamma} \subset \mathbb{H}$ such that $\chi(\boldsymbol{\gamma}) = (h_n)$ and that $\boldsymbol{\gamma} \cap S \neq \phi$ where $(\boldsymbol{\gamma} \cap S)^- = \partial_{h_m}(S)$ and $(\boldsymbol{\gamma} \cap S)^+ = \partial_{\overline{h}_{m+1}}(S)$. This geodesic is called a lift of χ and can be expressed as*

$$\ldots h_{m-1}\,\partial_{h_m}(S)\,h_{m+1}\,\ldots.$$

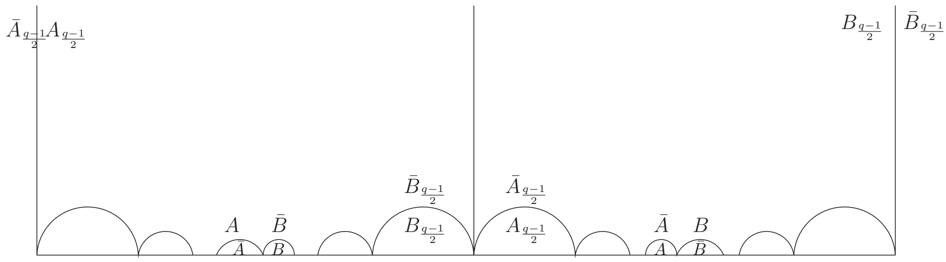

FIGURE 3. Fundamental regions showing the labels with respect to some of the generators.

(3) *For two sequences* $\chi\,,\chi' \in \Sigma_{\mathfrak{D}}$ *with* $\chi_n = \chi'_n$ *for all* $n \geq N$ *and a region* S *of* $\mathfrak{D}$ *such that*

$$\boldsymbol{\gamma} = \dots\, \chi_{N-1}\, \partial_{\chi_N}(S)\, \chi_{N+1}\, \cdots$$

and

$$\boldsymbol{\gamma}' = \dots\, \chi'_{N-1}\, \partial_{\chi'_N}(S)\, \chi'_{N+1}\, \cdots,$$

then $\boldsymbol{\gamma}^+ = \boldsymbol{\gamma}'^+$. *So we write* $\boldsymbol{\gamma}^+ = \partial_{\chi_N}(S)\, \chi_{N+1}\, \chi_{N+2}\, \dots$, *and for the given* $\boldsymbol{\gamma}$, *we write* $\boldsymbol{\gamma}^- = \partial_{\bar{\chi}_N}(\bar{\chi}_N S)\, \bar{\chi}_{N-1}\, \bar{\chi}_{N-2}\, \dots$. *Further, if* $m > n$, *then*

$$\partial_{\chi_n}(S)\, \chi_{n+1}\, \chi_{n+2}\, \cdots = \partial_{\chi_m}(\chi_n \chi_{n+1} \dots \chi_m S)\chi_{m+1} \dots .$$

(4) *If* $g = \chi_1 \dots \chi_k$ *is a cyclically reduced word in* H, *then*

$$\partial_{\chi_1}(Q)\, \dots \chi_k\, \chi_1 \dots \chi_k \cdots$$

is the positive fixed point of g. *Generally,*

$$\partial_{\psi 1}(Q)\psi_2 \dots \psi_r \overline{\chi_1 \cdots \chi_k}$$

is the fixed point of the word $w = \psi_1 \dots \psi_r \chi_1 \dots \chi_k \bar{\psi}_r \dots \bar{\psi}_1$ *whenever it is reduced, where by* $\overline{\chi_1 \dots \chi_k}$ *we mean an infinite periodic sequence of the terms* $\chi_1 \dots \chi_k$.

4.4. **Relation between $\mathfrak{T}$ and $\mathfrak{D}$ sequences.** From the definition of the fundamental regions P and Q, we can see that every $\mathfrak{D}$ region is divided into two $\mathfrak{T}$ regions by $h(I)$, where I is the imaginary axis and $h \in H$. We define the label φ for $\mathbf{I}$ on right-hand side and $\bar{\varphi}$ on the left-hand side; similarly, $h(\mathbf{I})$ is labelled φ and $h(\bar{\mathbf{I}})$ is labelled $\bar{\varphi}$. Let $\hat{\Sigma}_{\mathfrak{D}}$ be the set of reduced sequences from $\Pi_{-\infty}^{\infty} \hat{H}_0$, where $\hat{H}_0 = H_0 \cup \{\varphi, \bar{\varphi}\}$. In order to find the relation between $\Sigma_{\mathfrak{D}}$ and $\Sigma_{\mathfrak{T}}$, we first have to transform each sequence in $\Sigma_{\mathfrak{D}}$ into a sequence in $\hat{\Sigma}_{\mathfrak{D}}$. This can be done as follows, for every geodesic $\boldsymbol{\gamma} \subset \mathbb{H}$: let $\chi(\boldsymbol{\gamma}) \in \Sigma_{\mathfrak{D}}$. It is observed that $\hat{\chi}(\boldsymbol{\gamma})$ depends on $\chi(\boldsymbol{\gamma})$, more precisely every entry in $\hat{\chi}(\boldsymbol{\gamma})$ depends on the corresponding adjacent entries in $\chi(\boldsymbol{\gamma})$: let ψ be the map

$$\psi : \Sigma_{\mathfrak{D}}^2 \to \hat{\Sigma}_{\mathfrak{D}}^2 \cup \hat{\Sigma}_{\mathfrak{D}}^3 \text{ such that } \psi(h, h') = \hat{\chi}(\boldsymbol{\gamma}(h, h')),$$

where $\boldsymbol{\gamma} = \boldsymbol{\gamma}(h, h')$ is a geodesic segment contained entirely in the region Q with $\boldsymbol{\gamma}^- \in \partial_h(Q)$ and $\boldsymbol{\gamma}^+ \in \partial_{\bar{h}'}(Q)$. This is illustrated more clearly in the following example.

Example 4.3. Looking at Figure 4 we can see that

$$\psi(A, A) = A\varphi A \text{ and } \psi(A, B) = AB.$$

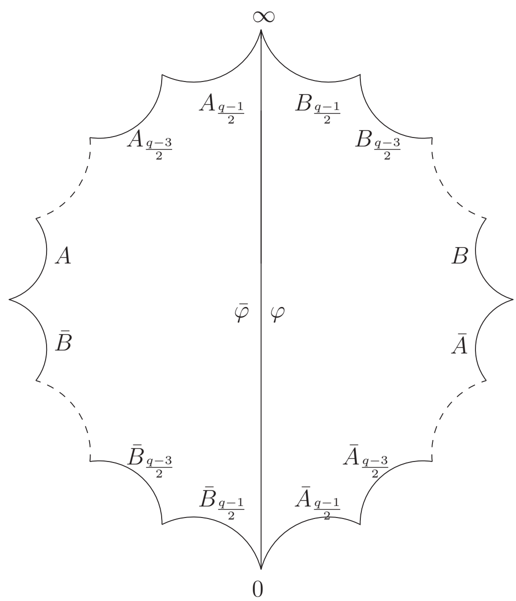

Figure 4. Fundamental regions showing the labels with respect to some of the generators.

From the map ψ we can find the unique sequence $\hat{\chi} = (\hat{\chi}_n) \in \hat{\Sigma}_{\mathfrak{D}}$ from any sequence $\chi = (\chi_n) \in \Sigma_{\mathfrak{D}}$. It is obvious that a sequence $\hat{\chi} = (\hat{\chi}_n)$ in $\hat{\Sigma}_{\mathfrak{D}}$ holds the same properties as that of the corresponding sequence $\chi = (\chi_n)$ in $\Sigma_{\mathfrak{D}}$.

Now for a relation from $\hat{\Sigma}_{\mathfrak{D}}$ to $\Sigma_{\mathfrak{T}}$, consider a mapping $\tau : \hat{\Sigma}^2_{\mathfrak{D}} \to \Sigma^1_{\mathfrak{T}}$ such that $\tau(h_1, h_2)$ is the $\mathfrak{T}$-symbol of the geodesic segment in P joining the side $\partial_{h_1}(P)$ to $\partial_{\bar{h}_2}(P)$. From this we can extend to a map $\tau : \hat{\Sigma}^n_{\mathfrak{D}} \to \Sigma^n_{\mathfrak{T}}$, and hence $\tau : \hat{\Sigma}_{\mathfrak{D}} \to \Sigma_{\mathfrak{T}}$. Consider the restriction in $\tau_h : \{h_1 \dots h_n \in \hat{\Sigma}^n_{\mathfrak{D}} : h_1 = h\} \to \Sigma^n_{\mathfrak{T}}$. Then τ_h is a bijection since the $\mathfrak{T}$-sequence determines the $\mathfrak{D}$-sequence uniquely with the fixed initial side.

