# NONLOCAL FINITENESS OF A $W$-GRAPH 

GEORGE LUSZTIG


#### Abstract

It is shown that the $W$-graph of an affine Weyl group of type $B_{2}$ (as defined by Kazhdan and Lusztig in Representations of Coxeter groups and Hecke algebras, Invent. Math. 53 (1979), 165-184) is not locally finite.


1. The purpose of this paper is to present an example of computation of the leading coefficients $\mu(y, w)$ of the polynomials $P_{y, w}$ of [KL], or equivalently, of the "inverse polynomials" $Q_{y, w}$ for an affine Weyl group.

We use the following method. We want to make use of the explicit formula [L] for the $P_{y, w}$ in the case where $y, w$ have maximal length in their double coset with respect to the finite Weyl group. For such $y$, $w$ we have $\mu(y, w)=0$ except in trivial cases, so this by itself does not give interesting examples of $\mu(y, w)$. It would be much more useful to be able to compute, instead, the polynomials $Q_{y, w}$ where $y, w$ have minimal length in their double coset with respect to the finite Weyl group. We show that these last polynomials can be directly related through a system of semilinear equations to the special polynomials $P_{y, w}$ above. These equations can sometimes be solved explicitly and we may hope to find in this way interesting examples of $\mu(y, w)$.

Consider the affine Weyl group of type $\tilde{B}_{2}$ with standard Coxeter generators $a, b, c$ where $a, c$ commute; set $p=a b a, q=c b a$. Using the method above, we shall obtain the following result:

$$
\begin{equation*}
\mu\left(p q^{n}, p q^{m}\right)=1 \text { if } m>n>0, m \text { even, } n \text { odd. } \tag{a}
\end{equation*}
$$

(Note that $\left\{p q^{n} \mid n=1,2, \ldots\right\}$ are distinct involutions and the length of $p q^{n}$ is $3 n+3$.) We see that the $W$-graph (see [KL, §1]) of our Coxeter group is not locally finite. This example suggests that the $W$-graphs of most affine Weyl groups are not locally finite. (They are locally finite for type $\tilde{A}_{1}, \tilde{A}_{2}$.)
2. Let $(W, S)$ be a Coxeter group ( $S$ is the set of simple reflections) and let $l$ : $W \rightarrow \mathbf{N}$ be the corresponding length function. Let $\mathcal{A}=\mathbf{Z}\left[v, v^{-1}\right]$ where $v$ is an indeterminate. For $f \in \mathcal{A}$, let $\operatorname{Res}_{v=0}(f) \in \mathbf{Z}$ denote the coefficient of $v^{-1}$ in $f$. Let ${ }^{-}: \mathcal{A} \rightarrow \mathcal{A}$ be the ring involution such that $\bar{v}=v^{-1}$. For $x, y \in W$, let $R_{x, y} \in \mathcal{A}$ be defined as in [KL, (2.0.a)] (where $q^{\frac{1}{2}}$ in loc. cit. is our $v$ ). We set $r_{x, y}=v^{l(x)-l(y)} R_{x, y}$. We have

$$
\begin{equation*}
r_{x, x}=1, \bar{r}_{x, y}=(-1)^{l(y)-l(x)} r_{x, y} \text { for any } x, y \in W \tag{a}
\end{equation*}
$$

[^0](see [KL, 2.1(i)]),
(b)
$$
r_{x, 1}=\delta_{x, 1}
$$
(c) $\quad v^{-l(x)-l(y)} r_{x, y}+v^{-l(s x)-l(y)} r_{s x, y}=v^{-l(x)-l(s y)} r_{x, s y}+v^{-l(s x)-l(s y)} r_{s x, s y}$,
(d)
$$
v^{l(x)+l(y)} r_{x, y}+v^{l(x)+l(s y)} r_{x, s y}=v^{l(s x)+l(y)} r_{s x, y}+v^{l(s x)+l(s y)} r_{s x, s y}
$$
(for any $x, y \in W$ and any $s \in S$ ). Note that (c) follows from [KL, (2.0.c)] and (d) follows by applying ${ }^{-}$to (c) and using (a). Moreover,
\[

$$
\begin{equation*}
r_{y, z} \neq 0 \Longrightarrow y \leq z \text { in the Bruhat order of } W \tag{e}
\end{equation*}
$$

