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LOCALLY COMPACT FLOWS ON CONNECTED MANIFOLDS

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Abstract. In this paper, we completely characterize locally compact flows G of homeomorphisms of connected manifolds M by proving that they are either circle groups or real groups. For $M = \mathbb{R}^m$, we prove that every recurrent element in G is periodic, and we obtain a generalization of the result of Yang [*Hilbert's fifth problem and related problems on transformation groups*, American Mathematical Society, Providence, RI, 1976, pp. 142–146.] by proving that there is no nontrivial locally compact flow on $\mathbb{R}^m$ in which all elements are recurrent.

1. Introduction and main results

Let $M = (M, d)$ be a metric space, and let $Homeo(M)$ be the group of all homeomorphisms of M equipped with the compact-open topology. In particular, $Homeo(M)$ is a Hausdorff group and every discrete subgroup of it is closed. We denote by id the identity map of M.

Let G be a subgroup of $Homeo(M)$. An element $g \in G$ is said to be *periodic* if there exists an integer $n \in \mathbb{N}^*$ such that $g^n = id$, and it is said to be *recurrent* if for every $\epsilon > 0$, there exists an integer $n \in \mathbb{N}^*$ such that

$$d(g^n(x), x) < \epsilon, \ \forall\ x \in M.$$

We denote by $Rec(G)$ the set of all recurrent elements of G, and by $P(G)$ the set of all periodic elements of G;

$$Rec(G) = \{g \in G \mid g \text{ is recurrent}\} \ ; \ P(G) = \{g \in G \mid g \text{ is periodic}\}.$$

The group G is said to be *torsion free* if $P(G) = \{id\}$, and it is said to be *torsion group* if $P(G) = G$. The group G is said to be *equicontinuous* if for every $x \in M$, for every $\epsilon > 0$, there exists $\eta > 0$ such that

$$\forall\ y \in M,\ d(x, y) < \eta \Longrightarrow d(g(x), g(y)) < \epsilon, \ \forall\ g \in G.$$

Let $g \in Homeo(M)$. We denote by $\langle g \rangle$ the subgroup generated by g;

$$\langle g \rangle = \{g^n \mid n \in \mathbb{Z}\}.$$

The monothetic group generated by g means the closure $\overline{\langle g \rangle}$ of $\langle g \rangle$ in $Homeo(M)$. The homeomorphism g is said to be *regular* if the subgroup $\langle g \rangle$ is equicontinuous, and it is said to be *regularly almost periodic* if for every $\epsilon > 0$, there exists an integer $n > 0$ such that

$$d(g^{mn}(x), x) < \epsilon, \ \forall\ x \in M, \ \forall\ m \in \mathbb{Z}.$$

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The *orbit* $G(x)$ of a point $x \in M$ under G is defined by

$$G(x) = \{g(x) \mid g \in G\}$$

A *flow* of homeomorphisms of M is a subgroup $G = \{h_t \mid t \in \mathbb{R}\}$ of $Homeo(M)$ satisfying the following conditions:

(1) $\phi : (t, x) \longmapsto h_t(x)$ is a continuous map from $\mathbb{R} \times M$ to M,
(2) $\phi(t_1, \phi(t_2, x)) = \phi(t_1 + t_2, x),\ \forall\, t_1,\ t_2 \in \mathbb{R},\ \forall\, x \in M$, and
(3) $\phi(0, x) = x,\ \forall\, x \in M$.

The flow G is said to be *locally compact* (resp. *compact*) if the group G is locally compact (resp. compact); it is said to be *periodic* if there exists $r \in \mathbb{R}^*$ such that $h_r = id$; and it is said to be *recurrent* if there exists a sequence $(t_k)_k$ in $\mathbb{R}_+$ such that $t_k \longrightarrow +\infty$ and $h_{t_k} \longrightarrow id$.

