REPRESENTATION THEORY
An Electronic Journal of the American Mathematical Society
Volume 1, Pages 207–279 (July 31, 1997)
S 1088-4165(97)00033-2

PERIODIC W-GRAPHS

G. LUSZTIG

Introduction

0.1. We begin by recalling the definition [KL1] of a W-graph. Let (W,S) be a Coxeter group.

Let $\mathcal{A}=\mathbf{Z}[v,v^{-1}]$ where v is an indeterminate. Let Y be a set. Assume that for each $y\in Y$ we are given a subset $\mathfrak{I}_y$ of S and that for any z,y in Y such that $\mathfrak{I}_z\not\subset\mathfrak{I}_y$ we are given an integer $\mu_{z,y}$ so that

$$\tau_s y=\begin{cases}-v^{-1}y, & \text{if } y\in Y, s\in\mathfrak{I}_y,\\ vy+\displaystyle\sum_{z\in Y;s\in\mathfrak{I}_z}\mu_{z,y}z, & \text{if } y\in Y, s\notin\mathfrak{I}_y\end{cases}$$

defines a representation of the braid group of W (or rather, of the corresponding Hecke algebra) on the free $\mathcal{A}$-module $\mathcal{A}[Y]$ with basis indexed by Y. (Here $s\in S$ and it is assumed that only finitely many terms of the sum above are non-zero.) In other words, for any $s\neq s'$ in S such that ss' has finite order m, we have $\underbrace{\tau_s\tau_{s'}\tau_s\dots}_{m\text{ factors}}=\underbrace{\tau_{s'}\tau_s\tau_{s'}\dots}_{m\text{ factors}}$. We then say that $(Y,(\mathfrak{I}_y)_{y\in Y},(\mu_{z,y}))$ is a *W-graph.*

0.2. One of the main results of [KL1] was a construction of a W-graph with $Y=W$ and with $\mathfrak{I}_y$ being the set of all $s\in S$ such that the length of sy is less than the length of y. Moreover, there is an induced W-graph structure on certain subsets of W (the “left cells” of W). In the case where W is finite, the W-graphs attached to the left cells come close to realizing all irreducible representations of the Hecke algebra. In particular, when W is a symmetric group $\mathfrak{S}_n$, all irreducible representations of the Hecke algebra arise from left cells.

If W is infinite, the left cells do not come even close to realizing the irreducible representations of the Hecke algebra. To remedy this, we give the following definition.

0.3. A W-graph is said to be *periodic* if the underlying set Y has a given free action of a finitely generated free abelian group $\mathcal{T}$ such that $\mathcal{T}\backslash Y$ is finite, $\mathfrak{I}_{ty}=\mathfrak{I}_y$ for all $t\in\mathcal{T},y\in Y$ and such that the μ-function is preserved by the action of $\mathcal{T}$. From such a graph we can obtain a family of finite dimensional representations of the Hecke algebra (over a field $\mathbf{K}$ of characteristic 0). Namely, for any ring homomorphism $\mathcal{A}[\mathcal{T}]\to\mathbf{K}$, we can form the finite dimensional $\mathbf{K}$-vector space

Received by the editors January 27, 1997 and, in revised form, June 9, 1997.
1991 *Mathematics Subject Classification.* Primary 20G99.
Supported in part by the National Science Foundation.

$\mathcal{A}[Y] \otimes_{\mathcal{A}[\mathcal{T}]} \mathbf{K}$ which is a module for the Hecke algebra, and is provided with a canonical decomposition as a direct sum of lines.

0.4. Assume that (W, S) is an affine Weyl group. The W-graphs attached to the left cells of W are not in general periodic. An example of a periodic W-graph is constructed, in this case, in [L1]. The associated family of finite representations of the affine Hecke algebra (see 0.3) is a version of the principal series representations. It is not the standard version, since in contrast with the standard version, it admits intertwining operators that are everywhere defined, without poles. This comes from the fact that the W-graph admits a as group of automorphisms not only $\mathcal{T}$ but W itself.

0.5. The aim of this paper is to construct a family of periodic W-graphs where (W, S) is an affine Weyl group. Namely, for any partial flag manifold of the algebraic group corresponding to W, we are seeking a W-graph Y which is periodic with respect to an action of $\mathcal{T} \cong \mathbf{Z}^r$ (where r is the dimension of the second homology space of the partial flag manifold) and the cardinal of $\mathcal{T} \backslash Y$ is the Euler characteristic of the partial flag manifold. This periodic W-graph should then give rise as in 0.3 to a family of representations of the affine Hecke algebra depending on r continuous parameters, which are generically irreducible of dimension $\chi(G/P)$ (a degenerate principal series).

Note however that we are not entirely successful in achieving our aim; namely, what we actually construct is a periodic W-graph in a wider sense (as defined in the Appendix) in which the local finiteness implicit in the definition 0.1 is not assumed. In fact, the definition of W-graph adopted in the rest of this paper is not the one in 0.1, but the slightly wider one, in the Appendix. This is most likely a temporary situation, since I believe that what we construct are W-graphs in the sense of 0.1; only the proof is missing for the time being. In the case of the full flag manifold, our W-graph reduces to the one in [L1]; in particular, in that case it is a W-graph in the sense of 0.1.

0.6. The set Y of vertices of our graph is the set of alcoves contained in a certain region Ξ of an euclidean space $\mathbf{R}^n$ ($n = \text{rank}(W)$) which is homeomorphic to $\mathbf{R}^r$ times a compact space.

Let M^K be the set of formal $\mathcal{A}$-linear combinations of elements in Y. In 4.6 we show that M^K is naturally a module over the affine Hecke algebra. In 4.13 we define two submodules $M^K_{\leq}, M^K_{\geq}$ of M^K by the requirement that the support of an element be bounded above (or below) with respect to a certain natural partial order on Y. In 4.14 we construct an involution b of $M^K_{\leq}$ which is antilinear with respect to the Hecke algebra structure. This is obtained by composing several simpler maps which may go out of $M^K_{\leq}$. Then in 11.2 we define, for each element $A \in Y$, a canonical b-invariant element $A^\flat \in M^K_{\leq}$ which is of the form A plus a (possibly infinite) linear combination of elements strictly smaller than A with coefficients of the form $c_{-1}v^{-1} + c_{-2}v^{-2} + \dots$. The integers c_{-1} then provide the main ingredient (the μ-function) in the definition of a W-graph and in 11.14 we do indeed show that a W-graph is obtained in this way. We expect, but cannot prove, that in fact the elements $A^\flat$ have finite support (they do, in the case $K = \emptyset$, studied in [L1], and in the case where G has rank 2). (This would imply that the W-graph we construct is locally finite.)

Much of the effort in §6-§10 is concerned with developing methods to prove a result (11.17) which, while doesn't show that $A^\flat$ has finite support, it points in that direction. Namely, we show that the polynomials $c_{-1}v^{-1}+c_{-2}v^{-2}+\dots$ above (in v^{-1}) have a universally bounded degree. The proof involves among other things, the use of the K-theoretic methods in [KL2] of a study of representations of affine Hecke algebras (there was no need for such methods in the case $K=\emptyset$ in [L1]).

0.7. Apart from the matters mentioned in the last paragraph, the methods used in this paper are completely elementary. Our results suggest a connection with geometry, namely it appears that our construction can be interpreted as providing a canonical basis of a certain equivariant K-homology group (see 13.15).

0.8. I am indebted to David Vogan for some valuable information on intertwining operators and also for his help with typesetting the figures.

Contents

1. Preliminaries

1.1. The basic reference for the results in this section is [B, Ch.5]; see also [L1, §1].

Let E be an affine euclidean space of finite dimension with a given set $\mathfrak{F}$ of hyperplanes. Let T be the vector space of translations of E. The natural action $E\times T\to E$ is denoted by $(e,x)\mapsto e+x$. Let $\mathcal{G}$ be the group of isometries of E; we regard $\mathcal{G}$ as acting on E on the right. Then T is naturally a subgroup of $\mathcal{G}$. For each $H\in\mathfrak{F}$, let $\sigma_H\in\mathcal{G}$ be the orthogonal reflection with fixed point set H. Let Ω be the subgroup of $\mathcal{G}$ generated by the $\sigma_H (H\in\mathfrak{F})$.

We assume that Ω is a discrete cocompact subgroup of $\mathcal{G}$, leaving stable the set $\mathfrak{F}$. (In [L1] we assumed also that the action of Ω on T is irreducible; that assumption was not essential. Therefore we feel free to quote results from [L1].)

Let $\mathcal{T}'$ be the set of all $x\in T$ such that the following holds: for any $H\in\mathfrak{F}$ we have $H+x\in\mathfrak{F}$. This is a lattice in T. Let $\mathcal{T}=T\cap\Omega$ (intersection in $\mathcal{G}$); this is a subgroup of finite index of $\mathcal{T}'$. For $H \in \mathfrak{F}$,

$$\mathrm{dir}(H) = \{x \in T | H + x = H\}$$

is a linear hyperplane in T. The set $\bar{\mathcal{F}}$ of linear hyperplanes in T of the form $\mathrm{dir}(H)$ for some $H \in \mathfrak{F}$ is finite with, say ν elements. A connected component of $T - \bigcup_{h\in\bar{\mathcal{F}}} h$ is called a *chamber*.

We assume that a chamber $\mathcal{C}^+$ has been chosen.

Let $\mathcal{C}^- = -\mathcal{C}^+$. Let I be the subset of $\bar{\mathcal{F}}$ consisting of the walls of $\mathcal{C}^+$. For $h \in \bar{\mathcal{F}}$, let $r_h : T \to T$ be the orthogonal reflection with fixed point set h. The subgroup of the orthogonal group of T generated by the r_i is a finite Coxeter group (a Weyl group) W^I with standard generators $\{r_i | i \in I\}$. We shall regard W^I as acting on T on the right. For any subset K of I we denote by W^K the parabolic subgroup of W^I generated by $\{r_i | i \in K\}$; let w_0^K be the longest element of W^K.

We have a natural homomorphism $\Omega \to W^I$ (with kernel $\mathcal{T}$) which for any $H \in \mathfrak{F}$ takes σ_H to r_h, where $h = \mathrm{dir}(H)$. If $\omega \in \Omega$ is mapped by this homomorphism to $w \in W^I$, then $(e + x)\omega = e\omega + xw$ for $e \in E, x \in T$.

The set of points of E that are not contained in any hyperplane in $\mathfrak{F}$ is a union of connected components called *alcoves*. The set of points of E that belong to exactly one hyperplane in $\mathfrak{F}$ is a union of connected components called *faces*.

Let X be the set of alcoves. It is known that Ω acts simply transitively on X. Let S be the set of Ω-orbits in the set of faces. Then S is finite. For any alcove A and any $s \in S$, there is a unique face $\delta_s(A)$ in the Ω-orbit s such that $\delta_s(A) \subset cl(A)$. (For any subset Y of E we denote by $cl(Y)$ the closure of Y in E.) If A is an alcove, we say that $H \in \mathfrak{F}$ is a wall of A if there exists $s \in S$ such that the face $\delta_s(A)$ is an open subset of H.

For $s \in S$, we define an involution $A \to sA$ of X as follows. Given an alcove A, we denote by sA the unique alcove $\neq A$ such that $\delta_s(A) = \delta_s(sA)$. The maps $A \to sA$ generate a group of permutations of X which is a Coxeter group (W, S) (an affine Weyl group). It acts simply transitively (on the left) on X and it commutes with the action of Ω on X.

A point $\epsilon \in E$ is said to be *special* of E if for any $H \in \mathfrak{F}$, there exists $H' \in \mathfrak{F}$ such that H' is parallel to H and $\epsilon \in H'$. It is known that there exist special points in E. If ϵ is a special point, let A_ϵ^+ (resp. A_ϵ^-) be the unique alcove contained in $\epsilon + \mathcal{C}^+$ (resp. in $\epsilon + \mathcal{C}^-$) and having ϵ in its closure. Let $\Omega_\epsilon = \{\omega \in \Omega | \epsilon\omega = \epsilon\}$. Let W_ϵ be the (parabolic) subgroup of W consisting of all $w \in W$ such that the closure of wA_ϵ^+ contains ϵ. There is a unique group isomorphism $j_\epsilon : \Omega \xrightarrow{\sim} W$ such that $j_\epsilon(\omega)A_\epsilon^+ = A_\epsilon^+\omega$ for all $\omega \in \Omega$. This restricts to an isomorphism $\Omega_\epsilon \xrightarrow{\sim} W_\epsilon$. The restriction of $\Omega \to W^I$ to Ω_ϵ is an isomorphism $\Omega_\epsilon \xrightarrow{\sim} W^I$.

For any subset K of I we denote by $\omega_{\epsilon,K}$ the element of Ω_ϵ corresponding to $w_0^K \in W^I$. We write ω_ϵ instead of $\omega_{\epsilon,I}$.

Note that $\mathcal{T}'$ acts (by translation) simply transitively on the set of special points in E. Moreover, if $x \in \mathcal{T}'$ and A is an alcove, then $A + x$ is an alcove.

For a special point $\epsilon \in E$ we define $D(\epsilon)$ to be the set of all alcoves $A \in X$ that contain ϵ in their closure. For $w \in W^I$ we set $A_{\epsilon,w} = A_\epsilon^-\omega$ where $\omega \in \Omega_\epsilon$ corresponds to w under the canonical isomorphism $\Omega_\epsilon \xrightarrow{\sim} W^I$. Then $w \mapsto A_{\epsilon,w}$ is a bijection $W^I \xrightarrow{\sim} D(\epsilon)$.

If E has dimension 0, then $\mathfrak{F} = \emptyset$, $\Omega = W = W^I = \{1\}$, X has exactly one element and S, I are empty.

1.2. If $H \in \mathfrak{F}$, then $E - H$ has two connected components E_H^+, E_H^-; E_H^+ meets $\epsilon + \mathcal{C}^+$ for any special point ϵ, and E_H^- is disjoint from $\epsilon + \mathcal{C}^+$ for some special point ϵ.

Let $A \in X$. Let $\mathcal{L}(A)$ be the subset of S consisting of those $s \in S$ such that $A \subset E_L^+, sA \subset E_L^-$, where L is the hyperplane in $\mathfrak{F}$ that supports the common face of A, sA.

For $A, B \in X$ and $H \in \mathfrak{F}$ we set

$$\tau_H(A,B) = \begin{cases} 1 & \text{if } A \subset E_H^-, B \subset E_H^+, \\ -1 & \text{if } A \subset E_H^+, B \subset E_H^-, \\ 0, & \text{otherwise .} \end{cases}$$

(For fixed A, B, the third alternative occurs for all but finitely many H.)

