

MINIMAL REPRESENTATIONS OF EXCEPTIONAL p -ADIC GROUPS

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Let F be a p -adic field. By assumption, F has characteristic zero. Let G be the F -rational points of a connected reductive F -group, and π an irreducible complex representation of G . In a sufficiently small neighborhood of the identity, the character of π may be viewed as a distribution on \mathfrak{g} , the Lie algebra of G . A result of Harish-Chandra states that this distribution has the form

$$f \mapsto \sum_{\mathcal{O}} c_{\mathcal{O}} \int \hat{f} \mu_{\mathcal{O}},$$

where \mathcal{O} runs over the nilpotent coadjoint orbits in \mathfrak{g}^* , $\mu_{\mathcal{O}}$ is a suitably normalized G -invariant measure on \mathcal{O} , and \hat{f} is the Fourier transform of f . Let \mathcal{O}_{\min} be a minimal non-trivial coadjoint orbit. Then π is said to be a minimal representation if $c_{\mathcal{O}} = 0$ whenever $\mathcal{O} \not\subset \overline{\mathcal{O}_{\min}}$.

Through the work of Kazhdan and Savin, minimal representations are known to exist when G is a split exceptional group of type E_6, E_7, E_8 , or G_2 ; in the case of G_2 , the representation actually lives on the three-fold cover of G . In this paper, I construct minimal representations for all other exceptional groups (not assumed split) over a p -adic field; in the case of F_4 , the representation lives on the two-fold cover. Our approach, which is similar to that of Kazhdan and Savin, may be summarized as follows. The group G has a maximal parabolic subgroup, P , whose unipotent radical, U , is a Heisenberg group. Using Weil's well-known theory, a Heisenberg representation, $\hat{\pi}_{\Psi}$, of U can be extended to $P^{\circ} = [P, P]$, or sometimes to a double cover. Let $\pi = \text{Ind}_{P^{\circ}}^G \hat{\pi}_{\Psi}$. The main task is to extend this representation to all of G . This can be done without great difficulty, except that we need to check a braid relation between a certain unitary geometric operator and the Fourier transform. This relation is especially non-trivial for groups of rank two.

Although we confine ourselves to the p -adic case, it would be easy to adapt the arguments to the real case. In particular, this could be done for the rank four form of E_8 . On the other hand, minimal representations have been studied substantially in the real case by other methods. See especially the recent work of Gross-Wallach and Brylinski-Kostant ([G-W] and [B-K] among others).

Let me describe the contents of this paper in somewhat greater detail. In the first section, which is of independent interest, we present a construction of Lie algebras in terms of rank three, central, simple Jordan algebras. In fact, we construct all Lie algebras in Freudenthal's "magic square". The method, which follows a suggestion of G. Savin, is different from the standard construction ("Tits second

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construction”), and is particularly well suited to studying exceptional p -adic groups because it gives all forms of the exceptional Lie algebras over a p -adic field. We have also included results on some dual pairs in these algebras. Further results on Jordan algebras, Lie algebras and dual pairs are contained in the appendix.

In section 2, we turn to the structure of exceptional p -adic groups. In particular, we give a lot of detail about the structure of the Heisenberg parabolic. We can say more when the Jordan algebra is reduced and hence G has rank greater than two. In section 3 we actually construct the minimal representation π . As indicated above, the crucial point is to check a certain braid relation. When G has rank larger than two this may be done directly. However, if G is the rank two form of E_6 , we must appeal to global techniques following an idea of Kazhdan. Also, we treat separately the case G of type F_4 since we are working on the double cover.

Finally, in section 4 we prove that the representations π are minimal. The method is similar to the one used by Savin in the split case, which itself is based on a remark of Kazhdan. The first step is to study the character of π restricted to a Borel subgroup. For this we can appeal to Howe’s Kirillov theory for solvable p -adic groups. Since we do not assume that G is quasi-split, this does not make sense for us. Instead, we study the character of π restricted to P and directly prove the results that we need.

Concurrent with this work, and with a somewhat different approach, Torasso [To] has proved that minimal representations exist for groups of rank larger than two.

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1. EXCEPTIONAL LIE ALGEBRAS

In this section we construct all forms of the exceptional Lie algebras over a p -adic field F . The starting point is a degree three Jordan algebra over F . The construction also works for other fields of characteristic not two or three (in particular, for Archimedean and global fields), but in general we do not get all forms.

1.1. Facts about Jordan algebras. Let \mathcal{J} be a degree three central simple Jordan algebra over F . If $A, B \in \mathcal{J}$, we will denote the Jordan multiplication by $A.B \in \mathcal{J}$. Recall that the Jordan product is commutative but not associative. It does satisfy

$$(1) \quad (A.A).(B.A) = ((A.A).B).A,$$

and is power associative. For any power-associative algebra, Jacobson has shown that there exists a “generic minimal polynomial”. In our case, given $A \in \mathcal{J}$, we get the polynomial

$$P_A(x) = x^3 - t(A)x^2 + R(A)x - n(A)I.$$

Here $t(A)$ and $n(A)$ are called the (generic) trace and norm, respectively,

$$R(A) = \frac{1}{2}[t(A)^2 - t(A.A)],$$

and I is the identity element in \mathcal{J} . It is a fact that $n(I) = 1$ and $t(I) = 3$.

The form $t(A^2)$ is homogeneous of degree 2. We denote the corresponding symmetric bilinear form with $\langle A, B \rangle$. That is,

$$\langle A, B \rangle = t(A.B).$$

The form $n(A)$ is homogeneous of degree 3, and we denote the corresponding symmetric trilinear form with $\langle A, B, C \rangle$. We have $\langle A, A, A \rangle = n(A)$. Finally, $R(A)$ is a homogeneous form of degree 2. Polarizing gives a symmetric bilinear form, $R(A, B)$. Written out,

$$R(A, B) = \frac{1}{2}[t(A)t(B) - t(A.B)].$$

One of the main properties of the generic minimal polynomial is that $P_A(A) = 0$. We will rewrite this in another form. First, let

$$A \times A = A.A - t(A)A + R(A)I.$$

Then, $P_A(A) = 0$ is the same as

$$(A \times A).A = n(A)I.$$

(The element $A \times A$ is the analog of the transpose of the matrix of cofactors in matrix algebra.) Polarizing $A \times A$ gives a symmetric bilinear map $\mathcal{J} \times \mathcal{J} \rightarrow \mathcal{J}$, denoted $A \times B$. Written out,

$$(2) \quad A \times B = A.B - S(A, B) + R(A, B)I$$

where

$$S(A, B) = 1/2[t(B)A + t(A)B].$$

Note that $t(A \times B) = R(A, B)$.

It is also useful to polarize the expression $(A \times A).A = n(A)I$. We get

$$(3) \quad (A \times B).C + (B \times C).A + (C \times A).B = 3\langle A, B, C \rangle I.$$

For a proof of the following lemma see [J].

Lemma 1. *Let $[A, B, C] = (A.B).C - A.(B.C)$. Then $t([A, B, C]) = 0$.*

Lemma 2. *$\langle A \times B, C \rangle = 3\langle A, B, C \rangle$. In particular, $\langle A \times B, C \rangle = \langle B \times C, A \rangle = \langle C \times A, B \rangle$.*

Proof. Using the expression for $A \times B$ and Lemma 1, it follows that that $\langle A \times B, C \rangle$ is symmetric. Now apply formula (3). \square

The main task of this section is to compute $(A \times B) \times C$.

Proposition 3.

$$(4) \quad 2(A \times B) \times C = (A.B).C - (B.C).A - (C.A).B + 1/2[\langle B, C \rangle A + \langle A, C \rangle B].$$

Proof. Applying formula (2) twice gives

$$\begin{aligned} 2(A \times B) \times C &= 2(A.B).C - 2S(A, B).C + 2R(A, B)C - 2S(A \times B, C) + 2R(A \times B, C)I. \end{aligned}$$

Using Lemma 2, $2R(A \times B, C) = R(A, B)t(C)I - 3\langle A, B, C \rangle$. Also, $2S(A \times B, C) = (A.B)t(C) - S(A, B)t(C) + R(A, B)t(C) + R(A, B)C$. Thus,

$$2(A \times B) \times C = 2(A.B).C - 2S(A, B).C + R(A, B)C - (A.B)t(C) + S(A, B)t(C) - 3\langle A, B, C \rangle.$$

On the other hand, let us polarize the expression $P_A(A) = 0$. We get

$$(A.B).C + (B.C).A + (C.A).B = t(A)B.C + t(B)A.C + t(C)A.B - R(A, B)C - R(B, C)A - R(C, A)B + 3\langle A, B, C \rangle.$$

Adding these expressions, we get

$$\begin{aligned} 2(A \times B) \times C &= (A.B).C - (B.C).A - (C.A).B + S(A, B)t(C) - R(B, C)A - R(C, A)B \\ &= (A.B).C - (B.C).A - (C.A).B + 1/2\langle A, C \rangle B + 1/2\langle B, C \rangle A. \quad \square \end{aligned}$$

1.2. Norm-preserving automorphisms. Let \mathcal{J} be as in section 1.1, and let \mathfrak{M} be the Lie algebra of linear maps $\mathcal{J} \rightarrow \mathcal{J}$ which preserve the norm up to a constant. That is, let \mathfrak{M} be the set of all linear maps $\mathfrak{m}: \mathcal{J} \rightarrow \mathcal{J}$ for which there exists a constant $\gamma_{\mathfrak{m}}$ so that

$$\langle \mathfrak{m}A, B, C \rangle + \langle A, \mathfrak{m}B, C \rangle + \langle A, B, \mathfrak{m}C \rangle = \gamma_{\mathfrak{m}} \langle A, B, C \rangle$$

for all $A, B, C \in \mathcal{J}$. Let $\mathfrak{M}_0 = \{\mathfrak{m} \in \mathfrak{M} | \gamma_{\mathfrak{m}} = 0\}$. Then \mathfrak{M}_0 is itself a Lie algebra and, in fact, \mathfrak{M}_0 is an ideal in \mathfrak{M} . Furthermore, \mathfrak{M}_0 is simple. For example, if \mathcal{J} is an exceptional Jordan algebra, then \mathfrak{M}_0 is of type E_6 .

Since the trace form is non-degenerate, given $\mathfrak{m} \in \mathfrak{M}$, there is another element $\mathfrak{m}' \in \mathfrak{M}$, which is characterized by

$$\langle \mathfrak{m}A, B \rangle + \langle A, \mathfrak{m}'B \rangle = 0.$$

Clearly, $(\mathfrak{m}')' = \mathfrak{m}$. The map $\mathfrak{m} \mapsto \mathfrak{m}'$ is analogous to the negative conjugate transpose (Cartan involution) in matrix algebra. Clearly, we can write $\mathfrak{M} = \mathfrak{M}^{(S)} \oplus \mathfrak{M}^{(A)}$ where $\mathfrak{M}^{(S)}$ are the symmetric elements, characterized by $\mathfrak{m} + \mathfrak{m}' = 0$, and $\mathfrak{M}^{(A)}$ are the anti-symmetric elements characterized by $\mathfrak{m} - \mathfrak{m}' = 0$. In the same way, we can split \mathfrak{M}_0 into symmetric and anti-symmetric pieces. It is easy to check that $\mathfrak{M}_0^{(A)} = \mathfrak{M}^{(A)}$ but $\mathfrak{M}_0^{(S)} \subsetneq \mathfrak{M}^{(S)}$. Furthermore, $\mathfrak{M}^{(A)}$ is a sub-Lie algebra. In case \mathcal{J} is exceptional, it is of type F_4 .

Proposition 4. *Let $\mathfrak{m} \in \mathfrak{M}_0$, $A, B \in \mathcal{J}$. Then*

$$(5) \quad \mathfrak{m}(A \times B) = \mathfrak{m}'A \times B + A \times \mathfrak{m}'B.$$

Proof. Pick $C \in \mathcal{J}$. Then $\langle \mathfrak{m}(A \times B), C \rangle = -\langle A, B, \mathfrak{m}'C \rangle$. Since $\mathfrak{m}' \in \mathfrak{M}_0$, we get that

$$\begin{aligned} \langle \mathfrak{m}(A \times B), C \rangle &= \langle \mathfrak{m}'A, B, C \rangle + \langle A, \mathfrak{m}'B, C \rangle \\ &= \langle (\mathfrak{m}'A \times B), C \rangle + \langle (A \times \mathfrak{m}'B), C \rangle. \end{aligned}$$

Since C was arbitrary, the proposition follows. \square

In addition to considering norm-preserving automorphisms, there are two other standard ways of associating Lie algebras to Jordan algebras. First, there is the space of derivations of the Jordan algebra, and second, there is the Lie algebra generated by the transformations $L_A: \mathcal{J} \rightarrow \mathcal{J}$ for all $A \in \mathcal{J}$, where L_A is given by $L_A(B) = A.B$. For the particular Jordan algebras that we are considering, all three essentially coincide. Nevertheless, it is sometimes useful to work in various realizations so we briefly sketch the situation.

Let \mathfrak{L} be the Lie algebra generated by the L_A . Let $L(A, B) = [L_A, L_B]$. Then it turns out that

$$(6) \quad [L(A, B), L_C] = L_{[B, C, A]}$$

(see [J] and Lemma 1). In particular, $\mathfrak{L} = \Sigma L_A \oplus \Sigma L(B, C)$, and $\mathfrak{D} = \Sigma L(B, C)$ forms a sub-Lie algebra. Furthermore, it can be checked that elements of \mathfrak{D} are derivations of \mathcal{J} , the so-called inner derivations. For our Jordan algebras, all derivations are inner. Finally, Lemma 1 says that $t([A, B, C]) = 0$ for all $A, B, C \in \mathcal{J}$. Thus, we can form a sub-Lie algebra of \mathfrak{L} by taking $\mathfrak{L}_0 = \Sigma L_A \oplus \mathfrak{D}$ where $t(A) = 0$.

We have the following identifications: $\mathfrak{M} = \mathfrak{L}$, $\mathfrak{M}_0 = \mathfrak{L}_0$, $\mathfrak{M}^{(A)} = \mathfrak{D}$, $\mathfrak{M}^{(S)} = \Sigma L_A$, and $\mathfrak{M}_0^{(S)} = \Sigma L_A$ with $t(A) = 0$.

Remark. These identifications imply that $L(A, B)' = L(A, B)$ and $L'_A = -L_A$.

We conclude this section with two formulas that we will need. The first can be proved directly from the defining relation for Jordan algebras (equation (1)).

$$(7) \quad L(A.B, C) + L(B.C, A) + L(C.A, B) = 0.$$

The next formula follows easily from (2) and (7):

$$(8) \quad L(A \times B, C) + L(B \times C, A) + L(C \times A, B) = 0.$$

1.3. Lie algebras containing Jordan algebras. We begin with a standard construction (see [Ko]). Let $\mathcal{J}, \mathfrak{M}$ be as before. By definition, \mathfrak{M} acts on \mathcal{J} . Let \mathcal{J}' be the dual representation. We have a Jordan algebra isomorphism $i: \mathcal{J}' \rightarrow \mathcal{J}$, which satisfies $\mathfrak{m}(i(\Gamma)) = i(\mathfrak{m}'\Gamma)$. Set $\mathfrak{h} = \mathfrak{M} \oplus \mathcal{J} \oplus \mathcal{J}'$. We will define a Lie algebra structure on \mathfrak{h} . The structure on \mathfrak{M} is given and the action of \mathfrak{M} on $\mathcal{J} \oplus \mathcal{J}'$ is as above. We declare that the bracket of any two elements of \mathcal{J} (or of \mathcal{J}') is zero. Finally, if $A \in \mathcal{J}$ and $\Gamma \in \mathcal{J}'$, then $[A, \Gamma] = -A \square i(\Gamma)$, where

$$(9) \quad A \square B = L_{A.B} + L(A, B) \in \mathfrak{M}$$

for any $A, B \in \mathcal{J}$. To check that \mathfrak{h} is a Lie algebra, we need only check the Jacobi identity:

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0.$$

By linearity, we only have to check the cases when x, y, z are elements of \mathfrak{M} , \mathcal{J} or \mathcal{J}' . If $x, y, z \in \mathfrak{M}$, it is already known. If two of them are in \mathfrak{M} then it is just an expression of the fact that $\mathcal{J} \oplus \mathcal{J}'$ is a Lie algebra representation of \mathfrak{M} . It is now easy to reduce to the cases $x, z \in \mathcal{J}$, $y \in \mathcal{J}'$ and $x \in \mathfrak{M}$, $y \in \mathcal{J}'$ and $z \in \mathcal{J}$. These amount to the two formulas:

$$(A \square B)(C) - (C \square B)(A) = 0$$

and

$$(10) \quad [A \square B, \mathfrak{m}] + (A \square \mathfrak{m}'(B)) + (\mathfrak{m}(A) \square B) = 0.$$

The first is easy. For the second formula, consider separately the cases \mathfrak{m} a derivation and $\mathfrak{m} = L_C$ for $C \in \mathcal{J}$. For derivations we can use the fact that $\mathfrak{m}' = \mathfrak{m}$ and the definition of derivation. For $\mathfrak{m} = L_C$ it is a consequence of (7) and (6).

Remark. It is a well-known fact that \mathfrak{h} is actually a simple Lie algebra; see [J] or [Ko].

Next, we give a similar but slightly more complicated construction of a Lie algebra (cf. [F]). Let \mathcal{J} and \mathfrak{M}_0 be as before, and let V be the standard three dimensional representation of $\mathfrak{sl}(3)$. We will use the notation $\mathfrak{g}_0 = \mathfrak{sl}(3) \oplus \mathfrak{M}_0$ and write $(\mathfrak{l}; \mathfrak{m})$ for a typical element. Then $V \otimes \mathcal{J}$ is in a natural way a representation of \mathfrak{g}_0 . Let $(V \otimes \mathcal{J})'$ be the dual representation. Set

$$(11) \quad \mathfrak{g} = \mathfrak{g}_0 \oplus (V \otimes \mathcal{J}) \oplus (V \otimes \mathcal{J})'.$$

We will define a Lie algebra structure on \mathfrak{g} . The structure on \mathfrak{g}_0 and the action of \mathfrak{g}_0 on $(V \otimes \mathcal{J}) \oplus (V \otimes \mathcal{J})'$ is the given one. We will freely identify $\bigwedge^2 V$ with V' and $\bigwedge^2 V'$ with V . Note that if $v, w \in V$ and $\phi \in V'$, then

$$(v \wedge w) \wedge \phi = \phi(v)w - \phi(w)v \in V.$$

Also, define $v \triangle \phi \in \mathfrak{sl}(3)$ by

$$(12) \quad (v \triangle \phi)(w) = 3\phi(w)v - \phi(v)w$$

for all $w \in V$.

Now we define the remaining brackets. In what follows, we will use lower case Roman letters for elements of V , lower case Greek letters for elements of V' , upper case Roman letters for elements of \mathcal{J} and upper case Greek letters for elements of \mathcal{J}' . We will also abuse notation by identifying elements of \mathcal{J} and \mathcal{J}' where convenient. Thus we write $A \square \Gamma$ instead of $A \square i(\Gamma)$ etc. The meaning will be clear from the context. The brackets are:

$$(13) \quad [v \otimes A, w \otimes B] = -2(v \wedge w) \otimes (A \times B) \in (V \otimes \mathcal{J})'$$

$$(14) \quad [\phi \otimes \Gamma, \psi \otimes \Lambda] = 2(\phi \wedge \psi) \otimes (\Gamma \times \Lambda) \in V \otimes \mathcal{J}$$

$$(15) \quad [v \otimes A, \phi \otimes \Gamma] = \left(\frac{\langle A, \Gamma \rangle}{3} v \triangle \phi; 2\phi(v)(A \square \Gamma - \langle A, \Gamma \rangle / 3L_I) \right) \in \mathfrak{g}_0.$$

Remark. In general, $A \square \Gamma \notin \mathfrak{M}_0$; the correction $-\langle A, \Gamma \rangle / 3L_I$ is precisely what is needed.

It remains to verify the Jacobi identity for triples of elements of \mathfrak{g} . If at least two of them are in \mathfrak{g}_0 , this is clear. Next, suppose that one element is in \mathfrak{g}_0 , one is in $V \otimes \mathcal{J}$ and one is in $(V \otimes \mathcal{J})'$. We must show that

$$[[v \otimes A, \phi \otimes \Gamma], (\mathfrak{l}, \mathfrak{m})] + [[\phi \otimes \Gamma, (\mathfrak{l}, \mathfrak{m})], v \otimes A] + [[(\mathfrak{l}, \mathfrak{m}), v \otimes A], \phi \otimes \Gamma] = 0.$$

The first term is:

$$\left(\frac{\langle A, \Gamma \rangle}{3} [v \triangle \phi, \mathfrak{l}]; 2\phi(v)[A \square \Gamma - \langle A, \Gamma \rangle / 3L_I, \mathfrak{m}] \right).$$

The second term is

$$\begin{aligned} [v \otimes A, \mathfrak{l}'(\phi) \otimes \Gamma + \phi \otimes \mathfrak{m}'(\Gamma)] &= \left(\frac{\langle A, \Gamma \rangle}{3} v \triangle \mathfrak{l}'(\phi); 2\mathfrak{l}'(\phi)(v)(A \square \Gamma - \langle A, \Gamma \rangle / 3L_I) \right) \\ &\quad + \left(\frac{\langle A, \mathfrak{m}'(\Gamma) \rangle}{3} v \triangle \phi; 2\phi(v)(A \square \mathfrak{m}'(\Gamma) - \langle A, \mathfrak{m}'(\Gamma) \rangle / 3L_I) \right). \end{aligned}$$

The third term is

$$\begin{aligned} [\mathfrak{l}(v) \otimes A + v \otimes \mathfrak{m}(A), \phi \otimes \Gamma] &= \left(\frac{\langle A, \Gamma \rangle}{3} \mathfrak{l}(v) \triangle \phi; 2\phi(\mathfrak{l}(v))(A \square \Gamma - \langle A, \Gamma \rangle / 3L_I)\right) \\ &\quad + \left(\frac{\langle \mathfrak{m}(A), \Gamma \rangle}{3} v \triangle \phi; 2\phi(v)(\mathfrak{m}(A) \square \Gamma - \langle \mathfrak{m}(A), \Gamma \rangle / 3L_I)\right). \end{aligned}$$

Thus the sum of the second and third terms is

$$\left(\frac{\langle A, \Gamma \rangle}{3} (v \triangle \mathfrak{l}'(\phi) + \mathfrak{l}(v) \triangle \phi); 2\phi(v)(A \square \mathfrak{m}'(\Gamma) + \mathfrak{m}(A) \square \Gamma)\right).$$

We see that the Jacobi identity in this case follows from equation (10) and

$$[v \triangle \phi, \mathfrak{l}] + (v \triangle \mathfrak{l}'(\phi)) + (\mathfrak{l}(v) \triangle \phi) = 0$$

which is easy.