Example 4.4. Again from Figure 4 we can see that if we fix one side, then the $\mathfrak{T}$-sequence gives a unique image under τ_h^{-1}:

$$\tau_A^{-1}(2) = A\varphi \text{ and } \tau_\varphi^{-1}(3) = \varphi A.$$

4.5. **Position of the endpoint of a geodesic using $\mathfrak{T}$-sequence.** Looking at only the first few entries of the $\mathfrak{T}$-sequence of a geodesic, we can give an interval in $\mathbb{R}$ where the endpoint of the geodesic lies. As we move onwards in the sequence, we get smaller and smaller intervals such that every next interval is nested in the preceding interval.

Let $\boldsymbol{\gamma}$ be a geodesic segment such that $\sigma(\boldsymbol{\gamma}) = i_1 \dots i_n \in \Sigma^n_{\mathfrak{T}}$. Let $\mathbf{s}$ be a side of $\mathfrak{T}$ such that $\boldsymbol{\gamma}^- \in \mathbf{s}$ and $\boldsymbol{\gamma} \wedge \mathbf{s} > 0$. Also let $\mathbf{s}_0 = \mathbf{s}, \mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_n$ be the sides cut by $\boldsymbol{\gamma}$ in order. Then we write $\mathbf{s}i_1 i_2 \dots i_n$ for the side $\mathbf{s}_n$. We define $Z[\mathbf{s}i_1 \dots i_n]$ for the interval in $\mathbb{R}$ bounded by the endpoints of $\mathbf{s}_n$. The endpoints of $Z[\mathbf{s}i_1 \dots i_n]$ are $Z[\mathbf{s}i_1 \dots i_n]^-$ and $Z[\mathbf{s}i_1 \dots i_n]^+$ such that $Z[\mathbf{s}i_1 \dots i_n]^- < Z[\mathbf{s}i_1 \dots i_n]^+$.

Lemma 4.5 (cf. [3, lemma 1.5.2])**.** *Let $\mathbf{s} = \mathbf{I}^{\pm}$, let $i_1 \dots i_n \in \Sigma_{\mathfrak{T}}^n$, and let $1 \leq i < j \leq q-1$. Then $Z[\mathbf{s}i_1 \dots i_n i]^- \geq Z[\mathbf{s}i_1 \dots i_n j]^+$, with equality if and only if $j = 1 + i$.*

Proof. As defined above let $\mathbf{s}_k = \mathbf{s}i_1 \dots i_k$ where $1 < k \leq n$. Let $Z[\mathbf{s}_k]$ be as defined above. We can see that $Z[\mathbf{s}_1] \supset Z[\mathbf{s}_2] \supset \cdots \supset Z[\mathbf{s}_k] \supset \dots$. Now consider the interval $Z[\mathbf{s}_n m]$, where $1 \leq m \leq (q-1)$. This implies that $Z[\mathbf{s}_n m] \subset Z[\mathbf{s}_n]$ for every m. Now there exists a unique $g \in \Gamma$ such that $g(\mathbf{s}_n) = \mathbf{I}$ and that $g(\mathbf{s}_n m)$ is the side of the region P with the $\mathfrak{T}$-symbol m for all $m \in \{1, 2, \dots, q-1\}$. From the definition of $\mathfrak{T}$-sequences, it is obvious that the smaller the symbol is, the farther the side is from the origin. The sides represented by two consecutive symbols intersect only at one point in $\partial\mathbb{H}$, which is the positive end of the side with the larger symbol and the negative end of the side with the smaller symbol. Therefore $Z[\mathbf{s}_n m]^- = Z[\mathbf{s}_n(m+1)]^+$. Hence the result. □

Throughout this paper we are dealing with symbolic sequences that converge to points on $\partial\mathbb{H}$. There are some important results in which we have to deal with the relationship between the endpoints of the intervals of the form $Z[\mathbf{r}]$, where $\mathbf{r}$ is one of the geodesics giving rise to the tessellation $\mathfrak{T}$. The task of finding out these relationships would be much easier if we could calculate these endpoints. The endpoints of the $\mathfrak{T}$-tessellation can be easily calculated using the translation S_q and the elliptic transformation E_q. We are only considering the case where $\mathbf{r}$ is of the form $\mathbf{I}\sigma_1\sigma_2\dots$, where $\sigma_j \in \{1, 2, \dots (q-1)\}$ and $j \in \mathbb{Z}^+$ is finite. The case where $\mathbf{r}$ is of the form $\bar{\mathbf{I}}\sigma_1\sigma_2\dots$ is analogous to the previous case.

Theorem 4.6. *For every $\mathbf{r}$ of the form described above, there exists a unique element $g \in G_q$ such that $Z[\mathbf{r}]^+ = g(\infty)$ and $Z[\mathbf{r}]^- = g(0)$.*

Proof. This theorem follows from the fact that every line $\mathbf{r}$ of the $\mathfrak{T}$-tessellation can be mapped back to the oriented side $\mathbf{I}$ by an element, say $h \in G_q$. Keeping in mind the orientation of the geodesics in the tessellation $\mathfrak{T}$, we can see $h(\mathbf{r}) = \mathbf{I}$ or $h^{-1}(\mathbf{I}) = \mathbf{r}$. Then h^{-1} is our required element g such that $h^{-1}(\infty) = Z[\mathbf{r}]^+$ and $h^{-1}(0) = Z[\mathbf{r}]^-$. Hence the result. □

Lemma 4.7. *Let $\mathbf{s}$ be the line of the form $\mathbf{I}j$, where $1 \leq j \leq (q-1)$. Then $E^{j-1}S(\infty) = Z[\mathbf{s}]^+$ and $E^{j-1}S(0) = Z[\mathbf{s}]^-$.*

Proof. It can easily be seen that when $\mathbf{s} = \mathbf{I}1$, then $S(\mathbf{I}) = \mathbf{s}$, and that $S(\infty) = Z[\mathbf{s}]^+$ and $S(0) = Z[\mathbf{s}]^-$. When $j > 1$, we can rotate the side $S(\mathbf{I})$ $j-1$ times by the elliptic transformation E_q in order to get to the side $\mathbf{I}j$. So we have $E_q^{j-1}S(\mathbf{I}) = \mathbf{I}j$. Now using Theorem 4.6, we get the result. □

For our convenience we denote the $\mathfrak{T}$-symbols for each pair of generators A_i and B_i from the sets, say $\Sigma(L_i, R_i)$, where $L_i = n$ and $R_i = \bar{n}$ and $n = v - i + 1$ for $i, n \in \{1, 2, \dots, v\}$, where v is as defined above. For example, the sequences for $A_1 = A$ and $B_1 = B$, when q is an odd integer, are from the set $\Sigma(\frac{q-1}{2}, \frac{q+1}{2}) = \Sigma(L_1, R_1)$, and that of $A_{\frac{q-1}{2}}$ and $B_{\frac{q-1}{2}}$ from the set $\Sigma(1, (q-1)) = \Sigma(L_{\frac{q-1}{2}}, R_{\frac{q+1}{2}})$. In general, $\Sigma(L_i, R_i) = \Sigma(n, \bar{n})$.

Lemma 4.8. *In terms of the $\mathfrak{T}$-sequence, the side* $\mathbf{I}L_i = E_q^{n-1}S(\mathbf{I})$ *whereas* $\mathbf{I}R_i = E_q^{\bar{n}-1}S(\mathbf{I})$, *where* $1 \leq i \leq \frac{q-1}{2}$ *and* $n+i = \frac{q+1}{2}$ *when q is odd and* $1 \leq i \leq \frac{q}{2}$ *and* $n+i = \frac{q+2}{2}$ *when q is even.*

Proof. This can be seen from the fact that for every i we have $L_i = n$ and $R_i = \bar{n}$. The result then follows from Lemma 4.7. □

We can calculate the values of $E_q^{n-1}S$ and $E_q^{\bar{n}-1}S$ in terms of n. We know the value of E_q^n from equation (2), so we can simplify to get the following:

$$E_q^{n-1}S = \begin{pmatrix} \sin\frac{n\pi}{q} & \sin\frac{(n+1)\pi}{q} \\ \sin\frac{(n-1)\pi}{q} & \sin\frac{n\pi}{q} \end{pmatrix}, \tag{5}$$

and using the fact that $n + \bar{n} = q$ we get

$$E_q^{\bar{n}-1}S = \begin{pmatrix} \sin\frac{n\pi}{q} & \sin\frac{(n-1)\pi}{q} \\ \sin\frac{(n+1)\pi}{q} & \sin\frac{n\pi}{q} \end{pmatrix}. \tag{6}$$

Lemma 4.9. *Let* $\mathbf{s} = \mathbf{I}\sigma_1\sigma_2\sigma_3\ldots\sigma_m$. *Then the transformation g mentioned in Theorem* 4.6 *is equal to the product* $\prod_{j=1}^m g_j$, *where*

$$g_j = \begin{cases} E_q^{n-1}S(\mathbf{I}), & \text{when } \sigma_j = L_i, \\ E_q^{\bar{n}-1}S(\mathbf{I}), & \text{when } \sigma_j = R_i, \end{cases}$$

where $\bar{n}$ is the inverse of n in $\mathbb{Z}_q$.