Following [L1, 1.4] we define a function $d : X \times X \to \mathbf{Z}$ by

$$d(A,B) = \sum_{H \in \mathfrak{F}} \tau_H(A,B).$$

We have $d(A,B) = \sum_{h \in \bar{\mathcal{F}}} d_h(A,B)$ where, for $h \in \bar{\mathcal{F}}$, we set

$$d_h(A,B) = \sum_{\substack{H \in \mathfrak{F} \\ \mathrm{dir}(H) = h}} \tau_H(A,B).$$

We have $d_h(A,A) = 0$; moreover, for $A, B, C \in X$, we have the additivity property

$$d_h(A,B) + d_h(B,C) + d_h(C,A) = 0.$$

Hence, $d(A,A) = 0$ and, for $A, B, C \in X$, we have the additivity property

$$d(A,B) + d(B,C) + d(C,A) = 0.$$

1.3. **Roots, coroots.** For any $h \in \bar{\mathcal{F}}$ there is a unique element $\alpha_h \in T$ such that, for any $H \in \mathfrak{F}$ with $\mathrm{dir}(H) = h$, properties (a)-(d) below hold:

(a) α_h is orthogonal to H;
(b) $H + \alpha_h \subset E_H^+$;
(c) $H + \frac{1}{2}\alpha_h \in \mathfrak{F}$;
(d) if $x \in \mathbf{R}$ satisfies $0 < x < \frac{1}{2}$, then $H + x\alpha_h \notin \mathfrak{F}$.

Then $\alpha_h \in \mathcal{T}$ and $\{\pm\alpha_h | h \in \bar{\mathcal{F}}\}$ is a root system in T with a set of simple roots $\{\alpha_i | i \in I\}$ (which is a $\mathbf{Z}$-basis of $\mathcal{T}$).

For $h \in \bar{\mathcal{F}}$, the coroot $\check{\alpha}_h : \mathcal{T} \to \mathcal{Z}$ corresponding to α_h satisfies the equality

(e) $\check{\alpha}_h(t) = d_h(A, A + t)$.

Let $\mathcal{T}^+ = \sum_{i \in I} \mathbf{N}\alpha_i$, $\quad \mathcal{T}_{\mathrm{dom}} = \{x \in \mathcal{T} | \check{\alpha}_i(x) \in \mathbf{N} \quad \forall i \in I\}$. Clearly, if $\epsilon \in E$ is special and $x \in \mathcal{T}$ then the following three conditions are equivalent: $x \in \mathcal{T}_{\mathrm{dom}}$; $\epsilon + x \in cl(\epsilon + \mathcal{C}^+)$; $A_\epsilon^+ x \subset \epsilon + \mathcal{C}^+$.

1.4. Let $A, B \in X$. Following [L1, 1.5], we say that $A \le B$ if there exists a sequence of alcoves $A = A_0, A_1, \ldots, A_n = B$ such that for any $j \in [1, n]$ we have $d(A_{j-1}, A_j) = 1$ and $A_j = A_{j-1}\sigma_{H_j}$ for some $H_j \in \mathfrak{F}$. Then $A \le B$ is a partial order in X. We write $A < B$ instead of $A \le B, A \ne B$. Note that $A < B$ implies $d(A,B) > 0$.

1.5. For any subset $K \subset I$ we set $\mathcal{T}^K = \sum_{i \in K} \mathbf{Z}\alpha_i \subset \mathcal{T}$.

2. Study of a K-alcove

2.1. Let K be a subset of I. Let $T_K \subset T$ be the intersection of the hyperplanes i where i runs through K; then T_K is a vector space of dimension $\dim(E) - |K|$. Let $E^K = E/T_K$, that is, the quotient space of E by the action of T_K (by translation); let $\rho : E \to E^K$ be the canonical map. We may regard E^K naturally as an affine euclidean space of dimension $|K|$; the distance between two points p, p' of E^K is the minimum distance (in E) between a point of $\rho^{-1}(p)$ and a point of $\rho^{-1}(p')$. The space of translations of E^K is T/T_K. Let $\mathfrak{F}^K$ be the subset of $\mathfrak{F}$ consisting of those H such that $H + T_K = H$. Let $\bar{\mathcal{F}}^K$ be the subset of $\bar{\mathcal{F}}$ consisting of hyperplanes of the form $\mathrm{dir}(H)$ with $H \in \mathfrak{F}^K$. Let Ω^K be the subgroup of Ω generated by the reflections σ_H with $H \in \mathfrak{F}^K$. Then $(E^K, T/T_K, \mathfrak{F}^K, \Omega^K)$ is like $(E, T, \mathfrak{F}, \Omega)$ in 1.1. In particular, we can define as in 1.1 (relative to E^K) special points, alcoves, faces, walls of alcoves in E^K. (In the case where $K = \{i\}, i \in I$, we write $\mathfrak{F}^i, \Omega^i$ instead of $\mathfrak{F}^{\{i\}}, \Omega^{\{i\}}$.)

Note that a point of E^K is special precisely when it is the image under ρ of a special point of E. The inverse image of an alcove of E^K under ρ is said to be a *K-alcove*. This is the same as a connected component of $E - \bigcup_{H \in \mathfrak{F}^K} H$. The walls of a K-alcove are by definition the inverse images under ρ of the walls of the corresponding alcove in E^K. Thus the walls of a K-alcove are hyperplanes in $\mathfrak{F}^K$. Clearly, the natural action of Ω^K on the set of K-alcoves is simply transitive.

2.2. *In the remainder of this section we assume that a special point $\tilde{\epsilon}$ of E^K has been fixed.*

For each $i \in K$ there is a unique hyperplane $H_i \in \mathfrak{F}^i$ such that $\rho^{-1}(\tilde{\epsilon}) \subset H_i$. We have automatically $\rho^{-1}(\tilde{\epsilon}) = \bigcap_{i \in K} H_i$.

There is a unique K-alcove Ξ such that $\Xi \subset \bigcap_{i \in K} E^+_{H_i}$ and such that $\rho^{-1}(\tilde{\epsilon}) \subset cl(\Xi)$. Then $H_i (i \in K)$ are among the walls of Ξ.

Let X_Ξ be the set of alcoves in X that are contained in Ξ.

2.3. Let $\mathbf{S}$ be the set of all special points $\epsilon \in E$ such that $\epsilon \in cl(\Xi)$. Let $\mathbf{S}_{\tilde{\epsilon}}$ be the set of all special points $\epsilon \in E$ such that $\rho(\epsilon) = \tilde{\epsilon}$. This set is non-empty since $\tilde{\epsilon}$ is a special point of E^K. We have $\mathbf{S}_{\tilde{\epsilon}} \subset \mathbf{S}$.

Let $\epsilon \in \mathbf{S}_{\tilde{\epsilon}}$. We define an isometry $\kappa_\epsilon : E \to E$ by

$$\kappa_\epsilon(\epsilon + x) = (\epsilon - x)\omega_{\epsilon,K} = \epsilon - x w_0^K$$

for all $x \in T$. The computation $\kappa_\epsilon(\kappa_\epsilon(\epsilon + x)) = \kappa_\epsilon(\epsilon - x w_0^K) = \epsilon + x w_0^K w_0^K = \epsilon + x$ for $x \in T$ shows that κ_ϵ is an involution.

Here are some simple properties of κ_ϵ.

(a) κ_ϵ maps the set $\mathfrak{F}$ into itself; hence it maps any alcove onto an alcove.
(b) κ_ϵ maps the set $\mathfrak{F}^K$ into itself; hence it maps any K-alcove onto a K-alcove.
(c) $\kappa_\epsilon(\Xi) = \Xi$. Hence κ_ϵ maps any alcove contained in Ξ onto an alcove contained in Ξ.
(d) If $H \in \mathfrak{F} - \mathfrak{F}^K$ and $H' = \kappa_\epsilon(H)$, then $\kappa_\epsilon(E^-_H) = E^+_{H'}, \kappa_\epsilon(E^+_H) = E^-_{H'}$.

Since $\omega_{\epsilon,K}$ clearly maps $\mathfrak{F}$ into itself and $\mathfrak{F}^K$ into itself, to verify (a) and (b) it suffices to check that the involution of E given by $\epsilon + x \mapsto \epsilon - x$ (with $x \in T$) maps $\mathfrak{F}$ into itself and $\mathfrak{F}^K$ into itself. This follows from the fact that this involution maps $H \in \mathfrak{F}$ to $H\sigma_{H'}$ where H' is the unique hyperplane in $\mathfrak{F}$ that contains ϵ and is parallel to H.

We verify (c). From the definition we see that κ_ϵ permutes among themselves the open half spaces $E^+_{H_i}$ with $i \in K$ (notation of 2.2). Hence $\kappa_\epsilon(\bigcap_{i\in K} E^+_{H_i}) = \bigcap_{i\in K} E^+_{H_i}$. It is clear that κ_ϵ maps $\mathbf{S}_{\tilde{\epsilon}}$ into itself. From the definition, Ξ is the unique K-alcove contained in $\bigcap_{i\in K} E^+_{H_i}$ and whose closure contains $\mathbf{S}_{\tilde{\epsilon}}$. By the previous argument $\kappa_\epsilon(\Xi)$ is again a K-alcove contained in $\bigcap_{i\in K} E^+_{H_i}$ and whose closure contains $\mathbf{S}_{\tilde{\epsilon}}$. By uniqueness, we have $\Xi = \kappa_\epsilon(\Xi)$. Thus (c) is verified. Now (d) follows easily from the definitions.

More generally, we can define the map $\kappa_\epsilon : E \to E$ for any $\epsilon \in \mathbf{S}$ (not necessarily in $\mathbf{S}_{\tilde{\epsilon}}$). Let Ξ' the K-alcove which is the image of Ξ under the involution $\epsilon+x \mapsto \epsilon-x$ with $x \in T$ and let ω be the unique element of Ω^K such that $\Xi'\omega = \Xi$. We then set $\kappa_\epsilon(\epsilon + x) = (\epsilon - x)\omega = \epsilon - x\omega$. (We have necessarily $\epsilon\omega = \epsilon$.) Properties (a),(b),(c) above continue to hold in this more general case.

2.4. For any $\epsilon \in \mathbf{S}$ we set $D_\Xi(\epsilon) = \{A \in D(\epsilon) | A \subset \Xi\}$. For example, if $\epsilon \in \mathbf{S}_{\tilde{\epsilon}}$, then $A^+_\epsilon \in D_\Xi(\epsilon)$. Let $A^!_\epsilon = \kappa_\epsilon(A^+_\epsilon) = A^-_\epsilon \omega_{\epsilon,K}$. By the results in 2.3, $A^!_\epsilon$ is an alcove contained in Ξ. Since $\kappa_\epsilon(\epsilon) = \epsilon$, we see that $A^!_\epsilon \in D_\Xi(\epsilon)$.

Lemma 2.5. *Let $\epsilon \in \mathbf{S}_{\tilde{\epsilon}}$.*

(a) *Let A be an alcove contained in Ξ such that $A \subset \epsilon + \mathcal{C}^+$. Then there exists a sequence $A_0, A_1, \dots, A_p$ of alcoves contained in Ξ such that $A_0 = A^+_\epsilon, A_p = A$ and such that the following holds. For any $n \in [1,p]$, there exists $s_n \in S$ such that $s_n \in \mathcal{L}(A_n)$ and $A_{n-1} = s_n A_n$. In particular, $A^+_\epsilon \le A$.*

(b) *Let B be an alcove contained in Ξ such that $B \subset \epsilon + \mathcal{C}^- w_0^K$. Then there exists a sequence $B_0, B_1, \dots, B_p$ of alcoves contained in Ξ such that $B_0 = A^!_\epsilon, B_p = B$ and such that the following holds. For any $n \in [1,p]$, there exists $s'_n \in S$ such that $s'_n \notin \mathcal{L}(B_n)$ and $B_{n-1} = s'_n B_n$. In particular, $B \le A^!_\epsilon$.*

We prove (a). Since $A \subset \epsilon + \mathcal{C}^+$, we have

$$A^+_\epsilon \subset E^-_H, A \subset E^+_H \qquad \text{(c)}$$

for any hyperplane H that separates A^+_ϵ from A. It follows that $d(A^+_\epsilon, A)$ is equal to the number of hyperplanes in $\mathfrak{F}$ that separate A^+_ϵ from A. In particular, $d(A^+_\epsilon, A) \ge 0$ and we have $d(A^+_\epsilon, A) = 0$ if and only if $A^+_\epsilon = A$.

We prove (a) by induction on $d = d(A^+_\epsilon, A)$. The case where $d = 0$ is trivial since in that case, $A^+_\epsilon = A$. Hence we may assume that $d \ge 1$ and that the result is known for $d-1$ instead of d. Since $A^+_\epsilon \ne A$, there exists a wall H of A such that A, A^+_ϵ are on different sides of H. By (c), we have $A^+_\epsilon \subset E^-_H, A \subset E^+_H$. Let $A' = A\sigma_H$. We have $A' = sA$ for a well defined $s \in S$ and $A' \subset E^-_H$. In particular, $s \in \mathcal{L}(A)$ and $d(A', A) = 1$. Now, for any wall H' of Ξ, the alcoves A^+_ϵ, A are on the same side of H' (since they are both contained in Ξ, which is an intersection of open half spaces defined by the walls of Ξ). It follows that $H \ne H'$, so that $A\sigma_H$ is on the same side of H' as A. We thus see that $A' \subset \Xi$. Similarly, for any $i \in I$, the alcoves A^+_ϵ, A are on the same side of $\epsilon + i$ (since they are both contained in $\epsilon + \mathcal{C}^+$, which is an intersection of open half spaces defined by the hyperplanes $\epsilon + i$, with $i \in I$). It follows that $H \ne \epsilon + i$, so that $A\sigma_H$ is on the same side of $\epsilon + i$ as A. We thus see that $A' \subset \epsilon + \mathcal{C}^+$. It follows that $A' \subset \Xi \cap (\epsilon + \mathcal{C}^+)$. Since $d(A', A) = 1$, we have $d(A^+_\epsilon, A') = d - 1$. The induction hypothesis is applicable to A'. Hence we can find a sequence $A_0, A_1, \dots, A_{p'}$ of alcoves contained in Ξ such that $A_0 = A^+_\epsilon, A_{p'} = A'$ and such that for any $n \in [1, p']$, there exists $s_n \in S$ such that $s_n \in \mathcal{L}(A_n)$ and $A_{n-1} = s_n A_n$. Then the sequence $A_0, A_1, \dots, A_{p'}, A$ is as required for A. This proves (a).