A point $x \in M$ is said to be *positively recurrent* (resp. *negatively recurrent*) under G if there exists a sequence $(t_k)_k$ in $\mathbb{R}$ such that $t_k \longrightarrow +\infty$ (resp. $t_k \longrightarrow -\infty$), and $h_{t_k}(x) \longrightarrow x$. The point x is said to be *recurrent* if it is either positively recurrent or negatively recurrent, and it is called *periodic* under G if $h_r(x) = x$ for some real $r \neq 0$.

In [2] (see also [6]), the authors showed that if G is a flow on a $2-cell$, then any recurrent point x under G is periodic under G. A natural question is : Is this result still true in higher dimension? A negative answer can be given by the flow $G = \{h_t \mid t \in \mathbb{R}\}$ defined on the product $S^1 \times S^1$ of $\mathbb{R}^4 \simeq \mathbb{C} \times \mathbb{C}$ by

$$h_t(z_1, z_2) = (e^{i2\pi\alpha t} z_1, e^{i2\pi t} z_2); \qquad (*)$$

where $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. This flow G is a group of isometries of $S^1 \times S^1$, and by compactness of $S^1 \times S^1$, every element h_t of G is recurrent since on compact spaces regular homeomorphisms are recurrent and isometries are always regular. It follows that G is recurrent, but it is not periodic. (Note here that it is easy to see that the recurrence of some element h_t of G ($t \neq 0$) implies the recurrence of G). Also, the point $(1, 1)$ is recurrent but it is not periodic.

In this paper, we show that for every locally compact flow G on a connected m-manifold M ($m \geq 1$), if G is recurrent then it is periodic, and recurrent points with locally compact orbits are periodic (Theorem 1.1). In all the paper, a m-manifold means a topological manifold of dimension m. Our first main result is the following theorem; where we completely characterize locally compact flows on connected m-manifolds for every $m \geq 1$.

Theorem 1.1. *Let G be a locally compact flow on a connected m-manifold M. Then the following hold.*

(1) *If G is recurrent, then it is periodic and it is a circle group.*
(2) *G is either a circle group or a real group.*
(3) *Every recurrent point x of M with locally compact orbit $G(x)$ is periodic.*

Remark 1.2.

(1) Without the condition "locally compact" for the flow G, the results of Theorem 1.1 fail to be true; it is sufficient to consider the flow defined by $(*)$ on $\mathbb{R}^4$ or on the product $S^1 \times S^1$ which is recurrent but not periodic. Indeed this flow is not locally compact because it is not closed. For proving this, take a sequence in the cyclic subgroup generated by the irrational rotation h_1 which converges to some periodic rotation $R \neq id$. Then there is no real t satisfying $h_t = R$.

(2) One can ask this question: Let G be a locally compact flow on a connected manifold M, then is it true that every recurrent point x of M is periodic?

In [12], the authors showed that every recurrent homeomorphism of the plane $\mathbb{R}^2$ is periodic. Is a recurrent homeomorphism of $\mathbb{R}^3$ periodic? it is still an open question. However, in this paper, we give partial answers. We show that a homeomorphism of $\mathbb{R}^3$ which is recurrent and regular is periodic. Indeed, for every integer $m \geq 1$, we show that if G is a compact Lie group of homeomorphisms of $\mathbb{R}^m$ or a locally compact flow on $\mathbb{R}^m$, then every recurrent element in G is periodic (Theorem 1.3).

Theorem 1.3.

(1) *Let G be a compact Lie group of homeomorphisms of $\mathbb{R}^m$, then Rec(G) = P(G).*
(2) *Let G be a locally compact flow on $\mathbb{R}^m$, then Rec(G) = P(G).*
(3) *A recurrent homeomorphism h of $\mathbb{R}^3$ is periodic if and only if h is regular.*

In all the paper, two topological groups G and T are isomorphic if there is a topological isomorphism between them; that is a group isomorphism which is also a homeomorphism. For a subset E of a metric space M, we denote by $\overline{E}$ its closure and by ∂E its boundary.