\]

For any $x, z \in W$ we have (by [KL, 2.1(ii)])

$$
\begin{equation*}
\sum_{y} r_{x, y} \bar{r}_{y, z}=\delta_{x, z} \tag{f}
\end{equation*}
$$

3. For $x \leq y$ in $W$, let $P_{x, y} \in \mathcal{A}$ be defined as in [KL, 1.1] (where $q^{\frac{1}{2}}$ in loc. cit. is our $v$ ). We set $p_{x, y}=v^{l(x)-l(y)} P_{x, y} \in \mathbf{Z}\left[v^{-1}\right]$ for $x \leq y$ and $p_{x, y}=0$ for all other $x, y$. We have

$$
\begin{equation*}
p_{x, x}=1, \quad p_{x, y} \in v^{-1} \mathbf{Z}\left[v^{-1}\right] \text { for } x<y \tag{a}
\end{equation*}
$$

$$
\begin{equation*}
\sum_{y} r_{x, y} p_{y, z}=\bar{p}_{x, z} \text { for any } x, z \text { in } W \tag{b}
\end{equation*}
$$

(see [KL, (2.2.a)]).
We define elements $q_{x, y} \in \mathbf{Z}\left[v^{-1}\right]$ (for any $x, y$ in $W$ ) by the system of equations

$$
\begin{equation*}
\sum_{z}(-1)^{l(x)+l(y)} p_{x, y} q_{y, z}=\delta_{x, z} \tag{c}
\end{equation*}
$$

(for any $x, z$ in $W$ ). We have
(d) $\quad q_{x, x}=1, \quad q_{x, y} \in v^{-1} \mathbf{Z}\left[v^{-1}\right]$ for $x<y$ and $q_{x, y}=0$ for all other $x, y$.

For any $x, z \in W$ we have

$$
\begin{gather*}
\sum_{y}(-1)^{l(x)-l(y)} q_{x, y} p_{y, z}=\delta_{x, z}  \tag{e}\\
\sum_{y} q_{x, y} r_{y, z}=\bar{q}_{x, z}
\end{gather*}
$$

As in [KL, 1.2], for $x, y \in W$, we set

$$
\mu(x, y)=\operatorname{Res}_{v=0}\left(p_{x, y}\right)=\operatorname{Res}_{v=0}\left(q_{x, y}\right) \in \mathbf{Z}
$$

4. If $I$ is a subset of $S$ we denote by $W_{I}$ the subgroup of $W$ generated by $I$. If $W_{I}$ is finite, we set

$$
\xi_{I}=\sum_{w \in W_{I}} v^{2 l(w)} \in \mathbf{Z}\left[v^{2}\right] .
$$

We now fix two subsets $I, I^{\prime}$ of $S$ such that $W_{I}$ and $W_{I^{\prime}}$ are finite. Let $\nu$ (resp. $\nu^{\prime}$ ) be the maximum length of an element of $W_{I}$ (resp. $W_{I^{\prime}}$ ). Let $X^{+}$be the set of $W_{I}-W_{I^{\prime}}$ double cosets in $W$. Each $\lambda \in X^{+}$contains a unique element $m_{\lambda}$ of minimal length and a unique element $M_{\lambda}$ of maximal length; we define

$$
\begin{gather*}
\xi_{\lambda}=\sum_{w \in \lambda} v^{2 l(w)} \in \mathbf{Z}\left[v^{2}\right]  \tag{a}\\
\pi_{\lambda}=\frac{\xi_{I}}{v^{\nu}} \frac{\xi_{I^{\prime}}}{v^{\nu^{\prime}}} \frac{v^{l\left(M_{\lambda}\right)+l\left(m_{\lambda}\right)}}{\xi_{\lambda}} \in \mathcal{A} \tag{b}
\end{gather*}
$$

