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CLASSIFICATION OF COMPACT COMPLEX HOMOGENEOUS SPACES WITH INVARIANT VOLUMES

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ABSTRACT. In this note we give a classification of compact complex homogeneous spaces with invariant volume.

1. Introduction

We call a $2n$-dimensional manifold M a *complex homogeneous space with invariant volume* if there is a complex structure and a nonzero $2n$-form on M such that a transitive Lie transformation group keeps both the complex structure and the $2n$-form invariant. There are many papers published in the direction of classification of such manifolds, e.g., [Bo], [DG1], [DG2], [DG3], [DN], [Gu1], [Gu2], [Ha1], [Hk], [HK], [Kz], [Mt1], [Mt2], [Wa2] and the references there (see also [BR], [Gu1], [Gu2], [Gu4], [Ti], [Wa1] for related topics involving compact complex homogeneous spaces). In this paper we will finish the classification in the compact case.

A major break-through in this direction became possible after the following two results were established. Firstly, the Hano-Kobayashi fibration of a compact complex homogeneous space with invariant volume (we might also call it the Ricci form reduction) is holomorphic and coincides with the anticanonical fibration (see [DG1]). Secondly, one can classify compact complex homogeneous spaces with invariant pseudo-Kähler structure (see [DG1], [Hk] and [Gu1], [Gu2], also [Gu4]).

The proof is much harder than the corresponding proof in the Kähler case in [Mt1]. Namely, in the Kähler case one can choose the transitive group to be compact. Then the isotropy group is a subgroup of an orthogonal group. In particular, both groups are reductive.

In [Hk] Huckleberry observed that one can handle the pseudo-Kähler case using methods from symplectic geometry. In particular, he applied here the construction of the moment map. In [Gu1], [Gu2] we observed that his method actually works for a compact complex homogeneous space with an invariant symplectic structure.

Huckleberry's method was used in [Gu1], [Gu2] to get a structure theorem for compact homogeneous complex manifolds with 2-cohomology classes ω such that $\omega^n \neq 0$ in the top cohomology. This generalized the result of [BR] for the Kähler case (one does not assume that the Kähler form is invariant).

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For a general compact complex homogeneous space with invariant volume, the symplectic method does not apply. However, our original method (see [HK], [Mt2] and [DG1]) gives a classification.

Main Theorem 1. *Every compact complex homogeneous space with an invariant volume form is a homogeneous complex torus bundle over the product of a projective rational homogeneous space and a parallelizable manifold. Conversely, every complex homogeneous manifold M of this kind admits a transitive real transformation Lie group G, acting on M by holomorphic transforms and preserving a volume form on M.*

For more details concerning the structure theorem, one might look at Sections 3, 4 and 5 of [Gu3]. We also note that every compact complex homogeneous space M with a 2-cohomology class such that its top power is nonzero in the top cohomology group, admits a transitive real Lie transformation group G, which acts on M by holomorphic transforms and preserves a volume form.

Our proof is better than the proof of both results in [DG1] and [Gu1].

In [Mt2] Matsushima considered the special case of a semisimple group action. He proved that if G/H is a compact complex homogeneous space with a G-invariant volume and if G is semisimple, then G/H is a holomorphic fiber bundle over a projective rational homogeneous space, the typical fiber being a complex parallelizable homogeneous space of a reductive complex Lie group.

Applying our Main Theorem 1 to this situation, we immediately see that the result of Matsushima can be generalized to the case when G is reductive. Moreover, we have the following stronger result.

Main Theorem 2. *Assume that G/H is a compact complex homogeneous space with a G-invariant volume and G is reductive. Then G/H is a holomorphic torus bundle over the product of a projective rational homogeneous space and a complex parallelizable homogeneous space of a semisimple complex Lie group.*

Even if we drop here the assumption that G/H has a G-invariant volume form, we still have a holomorphic fibration of G/H over a projective rational base with parallelizable fiber (see [BR], [Ti]). J. Hano [Ha2] proved that the fiber is of the form L/Γ, where Γ is a discrete cocompact subgroup of a reductive complex Lie group L. Our methods show that the converse is also true. Namely, we have the following theorem.

Main Theorem 3. *Suppose that a compact complex homogeneous space M admits a holomorphic fibration $\pi : M \to D$, where D is a projective rational homogeneous space. Assume that the typical fiber of π is of the form $F = L/\Gamma$, where L is a connected reductive complex Lie group, and Γ a discrete cocompact subgroup of L. Then any transitive effective complex Lie transformation group G, acting on M by holomorphic transforms, is reductive.*

Corollary. *Every compact complex homogeneous space is a holomorphic fiber bundle whose base is a compact complex homogeneous space of a reductive Lie group and whose typical fiber is a complex parallelizable homogeneous space of a nilpotent complex Lie group.*

Having in mind a classification of all compact complex homogeneous spaces as a goal, we hope to use the Corollary in our future research.

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