This lemma is a straightforward consequence of Lemma 4.8, and therefore leads to the following corollary.

Corollary 4.10. *For every* $\mathbf{s}$ *and every g_j defined in Lemma* 4.9, *we have* $Z[\mathbf{s}]^+ = \Pi_{j=1}^m g_j(\infty)$, *whereas* $Z[\mathbf{s}]^- = \Pi_{j=1}^m g_j(0)$.

From Lemma 4.8, we can now find all of the vertices of the Q region, which is the special fundamental domain of the surface $\Sigma_q = \mathbb{H}/H_q$.

Lemma 4.11. *The endpoints of the side* $\mathbf{I}L_i$ *are* $Z[\mathbf{I}L_i]^+ = \frac{\sin\frac{n\pi}{q}}{\sin\frac{(n-1)\pi}{q}}$ *and* $Z[\mathbf{I}L_i]^- = \frac{\sin\frac{(n+1)\pi}{q}}{\sin\frac{n\pi}{q}}$, *where n and i are as defined above. The endpoints of the side* $\mathbf{I}R_i$ *are* $Z[\mathbf{I}R_i]^+ = \frac{\sin\frac{n\pi}{q}}{\sin\frac{(n+1)\pi}{q}}$ *and* $Z[\mathbf{I}R_i]^- = \frac{\sin\frac{(n-1)\pi}{q}}{\sin\frac{n\pi}{q}}$. *Note that* $Z[\mathbf{I}L_i]^\pm \geq 1$ *and* $Z[\mathbf{I}R_i]^\pm \leq 1$, *with equality when q is odd and* $n = \frac{q-1}{2}$.

Proof. This can be seen by calculating the images of 0 and ∞ under the transformations $E_q^{n-1}S$ and $E_q^{\bar{n}-1}S$. The equality follows when q is odd and $n = \frac{q-1}{2}$, which gives us $Z[\mathbf{I}L_1]^- = \frac{\sin\frac{(n+1)\pi}{q}}{\sin\frac{n\pi}{q}} = \frac{\sin\frac{n\pi}{q}}{\sin\frac{(n+1)\pi}{q}} = Z[\mathbf{I}R_1]^+ = 1$. □

From this lemma, we can see that $Z[\mathbf{I}R_i]$ is the interval where the endpoints are just the reciprocals of the endpoints of $Z[\mathbf{I}L_i]$. This immediately gives us the following result.

Corollary 4.12. *For every* $i \in \{1, 2, \ldots, v\}$ *we have* $l(Z[\mathbf{I}L_i]) \geq l(Z[\mathbf{I}R_i])$, *where* $v = \frac{q-1}{2}$ *when q is an odd integer and* $v = \frac{q}{2}$ *when q is an even integer. Here* $l(Z[\mathbf{s}])$ *represents the length of the interval* $Z[\mathbf{I}]$ *on* $\partial\mathbb{H}$.

Lemma 4.13. *For every $i \in \{1,2,\dots,v\}$ we have $l(Z[\mathbf{I}L_i1]) > l(Z[\mathbf{I}L_iq])$ and $l(Z[\mathbf{I}R_iq]) > l(Z[\mathbf{I}R_i1])$, where $v = \frac{q-1}{2}$ when q is an odd integer and $v = \frac{q}{2}$ when q is an even integer.*

Proof. From Lemma 4.9, we have

$$g_1 = E_q^{n-1}S^2\begin{pmatrix} \sin\frac{n\pi}{q} & 2\sin\frac{n\pi}{q}\cos\frac{\pi}{q}+\sin\frac{(n+1)\pi}{q} \\ \sin\frac{(n-1)\pi}{q} & 2\sin\frac{(n-1)\pi}{q}\cos\frac{\pi}{q}+\sin\frac{n\pi}{q}\end{pmatrix}$$

and

$$g_2 = E_q^{n-1}SE_q^{q-1}S = \begin{pmatrix} 2\sin\frac{(n+1)\pi}{q}\cos\frac{\pi}{q}+\sin\frac{n\pi}{q} & \sin\frac{(n+1)\pi}{q} \\ 2\sin\frac{n\pi}{q}\cos\frac{\pi}{q}+\sin\frac{(n-1)\pi}{q} & \sin\frac{n\pi}{q}\end{pmatrix}$$

such that $g_1(\mathbf{I}) = \mathbf{I}L_i1$ and $g_2(\mathbf{I}) = \mathbf{I}L_iq$. Now the length of each interval can be easily calculated, that is,

$$l(Z[\mathbf{I}L_i1]) = g_1(\infty) - g_1(0) = \frac{\sin^2\frac{\pi}{q}}{2\sin^2\frac{(n-1)\pi}{q}\cos\frac{\pi}{q}+\sin\frac{n\pi}{q}\sin\frac{(n-1)\pi}{q}} \text{ and}$$

$$l(Z[\mathbf{I}L_iq]) = g_2(\infty) - g_2(0) = \frac{\sin^2\frac{\pi}{q}}{2\sin^2\frac{n\pi}{q}\cos\frac{\pi}{q}+\sin\frac{n\pi}{q}\sin\frac{(n-1)\pi}{q}}.$$

But since $2\sin^2\frac{n\pi}{q}\cos\frac{\pi}{q} > 2\sin^2\frac{(n-1)\pi}{q}\cos\frac{\pi}{q}$, we have $l(Z[\mathbf{I}L_i1]) > l(Z[\mathbf{I}L_iq])$. The second inequality $l(Z[\mathbf{I}R_i1]) < l(Z[\mathbf{I}R_iq])$ can be proved similarly. □

Now we try to see the relation between Euclidean lengths of sides of any fundamental regions bounded by $\mathfrak{T}$. First consider the elliptic transformation $F = \left(\begin{smallmatrix}\cos\pi/q & \sin\pi/q \\ -\sin\pi/q & \cos\pi/q\end{smallmatrix}\right)$, which fixes the point ι with rotation angle $2\pi/q$. We study its action on ∞.

Lemma 4.14. *Let $Z(z) = F(z) - z$. Then the function Z is increasing whenever $z \in \left(\frac{-1+\cos\frac{\pi}{q}}{\sin\frac{\pi}{q}}, \frac{1+\cos\frac{\pi}{q}}{\sin\frac{\pi}{q}}\right)$ and decreasing otherwise. Moreover, when q is an odd integer, then*

$$\frac{-1+\cos\frac{\pi}{q}}{\sin\frac{\pi}{q}} = F^{\frac{q-1}{2}}(\infty), \tag{7}$$

and when q is an even integer, then

$$F^{\frac{q}{2}}(\infty) = 0. \tag{8}$$

Proof. The function Z will be increasing when $Z'(z) > 0$ and decreasing when $Z' < 0$. This means that Z is increasing only when $|F'(z)| > 1$, that is, when

$$\frac{1}{\left(-z\sin\frac{\pi}{q}+\cos\frac{\pi}{q}\right)^2} > 1.$$

Therefore, $Z(z)$ is increasing when $\frac{-1+\cos\frac{\pi}{q}}{\sin\frac{\pi}{q}} < z < \frac{1+\cos\frac{\pi}{q}}{\sin\frac{\pi}{q}}$. Equation (8) is obvious. Equation (7) can be seen simply by using double angle identities on the $F^{\frac{q-1}{2}}(\infty)$. □

Lemma 4.15. *If $u > \frac{\cos\frac{\pi}{q}}{\sin\frac{\pi}{q}}$, we have $|F^n(u) - F^{n-1}(u)| > |F^{n+1}(u) - F^n(u)|$ when $n < v$, and $|F^n(u) - F^{n-1}(u)| < |F^{n+1}(u) - F^n(u)|$ when $v+1 < n < q$, where $v = \frac{q-1}{2}$ when q is odd and $v = \frac{q}{2}$ when q is even.*

Proof. To prove this claim, in view of Lemma 4.14, all we need to show is that $F^{\frac{q-1}{2}}(u) < \frac{-1+\cos\frac{\pi}{q}}{\sin\frac{\pi}{q}}$, whereas $F^{\frac{q+1}{2}}(u) > \frac{-1+\cos\frac{\pi}{q}}{\sin\frac{\pi}{q}}$ and $F^{q-1}(u) < \frac{1+\cos\frac{\pi}{q}}{\sin\frac{\pi}{q}}$.

