COMPLEX HYPERBOLIC TRIANGLE GROUPS
OF TYPE $[m, m, 0; 3, 3, 2]$

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ABSTRACT. In this paper we study discreteness of complex hyperbolic triangle groups of type $[m, m, 0; 3, 3, 2]$, i.e., groups of isometries of the complex hyperbolic plane generated by three complex reflections of orders $3, 3, 2$ in complex geodesics with pairwise distances $m, m, 0$. For fixed m, the parameter space of such groups is of real dimension one. We determine intervals in this parameter space that correspond to discrete and to non-discrete triangle groups.

1. INTRODUCTION

Complex hyperbolic triangle groups are groups of isometries of the complex hyperbolic plane generated by three complex reflections in complex geodesics. We will focus on the case of ultra-parallel groups, that is, the case where the complex geodesics are pairwise disjoint. Unlike real reflections, complex reflections can be of arbitrary order. If an ultra-parallel complex hyperbolic triangle group is generated by reflections of orders n_1, n_2, n_3 in complex geodesics C_1, C_2, C_3 with the distance between C_{k-1} and C_{k+1} equal to m_k for $k = 1, 2, 3$, then we say that the group is of type $[m_1, m_2, m_3; n_1, n_2, n_3]$. In this paper, we will study discreteness of ultra-parallel complex hyperbolic triangle groups of type $[m, m, 0; 3, 3, 2]$; i.e., two of the reflections are of order 3 and one is of order 2, the fixed point sets of order 3 reflections intersect on the boundary of the complex hyperbolic plane ($m_3 = 0$), and the other two distances between fixed point sets coincide ($m_1 = m_2$).

The deformation space of groups of type $[m, m, 0; 3, 3, 2]$ for a given m is of real dimension one; a group is determined up to an isometry by the angular invariant $\alpha \in [0, 2\pi]$. See section 2. Our main aim is to determine an interval in this one-dimensional deformation space such that for all values of the angular invariant in this interval the corresponding triangle group is discrete. The main result of the paper is the following proposition:

Proposition 1. *A complex hyperbolic triangle group of type $[m, m, 0; 3, 3, 2]$ with angular invariant α is discrete if*

$$m \geqslant \log_e(3) \quad and \quad \cos(\alpha) \leqslant -\frac{1}{2}.$$

In the previous works [WG, Mo, MPP], the authors considered cases where all three complex reflections are involutions. Ultra-parallel triangle groups of types $[m, m, 0; 2, 2, 2]$ and $[m, m, 2m; 2, 2, 2]$ have been considered in [WG], while groups of type $[m_1, m_2, 0; 2, 2, 2]$ have been considered in [MPP] and [Mo].

To prove Proposition 1, we use a version of Klein's combination theorem, adapted to the configurations in question. Two of the generating reflections share a fixed point on the boundary of the complex hyperbolic plane. We show that the ultra-parallel triangle group satisfies a compression property by carefully studying the structure of the stabilizer of this fixed point and of its subgroup of Heisenberg translations. The argument starts in a similar way to that for complex reflections of order 2. However for higher order complex reflections the rank of the group of Heisenberg translations is higher, leading to a quadratic optimisation problem over $\mathbb{Z}^2$ rather than $\mathbb{Z}$.

On the other hand we obtain the following non-discreteness result using a complex hyperbolic version of Shimizu's lemma:

Proposition 2. *A complex hyperbolic triangle group of type $[m, m, 0; 3, 3, 2]$ with angular invariant α is non-discrete if*

$$\cos(\alpha) > 1 - \frac{1}{12\sqrt{3}\cosh^2\left(\frac{m}{2}\right)}.$$

Combining these results, we see that there is a gap between the intervals of discreteness and non-discreteness. This is illustrated in Figure 1. The figure shows the (m, α)-space. The light grey box corresponds to discrete groups (Proposition 1). The black area corresponds to non-discrete groups (Proposition 2).

Ultra-parallel complex hyperbolic triangle groups of type $[m, m, 0; n_1, n_2, 2]$ with orders (n_1, n_2) other than $(2, 2)$ and $(3, 3)$ will be considered in [Po].
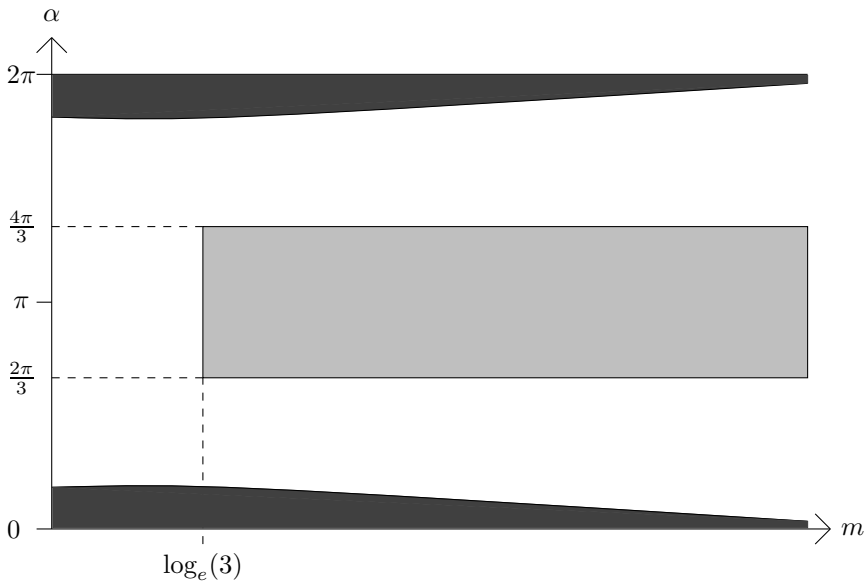

FIGURE 1. Discreteness and non-discreteness results in the (m, α)-space.

The paper is organised as follows: In section 2 we summarise the necessary background information on complex hyperbolic and Heisenberg geometry. We introduce the standard parametrisation for ultra-parallel $[m_1, m_2, 0; n_1, n_2, n_3]$-triangle groups in section 3. In section 4 we use the compression property to derive a discreteness condition for $[m_1, m_2, 0; n_1, n_2, n_3]$-groups. In section 5 we specialise the standard parametrisation to the case of ultra-parallel $[m, m, 0; 3, 3, 2]$-triangle groups. The fixed point sets of order 3 reflections intersect on the boundary of the complex hyperbolic plane. In section 6 we study the structure of the stabilizer of this intersection point. In section 7 we use the discreteness conditions from section 4 to give a proof of Proposition 1. In section 8 we use a version of Shimizu's lemma to show Proposition 2.

We use the following notation: For group elements A and B, their commutator is $[A, B] = A^{-1}B^{-1}AB$.

2. Background

In this section we will give a brief introduction to complex hyperbolic geometry; for further details see [Go, P10].

2.1. **Complex hyperbolic plane.** Let $\mathbb{C}^{2,1}$ be the 3-dimensional complex vector space equipped with a Hermitian form $\langle \cdot, \cdot \rangle$ of signature $(2, 1)$, e.g.,

$$\langle z, w \rangle = z_1 \bar{w}_1 + z_2 \bar{w}_2 - z_3 \bar{w}_3.$$

If $z \in \mathbb{C}^{2,1}$, then we know that $\langle z, z \rangle$ is real. Thus we can define subsets V_-, V_0, and V_+ of $\mathbb{C}^{2,1}$ as follows:

$$V_- = \{z \in \mathbb{C}^{2,1} \mid \langle z, z \rangle < 0\},$$
$$V_0 = \{z \in \mathbb{C}^{2,1} \backslash \{0\} \mid \langle z, z \rangle = 0\},$$
$$V_+ = \{z \in \mathbb{C}^{2,1} \mid \langle z, z \rangle > 0\}.$$

We say that $z \in \mathbb{C}^{2,1}$ is *negative*, *null*, or *positive* if z is in V_-, V_0, or V_+, respectively. Define a projection map $\mathbb{P}$ on the points of $\mathbb{C}^{2,1}$ with $z_3 \neq 0$ as

$$\mathbb{P} : z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \mapsto \begin{pmatrix} z_1/z_3 \\ z_2/z_3 \end{pmatrix} \in \mathbb{P}(\mathbb{C}^{2,1}).$$