Next, we prove (b). Let B be as in (b) and let $A = \kappa_\epsilon(B)$. Then A is an alcove contained in Ξ such that $A \subset \epsilon + \mathcal{C}^+$. Let $A_0, A_1, \dots, A_p$ a sequence of alcoves in Ξ attached to A as in (a). Let $B_u = \kappa_\epsilon(A_u)$ for $u = 0, 1, \dots, p$. Then $B_0, B_1, \dots, B_p$ are alcoves contained in Ξ and $B_0 = A^!_\epsilon, B_p = B$. Let $n \in [1, p]$. Then A_{n-1}, A_n have a common face and one is the mirror image of the other with respect to the hyperplane $H \in \mathfrak{F}$ containing that common face; moreover $A_{n-1} \subset E^-_H, A_n \subset E^+_H$. Since A_{n-1}, A_n are both contained in Ξ, we must necessarily have $H \in \mathfrak{F} - \mathfrak{F}^K$. Applying κ_ϵ, we see that B_{n-1}, B_n have a common face and one is the mirror image of the other with respect to the hyperplane $H' = \kappa_\epsilon(H) \in \mathfrak{F}$ containing that common face; moreover $B_{n-1} \subset \kappa_\epsilon(E^-_H), B_n \subset \kappa_\epsilon(E^+_H)$. Since $H \in \mathfrak{F} - \mathfrak{F}^K$, we have $\kappa_\epsilon(E^-_H) = E^+_{H'}, \kappa_\epsilon(E^+_H) = E^-_{H'}$. Thus, $B_{n-1} \subset E^+_{H'}, B_n \subset E^-_{H'}$. Denoting by s'_n the type of the common face of B_{n-1}, B_n, we see that $s'_n \notin \mathcal{L}(B_n)$ and $B_{n-1} = s'_n B_n$. This proves (b).

Lemma 2.6. (a) *Let A be an alcove contained in Ξ and let $H \in \mathfrak{F} - \mathfrak{F}^K$. Then there exists $\epsilon \in \mathbf{S}_{\tilde{\epsilon}}$ such that $A^+_\epsilon \subset E^-_H$ and $A \subset \epsilon + \mathcal{C}^+$.*

(b) *Let $A_1, A_2, \dots, A_p$ be a finite collection of alcoves contained in Ξ. Then there exists $\epsilon \in \mathbf{S}_{\tilde{\epsilon}}$ such that $A_u \subset \epsilon + \mathcal{C}^+$ for $u = 1, 2, \dots, p$.*

We prove (a). We set $\underline{\mathcal{T}} = \mathcal{T} \cap T_K$. This is a lattice in the $\mathbf{R}$-vector space T_K (since $\mathcal{T}$ is a lattice in T and T_K is the $\mathbf{R}$-subspace of E generated by a subset of $\mathcal{T}$).

Let ϵ' be a point in $\mathbf{S}_{\tilde{\epsilon}}$. For any $t \in \underline{\mathcal{T}}$, we have $\epsilon' + t \in \mathbf{S}_{\tilde{\epsilon}}$ and $A^+_{\epsilon'+t} = A^+_{\epsilon'} + t$. Hence it is enough to show that, for some $t \in \underline{\mathcal{T}}$, we have $A^+_{\epsilon'} + t \subset E^-_H$ and $A \subset \epsilon' + t + \mathcal{C}^+$.

For any $i \in I$, let $f_i : T \to \mathbf{R}$ be a linear form whose kernel is i and is such that $f_i(\mathcal{C}^+) \subset (0, \infty)$. Let $h = \mathrm{dir}(H)$ and let $g : T \to \mathbf{R}$ be a linear form such that $g^{-1}(0) = h, g(\mathcal{C}^+) \subset (0, \infty)$. Using the definitions we see that $g = \sum_{i \in I} c_i f_i$ where $c_i \in \mathbf{R}_{\geq 0}$ for all $i \in I$. Let $\tilde{f}_i, \tilde{g}$ be the restrictions of f_i, g to T_K. From the definitions we see that $\tilde{f}_i = 0$ for $i \in K$ and that $\tilde{f}_i$ (for $i \in I - K$) form an $\mathbf{R}$-basis of T_K. Hence $\tilde{g} = \sum_{i \in I-K} c_i \tilde{f}_i$. Since $H \in \mathfrak{F} - \mathfrak{F}^K$, we have $H + T_K = E$, so that $\tilde{g}$ is not identically zero; thus, we have $c_i > 0$ for some $i \in I - K$. Since $A^+_{\epsilon'}$ is a bounded subset of E and $H + T_K = E$, there exists $y_0 \in T_K$ such that $A^+_{\epsilon'} + y_0 \subset E^-_H$. Clearly, if $x \in T$ satisfies $g(x) < 0$, then $E^-_H + x \subset E^-_H$, hence $A^+_{\epsilon'} + y_0 + x \subset E^-_H$. We deduce that for $z \in T_K$ such that $\tilde{g}(z) < \tilde{g}(y_0)$ we have $A^+_{\epsilon'} + z \subset E^-_H$. Similarly, since A is a bounded subset of E and $i + T_K = T$ (for $i \in I - K$), there exists $x_i \in T_K$ such that $A - x_i \subset \epsilon' + f_i^{-1}(0, \infty)$. Clearly, if $x \in T$ satisfies $f_i(x) < 0$, then $f_i^{-1}(0, \infty) - x \subset f_i^{-1}(0, \infty)$, hence $A - x_i - x \subset \epsilon' + f_i^{-1}(0, \infty)$. We deduce that for $z \in T_K$ such that $\tilde{f}_i(z) < \tilde{f}_i(x_i)$ we have $A \subset \epsilon' + z + f_i^{-1}(0, \infty)$. The last inclusion holds also for $i \in K$ (for arbitrary $z \in T_K$); indeed, in this case, since $A \subset \Xi$ we have $A \subset \epsilon' + f_i^{-1}(0, \infty)$. On the other hand, we have $T_K + f_i^{-1}(0, \infty) \subset f_i^{-1}(0, \infty)$ since f_i is zero on T_K.

Since $\bigcap_{i \in I}(\epsilon' + z + f_i^{-1}(0, \infty)) = \epsilon' + z + \mathcal{C}^+$, we see that, if $z \in T_K$ satisfies $\tilde{g}(z) < \tilde{g}(y_0)$ and $\tilde{f}_i(z) < \tilde{f}_i(x_i)$ for all $i \in I - K$, then $A^+_{\epsilon'} + z \subset E^-_H$ and $A \subset \epsilon' + z + \mathcal{C}^+$. It is therefore enough to prove that the set

$$\{z \in T_K | \tilde{g}(z) < \tilde{g}(y_0), \quad \tilde{f}_i(z) < \tilde{f}_i(x_i) \text{ for all } i \in I - K\}$$

has a non-empty intersection with $\underline{\mathcal{T}}$. Hence, it is enough to show that, given real numbers $d_i (i \in I - K)$ and d', the set

$$P(d'; d_i) = \{z \in T_K | \tilde{g}(z) < d', \quad \tilde{f}_i(z) < d_i \text{ for all } i \in I - K\}$$

has a non-empty intersection with $\underline{\mathcal{T}}$. Since $P(d'; d_i) \subset P(\tilde{d}'; \tilde{d}_i)$ if $d' \leq \tilde{d}'$ and $d_i \leq \tilde{d}_i$ for $i \in I - K$, we see that it is enough to prove the assertion in the previous sentence assuming that $d' \leq 0$ and $d_i \leq 0$ for $i \in I - K$. We first show that $P(d'; d_i)$ is non-empty. In terms of the coordinates $z_i \in \mathbf{R}$ (with $i \in I - K$) given by $z_i = f_i(z)$, we can identify

$$P(d'; d_i) = \{(z_i) \in \mathbf{R}^{I-K} | \sum_{i \in I-K} c_i z_i < d', \quad z_i < d_i \text{ for all } i \in I - K\}.$$

This is clearly non-empty, since $c_i \geq 0$ for all $i \in I - K$ and $c_i > 0$ for some $i \in I - K$. Since the set $P(d'; d_i)$ is non-empty and obviously open in T_K, it must have non-empty intersection with the set $\underline{\mathcal{T}} \cup (\frac{1}{2}\underline{\mathcal{T}}) \cup (\frac{1}{3}\underline{\mathcal{T}}) \cup \dots$, which is dense in T_K since $\underline{\mathcal{T}}$ is a lattice in T_K. Thus, there exists $z \in P(d'; d_i)$ and $q \in \{1, 2, 3, \dots\}$ such that $qz \in \mathcal{T}$. Since $d' \leq 0$ and $d_i \leq 0$, we have $qd' \leq d'; qd_i \leq d_i$ hence $qP(d'; d_i) \subset P(d'; d_i)$. It follows that $qz \in P(d'; d_i) \cap \underline{\mathcal{T}}$. This proves (a).

The proof of (b) is entirely similar.

2.7. Recall that X_Ξ is a set of representatives for the Ω^K-orbits on X: for $A \in X$, we can find a unique $\omega \in \Omega^K$ such that $A\omega \in X_\Xi$. Setting $\pi(A) = A\omega$ we get a map $\pi : X \to X_\Xi$ whose restriction to X_Ξ is the identity. We have $\pi(A + \alpha_i) = \pi(A)$ for all $A \in X, i \in K$, since $A + \alpha_i$ is in the Ω_K-orbit of A.

Lemma 2.8. *Let $A \in X$ and $s \in S$. Let $H \in \mathfrak{F}$ be the hyperplane separating A from sA.*

(a) *If $H \in \mathfrak{F}^K$, then $\pi(sA) = \pi(A)$.*

(b) *If $H \notin \mathfrak{F}^K$, then $\pi(sA) = s\pi(A)$.*

Let $\omega \in \Omega^K$ be such that $A\omega \in X_\Xi$. Assume first that $H \in \mathfrak{F}^K$. Then $\sigma_H \in \Omega^K$ and $sA(\sigma_H\omega) = A\omega$ with $\sigma_H\omega \in \Omega^K$ shows that $\pi(sA) = \pi(A)$. Assume next that $H \notin \mathfrak{F}^K$. It is then enough to show that, if $B = A\omega$, then $sB \in X_\Xi$. Note that B, sB are separated by the hyperplane $H' = H\omega$. Since $\omega \in \Omega^K$, we have $H' \notin \mathfrak{F}^K$. Assume that $sB \notin X_\Xi$. Then H' must be a wall of Ξ. But any wall of Ξ is in $\mathfrak{F}^K$. This contradiction proves the lemma.

The following lemma will not be used. We include it since its proof serves as a model for the proof of 2.10.

Lemma 2.9. *Let A be an alcove contained in Ξ and let $H \in \mathfrak{F} - \mathfrak{F}^K$ be a hyperplane such that $A \subset E_H^+$. Let $A' = A\sigma_H$. Then $\pi(A') \leq A$.*

By 2.6, we can find $\epsilon \in \mathbf{S}_{\bar{\epsilon}}$ such that $A_\epsilon^+ \subset E_H^-$ and $A \subset \epsilon + \mathcal{C}^+$. Using 2.5, we can find a sequence $A_0, A_1, \dots, A_p$ of alcoves contained in Ξ such that

(a) $A_0 = A_\epsilon^+, A_p = A$;

(b) for any $n \in [1, p]$, there exists $s_n \in S$ such that $s_n \in \mathcal{L}(A_n)$ and $A_{n-1} = s_n A_n$.

Since $A_0 \subset E_H^-, A_p \subset E_H^+$, it follows that there exists $n \in [1, p]$ such that $A_{n-1} \subset E_H^-, A_n \subset E_H^+$. Since $A_{n-1} = s_n A_n$, it follows that $A_{n-1} = A_n \sigma_H$. From (a),(b) we have $A = s_p s_{p-1} \dots s_1 A_\epsilon^+$. We deduce that

$$\begin{aligned} A' &= s_p s_{p-1} \dots s_1 A_\epsilon^+ \sigma_H = s_p s_{p-1} \dots s_n A_{n-1} \sigma_H = s_p s_{p-1} \dots s_n A_n \\ &= s_p s_{p-1} \dots s_{n+1} A_{n-1} = s_p s_{p-1} \dots s_{n+1} s_{n-1} \dots s_1 A_\epsilon^+. \end{aligned}$$

Applying π and using repeatedly 2.8, we see that $\pi(A') = s_{k_q} s_{k_{q-1}} \dots s_{k_1} A_\epsilon^+$ where $\{k_1 < k_2 < \dots k_q\}$ is a subset of $\{1, 2, \dots, n-1\} \cup \{n+1, n+2, \dots, p\}$. From (b) we see that $d(A_{n-1}, A_n) = 1$ for all $n \in [0, p]$. By the additivity property of $d(,)$, we have $d(A_0, A_p) = \sum_{n=1}^p d(A_{n-1}, A_n) = p$ so that $d(A_\epsilon^+, s_p s_{p-1} \dots s_1 A_\epsilon^+) = p$. From this we deduce, using [L1, 3.4] that $s_{k_q} s_{k_{q-1}} \dots s_{k_1} A_\epsilon^+ \le s_p s_{p-1} \dots s_1 A_\epsilon^+ = A$ or equivalently, $\pi(A') \le A$. The lemma is proved.

Lemma 2.10. *Let A be an alcove contained in Ξ and let $i \in I$. We have $\pi(A - \alpha_i) \le A$.*

If $i \in K$, then $\pi(A - \alpha_i) = \pi(A) = A$ (see 2.7) so the result is trivial in this case. In the remainder of the proof we assume that $i \in I - K$. In this case, the proof will be similar to that of 2.9.

We can find (uniquely) $H', H'' \in \mathfrak{F}^i$ such that $E_{H'}^+ \cap E_{H''}^-$ contains A but contains no hyperplane in $\mathfrak{F}^i$. Let $H = H''\sigma_{H'} \in \mathfrak{F}^i$. Then $A - \alpha_i = A\sigma_{H'}\sigma_H$ and $A \subset E_{H'}^+ \subset E_H^+, A\sigma_{H'} \subset E_H^+ \cap E_{H'}^-$. By 2.6, we can find $\epsilon \in \mathbf{S}_{\tilde{\epsilon}}$ such that $A_\epsilon^+ \subset E_H^-$ and $A \subset \epsilon + \mathcal{C}^+$.

Using 2.5, we can find a sequence $A_0, A_1, \dots, A_p$ of alcoves contained in Ξ such that

(a) $A_0 = A_\epsilon^+, A_p = A$;
(b) for any $n \in [1, p]$, there exists $s_n \in S$ such that $s_n \in \mathcal{L}(A_n)$ and $A_{n-1} = s_n A_n$.

Since $A_0 \subset E_H^-, A_p \subset E_H^+$, it follows that there exists $n \in [1, p]$ such that $A_{n-1} \subset E_H^-, A_n \subset E_H^+$. Since $A_{n-1} = s_n A_n$, it follows that $A_{n-1} = A_n \sigma_H$. We have automatically $A_n \subset E_{H'}^-$. Since $A_n \subset E_{H'}^-, A_p \subset E_{H'}^+$, it follows that $n < p$ and there exists $m \in [n+1, p]$ such that $A_{m-1} \subset E_{H'}^-, A_m \subset E_{H'}^+$. Since $A_{m-1} = s_m A_m$, it follows that $A_{m-1} = A_m \sigma_{H'}$.