We know that $F^{\frac{q-1}{2}}(u) = \frac{u\sin\frac{\pi}{2q}+\cos\frac{\pi}{2q}}{-u\cos\frac{\pi}{2q}+\sin\frac{\pi}{2q}}$. Assume that $F^{\frac{q-1}{2}}(u) \geq \frac{-1+\cos\frac{\pi}{q}}{\sin\frac{\pi}{q}}$. Then we have $\frac{u\sin\frac{\pi}{2q}+\cos\frac{\pi}{2q}}{-u\cos\frac{\pi}{2q}+\sin\frac{\pi}{2q}} \geq \frac{-1+\cos\frac{\pi}{q}}{\sin\frac{\pi}{q}}$. We can see that $\sin\frac{\pi}{2q} - u\cos\frac{\pi}{2q} < 0$, because if not, then we have $u < \frac{\sin\frac{\pi}{2q}}{\cos\frac{\pi}{2q}} = \frac{1-\cos\frac{\pi}{q}}{\sin\frac{\pi}{q}} < \frac{\cos\frac{\pi}{q}}{\sin\frac{\pi}{q}}$, which is not possible since $u > \frac{\cos\frac{\pi}{q}}{\sin\frac{\pi}{q}}$. So, we get

$$u\sin\frac{\pi}{2q}\sin\frac{\pi}{q} + \cos\frac{\pi}{2q}\sin\frac{\pi}{q} \leq u\cos\frac{\pi}{2q} - \sin\frac{\pi}{2q} - u\cos\frac{\pi}{2q}\cos\frac{\pi}{q} + \sin\frac{\pi}{2q}\cos\frac{\pi}{q},$$

which simplifies to $\sin\left(\frac{\pi}{q} - \frac{\pi}{2q}\right) + \sin\frac{\pi}{2q} \leq u\cos\frac{\pi}{2q} - u\cos\left(\frac{\pi}{q} - \frac{\pi}{2q}\right) = 0$. This leads us to a contradiction. Hence our assumption is false. Therefore, $F^{\frac{q-1}{2}}(u) < \frac{-1+\cos\frac{\pi}{q}}{\sin\frac{\pi}{q}}$. This proves the inequality $|F^{n-1}(u) - F^n(u)| < |F^n(u) - F^{n+1}(u)|$ when $n < \frac{q-1}{2}$.

We now assume that $F^{\frac{q+1}{2}}(u) \leq \frac{-1+\cos\frac{\pi}{q}}{\sin\frac{\pi}{q}}$. This leads to the inequality $u \leq -\frac{\cos\frac{\pi}{q}}{\sin\frac{\pi}{q}}$. But this contradicts our hypothesis. Now assume $F^{q-1}(u) \geq \frac{1+\cos\frac{\pi}{q}}{\sin\frac{\pi}{q}}$. This gives $u \leq \frac{-1-\cos\frac{\pi}{q}}{\sin\frac{\pi}{q}}$, which is not true since $u > \frac{\cos\frac{\pi}{q}}{\sin\frac{\pi}{q}}$. This completes the proof. □

Corollary 4.16. *Let* $\mathbf{s}$ *be a side represented by either* $\mathbf{I}i_1\dots i_n$ *or* $\bar{\mathbf{I}}i_1\dots i_n$, *where* $i_1\dots i_n \in \Sigma^n_{\mathfrak{T}}$ *and* $n \in \mathbb{Z}$. *Then*

$$l(Z[\mathbf{s}1]) > l(Z[\mathbf{s}2]) > \cdots > l\left(Z\left[\mathbf{s}v\right]\right) \text{ and}$$
$$l\left(Z\left[\mathbf{s}\left(v+1\right)\right]\right) < \cdots < l(Z[\mathbf{s}(q-1)]),$$

where $v = \frac{q-1}{2}$ *when* q *is odd and* $v = \frac{q}{2}$ *when* q *is even.*

Proof. Let S be the region of $\mathfrak{T}$ with $\mathbf{s}$ as one of its sides such that the geodesic segment $\boldsymbol{\gamma}$ with symbolic sequence $\sigma(\boldsymbol{\gamma}) = \mathbf{I}i_1\dots i_n i_{n+1}$ crosses $\mathbf{s}$ and $\boldsymbol{\gamma}\wedge\mathbf{s} > 0$. As we know, there exists a $g \in \Gamma_q$ such that $gP = S$. Then gE_qg^{-1} is an elliptic element of order q with rotational angle $2\pi/q$ fixing a point $p = g(\cos\frac{\pi}{q} + \iota\sin\frac{\pi}{q})$ in S. The vertices of S are the rotation of $g(\infty)$ about the point p under the transformation gEg^{-1}.

Without loss of generality, we can translate and, if needed, expand the region S such that p maps to the point ι. Doing so gives us a new region S', and the vertices of this region can be obtained by rotating one of its vertices about the point ι with respect to the transformation F. From Figure 5, we can see that $Z[\mathbf{s}]^+ = Z[\mathbf{s}1]^+$ is now mapped to some point u and that $Z[\mathbf{s}]^- = Z[\mathbf{s}(q-1)]^-$ is mapped to $F(u)$. This is only true when $u > \frac{\cos\frac{\pi}{q}}{\sin\frac{\pi}{q}} = F^{q-1}(\infty)$, because otherwise the side connecting $F(u)$ with u will not be the image of the side $\mathbf{s}$ under translation and expansion. Also the interval $Z[\mathbf{s}i]$ in terms of the function F is mapped to the interval $[F^{\bar{i}}(u), F^{\bar{i}+1}(u)]$ where $1 \leq i \leq (q-1)$ and $\bar{i}$ is the inverse of i in $\mathbb{Z}_q$. This means $Z[\mathbf{s}i]^- = F^{\bar{i}}(u)$ and $Z[\mathbf{s}i]^+ = F^{\bar{i}+1}(u)$. We want to study the relative lengths of the intervals bounded by the endpoints of each of the sides of S. Since translations and expansions do not affect the relation between the lengths of the sides, we can study these relations on S'. Thus we can see that $l(Z[\mathbf{s}i]) < l(Z[\mathbf{s}(i+1)])$ if and only if $|F^{\bar{i}+1}(u) - F^{\bar{i}}(u)| < |F^{\bar{i}+1}(u) - F^{\bar{i}+2}(u)|$ and $l(Z[\mathbf{s}i]) > l(Z[\mathbf{s}(i+1)])$ if and only if $|F^{\bar{i}+1}(u) - F^{\bar{i}}(u)| > |F^{\bar{i}+1}(u) - F^{\bar{i}+2}(u)|$. Hence the result follows from Lemma 4.15. $\square$

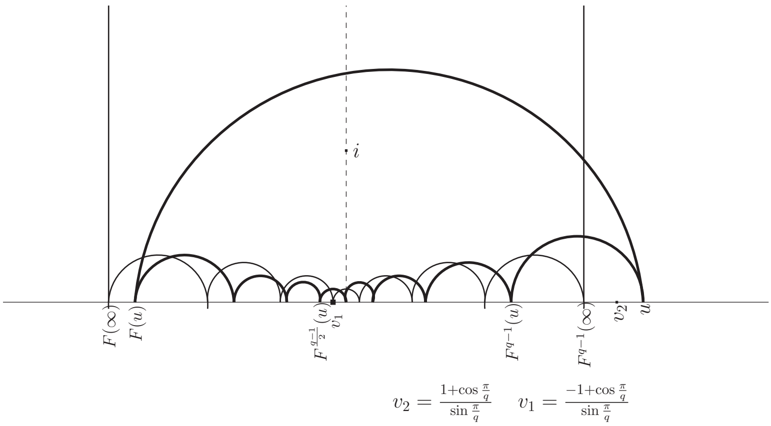

FIGURE 5. The bold lines represent sides of the S' region.

The following result is a modified version of Proposition 3.1 in [3]. It shows that the highest lift is in the position $\dots L_i \mathbf{I} L_i \dots$ or $\dots R_i \bar{\mathbf{I}} R_i \dots$.