2. Locally compact flows on connected manifolds

In this section, we prove Theorem 1.1.

Lemma 2.1. *Let G be a flow on a metric space M. Then the map*

$$\begin{array}{rccc} \alpha : & \mathbb{R} & \longrightarrow & G \\ & t & \longmapsto & h_t \end{array}$$

is continuous.

Proof. Let $(t_k)_k$ be a sequence in $\mathbb{R}$ such that $t_k \longrightarrow t$. We show that $h_{t_k} \longrightarrow h_t$. Let K be any compact subset in M. Let I be a compact neighborhood of t in $\mathbb{R}$. We know that the map

$$\begin{array}{rccc} \phi : & \mathbb{R} \times M & \longrightarrow & M \\ & (t,x) & \longmapsto & h_t(x) \end{array}$$

is continuous, then its restriction $\phi_{|I \times K}$ to the compact subset $I \times K$ is uniformly continuous. Then for every $\epsilon > 0$, there exists $\eta > 0$ such that for every $u,\ v \in I$ and for every $x,\ y \in K$;

$$|u - v| < \eta, \text{ and } d(x,y) < \eta \Longrightarrow d(h_u(x), h_v(y)) < \epsilon.$$

Since $t_k \longrightarrow t$, then for η there exists $k_0 > 0$ such that for every $k \geq k_0$, $|t_k - t| < \eta$, and $t_k \in I$, then

$$d(h_{t_k}(x), h_t(x)) < \epsilon,\ \forall\ x \in K.$$

Thus $d_K(h_{t_k}, h_t) \longrightarrow 0$ when $k \longrightarrow +\infty$. Therefore $h_{t_k} \longrightarrow h_t$ in G. So, α is continuous. □

In Proposition 2.2, we describe continuous flows on connected manifolds and their orbits.

Proposition 2.2. *Let G be a continuous flow on a connected manifold M, then the following hold:*

(1) *Either G is recurrent or G is isomorphic to $\mathbb{R}$.*
(2) *For every $x \in M$, either x is recurrent or the orbit $G(x)$ is homeomorphic to $\mathbb{R}$.*

Proof. For showing Item (1), assume that G is nonrecurrent, then G is nonperiodic and the map

$$\begin{array}{rccc} \alpha: & \mathbb{R} & \longrightarrow & G \\ & t & \longmapsto & h_t \end{array}$$

is bijective, it is also continuous by Lemma 2.1. For $t,\ t' \in \mathbb{R}$, $\alpha(t+t') = h_{t+t'} = \alpha(t)\alpha(t')$, then α is a homomorphism. For showing that $\alpha^{-1} : G \longrightarrow \mathbb{R}$ is continuous, we will prove that if $(h_{t_k})_k$ is a sequence in G converging to some element h_t of G then (t_k) converges to t. If this is not true, then (t_k) contains a subsequence $(t_{\varphi(k)})$ which either converges to ∞ or it converges to some real $r \neq t$ and G is periodic. Since G is assumed to be nonperiodic, then the first case holds. If $t_{\varphi(k)} \longrightarrow +\infty$, then G is recurrent since $h_{(t_{\varphi(k)}-t)} \longrightarrow id$. If $t_{\varphi(k)} \longrightarrow -\infty$, then $(t - t_{\varphi(k)}) \longrightarrow +\infty$ and $h_{(t_{\varphi(k)}-t)} \longrightarrow id$ implies $h_{(t-t_{\varphi(k)})} \longrightarrow id$ since G is a topological group, and so G is recurrent.