That is, provided $z_3 \neq 0$,

$$z = (z_1, z_2, z_3) \mapsto [z] = [z_1 : z_2 : z_3] = \left[\frac{z_1}{z_3} : \frac{z_2}{z_3} : 1 \right].$$

The *projective model* of the complex hyperbolic plane is defined to be the collection of negative lines in $\mathbb{C}^{2,1}$, and its boundary is defined to be the collection of null lines. That is,

$$H_{\mathbb{C}}^2 = \mathbb{P}(V_-) \quad \text{and} \quad \partial H_{\mathbb{C}}^2 = \mathbb{P}(V_0).$$

The metric on $H_{\mathbb{C}}^2$, called the *Bergman metric*, is given by the distance function ρ defined by the formula

$$\cosh^2 \left(\frac{\rho([z], [w])}{2} \right) = \frac{\langle z, w \rangle \langle w, z \rangle}{\langle z, z \rangle \langle w, w \rangle},$$

where $[z]$ and $[w]$ are the images of z and w in $\mathbb{C}^{2,1}$ under the projectivisation map $\mathbb{P}$. The group of holomorphic isometries of $H_{\mathbb{C}}^2$ with respect to the Bergman metric can be identified with the projective unitary group $\mathrm{PU}(2,1)$.

2.2. **Complex geodesics.** A *complex geodesic* is a projectivisation of a two-dimensional complex subspace of $\mathbb{C}^{2,1}$. Any complex geodesic is isometric to

$$\{[z:0:1] \mid z \in \mathbb{C}\}$$

in the projective model. Any positive vector $c \in V_+$ determines a two-dimensional complex subspace

$$\{z \in \mathbb{C}^{2,1} \mid \langle c, z \rangle = 0\}.$$

Projecting this subspace we obtain a complex geodesic

$$\mathbb{P}\left(\{z \in \mathbb{C}^{2,1} \mid \langle c, z \rangle = 0\}\right).$$

Conversely, any complex geodesic is represented by a positive vector $c \in V_+$, called a *polar vector* of the complex geodesic. A polar vector is unique up to multiplication by a complex scalar. We say that the polar vector c is *normalised* if $\langle c, c \rangle = 1$.

Let C_1 and C_2 be complex geodesics with normalised polar vectors c_1 and c_2, respectively. We call C_1 and C_2 *ultra-parallel* if they have no points of intersection in $H_{\mathbb{C}}^2 \cup \partial H_{\mathbb{C}}^2$, in which case

$$|\langle c_1, c_2 \rangle| = \cosh\left(\frac{1}{2}\operatorname{dist}(C_1, C_2)\right) > 1,$$

where $\operatorname{dist}(C_1, C_2)$ is the distance between C_1 and C_2. We call C_1 and C_2 *ideal* if they have a point of intersection in $\partial H_{\mathbb{C}}^2$, in which case $|\langle c_1, c_2 \rangle| = 1$ and $\operatorname{dist}(C_1, C_2) = 0$.

2.3. **Complex reflections.** For a given complex geodesic C, a *minimal complex hyperbolic reflection of order n* in C is the isometry ι_C in $\mathrm{PU}(2,1)$ of order n with fixed point set C given by

$$\iota(z) = -z + (1-\mu)\frac{\langle z, c \rangle}{\langle c, c \rangle}c,$$

where c is a polar vector of C and $\mu = \exp(2\pi i/n)$.

2.4. **Complex hyperbolic triangle groups.** A *complex hyperbolic triangle* is a triple (C_1, C_2, C_3) of complex geodesics in $H_{\mathbb{C}}^2$. A triangle (C_1, C_2, C_3) is a *complex hyperbolic ultra-parallel $[m_1, m_2, m_3]$-triangle* if the complex geodesics are ultra-parallel at distances $m_k = \operatorname{dist}(C_{k-1}, C_{k+1})$ for $k = 1, 2, 3$. We will allow $m_k = 0$ for some or all k. A *complex hyperbolic ultra-parallel $[m_1, m_2, m_3; n_1, n_2, n_3]$-triangle group* is a subgroup of $\mathrm{PU}(2,1)$ generated by complex reflections ι_k of order n_k in the sides C_k of a complex hyperbolic ultra-parallel $[m_1, m_2, m_3]$-triangle (C_1, C_2, C_3).

2.5. **Angular invariant.** The real dimension of the space of $[m_1, m_2, m_3]$-triangles for each fixed triple m_1, m_2, m_3 is equal to one. We can describe a parametrisation of the space of complex hyperbolic triangles in $H_{\mathbb{C}}^2$ by means of an angular invariant α. We define the *angular invariant* α of the triangle (C_1, C_2, C_3) by

$$\alpha = \arg\left(\prod_{k=1}^{3} \langle c_{k-1}, c_{k+1} \rangle\right),$$

where c_k is the normalised polar vector of the complex geodesic C_k. We use the following proposition, given in [Pra], which gives criteria for the existence of a triangle group in terms of the angular invariant.

Proposition 3. *An $[m_1, m_2, m_3]$-triangle in $H^2_{\mathbb{C}}$ is determined uniquely up to isometry by the three distances between the complex geodesics and the angular invariant α. For any $\alpha \in [0, 2\pi]$, an $[m_1, m_2, m_3]$-triangle with angular invariant α exists if and only if*

$$\cos(\alpha) < \frac{r_1^2 + r_2^2 + r_3^2 - 1}{2r_1 r_2 r_3},$$

where $r_k = \cosh(m_k/2)$.

For $m_3 = 0$ we have $r_3 = 1$, and the right hand side of the inequality in Proposition 3 is

$$\frac{r_1^2 + r_2^2}{2r_1 r_2} \geqslant 1,$$

so the condition on α is always satisfied; i.e., for any $\alpha \in [0, 2\pi]$ there exists an $[m_1, m_2, m_3]$-triangle with angular invariant α.

2.6. **Heisenberg group.** The boundary of the complex hyperbolic space can be identified with the *Heisenberg space*

$$\mathcal{N} = \mathbb{C} \times \mathbb{R} \cup \{\infty\} = \{(\zeta, \nu) \mid \zeta \in \mathbb{C}, \nu \in \mathbb{R}\} \cup \{\infty\}.$$

One homeomorphism taking $\partial H^2_{\mathbb{C}}$ to $\mathcal{N}$ is given by the stereographic projection:

$$[z_1 : z_2 : z_3] \mapsto \left(\frac{z_1}{z_2 + z_3}, \operatorname{Im}\left(\frac{z_2 - z_3}{z_2 + z_3} \right) \right) \quad \text{if } z_2 + z_3 \neq 0, \quad [0 : z : -z] \mapsto \infty.$$

The *Heisenberg group* is the Heisenberg space $\mathcal{N}$ with the group law

$$(\xi_1, \nu_1) * (\xi_2, \nu_2) = (\xi_1 + \xi_2, \nu_1 + \nu_2 + 2\operatorname{Im}(\xi_1 \bar{\xi}_2)).$$

The centre of $\mathcal{N}$ consists of elements of the form $(0, \nu)$ for $\nu \in \mathbb{R}$. The Heisenberg group is not abelian but is 2-step nilpotent. To see this, observe that

$$[(\xi_1, \nu_1), (\xi_2, \nu_2)] = (\xi_1, \nu_1)^{-1} * (\xi_2, \nu_2)^{-1} * (\xi_1, \nu_1) * (\xi_2, \nu_2) = (0, 4\operatorname{Im}(\xi_1 \bar{\xi}_2)).$$

Therefore the commutator of any two elements of $\mathcal{N}$ lies in the centre.

An alternative description of the Heisenberg group $\mathcal{N}$ is as the group of upper triangular matrices

$$\left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\}$$

with the operation of matrix multiplication. For any integer $k \neq 0$, the subgroup N_k generated by the matrices

$$a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad c = \begin{pmatrix} 1 & 0 & \frac{1}{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is a uniform lattice in $\mathcal{N}$ with the presentation

$$N_k = \langle a, b, c \mid [b, a] = c^k, \ [c, a] = [c, b] = 1 \rangle.$$

Moreover, any uniform lattice in $\mathcal{N}$ is isomorphic to N_k for some integer $k \neq 0$; see section 6.1 in [De].