Proposition 4.17. *Let $\sigma = (\sigma_n) \in PS(L_i, R_i)$ for a fixed i, and suppose that $\sigma_n = \sigma_{n+1} = L_i$ for some $n \in \mathbb{Z}$. Let $\mathbf{s}$ be a side of $\mathfrak{T}$, and let $m \in \mathbb{Z}$. Then*

$$\mathrm{ht}(\dots \sigma_n \mathbf{I} \sigma_{n+1} \dots) > \mathrm{ht}(\dots \sigma_m \mathbf{s} \sigma_{m+1} \dots),$$

unless $\sigma_m = \sigma_{m+1} = L_i$ and $\mathbf{s} = S_q^k \mathbf{I}$ for some $k \in \mathbb{Z}$, or $\sigma_m = \sigma_{m+1} = R_i$ and $\mathbf{s} = S_q^k \bar{\mathbf{I}}$ for some $k \in \mathbb{Z}$, where $S_q = \left(\begin{smallmatrix} 1 & w_q \\ 0 & 1 \end{smallmatrix}\right)$ and i and q is defined as above. We exclude the case when $i = 1$ for even values of q.

Proof. Fix a value of i, where $1 \le i \le \frac{q-1}{2}$. Let $\boldsymbol{\alpha}$ be the lift of the sequence $\dots \sigma_{n-1} \sigma_n \mathbf{I} \sigma_{n+1} \sigma_{n+2} \dots$, where $\sigma_n = \sigma_{n+1} = L_i$. Then $\boldsymbol{\alpha}^+ = \mathbf{I} L_i \sigma_{n+2} \dots$ and $\boldsymbol{\alpha}^- = \bar{\mathbf{I}} R_i \bar{\sigma}_{n-1} \dots$. It can be seen that $\boldsymbol{\alpha}^+ \in Z[\mathbf{I} L_i]$. Therefore from Lemma 4.5 we conclude that $\boldsymbol{\alpha}^+ > Z[\mathbf{I} L_i R_i]^-$. Similarly, $\boldsymbol{\alpha}^- \in Z[\bar{\mathbf{I}} R_i]$ and $\boldsymbol{\alpha}^- < Z[\bar{\mathbf{I}} R_i L_i]^+$. Hence,

$$\boldsymbol{\alpha}^+ - \boldsymbol{\alpha}^- > Z[\mathbf{I} L_i R_i]^- - Z[\bar{\mathbf{I}} R_i L_i]^+.$$

Now consider another curve $\boldsymbol{\beta}$ which is also a lift of the above sequence and is of the form $\dots L_i \mathbf{I} R_i \dots$. Then we can see that $\boldsymbol{\beta}^+ \le Z[\mathbf{I} R_i L_i]^+$ and $\boldsymbol{\beta}^- \ge Z[\bar{\mathbf{I}} R_i R_i]^-$. Assume that $1 \le i < \frac{q-1}{2}$. We will separately discuss the case when $i = \frac{q-1}{2}$. Then

$$\boldsymbol{\beta}^+ - \boldsymbol{\beta}^- \le Z[\mathbf{I} R_i L_i]^+ - Z[\bar{\mathbf{I}} R_i R_i]^-.$$

Thus we want to show that $\boldsymbol{\alpha}^+ - \boldsymbol{\alpha}^- > \boldsymbol{\beta}^+ - \boldsymbol{\beta}^-$. It is sufficient to see that

$$Z[\mathbf{I} R_i L_i]^+ - Z[\bar{\mathbf{I}} R_i R_i]^- < Z[\mathbf{I} L_i R_i]^- - Z[\bar{\mathbf{I}} R_i L_i]^+, \tag{9}$$

which is the same as

$$Z[\bar{\mathbf{I}} R_i L_i]^+ - Z[\bar{\mathbf{I}} R_i R_i]^- < Z[\mathbf{I} L_i R_i]^- - Z[\mathbf{I} R_i L_i]^+. \tag{10}$$

Since there is a symmetry about the imaginary axis, $Z[\bar{\mathbf{I}}R_iL_i]^+ - Z[\bar{\mathbf{I}}R_iR_i]^- = Z[\mathbf{I}L_iL_i]^+ - Z[\mathbf{I}L_iR_i]^-$. Now equation (10) becomes

$$Z[\mathbf{I}L_iL_i]^+ - Z[\mathbf{I}L_iR_i]^- < Z[\mathbf{I}L_iR_i]^- - Z[\mathbf{I}R_iL_i]^+. \tag{11}$$

Now we prove the following:

$$Z[\mathbf{I}L_iR_i]^- - Z[\mathbf{I}R_iL_i]^+ > Z[\mathbf{I}L_iR_i]^- - Z[\mathbf{I}R_i]^+ > Z[\mathbf{I}L_iL_i]^+ - Z[\mathbf{I}L_iR_i]^-, \tag{12}$$

as shown in Figure 6. This is done for $q = 5$ and $i = 1$ (that is, when $n = \frac{q-1}{2}$) by Series in [3] using direct computation. We know that $Z[\mathbf{I}L_iR_i]^- - Z[\mathbf{I}R_i]^+ < Z[\mathbf{I}L_iR_i]^- - Z[\mathbf{I}R_iL_i]^+$. Now suppose for a contradiction that

$$Z[\mathbf{I}L_iR_i]^- - Z[\mathbf{I}R_i]^+ \leq Z[\mathbf{I}L_iL_i]^+ - Z[\mathbf{I}L_iR_i]^-. \tag{13}$$

Then by using Corollary 4.10, for instance $Z[\mathbf{I}L_iL_i]^+ = E^{n-1}SE^{n-1}S(\infty)$, we can rewrite inequality (13) as follows:

$$\frac{2\sin^2\frac{n\pi}{q}\cos\frac{\pi}{q}}{\sin^2\frac{n\pi}{q}+\sin^2\frac{(n-1)\pi}{q}} - \frac{\sin\frac{n\pi}{q}}{\sin\frac{(n+1)\pi}{q}} \leq \frac{\sin^2\frac{n\pi}{q}+\sin\frac{(n-1)\pi}{q}\sin\frac{(n+1)\pi}{q}}{2\sin\frac{(n-1)\pi}{q}\sin\frac{n\pi}{q}} - \frac{2\sin^2\frac{n\pi}{q}\cos\frac{\pi}{q}}{\sin^2\frac{n\pi}{q}+\sin^2\frac{(n-1)\pi}{q}}.$$

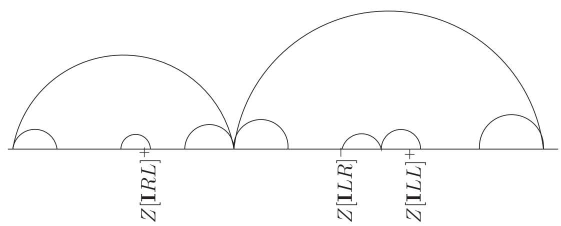

FIGURE 6. Magnified section of $\mathbb{H}$ showing the points on the tessellation lines of $\mathfrak{T}$ when $i = 1$ for any odd integer q.

First we prove that (13) is false for the case when $i = 1$ and q is an odd integer (that is, when $n = \frac{q-1}{2}$), in which case we have $\sin\frac{n\pi}{q} = \sin\frac{(n+1)\pi}{q} = \cos\frac{\pi}{2q}$ and $\sin\frac{(n-1)\pi}{q} = \cos\frac{3\pi}{2q}$. Thus in this case, the above inequality can be simplified into

$$8\cos^2\frac{\pi}{2q}\cos\frac{\pi}{q}\cos\frac{3\pi}{2q} \leq \cos^3\frac{\pi}{2q} + 3\cos^2\frac{\pi}{2q}\cos\frac{3\pi}{2q} + \cos\frac{\pi}{2q}\cos^2\frac{3\pi}{2q} + 3\cos^3\frac{3\pi}{2q}.$$

But the inequality is reversed and can be proved using the fact that $\cos\frac{3\pi}{2q} < \cos\frac{\pi}{q} < \cos\frac{\pi}{2q}$. So this inequality contradicts our assumption in (13).