Item (2) is true by the same argument of Item (1). □

Lemma 2.3 ([1])**.** *Let G be a locally compact flow on a connected m-manifold M, then G is a Lie group.*

Proposition 2.4. *Let G be a nontrivial locally compact flow on a connected manifold M, then either $\langle h_1\rangle$ is isomorphic to $\mathbb{Z}$ or G is periodic and it is a circle group.*

Proof. By Weil's Lemma [9, p. 215] either $\langle h_1\rangle$ is isomorphic to $\mathbb{Z}$ or the closure $H = \overline{\langle h_1\rangle}$ is compact. Assume that H is compact. Since G is assumed locally compact, then it is closed and we have $G = \{h_t\ :\ t \in \mathbb{R}\} = \{h_{n+r}\ :\ n \in \mathbb{Z},\ r \in [0,1]\} = \psi(H \times \alpha([0,1]))$ (α is the map of Lemma 2.1); where ψ is the map defined by

$$\begin{array}{rccc} \psi: & G \times G & \longrightarrow & G \\ & (f,g) & \longmapsto & fg. \end{array}$$

By continuity of ψ (since G is a topological group), we deduce that G is compact. It follows that every orbit $G(x)$ is compact. By [4, Proposition 1.12, p. 53 and (9.1) 3) p. 121], $G(x)$ is periodic and either $G(x) = \{x\}$ or $G(x)$ is homeomorphic to a circle. By Lemma 2.3 G is a compact connected Lie group, and so it is a torus group. Since every orbit $G(x)$ has dimension ≤ 1, by [10, Theorem 2, p. 246] $dim(G) \leq 1$. Hence G is a circle group. In particular, G contains a periodic element $f \neq id$; that is f satisfies $f^n = id$ for some integer $n \neq 0$. Since $f = h_t$ for some real $t \neq 0$, then $h_{nt} = id$; where $nt \neq 0$. Thus G is periodic. □

Lemma 2.5. *Let G be a flow on a connected manifold M. If G is recurrent, then there exists a sequence $(n_k)_k$ in $\mathbb{N}$ and a real $r \in [0,1]$ satisfying the following conditions :*

(1) *$(h_{n_k+r})_k$ is a sequence in $\overline{\langle h_1\rangle}$.*
(2) *$n_k \longrightarrow +\infty$ and $h_{n_k+r} \longrightarrow id$ when $k \longrightarrow +\infty$.*

Proof. Assume that G is recurrent, then there exists a sequence $(h_{t_k})_k$ in G such that $t_k \longrightarrow +\infty$ and $h_{t_k} \longrightarrow id$. For every t_k, there exists $n_k \in \mathbb{N}$ and $0 \leq r_k < 1$ such that $h_{t_k} = h_{n_k + r_k}$. Since $(r_k) \subset [0,1]$, then by compactness of $[0,1]$ we can assume that $r_k \longrightarrow r \in [0,1]$. By Lemma 2.1 $h_{r_k} \longrightarrow h_r$. Since $h_{t_k} \longrightarrow id$, then $h_{n_k} = (h_1)^{n_k} \longrightarrow h_{-r}$. So, $h_{-r} \in \overline{\langle h_1 \rangle}$ and $(h_{n_k+r})_k$ is a sequence in $\overline{\langle h_1 \rangle}$ converging to the identity map id. □

Proof of Theorem 1.1. (1) Assume that G is recurrent. By Proposition 2.4, it remains to show that if $\langle h_1 \rangle$ is isomorphic to $\mathbb{Z}$ then Item (1) is true. So, assume that $\langle h_1 \rangle$ is isomorphic to $\mathbb{Z}$. Then $\langle h_1 \rangle$ is a discrete subgroup of the Lie group G. By Lemma 2.5, there exists a sequence $(h_{n_k+r})_k$ in $\langle h_1 \rangle$ (since $\langle h_1 \rangle$ is closed) such that $h_{n_k+r} \longrightarrow id$; where $r \in [0,1]$. On the other hand, $\{id\}$ is open in $\langle h_1 \rangle$ since $\langle h_1 \rangle$ is discrete, then there exists an integer n_{k_0} such that $h_{n_{k_0}+r} = id$ and $n_{k_0} + r \neq 0$. So, G is periodic. In particular, G is compact, and so it is a circle group (see the proof of Proposition 2.4).

