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ON THE $\mathfrak{n}$-COHOMOLOGY OF LIMITS OF DISCRETE SERIES REPRESENTATIONS

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ABSTRACT. Using localization we translate the problem of computing the $\mathfrak{n}$-cohomology of limits of discrete series representations into a problem about the combinatorics of Weyl groups.

1. Introduction

In this article I will show how to reduce the problem of computing the $\mathfrak{n}$-cohomology of arbitrary limits of discrete series representations to some explicit combinatorics of Weyl groups, for $\mathfrak{n}$ the nilradical of a Borel containing a compact Cartan.

More precisely, I will write down very explicit complexes of vector spaces over $\mathbb{C}$ with bases consisting of certain subsets of the Weyl group and differentials given in these bases by explicit matrices of zeroes, ones and minus ones. The main result then says that the cohomology of these "combinatorial" complexes is the $\mathfrak{n}$-cohomology we were looking for. This recovers old results of Schmid [Sch76] and Williams [Wil88] concerning the discrete series and their nondegenerate limits respectively. In these cases the differentials of our combinatorial complexes all vanish. We also recover the vanishing result of Mirković [Mir90], Corollary 4.8, for degenerate limits under the hypothesis that $\mathfrak{n}$ is holomorphic.

Let me explain briefly how these results are obtained. The first step is to use localization to translate our problem into geometry. Since we are interested in singular central characters, this is not entirely straightforward. To describe the outcome I need some notations. Let $G \supset K \supset T$ be complexifications of our group, of a maximal compact subgroup and of a compact maximal torus. Let X be the flag variety, i.e. the variety of all Borel subgroups of G. Our discrete series becomes under localization the standard module corresponding to some closed K-orbit $i : Y \hookrightarrow X$.

Now $\mathfrak{n}$ is the nilradical of Lie$B$ for some Borel $B \supset T$. The singularity of the central character of our limit M of discrete series determines a parabolic subgroup $P \supset B$. Choosing also a possible eigenvalue λ for the action of Lie$T$ on the $\mathfrak{n}$-cohomology determines a subvariety $j : O \hookrightarrow X$ of the form $O = BgP/B \subset G/B = X$ for suitable $g \in G$. Now consider the category $\mathcal{P}_B(O)$ of perverse sheaves on O which are smooth along B-orbits. Let $\mathcal{A} \in \mathcal{P}_B(O)$ be the projective cover in

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this category of the simple standard object corresponding to the unique closed B-orbit in O. We will show that calculating $H^\bullet(\mathfrak{n}, M)_\lambda$ corresponds to the geometric problem of calculating the hypercohomology groups $\mathbb{H}^\bullet(Y, i^! j_* \mathcal{A})$.

Now $\mathcal{A}$ is not as untractable an object as the quick (but not very useful) definition given above suggests. It is known that $\mathcal{A}$ is selfdual and admits a filtration with each standard object of $\mathcal{P}_B(O)$ occurring exactly once as a subquotient. This filtration on $\mathcal{A}$ leads to some spectral sequence calculating the hypercohomology above. We write down its first stage (with even the differentials given explicitly) and prove that from the second stage on all differentials vanish. This then gives the main result.

This article owes its existence to H. Carayol who explained the problem to me and supplied illuminating examples as well as motivation [Car96].

2. Combinatorics

Let me describe in more detail the combinatorics of the result. I will give two descriptions, one in terms of geometry and one in terms of Coxeter groups. I start with the geometric one.

Given $O = BgP/B$ and a closed K-orbit Y, the combinatorics computing $\mathbb{H}^\bullet(Y, i^! j_* \mathcal{A})$ goes as follows: Call D the set of B-orbits in O and write $D = \bigcup_\nu D(\nu)$, where $D(\nu) \subset D$ denotes the subset of all orbits of dimension ν. Consider the free $\mathbb{C}$-vector space $\mathbb{C}D$ over D and give it the $\mathbb{Z}$-grading $(\mathbb{C}D)^\nu = \mathbb{C}D(\nu)$.

We make $\mathbb{C}D$ into a complex and need a differential. Let us say a four-tuple of orbits $(x; y, z; w)$ from D forms a "small square" if and only if $\{y, z\} = \{v \in D \mid x \subset \bar{v}, x \neq v, v \subset \bar{w}, v \neq w\}$, our orbits are all different from one another and $\dim x = \dim w - 2$. Now choose for every pair (x, y) of elements of D such that $x \subset \bar{y}$ and $\dim x = \dim y - 1$ a sign $s(x, y)$ such that for every small square $(x; y, z; w)$ the four signs multiply to $-1 = s(x, y)s(x, z)s(y, w)s(z, w)$. It is always possible to make such choices; see e.g. [BGG75]. We define the differential $d : (\mathbb{C}D)^\nu \to (\mathbb{C}D)^{\nu-1}$ to be given by the matrix $s(x, y)$, where it is understood that $s(x, y) = 0$ unless $x \subset \overline{y}$.

Now what we really need are the pieces of this complex "cut by Y in a fixed codimension". More precisely, for given $c \in \mathbb{N}$ put

$$D_Y^c = \{x \in D \mid x \text{ meets } Y, \text{ and } c \text{ is the codimension of } Y \cap x \text{ in } x\}.$$

We make $\mathbb{C}D_Y^c$ into a complex with $\mathbb{Z}$-grading $(\mathbb{C}D_Y^c)^\nu = \mathbb{C}(D_Y^c \cap D(\nu))$ and differential d given by the matrix $s(x, y)$. It is not immediately clear but true that $d^2 = 0$. We will prove that

$$\mathbb{H}^n(Y, i^! j_* \mathcal{A}) \cong \bigoplus_c H^{2c-n}(\mathbb{C}D_Y^c, d).$$

Let us reformulate this in combinatorial terms. Suppose given a finite crystallographic Coxeter group $\mathcal{W}$ with Bruhat order $\leq$ and length $l : \mathcal{W} \to \mathbb{N}$. Suppose $\mathcal{V} \subset \mathcal{W}$ is a subset containing the whole interval $\{z \in \mathcal{W} \mid x \leq z \leq y\}$ with any two elements x, y. We will associate to $\mathcal{V}$ a complex $C^\bullet\mathcal{V}$. As a graded vector space, we simply define $C^\nu\mathcal{V}$ to be the free vector space with basis $\{x \in \mathcal{V} \mid l(x) = \nu\}$. To define the differential, let us call a 4-tuple $(x; y, z; w)$ of elements of $\mathcal{W}$ a "small square" if and only if $\{y, z\} = \{v \mid x < v < w\}$ where $y \neq z$ and $l(x) = l(w) - 2$. Choose for every pair $x < y$ in $\mathcal{W}$ with $l(x) = l(y) - 1$ a sign $s(x, y)$ in such a way that for every small square $(x; y, z; w)$ the four signs multiply to $-1 = s(x, y)s(x, z)s(y, w)s(z, w)$. It is always possible to make such choices; see e.g. [BGG75]. Now we define the differential $d : C^\nu \mathcal{V} \to C^{\nu-1}\mathcal{V}$ to be given by the matrix $s(x,y)$, where we put $s(x,y) = 0$ if x and y are not comparable in the Bruhat order. As will be discussed in Lemma 8.1, we have $d^2 = 0$ and the complex $C^\bullet \mathcal{V}$ is independent of the choice of signs $s(x,y)$ up to isomorphism. We denote by $H^\bullet \mathcal{V}$ its homology.