For $i \in \{2, \ldots, (v-1)\}$ and $q > 5$, we consider the worst case scenario using Corollary 4.16, that is, when $Z[\mathbf{I}L_iL_i]^+ - Z[\mathbf{I}L_iR_i]^-$ is maximum, which only depends on i and happens when $i = (v-1)$ or equivalently when $n = 2$ because $l(Z[\mathbf{I}L_i])$ is the maximum when $i = (v-1)$ as compared to $i \in \{1, 2, \ldots, (v-2)\}$. So we get $Z[\mathbf{L}_{(v-1)}L_{(v-1)}]^+ = Z[\mathbf{I}L_{(v-1)}1]^-$ and $Z[\mathbf{L}_{(v-1)}R_{(v-1)}]^- = Z[\mathbf{I}L_{(v-1)}q]^+$. Keeping $i = (v-1)$ and varying q such that $Z[\mathbf{I}L_iR_i]^- - Z[\mathbf{I}R_iL_i]^+$ is minimum, we get the case when $q = 6$. This is because for every q the distance $Z[\mathbf{I}L_iR_i]^- - Z[\mathbf{I}R_iL_i]^+ > Z[\mathbf{I}L_i]^- - Z[\mathbf{I}R_i]^+ = \frac{\sin\frac{(n+1)\pi}{q}}{\sin\frac{n\pi}{q}} - \frac{\sin\frac{n\pi}{q}}{\sin\frac{(n+1)\pi}{q}}$ for all $1 < i \leq (v-1)$; see Figure 7. In the case of $i = (v-1)$ this becomes $\frac{\sin^2\frac{3\pi}{q}-\sin^2\frac{2\pi}{q}}{\sin\frac{3\pi}{q}\sin\frac{2\pi}{q}}$, which increases as q increases. Since the distance $Z[\mathbf{I}L_i]^- - Z[\mathbf{I}R_i]^+$ increases as q increases, then $Z[\mathbf{I}L_iR_i]^- - Z[\mathbf{I}R_iL_i]^+$ increases. So, by substituting $q = 6$ and $n = 2$ in (12) gives the following:

$$0.5292 > 0.2990 > 0.1443.$$

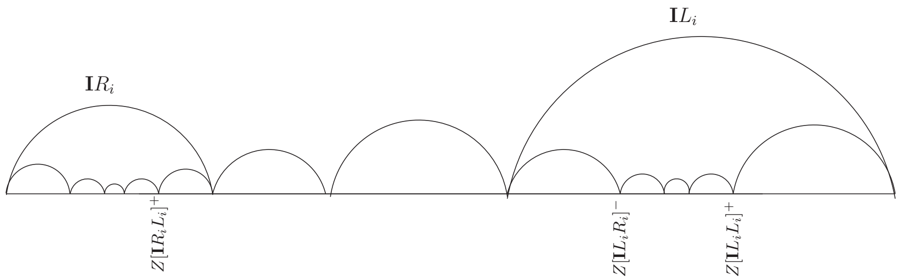

FIGURE 7. Magnified section of $\mathbb{H}$ showing the points on some tessellation lines of $\mathfrak{T}$ when $q = 6$ and $n = 2$ and $i = \frac{q-3}{2}$.

So, when $2 \leq i < \frac{q-3}{2}$, the term $Z[\mathbf{I}L_iL_i]^+ - Z[\mathbf{I}L_iR_i]^-$ increases since the length $l(Z[\mathbf{I}L_i])$ increases as i increases and at the same time $Z[\mathbf{I}L_iL_i]^+$ and $Z[\mathbf{I}L_iR_i]^-$ get closer to $Z[\mathbf{I}L_i]^+$ and $Z[\mathbf{I}L_i]^-$, respectively, until it reaches the maximum, that is, when $i = \frac{q-3}{2}$.

Keeping q fixed, the smallest value of $Z[\mathbf{I}L_iR_i]^- - Z[\mathbf{I}R_iL_i]^+$ is when $i = 1$ and q is an odd integer, in which case inequality (11) is already proved.

When $i = v$ and $q > 3$ (that is, when $n = 1$), we have a slightly different situation. Since the curve $\boldsymbol{\beta}$ is also a lift of the above sequence and is of the form $\dots L_i\mathbf{I}R_i \dots$, then we can see that $\boldsymbol{\beta}^+ \leq Z[\mathbf{I}R_iL_i]^+$, but instead of $Z[\bar{\mathbf{I}}R_iR_i]$ we take $\boldsymbol{\beta}^- \geq Z[\bar{\mathbf{I}}R_iL_i]^+$. This is because otherwise we would have the sequence of the form $\dots L_iL_i\mathbf{I}R_i \dots$; this means, for $i = \frac{q-1}{2}$, the side $\bar{\mathbf{I}}R_i$ is the translation $S_q^{-1}(\bar{\mathbf{I}})$. Therefore the sequence will be of the form $\dots L_iS_q^{-1}(\mathbf{I})L_iR_i \dots$, which is the case excluded by the assumption of this proposition. So now we need to prove that $\boldsymbol{\alpha}^+ - \boldsymbol{\alpha}^- > \boldsymbol{\beta}^+ - \boldsymbol{\beta}^-$. First note that

$$\boldsymbol{\beta}^+ - \boldsymbol{\beta}^- \leq Z[\mathbf{I}R_iL_i]^+ - Z[\bar{\mathbf{I}}R_iL_i]^+.$$

But since $Z[\mathbf{I}R_iL_i]^+ \leq Z[\mathbf{I}L_iR_i]^-$, we have

$$Z[\mathbf{I}R_iL_i]^+ - Z[\bar{\mathbf{I}}R_iL_i]^+ \leq Z[\mathbf{I}L_iR_i]^- - Z[\bar{\mathbf{I}}R_iL_i]^+ < \boldsymbol{\alpha}^+ - \boldsymbol{\alpha}^-.$$

This completes the proof. □

5. THE SURFACE $\mathbb{H}/H_q$ AND SYMMETRIES

As discussed earlier the quotient surface of $\mathbb{H}/G_q$ is topologically a sphere with two cone points of orders 2 and q. This sphere has a $2q$-fold cover, Σ_q, which is a genus $\frac{q-1}{2}$ surface with a puncture when q is an odd integer, whereas it is a genus $\frac{q-2}{2}$ surface with two cusps when q is even. The fundamental group of Σ_q is H_q, which is also a subgroup of G_q, generated by $(q-1)$ hyperbolic elements of G_q. For the group H_q there is a special set of generators $\{A_j, B_j | 1 \leq j \leq v\}$ where $v = \frac{q-1}{2}$ when q is odd and $v = \frac{q}{2}$ when q is even. We study some special features of the geodesics on the surface H_q, restricted to some classes which are the lifts of words formed by these special generators. In this section we discuss the fact that the surface Σ_q admits an involution. We also study the effect of this involution on a general geodesic as well as on some special geodesics.

5.1. **The quotient surface Σ_q.** We are interested in finding some of the properties of the surface Σ_q with respect to each of the commutators of the form $[A_i, B_i]$, where $i = 1, \ldots, v$ and $v = \frac{q-1}{2}$ when q is an odd integer and $v = \frac{q}{2}$ when q is an even integer. Each commutator separates the surface into two components. We study these complementary regions which will help us in later results. Let a_j and b_j be the nonseparating geodesics on Σ_q which are the projections of $\mathrm{Ax}(A_j)$ and $\mathrm{Ax}(B_j)$, where $1 \leq j \leq v$, and let g_j be the separating geodesic which is the projection of the axis of the commutator $[A_j, B_j]$, where $1 \leq j \leq v$.

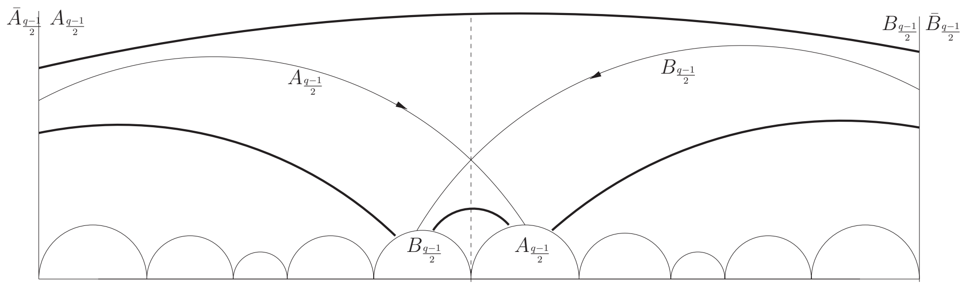

FIGURE 8. The highest commutator is shown by bold lines in the fundamental region for H_7.

Theorem 5.1. *For every pair of generators of the form (A_i, B_i), where $1 \leq i \leq \frac{q-1}{2}$ when q is an odd integer and $2 \leq i \leq \frac{q}{2}$ when q is even, there is a simple closed geodesic g_i on Σ_q represented by the commutator $[A_i, B_i]$. This geodesic divides the surface in two components, one of which is a torus T_i with a hole, where g_i is its boundary geodesic. Furthermore, this torus entirely contains the geodesics a_i and b_i.*

Proof. We consider the commutator $g_{\frac{q-1}{2}}$; all the rest of the cases are similar. Figure 8 shows the case when $q = 7$. In this figure we can see that the bold lines, which are the lift of g_i to the fundamental domain Q, bound a region in Q. Let $L_i = \{l_1, l_2, l_3, l_4\}$ be shown in Figure 9, where l_k have endpoints x_k and y_k for $1 \leq k \leq 4$. The opposite sides of Q contained in this region are identified under the quotient map $\pi : \mathbb{H} \mapsto \mathbb{H}/H_q$. From Figure 9 and from the construction of g_i, we can see that $\pi(y_k) = \pi(x_{k+1})$ for $1 \leq k \leq 3$ and $\pi(y_4) = \pi(x_1)$. According to this the geodesic segment $[x_2, y_3]$ is identified with the geodesic segment $[y_1, x_4]$, similarly the geodesic segment $[x_3, y_4]$ is identified with the geodesic segment $[y_2, x_1]$. So if we cut Q along the bold lines, the component bounded by it projects to a torus T_i with a hole for which g_i is the boundary geodesic. Clearly the remaining region of Q is also connected. This other component is a $(v-1)$-genus surface with a cusp and a hole. For the torus T_i, the group $\pi_1(T_i)$ is generated by the elements A_i and B_i, therefore the geodesics a_i and b_i lie entirely on it.