then $\bar{\pi}_{\lambda}=\pi_{\lambda}$.
From 2(c) we see that the function $y \rightarrow \sum_{x \in \lambda} v^{-l(x)-l(y)} r_{x, y}$ is constant on left
 double cosets in $W$. In particular, for $\lambda, \lambda^{\prime} \in X^{+}$, we have

$$
\begin{equation*}
\sum_{x \in \lambda ; y \in \lambda^{\prime}} v^{-l(x)+l(y)} r_{x, y}=\sum_{y \in \lambda^{\prime}} v^{2 l(y)} \sum_{x \in \lambda} v^{-l(x)-l\left(m_{\lambda^{\prime}}\right)} r_{x, m_{\lambda^{\prime}}} \tag{c}
\end{equation*}
$$

From 2(d) we see that the function $x \rightarrow \sum_{y \in \lambda^{\prime}} v^{l(x)+l(y)} r_{x, y}$ is constant on left
 double cosets in $W$. In particular, for $\lambda, \lambda^{\prime} \in X^{+}$, we have

$$
\begin{equation*}
\sum_{x \in \lambda ; y \in \lambda^{\prime}} v^{-l(x)+l(y)} r_{x, y}=\sum_{x \in \lambda} v^{-2 l(x)} \sum_{y \in \lambda^{\prime}} v^{l\left(M_{\lambda}\right)+l(y)} r_{M_{\lambda}, y} \tag{d}
\end{equation*}
$$

Comparing (c),(d) we see that

$$
\begin{equation*}
\xi_{\lambda^{\prime}} \sum_{x \in \lambda} v^{-l(x)-l\left(m_{\lambda^{\prime}}\right)} r_{x, m_{\lambda^{\prime}}}=\overline{\xi_{\lambda}} \sum_{y \in \lambda^{\prime}} v^{l\left(M_{\lambda}\right)+l(y)} r_{M_{\lambda}, y} \tag{e}
\end{equation*}
$$

5. Let $\leq$ be the partial order on $X^{+}$defined by $\lambda \leq \lambda^{\prime}$ if $M_{\lambda} \leq M_{\lambda^{\prime}}$. For $\lambda, \lambda^{\prime} \in X^{+}$, we set

$$
p_{\lambda, \lambda^{\prime}}=p_{M_{\lambda}, M_{\lambda^{\prime}}}, \quad q_{\lambda, \lambda^{\prime}}=q_{m_{\lambda}, m_{\lambda^{\prime}}}
$$

By [KL, 2.3.g], we have

$$
v^{l\left(M_{\lambda}\right)-l(x)} p_{x, M_{\lambda^{\prime}}}=p_{\lambda, \lambda^{\prime}}
$$

for all $x \in \lambda$ and similarly

$$
v^{l\left(x^{\prime}\right)-l\left(m_{\lambda^{\prime}}\right)} q_{m_{\lambda}, x^{\prime}}=q_{\lambda, \lambda^{\prime}}
$$

for all $x^{\prime} \in \lambda^{\prime}$. We set

$$
\begin{equation*}
a_{\lambda, \lambda^{\prime}}=\sum_{y \in \lambda^{\prime}}(-v)^{-l\left(M_{\lambda^{\prime}}\right)+l(y)} q_{M_{\lambda}, y} \tag{a}
\end{equation*}
$$

$$
\begin{equation*}
b_{\lambda, \lambda^{\prime}}=\sum_{z \in \lambda}(-v)^{l\left(m_{\lambda}\right)-l(z)} p_{z, m_{\lambda^{\prime}}} \tag{b}
\end{equation*}
$$