Let $\mathcal{W} \supset \mathcal{W}_K$ be the Weyl group and the compact Weyl group respectively. Let $\mathcal{S} \subset \mathcal{W}$ be the system of simple reflections determined by our Borel B. The singularity of the central character of our limit of discrete series determines in a way to be made more precise later a subset $\mathcal{S}_P \subset \mathcal{S}$. We let $\mathcal{W}_P \subset \mathcal{W}$ be the subgroup generated by $\mathcal{S}_P$.

The length $l(x)$ of $x \in \mathcal{W}$ can also be interpreted as the number of walls separating the antidominant Weyl chamber C from xC. We will also need the "K-length" $l_K : \mathcal{W} \to \mathbb{Z}$, where $l_K(x)$ denotes the number of compact walls separating C from xC.

To any triple (A, D, c) consisting of a left $\mathcal{W}_K$-orbit $A \subset \mathcal{W}$, a right $\mathcal{W}_P$-orbit $D \subset \mathcal{W}$ and an integer c we associate the set

$$\mathcal{V}(A, D, c) = \{x \in A \cap D \mid l(x) = l_K(x) + c\}.$$

We will express the $\mathfrak{n}$-cohomology of limits of discrete series as sums of suitable $H^\bullet \mathcal{V}(A, D, c)$.

To write down precise formulas we need more precise notations. Let C be the Weyl chamber on which all coroots $\alpha^\vee$ with α a root of B take *negative* values. Let $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$ be the Lie algebras of $G \supset B \supset T$ and let $\mathfrak{n} \subset \mathfrak{b}$ be the nilradical. Let $U = U(\mathfrak{g}) \supset Z$ be the universal enveloping algebra and its center. For any integral central character $\chi \in \mathrm{Max}Z$ let T^χ denote the translation functor from the trivial central character to χ. Define the $\mathfrak{g}$-module $L(Y, \chi) = T^\chi \Gamma(G/B, i_* \mathcal{O}_Y)$ where $i : Y \hookrightarrow G/B$ is the inclusion of a closed K-orbit and i_* denotes the direct image of $\mathcal{D}$-modules. Every limit of discrete series occurs among the $L(Y, \chi)$. We want to compute $H^n(\mathfrak{n}, L(Y, \chi))$.

Now this space admits a natural action by $\mathfrak{h}$. Let us normalize the Harish-Chandra homomorphism $\xi^\sharp : Z \to S(\mathfrak{h})$ by requiring $\xi^\sharp(z) - z \in U\mathfrak{n}$. Let $\xi : \mathfrak{h}^* \to \mathrm{Max}Z$ be the corresponding map on points. Its fibres are the orbits for the dot-action of $\mathcal{W}$ on $\mathfrak{h}^*$, given by $x \cdot \lambda = x(\lambda + \rho) - \rho$ with ρ the half-sum of the roots of $\mathfrak{n}$. For any $\mathfrak{g}$-module L which is annihilated by χ we have the generalized eigenspace decomposition $H^n(\mathfrak{n}, L) = \bigoplus_{\lambda \in \xi^{-1}(\chi)} H^n(\mathfrak{n}, L)_\lambda$. We want to establish for our problem the following solution.

Theorem 2.1. *The λ-weight space in the $\mathfrak{n}$-cohomology of the limit of discrete series $L(Y, \chi)$ is given by the formula*

$$H^n(\mathfrak{n}, L(Y, \chi))_\lambda = \bigoplus_c H^{2c + \dim Y - n} \mathcal{V}(A, D, c)$$

where $A = A(Y) = \{x \in \mathcal{W} \mid xB \in Y\}$ *and* $D = D(\lambda) = \{x \in \mathcal{W} \mid x^{-1}(\lambda + \rho) \in \overline{C}\}$.

Note that somewhat implicitly in this is hidden the statement that we have to take $\mathcal{W}_P$ as the isotropy group (for the linear $\mathcal{W}$-action) of the unique weight in $\overline{C} \cap (\xi^{-1}(\chi) + \rho)$. The summands for different c correspond to parts of various "weight" in the $\mathfrak{n}$-cohomology, as will be explained in the proof of the theorem at the very end of this paper.

I don't know how to further simplify the right hand side in general. However I want to remark on a special case: If $\mathcal{W}_K$ is generated by $\mathcal{W}_K \cap \mathcal{S}$, i.e. for "$\mathfrak{n}$

holomorphic", we see that c is constant on each A, and for this c the complex $C^\bullet\mathcal{V}(A,D,c)$ will look like the analogous complex for the group generated by all compact reflections with fixed point $\lambda+\rho$. It will thus have zero cohomology if there are any such reflections, i.e. in the degenerate case. This recovers Corollary 4.8 of [Mir90].

3. Preliminaries on $\mathfrak{n}$-cohomology

Let $\mathfrak{g} = \overline{\mathfrak{n}} \oplus \mathfrak{h} \oplus \mathfrak{n}$ be a complex reductive Lie algebra with a triangular decomposition. Put $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$. Let $U(\mathfrak{g}) \supset Z$ be the enveloping algebra of $\mathfrak{g}$ and its center. For any ideal $\chi \subset Z$ put $U_\chi = U(\mathfrak{g})/\chi U(\mathfrak{g})$.

Lemma 3.1. *For any U_χ-module M we have*

$$H^i(\mathfrak{n}, M) = \mathrm{Ext}^i_{U_\chi}(U_\chi \otimes_{U(\mathfrak{n})} \mathbb{C}, M).$$

Proof. To compute the $\mathfrak{n}$-cohomology $H^i(\mathfrak{n}, M)$ of an $\mathfrak{n}$-module M we have to take a free resolution $P_\bullet \twoheadrightarrow \mathbb{C}$ of the trivial representation $\mathbb{C}$ of $\mathfrak{n}$ and compute the cohomology of the complex $\mathrm{Hom}_{\mathfrak{n}}(P_\bullet, M)$. If M is a $\mathfrak{g}$-module, this coincides with the complex $\mathrm{Hom}_{\mathfrak{g}}(U(\mathfrak{g}) \otimes_{U(\mathfrak{n})} P_\bullet, M)$. If $\chi \subset Z$ is an ideal with $\chi M = 0$, we can identify our complex also with the complex $\mathrm{Hom}_{U_\chi}(U_\chi \otimes_{U(\mathfrak{n})} P_\bullet, M)$.

Let $\mathcal{W}$ be the Weyl group of $\mathfrak{g}$. Then by Chevalley's theorem we can find a homogeneous subspace $H \subset U(\mathfrak{h})$ such that the multiplication gives an isomorphism $U(\mathfrak{h})^{\mathcal{W}} \otimes H \xrightarrow{\sim} U(\mathfrak{h})$, and by Kostant's theorem for any such H the multiplication gives an isomorphism $U(\overline{\mathfrak{n}}) \otimes Z \otimes H \otimes U(\mathfrak{n}) \xrightarrow{\sim} U(\mathfrak{g})$. In particular U_χ is a free right $U(\mathfrak{n})$-module, thus the complex $U_\chi \otimes_{U(\mathfrak{n})} P_\bullet$ is a resolution of $U_\chi \otimes_{U(\mathfrak{n})} \mathbb{C}$ by free U_χ-modules. Hence the cohomology of the complex $\mathrm{Hom}_{U_\chi}(U_\chi \otimes_{U(\mathfrak{n})} P_\bullet, M)$ is $\mathrm{Ext}^\bullet_{U_\chi}(U_\chi \otimes_{U(\mathfrak{n})} \mathbb{C}, M)$. □