2.7. Chains.

A complex geodesic in $H_{\mathbb{C}}^2$ is homeomorphic to a disc, and its intersection with the boundary of the complex hyperbolic plane is homeomorphic to a circle. Circles that arise as the boundaries of complex geodesics are called *chains*.

There is a bijection between chains and complex geodesics. We can therefore, without loss of generality, talk about reflections in chains instead of reflections in complex geodesics.

Chains can be represented in the Heisenberg space; for more details see [Go]. Chains passing through ∞ are represented by vertical straight lines defined by $\zeta = \zeta_0$. Such chains are called *vertical*. The vertical chain C_{ζ_0} defined by $\zeta = \zeta_0$ has a polar vector

$$
c_{\zeta_0} = \begin{bmatrix} 1 \\ -\bar{\zeta}_0 \\ \bar{\zeta}_0 \end{bmatrix}.
$$

A chain not containing ∞ is called *finite*. A finite chain is represented by an ellipse whose vertical projection $\mathbb{C} \times \mathbb{R} \to \mathbb{C}$ is a circle in $\mathbb{C}$. The finite chain with centre $(\zeta_0, \nu_0) \in \mathcal{N}$ and radius $r_0 > 0$ has a polar vector

$$
\begin{bmatrix} 2\zeta_0 \\ 1 + r_0^2 - \zeta_0\bar{\zeta}_0 + i\nu_0 \\ 1 - r_0^2 + \zeta_0\bar{\zeta}_0 - i\nu_0 \end{bmatrix}
$$

and consists of all points $(\zeta, \nu) \in \mathcal{N}$ satisfying the equations

$$
|\zeta - \zeta_0| = r_0, \qquad \nu = \nu_0 - 2\operatorname{Im}(\zeta\bar{\zeta}_0).
$$

2.8. Heisenberg isometries.

We consider the space $\mathcal{N}$ equipped with the *Cygan metric*,

$$
\rho_0\left((\zeta_1, \nu_2), (\zeta_2, \nu_2)\right) = \left| |\zeta_1 - \zeta_2|^2 - i(\nu_1 - \nu_2) - 2i\operatorname{Im}(\zeta_1\bar{\zeta}_2) \right|^{1/2}.
$$

A *Heisenberg translation* $T_{(\xi,\nu)}$ by $(\xi, \nu) \in \mathcal{N}$ is given by

$$
(\zeta, \omega) \mapsto (\zeta + \xi, \omega + \nu + 2\operatorname{Im}(\xi\bar{\zeta})) = (\xi, \nu) * (\zeta, \omega)
$$

and corresponds to the following element in $\mathrm{PU}(2,1)$:

$$
\begin{pmatrix} 1 & \xi & \xi \\ -\bar{\xi} & 1 - \frac{|\xi|^2 - i\nu}{2} & -\frac{|\xi|^2 - i\nu}{2} \\ \bar{\xi} & \frac{|\xi|^2 - i\nu}{2} & 1 + \frac{|\xi|^2 - i\nu}{2} \end{pmatrix}.
$$

A special case is a vertical Heisenberg translation $T_{(0,\nu)}$ by $(0, \nu) \in \mathcal{N}$ given by

$$
(\zeta, \omega) \mapsto (\zeta, \omega + \nu).
$$

A *Heisenberg rotation* R_μ by $\mu \in \mathbb{C}$, $|\mu| = 1$ is given by

$$
(\zeta, \omega) \mapsto (\mu \cdot \zeta, \omega)
$$

and corresponds to the following element in $\mathrm{PU}(2, 1)$:

$$\begin{pmatrix} \mu & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

A minimal complex reflection ι_{C_φ} of order n in a vertical chain C_φ with polar vector

$$c_\varphi = \begin{bmatrix} 1 \\ -\bar{\varphi} \\ \bar{\varphi} \end{bmatrix}$$

is given by

$$(\zeta, \omega) \mapsto (\mu\zeta + (1 - \mu)\varphi, \omega - 2|\varphi|^2 \operatorname{Im}(1 - \mu) + 2\operatorname{Im}((1 - \mu)\bar{\varphi}\zeta))$$

and corresponds to the following element in $\mathrm{PU}(2, 1)$:

$$\begin{pmatrix} -\mu & -(1 - \mu)\varphi & -(1 - \mu)\varphi \\ -(1 - \mu)\bar{\varphi} & (1 - \mu)|\varphi|^2 - 1 & (1 - \mu)|\varphi|^2 \\ (1 - \mu)\bar{\varphi} & -(1 - \mu)|\varphi|^2 & -(1 - \mu)|\varphi|^2 - 1 \end{pmatrix},$$

where $\mu = \exp(2\pi i/n)$. The complex reflection ι_{C_φ} can be decomposed as a product of a Heisenberg translation and a Heisenberg rotation:

$$\iota_{C_\varphi} = R_\mu \circ T_{(\xi,\nu)} = T_{(\mu\xi,\nu)} \circ R_\mu,$$

where

$$\xi = (\bar{\mu} - 1)\varphi \quad \text{and} \quad \nu = -2|\varphi|^2 \cdot \operatorname{Im}(1 - \mu) = 2|\varphi|^2 \sin(2\pi/n).$$

Heisenberg translations, Heisenberg rotations, and complex reflections are isometries with respect to the Cygan metric. The group of all Heisenberg translations is isomorphic to $\mathcal{N}$. The group of all Heisenberg rotations $\{R_\mu \mid \mu \in \mathbb{C}, |\mu| = 1\}$ is isomorphic to $\mathrm{U}(1)$. The group of their products $\mathcal{N} \rtimes \mathrm{U}(1)$ contains all complex reflections.

2.9. **Products of reflections in chains.** What effect does the minimal complex reflection of order n in the vertical chain C_ζ have on another vertical chain, C_ξ, which intersects $\mathbb{C} \times \{0\}$ at ξ?

We calculate

$$\begin{pmatrix} -\mu & -(1 - \mu)\zeta & -(1 - \mu)\zeta \\ -(1 - \mu)\bar{\zeta} & (1 - \mu)|\zeta|^2 - 1 & (1 - \mu)|\zeta|^2 \\ (1 - \mu)\bar{\zeta} & -(1 - \mu)|\zeta|^2 & -(1 - \mu)|\zeta|^2 - 1 \end{pmatrix} \begin{bmatrix} 1 \\ -\bar{\xi} \\ \bar{\xi} \end{bmatrix} = \begin{bmatrix} -\mu \\ -(1 - \mu)\bar{\zeta} + \bar{\xi} \\ (1 - \mu)\bar{\zeta} - \bar{\xi} \end{bmatrix}.$$

This vector is a multiple of

$$\begin{bmatrix} 1 \\ (1 - \mu)\bar{\mu}\bar{\zeta} - \bar{\mu}\bar{\xi} \\ -(1 - \mu)\bar{\mu}\bar{\zeta} + \bar{\mu}\bar{\xi} \end{bmatrix} = \begin{bmatrix} 1 \\ -\overline{(\mu\xi - (\mu - 1)\zeta)} \\ \overline{(\mu\xi - (\mu - 1)\zeta)} \end{bmatrix},$$

which is the polar vector of the vertical chain that intersects $\mathbb{C} \times \{0\}$ at $\mu\xi - (\mu - 1)\zeta$.
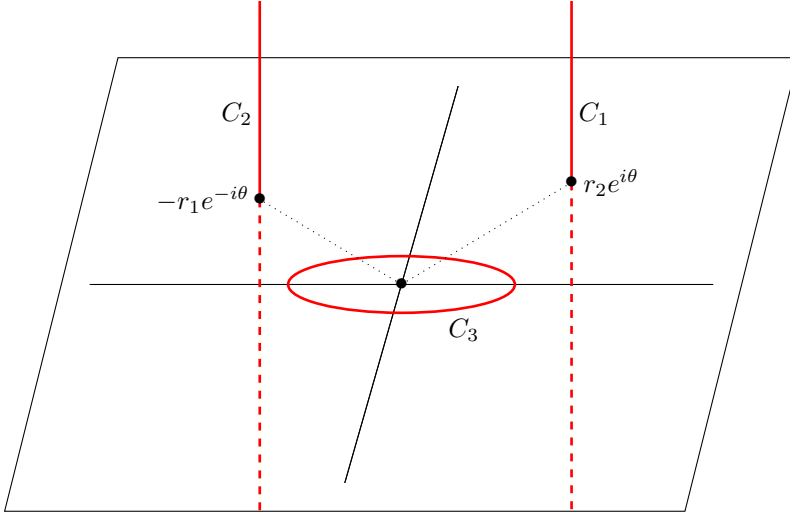
FIGURE 2. Chains C_1, C_2, and C_3 (figure from [MPP]).