Now let us take one element of \mathfrak{g}_0 and two of $V \otimes \mathcal{J}$ (two of $(V \otimes \mathcal{J})'$ is similar). This case is fairly simple and quickly reduces to the formulas

$$\mathfrak{m}'(A \times B) = \mathfrak{m}(A) \times B + A \times \mathfrak{m}(B)$$

and

$$\mathfrak{l}'(v \wedge w) = v \wedge \mathfrak{l}(w) + \mathfrak{l}(v) \wedge w.$$

The first is just a restatement of equation (5), and the second is just the fact that elements of $\mathfrak{sl}(3)$ have trace zero.

The next case is two elements from $V \otimes \mathcal{J}$ and one from $(V \otimes \mathcal{J})'$ (one from $V \otimes \mathcal{J}$ and two from $(V \otimes \mathcal{J})'$ is similar). We must show

$$[[v \otimes A, w \otimes B], \phi \otimes \Gamma] + [[w \otimes B, \phi \otimes \Gamma], v \otimes A] + [[\phi \otimes \Gamma, v \otimes A], w \otimes B] = 0.$$

The first term is $-4(\phi(v)w - \phi(w)v) \otimes ((A \times B) \times \Gamma)$. Using equation (4), this is

$$-(\phi(v)w - \phi(w)v) \otimes (\langle B, \Gamma \rangle A + \langle A, \Gamma \rangle B + 2((A.B).\Gamma - (B.\Gamma).A - (\Gamma.A).B)).$$

The second term is

$$\begin{aligned} &\frac{\langle B, \Gamma \rangle}{3} (3\phi(v)w - \phi(w)v) \otimes A \\ &\quad + v \otimes 2\phi(w)((B.\Gamma).A + B.(\Gamma.A) - \Gamma.(B.A) - \frac{\langle B, \Gamma \rangle}{3} A) \end{aligned}$$

and the third term is

$$\begin{aligned} &-\frac{\langle A, \Gamma \rangle}{3} (3\phi(w)v - \phi(v)w) \otimes B \\ &\quad - w \otimes 2\phi(v)((A.\Gamma).B + A.(\Gamma.B) - \Gamma.(B.A) - \frac{\langle A, \Gamma \rangle}{3} B). \end{aligned}$$

The sum is clearly zero.

Finally, suppose that all three elements are in $V \otimes \mathcal{J}$ ($(V \otimes \mathcal{J})'$ is similar), the Jacobi identity quickly reduces to the following two formulas:

$$u \triangle (v \wedge w) + w \triangle (u \wedge v) + v \triangle (w \wedge u) = 0$$

and

$$C \square (A \times B) + A \square (B \times C) + B \square (C \times A) = 3\langle A, B, C \rangle L_I.$$

The first one is routine; the second follows from equations (8) and (3).

Theorem 5. \mathfrak{g} is a simple Lie algebra.

Proof. The only thing that remains to be checked is the simplicity. Suppose that $\mathfrak{a} \subset \mathfrak{g}$ is a non-zero ideal. In particular, \mathfrak{a} is a representation of \mathfrak{g}_0 . But as a \mathfrak{g}_0 -module, \mathfrak{g} is the direct sum of four irreducible representations: $\mathfrak{sl}(3)$, \mathfrak{M}_0 , $(V \otimes \mathcal{J})'$ and $V \otimes \mathcal{J}$. Thus, \mathfrak{a} contains at least one of these spaces. However, it is clear from the formulas that an ideal containing an element from any of these must contain elements in all of them. It follows that $\mathfrak{a} = \mathfrak{g}$. \square

1.4. Dual pairs. Recall that if \mathfrak{a} and \mathfrak{b} are sub-Lie algebras of a Lie algebra \mathfrak{c} , then \mathfrak{a} and \mathfrak{b} are said to form a *dual pair* if they are mutual centralizers, that is, if the centralizer of \mathfrak{a} in \mathfrak{c} is \mathfrak{b} , and the centralizer of \mathfrak{b} in \mathfrak{c} is \mathfrak{a} . In this section, we will identify certain dual pairs in simple Lie algebras \mathfrak{g} from Theorem 5. Also, we will identify these Lie algebras as those in the final column of Freudenthal's magic square.

We begin by defining some sub-Lie algebras of \mathfrak{g} .

Let $\mathfrak{g}_2 \subset \mathfrak{g}$ be the Lie algebra generated by $\mathfrak{sl}(3)$, $V \otimes I$ and $V' \otimes I$. This is a Lie algebra of type G_2 . Pick a basis of simple roots for \mathfrak{g}_2 , say α long and β short. We assume that the embedding $S_\alpha: \mathfrak{sl}(2) \rightarrow \mathfrak{g}_2$ corresponding to α is given by

$$S_\alpha: \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{sl}(3) \subset \mathfrak{g}_2.$$

It is easy to see that we can choose the map $S_\beta: \mathfrak{sl}(2) \rightarrow \mathfrak{g}_2$ to satisfy

$$S_\beta: \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \mapsto \begin{pmatrix} -a & & \\ & 2a & \\ & & -a \end{pmatrix} \in \mathfrak{sl}(3).$$

We now choose an embedding of \mathfrak{h} into \mathfrak{g} . There are many natural choices corresponding to different $\mathcal{J} \subset V \otimes \mathcal{J}$. Let $v_2 = (0, 1, 0) \in V$. Then we take $\mathfrak{h} \subset \mathfrak{g}$ to be generated by \mathfrak{M}_0 , $v_2 \otimes \mathcal{J}$ and $v'_2 \otimes \mathcal{J}$. It is not difficult to check that with this choice, the image of the map $S_\beta: \mathfrak{sl}(2) \rightarrow \mathfrak{g}_2 \subset \mathfrak{g}$, lies in \mathfrak{h} . In fact, the Lie algebra $\mathfrak{M} \subset \mathfrak{h} \subset \mathfrak{g}$ is the direct sum of \mathfrak{M}_0 and the abelian Lie algebra of matrices

$$\mathfrak{a}_\beta = \left\{ \begin{pmatrix} -a & & \\ & 2a & \\ & & -a \end{pmatrix} \right\} \subset \mathfrak{sl}(3).$$

For future use, let us make two remarks. First, we note explicitly that $\mathfrak{h} \cap \mathfrak{sl}(3) = \mathfrak{a}_\beta$. Second, since $\mathfrak{M} = \mathfrak{M}_0 \oplus \mathfrak{a}_\beta$, there is an augmentation, d , on \mathfrak{M} given by $d: (\mathfrak{m}_0, \text{diag}(-a, 2a, -a)) \mapsto 3a$, for $\mathfrak{m}_0 \in \mathfrak{M}_0$. A more invariant definition is as follows. Let

$$\mathfrak{X}(t) = \begin{pmatrix} 0 & t & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{sl}(3).$$

Then

$$[\mathfrak{X}(t), \mathfrak{m}] = \mathfrak{X}(d(\mathfrak{m})t).$$

Obviously, $\mathfrak{M}_0 = \ker(d)$.

It is now easy to identify the dual pairs.

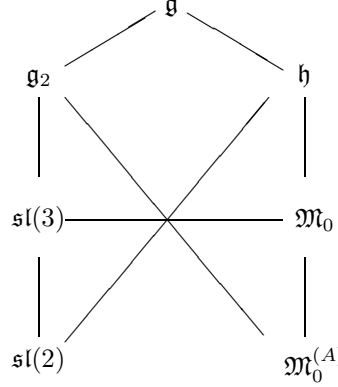


FIGURE 1. Dual Pairs.

Proposition 6. *The following dual pairs exist in \mathfrak{g} : $(\mathfrak{sl}(3), \mathfrak{M}_0)$, $(\mathfrak{g}_2, \mathfrak{M}_0^{(A)})$, and $(\mathfrak{sl}(2), \mathfrak{h})$ where this*

$$\mathfrak{sl}(2) = \left\{ \begin{pmatrix} a & 0 & b \\ 0 & 0 & 0 \\ c & 0 & -a \end{pmatrix} \right\} \subset \mathfrak{g}.$$

Remark. It is convenient to express this proposition with the “see-saw” formalism as in Figure 1.

Proposition 7. *For each degree three simple Jordan algebra, \mathcal{J} , there is a sequence of simple Lie algebras,*

$$\mathfrak{M}_0^{(A)} \subset \mathfrak{M}_0 \subset \mathfrak{h} \subset \mathfrak{g}$$

whose types correspond to a row of Freudenthal’s magic square. We get the first, second, third, and fourth rows when the dimension of \mathcal{J} is 6, 9, 15, or 27, respectively. Furthermore, in the p -adic case, as \mathcal{J} runs through the degree three central simple algebras, \mathfrak{g} runs through all forms of the algebras in the final column of the magic square.

Proof. The only things that are not standard are the statements relating to \mathfrak{g} . By counting dimensions, it is clear that we get forms of F_4, E_7 and E_8 when the dimension of \mathcal{J} is 6, 15 or 27. By a dimension count alone we can’t tell whether \mathfrak{g} is a form of E_6, B_6 or C_6 when $\dim \mathcal{J} = 9$. However, we have shown that in this case \mathfrak{g} has a dual pair of the form $(A_2 + A_2, A_2)$ and it is well-known that B_6 and C_6 can’t have dual pairs of this form. Thus, we get E_6 .

That we get all forms of the exceptional algebras (in the p -adic case) follows by simply counting and referring to the tables in [T2]. Indeed, it is enough to take \mathcal{J} to be either the Jordan algebra associated to a nine-dimensional division algebra, or else the Jordan algebra of 3×3 hermitian matrices over a composition algebra. It is easy to see that the \mathfrak{g} ’s we obtain are distinct and by [T2] this is all of them. \square

Remarks. (1) In the real case, we do not get all forms in this way. In particular, we do not get the compact forms.

(2) There exist other degree three central simple Jordan algebras besides those mentioned in the proof. However, if \mathcal{J} is reduced (see section 2.5) there exists a diagonal matrix, Γ , with entries in F , so that \mathcal{J} is the set of three by three matrices over the composition algebra which satisfy $x = \Gamma \bar{x} \Gamma^{-1}$. (See [Sch].)

A_1	A_2	C_3	F_4
A_2	$A_2 + A_2$	A_5	E_6
C_3	A_5	D_6	E_7
F_4	E_6	E_7	E_8

FIGURE 2. Freudenthal's Magic Square.

(3) By dropping the requirement that \mathcal{J} be central simple, we can construct many other Lie algebras including G_2 and 3D_4 . See the appendix.

2. THE GROUP G

2.1. Generalities and notations. We now begin the study of certain algebraic groups. They will be defined over a fixed p -adic field F , although, as in section 1, most of the results are also valid over Archimedean or global fields. We will abuse notation and write G both for an algebraic group defined over F and for the F -points of that group. Throughout the discussion we fix \mathcal{J} , a rank three central simple Jordan algebra over F .

In the last chapter, we associated to \mathcal{J} four Lie algebras,

$$\mathfrak{M}_0^{(A)} \subset \mathfrak{M}_0 \subset \mathfrak{h} \subset \mathfrak{g}.$$

Let $M_0^{(A)}$, M_0 , H and G be the corresponding simply connected groups (except that in the case $\dim \mathcal{J} = 6$ we take $M_0^{(A)}$ to be the adjoint group). Recall that

$$\mathfrak{g} = \mathfrak{sl}(3) \oplus \mathfrak{M}_0 \oplus (V \otimes \mathcal{J}) \oplus (V \otimes \mathcal{J})',$$

and the Lie algebra \mathfrak{g}_2 satisfies

$$\mathfrak{sl}(3) \subset \mathfrak{g}_2 \subset \mathfrak{g}.$$

Let G_2 be the group corresponding to \mathfrak{g}_2 .

We will need notations for various subgroups of $\mathrm{SL}(3) \subset G_2$. Let T be the diagonal subgroup of $\mathrm{SL}(3)$, and set

$$h(t_1, t_2) = \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_1^{-1}t_2^{-1} \end{pmatrix} \in T \subset \mathrm{SL}(3).$$

We write $h_\alpha(t) = h(t, t^{-1})$ and $h_\beta(t) = h(t^{-1}, t^2)$. The corresponding subgroups of T are h_α and h_β . Finally, we write $x(r)$, $y(s)$ and $z(t)$ for the elements

$$\begin{pmatrix} 1 & r & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

2	3	2	1
3	9	6	3
2	6	4	2
1	3	2	1

FIGURE 3. Orders of fundamental groups in magic square.

respectively. Often we will abuse notation and write $x(t)$ for the group consisting of all $x(t)$ etc.

Let v_1, v_2, v_3 be the standard basis of V ; v_1 corresponds to $(1, 0, 0)$, v_2 corresponds to $(0, 1, 0)$ and v_3 corresponds to $(0, 0, 1)$. The dual basis is denoted v'_i . On the Lie algebra level we can write

$$V \otimes \mathcal{J} = (v_1 \otimes \mathcal{J}) \oplus (v_2 \otimes \mathcal{J}) \oplus (v_3 \otimes \mathcal{J}) \subset \mathfrak{g},$$

and similarly

$$(V \otimes \mathcal{J})' = (v'_1 \otimes \mathcal{J}) \oplus (v'_2 \otimes \mathcal{J}) \oplus (v'_3 \otimes \mathcal{J}) \subset \mathfrak{g}.$$

The $v_i \otimes \mathcal{J}$ and $v'_i \otimes \mathcal{J}$ are abelian sub-Lie algebras of \mathfrak{g} . Denote the corresponding abelian unipotent subgroups of G by E_i and E'_i . Typical elements are written $E_i(A)$ and $E'_i(B)$ for $A, B \in \mathcal{J}$. Occasionally, we will write $E(v_1 \otimes A)$ instead of $E_1(A)$ etc.

Finally, let $M \subset G$ be the subgroup corresponding to the Lie algebra \mathfrak{M} , $\mathfrak{M}_0 \subset \mathfrak{M} \subset \mathfrak{g}$. It is clear that H is generated by M, E_2 and E'_2 and that $h_\beta \subset M$. Corresponding to the augmentation d on \mathfrak{M} there is a homomorphism $d: M \rightarrow F^*$. For example, $d(h_\beta(t)) = t^3$.

The following lemma is immediate.

Lemma 8. *T normalizes all of the groups E_i and E'_i , $i = 1, 2, 3$. In fact, if $r = \text{diag}(r_1, r_2, r_3) \in T$, then $rE_i(A)r^{-1} = E_i(r_i A)$, and $rE'_i(A)r^{-1} = E'_i(r_i^{-1} A)$.*

Proposition 9. (1) G_2 is generated by $\text{SL}(3)$ and the $E_i(I)$ and $E'_i(I)$ where I is the identity in \mathcal{J} .

(2) There are inclusions

$$M_0^{(A)} \subset M_0 \subset H \subset G.$$

These groups satisfy

- (a) $H \cap \text{SL}(3) = h_\beta \subset M$.
- (b) $M_0 \cap \text{SL}(3) = \{h_\beta(\mu) | \mu^3 = 1\}$.
- (c) $M_0^{(A)} \cap \text{SL}(3) = \{1\}$.

Moreover, $Z_G = Z_{M_0^{(A)}}$; that is, the centers of G and $M_0^{(A)}$ coincide.

For the proof of our proposition it will be convenient to record the orders of the fundamental groups of the the groups in the magic square. This is in Figure 3.

Proof. Statement (1) is clear. Also, because of the Lie algebra inclusions, to prove the inclusions in (2), we need only prove that we can take the groups to be simply connected. For this we may work over the algebraic closure. Since the root systems

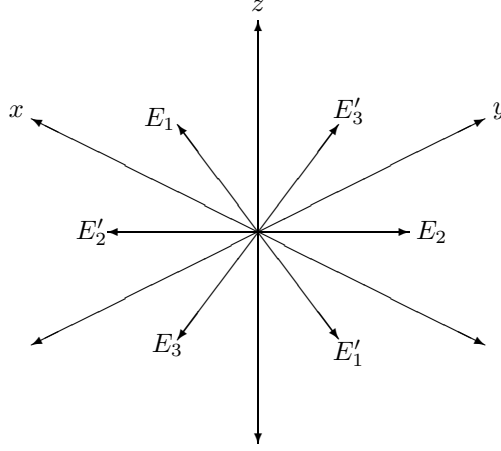


FIGURE 4

of M_0 and H are subsets of the root system of G , it is clear that, as G is simply connected, M_0 and H are also simply connected. For $M_0^{(A)}$ we argue case by case. Note that $M_0^{(A)}$ is the fixed points of an involution on M . F_4 is always simply connected. We have $\mathrm{Sp}(6) \hookrightarrow \mathrm{SL}(6)$, $\mathrm{SL}(3) \hookrightarrow \mathrm{SL}(3) \times \mathrm{SL}(3)$, $\mathrm{SO}(3) \hookrightarrow \mathrm{SL}(3)$. So $M_0^{(A)}$ is simply connected unless $\dim \mathcal{J} = 6$ in which case it is adjoint.

It is also clear from the above reasoning that $Z_{M_0^{(A)}} \subset Z_{M_0}$. But by proposition 6, $M_0^{(A)}$ centralizes G_2 . Also, M_0 and G_2 generate G . This proves that $Z_{M_0^{(A)}} \subset Z_G$. Comparing the orders of these groups, we see that $Z_{M_0^{(A)}} = Z_G$.

Next, we establish (a), (b) and (c). Recall from the last chapter that $\mathfrak{M} = \mathfrak{M}_0 \oplus \mathfrak{a}_\beta$ and $\mathfrak{h} \cap \mathfrak{sl}(3) = \mathfrak{a}_\beta$. Thus, both M and $H \cap \mathrm{SL}(3)$ contain the one-parameter subgroup corresponding to \mathfrak{a}_β . But this is exactly h_β . On the other hand, since H commutes with $z(t)$, $H \cap \mathrm{SL}(3) \subset h_\beta$. Hence (a). For part (b), we use the fact that M_0 and $\mathrm{SL}(3)$ centralize each other. This implies that $M_0 \cap \mathrm{SL}(3) \subset Z_{\mathrm{SL}(3)} = \{h_\beta(t) | t^3 = 1\}$. On the other hand, it is clear that $M_0 = \{m \in M | d(m) = 1\} \supset Z_{\mathrm{SL}(3)}$. This proves (b). Finally, for (c) we have $M_0^{(A)} \cap \mathrm{SL}(3) \subset Z_{M_0^{(A)}} \cap Z_{\mathrm{SL}(3)} = Z_G \cap Z_{\mathrm{SL}(3)}$. But by the lemma, this is trivial. \square

2.2. Parabolic subgroups. Consider the action of the one-parameter subgroup $h(t^2, t)$ on the Lie algebra \mathfrak{g} on the left. It induces a decomposition

$$\mathfrak{g} = \mathfrak{g}(-) \oplus \mathfrak{g}(0) \oplus \mathfrak{g}(+)$$

where $\gamma \in \mathfrak{g}(i)$ when $h(t^2, t)\gamma = t^i\gamma$, and i is zero, positive or negative, respectively. Let U_B be the unipotent subgroup of G corresponding to $\mathfrak{g}(+)$. Let L_B be the normalizer of U_B . Then $B = L_B U_B$ is a parabolic subgroup of G of corank 2. As follows immediately from Lemma 8, U_B is generated by $x(t), y(t), z(t), E_1, E_2$ and E'_3 , and $L_B = TM = h_\alpha M$. The root system of G relative to B is of type G_2 ; see Figure 4.

Let α and β be simple roots for $G_2 \subset G$ with α long. Then,

$$s_\alpha = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad s_\beta = E_2(I)E'_2(-I)E_2(I) \in G_2 \cap H$$

are representatives for the corresponding generators of the Weyl group of G_2 . It is clear that the Weyl group of G is generated by the Weyl group of M together with the images of s_α and s_β .

Let $P_\alpha, P_\beta \subset G$ be the parabolic subgroups generated by B and s_α, s_β , respectively. Write the Levi decompositions as $P_\alpha = L_\alpha U_\alpha$ and $P_\beta = L_\beta U_\beta$. These groups are easily identified. First,

$$L_\alpha = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} M \quad \text{and} \quad U_\alpha = y(t)z(t)E_1E_2E'_3.$$

Also, $L_\beta = h_\alpha H$ and $U_\beta = x(t)y(t)z(t)E_1E'_3$.

For us, P_β will be extremely important. To avoid cumbersome notation, we will usually just write $P = LU$ instead of $P_\beta = L_\beta U_\beta$, and just s for s_β . We will also need to consider $P^\circ = [P, P]$. Clearly, $P^\circ = L^\circ U$ where $L^\circ = [L, L]$ is semi-simple. In fact, $L^\circ = H$.

The unipotent group U is a Heisenberg group with center $Z = z(t)$. If we pick maximal abelian subgroups $W = x(t)E_1$ and $W' = E'_3y(t)$, then clearly, W, W' and Z generate U . It is well known that U/Z is a symplectic space with form given by $(u_1Z, u_2Z)_{U/Z} = c$ if $u_1u_2u_1^{-1}u_2^{-1} = z(c)$. Furthermore, if we abuse notation and use W and W' also for the images in U/Z , then W and W' are maximal isotropic subspaces. We will sometimes use the notation (t, A, B, u) , or (w, w') , for elements of U/Z .

We need to fix an identification of W' with W^* , the dual of W . If $E'_3(B)y(u)Z = w' \in W'$ and $x(t)E_1(A)Z = w \in W$, then let

$$w'(w) = \langle A, B \rangle + tu.$$

Proposition 10. *The symplectic form on U/Z may be written as*

$$((w_1, w'_1), (w_2, w'_2))_{U/Z} = w'_2(w_1) - w'_1(w_2).$$

Proof. It is enough to assume that $w'_1 = w_2 = 0$ and consider separately the cases $w_1 = x(t), w'_2 = y(t)$ and $w_1 = E_1(A), w'_2 = E'_3(B)$. That is, we must show that $x(t)y(s)x(-t)y(-s) = z(st)$ and $E_1(A)E'_3(B)E_1(-A)E'_3(-B) = z(\langle A, B \rangle)$. The first formula is simple, and the second follows from equation (15) in the last section. \square

Consider the right action of L on U . Recall that $Z \subset U$ is precisely the subgroup

$$\begin{pmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \subset \text{SL}(3).$$

Now clearly H acts trivially on Z . (See Proposition 6.) Moreover, since $h_\alpha(t) = \text{diag}(t, t^{-1}, 1) \in \text{SL}(3)$, right conjugation by $h_\alpha(t)$ acts as multiplication by t^{-1} on Z . This implies that the action of L on U factors to a map $p: L \rightarrow \text{GSp}(U/Z)$. Moreover, the image of H is in $\text{Sp}(U/Z)$. Now consider a Heisenberg representation of U . By the work of Weil, it extends to a representation of the semi-direct product

$\text{Mp}(U/Z) \ltimes U$. Our idea is to show that, when $\dim \mathcal{J}$ is odd, the metaplectic cover is split over $p(H)$ and thus we obtain a representation of $P^\circ = HU$. Before turning to this, however, we need to be very explicit about the action of H on U/Z and the action of $h_\alpha = P/P^\circ$ on P° .