Now we are interested in the $\mathfrak{n}$-cohomology of limits of discrete series. So consider for any $\chi \in \mathrm{Max} Z$ the category

$$\mathcal{M}_\chi = \{M \in \mathfrak{g}\text{-mod} \mid \forall\, m \in M,\ \exists n \gg 0 \text{ such that } \chi^n m = 0\}.$$

If ψ, $\chi \in \mathrm{Max}Z$ are integral, we define the translation functor $T_\psi^\chi : \mathcal{M}_\psi \to \mathcal{M}_\chi$ as usual. It comes with an adjointness $(T_\psi^\chi, T_\chi^\psi)$. Recall also the category

$$\mathcal{O} = \left\{ M \in \mathfrak{g}\text{-mod} \;\middle|\; \begin{array}{l} M \text{ is finitely generated,} \\ \text{locally finite over } \mathfrak{b} \text{ and} \\ \text{semisimple over } \mathfrak{h}. \end{array} \right\}$$

We put $\mathcal{O}_\psi = \mathcal{O} \cap \mathcal{M}_\psi$ for $\psi \in \mathrm{Max}Z$. We will also need its Z-diagonal version

$$\mathcal{O}'_\psi = \left\{ M \in \mathfrak{g}\text{-mod} \;\middle|\; \begin{array}{l} M \text{ is finitely generated,} \\ \text{locally finite over } \mathfrak{b} \text{ and} \\ \psi M = 0. \end{array} \right\}$$

Recall the Verma modules $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$ for $\lambda \in \mathfrak{h}^*$. We have $\mathrm{Ann}_Z M(\lambda) = \xi(\lambda)$. Now by [Soe89], 2.1, there exists for $\psi \in \mathrm{Max}Z$ regular an equivalence of categories

$$S = S_\psi : \mathcal{O}_\psi \xrightarrow{\sim} \mathcal{O}'_\psi$$

such that $SM(\lambda) \cong M(\lambda)$ for every $\lambda \in \xi^{-1}(\psi)$. Let us fix such equivalences. The following theorem will permit us to translate our problem of computing the $\mathfrak{n}$-cohomology of limits of discrete series into geometry.

Theorem 3.2. *Suppose $\psi, \chi \in \mathrm{Max} Z$ are integral and ψ is regular. Then for any $N \in U_\psi$-mod and $\lambda \in \xi^{-1}(\chi)$ we have*

$$H^i(\mathfrak{n}, T_\psi^\chi N)_\lambda \cong \mathrm{Ext}^i_{U_\psi}(ST_\chi^\psi M(\lambda), N).$$

Remark 3.3. For $\chi = \psi$ regular this boils down to the well-known formula

$$H^i(\mathfrak{n}, N)_\lambda \cong \mathrm{Ext}^i_{U_\chi}(M(\lambda), N).$$

Proof. We start with the equality

$$H^i(\mathfrak{n}, T_\psi^\chi N) = \mathrm{Ext}^i_{U_\chi}(U_\chi \otimes_{U(\mathfrak{n})} \mathbb{C}, T_\psi^\chi N) \tag{1}$$

from Lemma 3.1. Now we want to use the adjointness $(T_\chi^\psi, T_\psi^\chi)$. For this we need

Lemma 3.4. *Let T be any translation functor. Then the inclusion $\mathrm{Ann}_U M \subset \mathrm{Ann}_U N$ implies $\mathrm{Ann}_U TM \subset \mathrm{Ann}_U TN$.*

Proof. Left to the reader. □

Lemma 3.5. *Let $\psi, \chi \in \mathrm{Max} Z$ be integral and suppose ψ is regular. Then*

$$\mathrm{Ann}_U T_\chi^\psi M(\lambda) = U \mathrm{Ann}_Z T_\chi^\psi M(\lambda) \quad \textit{for all } \lambda$$

with $\chi M(\lambda) = 0$.

Proof. The inclusion $\supset$ is obvious. For the other inclusion note first that by the previous lemma we can assume without restriction $M(\lambda)$ simple. Then $T_\chi^\psi M(\lambda)$ has some simple Verma module as its unique simple quotient, call it $M(\mu)$. Consider now a Verma flag $T_\chi^\psi M(\lambda) = F_r \supset \ldots \supset F_1 \supset F_0 = 0$. We prove by induction from above on i that $\mathrm{Ann}_U(F_r/F_i) = U\mathrm{Ann}_Z(F_r/F_i)$. For $i = r-1$ we find $F_r/F_{r-1} \cong M(\mu)$ and our statement follows from a theorem of Duflo. Now we have $z \in \mathrm{Ann}_Z(F_r/F_i)$ if and only if $zF_r \subset F_i$. But it is clear that $\mathrm{Hom}_\mathfrak{g}(F_r, F_i/F_{i-1})$ is one-dimensional for every i. Since $\mathrm{Ann}_Z(F_r/F_{i-1})$ is the kernel of the obvious map $\mathrm{Ann}_Z(F_r/F_i) \to \mathrm{Hom}_\mathfrak{g}(F_r, F_i/F_{i-1})$, there exists $z_0 \in Z$ such that

$$\mathrm{Ann}_Z(F_r/F_i) = \mathrm{Ann}_Z(F_r/F_{i-1}) + \mathbb{C}z_0.$$

If $z_0(F_r) \subset F_{i-1}$, we are done. Otherwise z_0 has to induce an isomorphism of F_r/F_{r-1} onto the socle of F_i/F_{i-1}. Any $u \in \mathrm{Ann}_U(F_r/F_{i-1})$ annihilates F_r/F_i, hence can be written by induction in the form $u = u_1 + u_0 z_0$ with

$$u_1 \in U\mathrm{Ann}_Z(F_r/F_{i-1}), \qquad u_0 \in U.$$

But we deduce that u_0 annihilates the socle $M(\mu)$ of F_i/F_{i-1}, thus $u_0 = \tilde{u}_0 \tilde{z}_0$ with $\tilde{z}_0 \in \psi$ and $u = u_1 + \tilde{u}_0 \tilde{z}_0 z_0 \in U\mathrm{Ann}_Z(F_r/F_{i-1})$. □

Remarks 3.6.
1. This proof shows that the codimension in Z of the annihilator $\mathrm{Ann}_Z(T_\chi^\psi M(\lambda))$ is at most r alias the cardinality of the isotropy group W_λ of λ under the dot-action, $W_\lambda = \{w \in W \mid w(\lambda + \rho) = \lambda + \rho\}$.
2. The lemma and its proof use in fact neither integrality nor regularity. We just need that ψ lies on less walls than χ.

Let us put $\tilde{\psi} = \mathrm{Ann}_Z T_\chi^\psi M(\lambda)$ for any $\lambda \in \xi^{-1}(\chi)$. So $\tilde{\psi} \subset Z$ depends on χ, but by Lemma 3.4 it does not depend on the choice of $\lambda \in \xi^{-1}(\chi)$.