From (a),(b) we have $A = s_p s_{p-1} \dots s_1 A_\epsilon^+$. We deduce that

$$\begin{aligned}
A\sigma_{H'}\sigma_H &= s_p s_{p-1} \dots s_1 A_\epsilon^+ \sigma_{H'} \sigma_H \\
&= s_p s_{p-1} \dots s_m A_{m-1} \sigma_{H'} \sigma_H = s_p s_{p-1} \dots s_m A_m \sigma_H \\
&= s_p s_{p-1} \dots s_{m+1} A_{m-1} \sigma_H = s_p s_{p-1} \dots s_{m+1} s_{m-1} s_{m-2} \dots s_n A_{n-1} \sigma_H \\
&= s_p s_{p-1} \dots s_{m+1} s_{m-1} s_{m-2} \dots s_n A_n = s_p s_{p-1} \dots s_{m+1} s_{m-1} s_{m-2} \dots s_{n+1} A_{n-1} \\
&= s_p s_{p-1} \dots s_{m+1} s_{m-1} s_{m-2} \dots s_{n+1} s_{n-1} s_{n-2} \dots s_1 A_\epsilon^+.
\end{aligned}$$

Applying π and using repeatedly 2.8 we see that $\pi(A\sigma_{H'}\sigma_H) = s_{k_q} s_{k_{q-1}} \dots s_{k_1} A_\epsilon^+$ where $\{k_1 < k_2 < \dots k_q\}$ is a subset of

$$\{1, 2, \dots, n-1\} \cup \{n+1, n+2, \dots, m-1\} \cup \{m+1, m+2, \dots, p\}.$$

As in the proof of 2.9, we have $d(A_\epsilon^+, s_p s_{p-1} \dots s_1 A_\epsilon^+) = p$. From this we deduce, using [L1, 3.4], that $s_{k_q} s_{k_{q-1}} \dots s_{k_1} A_\epsilon^+ \le s_p s_{p-1} \dots s_1 A_\epsilon^+ = A$ or equivalently $\pi(A\sigma_{H'}\sigma_H) \le A$. The lemma is proved.

Lemma 2.11. *Let $A \in X$.*

(a) *There exist $A_1, A_2, \dots, A_p$ in X such that the following holds. If $B \in X$ satisfies $B \le A$, then $B = A_u - t$ for some $u \in [1, p]$ and some $t \in \mathcal{T}^+$. (See 1.3.)*
(b) *There exist $A'_1, A'_2, \dots, A'_p$ in X such that the following holds. If $B \in X$ satisfies $A \le B$, then $B = A'_u + t'$ for some $u \in [1, p]$ and some $t' \in \mathcal{T}^+$.*

Let Z be a Ω-orbit on the set of special points of E. For any $C \in X$ we can find a unique $\epsilon \in Z$ such that $B \in D(\epsilon)$ (see 1.1); we set $\tilde{C} = A_\epsilon^+$. We first verify the following statement.

(c) If $C, C' \in X$ satisfy $C \leq C'$, then $\tilde{C} = \tilde{C}' - t$ for some $t \in \mathcal{T}^+$.

By the definition of $\leq$, we may assume that there exists $H \in \mathfrak{F}$ such that $C = C'\sigma_H$, $d(C, C') = 1$. Let $\epsilon, \epsilon' \in Z$ be such that $C \in D(\epsilon), C' \in D(\epsilon')$. Then $\epsilon = \epsilon'\sigma_H$. Hence $\epsilon = \epsilon' - t$ where $t \in \mathcal{T}$ is an integer multiple of the root α_h where $h = \mathrm{dir}(H)$ (see 1.3). We claim that this is in fact a multiple by an integer ≥ 0. Indeed, we have $C' \subset E_H^+, C \subset E_H^-$ (otherwise, using [L1, 3.1] we would deduce that $d(C, C') < 0$); hence ϵ' is in the closure of E_H^+ and ϵ is in the closure of E_H^- and our claim follows. In particular, we have $t \in \mathcal{T}^+$. We have $A_\epsilon^+ = A_{\epsilon'-t}^+ = A_{\epsilon'} - t$ and (c) is proved.

Let ϵ' be a special point in Z such that $A \in D(\epsilon')$. We enumerate the elements of the finite set $D(\epsilon')$ as $A_1, A_2, \ldots, A_p$. Now let $B \in X$ be such that $B \leq A$. Let ϵ be a special point in Z such that $B \in D(\epsilon)$. By (c) we can find $t \in \mathcal{T}^+$ such that $\tilde{B} = \tilde{A} - t$. Then we also have $\epsilon = \epsilon' - t$. We can find a sequence $s_1, s_2, \ldots, s_q$ of elements of S such that $\tilde{B}, s_1\tilde{B}, s_2s_1\tilde{B}, \ldots, s_qs_{q-1}\ldots s_1\tilde{B} = B$ are all contained in $D(\epsilon)$. Applying translation by t we deduce that

$$\tilde{B} + t = \tilde{A}, s_1\tilde{B} + t = s_1\tilde{A}, s_2s_1\tilde{B} + t = s_2s_1\tilde{A}, \ldots,$$
$$s_qs_{q-1}\ldots s_1\tilde{B} + t = B + t = s_qs_{q-1}\ldots s_1\tilde{A}$$

are all contained in $D(\epsilon')$. Thus we have $B + t \in D(\epsilon')$. Hence $B = A_u - t$ for some $u \in [1, p]$. This proves (a).

We prove (b). We choose a special point ϵ in E. Applying (a) to $A\omega_\epsilon$, we see that there exist $A_1, A_2, \ldots, A_p \in X$ such that the following holds: if $B \in X$ satisfies $B\omega_\epsilon \leq A\omega\epsilon$ (or equivalently, $A \leq B$, see [L1, (1.5.1)]), then $B\omega_\epsilon = A_u - t$ for some $u \in [1, p]$ and some $t \in \mathcal{T}^+$. We set $A'_s = A_s\omega_\epsilon$ for $s \in [1, p]$ and $t' = -tw_0^I \in \mathcal{T}^+$. Then $B = A'_u + t'$. This proves (b). The lemma is proved.

2.12. Let $t \in \mathcal{T}'$. Since $\Xi + t$ is a K-alcove, there is a unique $\omega \in \Omega^K$ such that $\Xi + t = \Xi\omega^{-1}$ hence $cl(\Xi) + t = cl(\Xi)\omega^{-1}$. Let $\gamma_t : E \to E$ be defined by $e \mapsto (e+t)\omega$. By restriction to $cl(\Xi)$ we obtain a homeomorphism $cl(\Xi) \to cl(\Xi)$ denoted again by γ_t. (This is the composition of the homeomorphism $x \to x + t$ of $cl(\Xi)$ onto $cl(\Xi) + t$ with the homeomorphism $x \mapsto x\omega$ of $cl(\Xi) + t$ onto $cl(\Xi)$.) Since $x \mapsto x + t$ and $x \mapsto x\omega$ map $\mathfrak{F}$ into itself it follows that γ_t maps any alcove contained in Ξ onto an alcove contained in Ξ. Hence we have an induced permutation $\gamma_t : X_\Xi \to X_\Xi$.

We show that $t \mapsto \gamma_t$ is an action of the group $\mathcal{T}'$ on $cl(\Xi)$. Let $x \in cl(\Xi)$ and let $\omega, \omega' \in \Omega^K$ be such that $\gamma_t(x) = (x + t)\omega, \gamma_{t'}\gamma_t(x) = (\gamma_t(x) + t')\omega'$. Let $w \in W^K$ be the image of ω under $\Omega \to W^I$. We have $t'w^{-1} = t' + p$ where $p \in \mathcal{T}^K$ (it is enough to check this in the case where w is a generator $r_i, i \in K$ of W^I). Hence we have

$$\begin{aligned}\gamma_{t'}\gamma_t(x) &= \gamma_{t'}((x + t)\omega) = ((x + t)\omega + t')\omega' = (x + t + t'\omega^{-1})\omega\omega' \\ &= (x + t + t' + p)\omega\omega' = \gamma_{t+t'}(x).\end{aligned}$$

Note also that $\gamma_0(x) = x$. Our assertion follows.

From the definitions we see that, if $t \in \mathcal{T}^K$, then $\gamma_t : cl(\Xi) \to cl(\Xi)$ is the identity map. Hence $t \mapsto \gamma_t$ defines an action of $\mathcal{T}'/\mathcal{T}^K$ on $cl(\Xi)$ and on X_Ξ.

We have the following result.

(a) *Let A be an alcove contained in Ξ and let $t \in \mathcal{T}^+$. We have $\gamma_{-t}(A) \le A$.*
We argue by induction on $n = \sum_{i \in I-K} n_i$ where $t = \sum_{i \in I-K} n_i \alpha_i \mod \mathcal{T}^K$. If $n = 0$ then $t \in \mathcal{T}^K$ and $\gamma_{-t}A = A$ so that (a) holds. We may assume that $n \ge 1$. Then there exists $i \in I - K$ so that $n_i \ge 1$. Then $t' = t - \alpha_i \in \mathcal{T}^+$. By the induction hypothesis we have $\gamma_{-t'}(A) \le A$. We have

$$\gamma_{-t}(A) = \gamma_{-\alpha_i - t'}(A) = \gamma_{-\alpha_i}\gamma_{-t''}(A) \le \gamma_{-t''}(A)$$

where the last inequality follows from 2.10 applied to $\gamma_{-t''}(A)$ instead of A. Using the transitivity of $\le$ it follows that $\gamma_{-t}(A) \le A$ and (a) is proved.

We now verify the following statement.

(b) *If $x \in \mathbf{S}$ (see 2.3) and $t \in \mathcal{T}'$, then $\gamma_t(x) \in \mathbf{S}$. Moreover, $x \to \gamma_t(x)$ is a simply transitive action of $\mathcal{T}'/\mathcal{T}^K$ on $\mathbf{S}$.*

The first assertion and the transitivity in the second assertion follow from the fact that $\mathcal{T}'$ acts transitively on the set of special points of E. Now let $x \in \mathbf{S}, t \in \mathcal{T}'$ be such that $\gamma_t(x) = x$. It remains to prove that $t \in \mathcal{T}^K$. We have $x + t = x\omega$ for some $\omega \in \Omega^K$. We can find $t' \in \mathcal{T}^K$ such that $x\omega = x + t'$. Then from $x + t = x + t'$ we deduce $t = t'$ so that $t \in \mathcal{T}^K$.

The following result relates γ_t with the function $d : X \times X \to \mathbf{Z}$.

(c) *If A, B are contained in Ξ and $t \in \mathcal{T}'$, then $d(\gamma_t(A), \gamma_t(B)) = d(A, B)$.*
Clearly $d(A + t, B + t) = d(A, B)$ and $A + t, B + t$ are contained in the same K-alcove. Hence it suffices to verify the following statement.

If A, B are contained in the same K-alcove and $\omega \in \Omega^K$, then $d(A\omega, B\omega) = d(A, B)$.

We may assume that ω is one of the generators $\sigma_H (H \in \mathfrak{F}^i, i \in K)$ of Ω^K. The map $H' \to H'\sigma_H$ is a bijection between the set of hyperplanes in $\mathcal{H}$ that separate A from B and the set of hyperplanes in $\mathcal{H}$ that separate $A\sigma_H$ from $B\sigma_H$. It is then enough to show that corresponding hyperplanes have the same attached sign (used to define $d(A, B)$ or $d(A\sigma_H, B\sigma_H)$). If $H' \in \mathfrak{F}$ separates A from B, then $H' \notin \mathfrak{F}^K$ (since A, B are contained in the same K-alcove). In particular, $H' \notin \mathfrak{F}^i$. Hence from [L1, 1.2] it follows that $E^+_{H'}\sigma_H = E^+_{H'\sigma_H}, E^-_{H'}\sigma_H = E^-_{H'\sigma_H}$. Our assertion about signs follows and (c) is proved.

The following result relates γ_t with the maps κ_ϵ in 2.3.

(d) *Let $t \in \mathcal{T}'$ and let $\epsilon \in \mathbf{S}$. Let $\epsilon' = \gamma_t(\epsilon) \in \mathbf{S}$. We have $\gamma_t\kappa_\epsilon = \kappa_{\epsilon'}\gamma_t = \kappa_\epsilon\gamma_{-t} : E \to E$. Moreover, κ_ϵ is an involution of E.*

For $e, e' \in E$ we write $e \sim e'$ instead of "e, e' are in the same Ω^K-orbit".

Let $\omega, \omega' \in \Omega^K$ be such that $\gamma_t(e) = (e + t)\omega, \kappa_\epsilon(\epsilon + x) = (\epsilon - x)\omega'$ for all $e \in E, x \in T$. Let $w, w' \in W^I$ be the images of ω, ω' in W^I. Then $\kappa_\epsilon(\epsilon + x) = \epsilon - xw', \gamma_t(\epsilon + x) = \epsilon' + xw$ for $x \in T$.

Let $\omega_1 \in \Omega^K$ be such that $\gamma_{-t}(e) = (e - t)\omega_1$ for all $e \in E$; let $t_1 \in T$ be defined by $\epsilon\omega_1 = \epsilon + t_1$. Let w_1 be the image of ω_1 in W^I. For any $x \in T$, we have

$$\gamma_t\kappa_\epsilon(\epsilon + x) \sim \epsilon - xw' + t \sim \epsilon - x + t,$$

$$\kappa_{\epsilon'}\gamma_t(\epsilon + x) = \kappa_{\epsilon'}(\epsilon' + xw) = \epsilon' - xw = (\epsilon + t)\omega - xw \sim \epsilon + t - x,$$

$$\kappa_\epsilon\gamma_{-t}(\epsilon + x) = \kappa_\epsilon((\epsilon + x - t)\omega_1) = \kappa_\epsilon(\epsilon + t_1 + (x - t)w_1) \sim \epsilon - t_1 - (x - t)w_1$$
$$\sim \epsilon\omega_1 - t_1w_1 - (x - t)w_1^2 = \epsilon + t_1 - t_1w_1 - (x - t)w_1^2 \sim \epsilon - (x - t),$$

hence $\gamma_t\kappa_\epsilon(\epsilon + x) \sim \kappa_{\epsilon'}\gamma_t(\epsilon + x) \sim \kappa_\epsilon\gamma_{-t}(\epsilon + x)$.

If we choose $x \in T$ so that $\epsilon + x \in \Xi$ then, since $\gamma_t, \kappa_\epsilon, \kappa_{\epsilon'}$ map Ξ into itself, we have $\gamma_t\kappa_\epsilon(\epsilon+x) \in \Xi, \kappa_{\epsilon'}\gamma_t(\epsilon+x) \in \Xi, \kappa_\epsilon\gamma_{-t}(\epsilon+x) \in \Xi$. But two points of Ξ that are in the same Ω^K-orbit must be equal; hence $\gamma_t\kappa_\epsilon(\epsilon+x) = \kappa_{\epsilon'}\gamma_t(\epsilon+x) = \kappa_\epsilon\gamma_{-t}(\epsilon+x)$. Now $e \mapsto \gamma_t\kappa_\epsilon(e), e \mapsto \kappa_{\epsilon'}\gamma_t(e), e \mapsto \kappa_\epsilon\gamma_{-t}(e)$ are analytic on E and coincide on the open set Ξ; hence they coincide on E. This proves the first assertion of (d). To prove the second assertion of (d), we choose $t \in \mathcal{T}'$ so that $\gamma_t(\epsilon) = \epsilon' \in \mathbf{S}_{\tilde{\epsilon}}$ (see (b)). Then $\kappa_{\epsilon'}$ is an involution (see 2.3) hence $\kappa_\epsilon = \gamma_t^{-1}\kappa_{\epsilon'}\gamma_t$ is also an involution.