Clearly, $p_{\lambda, \lambda^{\prime}}, q_{\lambda, \lambda^{\prime}}, a_{\lambda, \lambda^{\prime}}, b_{\lambda, \lambda^{\prime}}$ are zero unless $\lambda \leq \lambda^{\prime}$ and are equal to 1 when $\lambda=\lambda^{\prime}$; when $\lambda<\lambda^{\prime}$ they belong to $v^{-1} \mathbf{Z}\left[v^{-1}\right]$ and

$$
\begin{equation*}
\operatorname{Res}_{v=0}\left(q_{\lambda, \lambda^{\prime}}\right)=\mu\left(m_{\lambda}, m_{\lambda^{\prime}}\right)=\operatorname{Res}_{v=0}\left(b_{\lambda, \lambda^{\prime}}\right) \tag{c}
\end{equation*}
$$

6. Using $3(\mathrm{c})$ and the definitions, we see that
(a)

$$
\sum_{\lambda^{\prime}}(-1)^{l\left(M_{\lambda}\right)-l\left(M_{\lambda^{\prime}}\right)} a_{\lambda, \lambda^{\prime}} p_{\lambda^{\prime}, \lambda^{\prime \prime}}=\delta_{\lambda, \lambda^{\prime \prime}}
$$

(b)

$$
\sum_{\lambda^{\prime}}(-1)^{l\left(m_{\lambda}\right)-l\left(m_{\lambda^{\prime}}\right)} q_{\lambda, \lambda^{\prime}} b_{\lambda^{\prime}, \lambda^{\prime \prime}}=\delta_{\lambda, \lambda^{\prime \prime}}
$$

for any $\lambda, \lambda^{\prime \prime} \in X^{+}$.
Proposition 7. For any $\lambda, \lambda^{\prime \prime} \in X^{+}$, we have
(a) $\sum_{\lambda^{\prime}} q_{\lambda, \lambda^{\prime}} \frac{1}{\pi_{\lambda^{\prime}}} \bar{p}_{\lambda^{\prime}, \lambda^{\prime \prime}}=\sum_{\lambda^{\prime}} \bar{q}_{\lambda, \lambda^{\prime}} \frac{1}{\pi_{\lambda^{\prime}}} p_{\lambda^{\prime}, \lambda^{\prime \prime}}$,
(b) $\sum_{\lambda^{\prime}} a_{\lambda, \lambda^{\prime}}(-1)^{l\left(m_{\lambda^{\prime}}\right)-l\left(M_{\lambda^{\prime}}\right)} \pi_{\lambda^{\prime}} \bar{b}_{\lambda^{\prime}, \lambda^{\prime \prime}}=\sum_{\lambda^{\prime}} \bar{a}_{\lambda, \lambda^{\prime}}(-1)^{l\left(m_{\lambda^{\prime}}\right)-l\left(M_{\lambda^{\prime}}\right)} \pi_{\lambda^{\prime}} b_{\lambda^{\prime}, \lambda^{\prime \prime}}$.

For any $\lambda, \lambda^{\prime}$ in $X^{+}$we set

$$
\begin{aligned}
& r_{\lambda, \lambda^{\prime}}=\sum_{z \in \lambda^{\prime}} v^{l(z)-l\left(M_{\lambda^{\prime}}\right)} r_{M_{\lambda}, z} \\
& \tilde{r}_{\lambda, \lambda^{\prime}}=\sum_{z \in \lambda} v^{-l(z)+l\left(m_{\lambda}\right)} r_{z, m_{\lambda^{\prime}}}
\end{aligned}
$$

Using these definitions and $3(\mathrm{~b}), 3(\mathrm{f})$, we deduce, for any $\lambda, \lambda^{\prime \prime}$ :
(c)

$$
\begin{aligned}
& \sum_{\lambda^{\prime}} r_{\lambda, \lambda^{\prime}} p_{\lambda^{\prime}, \lambda^{\prime \prime}}=\bar{p}_{\lambda, \lambda^{\prime \prime}} \\
& \sum_{\lambda^{\prime}} q_{\lambda, \lambda^{\prime}} \tilde{r}_{\lambda^{\prime}, \lambda^{\prime \prime}}=\bar{q}_{\lambda, \lambda^{\prime \prime}}
\end{aligned}
$$

(d)