This corresponds to rotating ξ around ζ through $\frac{2\pi}{n}$. So if we have a vertical chain C_ξ, the minimal complex reflection of order n in another vertical chain C_ζ rotates C_ξ as a set around C_ζ through $\frac{2\pi}{n}$ (but not pointwise).

2.10. **Bisectors and spinal spheres.** Unlike in the real hyperbolic space, there are no totally geodesic real hypersurfaces in $H_{\mathbb{C}}^2$. Acceptable substitutes are the metric bisectors. Let $z_1, z_2 \in H_{\mathbb{C}}^2$ be two distinct points. The *bisector equidistant* from z_1 and z_2 is defined as

$$\{z \in H_{\mathbb{C}}^2 \mid \rho(z_1, z) = \rho(z_2, z)\}.$$

The intersection of a bisector with the boundary of $H_{\mathbb{C}}^2$ is a smooth hypersurface in $\partial H_{\mathbb{C}}^2$ called a *spinal sphere*, which is diffeomorphic to a sphere. An example is the bisector

$$\mathcal{C} = \{[z : it : 1] \in H_{\mathbb{C}}^2 \mid |z|^2 < 1 - t^2, \ z \in \mathbb{C}, \ t \in \mathbb{R}\}.$$

Its boundary, the *unit spinal sphere*, can be described as

$$U = \{(\zeta, \nu) \in \mathcal{N} \mid |\zeta|^4 + \nu^2 = 1\}.$$

3. PARAMETRISATION OF COMPLEX HYPERBOLIC TRIANGLE GROUPS
OF TYPE $[m_1, m_2, 0; n_1, n_2, n_3]$

For $r_1, r_2 \geqslant 1$ and $\alpha \in (0, 2\pi)$, let C_1, C_2, and C_3 be the complex geodesics with respective polar vectors

$$c_1 = \begin{bmatrix} 1 \\ -r_2 e^{-i\theta} \\ r_2 e^{-i\theta} \end{bmatrix}, \quad c_2 = \begin{bmatrix} 1 \\ r_1 e^{i\theta} \\ -r_1 e^{i\theta} \end{bmatrix}, \quad \text{and} \quad c_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

where $\theta = (\pi - \alpha)/2 \in (-\pi/2, \pi/2)$. The type of triangle formed by C_1, C_2, C_3 is an ultra-parallel $[m_1, m_2, 0]$-triangle with angular invariant α, where $r_k = \cosh(m_k/2)$ for $k = 1, 2$.

Let ι_k be the minimal complex reflection of order n_k in the chain C_k for $k = 1, 2, 3$. The group $\langle \iota_1, \iota_2, \iota_3 \rangle$ generated by these three complex reflections is an ultra-parallel complex hyperbolic triangle group of type $[m_1, m_2, 0; n_1, n_2, n_3]$. Looking at the arrangement of the chains C_1, C_2, and C_3 in the Heisenberg space $\mathcal{N}$, the finite chain C_3 is the (Euclidean) unit circle in $\mathbb{C} \times \{0\}$, whereas C_1 and C_2 are vertical lines through the points $\varphi_1 = r_2 e^{i\theta}$ and $\varphi_2 = -r_1 e^{-i\theta}$, respectively; see Figure 2. For $k = 1, 2$, the reflection ι_k rotates any vertical chain as a set through $\frac{2\pi}{n_k}$ around C_k.

4. COMPRESSION PROPERTY

Let C_1, C_2, C_3 be chains in $\mathcal{N}$ as in the previous section. Let ι_k be the minimal complex reflection of order n_k in the chain C_k for $k = 1, 2, 3$. We will assume that $n_3 = 2$. To prove the discreteness of the group $\langle \iota_1, \iota_2, \iota_3 \rangle$ we will use the following version of Klein's combination theorem discussed in [WG]:

Proposition 4. *If there exist subsets U_1, U_2, and V in $\mathcal{N}$ with $U_1 \cap U_2 = \varnothing$ and $V \subsetneq U_1$ such that $\iota_3(U_1) = U_2$ and $g(U_2) \subsetneq V$ for all $g \neq \mathrm{Id}$ in $\langle \iota_1, \iota_2 \rangle$, then the group $\langle \iota_1, \iota_2, \iota_3 \rangle$ is a discrete subgroup of $\mathrm{PU}(2, 1)$. Groups with such properties are called compressing.*

Projecting the actions of complex reflections ι_1 and ι_2 to $\mathbb{C} \times \{0\}$ we obtain rotations j_1 and j_2 of $\mathbb{C}$ around $\varphi_1 = r_2 e^{i\theta}$ and $\varphi_2 = -r_1 e^{-i\theta}$ through $\frac{2\pi}{n_1}$ and $\frac{2\pi}{n_2}$, respectively. We will use Proposition 4 to prove the following lemma:

Lemma 1. *If $|f(0)| \geqslant 2$ for all $f \neq \mathrm{Id}$ in $\langle j_1, j_2 \rangle$ and $|h(0)| \geqslant 2$ for all vertical Heisenberg translations $h \neq \mathrm{Id}$ in $\langle \iota_1, \iota_2 \rangle$, then the group $\langle \iota_1, \iota_2, \iota_3 \rangle$ is discrete.*

Proof. Consider the unit spinal sphere

$$U = \{(\zeta, \nu) \in \mathcal{N} \mid |\zeta|^4 + \nu^2 = 1\}.$$

The complex reflection ι_3 in C_3 is given by

$$\iota_3([z_1 : z_2 : z_3]) = [-z_1 : z_2 : -z_3] = [z_1 : -z_2 : z_3].$$

The complex reflection ι_3 preserves the bisector

$$\mathcal{C} = \{[z : it : 1] \in H_{\mathbb{C}}^2 \mid |z|^2 < 1 - t^2, z \in \mathbb{C}, t \in \mathbb{R}\}$$

and hence preserves the unit spinal sphere U which is the boundary of the bisector $\mathcal{C}$. The complex reflection ι_3 interchanges the points $[0 : 1 : 1]$ and $[0 : -1 : 1]$ in $H_{\mathbb{C}}^2$, which correspond to the points $(0, 0)$ and ∞ in $\mathcal{N}$. Therefore, ι_3 leaves U invariant and switches the inside of U with the outside.

Let U_1 be the part of $\mathcal{N} \backslash U$ outside U, containing ∞, and let U_2 be the part inside U, containing the origin. Clearly

$$U_1 \cap U_2 = \varnothing \quad \text{and} \quad \iota_3(U_1) = U_2.$$

Therefore, if we find a subset $V \subsetneq U_1$ such that $g(U_2) \subsetneq V$ for all elements $g \neq \mathrm{Id}$ in $\langle \iota_1, \iota_2 \rangle$, then we will show that $\langle \iota_1, \iota_2, \iota_3 \rangle$ is discrete. Let

$$W = \{(\zeta, \nu) \in \mathcal{N} \mid |\zeta| = 1\}$$

be the cylinder consisting of all vertical chains through $\zeta \in \mathbb{C}$ with $|\zeta| = 1$. Let

$$W_1 = \{(\zeta, \nu) \in \mathcal{N} \mid |\zeta| > 1\} \quad \text{and} \quad W_2 = \{(\zeta, \nu) \in \mathcal{N} \mid |\zeta| < 1\}$$

be the parts of $\mathcal{N}\backslash W$ outside and inside the cylinder W, respectively. We have $U_2 \subset W_2$ and so $g(U_2) \subset g(W_2)$ for all $g \in \langle \iota_1, \iota_2 \rangle$. The set W_2 is a union of vertical chains. We know that elements of $\langle \iota_1, \iota_2 \rangle$ map vertical chains to vertical chains. There is also a vertical translation on the chain itself. Therefore, we look at both the intersection of the images of W_2 with $\mathbb{C} \times \{0\}$ and the vertical displacement of W_2.