Lemma 3.7. *The translations T_χ^ψ, T_ψ^χ restrict to functors*

$$\begin{array}{rcccl} T_\chi^\psi & : & U_\chi\text{-mod} & \to & U_{\tilde{\psi}}\text{-mod}, \\ T_\psi^\chi & : & U_{\tilde{\psi}}\text{-mod} & \to & U_\chi\text{-mod}. \end{array}$$

Proof. We only prove the second statement, the proof of the first being analogous and easier. Note that $T_\psi^\chi T_\chi^\psi M(\lambda)$ is just a direct sum of some copies of $M(\lambda)$, for every $\lambda \in \xi^{-1}(\chi)$. So

$$\begin{array}{rcl} N \in U_{\tilde{\psi}}\text{-mod} & \Rightarrow & \mathrm{Ann}_U N \supset \mathrm{Ann}_U T_\chi^\psi M(\lambda) = U\tilde{\psi} \\ & \Rightarrow & \mathrm{Ann}_U T_\psi^\chi N \supset \mathrm{Ann}_U T_\psi^\chi T_\chi^\psi M(\lambda) = U\chi \\ & \Rightarrow & T_\psi^\chi N \in U_\chi\text{-mod}. \quad \square \end{array}$$

Now if $P \in U_\chi$-mod is projective, so is $T_\chi^\psi P \in U_{\tilde{\psi}}$-mod. Indeed, $\mathrm{Hom}_{U_{\tilde{\psi}}}(T_\chi^\psi P, N) = \mathrm{Hom}_{U_\chi}(P, T_\psi^\chi N)$ is exact in $N \in U_{\tilde{\psi}}$-mod. So we can continue our equality from the beginning and get

$$H^i(\mathfrak{n}, T_\psi^\chi N) = \mathrm{Ext}^i_{U_\chi}(U_\chi \otimes_{U(\mathfrak{n})} \mathbb{C}, T_\psi^\chi N) \tag{1}$$

$$= \mathrm{Ext}^i_{U_{\tilde{\psi}}}(T_\chi^\psi(U_\chi \otimes_{U(\mathfrak{n})} \mathbb{C}), N). \tag{2}$$

To proceed further, we need

Lemma 3.8. *$T_\chi^\psi(U_\chi \otimes_{U(\mathfrak{n})} \mathbb{C})$ is free over $Z/\tilde{\psi}$.*

Proof. As explained in [MS95], 2.4, we have an isomorphism

$$U_\chi \otimes_{U(\mathfrak{n})} \mathbb{C} \cong U \otimes_{U(\mathfrak{b})} (U(\mathfrak{h})/U(\mathfrak{h})\xi^\sharp(\chi))$$

since both objects share a universal property. In addition by invariant theory we find that

$$U(\mathfrak{h})/U(\mathfrak{h})\xi^\sharp(\chi) = \bigoplus_{\xi(\lambda)=\chi} U(\mathfrak{h})/\tilde{\lambda}$$

where $\tilde{\lambda} \subset U(\mathfrak{h})$ are suitable ideals with radical $\sqrt{\tilde{\lambda}} = \ker \lambda \subset U(\mathfrak{h})$ and codimension $\dim_{\mathbb{C}}(U(\mathfrak{h})/\tilde{\lambda}) = |\mathcal{W}_\lambda|$ the cardinality of the isotropy group of λ under the dot-action. Let us define for any ideal $I \subset U(\mathfrak{h})$ the "thick Verma" $M(I) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} U(\mathfrak{h})/I$. Clearly $\mathrm{Ann}_Z M(I) = (\xi^\sharp)^{-1}(I)$. For any $\mathfrak{g}$-module E we have $E \otimes M(I) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (E \otimes U(\mathfrak{h})/I)$. If E is finite dimensional, its restriction to $\mathfrak{b}$ admits a filtration with one-dimensional subquotients k_ν where ν runs over the multiset $P(E) \subset \mathfrak{h}^*$ of weights of E. Thus $E \otimes M(I)$ has a filtration with subquotients $M(I+\nu)$ for ν running over $P(E)$, in obvious notations. In these terms we find

$$U_\chi \otimes_{U(\mathfrak{n})} \mathbb{C} = \bigoplus_{\xi(\lambda)=\chi} M(\tilde{\lambda})$$

and we deduce

$$T_\chi^\psi(U_\chi \otimes_{U(\mathfrak{n})} \mathbb{C}) = \bigoplus_{\xi(\lambda)=\chi} T_\chi^\psi M(\tilde{\lambda}).$$

In addition each of the summands has a filtration with subquotients of the form $M(\tilde{\mu})$ for suitable ideals $\tilde{\mu} \subset U(\mathfrak{h})$ of codimension $|W_\lambda|$ with radical $\sqrt{\tilde{\mu}} \in \xi^{-1}(\psi)$. Now it is certainly sufficient to show that these "thick Vermas" $M(\tilde{\mu})$ are free over $Z/\tilde{\psi}$. For this, note first that we have $\tilde{\psi} \subset \mathrm{Ann}_Z M(\tilde{\mu}) = (\xi^\sharp)^{-1}(\tilde{\mu})$. Now $\tilde{\psi} \subset Z$ has codimension at most $|\mathcal{W}_\lambda|$ by Remark 3.6 (1) following the proof of Lemma 3.5. On the other hand $(\xi^\sharp)^{-1}(\tilde{\mu}) \subset Z$ has the same codimension as $\tilde{\mu} \subset U(\mathfrak{h})$, since ξ is étale in $\sqrt{\mu}$, and this codimension we know already to be $|\mathcal{W}_\lambda|$. Comparing we find $\tilde{\psi} = (\xi^\sharp)^{-1}(\tilde{\mu})$. Now it is clear from the isomorphism

$$M(\tilde{\mu}) = U(\bar{\mathfrak{n}}) \otimes U(\mathfrak{h})/\tilde{\mu}$$

that this module is free over $Z/\tilde{\psi}$. $\square$

But this lemma means that a resolution of $T_\chi^\psi(U_\chi \otimes_{U(\mathfrak{n})} \mathbb{C})$ by free $U_{\tilde{\psi}}$-modules stays exact when we apply $\otimes_Z Z/\psi$. So we can go on and write

$$H^i(\mathfrak{n}, T_\psi^\chi N) = \mathrm{Ext}^i_{U_\chi}(U_\chi \otimes_{U(\mathfrak{n})} \mathbb{C}, T_\psi^\chi N) \tag{1}$$

$$= \mathrm{Ext}^i_{U_{\tilde{\psi}}}(T_\chi^\psi(U_\chi \otimes_{U(\mathfrak{n})} \mathbb{C}), N) \tag{2}$$

$$= \mathrm{Ext}^i_{U_\psi}(T_\chi^\psi(U_\chi \otimes_{U(\mathfrak{n})} \mathbb{C}) \otimes_Z Z/\psi, N) \tag{3}$$

if $\psi N = 0$, as assumed in our theorem. So to finish the proof of the theorem, we only have to show (in the notations from above):

Proposition 3.9. *$ST_\chi^\psi M(\lambda) \cong T_\chi^\psi(M(\tilde{\lambda})) \otimes_Z Z/\psi$.*

Proof. We first show that the dimensions of (generalized) $\mathfrak{h}$-weight spaces on both sides coincide. Indeed, both sides have a Verma flag with subquotients $M(\mu)$ where μ runs over the set

$$\Lambda = \{\mu \in \xi^{-1}(\psi) \mid T_\psi^\chi M(\mu) \cong M(\lambda)\}.$$

For the left hand side this is evident, for the right it follows from the existence of a filtration of $T_\chi^\psi M(\tilde{\lambda})$ by certain $M(\tilde{\mu})$ which appeared in the proof of Lemma 3.8. So we just have to establish a surjection $T_\chi^\psi M(\tilde{\lambda}) \twoheadrightarrow ST_\chi^\psi M(\lambda)$. Now $ST_\chi^\psi M(\lambda)$ has a unique simple quotient L, and $T_\psi^\chi L$ is the unique simple quotient $L(\lambda)$ of $M(\lambda)$. But by definition of $M(\tilde{\lambda})$ we can identify for all $M \in U_\chi$-mod our $\mathrm{Hom}_{\mathfrak{g}}(M(\tilde{\lambda}), M)$ with the generalized $\mathfrak{h}$-weight space in $M^{\mathfrak{n}}$ of weight λ. Since $T_\psi^\chi ST_\chi^\psi M(\lambda)$ is just a successive extension of copies of $M(\lambda)$, this means that the map

$$\begin{array}{c} \mathrm{Hom}_{\mathfrak{g}}(M(\tilde{\lambda}), T_\psi^\chi ST_\chi^\psi M(\lambda)) \\ \downarrow \\ \mathrm{Hom}_{\mathfrak{g}}(M(\tilde{\lambda}), T_\psi^\chi L) \end{array}$$

has to be a surjection, and taking the adjoint we find the surjection $T_\chi^\psi M(\tilde{\lambda}) \twoheadrightarrow ST_\chi^\psi M(\lambda)$ as needed.