(2) Follows from Proposition 2.2 and Item (1).

(3) Let $x \in M$ be a recurrent point under G such that the orbit $G(x)$ is locally compact. Assume that x is nonperiodic, then the stabilizer G_x is trivial. We will show that the map

$$\begin{array}{rccc} \varphi : & G & \longrightarrow & G(x) \\ & g & \longmapsto & g(x). \end{array}$$

is a homeomorphism. Clearly φ is a continuous bijection. So, it suffices to prove that φ is open. Let U be any open set in G and let $g \in U$. Let V be a compact neighborhood of e in G such that $V^{-1} = V$ and $gV^2 \subset U$.

Since G is second countable as a flow of homeomorphisms of a manifold, there exists a countable sequence (g_n) in G such that $G = \bigcup_n g_n V$. Since $G(x) = \bigcup_n g_n V(x)$ is locally compact, by Baire's Theorem some subset $g_{n_0} V(x)$ and so $V(x)$ has a nonempty interior. Then for some $v \in V$, $V(x)$ is a neighborhood of $v(x)$. Equivalently, $v^{-1}V(x)$ is a neighborhood of x. Since $gv^{-1}V(x) \subset gV^2(x) \subset U(x)$, then $U(x)$ is a neighborhood of $g(x)$. Hence $U(x)$ is open in $G(x)$. We conclude that G is homeomorphic to $G(x)$. Therefore, x is recurrent implies G is recurrent, and by Item (1), G is periodic, so x is periodic; which is a contradiction. We conclude that x is periodic. □

3. Recurrent homeomorphisms of $\mathbb{R}^m$

In this section, "$\|.\|$" means the Euclidean norm on $\mathbb{R}^m$.

In this section, we study which recurrent homeomorphisms of the Euclidean space $\mathbb{R}^m$ are periodic. We show that if G is a compact Lie subgroup of $Homeo(\mathbb{R}^m)$ or a locally compact flow on $\mathbb{R}^m$, then every recurrent element in G is periodic (Propositions 3.2 and 3.3).

Lemma 3.1. *Let f be a homeomorphism of $\mathbb{R}^m$ satisfying the following conditions:*

(i) *The closure $\overline{\langle f \rangle}$ is a compact Lie group.*
(ii) *There exists a real $c > 0$ such that $\|f(x) - x\| < c$, for all $x \in \mathbb{R}^m$.*

Then $f = id$.

Proof. Define a map g as follows:

$$g: \begin{array}{ccc} B & \longrightarrow & B \\ x & \longmapsto & \begin{cases} \varphi f \varphi^{-1}(x), & \text{if } x \in \mathrm{B}^o \\ x, & \text{if } x \in \partial B \end{cases} \end{array}$$

where B is the closed unit-ball of $\mathbb{R}^m$, and $\varphi : \mathbb{R}^m \to \mathrm{B}^o$ is the homeomorphism defined by $\varphi(x) = \frac{x}{1+\|x\|}$. Clearly g is continuous on B^o. For showing the continuity of g at $x_0 \in \partial B$, let $(x_k)_k$ be a sequence in B^o converging to x_0. It is easy to see that for every $x,\ y \in \mathbb{R}^m$,

$$\|\varphi(x) - \varphi(y)\| \leq \|x - y\| \frac{1 + 2\|y\|}{(1 + \|x\|)(1 + \|y\|)}. \quad (*)$$

For each $k \geq 0$, if we put $u_k = inf(\|\varphi^{-1}(x_k)\|, \|f(\varphi^{-1}(x_k))\|)$, then by $(*)$ and (ii), we obtain

$$\|g(x_k) - x_k\| \leq \frac{c(1 + 2u_k)}{1 + 2u_k + u_k^2} = \delta_k.$$

Since $x_k \longrightarrow x_0$, then $\|\varphi^{-1}(x_k)\| \longrightarrow +\infty$, and so by (ii), $\|f(\varphi^{-1}(x_k))\| \longrightarrow +\infty$. Hence $u_k \longrightarrow +\infty$ and $\delta_k \longrightarrow 0$, which implies that $g(x_k) \longrightarrow x_0$. Therefore, g is continuous. Since g is bijective, by compactness of B, g is a homeomorphism.