We can rewrite $4(\mathrm{e})$ in the form

$$
\xi\left(\lambda^{\prime}\right) v^{-l\left(m_{\lambda^{\prime}}\right)-l\left(m_{\lambda}\right)} \tilde{r}_{\lambda, \lambda^{\prime}}=\overline{\xi(\lambda)} v^{l\left(M_{\lambda}\right)+l\left(M_{\lambda^{\prime}}\right)} r_{\lambda, \lambda^{\prime}}
$$

or in the form
(e)

$$
\frac{1}{\pi_{\lambda^{\prime}}} \tilde{r}_{\lambda, \lambda^{\prime}}=\frac{1}{\pi_{\lambda}} r_{\lambda, \lambda^{\prime}}
$$

Using (c), we see that the left hand side of (a) is equal to

$$
\sum_{\lambda^{\prime}} q_{\lambda, \lambda^{\prime}} \frac{1}{\pi_{\lambda^{\prime}}} \sum_{\tilde{\lambda}} r_{\lambda^{\prime}, \tilde{\lambda}} p_{\tilde{\lambda}, \lambda^{\prime \prime}}
$$

and, using (e), to

$$
\sum_{\lambda^{\prime}, \tilde{\lambda}} q_{\lambda, \lambda^{\prime}} \frac{1}{\pi_{\tilde{\lambda}}} \tilde{r}_{\lambda^{\prime}, \tilde{\lambda}} p_{\tilde{\lambda}, \lambda^{\prime \prime}}
$$

Using (d), we see that this equals $\sum_{\tilde{\lambda}} \bar{q}_{\lambda, \tilde{\lambda}} \frac{1}{\pi_{\tilde{\lambda}}} p_{\tilde{\lambda}, \lambda^{\prime \prime}}$ which is the same as the right hand side of (a). Thus (a) is proved.

We prove (b). We can write (a) as an identity of matrices indexed by $X^{+} \times X^{+}$:

$$
\begin{equation*}
Q D^{-1} \bar{P}=\bar{Q} D^{-1} P \tag{f}
\end{equation*}
$$

where $Q=\left(q_{\lambda, \lambda^{\prime}}\right), P=\left(p_{\lambda, \lambda^{\prime}}\right), D$ is the diagonal matrix with entries $\pi_{\lambda}$; we agree that - applied to a matrix is obtained by applying - to each entry. Taking the inverse of both sides of ( f ), we obtain $P^{-1} D \bar{Q}^{-1}=\bar{P}^{-1} D Q^{-1}$. This is, up to signs, the same as (b) (we use $6(\mathrm{a}), 6(\mathrm{~b})$ ). The proposition is proved.
8. In the remainder of this paper we assume that $(W, S)$ is an irreducible affine Weyl group regarded as a Coxeter group. The set $X$ of elements of $W$ which have only finitely many conjugates, is a normal subgroup of $W$, which is free abelian, finitely generated, of finite index. We shall take $I=I^{\prime}=S-\left\{s_{0}\right\}$ where $s_{0} \in S$ is such $W$ is generated by $X$ and by $W_{I}$. There is a unique $\mathbf{Z}$-basis $\left\{\alpha_{s} \mid s \in I\right\}$ of $X$ and unique homomorphisms $\check{\alpha}_{s}: X \rightarrow \mathbf{Z}$ for $s \in I$ such that $s(x)=x-\check{\alpha}_{s}(x) \alpha_{s}$ for all $s \in I, x \in X$. (We write the group operation in $X$ as addition and we write $w(x)$ instead of $w x w^{-1}$ for $w \in W, x \in X$.) Let $R$ be the finite set consisting of all elements of $X$ of the form $w\left(\alpha_{s}\right)$ for various $s \in I$ and $w \in W_{I}$. Let $R^{+}=$ $R \cap \sum_{s \in I} \mathbf{N} \alpha_{s}$. Let $X^{+}=\left\{x \in X \mid \check{\alpha}_{s}(x) \geq 0 \quad \forall s \in I\right\}$. For any $\lambda \in X^{+}$, we set