This finishes the proof of the proposition and the theorem. □

4. Localization of our problem

Let $Z^+ \subset Z$ be the annihilator of the trivial representation $\mathbb{C}$ of $\mathfrak{g}$. By the famous localization theorem of [BB81], the action of G on its flag variety X induces an isomorphism $U/Z^+U \xrightarrow{\sim} \Gamma(X, \mathcal{D}_X)$ from U/Z^+U to the global (algebraic) differential operators on X, and the functor of global sections

$$\Gamma : \mathcal{D}_X\text{-mod} \to U/Z^+U\text{-mod}$$

is an equivalence of categories. The inverse functor is denoted Δ and called "localization". To apply this theory to our problem we have to understand the localization of the objects $ST_\chi^\psi M(\lambda)$ appearing in Theorem 3.2 for $\psi = Z^+$. Recall that here λ is an integral weight and $\chi = \xi(\lambda)$ its image in Max$Z$. Let us abbreviate $T_\chi^\psi = T_\chi$, $T_\psi^\chi = T^\chi$ for $\psi = Z^+$. Let us take $D = D(\lambda) \subset \mathcal{W}$ as in Theorem 2.1, $O = BDB/B$, $\mathcal{A} = \mathcal{A}(D) \in \mathcal{P}_B(O)$ the projective cover of the unique simple standard object of $\mathcal{P}_B(O)$ and $j : O \hookrightarrow G/B$ the inclusion.

We will freely use the Riemann-Hilbert correspondence to pass between regular holonomic $\mathcal{D}$-modules and perverse sheaves. We want to prove

Proposition 4.1. $\Delta ST_\chi M(\lambda) \cong j_! \mathcal{A}$.

Proof. This is a remodeling of the proof of Lemma 10 in [Soe89]. Note first that $j_! : \mathcal{P}_B(O) \to \mathcal{P}_B(G/B)$ is a fully faithful exact functor by [BBD82], and it is even fully faithful on higher Ext, since by e.g. [BGS96], 3.3, these coincide with the Ext in the derived categories of all $\mathcal{D}$-modules (or all sheaves, in the other picture). For $\{M_i \mid i \in I\}$ a collection of objects from some abelian category let $\langle M_i \mid i \in I \rangle$ denote the smallest full subcategory closed under kernels, cokernels and extensions containing the M_i. For $x \in \mathcal{W}$ let $\mathcal{M}_x \in \mathcal{P}_B(G/B)$ denote the localization of the Verma module $\mathcal{M}_x = \Delta M_x$ where we put $M_x = M(-x\rho - \rho)$. This is just the dual standard object $\mathcal{M}_x = j_{x!}\mathcal{O}_{BxB/B}$ corresponding to the cell $j_x : BxB/B \hookrightarrow G/B$. It is then clear that we have $j_! \mathcal{P}_B(O) = \langle \mathcal{M}_x \mid x \in D \rangle$ as subcategories of $\mathcal{P}_B(G/B)$.

Let $y \in D$ be the shortest element. Clearly $j_! \mathcal{A}$ is the projective cover of $\mathcal{M}_y$ in $\langle \mathcal{M}_x \mid x \in D \rangle$. We will be done if we show that $T_\chi M(\lambda)$ is the projective cover of M_y in the subcategory $\langle M_x \mid x \in D \rangle$ of $\mathcal{O}$, since under the functor ΔS this corresponds to the category $\langle \mathcal{M}_x \mid x \in D \rangle$ considered above.

Now $T_\chi M(\lambda)$ admits a filtration with its subquotients among the $M_x, x \in D$ (each appearing once), so it already lies in $\langle M_x \mid x \in D \rangle$. For any $M \in \langle M_x \mid x \in D \rangle$ we know that $T^\chi M$ is a direct sum of copies of $M(\lambda)$, since $T^\chi M_x \cong M(\lambda)$ for all $x \in D$, $\mathrm{End} M(\lambda) = \mathbb{C}$ and the $M(\lambda)$ admit no selfextensions in $\mathcal{O}$. For $M \in \langle M_x \mid x \in D \rangle$ we have

$$\begin{aligned} [M : M_y] &= \dim \mathrm{Hom}(M(\lambda), T^\chi M) \\ &= \dim \mathrm{Hom}(T_\chi M(\lambda), M) \end{aligned}$$

since this holds for $M = M_x, x \in D$, and since both sides are additive on short exact sequences. Thus indeed $T_\chi M(\lambda)$ is the projective cover of M_y in $\langle M_x \mid x \in D \rangle$ and the proposition is proved. □

Corollary 4.2. *$\mathcal{A}$ is selfdual.*

Proof. This is in fact a corollary of the preceding proof. Let us denote by $\mathbb{D}$ the Verdier duality functor on any variety. It is certainly sufficient to prove $j_! \mathbb{D}\mathcal{A} \cong j_! \mathcal{A}$. Now clearly the left hand side is the injective hull of $\mathcal{M}_y$ in $\langle \mathcal{M}_x \mid x \in D \rangle$. So we should prove the same thing for the right hand side, or equivalently prove that $T_\chi M(\lambda)$ is the injective hull of M_y in the subcategory $\langle M_x \mid x \in D \rangle$ of $\mathcal{O}$. But as before we find for any $M \in \langle M_x \mid x \in D \rangle$ the equality

$$\begin{aligned} [M : M_y] &= \dim \mathrm{Hom}(T^\chi M, M(\lambda)) \\ &= \dim \mathrm{Hom}(M, T_\chi M(\lambda)) \end{aligned}$$

and the statement follows. □

The proposition allows us to rewrite Theorem 3.2 in a more geometric form.

Corollary 4.3. *Let $\chi \in \mathrm{Max} Z$ be integral, $\lambda \in \xi^{-1}(\chi)$ and N a representation of $\mathfrak{g}$ such that $Z^+ N = 0$. Then*

$$H^n(\mathfrak{n}, T^\chi N)_\lambda \cong \mathrm{Ext}^n_{\mathcal{D}}(j_! \mathcal{A}, \Delta N).$$

Here $\mathrm{Ext}_{\mathcal{D}}$ stands for extensions in the category of all $\mathcal{D}$-modules and $\mathcal{A}$ is the $\mathcal{D}$-module depending on λ considered above.