Now, we show that g is regular. Since $\overline{\langle f \rangle}$ is compact, then by Ascoli's theorem f is regular. By the fact that the map φ satisfies the property

$$\|\varphi(u) - \varphi(v)\| \leq \|u - v\|,\ \forall\ u,\ v \in \mathbb{R}^m,$$

we deduce that g is regular at each point $x \in \mathrm{B}^o$. Now, we show that g is regular at $x_0 \in \partial B$. Let $W_{x_0} = B(x_0, \lambda) \cap B$ be a neighborhood of x_0 in B; where $B(x_0, \lambda)$ is an open ball in $\mathbb{R}^m$ centered at x_0. Let $A = G(\overline{W_{x_0}})$; where G is the closure of the group generated by g. Since $g = id$ on the boundary ∂B of B, there exists a subset F closed in B^o with boundary containing in ∂B such that $A = W_{x_0} \cup \overline{F}$. Since $\overline{\langle f \rangle}$ is compact, then $\varphi \overline{\langle f \rangle} \varphi^{-1}$ is a compact subgroup of $Homeo(\mathrm{B}^o)$, and so $G(F) = \varphi \overline{\langle f \rangle} \varphi^{-1}(F)$ is a closed subset of B^o and $G(\overline{F})$ is compact in B. Then $V_{x_0} = A \setminus G(\overline{F})$ is an open neighborhood of x_0 containing in W_{x_0} and it satisfies $g(V_{x_0}) = V_{x_0}$. Therefore, for every integer $n \in \mathbb{Z}$, $g^n(V_{x_0}) = V_{x_0} \subset W_{x_0}$. Hence g is regular at x_0, and so g is regular.

Since B is compact, then by [7, Corollary 14.3.2] g is almost periodic and by [7, Theorem 5.33, p. 55], there exists a sequence $(g_n)_n \subset \overline{\langle g \rangle}$ of regularly almost periodic elements g_n of $\overline{\langle g \rangle}$ such that $g_n \longrightarrow g$. By [7, Theorem 5.08, p. 50] every orbit under $\overline{\langle g_n \rangle}$ is 0-dimensional. For every $x \in \mathrm{B}^o$, $\overline{\langle g_n \rangle}(x) = \varphi \overline{\langle f_n \rangle} \varphi^{-1}(x)$; where $(f_n)_n$ is a sequence in $\overline{\langle f \rangle}$. Since $\overline{\langle f \rangle}$ is a compact Lie group, then by [8, Theorem 2.3, p. 14], $\overline{\langle f_n \rangle}(\varphi^{-1}(x))$ is a compact manifold. It follows that $\overline{\langle g_n \rangle}(x)$ is a compact 0-dimensional manifold, so it is finite. Moreover, for every $x \in \partial B$, $\overline{\langle g_n \rangle}(x) = \{x\}$. Then g_n is pointwise periodic, and by [10, Theorem p.224], g_n is periodic. Since g_n coincides with the identity on the boundary ∂B, then $g_n = id$ [11, Newman's theorem]; this is true for every integer n. Since $g_n \longrightarrow g$, then $g = id$; equivalently, $f = id$. □

Proposition 3.2 (Recurrent homeomorphisms in compact Lie groups)**.** *Let G be a compact Lie group of homeomorphisms of $\mathbb{R}^m$, then Rec(G) = P(G).*

Proof. Clearly $P(G) \subset Rec(G)$. For showing the converse inclusion, let $g \in G$ such that g is recurrent. Then there exists an integer $n \in \mathbb{N}^*$ such that

$$\|g^n(x) - x\| < 1, \ \forall\, x \in \mathbb{R}^m.$$

Since $g^n \in G$, then $\overline{\langle g^n \rangle}$ is a compact Lie group. So, by Lemma 3.1, $g^n = id$; that is g is periodic. Thus, $Rec(G) = P(G)$. □