2.3. Formulas. We record the *right* action of H on U/Z . Throughout, J denotes an arbitrary element of the Jordan algebra \mathcal{J} .

Claim 11. The action of $E_2(J)$ on U is as follows:

$$\begin{aligned} x(t)^{E_2(J)} &= x(t)E_1(tJ)E'_3(-tJ \times J)y(tn(J))z(t^2n(J)) \\ E_1(A)^{E_2(J)} &= E_1(A)E'_3(-2J \times A)y(\langle J \times J, A \rangle)z(\langle A \times A, J \rangle) \\ E'_3(A)^{E_2(J)} &= E'_3(A)y(-\langle J, A \rangle) \\ y(t)^{E_2(J)} &= y(t) \end{aligned}$$

Proof. These formulas follow from the standard commutation formulas and the formulas for the Lie-algebra bracket (equations 13 to 15). We give details starting from the bottom; refer to Figure 4. The last formula is obvious. Next,

$$\begin{aligned} E'_3(A)^{E_2(J)} &= E'_3(A) \exp([v'_3 \otimes A, v_2 \otimes J]) \\ &= E'_3(A)y(-\langle J, A \rangle) \end{aligned}$$

Similarly,

$$\begin{aligned} E_1(A)^{E_2(J)} &= E_1(A)E([v_1 \otimes A, v_2 \otimes J]) \\ &\quad \times \exp([v_1 \otimes A, v_2 \otimes J]/2)z\left(\frac{-1}{2}\langle A, [v_1 \otimes A, v_2 \otimes J] \rangle\right) \\ &= E_1(A)E'_3(-2J \times A)y(\langle J \times J, A \rangle)z(\langle A \times A, J \rangle) \end{aligned}$$

Finally,

$$x(t)^{E_2(J)} = x(t)E(\mu)E(\nu) \exp(\eta)z(\xi)$$

where

$$\begin{aligned} \mu &= [\mathfrak{X}(t), v_2 \otimes J] = tv_1 \otimes J \\ \nu &= [\mu, v_2 \otimes J]/2 = -tv'_3 \otimes J \times J \\ \exp(\eta) &= \exp([\nu, v_2 \otimes J]/6)) = y(t\langle J, J \times J \rangle/3) = y(tn(J)) \\ \xi &= \frac{-1}{2}(\langle \mu, \nu \rangle + t\eta) = t^2n(J) \quad \square \end{aligned}$$

Corollary 12. *The action of $E_2(J)$ on U/Z is given by*

$$\begin{aligned} (t, A, B, u)^{E_2(J)} \\ = (t, A + tJ, B - tJ \times J - 2J \times A, u + tn(J) + \langle J \times J, A \rangle - \langle J, B \rangle) \end{aligned}$$

It is important to introduce some notations which make this formula more transparent. Define $\mu_J: W \rightarrow W$ by $\mu_J(t, A) = (t, A + tJ)$. Also define $\nu_J: W' \rightarrow W'$ by $\nu_J(B, u) = (B, u - \langle J, B \rangle)$. Note that $\nu_J = (\mu^*)^{-1}$. Finally define a map $\hat{Q}_J: W \rightarrow W'$ by

$$\hat{Q}_J(t, A) = (-2A \times J - tJ \times J, -\langle J \times J, A \rangle - 2tn(J)).$$

Corollary 13.

$$(w, w')^{E_2(J)} = (\mu_J(w), \nu_J(w' + \hat{Q}_J(w))).$$

The next claim may be proved by the same sort of argument as the last.

Claim 14. The action of $E'_2(J)$ is as follows:

$$\begin{aligned} x(t)^{E'_2(J)} &= x(t) \\ E_1(A)^{E'_2(J)} &= E_1(A)x(\langle J, A \rangle) \\ E'_3(A)^{E'_2(J)} &= E'_3(A)E_1(-2A \times J)x(-\langle J \times J, A \rangle)z(-\langle A \times A, J \rangle) \\ y(t)^{E'_2(J)} &= y(t)E'_3(-tJ)E_1(tJ \times J)x(tn(J))z(-t^2n(J)) \end{aligned}$$

Claim 15.

$$\begin{aligned} x(t)^s &= y(t) \\ E_1(A)^s &= E'_3(A) \\ E'_3(A)^s &= E_1(-A) \\ y(t)^s &= x(-t) \end{aligned}$$

Proof. This follows immediately from the last two claims. For example, $y(t)^s = y(t)^{E_2(I)E'_2(-I)E_2(I)} = y(t)^{E'_2(-I)E_2(I)}$. But

$$y(t)^{E'_2(-I)} = y(t)E'_3(tI)E_1(tI)x(-t)z(t^2)$$

Thus,

$$\begin{aligned} y(t)^s &= y(t)E'_3(tI)y(-3t)E_1(tI)E'_3(-2tI)y(3t)z(3t^2) \\ &\quad \times x(-t)E_1(-tI)E'_3(tI)y(-t)z(t^2)z(t^2) \\ &= x(-t)z(-2\langle tI, tI \rangle + t^2)z(5t^2) \\ &= x(-t) \end{aligned}$$

The other formulas can be proved in a similar way. Alternatively, one may observe that the claim is obvious up to a possible factor of -1 which we have now checked. \square

Corollary 16. *The action of s on U/Z is given by*

$$(t, A, B, u)^s = (-u, -B, A, t)$$

Claim 17. Write the action of $m \in M$ on E_2 as $E_2(A)^m = E_2(A^m)$. Then the action of m on U is given by

$$\begin{aligned} x(t)^m &= x(d(m)t) \\ E_1(A)^m &= E_1(d(m)A^m) \\ E'_3(A)^m &= E'_3(d(m)^{-1}A^{sm s^{-1}}) \\ y(t)^m &= y(d(m)^{-1}t) \end{aligned}$$

Proof. The formula for the action of m on $x(t)$ follows immediately from one of our definitions of d on \mathfrak{M} ; see page 140. For the next formula, we have, by the first formula in Claim 11,

$$(x(-1)E_2(-A)x(1)E_2(A))^m = E_1(A)^m E'_3(-A \times A)^m y(n(A))^m z(n(A))^m.$$

On the other hand,

$$\begin{aligned} (x(-1)E_2(-A)x(1)E_2(A))^m &= x(-d(m))E_2(-A^m)x(d(m))E_2(A^m) \\ &= E_1(d(m)A^m)E'_3(-d(m)A^m \times A^m) \\ &\quad \times y(d(m)n(A^m))z(d(m)^2n(A^m)). \end{aligned}$$

This proves that $E_1(A)^m = E_1(d(m)A^m)$. Also, since m commutes with $z(t)$, $z(d(m)^2n(A^m)) = z(n(A))$, and hence $n(A^m) = d(m)^{-2}n(A)$. Using this, we get $y(t)^m = y(d(m)^{-1}t)$.

Finally, we apply Claim 15. First, $y(t)^m = (x(-t)^{m^s})^{s^{-1}} = y(d(m^s)t)$. Thus, $d(m^s) = d(m)^{-1}$. Next, $E'_3(A)^m = (E_1(-A)^{m^s})^{s^{-1}} = E'_3(d(m^s)A^{m^s})$ which equals $E'_3(d(m)^{-1}A^{m^s})$ by the last computation. \square

Corollary 18.

$$(t, A, B, u)^m = (d(m)t, d(m)A^m, d(m)^{-1}B^{sm^s^{-1}}, d(m)^{-1}u).$$

The next claim is very easy; in fact, it is mostly a special case of the last one. Nevertheless, it is convenient to state it separately.

Claim 19. $h_\beta(a) \in M$ acts via the following formulas.

$$\begin{aligned} E_2(A)^{h_\beta(a)} &= E_2(a^{-2}A) \\ x(t)^{h_\beta(a)} &= x(a^3t) \\ E_1(A)^{h_\beta(a)} &= E_1(aA) \\ E'_3(A)^{h_\beta(a)} &= E'_3(a^{-1}A) \\ y(t)^{h_\beta(a)} &= y(a^{-3}t) \end{aligned}$$

Corollary 20.

$$(t, A, B, u)^{h_\beta(a)} = (a^3t, aA, a^{-1}B, a^{-3}u).$$

Now we turn to the problem of the *left* action of h_α on P° . It is very simple to check that

Claim 21. h_α commutes with M . Also, $h_\alpha(a)$ maps

$$\begin{aligned} h_\alpha(a)E_2(A)h_\alpha(a)^{-1} &= E_2(a^{-1}A) \\ h_\alpha(a)x(t)h_\alpha(a)^{-1} &= x(a^2t) \\ h_\alpha(a)y(t)h_\alpha(a)^{-1} &= y(a^{-1}t) \\ h_\alpha(a)z(t)h_\alpha(a)^{-1} &= z(at) \\ h_\alpha(a)E_1(A)h_\alpha(a)^{-1} &= E_1(aA) \\ h_\alpha(a)E'_3(A)h_\alpha(a)^{-1} &= E'_3(A) \end{aligned}$$

Finally,

$$h_\alpha(a)sh_\alpha(a^{-1}) = h_\beta(a^{-1})s.$$

2.4. Quadratic forms. We begin with some notation. Let $c = \dim \mathcal{J}$. Let $\delta \in F^*/(F^*)^2$ be the discriminant of the non-degenerate bilinear form $\langle A, B \rangle$ on \mathcal{J} . Also, $(\cdot, \cdot)_F$ is the quadratic Hilbert symbol in F . Note that when discussing the Hilbert symbol we will assume that F does not have residue characteristic two, so that $(-1, -1)_F = 1$. On the other hand, this is mostly for convenience as the results have analogs in the residue characteristic two (or the real) case.

Recall that, for any $J \in \mathcal{J}$, in section 2.3 we defined a map $\hat{Q}_J: W \rightarrow W'$ by

$$\hat{Q}_J(t, A) = (-2A \times J - tJ \times J, -\langle J \times J, A \rangle - 2tn(J)).$$

We now wish to investigate the corresponding quadratic forms on W :

$$\begin{aligned} Q_J(t, A) &= \frac{1}{2} \hat{Q}_J(t, A)(t, A) = -\frac{1}{2} \langle A, 2A \times J + tJ \times J \rangle - \frac{t}{2} \langle J \times J, A \rangle - t^2 n(J) \\ &= -\langle J, A \times A \rangle - t \langle A, J \times J \rangle - t^2 n(J). \end{aligned}$$

We will write D_J and S_J for the discriminant and Hasse invariant of Q_J , respectively.

- Proposition 22.** (1) $tQ_J(t, A) = n(A) - n(A + tJ)$.
 (2) $D_J = (n(J)/2)^{c+1} \delta$ if $n(J) \neq 0$.
 (3) $S_J = S_I(\delta, n(J))_F^c (-1, n(J))_F^{c(c+1)/2}$ if $n(J) \neq 0$.

Proof. Part (1) is very simple; writing out the right-hand side we find

$$\begin{aligned} n(A) - n(A + tJ) &= \langle A, A, A \rangle - \langle A + tJ, A + tJ, A + tJ \rangle \\ &= -3t \langle J, A, A \rangle - 3t^2 \langle A, J, J \rangle - t^3 n(J) \\ &= -t \langle J, A \times A \rangle - t^2 \langle A, J \times J \rangle - t^3 n(J) \end{aligned}$$

which is the left-hand side. Before proving (2) and (3), notice that

Lemma 23. If $r = n(J) \neq 0$, then Q_J and rQ_I are equivalent as quadratic forms via the change of variables $(t, A) \mapsto (t, JA)$.

Proof. We wish to show that $rQ_I(t, A) = Q_J(t, JA)$. Clearly, it is enough to prove this when $t \neq 0$. However, by (1), this is the same as proving

$$r(n(A) - n(A + It)) = n(JA) - n(JA + Jt)$$

which is clear. \square

To prove (2), it only remains to compute the discriminant of Q_I . This form splits into a direct sum in the following way. Let \mathcal{J}_0 be the subspace of \mathcal{J} consisting of elements with trace zero. If $A \in \mathcal{J}$, we can write $A = A_0 + aI$ where $A_0 \in \mathcal{J}_0$. Now take (t, a, A_0) as coordinates on W . We have,

$$\begin{aligned} Q_I(t, a, A_0) &= -\text{Tr}(A \times A) - t \text{Tr}(A) - t^2 \\ &= \frac{1}{2} (\text{Tr}(A \cdot A) - \text{Tr}(A)^2) - 3ta - t^2 \\ &= \frac{1}{2} \text{Tr}(A_0^2) - (t^2 + 3ta + 3a^2). \end{aligned}$$

Clearly, the discriminant of $-(t^2 + 3ta + 3a^2)$ is 3. Thus, D_I is $3(1/2)^{c-1}$ (or, equivalently, $3(1/2)^{c+1}$) times the discriminant of $\text{Tr}(A_0^2)$. But δ is the discriminant of $\text{Tr}(A^2) = \text{Tr}(A_0^2) + 3a^2$. Thus, $D_I = (1/2)^{c+1} \delta$, and (2) is proved.

For the proof of (3), recall the definition of Hasse invariant in terms of Hilbert symbols. Let e_1, \dots, e_{c+1} be an orthogonal basis for W in terms of Q_I – or Q_J , it is the same by the lemma. Set $a_i = Q_I(e_i)$. Note that $\prod_i a_i = D_I$. By definition,

$$S_I = \prod_{i < j} (a_i, a_j)_F.$$

Thus,

$$\begin{aligned} S_J &= \prod_{i < j} (ra_i, ra_j)_F \\ &= \prod_{i < j} (a_i, a_j)_F (a_i, r)_F (a_j, r)_F (r, r)_F \\ &= S_I(D_I, r)_F^c (r, r)_F^{c(c+1)/2} \\ &= S_I(\delta, r)_F^c (1/2, r)_F^{c(c+1)/2} (r, -1)_F^{c(c+1)/2} \\ &= S_I(\delta, r)_F^c (r, -1)_F^{c(c+1)/2}. \end{aligned}$$

Here we have used properties of the Hilbert symbol as well as part (2). Part (3) is proved. \square

Using Proposition 22, we can get information on the γ -invariants of the quadratic forms Q_J . See [W] or [R] for the definition of γ . We will usually write $\gamma(J)$ instead of $\gamma(Q_J)$.

Corollary 24. *Suppose that c is odd. If $n(J) \neq 0$, then*

$$\frac{\gamma(J)}{\gamma(I)} = (\delta, n(J))_F (-1, n(J))_F^{(c+1)/2}.$$

Proof. Recall that γ is a homomorphism from the Witt ring of F , \mathcal{W}_F , to \mathbb{C}^* . Elements of \mathcal{W}_F are determined by three invariants: dimension, discriminant, and Hasse invariant. By assumption, the dimensions of Q_I and Q_J are the same and even. Also, by Proposition 22 part (2), the discriminants coincide. Finally, by part (3) of the same proposition,

$$S_J = S_I(\delta, n(J))_F (-1, n(J))_F^{(c+1)/2}.$$

Thus, as an element of \mathcal{W}_F , Q_J/Q_I has dimension zero, discriminant 1, and Hasse invariant $(\delta, n(J))_F (-1, n(J))_F^{(c+1)/2}$. But γ is known to be non-trivial when restricted to dimension zero forms of discriminant 1 [W, Proposition 4]. The corollary follows. \square

Corollary 25. *Suppose that c is odd and $n(J) \neq 0$. Then $Q_J \oplus Q_{-J}$ is trivial in \mathcal{W}_F . In particular, $\gamma(J)\gamma(-J) = 1$. Finally, $\gamma(I)^2 = (\delta, -1)_F$.*

Proof. We must show that the three invariants vanish on $Q_J \oplus Q_{-J}$. The only one that is not obvious is the Hasse invariant. The Hasse invariant of $Q_J \oplus Q_{-J}$ is $S_J S_{-J}(\delta, \delta)_F$. But $S_J S_{-J} = (\delta, -1)_F^c$, so the Hasse invariant is trivial.

For the final statement, we have $\gamma(I)^2 = \frac{\gamma(I)}{\gamma(-I)}$; now apply Corollary 24. \square

Corollary 26. *Suppose that c is odd and that for $i = 1, 2, 3$, $J_i \in \mathcal{J}$ satisfies $n(J_i) \neq 0$. Then,*

$$\gamma(J_1)\gamma(J_2)\gamma(J_3) = \gamma(-J_1 \cdot J_2 \cdot J_3).$$

Remark. Since $\gamma(J)$ depends only on $n(J)$, we do not need to specify the order of multiplication of the J_i .

Proof. By Corollaries 24 and 25,

$$\begin{aligned}\gamma(J_1)\gamma(J_2)\gamma(J_3) &= \gamma(I)^3(\delta, n(J_1.J_2.J_3))_F(-1, n(J_1.J_2.J_3))_F^{(c+1)/2} \\ &= \gamma(I)(\delta, -n(J_1.J_2.J_3))_F(-1, -n(J_1.J_2.J_3))_F^{(c+1)/2} \\ &= \gamma(-J_1.J_2.J_3). \quad \square\end{aligned}$$

2.5. When \mathcal{J} is reduced. In this section, we review some of the properties of reduced Jordan algebras and extend the discussion of the last two sections in the case \mathcal{J} is reduced.

A rank three Jordan algebra is *reduced* if it contains non-zero elements I_1, I_2, I_3 so that $I_i.I_i = I_i$, and $I_i.I_j = 0$ for $i \neq j$. It follows that $I_1 + I_2 + I_3 = I$. Set

$$\mathcal{J}_{i,i} = \{A \in \mathcal{J} | I_i.A = A\}.$$

Also, for $i \neq j$, set

$$\mathcal{J}_{i,j} = \{A \in \mathcal{J} | I_i.A = \frac{1}{2}A = I_j.A\}.$$

Obviously, $\mathcal{J}_{i,j} = \mathcal{J}_{j,i}$. It is a fact from the theory of Jordan algebras that

$$\mathcal{J} = \bigoplus_{i \leq j} \mathcal{J}_{i,j}.$$

In the next claim, the notation is that $A_{i,j} \in \mathcal{J}_{i,j}$ and i, j, k are all distinct. For the proof see [J].

Claim 27. For all $1 \leq i, j, k \leq 3$, the following hold.

$$\begin{array}{llll} I_i.A_{i,i} & = & A_{i,i} & I_i \times A_{i,i} & = & 0 \\ I_i.A_{i,j} & = & \frac{1}{2}A_{i,j} & I_i \times A_{i,j} & = & 0 \\ I_i.A_{j,j} & = & 0 & I_i \times A_{j,j} & = & \frac{1}{2}A_{k,k} \\ I_i.A_{j,k} & = & 0 & I_i \times A_{j,k} & = & -\frac{1}{2}A_{j,k}. \end{array}$$

Also, $\text{Tr}(A_{i,i})I_i = A_{i,i}$, and $\text{Tr}(A_{i,j}) = 0$.

Remark. The subgroup of G generated by $\text{SL}(3)$ and the $E_i(I_j)$ and $E'_i(I_j)$ for all choices of i, j is a split group of type D_4 .

For $1 \leq i \leq 3$, define

$$s_i = E_2(I_i)E'_2(-I_i)E_2(I_i).$$

It is easy to check that $[v_2 \otimes I_i, v'_2 \otimes I_j] = 0$ if $i \neq j$. Consequently, the s_i commute and $s = s_1 s_2 s_3$. Just as in section 2.3, one proves

Claim 28. The s_i act as follows:

$$\begin{aligned} x(t)^{s_i} &= E_1(tI_i) \\ E_1(A)^{s_i} &= x(-\langle A, I_i \rangle)E_1(2A.I_i - 2\langle A, I_i \rangle I_i)E'_3(-2A \times I_i) \\ E'_3(A)^{s_i} &= y(-\langle A, I_i \rangle)E'_3(2A.I_i - 2\langle A, I_i \rangle I_i)E_1(2A \times I_i) \\ y(t)^{s_i} &= E'_3(tI_i) \end{aligned}$$

For ease of notation, from now on we assume that the reduced Jordan algebra \mathcal{J} has the form 3×3 matrices over a composition algebra \mathcal{A} . As discussed at the end of section 1.4, although this does not include all reduced Jordan algebras, it is enough to give all forms of the p -adic exceptional groups. On the other hand, this assumption is not essential, as it would be straightforward to modify all of our results to include the general case.

If $A \in \mathcal{J}$, we will use the notation

$$A = \begin{pmatrix} a & d & \bar{e} \\ \bar{d} & b & f \\ e & \bar{f} & c \end{pmatrix}.$$

Here $A_{1,1} = a$, $A_{1,2} = d$ etc. Of course, $a, b, c \in F$ and $d, e, f \in \mathcal{A}$. The map $d \mapsto \bar{d}$ is the standard involution on \mathcal{A} . The norm and trace are the obvious ones. We can now write elements $x(t_1)E_1(A_1)E'_3(A_2)y(t_2)Z \in U/Z$ as

$$((t_1; a_1, b_1, c_1, d_1, e_1, f_1), (a_2, b_2, c_2, d_2, e_2, f_2; t_2)).$$

Corollary 29. *The action of the s_i on U/Z is as follows*

$$\begin{aligned} & ((t_1; a_1, b_1, c_1, d_1, e_1, f_1), (a_2, b_2, c_2, d_2, e_2, f_2; t_2))^{s_1} \\ & \quad = ((-a_1; t_1, c_2, b_2, d_1, e_1, -f_2), (t_2, -c_1, -b_1, d_2, e_2, f_1; -a_2)) \\ & ((t_1; a_1, b_1, c_1, d_1, e_1, f_1), (a_2, b_2, c_2, d_2, e_2, f_2; t_2))^{s_2} \\ & \quad = ((-b_1; c_2, t_1, a_2, d_1, -e_2, f_1), (-c_1, t_2, -a_1, d_2, e_1, f_2; -b_2)) \\ & ((t_1; a_1, b_1, c_1, d_1, e_1, f_1), (a_2, b_2, c_2, d_2, e_2, f_2; t_2))^{s_3} \\ & \quad = ((-c_1; b_2, a_2, t_1, -d_2, e_1, f_1), (-b_1, -a_1, t_2, d_1, e_2, f_2; -c_2)) \end{aligned}$$

Set

$$h_i(u) = E_2(uI_i)s_iE_2(u^{-1}I_i)s_i^{-1}E_2(uI_i)s_i^{-1}.$$

Using the last corollary and the known action of E_2 on U/Z , it is easy to compute the action of $h_i(u)$. Notice that $h_1(u)h_2(u)h_3(u) = h_\beta(u)$.