Elements of $\langle \iota_1, \iota_2 \rangle$ move the intersection of W_2 with $\mathbb{C} \times \{0\}$ by rotations j_1 and j_2 around $r_2 e^{i\theta}$ and $-r_1 e^{-i\theta}$ through $\frac{2\pi}{n_1}$ and $\frac{2\pi}{n_2}$, respectively. Provided that the interior of the unit circle is mapped completely off itself under all non-identity elements in $\langle j_1, j_2 \rangle$, then the same is true for W_2 and hence for U_2 under all elements in $\langle \iota_1, \iota_2 \rangle$ that are not vertical Heisenberg translations.

A vertical Heisenberg translation will shift W_2 and its images $g(W_2)$ vertically by the same distance; hence the same is true for U_2 and its images $g(U_2)$.

We choose V to be the union of all the images of U_2 under all non-vertical elements of $\langle \iota_1, \iota_2 \rangle$. This subset will satisfy the compressing conditions assuming that the interior of the unit circle is mapped off itself by any non-identity element in $\langle j_1, j_2 \rangle$ and that the interior of the unit spinal sphere U is mapped off itself by any non-identity vertical Heisenberg translation in $\langle \iota_1, \iota_2 \rangle$. Since the radius of the unit circle is preserved under rotations, we need to show that the origin is moved the distance of at least twice the radius of the circle:

$$|f(0)| \geqslant 2 \quad \text{for all } f \in \langle j_1, j_2 \rangle, \ f \neq \text{Id}.$$

Since vertical translations shift the spinal spheres vertically, we need to show that they shift by at least the height of the spinal sphere:

$$|h(0)| \geqslant 2 \quad \text{for all vertical Heisenberg translations } h \in \langle \iota_1, \iota_2 \rangle, \ h \neq \text{Id}.$$

We see that the conditions of this lemma ensure that the sets U_1, U_2, and V satisfy the conditions of Proposition 4.   □

5. PARAMETRISATION OF COMPLEX HYPERBOLIC TRIANGLE GROUPS OF TYPE $[m, m, 0; 3, 3, 2]$

We will now focus on the case of $[m_1, m_2, 0; n_1, n_2, n_3]$-groups with

$$m_1 = m_2 = m, n_1 = n_2 = 3, \quad \text{and} \quad n_3 = 2.$$

In this case the setting described in section 3 is as follows. We consider the following configuration of chains in $\mathcal{N}$: C_3 is the (Euclidean) unit circle in $\mathbb{C} \times \{0\}$, whereas C_1 and C_2 are vertical lines through the points $\varphi_1 = re^{i\theta}$ and $\varphi_2 = -re^{-i\theta}$, respectively, where $r = \cosh(m/2)$ and $\theta \in (-\pi/2, \pi/2)$. The type of triangle formed by C_1, C_2, C_3 is an ultra-parallel $[m, m, 0]$-triangle with angular invariant $\alpha = \pi - 2\theta \in (0, 2\pi)$. We will consider the ultra-parallel triangle group $\Gamma = \langle \iota_1, \iota_2, \iota_3 \rangle$ generated by the minimal complex reflections $\iota_1, \iota_2, \iota_3$ of orders $3, 3, 2$ in the chains C_1, C_2, C_3, respectively.

The description of complex reflections in section 2.8 in this case is as follows: The reflection ι_k for $k = 1, 2$ is given by

$$(\zeta, \omega) \mapsto (\mu\zeta + (1 - \mu)\varphi_k, \omega + 2|\varphi_k|^2 \operatorname{Im}(1 - \mu) + 2\operatorname{Im}((1 - \mu)\bar{\varphi}_k \zeta)),$$

where $\mu = \exp(2\pi i/3)$, and can be decomposed into a product of a Heisenberg translation and a Heisenberg rotation:

$$\iota_k = R_\mu \circ T_{(\xi_k, \nu_k)} = T_{(\mu \xi_k, \nu_k)} \circ R_\mu,$$

where

$$\xi_k = (\bar{\mu} - 1)\varphi_k \quad \text{and} \quad \nu_k = -2|\varphi_k|^2 \cdot \text{Im}(1 - \mu) = 2|\varphi_k|^2 \sin(2\pi/3).$$

For $k = 1, 2$, the reflection ι_k rotates any vertical chain as a set through $\frac{2\pi}{3}$ around C_k.

6. Subgroup of Heisenberg translations

Let $\Gamma = \langle \iota_1, \iota_2, \iota_3 \rangle$ be as in section 5. In this section we will consider the structure of the subgroup $E = \langle \iota_1, \iota_2 \rangle$ in more detail.

Proposition 5. *Let $\mathcal{T}$ be the subgroup of all Heisenberg translations in E. Every element of E can be written as a product of a Heisenberg translation and a power of ι_1. The group $\mathcal{T}$ is generated by the elements*

$$T_1 = \iota_2 \iota_1 \iota_2 \quad \text{and} \quad T_2 = \iota_1 \iota_1 \iota_2.$$

Let $H = [T_1, T_2] = (\iota_1 \iota_2)^3$. Every element of $\mathcal{T}$ is of the form $T_1^x T_2^y H^n$ for some $x, y, n \in \mathbb{Z}$. The elements T_1, T_2, H are Heisenberg translations by

$$(v_1, t_1) = (2r\sqrt{3} \cos(\theta) \cdot i, 12\sqrt{3} r^2 \cos^2(\theta)),$$
$$(v_2, t_2) = (r \cos(\theta) \cdot (3 + i\sqrt{3}), 12r^2 \sin(\theta) \cos(\theta)),$$
$$(0, \nu) = (0, 24r^2 \sqrt{3} \cos^2(\theta)),$$

respectively. The subgroup of vertical Heisenberg translations in E is an infinite cyclic group generated by H. The shortest non-trivial vertical translations in E are H and H^{-1}.

Proof. We can write every element in E as a word in the generators $\iota_1^{\pm 1}$ and $\iota_2^{\pm 1}$. Using the relations $\iota_1^{-1} = \iota_1^2$ and $\iota_2^{-1} = \iota_2^2$ we can rewrite it as a word in just ι_1 and ι_2. Consider the words $\iota_{k_1 k_2 k_3} = \iota_{k_1} \iota_{k_2} \iota_{k_3}$ of length 3. Using the decomposition $\iota_k = R_\mu \circ T_{(\xi_k, \nu_k)} = T_{(\mu \xi_k, \nu_k)} \circ R_\mu$ (section 5), we can write

$$\begin{aligned}
\iota_{k_1 k_2 k_3} &= (R_\mu \circ T_{(\xi_{k_1}, \nu_{k_1})}) \circ (R_\mu \circ T_{(\xi_{k_2}, \nu_{k_2})}) \circ (R_\mu \circ T_{(\xi_{k_3}, \nu_{k_3})}) \\
&= (R_\mu)^3 \circ T_{(\mu \xi_{k_1}, \nu_{k_1})} \circ T_{(\mu^2 \xi_{k_2}, \nu_{k_2})} \circ T_{(\xi_{k_3}, \nu_{k_3})} \\
&= T_{(\mu \xi_{k_1}, \nu_{k_1})} \circ T_{(\mu^2 \xi_{k_2}, \nu_{k_2})} \circ T_{(\xi_{k_3}, \nu_{k_3})};
\end{aligned}$$

hence $\iota_{k_1 k_2 k_3}$ is a Heisenberg translation. Let $f \in E = \langle \iota_1, \iota_2 \rangle$. We can write f as a product of some words of length 3 and one word of length at most 2. Moreover, using the relations $\iota_2 = \iota_{211} \cdot \iota_1$, $\iota_2^2 = \iota_{221} \cdot \iota_1^2$, $\iota_1 \iota_2 = \iota_{121} \cdot \iota_1^2$, and $\iota_2 \iota_1 = \iota_{211} \cdot \iota_1^2$, we can rewrite f as a product of some words of length 3 and a power of ι_1. Using the relations $\iota_1^3 = \iota_2^3 = \text{Id}$ we see that all words $\iota_{k_1 k_2 k_3}$ of length 3 can be expressed in terms of $T_1 = \iota_{212}$ and $T_2 = \iota_{112}$ as $\iota_{221} = T_2^{-1}$, $\iota_{122} = T_2 T_1^{-1}$, $\iota_{211} = T_1 T_2^{-1}$, and $\iota_{121} = T_2 T_1^{-1} T_2^{-1}$. Hence f can be written as a product of an element in $\langle T_1, T_2 \rangle$ and an element $w \in \langle \iota_1 \rangle$, and f is a Heisenberg translation if and only if $w = \text{Id}$. Therefore $\mathcal{T} = \langle T_1, T_2 \rangle$.