If N is a discrete series or more generally $N = \Gamma(X, i_* \mathcal{O}_Y)$ for $i : Y \hookrightarrow G/B$ the embedding of a smooth closed subvariety, we can rewrite this still further. First let us pass to the language of perverse sheaves. Let $\underline{Y}$ denote the constant sheaf on

Y. Then the $\mathcal{D}$-module $\mathcal{O}_Y$ corresponds to the perverse sheaf $\underline{Y}[\dim Y]$ and letting now $\mathrm{Hom}_{\mathcal{D}}$ denote homomorphisms in the derived category of sheaves we find

$$\begin{aligned} H^n(\mathfrak{n}, T^\chi N)_\lambda &= \mathrm{Hom}_{\mathcal{D}}(j_!\mathcal{A}, i_*\underline{Y}[n+\dim Y]) \\ &= \mathrm{Hom}_{\mathcal{D}}(i_!\underline{Y}, j_*\mathcal{A}[n-\dim Y]) \\ &= \mathrm{Hom}_{\mathcal{D}}(\underline{Y}, i^! j_*\mathcal{A}[n-\dim Y]) \\ &= \mathbb{H}^{n-\dim Y}(Y, i^! j_*\mathcal{A}) \end{aligned}$$

where we used the selfduality of $\mathcal{A}$ (Corollary 4.2) in the second step. For easy reference, we reformulate our overall findings as

Theorem 4.4. *For $Y \subset G/B$ an irreducible smooth closed subvariety, χ an integral central character, $\lambda \in \xi^{-1}(\chi)$ and $\mathcal{A}$ as above we have*

$$H^n(\mathfrak{n}, L(Y,\chi))_\lambda \cong \mathbb{H}^{n-\dim Y}(Y, i^! j_*\mathcal{A}).$$

Remark 4.5. In fact this holds more generally for arbitrary affine embeddings $i : Y \hookrightarrow G/B$ of smooth varieties Y, when we interpret $L(Y,\chi) = T^\chi\Gamma(X, i_*\mathcal{O}_Y)$.

5. Decomposition by codimension

To better understand the right hand side of Theorem 4.4, we break it into pieces. Let $Y \subset X$ be a closed K-orbit and $i : Y \hookrightarrow X$ the inclusion. By our assumptions $B \cap K$ is a Borel in K, each closed K-orbit Y in X is isomorphic to the flag manifold of K, and each B-orbit in X meets Y in a $(B \cap K)$-orbit or not at all. For $x \in \mathcal{W}$ let us define $c_Y(x) = c(x) \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ to be the codimension of $Y \cap \overline{BxB/B}$ in $\overline{BxB/B}$, resp. ∞ if $Y \cap \overline{BxB/B} = \emptyset$. We obtain a filtration of Y by the closed subsets $Y_c = \bigcup_{c(x)\leq c} Y \cap \overline{BxB/B}$, and it is clear that we have $Y_c = \bigcup_{c(x)\leq c} Y \cap BxB/B$ as well. Consider now the decomposition $Y_c = Y_{c-1} \cup (Y_c - Y_{c-1})$ into a closed subset and its open complement. Let i_c, j_c be the corresponding closed and open immersions and let a_c, resp. u_c, denote the embedding of any Y_c, resp. $Y_c - Y_{c-1}$, into Y. For a variety Y let $\mathcal{D}(Y)$ denote the bounded derived category with algebraically constructible cohomology of sheaves of $\mathbb{C}$-vector spaces on Y^{an}. Certainly for $\mathcal{F} \in \mathcal{D}(Y)$ we have distinguished triangles

$$(i_{c!} i_c^! a_c^! \mathcal{F}, a_c^! \mathcal{F}, j_{c*} j_c^* a_c^! \mathcal{F}).$$

If we apply k_* (where $k : Y \to \mathrm{pt}$ is the constant map to a point) and rewrite somewhat the result, we get distinguished triangles

$$(k_* a_{c-1}^! \mathcal{F}, k_* a_c^! \mathcal{F}, k_* u_c^! \mathcal{F})$$

in $\mathcal{D}(\mathrm{pt})$. The point of this section is to prove that in case $\mathcal{F} = i^! j_* \mathcal{A}$ with $\mathcal{A} = \mathcal{A}(D)$ as above, all these triangles are in fact split short exact sequences, thus leading to a direct sum decomposition.

Lemma 5.1. $\mathbb{H}^\bullet i^! j_* \mathcal{A} = \bigoplus_c \mathbb{H}^\bullet u_c^! i^! j_* \mathcal{A}.$

Proof. To see this, we need weights. As is proved in [BGS96], the antidominant projective $\mathcal{A}(\mathcal{W})$ on the whole of G/B admits a mixed structure (in the sense of mixed Hodge modules, say) such that it admits a filtration where the successive subquotients are isomorphic to

$$j_{z*}\underline{BzB/B}[l(z)]$$

with z running over the Weyl group. Here for Z a variety we let $\underline{Z}$ denote the constant sheaf with trivial mixed structure. From the geometric constructions in

[Soe89], Lemmas 8 and 9, we deduce that similarily $j_*\mathcal{A}$ admits a mixed structure with filtrations by subquotients $j_{z*}\underline{BzB/B}[l(z)]$ where z runs over D. In general, if $a : Y \to Z$ is a morphism of smooth varieties, then $a^!\underline{Z} = \underline{Y}[-2c](-c)$ with $c = \dim Z - \dim Y$ the codimension.

Using base change, we see that for $z \in D$ all

$$\mathbb{H}^\nu u_c^! i^! j_{z*}\underline{BzB/B}[l(z)]$$

are pure of weight $2c$ (or zero). Whence all $\mathbb{H}^\nu u_c^! i^! j_*\mathcal{A}$ are pure of weight $2c$, and the Lemma follows. □

6. Investigation of some spectral sequences

To investigate the pieces $\mathbb{H}^\bullet u_c^! i^! j_*\mathcal{A}$ we begin with some generalities. Let Z be a variety, and $Z = Z_{-\infty} \supset \ldots \supset Z_p \supset Z_{p+1} \supset \ldots \supset Z_n = \emptyset$ a (finite) filtration by closed subvarieties. Let $i_p : Z_p - Z_{p+1} \hookrightarrow Z$ denote the inclusion. Let $\mathcal{F}$ be a complex of sheaves on Z. As explained in [BGS96], 3.4, the hypercohomology $\mathbb{H}^n\mathcal{F}$ is the limit of a spectral sequence with E_1-term $E_1^{p,q} = \mathbb{H}^{p+q} i_p^! \mathcal{F}$. Its differential $E_1^{p,q} \to E_1^{p+1,q}$ can be described as follows: Consider the decomposition of $Z_p - Z_{p+2}$ into an open and a closed subset

$$Z_p - Z_{p+1} \overset{u}{\hookrightarrow} Z_p - Z_{p+2} \overset{a}{\hookleftarrow} Z_{p+1} - Z_{p+2}$$

and let $j : Z_p - Z_{p+2} \hookrightarrow Z$ be the inclusion. Then we have a distinguished triangle $(a_! a^! j^! \mathcal{F}, j^! \mathcal{F}, u_* u^* j^! \mathcal{F}) = (a_* i_{p+1}^! \mathcal{F}, j^! \mathcal{F}, u_* i_p^! \mathcal{F})$ giving rise to boundaries $\mathbb{H}^n i_p^! \mathcal{F} \to \mathbb{H}^{n+1} i_{p+1}^! \mathcal{F}$ which are the boundaries of our spectral sequence.