Corollary 30.

$$\begin{aligned} & ((t_1; a_1, b_1, c_1, d_1, e_1, f_1), (a_2, b_2, c_2, d_2, e_2, f_2; t_2))^{h_1(u)} \\ & \quad = ((ut_1; u^{-1}a_1, ub_1, uc_1, d_1, e_1, uf_1), (ua_2, u^{-1}b_2, u^{-1}c_2, d_2, e_2, u^{-1}f_2; u^{-1}t_2)) \\ & ((t_1; a_1, b_1, c_1, d_1, e_1, f_1), (a_2, b_2, c_2, d_2, e_2, f_2; t_2))^{h_2(u)} \\ & \quad = ((ut_1; ua_1, u^{-1}b_1, uc_1, d_1, ue_1, f_1), (u^{-1}a_2, ub_2, u^{-1}c_2, d_2, u^{-1}e_2, f_2; u^{-1}t_2)) \\ & ((t_1; a_1, b_1, c_1, d_1, e_1, f_1), (a_2, b_2, c_2, d_2, e_2, f_2; t_2))^{h_3(u)} \\ & \quad = ((ut_1; ua_1, ub_1, u^{-1}c_1, ud_1, e_1, f_1), (u^{-1}a_2, u^{-1}b_2, uc_2, u^{-1}d_2, e_2, f_2; u^{-1}t_2)) \end{aligned}$$

Finally, it is convenient to record the action of the s_i on E_2 . We give the action of s_1 ; the others are similar.

Claim 31.

$$\begin{aligned}
E_2(A_{1,1})^{s_1} &= E'_2(A_{1,1}) \\
E_2(A_{2,2})^{s_1} &= E_2(A_{2,2}) \\
E_2(A_{3,3})^{s_1} &= E_2(A_{3,3}) \\
E_2(A_{1,2})^{s_1} &= \exp(L_{A_{1,2}} + 2L(I_1, A_{1,2})) \in M_0 \\
E_2(A_{1,3})^{s_1} &= \exp(L_{A_{1,3}} + 2L(I_1, A_{1,3})) \in M_0 \\
E_2(A_{2,3})^{s_1} &= E_2(A_{2,3})
\end{aligned}$$

We need a result like Proposition 22 except with I_i in place of I . Let $q_{\mathcal{A}}$ denote the norm form on \mathcal{A} , $q_{\mathcal{A}}(d) = d\bar{d}$, and let $\tilde{\delta}$ be the discriminant of $q_{\mathcal{A}}$. Also, recall that $c = \dim \mathcal{J}$. Then clearly $\dim \mathcal{A} = c/3 - 1$.

Lemma 32. $\delta = 2^{c/3-1}\tilde{\delta}$.

Proof. A simple computation shows that

$$\langle A, A \rangle = a^2 + b^2 + c^2 + 2(q_{\mathcal{A}}(d) + q_{\mathcal{A}}(e) + q_{\mathcal{A}}(f)).$$

Thus, $\delta = 2^{\dim \mathcal{A}}\tilde{\delta} = 2^{c/3-1}\tilde{\delta}$. □

We state the next proposition in terms of I_1 for convenience; the analogous result for I_2 and I_3 also holds.

Proposition 33. (1) $Q_{I_1}(t; a, b, c, d, e, f) = -bc + q_{\mathcal{A}}(f)$.
(2) $D_{kI_1} = -(k/2)^{c/3+1}\delta$ for any non-zero $k \in F$.
(3) $S_{kI_1} = S_{I_1}(\delta, k)_F^c(-1, k)_F^{c(c+1)/2}$.

Proof. Part (1) is a simple computation:

$$\begin{aligned}
Q_{I_1}(t, A) &= -\text{Tr}(I_1 \cdot A \times A) \\
&= -\text{Tr}(I_1 \cdot (A \cdot A)) + \text{Tr}(A) \text{Tr}(A \cdot I_1) - 1/2 \text{Tr}(A)^2 + 1/2 \text{Tr}(A \cdot A) \\
&= -(a^2 + q_{\mathcal{A}}(d) + q_{\mathcal{A}}(e)) + (a + b + c)(a) - 1/2(a + b + c)^2 \\
&\quad + 1/2(a^2 + b^2 + c^2 + 2q_{\mathcal{A}}(d) + 2q_{\mathcal{A}}(e) + 2q_{\mathcal{A}}(f)) \\
&= -bc + q_{\mathcal{A}}(f).
\end{aligned}$$

This proves part (1) and that $D_{I_1} = -\tilde{\delta}$. Since it is clear that $Q_{kI_1} = kQ_{I_1}$, part (2) follows from the lemma. Finally, using the same argument as for part (3) of Proposition 22, we get that

$$S_{kI_1} = S_{I_1}(D_{I_1}, k)_F^{c/3}(k, k)_F^{c/3(c/3+1)/2}.$$

By statement (2), $(D_{I_1}, k)_F^{c/3} = (\delta, k)_F^{c/3}(1/2, k)_F^{c/3(c/3+1)}(-1, k)_F^{c/3}$ which equals $(\delta, k)_F^c(-1, k)_F^c$. Thus,

$$\begin{aligned}
S_{kI_1} &= S_{I_1}(\delta, k)_F^c(-1, k)_F^c(-1, k)_F^{c(c+3)/2} \\
&= S_{I_1}(\delta, k)_F^c(-1, k)_F^{c(c+1)/2}.
\end{aligned}$$

Statement (3) is proved. □

Just as with Proposition 22, there are corollaries about γ invariants.

Corollary 34. *Suppose that c is odd. If $k \neq 0$, then*

$$\frac{\gamma(kI_i)}{\gamma(I_i)} = (\delta, k)_F(-1, k)_F^{(c+1)/2}$$

Corollary 35. *Suppose that c is odd and $k \neq 0$. Then $Q_{kI_i} \oplus Q_{-kI_i}$ is trivial in \mathcal{W}_F . In particular, $\gamma(kI_i)\gamma(-kI_i) = 1$. Finally, $\gamma(I_i)^2 = (\delta, -1)_F$.*

Corollary 36. *Suppose that c is odd and that $a, b, c \in F$ are non-zero. Then,*

$$\gamma(aI_i)\gamma(bI_i)\gamma(cI_i) = \gamma(-abcI_i)$$

We conclude this section with a version of the Bruhat decomposition for H . The proof is straightforward.

Proposition 37. *Set $\mathcal{P} = ME_2$. If H has rank one, then $H = \mathcal{P} \cup \mathcal{P}s\mathcal{P}$, disjoint union. Otherwise, \mathcal{J} is reduced and $H = \mathcal{P} \cup \mathcal{P}s_1\mathcal{P} \cup \mathcal{P}s_1s_2\mathcal{P} \cup \mathcal{P}s_1s_2s_3\mathcal{P}$.*

3. THE REPRESENTATION

3.1. Representation of P . As we have seen, the action of H on U/Z leads to a map $p: H \mapsto \mathrm{Sp}(U/Z)$. It is easy to see that the kernel of p is precisely Z_G . Since U is a Heisenberg group, for each fixed character Ψ , U has a canonical unitary representation. We can realize the Heisenberg representation on $\hat{V} = L^2(W)$. Denote the action by $\hat{\pi}_\Psi$. By the well known theory of the Weil representation, using the map p , $\hat{\pi}_\Psi$ extends naturally to a projective representation of $P^\circ = HU$. As we will see, the cocycle corresponding to this projective representation is trivial if and only if $c = \dim \mathcal{J}$ is odd. The analysis in each case will be similar. Nevertheless, to simplify the discussion, the case of c even (which leads to G of type F_4) will be discussed in section 3.5. Until then, we *assume c is odd*.

We will need some terminology from [R]. In Theorem 3.5 of [R], Rao gives formulas which define a projective representation of $\mathrm{Sp}(U/Z)$. This is the *standard Weil representation*. His notation is $r(\sigma)$ for the operator corresponding to $\sigma \in \mathrm{Sp}(U/Z)$. The *multiplier* of the standard Weil representation are the numbers $c(\sigma_1, \sigma_2)$ which satisfy $r(\sigma_1)r(\sigma_2) = c(\sigma_1, \sigma_2)r(\sigma_1\sigma_2)$. In section 5 of [R], Rao defines *normalization factors*, $m(\sigma)$, which have the property that the multiplier of the normalized representation, $\hat{r}(\sigma) = m(\sigma)r(\sigma)$, takes values in ± 1 .

The next lemma follows from [R] Theorem 4.1 and some simple calculations.

Lemma 38. *The multiplier, $c(\cdot, \cdot)$, of the standard Weil representation restricted to $p(H)$ has the following properties.*

- (1) *Suppose that $r, r_1, r_2 \in \mathcal{P} = ME_2$ and that η_1, η_2 are arbitrary. Then $c(r_1\eta_1r, r^{-1}\eta_2r_2) = c(\eta_1, \eta_2)$.*
- (2) *$c(\eta, 1) = c(1, \eta) = 1$ for η arbitrary.*
- (3) *$c(\sigma_1m, \sigma_2) = 1$ where $\sigma_1, \sigma_2 \in \{s^{\pm 1}, s_i^{\pm 1}, (s_i s_j)^{\pm 1}\}$ and $m \in M$.*
- (4) *$c(s^{-1}E_2(J), s) = \gamma(J)$.*
- (5) *If \mathcal{J} is reduced, $c(s_i^{-1}E_2(kI_i), s_i) = \gamma(kI_i)$.*

In the next proposition, we define normalization factors which eliminate the multiplier entirely.

Proposition 39. *Suppose that $c = \dim \mathcal{J}$ is odd. Then, the cocycle corresponding to the projective representation $\hat{\pi}_\Psi$ splits over H . In particular, $\hat{\pi}_\Psi$ extends to a representation of P° .*

Furthermore, the action of H on $L^2(W)$ is related to the standard Weil representation by the following normalization factors:

$$\phi(mE_2(A)) = (d(m), \delta)_F(d(m), -1)_F^{(c+1)/2},$$

$\phi(s) = \gamma(I)$ and, if \mathcal{J} is reduced, $\phi(s_i) = \gamma(I_i)$ and $\phi(s_i s_j) = \gamma(I_i) \gamma(I_j)$ for $i \neq j$. Also, $\phi(m_1 E_2(A_1) \sigma m_2 E_2(A_2)) = \phi(m_1 m_2) \phi(\sigma)$ for $\sigma = s, s_i$ or $s_i s_j$.

Remark. ϕ is well defined on all of H by Proposition 37.

For the proof, we will need the following lemma which follows from the work of Steinberg and is well known; see e.g. [S1].

Lemma 40. *Let $G_1 \subset G_2$ be simple, split, simply connected groups with maximal tori $T_1 \subset T_2$. If α is a root of G_i , normalize the killing form, $\langle \cdot, \cdot \rangle_i$, so that $\langle \alpha, \hat{\alpha} \rangle_i = 2$. Then, the non-trivial degree n cover of G_2 splits over G_1 if and only if $\langle \cdot, \cdot \rangle_2|_{T_1} = nm \langle \cdot, \cdot \rangle_1$ for some integer m .*

We will apply the lemma in the case of the metaplectic cover of $\mathrm{Sp}(W)$, and various $\mathrm{SL}(2) \subset \mathrm{Sp}(W)$.

Proof. Let us consider the $\mathrm{SL}(2) \subset H$ corresponding to β , that is, generated by $E_2(kI)$ and $E'_2(kI)$ and with torus $h_\beta(t)$. We find that p restricted to this $\mathrm{SL}(2)$ is injective. Using Lemma 40 and our formulas for the action of $h_\beta(t)$, it is now easy to see that the metaplectic cover splits over this subgroup. Note that this fails when $\dim \mathcal{J} = 6$.

Next, we compute the normalizing factors for this subgroup. Lemma 38 and section 5 of [R] imply that $\phi|_{E_2(kI)}$ is a homomorphism $F \rightarrow \mathbb{Q}/\mathbb{Z}$. Thus, $\phi(E_2(kI)) = 1$. Also, since $s = E_2(I)s^{-1}E_2(I)sE_2(I)$, $\phi(s) = \phi(s^{-1})\phi(s)\gamma(I) = \gamma(I)$ by Lemma 38. It remains to compute $\phi(h(t))$. Using the relation

$$h(t)s = E_2(tI)s^{-1}E_2(t^{-1}I)sE_2(tI)$$

as well as Lemma 38, we see that

$$\phi(h(t))\phi(s) = \phi(s^{-1})\phi(s)c(s^{-1}E_2(t^{-1}I), s) = \gamma(t^{-1}I).$$

Thus,

$$\phi(h(t)) = \frac{\gamma(t^{-1}I)}{\gamma(I)}.$$

Hence, by Corollary 24,

$$\phi(h(t)) = (\delta, t^{-3})_F(-1, t^{-3})_F^{(c+1)/2}.$$

If \mathcal{J} is reduced, similar reasoning, using Lemma 40 and Corollary 34, shows that the metaplectic cover splits over the corresponding $\mathrm{SL}(2)$'s – that is, those generated by $E_2(kI_i)$ and $E'_2(kI_i)$ – and that $\phi(s_i) = \gamma(I_i)$ and $\phi(h_i(t)) = (\delta, t)_F(-1, t)_F^{(c+1)/2}$.

To complete the argument, we must show that for any $\eta_1, \eta_2 \in H$,

$$(16) \quad \phi(\eta_1)\phi(\eta_2)c(\eta_1, \eta_2) = \phi(\eta_1\eta_2)$$

In the case that H has rank one, Lemma 38 and Proposition 37 imply that it is enough to consider $\eta_1 = s^{-1}E_2(B)$ with B invertible, and $\eta_2 = s$. We have,

$$\eta_1\eta_2 = E_2(B^{-1})sm_BE_2(B^{-1})$$

with $d(m_B) = n(B)$. Thus, we need to check that

$$\gamma(B) = \gamma(I)(n(B), \delta)_F(n(B), -1)_F^{(c+1)/2}$$

which is correct by Proposition 22.

If H does not have rank 1, we can still use Lemma 38 and Proposition 37 to reduce the problem, but there are more cases to check. This can be done, but it is rather tedious. Instead, we argue as follows. It is a result of Prasad and Raghunathan ([Pr-R] Theorem 9.5) that if a cocycle on a simply connected group is trivial on an $\mathrm{SL}(2)$ corresponding to a long root, then it is trivial on the whole group. It follows that the cocycle must be trivial on H . Then, since equation (16) must hold, it follows from the calculations above that ϕ is as we claimed. \square

Remark. Since Lemma 40 is true for any local field of characteristic zero, so is the fact that $\hat{\pi}_\Psi$ extends to a representation of P° .

Now that we have explicit knowledge of the splitting, we can easily deduce formulas for $\hat{\pi}_\Psi$ from the standard Weil representation in [R]. The vector space is $\hat{V} = L^2(W)$; we will take $(r, R) \in W$ as the variable. Also, we write (a, A) and (B, b) for $x(a)E_1(A)$ and $E'_3(B)y(b)$, respectively.

Proposition 41. *The representation $\hat{\pi}_\Psi: P^\circ \rightarrow$ unitary operators on \hat{V} is given by the following formulas:*

$$\begin{aligned} \hat{\pi}_\Psi(a, A)\mathbf{f}(r, R) &= \mathbf{f}(r + a, R + A) \\ \hat{\pi}_\Psi(B, b)\mathbf{f}(r, R) &= \Psi(\langle R, B \rangle + rb)\mathbf{f}(r, R) \\ \hat{\pi}_\Psi(z(t))\mathbf{f}(r, R) &= \Psi(t)\mathbf{f}(r, R) \\ \hat{\pi}_\Psi(m)\mathbf{f}(r, R) &= (d(m), \delta)_F(d(m), -1)_F^{(c+1)/2} |d(m)|^{(c/3+1)/2} \mathbf{f}(d(m)r, d(m)R^m) \\ \hat{\pi}_\Psi(h_\beta(t))\mathbf{f}(r, R) &= (t, \delta)_F(t, -1)_F^{(c+1)/2} |t|^{(c+3)/2} \mathbf{f}(t^3r, tR) \\ \hat{\pi}_\Psi(E_2(J))\mathbf{f}(r, R) &= \Psi(Q_J(r, R))\mathbf{f}(r, R + rJ) \\ \hat{\pi}_\Psi(s)\mathbf{f}(r, R) &= \gamma(I) \int_{F \times \mathcal{J}} \Psi(\langle \tilde{R}, R \rangle + r\tilde{r}) \mathbf{f}(\tilde{r}, \tilde{R}) d\tilde{r} d\tilde{R} \end{aligned}$$

Remark. We should be more precise about the meaning of the integral operator $\hat{\pi}_\Psi(s)$ (essentially the Fourier transform). This integral converges on Schwartz-Bruhat functions but need not converge on a general L^2 function. However, since the Schwartz-Bruhat functions are dense in L^2 , the operator has a canonical extension. Henceforth, we will use this convention without comment.

If \mathcal{J} is reduced, we use the notation $R = (a, b, c, d, e, f)$ where $a, b, c \in F$ and $d, e, f \in \mathcal{A}$ as in section 2.5. Also, it is useful to have notations for those elements of \mathcal{J} which commute with I_i . We have $C_1 = (0, b, c, 0, 0, f)$, $C_2 = (a, 0, c, 0, e, 0)$, and $C_3 = (a, b, 0, d, 0, 0)$. We usually view $C_i \in \mathcal{J}$ as an element of W . Then, $C_i^{s_i} \in \mathcal{J} \subset W'$. We write the corresponding map on elements of \mathcal{J} as $C_i \mapsto C'_i$. For example, $C'_1 = (0, -c, -b, 0, 0, f)$. Finally, set $C_1^{(\nu)} = (0, \nu_b, \nu_c, 0, 0, \nu_f)$ etc.

Proposition 42. *If \mathcal{J} is reduced,*

$$\hat{\pi}(s_1)\mathbf{f}(r, a, b, c, d, e, f) = \gamma(I_1) \int_{F \times F \times \mathcal{A}} \Psi(\langle C_1^{(\nu)}, C'_1 \rangle) \mathbf{f}(-a, r, \nu_b, \nu_c, d, e, \nu_f) dC_1^{(\nu)}$$

$$\hat{\pi}(s_2)\mathbf{f}(r, a, b, c, d, e, f) = \gamma(I_2) \int_{F \times F \times \mathcal{A}} \Psi(\langle C_2^{(\nu)}, C_2' \rangle) \mathbf{f}(-b, \nu_a, r, \nu_c, d, \nu_e, f) dC_2^{(\nu)}$$

$$\hat{\pi}(s_3)\mathbf{f}(r, a, b, c, d, e, f) = \gamma(I_3) \int_{F \times F \times \mathcal{A}} \Psi(\langle C_3^{(\nu)}, C_3' \rangle) \mathbf{f}(-c, \nu_a, \nu_b, r, \nu_d, e, f) dC_3^{(\nu)}$$

Set $\pi = \text{Ind}_{P^\circ}^P \hat{\pi}_\psi$. The representation π may be realized on $V = L^2(W \times F)$; here are the formulas for the action.

Proposition 43. *The representation $\pi: P \rightarrow$ unitary operators on V is given by the following formulas:*

$$\pi(h_\alpha(t))\mathbf{f}(r, R, u) = \mathbf{f}(r, R, tu)$$

$$\pi(a, A)\mathbf{f}(r, R, u) = \mathbf{f}(r + u^2a, R + uA, u)$$

$$\pi(B, b)\mathbf{f}(r, R, u) = \Psi(\langle R, B \rangle + rbu^{-1})\mathbf{f}(r, R, u)$$

$$\pi(z(t))\mathbf{f}(r, R, u) = \Psi(ut)\mathbf{f}(r, R, u)$$

$$\pi(m)\mathbf{f}(r, R, u) = (d(m), \delta)_F(d(m), -1)_F^{(c+1)/2} |d(m)|^{(c/3+1)/2} \mathbf{f}(d(m)r, d(m)R^m, u)$$

$$\pi(h_\beta(t))\mathbf{f}(r, R, u) = (t, \delta)_F(t, -1)_F^{(c+1)/2} |t|^{(c+3)/2} \mathbf{f}(t^3r, tR, u)$$

$$\pi(E_2(J))\mathbf{f}(r, R, u) = \Psi(Q_{u^{-1}J}(r, R))\mathbf{f}(r, R + ru^{-1}J, u)$$

$$\pi(s)\mathbf{f}(r, R, u) = \gamma(uI)|u|^{-(c+3)/2} \int_{F \times \mathcal{J}} \Psi(\langle \tilde{R}, u^{-1}R \rangle + u^{-3}r\tilde{r})\mathbf{f}(\tilde{r}, \tilde{R}, u)d\tilde{r}d\tilde{R}$$

Furthermore, π is irreducible. In fact, π restricted to $h_\alpha U$ is irreducible.

Remark. Observe that $Q_{u^{-1}J}(r, R) = u^{-1}Q_J(u^{-1}r, R)$.

Proof. The formulas follow immediately from Proposition 41 and the formulas for the left action of h_α on P° (see Claim 21) and Corollary 24.