The following result is a variant of 2.6(a).

(e) *Let $A \in X_\Xi$ and let $\epsilon' \in \mathbf{S}_{\tilde{\epsilon}}$. There exists $t' \in \mathcal{T}^+$ such that $A^+_{\epsilon'} \le \gamma_{t'}A$.* By 2.6(a) and its proof we can find $\epsilon \in \mathbf{S}_{\tilde{\epsilon}}$ such that $A \subset \epsilon + \mathcal{C}^+$ and such that $\epsilon' = \epsilon + t$ where $t \in \mathcal{T} \cap T_K$. Let $A_0, A_1, \dots, A_p$ be a sequence as in 2.5(a). We set $A'_s = \gamma_t(A_s)$ for $s \in [0,p]$. This is a sequence of alcoves in X_Ξ. Since γ_t is an isometry of E preserving $\mathcal{H}$, for any $s \in [1,p]$, A'_{s-1}, A'_s are symmetric with respect to some hyperplane in $\mathcal{H}$; moreover, by (c), we have $d(A'_{s-1}, A'_s) = d(A_{s-1}, A_s) = 1$. It follows that $A'_0 \le A'_p$ or equivalently $\gamma_t(A^+_\epsilon) \le \gamma_t A$. Since $t \in T_K$, the translation by t maps Ξ into itself; hence it coincides with γ_t; in particular, $\gamma_t(A^+_\epsilon) = A^+_{\epsilon+t} = A^+_{\epsilon'}$. Next we can find $t_1 \in \mathcal{T}^+$ so that $t + t_1 \in \mathcal{T}^+$. By 2.12(a) we have $\gamma_t A \le \gamma_{t_1}\gamma_t(A) = \gamma_{t+t_1}(A)$. Thus, $A^+_{\epsilon'} \le \gamma_{t+t_1}(A)$. This proves (e).

Next we prove the following result.

(f) *The action of $\mathcal{T}/\mathcal{T}^K$ on X_Ξ (restriction of the action of $\mathcal{T}'/\mathcal{T}^K$) is free and, for any $\epsilon \in \mathbf{S}$, the set $D_\Xi(\epsilon)$ is a set of representatives for the orbits of $\mathcal{T}/\mathcal{T}^K$ on X_Ξ.*

Assume that $t \in \mathcal{T}, A \in X_\Xi$ satisfy $\gamma_t(A) = A$. Then $A + t = A\omega$ for some $\omega \in \Omega^K$. Since the action of Ω on X is free, it follows that $t \in \mathcal{T} \cap \Omega^K = \mathcal{T}^K$ and our first assertion is proved.

Now let us fix $\epsilon \in \mathbf{S}$. If $B \in X_\Xi$, then there exists a special point $\epsilon' \in E$ such that $\epsilon' = \epsilon + t$ for some $t \in \mathcal{T}$ and $\epsilon' \in cl(B)$. Then $\epsilon' = \gamma_t(\epsilon)$. (Indeed, $\gamma_t(\epsilon)$ and $\epsilon + t$ are in the same Ω^K-orbit and are both in $cl(\Xi)$ hence they coincide.) From $\epsilon' \in cl(B)$ we deduce that $\epsilon = \gamma_{-t}\epsilon' \in cl(\gamma_{-t}B)$ hence $\gamma_{-t}B \in D_\Xi(\epsilon)$. Thus any $\mathcal{T}/\mathcal{T}^K$ orbit on X_Ξ meets $D_\Xi(\epsilon)$.

Finally, assume that $A, A' \in D_\Xi(\epsilon)$ are in the same $\mathcal{T}/\mathcal{T}^K$ orbit. Then $A' = \gamma_t A$ for some $t \in \mathcal{T}$. Since $\epsilon \in cl(A)$, we have $\gamma_t(\epsilon) \in cl(\gamma_t A) = cl(A')$. We have also $\epsilon \in cl(A')$. Thus, both $\gamma_t(\epsilon), \epsilon$ (which are special points of E in the same Ω-orbit) belong to $cl(A')$. But this implies $\gamma_t(\epsilon) = \epsilon$. Now using (b) we deduce that $t \in \mathcal{T}^K$. This completes the proof of (f).

Lemma 2.13. (a) *Let $A \in X_\Xi$. There exist $A_1, A_2, \dots, A_p$ in X_Ξ such that the following holds. If $B \in X_\Xi$ satisfies $B \le A$, then $B = \gamma_{-t}A_u$ for some $u \in [1,p]$ and some $t \in \mathcal{T}^+$.*

(b) *Let $A_1, A_2, \dots, A_p$ in X_Ξ. There exists $A \in X_\Xi$ such that the following holds. If $B \in X_\Xi$ satisfies $B = \gamma_{-t}A_u$ for some $u \in [1,p]$ and some $t \in \mathcal{T}^+$, then $B \le A$.*

(c) *Let $A \in X_\Xi$. There exist $A_1, A_2, \dots, A_p$ in X_Ξ such that the following holds. If $B \in X_\Xi$ satisfies $A \le B$, then $B = \gamma_t A_u$ for some $u \in [1,p]$ and some $t \in \mathcal{T}^+$.*

(d) *Let $A_1, A_2, \dots, A_p$ in X_Ξ. There exists $A \in X_\Xi$ such that the following holds. If $B \in X_\Xi$ satisfies $B = \gamma_t A_u$ for some $u \in [1,p]$ and some $t \in \mathcal{T}^+$, then $A \le B$.*

We prove (a). By 2.11(a) we can find $A'_1, A'_2, \dots A'_p$ in X such that the following holds. If $B \in X$ satisfies $B \le A$, then $B = A'_u - t$ for some $u \in [1,p]$ and some $t \in \mathcal{T}^+$. In particular this holds if $B \in X_\Xi$ satisfies $B \le A$. Let $A_1 = \pi(A'_1), A_2 =$

$\pi(A'_2), \dots, A_p = \pi(A'_p)$. We can write $A'_u = A_u\omega^{-1}$ where $\omega \in \Omega^K$. Let $w \in W^K$ be the image of ω under the canonical map $\Omega \to W^I$. We then have

$$\begin{aligned} B &= \pi(B) = \pi(A'_u - t) = \pi(A_u\omega^{-1} - t) = \pi(A_u - tw)\omega^{-1}) = \pi(A_u - tw) \\ &= \pi(A_u - t - t_1) = \pi(A_u - t) = \gamma_{-t}(A_u). \end{aligned}$$

Here we have used $tw = t + t_1$ for some $t_1 \in \mathcal{T}^K$. This proves (a).

We prove (b). In the setup of (b) let $B \in X_\Xi$ be such that $B = \gamma_{-t}A_u$ for some $u \in [1, p]$ and some $t \in \mathcal{T}^+$. By 2.12(a) we have $B \le A_u$. It remains to show that there exists $A \in X_\Xi$ such that $A_u \le A$ for $u = 1, 2, \dots, p$. Let $\epsilon \in \mathbf{S}_{\tilde{\epsilon}}$ and let $B_u = \kappa_\epsilon(A_u)$ for $u = 1, 2, \dots, p$. By 2.6(b) we can find $\epsilon' \in \mathbf{S}_{\tilde{\epsilon}}$ such that $B_u \subset \epsilon' + \mathcal{C}^+$ for $u = 1, 2, \dots, p$. Applying κ_ϵ we deduce that $A_u \subset \epsilon'' + \mathcal{C}^- w_0^K$ for $u = 1, 2, \dots, p$, where $\epsilon'' = \kappa_\epsilon(\epsilon') \in \mathbf{S}_{\tilde{\epsilon}}$. By 2.5(b) we have $A_u \le A$ where $A = A^!_{\epsilon''}$. This proves (b).

Now (c) is deduced from 2.11(b) in the same way as (a) was deduced from 2.11(a).

We prove (d). In the setup of (d) let $B \in X_\Xi$ be such that $B = \gamma_t A_u$ for some $u \in [1, p]$ and some $t \in \mathcal{T}^+$. By 2.12(a) we have $A_u \le B$. It remains to show that there exists $A \in X_\Xi$ such that $A \le A_u$ for $u = 1, 2, \dots, p$. By 2.6(b) we can find $\epsilon \in \mathbf{S}_{\tilde{\epsilon}}$ such that $A_u \subset \epsilon + \mathcal{C}^+$ for $u = 1, 2, \dots, p$. By 2.5(a) we have $A^+_\epsilon \le A_u$ for $u = 1, 2, \dots, p$. Thus we can take $A = A^+_\epsilon$. This proves (d).

The lemma is proved.

2.14. We define $d_K : X \times X \to \mathbf{Z}$ by $d_K(A, B) = \sum_{h \in \bar{\mathcal{F}}^K} d_h(A, B)$. We have $d_K(A, A) = 0$; moreover, for $A, B, C \in X$, we have $d_K(A, B) + d_K(B, C) + d_K(C, A) = 0$.

3. The module M and intertwining operators

3.1. Let $\mathcal{A} = \mathbf{Z}[v, v^{-1}]$ where v is an indeterminate. The Hecke algebra $\mathcal{H}$ associated to the affine Weyl group W is the associative $\mathcal{A}$-algebra which, as an $\mathcal{A}$-module, has basis elements $\tilde{T}_w$ (one for each $w \in W$) and multiplication defined by the rules

$$(\tilde{T}_s + v^{-1})(\tilde{T}_s - v) = 0, \quad (s \in S),$$
$$\tilde{T}_w \tilde{T}_{w'} = \tilde{T}_{ww'} \text{ if } l(w) + l(w') = l(ww');$$

here, l is the standard length function on W.

3.2. Let M be the set of all functions $X \to \mathcal{A}$. If $m \in M$ we shall generally denote by $m_A \in \mathcal{A}$ the value of m at $A \in X$ and we write $m = \sum_{A \in X} m_A A$. We regard M as an $\mathcal{A}$-module in the obvious way. For $m \in M$, we set $\operatorname{supp}(m) = \{A \in X | m_A \ne 0\}$ (the *support* of m).

A family $(m^\lambda)_{\lambda \in \Lambda}$ of elements of M is said to be *locally finite* if, for any $A \in X$, the set $\{\lambda \in \Lambda | A \in \operatorname{supp}(m^\lambda)\}$ is finite. In this case, the sum $\sum_{\lambda \in \Lambda} m^\lambda$ is a well defined element of M, namely $\sum_{A \in X}(\sum_{\lambda \in \Lambda} m^\lambda_A)A$.

For any $s \in S$ we define $\tilde{T}_s : M \to M$ by $\tilde{T}_s(\sum_A m_A A) = \sum_A m_A \tilde{T}_s A$ where

$$\tilde{T}_s A = \begin{cases} sA, & \text{if } s \notin \mathcal{L}(A), \\ sA + (v - v^{-1})A, & \text{if } s \notin \mathcal{L}(A). \end{cases}$$

Note that the family $(m_A \tilde{T}_s A)_{A \in X}$ is locally finite, so that the sum $\sum_A m_A \tilde{T}_s A$ is defined. One checks that this defines an $\mathcal{H}$-module structure on M.

Let $i \in I$. A function $A \mapsto m_A$ from X to $\mathcal{A}$ is said to be i-bounded above (resp. i-bounded below) if for any $A \in X$ there exists $n_A \in \mathbf{Z}$ such that $m_{A+n\alpha_i} = 0$ for all integers $n \geq n_A$ (resp. $n \leq n_A$).

Let $M_{i,\leq}$ (resp. $M_{i,\geq}$) be the set of all elements $m \in M$ such that $A \mapsto m_A$ is i-bounded above (resp. i-bounded below). It is clear that $M_{i,\leq}$ and $M_{i,\geq}$ are $\mathcal{H}$-submodules of M.

3.3. **The map θ_H.** Here (and until the end of 3.7) we fix $i \in I$ and $H \in \mathfrak{F}^i$.

If $A \in X$, the Ω^i-orbit of A is $(A^z)_{z\in\mathbf{Z}}$ where $A^0 = A\sigma_H$ and $d_i(A^z, A^{z+1}) = 1$ for all $z \in \mathbf{Z}$. (These conditions define uniquely A^z for $z \in \mathbf{Z}$.) We set

(a)

$$\begin{aligned}\theta_H A &= v^{-1}(A^0 - v^{-1}A^1 + v^{-2}A^2 - v^{-3}A^3 + \dots) + (A^1 - v^{-1}A^2 + v^{-2}A^3 - \dots)\\ &= v^{-1}A^0 + \sum_{n=1}^{\infty}(-1)^{n-1}(v^{-n+1} - v^{-n-1})A^n \in M.\end{aligned}$$

If $m \in M_{i,\leq}$, then the family of elements $(m_A\theta_H A)_{A\in X}$ in M is locally finite. Hence the infinite sum $\sum_A m_A\theta_H A$ makes sense as an element of M. We then set $\theta_H m = \sum_A m_A\theta_H A$. This is in fact an element of $M_{i,\geq}$.

Let $(m_\lambda)_{\lambda\in\Lambda}$ be a locally finite family of elements of M. We say that this family is i-bounded above (resp. i-bounded below) if for any $A \in X$ there exists $n_A \in \mathbf{Z}$ such that $A+n\alpha_i \notin \mathrm{supp}(m_\lambda)$ for all $\lambda \in \Lambda$ and all integers $n \geq n_A$ (resp. $n \leq n_A$). Using the definitions, we see that the following "continuity property" of θ_H holds.

If $(m_\lambda)_{\lambda\in\Lambda}$ is locally finite and i-bounded above, then $(\theta_H(m_\lambda))_{\lambda\in\Lambda}$ is locally finite and i-bounded below and $\theta_H(\sum_\lambda m_\lambda) = \sum_\lambda \theta_H(m_\lambda)$. It is easy to see that $\theta_H : M_{i,\leq} \to M_{i\geq}$ is an isomorphism; its inverse is given by

$$A \mapsto vA^0 + \sum_{n=1}^{\infty}(-1)^n(v^{n+1} - v^{n-1})A^{-n}.$$

Lemma 3.4. *Let $A \in X$ and let $\tilde{A} \in X$ be the alcove in the Ω^i-orbit of A such that $d_i(\tilde{A}, A) = 1$. We have $\theta_H(A + v^{-1}\tilde{A}) = \tilde{A}\sigma_H + v^{-1}A\sigma_H$,*

With our earlier notation we have

$$\theta_H(A) = v^{-1}A^0 + \sum_{n=1}^{\infty}(-1)^{n-1}(v^{-n+1} - v^{-n-1})A^n,$$

$$\theta_H(v^{-1}\tilde{A}) = v^{-2}A^1 + \sum_{n=2}^{\infty}(-1)^n(v^{-n+1} - v^{-n-1})A^n.$$

Adding, we get $\theta_H(A + v^{-1}\tilde{A}) = v^{-1}A^0 + A^1 = \tilde{A}\sigma_H + v^{-1}A\sigma_H$. The lemma is proved.