Let $H = [T_1, T_2] \in \mathcal{T}$. As a commutator of two Heisenberg translations, the element H is a vertical Heisenberg translation and lies in the centre of $\mathcal{N}$; hence $[H, T_1] = [H, T_2] = 1$. Direct computation shows that $T_2 H T_2^{-1} = (\iota_1 \iota_2)^3$. On

the other hand, $T_2H = HT_2$ implies that $T_2HT_2^{-1} = HT_2T_2^{-1} = H$; hence $H = (\iota_1\iota_2)^3$. Using the relations $HT_1 = T_1H$, $HT_2 = T_2H$, and $T_2T_1 = T_1T_2H^{-1}$, every element of $\mathcal{T}$ can be written in the form $T_1^x T_2^y H^n$ for some $x, y, n \in \mathbb{Z}$. The elements T_1 and T_2 are Heisenberg translations by (v_1, t_1) and (v_2, t_2), respectively. The commutator $H = [T_1, T_2]$ is a vertical Heisenberg translation by $\nu = 4\operatorname{Im}(v_1\bar{v}_2)$. We determine (v_k, t_k) and ν by direct computation. Projection to $\mathbb{C}$ maps H to the identity, T_k to the Euclidean translation by v_k, and $T_1^x T_2^y H^n$ to the Euclidean translation by $xv_1 + yv_2$. Hence $T_1^x T_2^y H^n$ is a vertical translation if and only if $x = y = 0$, i.e., if it is a power of H. Therefore the subgroup of vertical Heisenberg translations in E is generated by H. □

Remark. The group $\mathcal{T}$ has the presentation

$$\mathcal{T} = \langle T_1, T_2, H \mid [T_1, T_2] = H, \ [H, T_1] = [H, T_2] = 1 \rangle$$

and is isomorphic to the uniform lattice N_1 as defined in section 2.6.

Remark. An alternative approach to the understanding of the structure of the subgroup $E = \langle \iota_1, \iota_2 \rangle$ is to use the classification of almost-crystallographic groups by Dekimpe [De]. An *almost-crystallographic group* is a uniform discrete subgroup E of $G \rtimes C$, where G is a connected, simply connected nilpotent Lie group and C is a maximal compact subgroup of $\operatorname{Aut}(G)$. As a discrete subgroup of $\mathcal{N} \rtimes \operatorname{U}(1)$ (see section 2.8), the group E is an almost-crystallographic group with $G = \mathcal{N}$ and $\operatorname{U}(1) \subset C \subset \operatorname{Aut}(\mathcal{N})$. The projection of $E = \langle \iota_1, \iota_2 \rangle$ to $\mathbb{C}$ is a wallpaper group $Q = \langle j_1, j_2 \rangle$, where j_k is the rotation of $\mathbb{C}$ around φ_k through $2\pi/3$ obtained by projecting ι_k to $\mathbb{C}$. The wallpaper group $Q = \langle j_1, j_2 \rangle$ is generated by two order 3 rotations and has a presentation

$$\langle j_1, j_2 \mid j_1^3 = j_2^3 = (j_1 j_2)^3 = 1 \rangle.$$

The standard notation for this wallpaper group is **p3**; see for example [BB]. In the classification of three-dimensional almost-crystallographic groups in section 7.1 of [De], the wallpaper group **p3** appears in case 13 on page 164. In this case the group E is generated by elements a, b, c, α with relations

$$[b, a] = c^{k_1}, \ [c, a] = [c, b] = [c, \alpha] = 1, \ \alpha a = b\alpha c^{k_2}, \ \alpha b = a^{-1}b^{-1}\alpha c^{k_3}, \ \alpha^3 = c^{k_4}.$$

We consider the generators $\iota_1 = \alpha$ and $\iota_2 = \alpha a$ so that $\alpha^3 = (\alpha a)^3 = 1$. The hypothesis $\alpha^3 = 1$ implies $k_4 = 0$. The hypothesis $(\alpha a)^3 = 1$ can be rewritten as

$$1 = (\alpha a)^3 = \alpha a\alpha(a\alpha)a = \alpha a\alpha(b^{-1}\alpha)b^{-1}ac^{k_3} = \alpha a(\alpha\alpha)a^{-1}b^{-1}ac^{k_2+k_3}$$
$$= (ba^{-1}b^{-1}a)c^{2k_2+k_3} = b([b, a])^{-1}b^{-1}c^{2k_2+k_3} = c^{-k_1+2k_2+k_3};$$

hence $-k_1 + 2k_2 + k_3 = 0$. The translations T_1 and T_2 in Proposition 5 are

$$T_1 = \iota_2\iota_1\iota_2 = (\alpha a)\alpha^2 a = bac^{k_2} \quad \text{and} \quad T_2 = \iota_1\iota_1\iota_2 = \alpha^3 a = a.$$

Their commutator is

$$H = [T_1, T_2] = (bac^{k_2})^{-1}a^{-1}(bac^{k_2})a = a^{-1}b^{-1}a^{-1}baa = a^{-1}[b, a]a = c^{k_1}.$$

On the other hand, the kernel of the map $E = \langle \iota_1, \iota_2 \rangle \to \langle j_1, j_2 \rangle$ given by $\iota_1 \mapsto j_1$, $\iota_2 \mapsto j_2$ is generated by $(\iota_1\iota_2)^3$. We calculate

$$(\iota_1\iota_2)^3 = (\alpha^2 a)^3 = \alpha^2 a\alpha^2(a\alpha^2)a = \alpha^2 a(\alpha b)ac^{k_2} = \alpha^2 b^{-1}(\alpha a)c^{k_2+k_3} = c^{2k_2+k_3}.$$

Using $-k_1 + 2k_2 + k_3 = 0$ we can rewrite this as $(\iota_1\iota_2)^3 = c^{k_1}$. Hence the element $H = [T_1, T_2] = (\iota_1\iota_2)^3 = c^{k_1}$ is the shortest vertical Heisenberg translation in $E = \langle \iota_1, \iota_2 \rangle$.

7. Proof of Proposition 1

Let $\Gamma = \langle \iota_1, \iota_2, \iota_3 \rangle$ be as in section 5. In this section we will use Lemma 1 to find conditions for the group Γ to be discrete.

Proof. We need to check that the conditions of Lemma 1 are satisfied. Note that $m \geqslant \log_e(3)$ implies that

$$r = \cosh\left(\frac{m}{2}\right) \geqslant \frac{2}{\sqrt{3}}.$$

We first check that $|h(0)| \geqslant 2$ for all vertical Heisenberg translations $h \neq \mathrm{Id}$ in $\langle \iota_1, \iota_2 \rangle$. Any vertical translation in $\langle \iota_1, \iota_2 \rangle$ is a power of the vertical translation H by $(0, 24r^2\sqrt{3}\cos^2(\theta))$. We need the displacement of each vertical translation H^n, $n \neq 0$, to be at least the height of the spinal sphere, i.e.,

$$24r^2\sqrt{3}\cos^2(\theta) \geqslant 2 \iff r^2\cos^2(\theta) \geqslant \frac{\sqrt{3}}{36}.$$

The hypothesis $\cos(\alpha) \leqslant -\frac{1}{2}$ for $\alpha \in (0, 2\pi)$ implies that $\frac{2\pi}{3} \leqslant \alpha \leqslant \frac{4\pi}{3}$ and hence $|\theta| = \left|\frac{\pi - \alpha}{2}\right| \leqslant \frac{\pi}{6}$. For $\cos(\theta) \geqslant \frac{\sqrt{3}}{2}$ and $r \geqslant \frac{2}{\sqrt{3}}$ we have

$$r^2\cos^2(\theta) \geqslant 1 > \frac{\sqrt{3}}{36};$$

hence the condition $|h(0)| \geqslant 2$ is satisfied for all vertical translations $h \neq \mathrm{Id}$ in $\langle \iota_1, \iota_2 \rangle$.