We now prove that $\pi|_{h_\alpha U}$ is irreducible. Clearly, $\pi = \text{Ind}_U^{h_\alpha U} \rho$ where ρ is the unique irreducible representation of U with central character Ψ . (Recall that U is a Heisenberg group.) Thus, as follows from Mackey's theory of representations of semi-direct products, if ρ_t is the representation of U given by $\rho_t(u) = \rho(h_\alpha(t)uh_\alpha(t^{-1}))$, then it is enough to show that for each $t \in F$, the ρ_t are distinct. But ρ_t has central character Ψ_t where $\Psi_t(x) = \Psi(tx)$. \square

Corollary 44. *The representation π is independent of Ψ .*

Proof. Let (V_c, π_c) be the representation with $\Psi(x)$ replaced by $\Psi_c(x) = \Psi(cx)$. We must define a map $\Omega: V \rightarrow V_c$ which intertwines the action of P . By checking the formulas, it is easy to see that the map

$$\Omega(\mathbf{f}(r, R, u)) = \mathbf{f}(c^2r, cR, cu)$$

works. Note that the definition of γ depends upon the choice of Ψ . \square

It is easy to check that $h_\alpha(u)s_i h_\alpha(u^{-1}) = h_i(u^{-1})s_i$. Thus, using Corollary 30 and Corollary 34, we can deduce formulas for the $\pi(s_i)$ from those of $\hat{\pi}(s_i)$. We get

Proposition 45. *If \mathcal{J} is reduced,*

$$\begin{aligned} & \pi(s_1)\mathbf{f}(r, a, b, c, d, e, f, u) \\ &= \gamma(uI_1)|u|^{-(c/3+1)/2} \int_{F \times F \times \mathcal{A}} \Psi(\langle C_1^{(\nu)}, u^{-1}C_1' \rangle) \mathbf{f}(-ua, u^{-1}r, \nu_b, \nu_c, d, e, \nu_f, u) dC_1^{(\nu)} \end{aligned}$$

$$\begin{aligned}
& \pi(s_2)\mathbf{f}(r, a, b, c, d, e, f, u) \\
&= \gamma(uI_2)|u|^{-(c/3+1)/2} \int_{F \times F \times \mathcal{A}} \Psi(\langle C_2^{(\nu)}, u^{-1}C_2' \rangle) \mathbf{f}(-ub, \nu_a, u^{-1}r, \nu_c, d, \nu_e, f, u) dC_2^{(\nu)} \\
& \pi(s_3)\mathbf{f}(r, a, b, c, d, e, f, u) \\
&= \gamma(uI_3)|u|^{-(c/3+1)/2} \int_{F \times F \times \mathcal{A}} \Psi(\langle C_3^{(\nu)}, u^{-1}C_3' \rangle) \mathbf{f}(-uc, \nu_a, \nu_b, u^{-1}r, \nu_d, e, f) dC_3^{(\nu)}
\end{aligned}$$

3.2. Representation of G . Our goal now is to extend π to a representation of G . We continue to assume that c is odd. The corresponding results for c even are in section 3.5.

We begin by restricting π to B and then trying to extend it to a representation of P_α .

Lemma 46. P_α is generated by B and s_α with the following relations:

- (1) $s_\alpha y(t) = z(t)s_\alpha$,
- (2) $s_\alpha E_2(A) = E_1(A)s_\alpha$,
- (3) $s_\alpha E_3'(A) = E_3'(A)s_\alpha$,
- (4) $s_\alpha m = mh_\alpha(d(m))s_\alpha$,
- (5) $s_\alpha h_\alpha(t) = h_\alpha(t^{-1})s_\alpha$,
- (6) $s_\alpha^2 = h_\alpha(-1)$,
- (7) $x(t)s_\alpha x(t^{-1})s_\alpha^{-1}x(t) = h_\alpha(t)s_\alpha$.

Proposition 47. There is a unique extension of π from B to P_α . It is given by

$$(17) \quad \pi(s_\alpha)\mathbf{f}(r, R, u) = \Psi\left(\frac{n(R)}{r}\right) \mathbf{f}(-r, R, -r/u).$$

Proof. First, we verify that equation (17) does define an extension of π to P_α . For this, we must check relations (1)-(7) of Lemma 46. The hardest of these are (2) and (7). We will check these and leave the rest as an easy exercise.

For (2), we get

$$\begin{aligned}
\pi(s_\alpha E_2(A))\mathbf{f}(r, R, u) &= \Psi\left(\frac{n(R)}{r}\right) \Psi\left(-\frac{uQ_A(u, R)}{r}\right) \mathbf{f}(-r, R + uA, -r/u) \\
&= \Psi\left(\frac{n(R)}{r}\right) \Psi\left(-\frac{n(R) - n(R + uA)}{r}\right) \mathbf{f}(-r, R + uA, -r/u)
\end{aligned}$$

using Proposition 22 part (1)

$$= \Psi\left(\frac{n(R + uA)}{r}\right) \mathbf{f}(-r, R + uA, -r/u).$$

On the other hand,

$$\pi(E_1(A)s_\alpha)\mathbf{f}(r, R, u) = \Psi\left(\frac{n(R + uA)}{r}\right) \mathbf{f}(-r, R + uA, -r/u),$$

so (2) is proved.

For (7),

$$\begin{aligned}
& \pi(x(t)s_\alpha x(t^{-1})s_\alpha^{-1}x(t))\mathbf{f}(r, R, u) \\
&= \pi(x(t)s_\alpha x(t^{-1}))\Psi\left(\frac{n(R)}{r}\right) \mathbf{f}\left(-r + \frac{r^2 t}{u^2}, R, \frac{r}{u}\right)
\end{aligned}$$

$$\begin{aligned}
&= \pi(x(t)s_\alpha)\Psi\left(\frac{n(R)}{r+u^2t^{-1}}\right)\mathbf{f}\left(-r-u^2t^{-1}+\frac{(r+u^2t^{-1})^2t}{u^2}, R, \frac{r+u^2t^{-1}}{u}\right) \\
&= \pi(x(t)s_\alpha)\Psi\left(\frac{n(R)}{r+u^2t^{-1}}\right)\mathbf{f}\left(\frac{r^2t}{u^2}+r, R, \frac{r}{u}+\frac{u}{t}\right) \\
&= \pi(x(t))\Psi\left(\frac{n(R)}{r}\right)\Psi\left(\frac{n(R)}{-r+r^2/tu^2}\right)\mathbf{f}(u^2t-r, R, u-\frac{r}{ut}) \\
&= \pi(x(t))\Psi\left(\frac{n(R)}{r}\right)\Psi\left(\frac{tu^2n(R)}{r(r-tu^2)}\right)\mathbf{f}(u^2t-r, R, u-\frac{r}{ut}) \\
&= \pi(x(t))\Psi\left(\frac{n(R)}{r-tu^2}\right)\mathbf{f}(u^2-r, R, u-\frac{r}{ut}) \\
&= \Psi\left(\frac{n(R)}{r}\right)\mathbf{f}\left(-r, R, u-\frac{r+u^2t}{ut}\right) \\
&= \Psi\left(\frac{n(R)}{r}\right)\mathbf{f}\left(-r, R, -\frac{r}{ut}\right).
\end{aligned}$$

On the other hand,

$$\pi(h_\alpha(t)s_\alpha)\mathbf{f}(r, R, u) = \Psi\left(\frac{n(R)}{r}\right)\mathbf{f}\left(-r, R, -\frac{r}{ut}\right),$$

so (7) is proved.

It remains to prove that this extension of π to P_α is unique. The key is the following lemma. Let $l: F^* \rightarrow F^*/(F^*)^3$ be the canonical projection. Then, for $\delta \in F^*/(F^*)^3$, let $F_\delta = \{x \in F \mid l(x) \in \delta\}$.

Lemma 48. π restricted to TU_α decomposes into the following direct sum of inequivalent irreducible representations of TU_α :

$$\pi = \bigoplus_{\delta \in F^*/(F^*)^3} V_\delta$$

where $V_\delta = \{\mathbf{f} \in L^2(F_\delta \times W \times F)\} \subset V$.

We will prove the lemma after using it to complete the proof of Proposition 47.

Let Σ be the operator $\pi(s_\alpha)$ from the statement of the proposition, and suppose that Σ_0 is another unitary operator on V which extends π . Since TU_α is invariant under conjugation by s_α , it follows from Schur's lemma and Lemma 48 that, for each $\delta \in F^*/(F^*)^3$, Σ and Σ_0 preserve V_δ and are equal up to multiplication by some constant c_δ . Furthermore, relation (6) in Lemma 46 implies that $c_\delta = \pm 1$.

To prove that in fact each $c_\delta = 1$, we use relation (7) from Lemma 46. Actually, let us re-write it as

$$(18) \quad x(t)s_\alpha x(t^{-1})s_\alpha^{-1} = h_\alpha(t)s_\alpha x(t^{-1}).$$

I claim that we can find an $\mathbf{f} \in V_\delta$ and $t \in F$ so that $\pi(x(t^{-1}))\mathbf{f} \in V_\delta$ and also $\pi(x(t^{-1})s_\alpha^{-1})\mathbf{f} \in V_\delta$. This would complete the proof because by applying both sides of equation (18) – with this t – to \mathbf{f} , we see that $c_\delta^2 = c_\delta$. Thus, $c_\delta = 1$ and $\Sigma = \Sigma_0$.

To see that there exists such a t and \mathbf{f} , let ϵ be a small number. Certainly there is a Schwartz function $\mathbf{f} \in V_\delta$ which satisfies $\mathbf{f}(r, R, u) \neq 0$ only if $\epsilon < |r| < 1/\epsilon$ and $\epsilon < |u| < 1/\epsilon$. Notice that $\Sigma^{-1}(\mathbf{f})$ (or $\Sigma_0^{-1}(\mathbf{f})$) has the same property except with ϵ^2 in place of ϵ . Suppose that \mathbf{f} satisfies this property. We wish to show that we can find t so large that $\mathbf{f}(r+t^{-1}u^2, R, u) \in V_\delta$. We can certainly take t large

enough so that the support is contained in $\epsilon < |r|$. Furthermore, taking t larger if necessary, we may assume that whenever this function is non-zero, $|t^{-1}u^2r^{-1}|$ is so small that $1 + t^{-1}u^2r^{-1}$ is a cube. Thus, $r + t^{-1}u^2 \in \delta$ implies $r \in \delta$. In other words, $\pi(x(t^{-1}))\mathbf{f}$ and $\pi(x(t^{-1})s_\alpha^{-1})\mathbf{f}$ are in V_δ .

This completes the proof of Proposition 47. \square

We now prove Lemma 48.

Proof. It is clear from the formulas that π restricted to TU_α is the direct sum of the V_δ . We must show that the V_δ are irreducible and inequivalent. Consider V_δ restricted to U_α . For each $b, c \in F$, I will define a quotient of V_δ in the space $L^2(\mathcal{J})$; evaluate smooth functions at $r = b, u = c$, and then extend by continuity. Denote this unitary representation of U_α by $\chi_{b,c}$. Of course, by the definition of V_δ , $\chi_{b,c}$ is non-zero exactly when $b \in F_\delta$. Furthermore, it is easy to check that the non-zero $\chi_{b,c}$ are distinct; just restrict to $K = y(s)E_1E'_3Z \subset U_\alpha$ which is the direct sum of an abelian group $y(s)$ and a Heisenberg group. The representation $\chi_{b,c}$ restricted to K is the tensor product of the character $\Psi_{bc^{-1}}$ of $y(s)$ and the Heisenberg representation with central character Ψ_c . This also proves that the $\chi_{b,c}$ are irreducible.

Now, T acts on U_α by conjugation and thus acts on the space of irreducible representations of U_α . More precisely, if $\tau \in T$ and $m \in U_\alpha$, $\chi_{b,c}^\tau(m) = \chi_{b,c}(m^\tau)$. I claim that if $\tau = h_\alpha(t_1)h_\beta(t_2)$ then there is an equivalence of unitary representations:

$$\chi_{b,c}^\tau = \chi_{t_2^3b, t_1c}.$$

The proof of this fact is straightforward but tedious so I will omit the details. The point is that both representations may be realized in $L^2(\mathcal{J})$ and formulas for the actions are easily deduced from the formulas for π and the action of T on U_α (section 2.2); one checks that the formulas coincide after making the change of variable $\mathbf{f}(J) \mapsto \mathbf{f}(t_2J)$. This proves, in particular, that the set of $\chi_{b,c}$ with $b \in \delta$ forms a single orbit under the action of T .

Thus we see that V_δ restricted to U_α is the direct integral of irreducible representations in a single T -orbit. In other words, V_δ restricted to U_α is identified with $\text{Ind}_{T^0U_\alpha}^{TU_\alpha} \chi_{b_0, c_0}$ restricted to U_α , where T^0 is the stabilizer of χ_{b_0, c_0} . By Mackey's theory, this shows that the V_δ are irreducible representations of TU_α . It also proves that they are inequivalent because their restrictions to U_α are distinct.

Lemma 48 is proved. \square

We have the following corollary of Proposition 47.

Corollary 49. *There is at most one representation of G whose restriction to P coincides with π .*

Theorem 50. *Suppose that $c = \dim \mathcal{J}$ is odd. There exists a unique unitary representation, π , of G so that π restricted to P is given by Propositions 43 and 45, and π restricted to P_α is given by Proposition 47.*

The uniqueness is clear. Here is a strategy for proving existence. It follows from [T1, section 13], that G is the free product of P and P_α amalgamated by B , and subject to the following braid relations. In the rank two case

$$(19) \quad (s_\alpha s_\beta)^3 = (s_\beta s_\alpha)^3,$$

and in the reduced case

$$(20) \quad s_\alpha s_i s_\alpha = s_i s_\alpha s_i,$$

for $i = 1, 2, 3$. Since our representations of P and of P_α coincide on $P \cap P_\alpha = B$, to prove Theorem 50 it suffices to establish that the operators $\pi(s_\alpha)$ and $\pi(s_\beta)$ satisfy equation (19) (or (20)).

This strategy works when \mathcal{J} is reduced. In the next section, we prove that π satisfies equation (20) and so prove Theorem 50 in this case. This will make essential use of Weil's computation of the Fourier transform of a quadratic function. Unfortunately, I do not know how to check directly that π satisfies equation (19); this would seem to require understanding the Fourier transform of *cubic* functions (such as $\Psi(n(R))$). Therefore, we use an indirect method for proving the existence part of Theorem 50 in the rank two case. The highly non-trivial braid relation is then a consequence of the theorem.

3.3. Braid relation (reduced case).

Theorem 51. *For $i = 1, 2, 3$, there is an equivalence of unitary operators*

$$\pi(s_\alpha)\pi(s_i)\pi(s_\alpha) = \pi(s_i)\pi(s_\alpha)\pi(s_i).$$

The argument is essentially the same for each $i = 1, 2, 3$ so we just give the case $i = 1$. For convenience, we will write the operators $\pi(s_\alpha)$ and $\pi(s_1)$ as A and B , respectively. Thus, we must prove that

$$ABA = BAB.$$

The main part of the proof turns out to be the following theorem.

Theorem 52. *Set*

$$\Phi(r, R, u) = \gamma\left(-\frac{r}{u}I_1\right)|r/u|^{-(c/3+1)/2}\Psi\left(\frac{n(R)}{r}\right).$$

Then $B(\Phi) = \Phi$.

For the proof we will need some notation and a lemma. We use the notation defined before the statement of Proposition 42 except we write just C instead of C_1 etc. Also, set $D = (0, q_A(d), q_A(e), 0, 0, d, e)$. It is straightforward to prove

Lemma 53.

$$\begin{aligned} \frac{n(R)}{r} &= Q_{-\frac{a}{r}I_1}(1, C - 1/aD) \\ &= Q_{-\frac{a}{r}I_1}(1, C' - 1/aD') \end{aligned}$$

and

$$\frac{n(R)}{r} = 1/r\langle C, D' \rangle + Q_{-\frac{a}{r}I_1}(1, C).$$

Proof (Theorem 52). The main point is to use Weil's computation of the Fourier transform of a quadratic function. As a matter of notation, we will write the operator B as an integral (see remark on page 156). Alternately, we could write it out in terms of \mathcal{F} , the Fourier transform.

Using the second part of the lemma,

$$\begin{aligned}
B(\Phi(r, R, u)) &= \gamma(uI_1)|u|^{-(c/3+1)/2}\gamma(aI_1)|a|^{-(c/3+1)/2} \\
&\quad \times \int_{C^{(\nu)}} \Psi(\langle C^{(\nu)}, u^{-1}C' \rangle) \Psi(-1/ua\langle C^{(\nu)}, D' \rangle) \Psi(Q_{\frac{r}{au^2}I_1}(1, C^{(\nu)})) dC^{(\nu)} \\
&= \gamma(uI_1)\gamma(aI_1)|ua|^{-(c/3+1)/2} \\
&\quad \times \int_{C^{(\nu)}} \Psi(\langle C^{(\nu)}, 1/u(C' - 1/aD') \rangle) \Psi(Q_{\frac{r}{au^2}I_1}(1, C^{(\nu)})) dC^{(\nu)}.
\end{aligned}$$

By the definition of γ (i.e. Weil's computation)

$$\begin{aligned}
&= \gamma(uI_1)\gamma(aI_1)\gamma(r/au^2I_1)|ua|^{-(c/3+1)/2}|r/au^2|^{-(c/3+1)/2} \\
&\quad \times \Psi(Q_{-\frac{au^2}{r}I_1}(1, 1/u(C' - 1/aD')))
\end{aligned}$$

using Corollary 36

$$= \gamma(-r/uI_1)|r/u|^{-(c/3+1)/2}\Psi(Q_{-\frac{a}{r}I_1}(1, C' - 1/aD'))$$

by the lemma

$$= \Phi(r, R, u). \quad \square$$

Proof (Theorem 51). Let $S(W \times F^*)$ be the space of Schwartz-Bruhat functions. Then $S(W \times F^*) \subset V$ is dense. Thus, to show that $ABA = BAB$, it is enough to show that $ABA(\mathbf{f}) = BAB(\mathbf{f})$ for $\mathbf{f} \in S(W \times F^*)$. The point is that for such \mathbf{f} the integral formula for B is valid (see the remark on page 156). Furthermore, it is easy to see that the integral formula is valid for B applied to $A(\mathbf{f})$. Now we compute:

$$ABA(\mathbf{f}) = \int_{C^{(\nu)}} K\mathbf{f}(-ra/u, u, \nu_b, \nu_c, d, e, \nu_f, a) dC^{(\nu)}$$

where

$$\begin{aligned}
K &= \Psi\left(\frac{n(R)}{r}\right) \gamma\left(-\frac{r}{u}I_1\right) |r/u|^{-(c/3+1)/2} \Psi(-\langle C^{(\nu)}, u/rC' \rangle) \\
&\quad \times \Psi(Q_{-\frac{u^2}{ra}I_1}(1, C^{(\nu)} - 1/uD)) \\
&= \gamma\left(-\frac{r}{u}I_1\right) |r/u|^{-(c/3+1)/2} \Psi(Q_{-\frac{a}{r}I_1}(1, ua^{-1}C^{(\nu)} - 1/aD)) \\
&\quad \times \Psi(Q_{-\frac{a}{r}I_1}(1, C - 1/aD)) \Psi(-a/r\langle ua^{-1}C^{(\nu)}, C' \rangle) \\
&= \gamma\left(-\frac{r}{u}I_1\right) |r/u|^{-(c/3+1)/2} \Psi(Q_{-\frac{a}{r}I_1}(1, C + ua^{-1}C^{(\nu)} - 1/aD)) \\
&= \Phi(r, R + ua^{-1}C^{(\nu)}, u).
\end{aligned}$$

To compute $BAB(\mathbf{f})$, first note that

$$\begin{aligned}
AB(\mathbf{f}) &= \Psi\left(\frac{n(R)}{r}\right) \gamma\left(-\frac{r}{u}I_1\right) |r/u|^{-(c/3+1)/2} \\
&\quad \times \int_{C^{(\nu)}} \Psi(-\langle C^{(\nu)}, u/rC' \rangle) \mathbf{f}(ra/u, u, \nu_b, \nu_c, d, e, \nu_f, -r/u) dC^{(\nu)}.
\end{aligned}$$

Thus

$$\begin{aligned} & BAB(\mathbf{f}) \\ &= \gamma(uI_1)|u|^{-(c/3+1)/2} \gamma(aI_1) \int_{C^{(\xi)}} \Psi(\langle C^{(\xi)}, u^{-1}C' \rangle) \Psi(Q_{\frac{r}{u^2a}I_1}(1, C^{(\xi)} - 1/uD)) \\ &\quad \times \left[\int_{C^{(\nu)}} \Psi(\langle C^{(\nu)}, a(C^{(\xi)})' \rangle) \mathbf{f}(-ra/u, u, \nu_b, \nu_c, d, e, \nu_f, a) dC^{(\nu)} \right] dC^{(\xi)} \end{aligned}$$

Reversing the order of integration,

$$BAB(\mathbf{f}) = \int_{C^{(\nu)}} L\mathbf{f}(-ra/u, u, \nu_b, \nu_c, d, e, \nu_f, a) dC^{(\nu)}$$

where

$$\begin{aligned} L &= \gamma(uI_1)|u|^{-(c/3+1)/2} \int_{C^{(\xi)}} \Psi(\langle C^{(\xi)}, u^{-1}(C + u/aC^{(\nu)})' \rangle) \\ &\quad \times \gamma(aI_1) \Psi(Q_{\frac{r}{u^2a}I_1}(1, C^{(\xi)} - 1/uD)) dC^{(\xi)} \\ &= B(\Phi)(r, R + ua^{-1}C^{(\nu)}, u). \end{aligned}$$

This procedure makes sense as long as we interpret the last integral as a notation for a certain Fourier transform (see the proof of Theorem 52). Now, since $B(\Phi) = \Phi$, this proves that $L = K$ and thus $ABA(\mathbf{f}) = BAB(\mathbf{f})$. \square

Theorem 50 in the reduced case is now proved.

3.4. Rank two case. In this section, we prove the existence part of Theorem 50 in the case \mathcal{J} is the Jordan algebra coming from a nine-dimensional division algebra, and hence G is the rank two form of E_6 . We use a wonderful trick of Kazhdan. The idea is to use Langlands analytic continuation of Eisenstein series to prove the existence of a certain representation *globally*. Kazhdan proves that if the restriction of an automorphic representation to P is equivalent to π for at least one place, then it is equivalent at every place. Since we already know locally that there is at most one representation with the correct restriction (Corollary 49), it follows that the one constructed this way must satisfy the requirements of Theorem 50.

Since we are now working globally, our notation will be somewhat different than in other sections. Let K be a global field of characteristic zero and Σ the set of places of K . If $\nu \in \Sigma$, K_ν denotes the completion of K at ν . Fix an odd-dimensional rank three central simple Jordan algebra, \mathcal{J}_K , over K . (As indicated above, we are most interested in the case \mathcal{J}_K a nine-dimensional division algebra over K .) Let \underline{G} be the algebraic K -group corresponding to \mathcal{J}_K (see sections 1 and 2). It is easy to see ([J] V.7 and [Sch] II.2 and IV.2) that the corresponding localized Jordan algebras, \mathcal{J}_{K_ν} , are also rank three central simple Jordan algebras of odd dimension. It follows that, for each $\nu \in \Sigma$ (including Archimedean places), $\underline{G}(K_\nu)$ is one of the groups considered in section 2.