Lemma 3.5. *$\theta_H : M_{i,\leq} \to M_{i,\geq}$ is an $\mathcal{H}$-linear map.*

Lemma 3.4 shows that θ_H, restricted to the $\mathcal{A}$-submodule of M generated by the elements $(A+v^{-1}\tilde{A})$ as in 3.4, coincides with the map θ_H defined in [L1, 2.4]. (But note that the basis elements A considered in [L1] differ by powers of v from those considered here.) Then, from [L1, 2.4] we deduce that, for any $s \in S$, the maps $\theta_H\tilde{T}_s$ and $\tilde{T}_s\theta_H$ from $M_{i,\leq}$ to $M_{i,\geq}$ coincide on all elements of the form $(A+v^{-1}\tilde{A})$ with $A \in X$ and $\tilde{A}$ as in 3.4. Clearly, any element $m \in M_{i,\leq}$ can be written uniquely as an infinite sum $\sum_{A\in X} g_A(A + v^{-1}\tilde{A})$ where $A \mapsto g_A \in \mathcal{A}$ is i-bounded above.

Then $(g_A(A+v^{-1}\tilde{A}))_{A\in X}$ is a locally finite family of elements which is i-bounded above. Using the continuity property of θ_H and the analogous continuity property of $\tilde{T}_s$ we deduce that $\theta_H \tilde{T}_s(m) = \tilde{T}_s \theta_H(m)$. The lemma is proved.

3.6. For any special point $\epsilon \in E$, we set $e_\epsilon = \sum_{A\in D(\epsilon)} v^{-d(A,A_\epsilon^+)} A \in M$. Note that $\mathrm{supp}(e_\epsilon)$ is a finite set. Hence e_ϵ belongs both to $M_{i,\leq}$ and to $M_{i,\geq}$.

Lemma 3.7. $\theta_H(e_\epsilon) = e_{\epsilon\sigma_H}$.

Let $\epsilon' = \epsilon\sigma_H$. Let $H_1 \in \mathfrak{F}^i$ be defined by the condition that $\epsilon \in H_1$ and let similarly $H_1' \in \mathfrak{F}^i$ be defined by the condition that $\epsilon' \in H_1'$. Let $Y(\epsilon)$ be the set of all $A \in D(\epsilon)$ such that $d_i(A\sigma_{H_1}, A) = 1$. We then have also $d(A\sigma_{H_1}, A) = 1$. (See [L1, 2.5].) It follows that

$$e_\epsilon = \sum_{A\in Y(\epsilon)} v^{-d(A,A_\epsilon^+)} (A + v^{-1} A\sigma_{H_1}).$$

Using 3.4, we see that

$$\begin{aligned}\theta_H(e_\epsilon) &= \sum_{A\in Y(\epsilon)} v^{-d(A,A_\epsilon^+)} \theta_H(A + v^{-1}A\sigma_{H_1}) \\ &= \sum_{A\in Y(\epsilon)} v^{-d(A,A_\epsilon^+)} \theta_H(A\sigma_{H_1}\sigma_H + v^{-1}A\sigma_H).\end{aligned}$$

Now $\sigma_{H_1}\sigma_H$ is a translation by an element $x \in \mathcal{T}$, so that

$$A\sigma_{H_1}\sigma_H + v^{-1}A\sigma_H = (A+x) + v^{-1}(A+x)\sigma_{H_1'}.$$

Moreover, $A \mapsto A + x$ defines a bijection $Y(\epsilon) \xrightarrow{\sim} Y(\epsilon')$ and $d(A, A_\epsilon^+) = d(A+x, A_\epsilon^+ + x) = d(A+x, A_{\epsilon'}^+)$. Hence

$$\theta_H(e_\epsilon) = \sum_{A'\in Y(\epsilon')} v^{-d(A',A_{\epsilon'}^+)} (A' + v^{-1}A'\sigma_{H_1'}) = e_{\epsilon'}.$$

The lemma is proved.

3.8. Let U be a subset of $\mathcal{T}$. Let $\mathcal{S}(U)$ be the set consisting of all subsets of X that are finite unions of subsets of form $\bigcup_{x\in U}(A+x)$ for various $A \in X$.

Let $M(U)$ be the set of all $m \in M$ such that $\mathrm{supp}(m)$ is contained in some subset of X which is in $\mathcal{S}(U)$. In other words, $m \in M(U)$ means that there exist $A_1, A_2, \ldots, A_n$ in X such that

$$\mathrm{supp}(m) \subset \bigcup_{k\in[1,n];x\in U} (A_k + x).$$

It is clear that $M(U)$ is an $\mathcal{H}$-submodule of $\mathcal{H}$.

Now let U, U' be two subsets of $\mathcal{T}$. A group homomorphism $c : M(U) \to M(U')$ is said to be *continuous* if the following holds.

Let $(F_\lambda)_{\lambda\in\Lambda}$ be a family of elements of $M(U)$ which is locally finite and is such that $\bigcup_{\lambda\in\Lambda} \mathrm{supp}(F_\lambda)$ is contained in some $P \in \mathcal{S}(U)$. Then $(c(F_\lambda))_{\lambda\in\Lambda}$ is a locally finite family of elements of $M(U')$ such that $\bigcup_{\lambda\in\Lambda} \mathrm{supp}(c(F_\lambda))$ is contained in some $P' \in \mathcal{S}(U')$. Moreover, $c(\sum_\lambda F_\lambda) = \sum_\lambda c(F_\lambda)$.

3.9. We fix a subset K of I. Let $j_1, j_2, \dots, j_q$ be a sequence in K such that $r_{j_q} \dots r_{j_2} r_{j_1}$ is a reduced expression for w_0^K in W^I. Let $i_1, i_2, \dots, i_p$ be a sequence in I such that $r_{i_p} \dots r_{i_2} r_{i_1} r_{j_q} \dots r_{j_2} r_{j_1}$ is a reduced expression in W^I.

Lemma 3.10. *Fix $k \in [1,p]$ and $H \in \mathfrak{F}^{i_k}$. Let*

$$U = \sum_{i \in I-K} (-\mathbf{N})\alpha_i r_{i_1} r_{i_2} \dots r_{i_{k-1}} + \sum_{i \in K} \mathbf{Z}\alpha_i r_{i_1} r_{i_2} \dots r_{i_{k-1}},$$

$$U' = \sum_{i \in I-K} (-\mathbf{N})\alpha_i r_{i_1} r_{i_2} \dots r_{i_k} + \sum_{i \in K} \mathbf{Z}\alpha_i r_{i_1} r_{i_2} \dots r_{i_k}.$$

Then
(i) $M(U) \subset M_{i_k, \le}$.
(ii) $M(U') \subset M_{i_k, \ge}$.
(iii) $\theta_H : M_{i_k, \le} \to M_{i_k, \ge}$ *carries* $M(U)$ *into* $M(U')$.

For any $A \in X$ we denote by $(A^z)_{z \in \mathbf{Z}}$ the sequence defined in terms of A and H as in 3.3 (for i_k instead of i). Note that $A^{z+2} = A^z + \alpha_k$ for all $z \in \mathbf{Z}$.

Let $m \in M(U)$. There exist $A_1, A_2, \dots, A_n$ in X such that

(a) $$m_A \neq 0 \implies A = A_u - \sum_{i \in I} g_i \alpha_i r_{i_1} r_{i_2} \dots r_{i_{k-1}}$$

for some $u \in [1,n]$ and $g_i \in \mathbf{Z}$ for $i \in K$, $g_i \in \mathbf{N}$ for $i \in I-K$. Applying σ_H and using $A^0 = A\sigma_H, A_u^0 = A_u \sigma_H$ we see that

(b) $$m_A \neq 0 \implies A^0 = A_u^0 - \sum_{i \in I} g_i \alpha_i r_{i_1} r_{i_2} \dots r_{i_{k-1}} r_{i_k}$$

for some $u \in [1,n]$ and $g_i \in \mathbf{Z}$ for $i \in K$, $g_i \in \mathbf{N}$ for $i \in I-K$. From the definitions,

$$A^0 = A_u^0 + x, x \in \mathcal{T} \implies A^1 = A_u^1 + x.$$

Hence

(c) $$m_A \neq 0 \implies A^1 = A_u^1 - \sum_{i \in I} g_i \alpha_i r_{i_1} r_{i_2} \dots r_{i_{k-1}} r_{i_k}$$

for some $u \in [1,n]$ and $g_i \in \mathbf{Z}$ for $i \in K$, $g_i \in \mathbf{N}$ for $i \in I-K$. Next we note that the assumptions of the lemma imply

(d) $$\alpha_{i_k} r_{i_{k-1}} \dots r_{i_2} r_{i_1} = \sum_{i \in I} c_i \alpha_i$$

where $c_i \in \mathbf{N}$ for all $i \in I$. Assume that $c_i = 0$ for all $i \in I-K$. Using again the assumptions of the lemma, we see that $\alpha_{i_k} r_{i_{k-1}} \dots r_{i_2} r_{i_1} r_{j_q} \dots r_{j_2} r_{j_1} \in \sum_{i \in I} \mathbf{N}\alpha_i$. On the other hand, $\alpha_{i_k} r_{i_{k-1}} \dots r_{i_2} r_{i_1} r_{j_q} \dots r_{j_2} r_{j_1} = (\sum_{i \in K} c_i \alpha_i) w_0^K \in -\sum_K \mathbf{N}\alpha_i$ since, for any $i \in K$, we have $\alpha_i w_0^K = -\alpha_{i'}$ for some $i' \in K$. It follows that

$$\alpha_{i_k} r_{i_{k-1}} \dots r_{i_2} r_{i_1} r_{j_q} \dots r_{j_2} r_{j_1} = 0.$$

This is a contradiction. We have thus proved that $c_i > 0$ for some $i \in I-K$.

Applying $r_{i_1} r_{i_2} \dots r_{i_{k-1}}$ to (d), we deduce that

(d′) $$\alpha_{i_k} = \sum_{i \in I} c_i \alpha_i r_{i_1} r_{i_2} \dots r_{i_{k-1}}, \quad \alpha_{i_k} = -\sum_{i \in I} c_i \alpha_i r_{i_1} r_{i_2} \dots r_{i_{k-1}} r_{i_k}.$$

We show that $m \in M_{i_k,\le}$. Let $A \in X, y \in \mathbf{N}$ be such that $m_{A+y\alpha_{i_k}} \ne 0$. From (a), we have $A+y\alpha_{i_k} = A_u - \sum_{i\in I} g_i\alpha_i r_{i_1}r_{i_2}\dots r_{i_{k-1}}$ for some $u \in [1,n]$ and $g_i \in \mathbf{Z}$ for $i \in K$, $g_i \in \mathbf{N}$ for $i \in I-K$. Using (d′), we can rewrite this as

$$\text{(e)} \qquad A+\sum_{i\in I} yc_i\alpha_i r_{i_1}r_{i_2}\dots r_{i_{k-1}} = A_u - \sum_{i\in I} g_i\alpha_i r_{i_1}r_{i_2}\dots r_{i_{k-1}}.$$

Since $\{\alpha_i r_{i_1}r_{i_2}\dots r_{i_{k-1}} | i \in I\}$ is a basis of T, and A, A_u are bounded subsets of E, there exists $N_{A,u} > 0$ independent of y such that $yc_i + g_i \le N_{A,u}$ for all $i \in I$. We can choose $i \in I-K$ such that $c_i \ge 1$; since for that i we have $g_i \ge 0$, it follows that $y \le yc_i + g_i$ hence $y \le N_{A,u}$. Since u takes only finitely many values, we see that y is bounded above. Thus we have proved that $m \in M_{i_k,\le}$. Hence (i) is proved.

We prove (ii). Let $m' \in M(U')$. There exist $A'_1, A'_2, \dots, A'_n$ in X such that

$$\text{(f)} \qquad m'_A \ne 0 \implies A = A'_u - \sum_{i\in I} g'_i\alpha_i r_{i_1}r_{i_2}\dots r_{i_k}$$

for some $u \in [1,n]$ and $g'_i \in \mathbf{Z}$ for $i \in K$, $g'_i \in \mathbf{N}$ for $i \in I-K$. Let $A \in X$ and let $y \in \mathbf{N}$ be such that $m'_{A-y\alpha_{i_k}} \ne 0$. From (f) we have

$$A - y\alpha_{i_k} = A'_u - \sum_{i\in I} g'_i\alpha_i r_{i_1}r_{i_2}\dots r_{i_k}$$

for some $u \in [1,n]$ and $g'_i \in \mathbf{Z}$ for $i \in K$, $g'_i \in \mathbf{N}$ for $i \in I-K$. Using (d′), we obtain

$$A+\sum_{i\in I} yc_i\alpha_i r_{i_1}r_{i_2}\dots r_{i_k} = A'_u - \sum_{i\in I} g'_i\alpha_i r_{i_1}r_{i_2}\dots r_{i_k}.$$

Since $\{\alpha_i r_{i_1}r_{i_2}\dots r_{i_k} | i \in I\}$ is a basis of T, we see as in the earlier argument that y must be bounded above (for fixed A). Thus, $m' \in M_{i_k,\ge}$ and (ii) is proved.

We prove (iii). Assume that $m_A \ne 0$ and that $B \in X$ appears with non-zero coefficient in $\theta_H A$. Then $B = A^z$ for some $z \in \mathbf{N}$; hence there exists $y \in \mathbf{N}$ such that either $B = A^0 + y\alpha_{i_k}$ or $B = A^1 + y\alpha_{i_k}$. From (b),(c) we deduce that

$$B = A^z_u + y\alpha_{i_k} - \sum_{i\in I} g_i\alpha_i r_{i_1}r_{i_2}\dots r_{i_k}$$

for some $u \in [1,n]$, some $z \in \{0,1\}$, some $y \in \mathbf{N}$ and some $g_i \in \mathbf{Z}$ (with $g_i \in \mathbf{N}$ for $i \in I-K$). Using (d′), we deduce that $B = A^z_u - \sum_{i\in I}(g_i + yc_i)\alpha_i r_{i_1}r_{i_2}\dots r_{i_k}$. For $i \in I-K$ we have $g_i + yc_i \in \mathbf{N}$. Hence $\theta_H(m) \in M(U')$. The lemma is proved.

Lemma 3.11. *In the setup of 3.10, $\theta_H : M(U) \to M(U')$ is continuous.*

The proof is a refinement of the argument in the proof of 3.10(iii).