$$
W_{I}^{\lambda}=\left\{w \in W_{I} \mid w(\lambda)=\lambda\right\}
$$

In our case, $\pi_{\lambda}$ of $4(\mathrm{~b})$ can be rewritten as follows:

$$
\pi_{\lambda}=v^{-\nu_{\lambda}} \sum_{w \in W_{I}^{\lambda}} v^{2 l(w)}
$$

where $\nu_{\lambda}$ is the number of reflections of $W_{I}^{\lambda}$.
There is a 1-1 correspondence between $X^{+}$and the set of $W_{I}-W_{I}$ double cosets in $W$ (to an element of $X^{+}$corresponds the unique double coset containing it). We identify in this way $X^{+}$with the set of $W_{I}-W_{I}$ double cosets in $W$, thus reconciling the present notation with the notation $X^{+}$in $\S 4$.
9. Let $X^{\prime}$ be the subgroup of $\mathbf{Q} \otimes X$ consisting of all $x$ such that $\check{\alpha}_{s}(x) \in \mathbf{Z}$ for all $s \in I$. We have $X \subset X^{\prime}$ and the action of $W_{I}$ on $X$ extends uniquely to a linear action of $W_{I}$ on $X^{\prime}$. For any subset $\mathbf{i} \subset R^{+}$, we set $\alpha_{\mathbf{i}}=\sum_{\alpha \in \mathbf{i}} \alpha \in X$. It is known that $\alpha_{R^{+}}=2 \rho$ where $\rho \in X^{\prime}$ and that $w(\rho)-\rho \in X$ for any $w \in W_{I}$. Let $\Gamma$ be the free $\mathcal{A}$-module with basis $Z_{\lambda},\left(\lambda \in X^{+}\right)$. We define elements $Z_{\lambda} \in \Gamma$ for all $\lambda \in X$ (not just for $\lambda \in X^{+}$) by setting

$$
Z_{\lambda}=(-1)^{l(w)} Z_{w(\lambda+\rho)-\rho}
$$

if $w(\lambda+\rho)-\rho \in X^{+}$, for some $w \in W_{I}$ (necessarily unique),

$$
Z_{\lambda}=0
$$

otherwise.
Theorem 10.

$$
v^{-\nu_{\lambda^{\prime}}} \pi_{\lambda^{\prime}} \sum_{\lambda \in X^{+}} a_{\lambda, \lambda^{\prime}} Z_{\lambda}=\sum_{\mathbf{i} ; \mathbf{i} \subset R^{+}}\left(-v^{2}\right)^{-|\mathbf{i}|} Z_{\lambda^{\prime}-\alpha_{\mathbf{i}}}
$$

(equality in $\Gamma$ ) holds for any $\lambda^{\prime} \in X^{+}$; here, $a_{\lambda, \lambda^{\prime}}$ are as in $\S 5$.

This is obtained by assembling several identities in [L] in the affine Hecke algebra (notation of [L]):

$$
J_{\rho}\left(v^{-l\left(p_{\lambda^{\prime}}\right)} K_{\lambda^{\prime}}\right)=\frac{1}{\overline{\mathcal{P}}_{\lambda^{\prime}}} \sum_{\mathbf{i}}\left(-v^{2}\right)^{-|\mathbf{i}|} J_{\lambda^{\prime}+\rho-\alpha_{\mathbf{i}}}
$$

([L, 6.7]);

$$
J_{\rho} C_{\lambda}^{\prime}=J_{\lambda+\rho}
$$

([L, 6.9]);

$$
C_{\lambda}^{\prime}=\sum_{\lambda^{\prime} \in X^{+}} p_{\lambda^{\prime}, \lambda} v^{-l\left(p_{\lambda^{\prime}}\right)} K_{\lambda^{\prime}}
$$