We will write G_K for $\underline{G}(K)$ and G_ν for $\underline{G}(K_\nu)$. Similarly, if \mathbb{A} is the ring of adèles of K then $G_{\mathbb{A}} = \underline{G}(\mathbb{A})$. The same conventions hold for the subgroups $\underline{H}, \underline{P}, \underline{U}$ etc. In this section, Ψ denotes a choice of non-trivial additive character of \mathbb{A} , trivial on K . We have $\Psi = \bigotimes_\nu \Psi_\nu$. Since U_ν is a Heisenberg group, it has a canonical irreducible representation with central character Ψ_ν which we have seen extends to a representation of P_ν° (see the remark following the proof of Proposition 39). In previous sections, we were concerned with the Hilbert space version of this

representation and denoted it by $\hat{\pi}_{\Psi_\nu}$. Here we will also be concerned with the corresponding smooth representation; call it ρ_{Ψ_ν} .

We begin with a result of Weil [W]. Recall that $z(t)$ parametrizes the center of the Heisenberg group \underline{U} .

Lemma 54. *Let (ρ_Ψ, W_Ψ) be the natural representation of $U_\mathbb{A}$ in the space of smooth functions, $f: U_\mathbb{A} \rightarrow \mathbb{C}$, which satisfy $f(\gamma z(a)u) = f(u)\Psi(a)$ for $\gamma \in U_K$, $a \in \mathbb{A}$ and $u \in U_\mathbb{A}$. Then ρ_Ψ is isomorphic to the restricted tensor product of the ρ_{Ψ_ν} . In particular, ρ_Ψ extends to a representation of $P_\mathbb{A}^\circ$.*

It is necessary to be more precise about the action of $H_\mathbb{A}$, the Levi component of $P_\mathbb{A}^\circ$.

Lemma 55. *Suppose that $h \in H_\mathbb{A}$ and f is a vector in ρ_Ψ . Then,*

$$\rho_\Psi(h)f(u) = f(u^h).$$

Proof. On the one hand, this formula respects the action of $U_\mathbb{A}$. On the other hand, the extension of ρ_Ψ to $P_\mathbb{A}^\circ$ is unique. \square

Corollary 56. *If $\gamma \in P_K^\circ$ then*

$$\rho_\Psi(\gamma)f(u) = f(\gamma^{-1}u\gamma).$$

Here is Kazhdan's rigidity result ([K]). For the convenience of the reader, we indicate the proof.

Suppose that for each $\nu \in \Sigma$, $\sigma_\nu^{(2)}$ is an irreducible unitary representation of G_ν ; we write just σ_ν for the corresponding smooth representation. Suppose further that for almost all ν , σ_ν has a vector fixed under the hyperspecial maximal compact subgroup (which exists at all but finitely many places). Then it makes sense to consider the restricted tensor product $\sigma = \bigotimes_\nu \sigma_\nu$. It is an irreducible representation of $G_\mathbb{A}$.

Proposition 57. *Let $(\sigma, V) = \bigotimes_\nu \sigma_\nu$ be as above. Assume that σ has a non-trivial G_K -equivariant functional, and that for at least one place, ν_1 , σ_{ν_1} restricted to P_{ν_1} is equivalent to $\rho_{\Psi_{\nu_1}}$. Then at every place ν , σ_ν restricted to P_ν is equivalent to ρ_{Ψ_ν} .*

Proof. Clearly, to prove the proposition it is enough to construct an $P_\mathbb{A}$ -equivariant embedding, R , from (σ, V) to $\{f: P_\mathbb{A} \rightarrow W_\Psi | f(p^\circ p) = \rho_\Psi(p^\circ)f(p) \forall p^\circ \in P_\mathbb{A}^\circ\}$. By assumption, there is a non-zero G_K -invariant map $T: (\sigma, V) \rightarrow \mathbb{C}$. We can now define R by

$$R(v)(p)(u) = \int_{\mathbb{A}/K} \Psi(-a)T(\sigma(z(a)up)v)da$$

for $v \in V$, $p \in P_\mathbb{A}$ and $u \in U_\mathbb{A}$. It is clear that $R(v)(p) \in W_\Psi$. It remains to check two things. First, we must show that R is injective, and second that for any $v \in V$, $p^\circ \in P_\mathbb{A}^\circ$, and $p \in P_\mathbb{A}$,

$$(21) \quad R(v)(p^\circ p) = \rho_\Psi(p^\circ)R(v)(p).$$

Note that equation (21) is obvious if $p^\circ \in U_\mathbb{A} \subset P_\mathbb{A}^\circ$.

I claim that the kernel of R is precisely the set of v which are invariant by $\sigma(z(a))$ for all $a \in \mathbb{A}$ and, in particular, $\text{Ker}(R)$ is invariant by $P_\mathbb{A}$. This would imply that $\text{Ker}(R)$ is actually trivial, because otherwise it would contradict our assumptions about σ_{ν_1} . Now, it is obvious from the definition that $R(v) = 0$ for all v with this property. We now prove the converse. Pick $x \in K^*$. It is possible to choose $\gamma \in P_K$

so that $z(a)^\gamma = z(ax^{-1})$ (see remarks following Proposition 10 or Claim 21). By assumption,

$$0 = R(v)(\gamma p)(1) = \int_{\mathbb{A}/K} \Psi(-a) T(\sigma(z(a)\gamma p)v) da$$

since T is G_K -equivariant

$$\begin{aligned} &= \int_{\mathbb{A}/K} \Psi(-a) T(\sigma(\gamma^{-1}z(a)\gamma p)v) da \\ &= \int_{\mathbb{A}/K} \Psi(-ax) T(\sigma(z(a)p)v) da. \end{aligned}$$

Since every nontrivial character of \mathbb{A}/K has the form $\tilde{\Psi}(a) = \Psi(ax)$, this implies that $T(\sigma(z(a)p)v)$ is independent of a . Furthermore, since P normalizes $z(a)$, we have that $T(\sigma(pz(a))v) = T(\sigma(p)v)$ for all $a \in \mathbb{A}$ and $p \in P_{\mathbb{A}}$. I claim that this implies $\sigma(z(a))v = v$. It is equivalent to the following lemma.

Lemma 58. *The map $V \rightarrow$ functions on $P_{\mathbb{A}}$, given by $v \mapsto T(\sigma(p)v)$, is injective.*

Remark. We will actually prove the following equivalent statement: For any nonzero element $v \in V$, there is an element $p \in P_{\mathbb{A}}$ so that $T(\sigma(p)v) \neq 0$.

Proof. Suppose that $v \in V$ is such that $T(\sigma(p)v) = 0$ for all $p \in P_{\mathbb{A}}$. Then, because T is G_K -equivariant, $T(\sigma(\gamma)v) = 0$ for all $\gamma \in G_K$. But $G_K P_{\mathbb{A}}$ is dense in $G_{\mathbb{A}}$. Thus, $T(\sigma(g)v) = 0$ for all $g \in G_{\mathbb{A}}$. The set of such v clearly forms a $G_{\mathbb{A}}$ -invariant subspace of V which is not all of V because T is not zero. By irreducibility, $v = 0$. \square

We now turn to the verification of equation (21). If $p^\circ \in U_{\mathbb{A}} \subset P_{\mathbb{A}}^\circ$, it is obvious. In particular, fixing $p \in P_{\mathbb{A}}$, R gives a map from V to W_{Ψ} which is a $U_{\mathbb{A}}$ -homomorphism, and so in particular, a U_{ν_1} -homomorphism; call it r . But since there is a unique extension of ρ_{ν_1} from U_{ν_1} to $P_{\nu_1}^\circ$, our assumptions imply that r must be actually be a $P_{\nu_1}^\circ$ -homomorphism. In particular, equation (21) holds for $p^\circ \in P_{\nu_1}^\circ$.

Next, suppose that $\gamma \in P_K^\circ$. Then

$$R(v)(\gamma p)(u) = \int_{\mathbb{A}/K} \Psi(-a) T(\sigma(z(a)u\gamma p)v) da$$

since T is G_K equivariant and P° commutes with Z ,

$$\begin{aligned} &= \int_{\mathbb{A}/K} \Psi(-a) T(\sigma(z(a)(\gamma^{-1}u\gamma)p)v) da \\ &= \rho_{\Psi}(\gamma) R(v)(p)(u) \end{aligned}$$

by Corollary 56.

We have now proved that equation (21) holds for all $p^\circ \in H_{\nu_1} H_K U_{\mathbb{A}}$. By strong approximation, $H_{\nu_1} H_K$ is dense in $H_{\mathbb{A}}$. Thus, equation (21) holds for all of $P_{\mathbb{A}}^\circ$. \square

Recall that we wish to use Proposition 57 to prove Theorem 50 in the case of the rank two form of E_6 . Suppose that $\nu_0 \in \Sigma$ satisfies $K_{\nu_0} = F$. Pick a nine-dimensional division algebra, D , over K which is ramified at ν_0 . Let \mathcal{J}_D be the Jordan algebra over K obtained by modifying the multiplication on D in the usual way (namely, $a.b = \frac{1}{2}(ab + ba)$), and suppose that \underline{G} is the simply connected

algebraic group over K obtained from \mathcal{J}_D using the method of sections 1 and 2. With these notations, our goal is to prove the existence part of Theorem 50 for G_{ν_0} .

Let Δ be the modulus character on P , and set $I(s) = \text{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} \Delta^s$; we use normalized induction. In [Ru2] it is proved that the corresponding Eisenstein series has a simple pole at $s = 7/22$. Furthermore, the residue representation, θ_D , satisfies the assumptions of Proposition 57 with ν_1 any finite place where D is split. It follows that $(\theta_D)_{\nu_1}$ (really its unitary closure) is an irreducible representation of G_{ν_0} which agrees with π on P_{ν_0} . Theorem 50 is proved.

3.5. The case of F_4 . In this section we construct the representation π for G derived from the Jordan algebra of three by three symmetric matrices over F under the product $M.N = 1/2(MN + NM)$. In particular, \mathcal{J} is reduced and $c = \dim \mathcal{J} = 6$. The group is split of type F_4 . Note that F_4 has no fundamental group.

Just as in the odd case, the unipotent group U is a Heisenberg group and the action of H on U/Z leads to a map $H \rightarrow \text{Sp}(U/Z)$. Thus, by the theory of the Heisenberg representation, there is a canonical projective representation, $\hat{\pi}_\Psi$, of $P^\circ = HU$; it may be realized on $\hat{V} = L^2(W)$.

Lemma 59. *The extension of H corresponding to $\hat{\pi}_\Psi$ is not split.*

Proof. It is enough to show that the cover is not split over the $\text{SL}(2) \subset H$ corresponding to β , that is, generated by $E_2(kI)$ and $E'_2(kI)$. We can view this $\text{SL}(2)$ as a subset of $\text{Sp}(U/Z)$ by composing with the map $H \rightarrow \text{Sp}(U/Z)$. It is then part of a dual pair with a certain odd-dimensional unitary group. (Note that $\dim \mathcal{J}$ even implies that $\dim W$ odd.) It is well known that the cover does not split in this case; see [M-V-W, Theorem 3.III.1]. \square

On the other hand, it was proved by Weil that $\hat{\pi}_\Psi$ may be normalized so that the cocycle takes values in ± 1 . Explicit formulas for the normalization are in section 5 of [R]. Our notation will be $\Gamma(x)$ and $\gamma(1)$ for what Rao would call $\gamma_F(x, \frac{1}{2}\Psi)$ and $\gamma_F(\frac{1}{2}\Psi)$, respectively. See also the remark following the proof of Corollary 68 below.

Proposition 60. *The following formulas define a projective representation $P^\circ \rightarrow$ unitary operators on \hat{V} whose cocycle takes values in ± 1 .*

$$\begin{aligned} \hat{\pi}_\Psi(a, A)\mathbf{f}(r, R) &= \mathbf{f}(r + a, R + A) \\ \hat{\pi}_\Psi(B, b)\mathbf{f}(r, R) &= \Psi(\langle R, B \rangle + rb)\mathbf{f}(r, R) \\ \hat{\pi}_\Psi(z(t))\mathbf{f}(r, R) &= \Psi(t)\mathbf{f}(r, R) \\ \hat{\pi}_\Psi(m)\mathbf{f}(r, R) &= \Gamma(d(m))^{-1}|d(m)|^{3/2}\mathbf{f}(d(m)r, d(m)R^m) \\ \hat{\pi}_\Psi(h_\beta(t))\mathbf{f}(r, R) &= \Gamma(t)^{-1}|t|^{9/2}\mathbf{f}(t^3r, tR) \\ \hat{\pi}_\Psi(E_2(J))\mathbf{f}(r, R) &= \Psi(Q_J(r, R))\mathbf{f}(r, R + rJ) \\ \hat{\pi}_\Psi(s)\mathbf{f}(r, R) &= \gamma(1)^{-7}\Gamma(-1)^{-1} \int_{F \times \mathcal{J}} \Psi(\langle \tilde{R}, R \rangle + r\tilde{r})\mathbf{f}(\tilde{r}, \tilde{R})d\tilde{r}d\tilde{R} \end{aligned}$$

Also,

$$\hat{\pi}(s_1)\mathbf{f}(r, a, b, c, d, e, f) = \gamma(1)^{-3} \int_{F \times F \times \mathcal{A}} \Psi(\langle C_1^{(\nu)}, C_1' \rangle)\mathbf{f}(-a, r, \nu_b, \nu_c, d, e, \nu_f) dC_1^{(\nu)}$$

$$\begin{aligned}\hat{\pi}(s_2)\mathbf{f}(r, a, b, c, d, e, f) &= \gamma(1)^{-3} \int_{F \times F \times \mathcal{A}} \Psi(\langle C_2^{(\nu)}, C_2' \rangle) \mathbf{f}(-b, \nu_a, r, \nu_c, d, \nu_e, f) dC_2^{(\nu)} \\ \hat{\pi}(s_3)\mathbf{f}(r, a, b, c, d, e, f) &= \gamma(1)^{-3} \int_{F \times F \times \mathcal{A}} \Psi(\langle C_3^{(\nu)}, C_3' \rangle) \mathbf{f}(-c, \nu_a, \nu_b, r, \nu_d, e, f) dC_3^{(\nu)}\end{aligned}$$

It is useful to note explicitly

Corollary 61.

$$\begin{aligned}\hat{\pi}_\Psi(h_1(t))\mathbf{f}(r, a, b, c, d, e, f) &= \Gamma(t)^{-1} |t|^{3/2} \mathbf{f}(tr, t^{-1}a, tb, tc, d, e, tf) \\ \hat{\pi}_\Psi(h_2(t))\mathbf{f}(r, a, b, c, d, e, f) &= \Gamma(t)^{-1} |t|^{3/2} \mathbf{f}(tr, ta, t^{-1}b, tc, d, te, f) \\ \hat{\pi}_\Psi(h_3(t))\mathbf{f}(r, a, b, c, d, e, f) &= \Gamma(t)^{-1} |t|^{3/2} \mathbf{f}(tr, ta, tb, t^{-1}c, td, e, f)\end{aligned}$$

Note that $\hat{\pi}_\Psi(s_1)\hat{\pi}_\Psi(s_2)\hat{\pi}_\Psi(s_3) = \hat{\pi}(s)_\Psi(-1, -1)_F = \hat{\pi}_\Psi(s)$ and

$$\hat{\pi}_\Psi(h_1(t))\hat{\pi}_\Psi(h_2(t))\hat{\pi}_\Psi(h_3(t)) = \hat{\pi}_\Psi(h_\beta(t))(-1, t)_F.$$

It is a fact that every split, simply connected group over a p -adic field has a unique two-fold cover. We will write \tilde{G} for the double cover of G . According to [St], the universal central extension of G is generated by symbols $X_\eta(t)$, for η a root and $t \in F$, which are additive in t and satisfy the Serre relations (here we are assuming $\text{rank } G > 1$). Set $w_\eta(t) = X_\eta(t)X_{-\eta}(-t^{-1})X_\eta(t)$ and $m_\eta(t) = w_\eta(t)w_\eta(-1)$. Matsumoto shows in [Ma] that if we require also that $m(t)m(r) = (t, r)_F m(rs)$ for long roots η , then we obtain a complete set of generators and relations for \tilde{G} .

We will view the double cover of G in terms of these generators and relations. Restricting to $H \subset G$, we obtain a double cover, \tilde{H} , of H . It is easy to check that it is not split. We also have the double cover of $H \subset P^\circ$ defined by the cocycle in Proposition 60. Of course, these coverings coincide. On the other hand, it is not immediately clear how to relate the covering defined by generators and relations with the one defined by the representation. Put another way, we wish to write formulas for the representation $\hat{\pi}_\Psi$ in terms of the generators of Steinberg-Matsumoto.

Let $\theta: \tilde{P}^\circ \rightarrow P^\circ$ be the natural projection.

Proposition 62. *$\hat{\pi}_\Psi$ defines an ordinary representation of \tilde{P}° . It satisfies the following formulas: $\hat{\pi}_\Psi(m_i(t)) = (-1, t)_F \hat{\pi}_\Psi(h_i(t))$ and $\hat{\pi}_\Psi(w_i(1)) = \hat{\pi}_\Psi(s_i)$ for $i = 1, 2, 3$. Also, if $p \in M_0 U_B$ (notation from 2.2) is some $X_\eta(t), w_\eta(1)$ or $m_\eta(t)$, then $\hat{\pi}_\Psi(p) = \hat{\pi}_\Psi(\theta(p))$ as given in Proposition 60.*

Proof. Lemma 5.4 of [Ma] implies that if $c_\eta(r, t)$ is the cocycle corresponding to a root η (i.e. $m_\eta(r)m_\eta(t) = c_\eta(r, t)m_\eta(rt)$), then

$$c_\eta(r, t)^{\langle \eta, \delta \rangle} = c_\delta(r, t)^{-\langle \delta, \eta \rangle}.$$

Here $\langle \eta, \delta \rangle = \frac{2(\eta, \delta)}{(\delta, \delta)}$. Using this it is easy to see that \tilde{H} splits over M_0 . Thus, the portion of the proposition concerning M_0 is clear. Also, \tilde{P}° splits over U_B so this is no problem.

Finally, we consider the $w_i(1)$ and $m_i(t)$. Clearly, $\hat{\pi}_\Psi(m_i(t)) = (-1, t)_F^\epsilon \hat{\pi}_\Psi(h_i(t))$ where ϵ is zero or one. Similarly, $\hat{\pi}_\Psi(w_i(1)) = (-1, -1)_F^\epsilon \hat{\pi}_\Psi(s_i)$. But $(-1, -1)_F =$

1; this proves the claim about $w_i(1)$. Furthermore,

$$\begin{aligned}\hat{\pi}_\Psi(m_i(t)) &= \hat{\pi}_\Psi(X_i(t))\hat{\pi}_\Psi(w_i(1))\hat{\pi}_\Psi(X_i(t^{-1}))\hat{\pi}_\Psi(w_i(-1))\hat{\pi}_\Psi(X_i(t)) \\ &= \hat{\pi}_\Psi(E_2(tI_i))\hat{\pi}_\Psi(s_i)\hat{\pi}_\Psi(E_2(t^{-1}I_i))\hat{\pi}_\Psi(s_i^{-1})\hat{\pi}_\Psi(E_2(tI_i)) \\ &= \hat{\pi}_\Psi(h_i(t))\kappa(s_iE_2(tI_1), s_i^{-1})\end{aligned}$$

where $\kappa(\cdot, \cdot)$ is the cocycle from Proposition 60. But, an easy computation using Theorem 5.3 of [R] shows that $\kappa(s_iE_2(tI_1), s_i^{-1}) = (-1, t)_F$. Proposition 62 is proved. \square

Remark. It is interesting to compare the case of F_4 with that of G_2 (see [S1]). In both cases the minimal representation lives on a central extension of the linear group, but the extensions arise for different reasons. As we have seen, for F_4 the metaplectic cover does not split over the image of H in $\mathrm{Sp}(U/Z)$. Thus, we need a double cover already for $\hat{\pi}_\Psi$. But for G_2 , the corresponding Jordan algebra is one dimensional (see the appendix) so Proposition 39 applies. Hence, in this case $\hat{\pi}_\Psi$ lives on the linear group P° and the issues in Propositions 60 and 62 do not arise. The reason that a central extension is required for the minimal representation of G_2 is related to the braid relation.

We will continue to work with \tilde{G} and its subgroups in terms of the Steinberg-Matsumoto generators. However, in order to emphasize the parallels between the case of F_4 and the case of an odd-dimensional Jordan algebra, we refer to the generators by their canonical projections. For example, since $s_i = \theta(w_i(1))$, we will write s_i instead of $w_i(1)$. Similarly, we will write $h_i(t)$ and $E_2(tI_i)$ instead of $m_i(t)$ and $X_i(t)$. Finally, since $s = \theta(w_1(1)w_2(1)w_3(1))$ and $h_\beta(r) = \theta(m_1(r)m_2(r)m_3(r))$, we will write s and $h_\beta(r)$ instead of $w_1(1)w_2(1)w_3(1)$ and $m_1(r)m_2(r)m_3(r)$.

Set $\pi = \mathrm{Ind}_{\tilde{P}^\circ}^{\tilde{P}} \hat{\pi}_\Psi$. The representation π may be realized on $V = L^2(W \times F)$, and, if we identify $\tilde{P}/\tilde{P}^\circ$ with $h_\alpha(u)$, it is easy to write formulas for the action. The key point is to understand the commutators $[h_\alpha(u), \cdot]$. This we can do using the formulas in [Ma]. In particular, $[h_\alpha(u), h_\alpha(t)] = (u, t)_F$. Also, $h_i(u)s = h_i(u^{-1})s_ih_\alpha(u)$ and $h_\alpha(u)s = h_\beta(u^{-1})sh_\alpha(u)$.