Let $(F_\lambda)_{\lambda\in\Lambda}$ be a family of elements of $M(U)$ which is locally finite and is such that $\bigcup_{\lambda\in\Lambda} \operatorname{supp}(F_\lambda)$ is contained in some $P \in \mathcal{S}(U)$. Then there exist $A_1, A_2, \dots, A_n$ in X such that

(a) *if $A \in \operatorname{supp}(F_\lambda)$ for some λ, then $A = A_u - \sum_{i\in I} g_i\alpha_i r_{i_1}r_{i_2}\dots r_{i_{k-1}}$ for some $u \in [1,n]$ and $g_i \in \mathbf{Z}$ for $i \in K$, $g_i \in \mathbf{N}$ for $i \in I-K$.*

From this we deduce, as in the proof of 3.10:

(b) *if $A \in \operatorname{supp}(F_\lambda)$ for some λ, then for $z \in \{0,1\}$ we have*

$$A^z = A^z_u - \sum_{i\in I} g_i\alpha_i r_{i_1}r_{i_2}\dots r_{i_k}$$

for some $u \in [1,n]$ and $g_i \in \mathbf{Z}$ for $i \in K$, $g_i \in \mathbf{N}$ for $i \in I-K$.
We fix $B \in X$ such that $B \in \mathrm{supp}(\theta_H(F_\lambda))$ for some $\lambda \in \Lambda$. Then there exists $A \in \mathrm{supp}(F_\lambda)$ and $z \in \mathbf{N}$ such that $B = A^z$. Hence there exists $A \in \mathrm{supp}(F_\lambda), z \in \{0,1\}$ and $y \in \mathbf{N}$ such that $B = A^z + y\alpha_{i_k}$. From (b) we deduce that $B = A^z_u + y\alpha_{i_k} - \sum_{i\in I} g_i\alpha_i r_{i_1} r_{i_2} \dots r_{i_k}$ for some $u \in [1,n]$, some $z \in \{0,1\}$, some $y \in \mathbf{N}$ and some $g_i \in \mathbf{Z}$ (with $g_i \in \mathbf{N}$ for $i \in I-K$). As in the proof of 3.10, we deduce

$$B = A^z_u - \sum_{i\in I}(g_i + yc_i)\alpha_i r_{i_1} r_{i_2} \dots r_{i_k}.$$

For $i \in I-K$ we have $g_i + yc_i \in \mathbf{N}$. This shows that $\bigcup_\lambda \mathrm{supp}(\theta_H(F_\lambda)) \subset P'$ where

$$P' = \bigcup_{u\in[1,n], z\in\{0,1\}, x\in U'} (A^z_u + x) \in \mathcal{S}(U').$$

For any $A \in X$, let $Z(A) = \{\lambda \in \Lambda | A \in \mathrm{supp}(F_\lambda)\}$. Let R be the set of all triples (A,z,y) where $A \in X, z \in \{0,1\}, y \in \mathbf{N}$ are such that $Z(A) \neq \emptyset$ and $B = A^z + y\alpha_{i_k}$. We show that R is a finite set. Let $(A,z,y) \in R$. Since (b) is applicable to A, we see that

$$B - y\alpha_{i_k} = A^z_u - \sum_{i\in I} g_i\alpha_i r_{i_1} r_{i_2} \dots r_{i_k}$$

for some $u \in [1,n]$ and $g_i \in \mathbf{Z}$ for $i \in K$, $g_i \in \mathbf{N}$ for $i \in I-K$. Using 3.10(d′), we can rewrite this as

$$B + \sum_{i\in I} yc_i\alpha_i r_{i_1} r_{i_2} \dots r_{i_k} = A^z_u - \sum_{i\in I} g_i\alpha_i r_{i_1} r_{i_2} \dots r_{i_k}$$

for some $u \in [1,n]$ and $g_i \in \mathbf{Z}$ for $i \in K$, $g_i \in \mathbf{N}$ for $i \in I-K$. From this we see as in the proof of 3.10(i) that y is bounded above by a constant depending only on B. Hence, when (A,z,y) runs through R, the coordinate y takes only finitely many values. From the equation $B = A^z + y\alpha_{i_k}$ it follows that A^z also takes only finitely many values. Since $z \in \{0,1\}$, it follows that A takes only finitely many values, say $A(1), A(2), \dots, A(f)$. In particular, R is finite.

Now let $Y = \{\lambda \in \Lambda | B \in \mathrm{supp}(\theta_H(F_\lambda))\}$. For any $\lambda \in Y$, we consider the set $R(\lambda)$ consisting of all triples $(A,b,y) \in R$ such that $\lambda \in Z(A)$. As we have seen above, $R(\lambda) \neq \emptyset$ for any $\lambda \in Y$. It follows that $Y \subset Z(A(1)) \cup Z(A(2)) \cup \dots \cup Z(A(f))$ which is a finite set since each $Z(A)$ is finite. Thus, Y is finite. We have thus proved that $(\theta_H(F_\lambda))_{\lambda\in\Lambda}$ is a locally finite family of elements of $M(U')$.

Finally, we prove the equality

$$\text{(c)} \qquad \theta_H(\sum_\lambda F_\lambda) = \sum_\lambda \theta_H(F_\lambda).$$

We now fix $B \in X$ and we use the notation relative to B in the earlier part of the proof. We have

$$\sum_{\lambda\in Y} \theta_H(F_\lambda)_B = \sum_{\lambda\in Z(A(1))\cup\dots\cup Z(A(f))} \theta_H(F_\lambda)_B = \theta_H(\sum_{\lambda\in Z(A(1))\cup\dots\cup Z(A(f))} F_\lambda)_B.$$

It is then enough to show that $\theta_H(\sum_{\lambda\notin Z(A(1))\cup\dots\cup Z(A(f))} F_\lambda)_B = 0$. If this were not true there would exist $\lambda \notin Z(A(1)) \cup \dots \cup Z(A(f))$ and $A \in X$ such that $A \in \mathrm{supp}(F_\lambda)$ and such that B appears with non-zero coefficient in $\theta_H A$ (that is,

$B = A^z$ for some $z \in \mathbf{N}$). But this contradicts the definition of $A(1), \dots, A(f)$. The lemma is proved.

3.12. We preserve the notation of 3.9. Let $H_1 \in \mathfrak{F}^{i_1}, H_2 \in \mathfrak{F}^{i_2}, \dots, H_p \in \mathfrak{F}^{i_p}$. Applying p times 3.10, we see that the maps

$$M\left(\sum_{i\in I-K}(-\mathbf{N})\alpha_i + \sum_{i\in K}\mathbf{Z}\alpha_i\right) \xrightarrow{\theta_{H_1}} M\left(\sum_{i\in I-K}(-\mathbf{N})\alpha_i r_{i_1} + \sum_{i\in K}\mathbf{Z}\alpha_i r_{i_1}\right)$$
$$\xrightarrow{\theta_{H_2}} M\left(\sum_{i\in I-K}(-\mathbf{N})\alpha_i r_{i_1} r_{i_2} + \sum_{i\in K}\mathbf{Z}\alpha_i r_{i_1} r_{i_2}\right)$$
$$\xrightarrow{\theta_{H_3}} \dots \xrightarrow{\theta_{H_p}} M\left(\sum_{i\in I-K}(-\mathbf{N})\alpha_i r_{i_1} r_{i_2} \dots r_{i_p} + \sum_{i\in K}\mathbf{Z}\alpha_i r_{i_1} r_{i_2} \dots r_{i_p}\right)$$

are well defined homomorphisms of $\mathcal{H}$-modules. Their composition is a homomorphism of $\mathcal{H}$-modules

$$\text{(a)} \quad \begin{aligned}\theta_{H_1,H_2,\dots,H_p} : M&\left(\sum_{i\in I-K}(-\mathbf{N})\alpha_i + \sum_{i\in K}\mathbf{Z}\alpha_i\right)\\ &\to M\left(\sum_{i\in I-K}(-\mathbf{N})\alpha_i r_{i_1} r_{i_2} \dots r_{i_p} + \sum_{i\in K}\mathbf{Z}\alpha_i r_{i_1} r_{i_2} \dots r_{i_p}\right).\end{aligned}$$

This homomorphism is continuous (in the sense of 3.8); this follows by applying p times 3.11. Using p times 3.7, we see that, for any special point $\epsilon \in E$ we have

$$\text{(b)} \qquad \theta_{H_1,H_2,\dots,H_p}(e_\epsilon) = e_{\epsilon\sigma_{H_1}\sigma_{H_2}\dots\sigma_{H_p}}.$$

3.13. Let $j_1, j_2, \dots, j_q$ be as in 3.9 and let $i_1, i_2, \dots, i_p$ in I so that $r_{i_p} \dots r_{i_2} r_{i_1}$ is a reduced expression for $w_0^I w_0^K$. Moreover, we choose a special point $\epsilon \in E$ and take $H_1 = \epsilon + i_1, H_2 = \epsilon + i_2, \dots, H_p = \epsilon + i_p$. Then the previous discussion is applicable. We use the notation

$$U_K^- = \sum_{i\in I-K}(-\mathbf{N})\alpha_i + \sum_{i\in K}\mathbf{Z}\alpha_i \subset \mathcal{T}, \quad U_K^+ = \sum_{i\in I-K}\mathbf{N}\alpha_i + \sum_{i\in K}\mathbf{Z}\alpha_i.$$

Let $i \mapsto i^*$ be the involution of I given by $\alpha_i w_0^I = -\alpha_{i^*}$. Let K^* be the image of K under this involution. For $i \in I - K$, we have $\alpha_i w_0^K = \alpha_i + \sum_{j\in K} c_{ij}\alpha_j$ for some integers c_{ij}. Aplying w_0^I, we obtain $\alpha_i w_0^K w_0^I = -\alpha_{i^*} + \sum_{j\in K} c_{ij}\alpha_{j^*}$. On the other hand, for $i \in K$, we have $\alpha_i w_0^K = -\alpha_{i^\diamond}$ where $i \mapsto i^\diamond$ is a well defined involution on K. Applying w_0^I, we obtain $\alpha_i w_0^K w_0^I = \alpha_{i^{\diamond *}}$. It follows that

$$\begin{aligned}&\sum_{i\in I-K}(-\mathbf{N})\alpha_i r_{i_1} r_{i_2} \dots r_{i_p} + \sum_{i\in K}\mathbf{Z}\alpha_i r_{i_1} r_{i_2} \dots r_{i_p}\\ &\quad = \sum_{i\in I-K}(-\mathbf{N})\alpha_i w_0^K w_0^I + \sum_{i\in K}\mathbf{Z}\alpha_i w_0^K w_0^I = \sum_{i\in I-K^*}\mathbf{N}\alpha_i + \sum_{i\in K^*}\mathbf{Z}\alpha_i = U_{K^*}^+.\end{aligned}$$

Hence the homomorphism 3.12(a) becomes in our case a (continuous) homomorphism of $\mathcal{H}$-modules

$$\text{(a)} \qquad \theta_{H_1,H_2,\dots,H_p} : M(U_K^-) \to M(U_{K^*}^+).$$

Lemma 3.14. *We preserve the setup in 3.13. Let ϵ' be a special point in E. We have*

$$\theta_{H_1,H_2,\dots,H_p}(e_{\epsilon'}) = e_{\epsilon'\omega_{\epsilon,K}\omega_\epsilon}.$$

Using p times 3.7 we have

$$\theta_{H_1,H_2,\dots,H_p}(e_{\epsilon'}) = \theta_{\epsilon+i_p}\dots\theta_{\epsilon+i_2}\theta_{\epsilon+i_1}(e_{\epsilon'}) = e_{\epsilon'\sigma_{\epsilon+i_1}\sigma_{\epsilon+i_2}\dots\sigma_{\epsilon+i_p}} = e_{\epsilon'\omega_{\epsilon,K}\omega_\epsilon}.$$

The lemma is proved.

3.15. Let $^- : \mathcal{A} \to \mathcal{A}$ be the ring involution which sends v^n to v^{-n} for any $n \in \mathbf{Z}$. Let $^- : \mathcal{H} \to \mathcal{H}$ be the ring involution which sends $v^n\tilde{T}_w$ to $v^{-n}\tilde{T}_{w^{-1}}^{-1}$ for any $n \in \mathbf{Z}$ and $w \in W$. If M_1, M_2 are $\mathcal{H}$-modules, a group homomorphism $\chi : M_1 \to M_2$ is said to be *$\mathcal{H}$-antilinear* if $\chi(hm) = \bar{h}\chi(m)$ for all $h \in \mathcal{H}, m \in M_1$.

For any special point $\epsilon \in E$ we define a map $\phi_\epsilon : M \to M$ by

$$\phi_\epsilon(\sum_{A\in X} m_A A) = \sum_{A\in X} v^{-\nu}\overline{m_A}A\omega_\epsilon.$$

We show that

(a) $$\phi_\epsilon : M \to M \text{ is } \mathcal{H}\text{-antilinear.}$$

(Compare [L1, 2.10].) Let $s \in S$ and $A \in X$. It is enough to show that $\phi_\epsilon(\tilde{T}_sA) = \tilde{T}_s^{-1}\phi_\epsilon(A)$. Using the fact that $s \in \mathcal{L}(A) \iff s \notin \mathcal{L}(A\omega_\epsilon)$ we see that

$$\tilde{T}_s\phi_\epsilon(\tilde{T}_sA) = \tilde{T}_s\phi_\epsilon(sA) = v^{-\nu}\tilde{T}_s(sA\omega_\epsilon) = v^{-\nu}A\omega_\xi = \phi_\epsilon(A)$$

if $s \notin \mathcal{L}(A)$ and

$$\begin{aligned}\tilde{T}_s\phi_\epsilon(\tilde{T}_sA) &= \tilde{T}_s\phi_\epsilon(sA + (v - v^{-1})A) = v^{-\nu}\tilde{T}_s(sA\omega_\epsilon + (v^{-1} - v)A\omega_\epsilon)\\ &= v^{-\nu}(A\omega_\epsilon + (v - v^{-1})sA\omega_\epsilon + (v^{-1} - v)sA\omega_\epsilon) = v^{-\nu}A\omega_\epsilon = \phi_\epsilon(A)\end{aligned}$$

if $s \in \mathcal{L}(A)$.

Lemma 3.16. *If ϵ, ϵ' are special points in E, then $\phi_\epsilon(e_{\epsilon'}) = e_{\epsilon'\omega_\epsilon}$.*

Note that $A \mapsto A' = A\omega_\epsilon$ is a bijection between $D(\epsilon')$ and $D(\epsilon'')$ where $\epsilon'' = \epsilon'\omega_\epsilon$; it carries $A^+_{\epsilon'}$ to $A^-_{\epsilon''}$. Hence

$$\begin{aligned}\phi_\epsilon(e_{\epsilon'}) &= \phi_\epsilon(\sum_{A\in D(\epsilon')} v^{-d(A,A^+_{\epsilon'})}A)\\ &= v^{-\nu}\sum_{A\in D(\epsilon')} v^{d(A,A^+_{\epsilon'})}A\omega_\epsilon = v^{-\nu}\sum_{A'\in D(\epsilon'')} v^{d(A'\omega_\epsilon,A^+_{\epsilon'})}A'.\end{aligned}$$

It remains to show that $d(A'\omega_\epsilon, A^+_{\epsilon'}) - \nu = -d(A', A^+_{\epsilon''})$. The left hand side equals

$$-d(A', A^+_{\epsilon'}\omega_\epsilon) - \nu = -d(A', A^-_{\epsilon''}) - d(A^-_{\epsilon''}, A^+_{\epsilon''}) = -d(A', A^+_{\epsilon''}).$$

The lemma is proved.