Now we go to a much more special situation. Namely Z should admit a stratification by affine spaces $Z_v \cong \mathbb{A}^{l(v)}, v \in I$ for I some index set such that $Z_p = \bigcup_{l(v) \le -p} Z_v$. We suppose furthermore that $\mathcal{F}$ is actually a perverse sheaf and admits a filtration with subquotients isomorphic to the standard objects $j_{v*}\underline{Z}_v[l(v)]$ where $j_v : Z_v \hookrightarrow Z$ is the inclusion. To simplify the exposition let us assume in addition that each standard module occurs at most once as a subquotient of $\mathcal{F}$, and let $V \subset I$ be the parameters of those standard modules which actually occur. By rearranging our filtration if necessary, we can find a filtration

$$\mathcal{F} = \mathcal{F}_{-\infty} \supset \ldots \supset \mathcal{F}_0 \supset \mathcal{F}_1 = 0$$

such that $\mathcal{F}_p/\mathcal{F}_{p+1} = \bigoplus_{l(v)=-p} \mathcal{N}_v$ where each $\mathcal{N}_v$ is isomorphic to $j_{v*}\underline{Z}_v[l(v)]$. Note that since between different standard objects with same $l(v)$ there are no nonzero homomorphisms, the $\mathcal{N}_v$ are well defined as subobjects of $\mathcal{F}_p/\mathcal{F}_{p+1}$. There is however no canonical isomorphism between $\mathcal{N}_v$ and $j_{v*}\underline{Z}_v[l(v)]$.

The object $\mathcal{F}$ determines for every $v, w \in V$ with $l(v) = l(w) + 1$ an extension of perverse sheaves $e_{w,v}(\mathcal{F}) \in \mathrm{Hom}_{\mathcal{D}}(\mathcal{N}_v, \mathcal{N}_w[1])$, namely the extension "realized" in the short exact sequence $\mathcal{F}_{p+1}/\mathcal{F}_{p+2} \hookrightarrow \mathcal{F}_p/\mathcal{F}_{p+2} \to \mathcal{F}_p/\mathcal{F}_{p+1}$. Let us get back to our spectral sequence. Note that

$$i_p^! \mathcal{F} = i_p^! (\mathcal{F}_p/\mathcal{F}_{p+1}) = \bigoplus_{l(v)=-p} i_p^! \mathcal{N}_v.$$

Now $\mathbb{H}^n i_p^! \mathcal{N}_v = \mathbb{H}^n \mathcal{N}_v$ is one-dimensional if $n = l(v)$, and zero else. Thus $\mathbb{H}^{p+q} i_p^! \mathcal{F} = 0$ for $q \neq 0$, whereas $\mathbb{H}^p i_p^! \mathcal{F} = \bigoplus_{l(v)=-p} \mathbb{H}^p \mathcal{N}_v$. So our spectral sequence lives just on one line and therefore degenerates at the E_2-stage. Furthermore we see that its differential

$$\bigoplus_{l(v)=-p} \mathbb{H}^p\mathcal{N}_v \to \bigoplus_{l(w)=-p-1} \mathbb{H}^{p+1}\mathcal{N}_w$$

is just given by the matrix $e_{w,v}(\mathcal{F})$ defined above.

7. Restricting the spectral sequence

In addition to the assumptions of the previous section suppose now we are given a locally closed subvariety $u : U \hookrightarrow Z$ (think of $U = Y_c - Y_{c-1}$) such that the nonempty intersections $U_v = Z_v \cap U$ form a stratification of U by affine spaces $U_v = \mathbb{A}^{l(v)-c}$ for some fixed $c \geq 0$. We let $J = \{v \in I \mid Z_v \cap U \neq \emptyset\}$ be the subset of I parametrizing this stratification of U, and let $\hat{\jmath}_v : U_v \hookrightarrow U$ be the inclusion of the strata. Then by base change

$$u^! j_{v*}\underline{Z}_v[l(v)] = \begin{cases} \hat{\jmath}_{v*}\underline{U}_v[l(v) - 2c](-c) & \text{if } U \cap Z_v \neq \emptyset; \\ 0 & \text{else.} \end{cases}$$

In particular, if $\mathcal{F}$ is as before a perverse sheaf on Z which can be written as a successive extension of standard objects $\mathcal{N}_v$ with $v \in V$, then $u^!\mathcal{F}[c]$ is a perverse sheaf on U and a successive extension of the standard objects $u^!\mathcal{N}_v$ with $v \in V \cap J$. Furthermore $\mathbb{H}^p\mathcal{N}_v = \mathbb{H}^{p+c}u^!\mathcal{N}_v[c](c)$ canonically for $v \in V \cap J$. Indeed any choice of an isomorphism $\mathcal{N}_v \cong j_{v*}\underline{Z}_v$ leads to an identification of both sides with $\mathbb{C}$, and the composition of these two identifications is independent of the chosen isomorphism. The problem now is to determine for $v, w \in V \cap J$ with $l(v) = -p$, $l(w) = -p - 1$ the factor $a_{w,v} \in \mathbb{C}$ such that the diagram

$$\begin{array}{ccc} \mathbb{H}^p\mathcal{N}_v & \longrightarrow & \mathbb{H}^{p+1}\mathcal{N}_w \\ \| & & \| \\ \mathbb{H}^{p+c}u^!\mathcal{N}_v[c](c) & \longrightarrow & \mathbb{H}^{p+1+c}u^!\mathcal{N}_w[c](c) \end{array}$$

commutes when we take as the upper horizontal $e_{w,v}(\mathcal{F})$ and as the lower one $a_{w,v}e_{w,v}(u^!\mathcal{F}[c])$. I want to explain why $a_{w,v}$ is one, if U is in a suitable sense transversal to the stratification on Z. More precisely we need

Proposition 7.1. *Let $v, w \in V \cap J$ be given with $l(v) = l(w) + 1$. Suppose that $Z_{v,w} := Z_v \cup Z_w$ and $U_{v,w} := U_v \cup U_w$ are open smooth subvarieties of $\overline{Z_v}$ and $\overline{U_v}$ respectively. Suppose further that Z_w and $U_{v,w}$ meet transversally in some point $x \in U_w$, i.e. $T_xU_w = T_xZ_w \cap T_xU_{v,w}$. Then $a_{w,v} = 1$.*

Remark 7.2. Since the codimension of T_xU_w in $T_xU_{v,w}$ is one, the transversality condition could as well be written $T_xU_{v,w} \not\subset T_xZ_w$.

Proof. Take $E = \{z \in \mathbb{C} \mid |z| < 1\}$. It is possible to find a holomorphic embedding $d : E \hookrightarrow U_{v,w}$ such that $d(0) = x$ and that the image of T_0E and T_xU_w together span $T_xU_{v,w}$; in words, the image of E in $U_{v,w}$ is transversal to U_w. By our assumptions on the geometry the composition $E \hookrightarrow U_{v,w} \hookrightarrow Z_{v,w}$ has analogously an image in $Z_{v,w}$ which is transversal to Z_w. We will also denote by d the composition $E \hookrightarrow$

$U_{v,w} \hookrightarrow U$. Put $p = -l(v)$ and $q = -p - 1$. Then I claim a commutative diagram

$$\begin{array}{ccc} \mathbb{H}^p \mathcal{N}_v & \stackrel{e_{w,v}(\mathcal{F})}{\longrightarrow} & \mathbb{H}^{p+1} \mathcal{N}_w \\ \| & & \| \\ \mathbb{H}^{-1} d^! u^! \mathcal{N}_v[q](q) & \stackrel{e_{w,v}(d^! u^! \mathcal{F}[q])}{\longrightarrow} & \mathbb{H}^0 d^! u^! \mathcal{N}_w[q](q) \\ \| & & \| \\ \mathbb{H}^{p+c} u^! \mathcal{N}_v[c](c) & \stackrel{e_{w,v}(u^! \mathcal{F}[c])}{\longrightarrow} & \mathbb{H}^{p+c+1} u^! \mathcal{N}_w[c](c) \end{array}$$

where all four vertical equalities are analogs of our previous canonical isomorphism in an analytic situation, and the composition of two vertical equalities is just exactly our previous canonical isomorphism. To check the commutativity of both little squares, then, is work in local coordinates, and commutativity of the big square is just our claim. I have good reasons to leave the details to the reader. □

8. Proof of the main result

In the first sections we transformed the problem of computing the $\mathfrak{n}$-cohomology of limits of discrete series representations into the geometric problem of computing the hypercohomology groups $\mathbb{H}^\bullet(u_c^! i^! j_* \mathcal{A})$ for $\mathcal{A} = \mathcal{A}(D)$ as in Proposition 4.1. Now we attack this problem with the methods developed in the two preceding sections.