Proposition 63. *The representation $\pi: \tilde{P} \rightarrow$ unitary operators on V is given by the following formulas:*

$$\begin{aligned}\pi(h_\alpha(t))\mathbf{f}(r, R, u) &= (t, u)_F \mathbf{f}(r, R, tu) \\ \pi(a, A)\mathbf{f}(r, R, u) &= \mathbf{f}(r + u^2a, R + uA, u) \\ \pi(B, b)\mathbf{f}(r, R, u) &= \Psi(\langle R, B \rangle + rbu^{-1})\mathbf{f}(r, R, u) \\ \pi(z(t))\mathbf{f}(r, R, u) &= \Psi(ut)\mathbf{f}(r, R, u) \\ \pi(m)\mathbf{f}(r, R, u) &= (d(m), u)_F \Gamma(d(m))^{-1} |d(m)|^{(c/3+1)/2} \mathbf{f}(d(m)r, d(m)R^m, u) \\ \pi(h_\beta(t))\mathbf{f}(r, R, u) &= (t, u)_F \Gamma(t)^{-1} |t|^{9/2} \mathbf{f}(t^3r, tR, u) \\ \pi(E_2(J))\mathbf{f}(r, R, u) &= \Psi(Q_{u^{-1}J}(r, R))\mathbf{f}(r, R + ru^{-1}J, u) \\ \pi(s)\mathbf{f}(r, R, u) &= \gamma(1)^{-7} \Gamma(-1)^{-1} \Gamma(u)^{-1} |u|^{-(c+3)/2} \\ &\quad \int_{F \times \mathcal{J}} \Psi(\langle \tilde{R}, u^{-1}R \rangle + u^{-3}r\tilde{r})\mathbf{f}(\tilde{r}, \tilde{R}, u) d\tilde{r} d\tilde{R}\end{aligned}$$

Also,

$$\begin{aligned}\pi(s_1)\mathbf{f}(r, a, b, c, d, e, f, u) &= (-1, u)_F \gamma(1)^{-3} \Gamma(u)^{-1} |u|^{-(c/3+1)/2} \\ &\quad \times \int_{F \times F \times \mathcal{A}} \Psi(\langle C_1^{(\nu)}, u^{-1} C_1' \rangle) \mathbf{f}(-ua, u^{-1}r, \nu_b, \nu_c, d, e, \nu_f, u) dC_1^{(\nu)} \\ \pi(s_2)\mathbf{f}(r, a, b, c, d, e, f, u) &= (-1, u)_F \gamma(1)^{-3} \Gamma(u)^{-1} |u|^{-(c/3+1)/2} \\ &\quad \times \int_{F \times F \times \mathcal{A}} \Psi(\langle C_2^{(\nu)}, u^{-1} C_2' \rangle) \mathbf{f}(-ub, \nu_a, u^{-1}r, \nu_c, d, \nu_e, f, u) dC_2^{(\nu)} \\ \pi(s_3)\mathbf{f}(r, a, b, c, d, e, f, u) &= (-1, u)_F \gamma(1)^{-3} \Gamma(u)^{-1} |u|^{-(c/3+1)/2} \\ &\quad \times \int_{F \times F \times \mathcal{A}} \Psi(\langle C_3^{(\nu)}, u^{-1} C_3' \rangle) \mathbf{f}(-uc, \nu_a, \nu_b, u^{-1}r, \nu_d, e, f) dC_3^{(\nu)}\end{aligned}$$

and

$$\begin{aligned}\pi(h_1(t))\mathbf{f}(r, a, b, c, d, e, f, u) &= \Gamma(t)^{-1} |t|^{3/2} \mathbf{f}(tr, t^{-1}a, tb, tc, d, e, tf, u) \\ \pi(h_2(t))\mathbf{f}(r, a, b, c, d, e, f, u) &= \Gamma(t)^{-1} |t|^{3/2} \mathbf{f}(tr, ta, t^{-1}b, tc, d, te, f, u) \\ \pi(h_3(t))\mathbf{f}(r, a, b, c, d, e, f, u) &= \Gamma(t)^{-1} |t|^{3/2} \mathbf{f}(tr, ta, tb, t^{-1}c, td, e, f, u)\end{aligned}$$

Furthermore, π is independent of Ψ and irreducible. In fact, π restricted to $h_\alpha U$ is irreducible.

This proposition follows from the arguments of Proposition 43 and Corollary 44.

Next, we wish to extend π to \tilde{G} . Just as in section 3.2, the first step is to extend it from \tilde{B} to \tilde{P}_α . Note that, although G is split, in our notation B is not the Borel subgroup (see 2.2).

Proposition 64. *There is a unique extension of π from \tilde{B} to \tilde{P}_α . It is given by*

$$(22) \quad \pi(s_\alpha)\mathbf{f}(r, R, u) = (-r, u)_F (r, -1)_F \Psi\left(\frac{n(R)}{r}\right) \mathbf{f}(-r, R, -r/u).$$

Proof. It follows from the formulas in section 5 of [Ma] that Lemma 46 continues to hold in the context of two-fold covers. That is, \tilde{P}_α is generated by \tilde{B} and s_α subject to the relations (1)-(7) listed in the lemma. Thus, to check that equation (22) defines an extension of π , we must check these relations. Of course, except for factors of ± 1 , we have already done so in the proof of Proposition 47. Thus, we need to check only that the factors work out. The hardest are (4) and (7). We leave the others as an exercise.

For relation (4), the factors on the left side are $(-r, u)_F (r, -1)_F (d(m), -r/u)_F$. On the right side we get

$$\begin{aligned}(d(m), u)_F (d(m), u)_F (-d(m)r, d(m)u)_F (d(m)r, -1)_F \\ = (-r, d(m)u)_F (d(m), u)_F (d(m), d(m))_F (d(m), -1)_F (r, -1)_F \\ = (-r, d(m)u)_F (d(m), u)_F (r, -1)_F \\ = (-r, u)_F (d(m), -ru)_F (r, -1)_F.\end{aligned}$$

The right-hand side of relation (7) gives $(t, u)_F (-r, ut)_F (r, -1)_F$. The left-hand side we work out in stages. First, since $s_\alpha^{-1}x(t) = h_\alpha(-1)s_\alpha x(t)$ we get the term $(u, -1)_F (-r, -u)_F (r, -1)_F$. Next, $x(t^{-1})$ gives

$$(u, -1)_F (-(r + u^2/t), -u)_F (r + u^2/t, -1)_F.$$

Including s_α we get

$$\begin{aligned} & (-r, u)_F(r, -1)_F(-r/u, -1)_F(r - r^2/u^2t, r/u)_F(-r + r^2/u^2t, -1)_F \\ &= (-r, u)_F(r, -1)_F(-r/u, -1)_F(r, r/u)_F(1 - r/u^2t, r/u)_F(-r, -1)_F(1 - r/u^2t, -1)_F. \end{aligned}$$

This simplifies to $(1 - r/u^2t, r/u)_F(1 - r/u^2t, -1)_F$. Finally, adding $x(1)$ gives

$$\begin{aligned} & (1 - (r + u^2t)/u^2t, (r + u^2t)/u)_F(1 - (r + u^2t)/u^2t, -1)_F \\ &= (-r/u^2t, r/u(1 + u^2t/r))_F(r/u^2t, -1)_F \\ &= (-rt, ru)_F(rt, -1)_F \\ &= (t, u)_F(t, r)_F(-r, u)_F(r, -1)_F(t, -1)_F \\ &= (t, u)_F(-r, tu)_F(r, -1)_F. \end{aligned}$$

The existence part of Proposition 64 is now proved. The uniqueness follows from the same arguments as in Proposition 47. \square

We can now state the main result of this section.

Theorem 65. *There exists a unique unitary representation, π , of \tilde{G} so that π restricted to \tilde{P} is given by Proposition 63 and π restricted \tilde{P}_α is given by Proposition 64.*

Just as in section 3.3 it is enough to prove

Theorem 66. *For $i = 1, 2, 3$ there is an equivalence of unitary operators*

$$\pi(s_\alpha)\pi(s_i)\pi(s_\alpha) = \pi(s_i)\pi(s_\alpha)\pi(s_i).$$

Since the argument is essentially the same for each $i = 1, 2, 3$, we just give the case $i = 1$. Let A and B be the operators $\pi(s_\alpha)$ and $\pi(s_1)$, respectively. We wish to show that

$$ABA = BAB.$$

Lemma 67. *Set*

$$\tilde{\Phi} = |r/u|^{-3/2}\Psi\left(\frac{n(R)}{r}\right).$$

Then

$$B(\tilde{\Phi}) = (-1, u)_F\gamma(1)^{-3}\Gamma(u)^{-1}\gamma\left(\frac{r}{au^2}I_1\right)\tilde{\Phi}.$$

The proof is just the easy part of the argument in Theorem 52.

Corollary 68. $B(\tilde{\Phi}) = \Gamma(-ura)(-u, -ra)_F\tilde{\Phi}.$

Proof. By Proposition 33, the quadratic form Q_{kI_1} has the form $Q_{kI_1}(b, c, f) = -kbc + kf^2$. Thus,

$$(23) \quad \gamma\left(\frac{r}{au^2}I_1\right)\gamma(1)^{-3} = (ra, -1)_F\Gamma(-ra).$$

Furthermore, $\Gamma(u)^{-1} = (-1, u)_F\Gamma(u)$. Hence by the lemma,

$$B(\tilde{\Phi}) = (ra, -1)_F\Gamma(u)\Gamma(-ra)\tilde{\Phi}.$$

But $\Gamma(u)\Gamma(-ra) = \Gamma(-ura)(u, -ra)_F$. Thus, we have reduced the corollary to the statement $(ra, -1)_F(u, -ra)_F = (-u, -ra)_F$ which is clear. \square

Remark. There is one subtle point about equation (23) that should be noted. Since in his definition of the normalization factors, Rao includes a $\frac{1}{2}$, it seems that we need to have $Q_{kI_1}(b, c, f) = -kbc + \frac{1}{2}kf^2$ to make things work. However, this is not the case. The confusion arises because we chose the identification of W' with W^* given by $w'(w) = \langle R, J \rangle + tu$ where $w = (t, R) \in W$ and $w' = (J, u) \in W'$. In the case of F_4 this leads to an inner product on U/Z given by

$$\begin{aligned} &((t_1; a_1, b_1, c_1, d_1, e_1, f_1), (a_2, b_2, c_2, d_2, e_2, f_2; t_2)) \\ &= t_1 t_2 + a_1 a_2 + b_1 b_2 + c_1 c_2 + 2(d_1 d_2 + e_1 e_2 + f_1 f_2). \end{aligned}$$

The definition of the inner product used in [R] does not include the factor of two.

Proof (Theorem 66). Just as in the proof of Theorem 51, it is easy to prove that ABA is an integral operator with kernel $(r, u)_F(a, ru)_F \Gamma(-ru)^{-1} \gamma(1)^{-3} \tilde{\Phi}$.

On the other hand, one checks that BAB is the same operator except with kernel $(a, u)_F \Gamma(a)^{-1} \gamma(1)^{-3} B(\tilde{\Phi})$. Thus, by Corollary 68, it is enough to prove that

$$(r, u)_F(a, ru)_F \Gamma(-ru)^{-1} \gamma(1)^{-3} = (a, u)_F \Gamma(a)^{-1} \gamma(1)^{-3} \Gamma(-ura)(-u, -ra)_F.$$

Equivalently,

$$(r, au)_F = (-u, -ra)_F(-ru, -1)_F \Gamma(-ura) \Gamma(a)^{-1} \Gamma(-ru)^{-1}.$$

The right side equals $(-u, -ra)_F(-ru, -1)_F(a, -ru)_F = (-u, -ra)_F(-ru, -a)_F = (-u, r)_F(r, -a)_F = (r, au)_F$ which is the left-hand side. Theorem 66 is proved. \square

This completes the proof of Theorem 65.

4. MINIMALITY

In this section we prove that the representation π constructed in section 3 is minimal. As was pointed out by Kazhdan, this is morally clear because, by results of Howe, π restricted to a Borel subgroup has the character that you expect. If G is split (and simply laced), this was made into a proof by Savin in [S2].

Essentially the same argument can be given in general. Of course, G may not have a Borel subgroup so Howe's Kirillov theory for solvable groups is not directly applicable. It turns out that this is not necessary. In section 2 we prove directly that π restricted to P has the expected character. Then we present the proof of minimality closely following [S2] except working with P in place of a Borel subgroup.

Remark. Although in this chapter we use the language of linear groups, virtually the same arguments apply to F_4 as well.

4.1. Definitions. Let F be a p -adic field. Let $G = G(F)$ be the F -rational points of a connected reductive group defined over F . Let \mathfrak{g} be the Lie algebra of G , and \mathfrak{g}^* the dual of \mathfrak{g} . There are two topologies on these spaces: Zariski and p -adic. Unless otherwise indicated we will work with the p -adic topology. We will now define congruence subgroups of G . Since G is an algebraic F -group, there is an F -rational injection $G(\bar{F}) \rightarrow \mathrm{GL}_n(\bar{F})$ for some n , and a corresponding F -rational injection $\mathfrak{g}(\bar{F}) \rightarrow M_n(\bar{F})$. Let I_n be the identity in GL_n , $\mathcal{R} \subset F$ the ring of integers and $\varpi \in \mathcal{R}$ a uniformizing parameter. Then, for r a positive integer, set $G_r = G \cap (I_n + \varpi^r M_n(\mathcal{R}))$; cf. [Pl-R] section 3.1. As is well known (see [H1]) when r is large enough, \log provides a homeomorphism from G_r to an \mathcal{R} -module $\mathfrak{g}_r \subset \mathfrak{g}$ which is closed under the bracket operation.

Let \mathcal{S}_G be the space of Schwartz-Bruhat functions on G ; that is, \mathcal{S}_G is the space of locally constant compactly supported (complex valued) functions on G . Also, let dg be (a choice of) Haar measure on G . If (ρ, V) an irreducible representation of G , define the operator $\rho(f)$ on V for each $f \in \mathcal{S}_G$ by

$$\rho(f) = \int_G f(g)\rho(g)dg.$$

It is known that since ρ is irreducible, $\rho(f)$ is a finite rank operator. Thus, we can define the character of ρ by

$$\Theta_\rho(f) = \text{Tr}(\rho(f)).$$

It is a distribution on G . Note that this trace is the same whether we take for ρ an irreducible Hilbert space representation, or the corresponding smooth representation.

Choose r so large that \log gives a homeomorphism from G_r to $\mathfrak{g}_r \subset \mathfrak{g}$. If $f \in \mathcal{S}_G$ is supported in G_r , then we may view it as a function on \mathfrak{g} . Let \hat{f} be the Fourier transform of f with respect to the Killing form. It is a function on \mathfrak{g}^* . In this situation, Harish-Chandra [H-C] has proved that there are numbers, $c_{\mathcal{O}}$, indexed by the nilpotent coadjoint orbits $\mathcal{O} \subset \mathfrak{g}^*$, so that if r is large enough,

$$\Theta_\rho(f) = \sum_{\mathcal{O}} c_{\mathcal{O}} \int \hat{f} \mu_{\mathcal{O}}.$$

Here $\mu_{\mathcal{O}}$ is a suitably normalized G -invariant positive measure on \mathcal{O} . It is convenient to write simply

$$\Theta_\rho = \sum_{\mathcal{O}} c_{\mathcal{O}} \hat{\mu}_{\mathcal{O}}.$$

The numbers $c_{\mathcal{O}}$ are obviously invariants of ρ . Two less refined invariants are the Gelfand-Kirillov dimension of ρ ,

$$\text{GK}(\rho) = \max_{\{\mathcal{O} | c_{\mathcal{O}} \neq 0\}} \frac{\dim \mathcal{O}}{2},$$

and the wave front set,

$$\text{WF}(\rho) = \bigcup_{\{\mathcal{O} | c_{\mathcal{O}} \neq 0\}} \overline{\mathcal{O}} \subset \mathfrak{g}^*.$$

Here $\overline{\mathcal{O}}$ is the closure of \mathcal{O} .

A nilpotent coadjoint orbit, \mathcal{O}_{\min} , is minimal if $\overline{\mathcal{O}_{\min}} = \mathcal{O}_{\min} \cup \{0\}$. A representation is called minimal if its wave front set is the closure of a minimal orbit. If G has a unique minimal orbit, then it is equivalent to define a minimal representation to be one with smallest possible (positive) Gelfand-Kirillov dimension.

We now specialize to G as in section 2. I will show that G has a unique minimal orbit. It may be characterized as follows. Let

$$\omega = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in \mathfrak{sl}(3) \subset \mathfrak{g}.$$

Identifying \mathfrak{g} with \mathfrak{g}^* using the Killing form, we may view ω as an element of \mathfrak{g}^* .

Lemma 69. *G has a unique minimal orbit, namely $\mathcal{O}_{\min} = \text{Ad}^*(G)\omega$.*

Proof. First, it is easy to see that over the algebraic closure there is a unique minimal orbit and it is generated by ω . (For example, one can adapt the argument in [C-M] Chapter 4.) Thus, if u is in a minimal orbit, then it is conjugate to ω over \bar{F} . We will be done if we prove that u is *rationally* conjugate to ω .

This is a problem in Galois cohomology as follows. Let Γ be the Galois group of \bar{F}/F and let $C(\bar{F})$ be the centralizer of ω in $G(\bar{F})$. We know that there exists $g \in G(\bar{F})$ with $u = g\omega g^{-1}$. Since u and ω are rational, if $\sigma \in \Gamma$ we have $u = \sigma(g)\omega\sigma(g^{-1})$. It follows that $g^{-1}\sigma(g) \in C(\bar{F})$. In fact, it is easy to see that the map $\rho: \Gamma \rightarrow C(\bar{F})$ given by $\rho(\sigma) = g^{-1}\sigma(g)$ is a 1-cocycle.

I claim that ρ is a coboundary. This would mean that there exists an element $a \in C(\bar{F})$ so that $\rho(\sigma) = a^{-1}\sigma(a)$. That is, $a^{-1}\sigma(a) = g^{-1}\sigma(g)$ for all $\sigma \in \Gamma$, and so $ga^{-1} = \sigma(ga^{-1})$ for all $\sigma \in \Gamma$. In other words, $ga^{-1} \in G(F)$. Since $ga^{-1}\omega ag^{-1} = g\omega g^{-1} = u$, this means that ω and u are rationally conjugate.

It remains to prove that ρ is a coboundary. Certainly it is enough to prove that $H^1(\Gamma, C(\bar{F}))$ is trivial. To do this, first recall that when computing H^1 , we can replace any group with its reductive part (see [Pl-R] Proposition 2.9.). Now it is easy to see that the reductive part of $C(\bar{F})$ is, in the notation of section 2.1, $H(\bar{F})$. Consequently, $H^1(\Gamma, C(\bar{F})) = H^1(\Gamma, H(\bar{F}))$. Next, since H is semi-simple and simply connected (Proposition 9), Theorem 6.4 of [Pl-R] implies $H^1(\Gamma, H(\bar{F})) = 1$. \square

Remark. The preceding argument used, via Proposition 9, the fact that we took G simply connected. However, it is straightforward to prove that if a simply connected group has a unique minimal orbit, then so does any isogenous group.

Let \mathfrak{p} be the Lie algebra of P . We can identify \mathfrak{p}^* with $\bar{\mathfrak{p}}$ (the opposite parabolic) and consider $\omega \in \mathfrak{p}^*$. Let \mathcal{O}_P be the coadjoint P -orbit containing ω . Corresponding to the injection $\mathfrak{p} \rightarrow \mathfrak{g}$ there is a map $\kappa: \mathfrak{g}^* \rightarrow \mathfrak{p}^*$; κ can be identified with the natural projection $\mathfrak{g} = \bar{\mathfrak{p}} \oplus \mathfrak{u} \rightarrow \bar{\mathfrak{p}}$.

It is easy to check that $\dim \mathcal{O}_{\min} = \dim \mathcal{O}_P$. Since κ is P -equivariant, this implies

Lemma 70. κ is a P -equivariant bijection between an open set in \mathcal{O}_{\min} and \mathcal{O}_P .

4.2. Parabolic theory. Let $\sigma = \pi|_P$. Set $P_r = P \cap G_r$ and $\mathfrak{p}_r = \mathfrak{p} \cap \mathfrak{g}_r$. The goal of this section is to prove the following result.

Proposition 71. *There is an explicit constant r so that if h is a function on \mathfrak{p}_r such that $\text{supp } \hat{h} \cap \mathcal{O}_P$ is compact, then $\sigma(h)$ is trace class and*

$$\text{Tr } \sigma(h) = \int_{\mathcal{O}_P} \hat{h} \mu_{\mathcal{O}_P}.$$

Here $\mu_{\mathcal{O}_P}$ is a suitably normalized P -invariant measure on \mathcal{O}_P .

If P were a solvable group, such as a Borel subgroup, then this proposition would be valid for *any* irreducible representation, and is due to Howe [H2]. We are not generalizing his results to more general groups (it would be false) but rather claiming that it works for our one specific representation.

Let $P = LU \subset G$ as earlier. Pick r so large that G_r is sufficiently small in the sense of [H1, Theorem 1.1]. We will explain what this means. For ease of notation, let $K = G_r$ and $\mathcal{K} = \mathfrak{g}_r$. Howe defines a constant β , and the condition is that $[\mathcal{K}, \mathcal{K}] \subseteq \varpi^{\beta+1}\mathcal{K}$. Here ϖ is a uniformizer in F .

Choosing r larger if necessary, we may assume that P and K are in good position, and that there is a $\lambda \in L$ strictly dominant with respect to the pair (P, K) . This means that if $K_L = L \cap K$, $K_U = U \cap K$, and $K_{\overline{U}} = \overline{U} \cap K$, then $K = K_{\overline{U}}K_LK_U$. Also, if $U_n = \lambda^{-n}K_U\lambda^n$, then $\text{Ad}(\lambda)$ normalizes K_L and $\bigcup_n U_n = U$. Thus, it makes sense to define $C_n = K_LU_n$, and $C_\infty = \bigcup_n C_n = K_LU$.

Remark. Using the formulas in section 2.3 it is easy to find a such a λ . For example, pick $a \in F^*$ very large. Then $\lambda = h_\alpha(a^{-2})h_\beta(a^{-1})$ works.