Note that we exclude the case when $i = 1$ and q is even, because in this case we have seen that $A_1 = \bar{B}_1$, which implies that $[A_1, B_1]$ is identity matrix. Hence the result. □

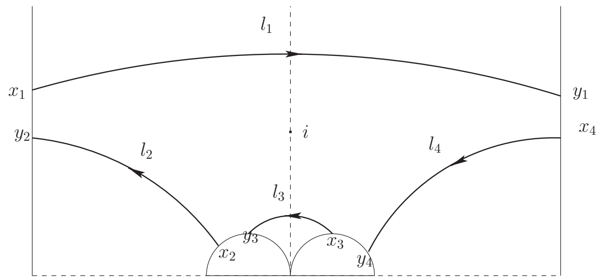

FIGURE 9. The bold lines show the lift L_v of the geodesic g_v.

5.2. **Hyper-elliptic involution.** Following the definitions from [2], a *hyper-elliptic involution* of a surface Σ is an order 2 conformal automorphism which is also an isometry of the surface. If such an involution exists, then it is unique [2]. Any surface admitting such an involution is called a *hyper-elliptic surface.* The existence of a hyper-elliptic involution on a genus two surface is well known; see Haas [2]. When there are one or two cusps on a surface with genus $g \geq 2$ no such involution exists in general. But in the case of Σ_q, where the surface is equipped with a special hyperbolic structure, one can prove that this surface has a hyper-elliptic involution ζ. We claim that ζ is induced by the isometry $J_q : \mathbb{H} \to \mathbb{H}$, defined by $z \mapsto -\frac{1}{z}$ which fixes the fundamental domain Q. The involution ζ fixes q points on the surface and an additional cusp only when q is an odd integer. We prove the existence of this involution in the following theorem.

Theorem 5.2. *Let ζ be the isometry of the surface Σ_q induced by the isometry J_q of the hyperbolic plane $\mathbb{H}$. Then ζ is the hyperelliptic involution of Σ_q which fixes exactly $q+1$ points when q is an odd integer and q points when q is an even integer on the surface, called the Weierstrass point.*

In the following theorem, we discuss the behaviour of some special geodesics.

Theorem 5.3. *Let ζ be the homeomorphism of the surface Σ_q induced by the mapping J_q. Let v be $\frac{q-1}{2}$ when q is an odd integer and $\frac{q}{2}$ when q is even. Then:*

(1) *ζ maps the nonseparating geodesics a_j and b_j to themselves reversing orientation, for every $1 \leq j \leq v$.*
(2) *Any such nonseparating simple closed geodesic passes through exactly two of the Weierstrass points.*
(3) *The separating simple closed geodesics g_j remain invariant under the involution ζ, where $1 \leq j \leq v$, with the exception of the case when $j = 1$ and q is even.*
(4) *The separating simple closed geodesics g_j do not pass through any of the Weierstrass points.*

5.3. **Structure of the special commutators on Σ_q.** From the above discussions about the properties of each geodesic g_j, we now discuss the structure and position of these geodesics on the surface Σ_q. We can now gather all the information:

(1) From Theorem 5.3 we have proved that the involution ζ preserves the geodesic g_j where $1 \leq j \leq v$.

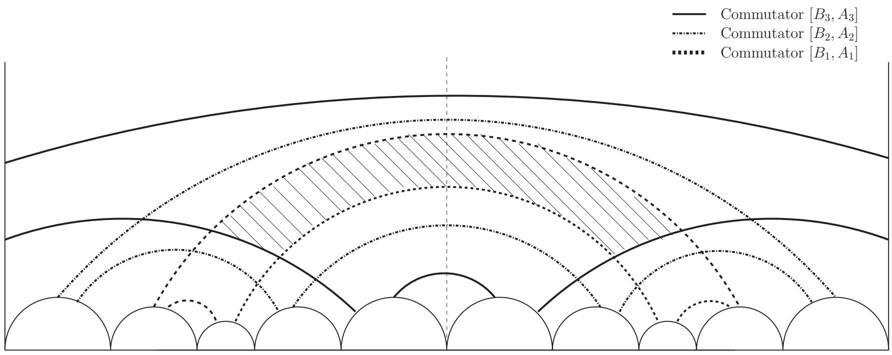

FIGURE 10. Figure showing how all g_j intersect each other. The shaded region is common in all regions bounded by each commutator.

(2) From Theorem 5.1, each of the geodesics g_j separates the surface into two components one of which is a torus T_j with a hole.
(3) We have seen from Theorem 5.3 (2) that every a_j and b_j, where $1 \leq j \leq v$, passes through the projection of the point ι on the surface Σ_q, and since $\pi_1(T_j) = \langle a_j, b_j \rangle$, then $\pi(\iota) \in T_j$ for all j. We call this point the *common Weierstrass point.*
(4) From Theorem 5.3 (3), we have seen that L_j bounds a region in Q which projects to T_j for every $1 \leq j \leq \frac{q-1}{2}$ when q is an odd integer and for every $1 < j \leq \frac{q}{2}$ when q is an even integer. In part (3) of the above, we concluded that there is a common Weierstrass point in every T_j. Since we have a finite number of the embedded tori with hole T_j, this implies that there is a small region containing the point $\pi(i)$ which is common in every such torus with a hole T_j. This means that all the commutators intersect transversally with each other as seen in Figure 10.

Using all of the above observations, we conclude that the geodesic g_j must be positioned as shown in Figure 11. This figure shows an example of geodesics g_1 and g_2 on the surface Σ_5, with the embedded tori with holes T_1 and T_2. As we know, $\mathfrak{h}(a_2) = \mathfrak{h}(b_2) > \mathfrak{h}(a_1) = \mathfrak{h}(b_1)$, and $T_1 = \langle a_1, b_1 \rangle$ and $T_2 = \langle a_2, b_2 \rangle$. So obviously the commutator g_2 must cut the surface in such a way that the shortest distance of g_2 to the cusp is smaller than $\mathfrak{h}(g_1)$; see Figure 10. Therefore, the boundary of T_2 is higher than that of T_1, which means that T_2 is closer to the cusp than T_1. Each of the tori contains the common Weierstrass point as well as two more Weierstrass points such that no Weierstrass point occurs in both T_1 and T_2 other than the common Weierstrass point. In T_1 (resp., T_2) the geodesics a_1 and b_1 (resp., a_2 and b_2) pass through the common Weierstrass point and either of the remaining two points. The other component cut out by g_2 is a torus with a hole and a cusp such that it contains the remaining two Weierstrass points not lying on the geodesics a_2 and b_2.

6. Calculating heights of geodesics

One way of calculating the height of a geodesic on the surface Σ_q is by finding its respective symbolic sequences. We have already seen that there is a bijection τ_φ

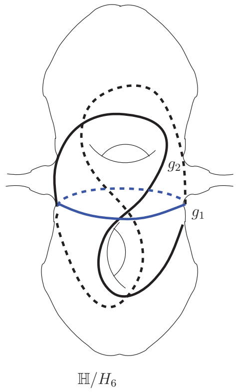

FIGURE 11. Bold lines showing geodesics g_i. Left figure shows g_2 for an even value of q, whereas the figure on the right shows g_1 as well as q_2 for an odd value of q.

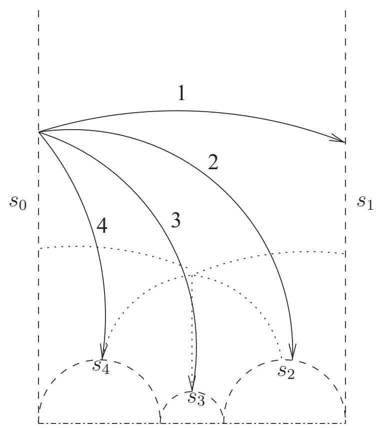

FIGURE 12. Bold lines indicating the symbols associated with crossing geodesics through the pentagon for $q = 5$

between the set of all $\mathfrak{T}$-sequences $\Sigma_{\mathfrak{T}}$ and the set of all reduced $\mathfrak{O}$-sequences $\Sigma_{\mathfrak{O}}$ with initial side φ or $\bar{\varphi}$.