Let us first have a look at the spectral sequence computing $\mathbb{H}^\bullet(j_*\mathcal{A})$. Note that $j_*\mathcal{A}$ is an object as the $\mathcal{F}$ considered in Section 6, with corresponding $\mathcal{N}_x$ for $x \in D$. I claim that for $x, y \in D$ with $x < y$ and $l(y) = l(x) + 1$, the induced map $e_{x,y} = e_{x,y}(j_*\mathcal{A})$ from $\mathbb{H}^{-l(y)}\mathcal{N}_y$ to $\mathbb{H}^{-l(x)}\mathcal{N}_x$ does not vanish. Indeed, consider the locally closed subset $V = ByB/B \cup BxB/B$ of the flag manifold and let $a : V \to G/B$ be the inclusion. Since Schubert varieties are normal (see e.g. [Jan83] II, 14.15), our V is smooth. Then $a_! \underline{V}[l(y)]$ is just the cokernel of the inclusion of the !-standard objects corresponding to the Bruhat cells of x and y, thus

$$\begin{aligned} 0 &= \operatorname{Hom}_{\mathcal{D}}(a_! \underline{V}[l(y)], j_*\mathcal{A}) \\ &= \mathbb{H}^{-l(y)} a^! j_* \mathcal{A}. \end{aligned}$$

On the other hand, if we try to compute the hypercohomology of $a^! j_* \mathcal{A}$ by restricting the spectral sequence as in Section 7, we find that it is the cohomology of the complex

$$\ldots 0 \longrightarrow \mathbb{H}^{-l(y)}\mathcal{N}_y \xrightarrow{e_{x,y}} \mathbb{H}^{-l(x)}\mathcal{N}_x \longrightarrow 0 \ldots$$

and hence $e_{x,y} \neq 0$.

Furthermore it is clear that for every small square $(x; y, w; z)$ from D we have $e_{x,y} \circ e_{y,z} = -e_{x,w} \circ e_{w,z}$ as maps from $\mathbb{H}^{-l(z)}\mathcal{N}_z$ to $\mathbb{H}^{-l(x)}\mathcal{N}_x$, since the square of the differential of our spectral sequence computing the hypercohomology of $j_*\mathcal{A}$ has to vanish. These two properties already determine our spectral sequence up to isomorphism. In fact, we have more generally

Lemma 8.1. *Let $(\mathcal{W}, \mathcal{S})$ be a finite crystallographic Coxeter group, $l : \mathcal{W} \to \mathbb{N}$ its length function. Let $\mathcal{W}(i) = \{x \in \mathcal{W} \mid l(x) = i\}$ and let $\mathbb{C}\mathcal{W}(i)$ be the free vector space with basis $\mathcal{W}(i)$ over $\mathbb{C}$. Suppose we are given for any $x < y$ with $l(y) = l(x) + 1$ a constant $e_{x,y} \in \mathbb{C}^\times$ such that $e_{x,y}e_{y,w} = -e_{x,z}e_{z,w}$ for every small square $(x; y, z; w)$ from $\mathcal{W}$.*

Then the matrix of all $e_{y,x}$ determines a differential on the graded space $\bigoplus \mathbb{C}\mathcal{W}(i)$ making it into a complex, and any other choice of constants $e'_{y,x}$ leads to an isomorphic complex.

Proof. I am quite unhappy to know only a silly overkill proof of this purely combinatorial result. However, one can argue as follows:

Any choice of $e_{y,x}$ determines a differential for a BGG-resolution of the trivial one-dimensional $\mathfrak{g}$-module

$$\ldots \to C^1 \to C^0 \to \mathbb{C}$$

with $C^i = \bigoplus_{x \in \mathcal{W}(i)} M(x \cdot 0)$. But any two such resolutions are isomorphic to a canonical one, where the C^i are rather given inductively as sums of Verma modules with highest weight spaces certain one-dimensional weight spaces of $\ker(C^{i-1} \to C^{i-2})$.

Now an isomorphism between the complexes coming from the $e_{y,x}$ and certain other $e'_{y,x}$ is necessarily given by scalars $a_x \in \mathbb{C}^\times$ on $M(x \cdot 0)$. These scalars then give the isomorphisms of complexes claimed by the Lemma. □

Now we restrict the spectral sequence calculating $\mathbb{H}^\bullet(j_*\mathcal{A})$ via $u_c^! i^!$. We want to apply Proposition 7.1 and have to show some transversality of K-orbits with respect to the stratification by Bruhat cells. For $v \in W$ let $X_v = BvB/B$ denote the corresponding Bruhat cell. If $w \in W$ is given such that $v > w$ and $l(v) = l(w) + 1$, then $X_{v,w} := X_v \cup X_w$ is open smooth in $\overline{X_v}$, as follows from the normality of Schubert varieties. If $U = Y_c - Y_{c-1}$ meets both X_v and X_w, the intersections $U_v = U \cap X_v$ and $U_w = U \cap X_w$ are again neighbouring Bruhat cells in Y (which is isomorphic to the flag variety of K), whence $U_{v,w} = U_v \cup U_w$ is also open smooth in $\overline{U_v}$. So to apply Proposition 7.1 to our situation we just need

Lemma 8.2. *$T_x U_{v,w}$ is not contained in $T_x X_w$, for some $x \in U_w$.*

Proof. Certainly $T_x U_{v,w}$ lies in $T_x Y$, so it will be sufficient to show $T_x X_w \cap T_x Y = T_x U_w$. Here X_w, Y and U_w are respectively the orbit of x under B, K and $B \cap K$. We may choose x as the fixed point of our maximal torus T which lies in B and K. Then the isotropy group G_x of x in G is also a Borel subgroup above T. We can identify $T_x(G/B) = \mathrm{Lie}G/\mathrm{Lie}G_x$ and our tangent spaces above become just the images of $\mathrm{Lie}B$, $\mathrm{Lie}K$ and $\mathrm{Lie}(B \cap K) = \mathrm{Lie}B \cap \mathrm{Lie}K$ under the projection $\mathrm{Lie}G \twoheadrightarrow \mathrm{Lie}G/\mathrm{Lie}G_x$. We can thus check the equality $T_x X_w \cap T_x Y = T_x U_w$ root space by root space, and the Lemma is proven. □

We now prove the main Theorem 2.1.

Proof. We have

$$\begin{aligned} \mathbb{H}^n(\mathfrak{n}, L(Y,\chi))_\lambda &= \mathbb{H}^{n-\dim Y}(Y, i^! j_* \mathcal{A}) \text{ by Theorem 4.4,} \\ &= \textstyle\bigoplus_c \mathbb{H}^{n-\dim Y - c}(u_c^! i^! j_* \mathcal{A}[c]) \text{ by Lemma 5.1,} \end{aligned}$$

and the spectral sequence computing $\mathbb{H}^n(u_c^! i^! j_* \mathcal{A}[c])$ is just the complex $C^{c-n}\mathcal{V}(A, D, c)$ by the last three sections. □

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