We will now check that $|f(0)| \geqslant 2$ for all $f \neq \mathrm{Id}$ in $\langle j_1, j_2 \rangle$. We can write every element f in $\langle j_1, j_2 \rangle$ as a word in the generators j_1 and j_2. Figure 3 shows the points $f(0)$ for all words f of length up to 6 in the case $r = 1$ and $\theta = 0$:
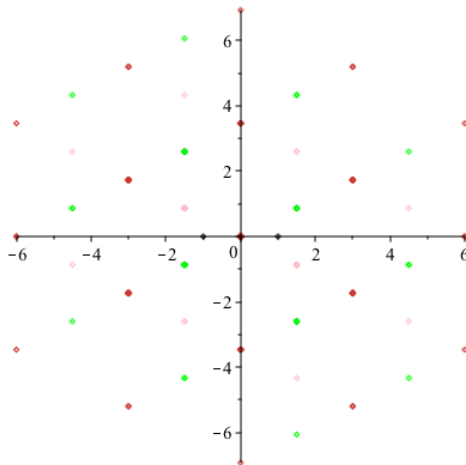

FIGURE 3. Points $f(0)$ for all words f up to length 6.

The group $\langle j_1, j_2 \rangle$ is the projection to $\mathbb{C}$ of the group $E = \langle \iota_1, \iota_2 \rangle$. For $k = 1, 2$, projecting ι_k to $\mathbb{C}$, we obtain a rotation j_k of $\mathbb{C}$ through $\frac{2\pi}{3}$ around φ_k. These rotations are given by $j_k(z) = \mu \cdot z + (1 - \mu) \cdot \varphi_k$, where $\mu = \exp(2\pi i / 3)$. According to Proposition 5, every element of E is of the form $T_1^x T_2^y H^n \iota_1^\ell$ for some $x, y, n \in \mathbb{Z}$ and $\ell \in \{0, 1, 2\}$, where $T_1 = \iota_2 \iota_1 \iota_2$, $T_2 = \iota_1 \iota_1 \iota_2$, and $H = [T_1, T_2]$ are Heisenberg translations by (v_1, t_1), (v_2, t_2), and $(0, \nu)$, respectively, and

$$v_1 = 2r\sqrt{3}\cos(\theta) \cdot i, \quad v_2 = r\cos(\theta) \cdot (3 + i\sqrt{3}).$$

Projection to $\mathbb{C}$ maps H to the identity, T_k to the Euclidean translation by v_k, $T_1^x T_2^y H^n$ to the Euclidean translation by $xv_1 + yv_2$, and ι_1 to the rotation j_1. Therefore every element of $\langle j_1, j_2 \rangle$ is a product of a translation by $xv_1 + yv_2$ for some $x, y \in \mathbb{Z}$ and a rotation j_1^ℓ for some $\ell \in \{0, 1, 2\}$. Hence every point in the orbit of 0 under $\langle j_1, j_2 \rangle$ is of the form $p + xv_1 + yv_2$, where $x, y \in \mathbb{Z}$ and

$$p \in \{0, j_1(0), j_1^2(0)\} = \{0, (1 - \mu)\varphi_1, (1 - \bar{\mu})\varphi_1\}.$$

Using $|v_1|^2 = |v_2|^2 = 2\operatorname{Re}(v_1 \bar{v}_2) = 12r^2 \cos^2(\theta)$, we calculate

$$
\begin{aligned}
&|p + xv_1 + yv_2|^2 \\
&= x^2 |v_1|^2 + y^2 |v_2|^2 + 2xy \operatorname{Re}(v_1 \bar{v}_2) + 2x \operatorname{Re}(p\bar{v}_1) + 2y \operatorname{Re}(p\bar{v}_2) + |p|^2 \\
&= 12r^2 \cos^2(\theta) \cdot (x^2 + xy + y^2) + 2x \operatorname{Re}(p\bar{v}_1) + 2y \operatorname{Re}(p\bar{v}_2) + |p|^2.
\end{aligned}
$$

We make a coordinate change $u = y - x$ and $v = x + y$; that is, $x = (v - u)/2$ and $y = (u + v)/2$. Points $(x, y) \in \mathbb{Z}^2$ are mapped to points $(u, v) \in \mathbb{Z}^2$ with $u \equiv v \bmod 2$. We obtain

$$
\begin{aligned}
|p + xv_1 + yv_2|^2 &= 3r^2 \cos^2(\theta) \cdot (u^2 + 3v^2 - 2au - 6bv + a^2 + 3b^2) \\
&= 3r^2 \cos^2(\theta) \cdot ((u - a)^2 + 3(v - b)^2),
\end{aligned}
$$

where

$$a = \frac{\operatorname{Re}(p(\bar{v}_1 - \bar{v}_2))}{6r^2 \cos^2(\theta)} = -\frac{\operatorname{Re}\left(p(3 + i\sqrt{3})\right)}{6r\cos(\theta)},$$

$$b = -\frac{\operatorname{Re}(p(\bar{v}_1 + \bar{v}_2))}{18r^2 \cos^2(\theta)} = -\frac{\operatorname{Re}\left(p(1 - i\sqrt{3})\right)}{6r\cos(\theta)}$$

and

$$a^2 + 3b^2 = \frac{|p|^2}{3r^2 \cos^2(\theta)}.$$

Our aim is to show that $|p + xv_1 + yv_2|^2 \geqslant 3r^2$ for all $(x, y) \in \mathbb{Z}^2$, excluding the case $p = 0$, $x = y = 0$ that corresponds to $f = \operatorname{Id}$. This is equivalent to $(u - a)^2 + 3(v - b)^2 \geqslant \sec^2(\theta)$ for all $(u, v) \in \mathbb{Z}^2$ with $u \equiv v \bmod 2$, excluding the case $a = b = u = v = 0$. Note that this inequality is always satisfied if $|u - a| \geqslant \sec(\theta)$ or $|v - b| \geqslant \sec(\theta)/\sqrt{3}$, so we only need to check that

$$g(u, v) = (u - a)^2 + 3(v - b)^2 - \sec^2(\theta) \geqslant 0$$

for all $(u, v) \in \mathbb{Z}^2$ with $u \equiv v \bmod 2$ inside the bounding box

$$\left(a - \sec(\theta), a + \sec(\theta)\right) \times \left(b - \frac{\sec(\theta)}{\sqrt{3}}, b + \frac{\sec(\theta)}{\sqrt{3}}\right).$$

In the following table we list the values of a, b, and $a^2 + 3b^2$ in terms of $t = \tan(\theta)$ and $\mu = \exp(2\pi i/3) = -\frac{1-i\sqrt{3}}{2}$ for $w \in \{\mathrm{Id}, j_1, j_1^2\}$:

w	$p = w(0)$	a	b	$a^2 + 3b^2$
Id	0	0	0	0
j_1	$(1 - \mu) \cdot \varphi_1$	-1	$-\frac{t}{\sqrt{3}}$	$t^2 + 1$
j_1^2	$(1 - \bar{\mu}) \cdot \varphi_1$	$\frac{1}{2}(t\sqrt{3} - 1)$	$-\frac{1}{6}(3 + t\sqrt{3})$	$t^2 + 1$

Under the assumption that $|\theta| \leqslant \frac{\pi}{6}$ we have $t = \tan(\theta) \in [-d, d]$ and $\sec(\theta) \in [1, 2d]$, where $d = 1/\sqrt{3} \approx 0.577$. In each of the three cases we list the bounds on a and b and the size of the bounding box:

$$(\min(a) - 2d, \max(a) + 2d) \times (\min(b) - 2/3, \max(b) + 2/3).$$

We then calculate

$$g(u, v) = (u - a)^2 + 3(v - b)^2 - \sec^2(\theta)$$
$$= u^2 + 3v^2 - 2au - 6bv + (a^2 + 3b^2) - (t^2 + 1)$$

and check that $g(u, v) \geqslant 0$ for all $(u, v) \in \mathbb{Z}^2$ with $u = v \bmod 2$ inside the bounding box.