Theorem 6.1. *For each symbol j we associate a unique transformation $S_q^{-1}E_q^{-(j-1)}$, for every $1 \leq j \leq (q-1)$, such that it takes any point on the side $\mathbf{s}_j$ of the region P to the side s_0.*

This theorem is the reverse of Lemma 4.7; see Figure 12.

Proof. Consider a geodesic segment $\boldsymbol{\gamma}_i$ of a geodesic $\boldsymbol{\gamma}$, and assume that this segment lies in the region P with $\boldsymbol{\gamma}_i^-$ lying on the line labelled $\mathbf{s}_0$ such that $\boldsymbol{\gamma}_i \wedge \mathbf{s}_0 > 0$. As explained earlier, the main idea is to pull back the endpoint $\boldsymbol{\gamma}_i^+$ of the geodesic segment from the side, say $\mathbf{s}_j$, of P to the side s_0. We need to find the unique transformation that performs this task for each label in the set $\{1, 2, \ldots, (q-1)\}$. For this, all we have to do is use the translation transformation $S_q : z \mapsto z + \omega_q$

and rotational transformation E_q, which rotates at an angle of π/q in the clockwise manner about the point $\cos\frac{\pi}{q} + i\sin\frac{\pi}{q}$. It also rotates the lines of P in the order $\mathbf{s}_0 \to \mathbf{s}_1 \to \mathbf{s}_2 \to \cdots \to \mathbf{s}_{q-1} \to \mathbf{s}_0$. For the symbol 1, the point $\boldsymbol{\gamma}_i^+$ lies on the line labelled $\mathbf{s}_1$ which is parallel to $\mathbf{s}_0$ and at a distance ω from it. Clearly, the point $S^{-1}(\boldsymbol{\gamma}_i^+)$ lies on the side $\mathbf{s}_0$, and since it is a translation of the point $\boldsymbol{\gamma}_i^+$ the height is preserved. For symbol 2, we have to first rotate $\boldsymbol{\gamma}_i^+$ from $\mathbf{s}_2$ to $\mathbf{s}_1$ and then translate from $\mathbf{s}_1$ to $\mathbf{s}_0$. So the transformation taking $\boldsymbol{\gamma}_i^+$ from $\mathbf{s}_2$ to $\mathbf{s}_0$ is $S_q^{-1}E_q^{-1}$. Similarly, we can find the unique transformations for all the other symbols. $\square$

These symbols are used only for the fundamental domain P for the group Γ_q. We need to find the heights of geodesics on the surface $\mathbb{H}/H_q$, that is, we want to calculate the heights of a geodesic with respect to the fundamental domain Q. In order to find the essential height, we also have to consider the reflection Ω of geodesics about the line $\mathbf{I}$, also referred to as $\mathbf{s}_0$. The effect of Ω on the symbols is that it interchanges the symbols from $n \longleftrightarrow \bar{n}$, for every $n \in \{1, 2, \ldots, (q-1)/2\}$. The new geodesic, which may or may not be the same, has the same height on the surface as that of the original one since reflection preserves heights.

Theorem 6.2. *The height of any closed geodesic, intersecting the projection of the line* $\mathbf{I}$*, on the surface* $\mathbb{H}/H_q$ *with known symbolic sequence can be calculated with the help of its* $\mathfrak{O}$*-sequence.*

Proof. We know that the symbolic sequence of a closed geodesic is periodic. Let $\boldsymbol{\gamma}$ be a closed geodesic on Σ_q. Consider the word $X \in H_q^* = \{A_i, B_i, \bar{A}_i, \bar{B}_i : 1 \leq i \leq v\}$ that forms a period of the $\mathfrak{O}$-sequence, starting from the first symbol with initial point on $\mathbf{I}$ oriented from left to right. Multiply the matrices that correspond to each of the $\mathfrak{O}$-symbols, where these matrices belong to the set H_q^*, in such a way that every next matrix is multiplied on the right side. Once the period is completed, we get a primitive matrix representing its axis with the same symbolic sequence. This means that the axis of this matrix is a lift of $\boldsymbol{\gamma}$ to $\mathbb{H}$. We can now easily find the height of this lift. Since it is a periodic sequence the choice of our starting symbol varies. This means that any cyclic permutation of the word X also gives a lift of the geodesic $\boldsymbol{\gamma}$. Here by the starting symbol we mean the symbol with initial point on the line $\mathbf{I}$. Now since we want to find the essential height of the geodesic $\boldsymbol{\gamma}$ on the surface Σ_q, we have to consider the heights of all the possible lifts of this geodesic to $\mathbb{H}$. Taking the maximum of these heights will give the essential height. The heights corresponding to all the lifts can be calculated by finding the heights of the all possible cyclic permutations of the word X and its inverse.

Since we have defined the assignment of each symbol only when oriented from left to right, we also have to find some method to deal with the possibilities when oriented from right to left. So here we apply Ω, which interchanges the symbols and may change the word as well. But the new geodesic still has the same height, on the surface, as the original one. So again for this new set of symbols we find heights of all possible lifts. And the greatest value among these heights is the essential height of the original geodesic, or the height of it on the surface $\mathbb{H}/H_q$. $\square$

Lemma 6.3. *Let* $\chi = \{\chi_k\} \in \Sigma_{\mathfrak{O}}$ *be a periodic sequence such that* $\chi_k \in \{A_i, \bar{B}_i\}$ *and* X *is a word which corresponds to its period. Then the essential height of* χ *is realized when* X *is of the form* $\bar{B}_i x A_i$ *where* x *is a word in* A_i *and* $\bar{B}_i$*.*

Proof. From Proposition 4.17, we know that for any σ sequence such that $\sigma_k \in \{L_i, R_i\}$, the highest lift of σ occurs when we have a sequence of the form $\dots L_i \mathbf{I} L_i \dots$, or in terms of $\mathfrak{O}$-sequence, $\dots A_i \mathbf{I} \bar{B}_i \dots$. Now if σ is periodic, then the period of this lift is $L_i\sigma_2' \dots \sigma_m' L_i$ or $\tau_\varphi^{-1}(L_i\sigma_2' \dots \sigma_m' L_i) = \varphi \bar{B}_i \chi_2' \dots \chi_{m+1}' A_i$, which can be written in the form $\bar{B}_i x A_i$ after ignoring the symbol φ, where x is a word consisting only of A_i and $\bar{B}_i$. □

Example 6.4. To explain Lemma 6.3, we use the following examples:

(1) The highest lift of the symbolic sequence of $A_iB_iA_i\bar{B}_i$ is a periodic sequence with repeating blocks of the form $[\bar{B}_iA_iB_iA_i]$.

(2) The height corresponding to the word $A_iB_i\bar{A}_i\bar{B}_i$ is less than that of the word $\bar{B}_i\bar{A}_iB_iA_i$. Since both are inverses of each other the essential height of the geodesic corresponding to this word is the height given by the word $\bar{B}_i\bar{A}_iB_iA_i$. It is calculated to be 1.3969 when $q = 5$ and $i = 1$.

In [5] we used this method to calculate the essential heights of the geodesics on the surface Σ_q, where q is an odd integer greater than 3, and found some important results about its height spectrum. We have proved that unlike the height spectrum of a once punctured torus, the heights of simple closed geodesics and that of a nonsimple closed geodesics do not lie on separate intervals. In fact, there are some very small intervals on the height spectrum where we can find the heights of simple closed geodesics as well as the heights of finitely intersecting closed geodesics.

Acknowledgment

I am thankful for the support and guidance of Professor Caroline Series. Most of the work in this article is part of my PhD thesis under the supervision of Professor Series.

References

[1] Andrew Haas and Caroline Series, *The Hurwitz constant and Diophantine approximation on Hecke groups*, J. London Math. Soc. (2) **34** (1986), no. 2, 219–234, DOI 10.1112/jlms/s2-34.2.219. MR856507

[2] Andrew Haas and Perry Susskind, *The geometry of the hyperelliptic involution in genus two*, Proc. Amer. Math. Soc. **105** (1989), no. 1, 159–165, DOI 10.2307/2046751. MR930247

[3] Caroline Series, *The Markoff spectrum in the Hecke group G_5*, Proc. London Math. Soc. (3) **57** (1988), no. 1, 151–181, DOI 10.1112/plms/s3-57.1.151. MR940433

[4] Caroline Series, *Symbolic dynamics and Diophantine equations*, Number theory and dynamical systems (York, 1987), London Math. Soc. Lecture Note Ser., vol. 134, Cambridge Univ. Press, Cambridge, 1989, pp. 49–67, DOI 10.1017/CBO9780511661983.005. MR1043705

[5] K. Farooq, *Properties of quotient surface $\mathbb{H}/H_q$ for any odd integer $q \geq 5$*, preprint.

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