Lemma 72. *The groups C_n satisfy the assumptions of [H1, Theorem 1.1].*

Proof. Let \mathcal{K} and \mathcal{C}_n be the subalgebras of \mathfrak{g} corresponding to K and C_n , respectively. By assumption, $[\mathcal{K}, \mathcal{K}] \subseteq \varpi^{\beta+1}\mathcal{K}$. We must show that $[\mathcal{C}_n, \mathcal{C}_n] \subseteq \varpi^{\beta+1}\mathcal{C}_n$. Suppose that $k, k' \in \mathcal{K}_L$ and $k_U, k'_U \in \mathcal{K}_U$. Then,

$$\begin{aligned} & [k + \lambda^{-n}k_U\lambda^n, k' + \lambda^{-n}k'_U\lambda^n] \\ &= [k, k'] + \lambda^{-n}[k_U, k'_U]\lambda^n + \lambda^{-n}k_U\lambda^n k' + k\lambda^{-n}k'_U\lambda^n - \lambda^{-n}k'_U\lambda^n k - k'\lambda^{-n}k_U\lambda^n \\ &= [k, k'] + \lambda^{-n}[k_U, k'_U]\lambda^n + \lambda^{-n}[\text{Ad}(\lambda^n)(k), k'_U]\lambda^n + \lambda^{-n}[k_U, \text{Ad}(\lambda^n)(k')]\lambda^n. \end{aligned}$$

Since the adjoint action of λ preserves \mathcal{K}_L , the lemma follows. \square

In the same way that Howe proves Proposition 1.1 of [H2], we can now apply Theorem 1.1 of [H1] to the C_n and take an inductive limit. Set $\mathcal{C}_\infty = \bigcup_n \mathcal{C}_n$. We get the following:

Proposition 73. *Let $\hat{\mathcal{C}}_\infty$ be the Pontryagin dual of \mathcal{C}_∞ . Then*

- (1) *There is a natural homeomorphism $B: \hat{\mathcal{C}}_\infty / \text{Ad}^* C_\infty \rightarrow \hat{\mathcal{C}}_\infty$ where $\hat{\mathcal{C}}_\infty$ is the space of irreducible unitary representations of C_∞ suitably topologized.*
- (2) *If ρ is an irreducible representation of C_∞ , and f is a Schwartz-Bruhat function on C_∞ , then the character of ρ , $\text{Ch}_\rho: f \mapsto \text{Tr}(\rho(f))$, makes sense.*
- (3) *If $\rho = B(\mathcal{O})$ for some $\text{Ad}^* C_\infty$ -orbit $\mathcal{O} \subset \hat{\mathcal{C}}_\infty$, then $\text{Ch}_\rho(f) = \int_{\mathcal{O}} \hat{f}_1 \mu_{\mathcal{O}}$ where $f_1 = f \circ \exp$ is a Schwartz-Bruhat function on \mathcal{C}_∞ , and $\mu_{\mathcal{O}}$ is the unique C_∞ -invariant measure on \mathcal{O} suitably normalized.*

By choosing a character of F , we can identify \mathfrak{p}^* with $\hat{\mathfrak{p}}$. Then, corresponding to the injection $\mathcal{C}_\infty \subset \mathfrak{p}$, there is a projection $\lambda: \mathfrak{p}^* \rightarrow \hat{\mathcal{C}}_\infty$. It is easy to see that, using λ , we can view ω as an element of $\hat{\mathcal{C}}_\infty$. Let \mathcal{O}_∞ be the coadjoint C_∞ -orbit containing ω . Also, let $R = \{x \in F^* | h_\alpha(x) \in K\}$ and suppose that $\{x_i\}$, $i \in \mathbb{Z}$ is a set of coset representatives for F^*/R . Then it is clear that

$$(24) \quad \lambda(\mathcal{O}_P) = \bigcup_i \text{Ad}^*(h_\alpha(x_i))\mathcal{O}_\infty.$$

Lemma 74. *Let $\rho = B(\mathcal{O}_\infty)$. Then σ restricted to C_∞ is the direct sum of the representations $\text{Ad}^*(h_\alpha(x_i))\rho$.*

Proof. Recall how $\sigma = \pi|_P$ was constructed. Begin with the Heisenberg representation of U corresponding to a non-trivial additive character of F , Ψ . As we saw in Corollary 44, σ is independent of this choice. The Heisenberg representation extends uniquely to a representation of $P^\circ = HU$ which we called $\hat{\pi}$. Finally, induce $\hat{\pi}$ from P° to P . Note that P/P° may be identified with h_α .

It is now clear that σ restricted to C_∞ may be constructed as follows. Extend the Heisenberg representation to K_HU where $K_H = K \cap H$; call it $\hat{\rho}$. Next induce

from $K_H U$ to $h_\alpha K_H U$, and finally restrict back to $h_\alpha(R)K_H U = C_\infty$. Thus by Mackey's theory,

$$\sigma|_{C_\infty} = \bigoplus \text{Ad}^*(h_\alpha(x_i)) \text{Ind}_{K_H U}^{C_\infty} \hat{\rho}.$$

Let $\tau = \text{Ind}_{K_H U}^{C_\infty} \hat{\rho}$. It remains to check that $\rho = \tau$. Recall that $W' \subset U$ is a maximal isotropic subspace. Let $A \subset C_\infty$ be the subgroup generated by $z(t), W'$ and K_H , and let \mathcal{A} be its Lie algebra. Then $\{\omega\} \in \hat{\mathcal{A}}$ is a coadjoint A -orbit. The corresponding representation of A is the character $z(t)w'k \mapsto \Psi(t)$. It follows from the theory of the Heisenberg representation that $\tau = \text{Ind}_A^{C_\infty} \Psi$.

On the other hand, $\omega \in \hat{\mathcal{A}}$ is obviously in the image of \mathcal{O}_∞ under the canonical map $\hat{\mathcal{C}}_\infty \rightarrow \hat{\mathcal{A}}$. Since the map B from Proposition 73 is natural, this means that there is a map from $\rho|_A$ to Ψ . Thus by Frobenius reciprocity,

$$\rho \subset \text{Ind}_A^{C_\infty} \Psi = \tau.$$

Hence, to show $\tau \cong \rho$, it enough to check that τ is irreducible. This follows from the arguments in Proposition 43. \square

Finally, we can prove Proposition 71. We are assuming that the support of h is contained in $\mathfrak{p}_r \subset \mathcal{C}_\infty$. Thus, by Proposition 73, Lemma 74, and equation (24),

$$\begin{aligned} \text{Tr } \sigma(h) &= \sum_i \int_{\text{Ad}^*(h_\alpha(x_i))\mathcal{O}_\infty} \hat{h}\mu_i \\ &= \int_{\lambda(\mathcal{O}_P)} \hat{h}\mu. \end{aligned}$$

Note that only finitely many terms of the sum are nonzero because $\text{supp } \hat{h} \cap \mathcal{O}_P$ is compact. Moreover, because we are dealing only with functions on \mathcal{C}_∞ , we can normalize the measure on \mathcal{O}_P so that

$$\int_{\lambda(\mathcal{O}_P)} \hat{h}\mu = \int_{\mathcal{O}_P} \hat{h}\mu_{\mathcal{O}_P}.$$

Proposition 71 is proved.

4.3. Proof of Minimality.

Theorem 75. π is a minimal representation of G .

For any nilpotent coadjoint orbit \mathcal{O} , set $2d(\mathcal{O}) = \dim \mathcal{O}$. Let $\mathbf{d} = \text{GK}(\pi)$ and $d_0 = d(\mathcal{O}_{\min})$. Since \mathcal{O}_{\min} is the unique minimal coadjoint G -orbit, to prove Theorem 75 it is enough to show that $\mathbf{d} = d_0$.

Of course, the main idea is to find functions on G whose support is very close to P so that we can relate Θ_π and $\text{Tr}(\sigma)$. First some notation. Set $\overline{U}_n = \overline{U} \cap G_n$, and let $\chi_{\overline{U}_n}$ be the characteristic function of \overline{U}_n normalized to have total integral one. Also, recall the notation introduced after Proposition 71, $K = G_r$ and $K = K_{\overline{U}}K_LK_U$. So, for $x \in K$, we can write $x = zy$ where $z \in \overline{U}$ and $y \in P$. Now, if $h(y)$ is a Schwartz-Bruhat function on P which vanishes outside of P_r , we define $f_n(x) = \chi_{\overline{U}_n}(z) \otimes h(y)$; f_n vanishes outside of K .

Lemma 76. *There is an integer n_0 so that for $n \geq n_0$*

$$\pi(f_n) = \sigma(h)\pi(\chi_n).$$

Here χ_n is the characteristic function of G_n with total integral one.

Proof. Since h is locally constant and with compact support contained in P_r , we can choose n_0 so large that for all $n \geq n_0$ there are constants c_i and elements $p_i \in P_r$ (depending on n) so that

$$h(y) = \sum c_i \chi_{P_n}(p_i^{-1}y).$$

Thus,

$$f_n(x) = \sum c_i \chi_n(p_i^{-1}x).$$

Now,

$$\sigma(h)\pi(\chi_n) = \left(\sum c_i \pi(\chi_{P_n})\pi(p_i) \right) \pi(\chi_n).$$

But $p_i \in P_r \subset P_1$ so it normalizes G_n . Thus,

$$\begin{aligned} \sigma(h)\pi(\chi_n) &= \sum c_i \pi(\chi_{P_n})\pi(\chi_n)\pi(p_i) \\ &= \sum c_i \pi(\chi_n)\pi(p_i) \\ &= \pi(f_n). \quad \square \end{aligned}$$

Corollary 77. *Suppose that $\text{supp } \hat{h} \cap \mathcal{O}_P$ is compact. Then*

$$\lim_{n \rightarrow \infty} \Theta_\pi(f_n) = \text{Tr}(\sigma(h)).$$

Remark. The assumptions of the corollary imply that $\hat{h}(0) = 0$ and thus $\hat{f}_n(0) = 0$.

Lemma 78. (1) $\hat{f}_n = \hat{\chi}_{\overline{U}_n} \otimes \hat{h}$.

(2) $\hat{\chi}_{\overline{U}_{n+1}} \geq \hat{\chi}_{\overline{U}_n} \geq 0$.

(3) *Suppose that $\hat{h} \geq 0$ and $\text{supp } \hat{h} \cap \mathcal{O}_P \neq \emptyset$. Then if \mathcal{O} is any non-zero nilpotent coadjoint orbit,*

$$\hat{\mu}_{\mathcal{O}}(f_n) > 0.$$

Remark. It is clear that there exist h satisfying the hypotheses of both part (3) and Corollary 77.

Proof. Parts (1) and (2) are obvious. Also, they immediately imply that $\hat{\mu}_{\mathcal{O}}(f_n) \geq 0$. To prove (3), it remains to check that the support of \hat{f}_n intersects \mathcal{O} . Since both sets are open, it is enough to check that the support of \hat{f}_n intersects $\overline{\mathcal{O}}$. But since G has a unique minimal orbit,

$$\overline{\mathcal{O}} \supset \mathcal{O}_{\min} \supset \mathcal{O}_P. \quad \square$$

The following result is proved in [M-W].

Proposition 79 (Moeglin-Waldspurger). *If $\text{GK}(\rho) = \mathbf{d}$ and \mathcal{O} is a nilpotent coadjoint orbit of dimension $2\mathbf{d}$, then $c_{\mathcal{O}}$ is a non-negative integer.*

It follows from Lemma 78 and Proposition 79 that we can choose h so that

$$(25) \quad \sum_{\dim \mathcal{O}=2\mathbf{d}} c_{\mathcal{O}} \hat{\mu}_{\mathcal{O}}(f_n)$$

is bounded away from zero. The strategy of the proof is to show that if $\mathbf{d} > d_0$ then the limit as $n \rightarrow \infty$ of expression (25) is zero. Thus, $\mathbf{d} = d_0$.

The essential idea of Savin is to consider not just the single family of functions f_n , but several related families in order to separate the orbital integrals of different dimensions. We need the following result from [H-C, Lemma 3].

Proposition 80 (Harish-Chandra). *Let $g(x^*) \in \mathcal{S}_{\mathfrak{g}^*}$, and let \mathcal{O} be a nilpotent coadjoint orbit of dimension $2d(\mathcal{O})$. Then*

$$\int g(r^{-1}x^*)\mu_{\mathcal{O}} = |r|^{d(\mathcal{O})} \int g(x^*)\mu_{\mathcal{O}}$$

for any $r \in F^*$.

Remark. Here $\mathcal{S}_{\mathfrak{g}^*}$ is the set of Schwartz-Bruhat functions on \mathfrak{g}^* .

For any $f \in \mathcal{S}_{\mathfrak{g}}$, define $f^{(r)}(x) = |r|^{\dim \mathfrak{g}} f(rx)$. Then it is easy to check that $\widehat{f^{(r)}}(x^*) = \widehat{f}(r^{-1}x^*)$. Thus,

Corollary 81. *Let f and \mathcal{O} be as above. Then $\hat{\mu}_{\mathcal{O}}(f^{(r)}) = |r|^{d(\mathcal{O})} \hat{\mu}_{\mathcal{O}}(f)$.*

It is easy to see that $f_n^{(r)} = \chi_{\overline{U}_n}^{(r)} \otimes h^{(r)}$, where these functions are defined in the analogous manner. Furthermore, if the residue field of F has q elements and $|r| = q^k$, then $\chi_{\overline{U}_n}^{(r)} = \chi_{\overline{U}_{n+k}}$. It follows that Corollary 77 holds for $f_n^{(r)}$. That is,

$$\lim_{n \rightarrow \infty} \Theta_{\pi}(f_n^{(r)}) = \text{Tr}(\sigma(h^{(r)})).$$

Of course, using Corollary 81

$$\begin{aligned} \Theta_{\pi}(f_n^{(r)}) &= \sum_{k=d_0}^{\mathbf{d}} \sum_{d(\mathcal{O})=k} c_{\mathcal{O}} \hat{\mu}_{\mathcal{O}}(f_n^{(r)}) \\ &= \sum_{k=d_0}^{\mathbf{d}} \sum_{d(\mathcal{O})=k} |r|^k c_{\mathcal{O}} \hat{\mu}_{\mathcal{O}}(f_n). \end{aligned}$$

(Recall that $\hat{f}_n(0) = 0$.) Furthermore, since $\mu_{\mathcal{O}_P}$ is up to a constant the restriction of $\mu_{\mathcal{O}_{\min}}$ to \mathcal{O}_P (see Lemma 70), statements analogous to Proposition 80 and Corollary 81 hold for $\mu_{\mathcal{O}_P}$. Thus, by Proposition 71,

$$\text{Tr}(\sigma(h^{(r)})) = |r|^{d_0} \text{Tr}(\sigma(h)).$$

This proves that

$$(26) \quad \lim_{n \rightarrow \infty} \sum_{k=d_0}^{\mathbf{d}} \sum_{d(\mathcal{O})=k} |r|^k c_{\mathcal{O}} \hat{\mu}_{\mathcal{O}}(f_n) = |r|^{d_0} \text{Tr}(\sigma(h))$$

for all $r \in F^*$. For each $d_0 \leq i \leq \mathbf{d}$, pick r_i so that $|r_i| \neq |r_j|$ for $i \neq j$. Next, rewrite the corresponding family of expressions (26) as follows. Let \mathbf{T} be the $(\mathbf{d} - d_0) \times (\mathbf{d} - d_0)$ matrix with $\mathbf{T}_{i,j} = |r_j|^i$, $d_0 \leq i, j \leq \mathbf{d}$. Also, \mathbf{X}^n is the $(\mathbf{d} - d_0)$ -vector given by

$$(\mathbf{X}^n)_i = \begin{cases} \sum_{d(\mathcal{O})=i} c_{\mathcal{O}} \hat{\mu}_{\mathcal{O}}(f_n) & \text{if } i > d_0, \\ (c_{\mathcal{O}_{\min}} \hat{\mu}_{\mathcal{O}_{\min}}(f_n)) - \text{Tr}(\sigma(h)) & \text{if } i = d_0. \end{cases}$$

With this notation, we can write

$$\lim_{n \rightarrow \infty} \mathbf{T} \mathbf{X}^n = 0.$$

However, $\det \mathbf{T}$ is easily seen to be non-zero; it is a Vandermonde determinant. Thus, $\lim_{n \rightarrow \infty} \mathbf{X}^n = 0$. But we have already shown (equation (25)) that

$$\sum_{d(\mathcal{O})=\mathbf{d}} c_{\mathcal{O}} \hat{\mu}_{\mathcal{O}}(f_n)$$

is bounded away from zero. Thus, $\mathbf{d} = d_0$. In other words, π is a minimal representation. Theorem 75 is proved.

APPENDIX A. JORDAN ALGEBRAS AND LIE ALGEBRAS

In this appendix, we extend the results of section 1 in two ways. First, we construct more Lie algebras. Second, we give a technique for constructing large tables of dual pairs in exceptional Lie algebras.

In section 1, we associated a Lie algebra to any central, simple, rank three Jordan algebra, \mathcal{J} , over a field, F . When F is a p -adic field, we observed that these include all Lie algebras of type E_6, E_7, E_8 and F_4 . However, the construction actually uses only the fact that \mathcal{J} has rank three. By considering \mathcal{J} 's which are not necessarily central simple, we obtain a wider class of Lie algebras. For example, if E is a cubic extension of F and $\mathcal{J} = E$ as a vector space, with multiplication $a.b = \frac{1}{2}(ab + ba)$, then we obtain a quasi-split rank two form of D_4 ("triality D_4 "). Also, $\mathcal{J} = F$ with norm $x \mapsto x^3$ and trace $x \mapsto 3x$, gives G_2 .

One general way to obtain a rank three Jordan algebra is as the direct sum of a trivial rank one Jordan algebra with a central simple rank two Jordan algebra. It is a fact that every central simple rank two Jordan algebra is the Jordan algebra of a symmetric bilinear form. These have the form $V \times F1$ where V is a vector space equipped with a quadratic form, (\cdot, \cdot) . The multiplication is given by $v.v' = (v, v')1$ for $v, v' \in V$, and $a1.b1 = ab1$ for $a, b \in F$. Note that $V \times F1$ embeds in the Clifford algebra of V as the tensors of degree less than or equal to one; here, of course, the associative Clifford algebra is made into a Jordan algebra in the usual way.

It is easy to check that if $d = \dim V$ is odd, then the Lie algebra coming from the Jordan algebra $F \oplus (V \times F1)$ is a form of $D_{(d+1)/2+3}$. Similarly, if d is even, we get $B_{d/2+3}$. Of course, these orthogonal Lie algebras may be constructed in other ways. On the other hand, this construction makes it easy to define embeddings of orthogonal algebras into exceptional algebras as discussed below.

Recall from section 1 that the Lie algebra \mathfrak{M} is generated by the L_A for $A \in \mathcal{J}$. Using this, one can check that if $\mathcal{I} \subset \mathcal{J}$ are two rank three Jordan algebras, then $\mathfrak{M}(\mathcal{I}) \subset \mathfrak{M}(\mathcal{J})$. More generally, any of the Lie algebras associated to \mathcal{I} constructed in section 1, $\mathfrak{g}(\mathcal{I}), \mathfrak{h}(\mathcal{I}), \mathfrak{M}_0(\mathcal{I}), \mathfrak{M}_0^{(A)}(\mathcal{I})$, are sub-Lie algebras of the corresponding algebras associated to \mathcal{J} . Furthermore, it is easy to prove the following result (cf. [S2, Theorem 7.3]).

Proposition 82. *Suppose that $\mathcal{I} \subset \mathcal{J}$ are rank three Jordan algebras. Let $\mathfrak{L} = \{\mathfrak{m} \in \mathfrak{M}_0(\mathcal{J}) \mid m(A) = 0 \ \forall A \in \mathcal{I}\}$. Assume that $(\mathfrak{M}_0(\mathcal{I}), \mathfrak{L})$ is a dual pair in $\mathfrak{M}_0(\mathcal{J})$. Then $(\mathfrak{h}(\mathcal{I}), \mathfrak{L})$ and $(\mathfrak{g}(\mathcal{I}), \mathfrak{L})$ are dual pairs in $\mathfrak{h}(\mathcal{J})$ and $\mathfrak{g}(\mathcal{J})$, respectively.*

The proposition leads to the construction of many dual pairs. For example, consider the following set of inclusions of Jordan algebras. Let \mathcal{J}_0 be the exceptional Jordan algebra (three by three hermitian matrices over the octonions). Next let \mathcal{J}_i , $i = 1, 2, 3$ be three by three hermitian matrices over the quaternions, a quadratic algebra, and the field itself, respectively. It is clear that we can set things up so

							A_1
						F	A_2
				finite	A_1	G_2	
			finite	F^2	A_1^3	D_4	
		finite	A_1	A_2	C_3	F_4	
	finite	F^2	A_2	$A_2 + A_2$	A_5	E_6	
F	A_1	A_1^3	C_3	A_5	D_6	E_7	
A_1	A_2	G_2	D_4	F_4	E_6	E_7	E_8

FIGURE 5. Table of dual pairs.

that $\mathcal{J}_i \supset \mathcal{J}_j$ when $i < j$. Next, take \mathcal{J}_4 to be the diagonal matrices in \mathcal{J}_3 , and \mathcal{J}_5 the elements $\text{diag}(a, a, a)$. The corresponding inclusions of Lie the algebras $\mathfrak{g}(\mathcal{J}_i)$ are

$$G_2 \subset D_4 \subset F_4 \subset E_6 \subset E_7 \subset E_8.$$

We could now write a large see-saw diagram for the corresponding dual pairs in E_8 , another for E_7 and so on (special care is needed for \mathcal{J}_5 as the proposition does not apply). Instead, we will represent this information in Figure 5. The last column represents the inclusions above together with $A_1 \subset A_2 \subset G_2$ (if we wish, we can view A_2 as coming from the “zero” Jordan algebra). The rows record the dual pairs. Thus, in E_8 , A_1 pairs with E_7 , A_2 pairs with E_6 , G_2 pairs with F_4 and so on. Similarly for the other rows. For example, A_2 is paired with A_5 in E_7 and with $A_2 + A_2$ in E_6 . Furthermore, if we delete the last column, the diagram has the same meaning. Thus, for example, $(A_1)^3$ pairs with D_4 in E_7 and with $(A_1)^3$ in D_6 . The same phenomenon occurs if we delete the last two columns.

Remarks. (1) One astonishing feature of this particular collection is its symmetry. I can not explain why it is true.

(2) The word “finite” in the figure means that the dual pair in question involves a finite group (and so may not exist on the Lie algebra level).

Using our construction of orthogonal Lie algebras above, we can get many more dual pairs in exceptional Lie algebras. For example, two by two hermitian matrices over the quaternions are easily seen to form a Jordan algebra of a symmetric bilinear

form of dimension 5. Thus, considering matrices of the form

$$\begin{pmatrix} a & d & 0 \\ \bar{d} & b & 0 \\ 0 & 0 & c \end{pmatrix},$$

we can embed D_6 into E_7 . Similarly, if we set $a = b$, we get B_5 . In this manner, one can construct long inclusions of rank three Jordan algebras and thus large diagrams similar to Figure 5. In general, of course, they will not be symmetric. Nevertheless, it is easy to see that we can enlarge Figure 5 by including B_4 between D_4 and F_4 , and B_3 between G_2 and F_4 . The resulting diagram is still symmetric.

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