([L, 6.10, 6.13]) and using $6(\mathrm{a})$. (We take $\Gamma$ to be the $\mathcal{A}$-submodule of the affine Hecke algebra spanned by $Z_{\lambda}=J_{\lambda+\rho}$ with $\lambda \in X^{+}$.) In our case, we have $(-1)^{l\left(M_{\lambda}\right)-l\left(M_{\lambda^{\prime}}\right)}=1$ for $\lambda, \lambda^{\prime}$ in $X^{+}$.
Corollary 11. For any $x \in X$ we set

$$
\Phi(x)=\sum_{\mathbf{i} ; \mathbf{i} \subset R^{+} ; \alpha_{\mathbf{i}}=x}\left(-v^{2}\right)^{-|\mathbf{i}|}
$$

For any $\lambda, \lambda^{\prime}$ in $X^{+}$, we have

$$
a_{\lambda, \lambda^{\prime}}=\frac{v^{\nu_{\lambda^{\prime}}}}{\pi_{\lambda^{\prime}}} \sum_{w \in W_{I}}(-1)^{l(w)} \Phi\left(\lambda^{\prime}+\rho-w(\lambda+\rho)\right) .
$$

12. It is likely that, in the case where $\check{\alpha}_{s}\left(\lambda^{\prime}\right) \geq 1$ for all $s \in I$ and $\lambda \in X^{+}$, we have

$$
b_{\lambda, \lambda^{\prime}}=(-1)^{l\left(m_{\lambda}-l\left(m_{\lambda^{\prime}}\right)\right.} \frac{1}{\pi_{\lambda}} \sum_{w \in W_{I}}(-1)^{l(w)} \Phi\left(w\left(\lambda^{\prime}-\rho\right)-(\lambda-\rho)\right)
$$

13. Now let $(W, S)$ be an affine Weyl group of type $\tilde{B}_{2}$. Write $I=\{1,2\}$ so that $\check{\alpha}_{2}\left(\alpha_{1}\right)=-1, \check{\alpha}_{1}\left(\alpha_{2}\right)=-2$. We solve the system of semilinear equations $7(\mathrm{~b})$ with unknowns $b_{\lambda^{\prime}, \lambda^{\prime \prime}}$ for fixed $\lambda^{\prime \prime}=t \alpha_{1}+t \alpha_{2}$, where $t \geq 3$ and the quantities $a_{\lambda, \lambda^{\prime}}$ are given by $\S 11$. (That system has a unique solution subject to the requirement that $b_{\lambda^{\prime}, \lambda^{\prime \prime}}$ is zero unless $\lambda^{\prime} \leq \lambda^{\prime \prime}$, is equal to 1 when $\lambda^{\prime}=\lambda^{\prime \prime}$ and belongs to $v^{-1} \mathbf{Z}\left[v^{-1}\right]$ when $\lambda^{\prime}<\lambda^{\prime \prime}$.) We find

$$
\begin{aligned}
b_{t \alpha_{1}+(t-1) \alpha_{2}, t \alpha_{1}+t \alpha_{2}} & =v^{-3}+v^{-1} \\
b_{s \alpha_{1}+(s-1) \alpha_{2}, t \alpha_{1}+t \alpha_{2}} & =-v^{-5}+v^{-1}
\end{aligned}
$$

for $s=t-1, t-2, \ldots, 3$. It follows that $\operatorname{Res}_{v=0}\left(b_{s \alpha_{1}+(s-1) \alpha_{2}, t \alpha_{1}+t \alpha_{2}}\right)=1$, for $s=t, t-1, \ldots, 3$. Using $5(\mathrm{c})$, this yields formula 1(a).

## References

[KL] D. Kazhdan and G. Lusztig, Representations of Coxeter groups and Hecke algebras, Invent. Math. 53 (1979), 165-184. MR 81j:20066
[L] G. Lusztig, Singularities, character formulas and a q-analog of weight multiplicities, Astérisque 101-102 (1983), 208-229. MR 85m: 17005

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

E-mail address: gyuri@math.mit.edu


[^0]:    Received by the editors August 13, 1996 and, in revised form, August 21, 1996.
    1991 Mathematics Subject Classification. Primary 20G99.
    Supported in part by the National Science Foundation.