- $w = \mathrm{Id}$, $a = b = 0$: The bounding box

$$(-2d, 2d) \times (-2/3, 2/3) \subset (-2, 2) \times (-1, 1)$$

contains only one point $(u, v) \in \mathbb{Z}^2$ with $u = v \bmod 2$, the point $(u, v) = (0, 0)$, which corresponds to the excluded case $f = \mathrm{Id}$.
- $w = j_1$, $a = -1$, $b = -t/\sqrt{3} \in [-1/3, 1/3]$: The bounding box

$$(-1 - 2d, -1 + 2d) \times (-1, 1) \subset (-3, 1) \times (-1, 1)$$

contains points $(0, 0)$ and $(-2, 0)$. The function

$$g(u, v) = u^2 + 3v^2 + 2u + 2tv\sqrt{3} + (t^2 + 1) - (t^2 + 1)$$
$$= u^2 + 3v^2 + 2u + 2tv\sqrt{3}$$

is non-negative: $g(0, 0) = g(-2, 0) = 0$.
- $w = j_1^2$, $a = \frac{1}{2}(t\sqrt{3} - 1) \in [-1, 0]$, $b = -\frac{1}{6}(3 + t\sqrt{3}) \in [-2/3, -1/3]$: The bounding box

$$(-1 - 2d, 2d) \times (-4/3, 1/3) \subset (-3, 2) \times (-2, 1)$$

contains points $(1, -1)$, $(0, 0)$, $(-1, -1)$, and $(-2, 0)$. The function

$$g(u, v) = u^2 + 3v^2 - u(t\sqrt{3} - 1) + v(3 + t\sqrt{3}) + (t^2 + 1) - (t^2 + 1)$$
$$= u^2 + 3v^2 + u + 3v - (u - v)t\sqrt{3}$$

is non-negative:

$$g(0, 0) = g(-1, -1) = 0, \ g(1, -1) = 2 - 2t\sqrt{3} \geqslant 0, \ g(-2, 0) = 2 + 2t\sqrt{3} \geqslant 0.$$

In all cases we have shown that $g(u, v) \geqslant 0$, hence $|p + xv_1 + yv_2|^2 \geqslant 3r^2$. Under the assumption that $m \geqslant \log_e(3)$ we have $3r^2 = 3 \cosh^2\left(\frac{m}{2}\right) \geqslant 4$. Therefore $|f(0)| \geqslant 2$ for all $f \neq \mathrm{Id}$ in $\langle j_1, j_2 \rangle$. Hence all conditions of Lemma 1 are satisfied, and we can conclude that the group $\langle \iota_1, \iota_2, \iota_3 \rangle$ is discrete. $\square$

8. Proof of Proposition 2

Let $\Gamma = \langle \iota_1, \iota_2, \iota_3 \rangle$ be an ultra-parallel $[m, m, 0; 3, 3, 2]$-triangle group as in section 5. In this section we will use the following complex hyperbolic version of Shimizu's lemma introduced in [P92, P94, P97] to find conditions for the group Γ not to be discrete.

Lemma 2. *Let Γ be a discrete subgroup of* $\mathrm{PU}(2, 1)$. *Let $g \in \Gamma$ be a Heisenberg translation by (ξ, ν) and let $h = (h_{ij})_{1 \leqslant i, j \leqslant 3} \in \Gamma$ be an element that satisfies $h(\infty) \neq \infty$. Then*

$$r_h^2 \leqslant \rho_0(g(h^{-1}(\infty)), h^{-1}(\infty))\rho_0(g(h(\infty)), h(\infty)) + 4\,|\xi|^2\,,$$

where ρ_0 is the Cygan metric on $\mathcal{N}$ and

$$r_h = \sqrt{\frac{2}{|h_{22} - h_{23} + h_{32} - h_{33}|}}$$

is the radius of the isometric sphere of h.

We will now prove Proposition 2:

Proof. We will apply Lemma 2 to the vertical Heisenberg translation $g = (\iota_1 \iota_2)^3$ and the element $h = \iota_3$ in $\Gamma = \langle \iota_1, \iota_2, \iota_3 \rangle$. The matrix of the element $h = \iota_3 = \iota_3^{-1}$ is

$$h = h^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The radius of the isometric sphere of h is $r_h = 1$. To calculate $h(\infty)$ we first map ∞ from the Heisenberg space to the boundary of complex hyperbolic 2-space. That is,

$$\infty \mapsto [0 : 1 : -1] \in \partial H_{\mathbb{C}}^2.$$

We apply h to this point:

$$h([0 : 1 : -1]) = [0 : 1 : 1] \in \partial H_{\mathbb{C}}^2.$$

Note that $h(\infty) \neq \infty$. Mapping this point back to the Heisenberg space,

$$[0 : 1 : 1] \mapsto (0, 0) \in \mathcal{N}.$$

For a vertical Heisenberg translation g, we have $\xi = 0$ and $\rho_0(g(\zeta, \omega), (\zeta, \omega)) = |\nu|^{\frac{1}{2}}$ for all $(\zeta, \omega) \in \mathcal{N}$. Substituting these values into the inequality given in Lemma 2, we obtain that if $|\nu| < 1$, then the group is not discrete. From Proposition 5 we know that $g = (\iota_1 \iota_2)^3$ is a vertical Heisenberg translation by $(0, \nu)$ with $\nu = 24\sqrt{3}r^2 \cos^2(\theta)$. Hence the group Γ is not discrete if

$$\cos^2(\theta) < \frac{1}{24\sqrt{3}r^2}.$$

Using $\cos(\alpha) = 1 - 2\cos^2(\theta)$, we conclude that the group Γ is not discrete provided that

$$\cos(\alpha) > 1 - \frac{1}{12\sqrt{3}r^2} = 1 - \frac{1}{12\sqrt{3}\cosh^2\left(\frac{m}{2}\right)}. \qquad \square$$

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References

[BB] Harold Brown, Rolf Bülow, Joachim Neubüser, Hans Wondratschek, and Hans Zassenhaus, *Crystallographic groups of four-dimensional space*, Wiley Monographs in Crystallography, Wiley-Interscience [John Wiley & Sons], New York-Chichester-Brisbane, 1978. MR0484179

[De] Karel Dekimpe, *Almost-Bieberbach groups: affine and polynomial structures*, Lecture Notes in Mathematics, vol. 1639, Springer-Verlag, Berlin, 1996. MR1482520

[Go] William M. Goldman, *Complex hyperbolic geometry*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1999. MR1695450

[Mo] Andrew Monaghan, *Complex hyperbolic triangle groups*, ProQuest LLC, Ann Arbor, MI. Thesis (Ph.D.)–The University of Liverpool (United Kingdom), 2013. MR3389497

[MPP] Andrew Monaghan, John R. Parker, and Anna Pratoussevitch, *Discreteness of ultra-parallel complex hyperbolic triangle groups of type* $[m_1, m_2, 0]$, J. Lond. Math. Soc. (2) **100** (2019), no. 2, 545–567, DOI 10.1112/jlms.12227. MR4017154

[P92] John R. Parker, *Shimizu's lemma for complex hyperbolic space*, Internat. J. Math. **3** (1992), no. 2, 291–308, DOI 10.1142/S0129167X92000096. MR1146815

[P94] John R. Parker, *On Ford isometric spheres in complex hyperbolic space*, Math. Proc. Cambridge Philos. Soc. **115** (1994), no. 3, 501–512, DOI 10.1017/S0305004100072261. MR1269935

[P97] John R. Parker, *Uniform discreteness and Heisenberg translations*, Math. Z. **225** (1997), no. 3, 485–505, DOI 10.1007/PL00004315. MR1465903

[P10] J. R. Parker, *Notes on complex hyperbolic geometry*, lecture notes, 2010.

[Po] S. Povall, *Ultra-parallel complex hyperbolic triangle groups*, Ph.D. thesis, University of Liverpool, 2019.

[Pra] Anna Pratoussevitch, *Traces in complex hyperbolic triangle groups*, Geom. Dedicata **111** (2005), 159–185, DOI 10.1007/s10711-004-1493-0. MR2155180

[Sch] Richard Evan Schwartz, *Complex hyperbolic triangle groups*, Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002), Higher Ed. Press, Beijing, 2002, pp. 339–349. MR1957045

[WG] Justin Olav Wyss-Gallifent, *Complex hyperbolic triangle groups*, ProQuest LLC, Ann Arbor, MI. Thesis (Ph.D.)–University of Maryland, College Park, 2000. MR2700554

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