Proposition 3.3 (Recurrent homeomorphisms in locally compact flows)**.**

(1) *Let G be a locally compact flow on $\mathbb{R}^m$, then $Rec(G) = P(G)$.*
(2) *Let $f : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ be a recurrent homeomorphism of $\mathbb{R}^m$. If f is embeddable in a locally compact flow, then f is periodic.*

Proof. (1) Let $f \in Rec(G)$. Then $\overline{\langle f \rangle}$ is a locally compact monothetic subgroup of G. Then, either $\langle f \rangle$ is isomorphic to $\mathbb{Z}$ or $\overline{\langle f \rangle}$ is compact.

If $\langle f \rangle$ is isomorphic to $\mathbb{Z}$, then $\langle f \rangle$ is discrete and $\{id\}$ is open in $\langle f \rangle$. Since f is recurrent, then there exists a sequence $(f^{n_k})_k$ in $\langle f \rangle$ such that $f^{n_k} \longrightarrow id$ and $n_k \longrightarrow +\infty$ when $k \longrightarrow +\infty$. Then, there exists $k_0 > 0$ such that for every $k \geq k_0$, $f^{n_k} \in \{id\}$, which implies that $f^n = id$ for some integer $n \in \mathbb{N}^*$. So, f is periodic.

Now, assume that $\overline{\langle f \rangle}$ is compact. Then $\overline{\langle f \rangle}$ is a compact Lie subgroup since G is a Lie group (Lemma 2.3), and by Proposition 3.2, f is periodic. Hence $Rec(G) = P(G)$.

(2) If f is embedded in a locally compact flow G, then $f \in Rec(G)$, and by (1) f is periodic. □

In Corollary 3.4, we obtain two important results for flows G of homeomorphisms of $\mathbb{R}^m$. The first one says that if all elements in G are recurrent then G contains no periodic element, and the second one says that if G is locally compact and every element in G is recurrent then G must be trivial. This last result is a generalization of the result of Yang [14] saying that compact groups of homeomorphisms of manifolds in which every element is periodic are finite.

Corollary 3.4. *Let G be a flow on $\mathbb{R}^m$ such that $Rec(G) = G$, then the following hold.*

(1) $P(G) = \{id\}$.
(2) *If G is locally compact, then $G = \{id\}$.*

Proof. (1) Assume that $P(G) \neq \{id\}$, then there exists a periodic element $g = h_{t_0} \in G\backslash\{id\}$. So, $g^q = id$ for some integer $q > 0$, and $h_{qt_0} = id$; where $r = qt_0 \neq 0$. Then $G = \{h_t \mid 0 \leq t < r\}$. By Lemma 2.1, $G = \alpha([0, r])$ is compact. Then by Lemma 2.3, G is a compact Lie group, and by Proposition 3.2, $Rec(G) = P(G) = G$. Then, by [14], G is finite. But G is connected, so $G = \{id\}$ which contradicts the assumption that $P(G) \neq \{id\}$. Thus $P(G) = \{id\}$.

(2) Follows from Item (1) and Proposition 3.3.(1). □

Proof of Theorem 1.3. (1) Follows from Proposition 3.2.

(2) Follows from Proposition 3.3.

(3) Let $h : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a recurrent homeomorphism which is moreover regular. Then for every orbit O_x ($x \in \mathbb{R}^3$), the closure $\overline{O_x}$ is minimal, and since h is recurrent, any point x is almost periodic [7, Theorem 7.05]. It follows that $\overline{O_x}$ is compact. Then by Ascoli's theorem, $G = \overline{\langle h \rangle}$ is a compact group, and by [13], G is a Lie group. So, by (1), h is periodic. □

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