Lemma 3.17. *In the setup of 3.13, $\phi_\epsilon : M \to M$ restricts to a continuous homomorphism $\phi_\epsilon : M(U^+_{K^*}) \to M(U^-_K)$.*

For any $A \in X$, we have $(A+U^+_{K^*})\omega_\epsilon = A\omega_\epsilon + U^-_K$ since w_0^I maps $\{\alpha_i | i \in I - K^*\}$ onto $\{-\alpha_i | i \in I - K\}$ and $\{\alpha_i | i \in K^*\}$ onto $\{-\alpha_i | i \in K\}$. Thus, ϕ_ϵ restricts to a homomorphism as shown. The continuity is obvious.

Proposition 3.18. *In the setup of 3.13, the composition of the maps $\theta_{H_1,H_2,\dots,H_p}$ in 3.13*(a) *and ϕ_ϵ in 3.17 is a continuous $\mathcal{H}$-antilinear map $\phi_\epsilon\theta_{H_1,H_2,\dots,H_p} : M(U_K^-) \to M(U_K^-)$ which takes $e_{\epsilon'}$ to $e_{\epsilon'\omega_{\epsilon,K}}$ for any special point $\epsilon' \in E$.*

This follows from 3.13(a), 3.17, 3.15(a), 3.14, 3.16.

4. The antilinear involution $b : M_{\le}^K \to M_{\le}^K$

4.1. *In the remainder of this paper, we fix a subset K of I, a special point $\tilde{\epsilon} \in E^K$ and the K-alcove Ξ as in 2.2.*

Lemma-Definition 4.2. *$\tilde{M}^K$ is the $\mathcal{A}$-submodule of M consisting of elements $m \in M$ that satisfy the equivalent conditions* (a),(b),(c) *below.*

(a) *$m_{A\omega} = v^{d_K(A,A\omega)} m_A$ for any $A \in X$ and any $\omega \in \Omega^K$.*
(b) *$m_{A\omega} = v^{d_K(A,A\omega)} m_A$ for any $A \in X_\Xi$ and any $\omega \in \Omega^K$.*
(c) *$m_{A\omega} = v^{d_K(A,A\omega)} m_A$ for any $A \in X$ and any $\omega \in \Omega^K$ of the form σ_H for some $H \in \mathfrak{F}^i, i \in K$.*

The equivalence of (a),(b) follows from the implication

$$m_{A\omega} = v^{d_K(A,A\omega)} m_A,\ m_{A\omega\omega'} = v^{d_K(A,A\omega\omega')} m_A \implies m_{A\omega\omega'} = v^{d_K(A\omega,A\omega\omega')} m_{A\omega}$$

using

$$v^{d_K(A,A\omega\omega')} v^{-d_K(A,A\omega)} = v^{d_K(A\omega,A\omega\omega')}$$

and the fact that X_Ξ is a set of representatives for the Ω^K-orbits on X.

The equivalence of (a),(c), follows from the implication

$$m_{A\omega} = v^{d_K(A,A\omega)} m_A,\ m_{A\omega\omega'} = v^{d_K(A\omega,A\omega\omega')} m_{A\omega} \implies m_{A\omega\omega'} = v^{d_K(A,A\omega\omega')} m_A.$$

(We use the fact that the group Ω^K is generated by $\{\sigma_H | H \in \mathfrak{F}^i, i \in K\}$.)

Proposition 4.3. *$\tilde{M}^K$ is an $\mathcal{H}$-submodule of M. Moreover, if for any $A \in X$ we set $P_A = \sum_{\omega \in \Omega^K} v^{d_K(A,A\omega)} A\omega \in \tilde{M}^K$, then for any $A \in X, s \in S$ we have*

(a) *$\tilde{T}_s P_A = P_{sA}$ if $s \notin \mathcal{L}(A), L \notin \mathfrak{F}^K$,*
(b) *$\tilde{T}_s P_A = P_{sA} + (v - v^{-1}) P_A$ if $s \in \mathcal{L}(A), L \notin \mathfrak{F}^K$,*
(c) *$\tilde{T}_s P_A = v P_A$ if $L \in \mathfrak{F}^K$, where $L \in \mathfrak{F}$ is the hyperplane that separates A from sA.*
(d) *For any $A \in X$ and $\omega' \in \Omega^K$ we have $P_{A\omega'} = v^{-d_K(A,A\omega')} P_A$.*

In the proof we shall use the following result.

Lemma 4.4. *Let $A \in X, s \in S$ and let $L \in \mathfrak{F}$ be the hyperplane separating A from sA.*

(a) *Let $i \in I$ and $H \in \mathfrak{F}^i$ be such that $s \in \mathcal{L}(A), s \notin \mathcal{L}(A\sigma_H)$. Then $L \in \mathfrak{F}^i$.*
(b) *Assume that $L \notin \mathfrak{F}^K$. Let $\omega \in \Omega^K$. Then we have $s \in \mathcal{L}(A)$ if and only if $s \in \mathcal{L}(A\omega)$.*

We prove (a). Assume that $L \notin \mathfrak{F}^i$. Since $s \in \mathcal{L}(A), s \notin \mathcal{L}(A\sigma_H)$, we have $A \subset E_L^+$ and $A\sigma_H \subset E_{L\sigma_H}^-$. We deduce that $A\sigma_H \subset E_L^+\sigma_H$. Thus, $E_{L\sigma_H}^-$ has non-empty intersection with $E_L^+\sigma_H$. The assumption $L \notin \mathfrak{F}^i$ implies by [L1, 1.2] that $E_L^+\sigma_H = E_{L\sigma_H}^+$. Hence $E_{L\sigma_H}^-$ has non-empty intersection with $E_{L\sigma_H}^+$. This is a contradiction; (a) is proved.

We prove (b). We can write $\omega = \sigma_{H_1}\sigma_{H_2}\dots\sigma_{H_n}$ where $H_k \in \mathfrak{F}^{i_k}$ where $i_1, i_2, \dots, i_n$ are in K. We argue by induction on n. When $n = 0$ there is nothing to prove. When $n = 1$, the result follows from (a). Assume now that $n > 1$. Let $\omega' = \sigma_{H_1}\sigma_{H_2}\dots\sigma_{H_{n-1}}$. Note that the hyperplane separating $A\omega'$ from $sA\omega'$ is $L\omega'$ which is not in $\mathfrak{F}^K$. Hence, by (a), the conditions $s \in \mathcal{L}(A\omega'), s \in \mathcal{L}(A\omega'\sigma_{H_n}) = \mathcal{L}(A\omega)$ are equivalent. By the induction hypothesis, the conditions $s \in \mathcal{L}(A), s \in \mathcal{L}(A\omega')$ are equivalent. It follows that the conditions $s \in \mathcal{L}(A), s \in \mathcal{L}(A\omega)$ are equivalent. The lemma is proved.

4.5. **Proof of Proposition 4.3.** We first show that, if $L \notin \mathfrak{F}^K$, then

(a) $$d_K(A, A\omega) = d_K(sA, sA\omega)$$

for any $\omega \in \Omega^K$. Indeed, we have $d_K(sA, sA\omega) = -d_K(A, sA) + d_K(A, A\omega) + d_K(A\omega, sA\omega)$ hence it is enough to show that $d_K(A, sA) = d_K(A\omega, sA\omega)$. But in fact $d_K(A, sA) = 0$ since the hyperplane separating A, sA is not in $\mathfrak{F}^K$ and $d_K(A\omega, sA\omega) = 0$ since the hyperplane separating $A\omega, sA\omega$ is not in $\mathfrak{F}^K$.

Assume that we are in the setup of 4.3(a). Using 4.4(b), we have

$$\tilde{T}_s P_A = \sum_{\omega\in\Omega^K} v^{d_K(A,A\omega)} sA\omega = P_{sA}.$$

Assume that we are in the setup of 4.3(b). Using 4.4(b), we have

$$\tilde{T}_s P_A = \sum_{\omega\in\Omega^K} v^{d_K(A,A\omega)} sA\omega + (v - v^{-1}) \sum_{\omega\in\Omega^K} v^{d_K(A,A\omega)} A\omega = P_{sA} + (v - v^{-1})P_A.$$

In the setup of 4.3(c), we have $sA = A\omega_1$ for some involution $\omega_1 \in \Omega^K$. We can partition Ω^K into two element subsets $\{\omega, \omega_1\omega\}$. Hence it is enough to prove that

$$\tilde{T}_s(v^{d_K(A,A\omega)} A\omega + v^{d_K(A,A\omega_1\omega)} A\omega_1\omega) = v(v^{d_K(A,A\omega)} A\omega + v^{d_K(A,A\omega_1\omega)} A\omega_1\omega)$$

or, equivalently, that $\tilde{T}_s(A\omega + v^{d_K(A\omega,sA\omega)} sA\omega) = v(A\omega + v^{d_K(A\omega,sA\omega)} sA\omega)$ for any $\omega \in \Omega^K$. But this follows immediately from the definition of $\tilde{T}_s$ since $d_K(A\omega, sA\omega)$ is 1 if $s \notin \mathcal{L}(A\omega)$ and is -1 if $s \in \mathcal{L}(A\omega)$. (Note that the hyperplane separating $A\omega, sA\omega$ is in $\mathfrak{F}^K$.) Thus, 4.3(a),(b),(c) are verified.

Next, 4.3(d) follows immediately from the definitions. It remains to verify the first assertion of 4.3. This follows from the earlier part of the proof, since any element of $\tilde{M}^K$ can be written uniquely in the form $\sum_A f_A P_A$ where A runs over over X_Ξ and $f_A \in \mathcal{A}$. Proposition 4.3 is proved.

4.6. Let M^K be the set of all functions $X_\Xi \to \mathcal{A}$. If $m \in M^K$ we shall generally denote by $m_A \in \mathcal{A}$ the value of m at $A \in \Xi$ and we write $m = \sum_{A\in\Xi} m_A A$. We regard M^K as an $\mathcal{A}$-module in the obvious way. Let

(a) $$\mathrm{res}_K : \tilde{M}^K \xrightarrow{\sim} M^K$$

be the map given by $\sum_{A\in X} m_A A \mapsto \sum_{A\in X_\Xi} m_A A$ (or restriction of functions).

From the definition of $\tilde{M}^K$ based on 4.2(b), it follows that res_K is an isomorphism of $\mathcal{A}$-modules, since X_Ξ is a set of representatives for the Ω^K-orbits on X. It follows that there is a unique $\mathcal{H}$-module structure on M^K such that res_K is an isomorphism of $\mathcal{H}$-modules.

Lemma 4.7. *The following holds in the $\mathcal{H}$-module M^K. Let $s \in S$ and let $A \in X_\Xi$. Let $H \in \mathfrak{F}$ be the hyperplane that separates A from sA.*

(a) *If $s \notin \mathcal{L}(A)$ and H is not a wall of Ξ, then $\tilde{T}_s A = sA$.*

(b) *If $s \in \mathcal{L}(A)$ and H is not a wall of Ξ, then $\tilde{T}_s A = sA + (v - v^{-1})A$.*

(c) *If H is a wall of Ξ, then $\tilde{T}_s A = vA$.*

We apply res_K to the identities in 4.3(a),(b),(c) with $A \in \Xi$. Note that $\mathrm{res}_K(P_A) = A$ for $A \in X_\Xi$. The lemma follows.

4.8. The *support* of an element $m \in M^K$ is the subset

$$\mathrm{supp}(m) = \{A \in X_\Xi | m_A \neq 0\}$$

of X_Ξ. (Compare 3.2.)

A family $(m^\lambda)_{\lambda \in \Lambda}$ of elements of M^K is said to be *locally finite* if, for any $A \in X_\Xi$, the set $\{\lambda \in \Lambda | A \in \mathrm{supp}(m^\lambda)\}$ is finite. In this case, the sum $\sum_{\lambda \in \Lambda} m^\lambda$ is a well defined element of M, namely $\sum_{A \in X_\Xi} (\sum_{\lambda \in \Lambda} m^\lambda_A) A$. (Compare 3.2.)

Let $M^K_\leftarrow$ be the set of all $m \in M^K$ such that there exist $A_1, A_2, \dots, A_p$ in X_Ξ with $\mathrm{supp}(m) \subset \{\gamma_{-t}(A_u) | u \in [1,p], t \in \mathcal{T}^+\}$. Let $M^K_\rightarrow$ be the set of all $m \in M^K$ such that there exist $A_1, A_2, \dots, A_p$ in X_Ξ with $\mathrm{supp}(m) \subset \{\gamma_t(A_u) | u \in [1,p], t \in \mathcal{T}^+\}$. Let

$$\tilde{M}^K_\leftarrow = \tilde{M}^K \cap M(U_K^-), \quad \tilde{M}^K_\rightarrow = \tilde{M}^K \cap M(U_K^+).$$

From 4.3 it follows that

(a) *$\tilde{M}^K_\leftarrow, \tilde{M}^K_\rightarrow$ are $\mathcal{H}$-submodules of M.*

We show that the isomorphism $\mathrm{res}_K : \tilde{M}^K \to M^K$ (see 4.6) restricts to isomorphisms

$$\text{(b)} \qquad \tilde{M}^K_\leftarrow \xrightarrow{\sim} M^K_\leftarrow, \quad \tilde{M}^K_\rightarrow \xrightarrow{\sim} M^K_\rightarrow.$$

Let $\tilde{m} \in \tilde{M}^K$ and let $m = \mathrm{res}_K(\tilde{m}) \in M^K$. Assume first that $\tilde{m} \in \tilde{M}^K_\leftarrow$. Then there exist $A_1, A_2, \dots, A_p$ in X such that any $A \in \mathrm{supp}(\tilde{m})$ is of the form

$$A = A_u - \sum_{i \in I-K} n_i \alpha_i + \sum_{i \in K} z_i \alpha_i$$

where $u \in [1,p], n_i \in \mathbf{N}, z_i \in \mathbf{Z}$. Now let $A \in \mathrm{supp}(m)$. Then $A \in \mathrm{supp}(\tilde{m})$ hence it is of the form above. Let $B_u = \pi(A_u)$. Since $\pi(A) = A$, we have

$$A = \pi(A_u - \sum_{i \in I-K} n_i \alpha_i) = \gamma_{-t}(B_u)$$

where $t = \sum_{i \in I-K} n_i \alpha_i \in \mathcal{T}^+$. This shows that $m \in M^K_\leftarrow$.

Conversely, assume that $m \in M^K_\leftarrow$. Then there exist $B_1, B_2, \dots, B_p$ in X_Ξ such that any $B \in \mathrm{supp}(m)$ is of the form $\gamma_{-t}(B_u)$ where $u \in [1,p]$ and $t \in \mathcal{T}^+$. Now let $A \in \mathrm{supp}(\tilde{m})$. Then there exists $B \in \mathrm{supp}(m)$ and $\omega \in \Omega^K$ such that $A = B\omega$. Thus $A = \gamma_{-t}(B_u)\omega$ for some $u \in [1,p]$, some $t \in \mathcal{T}^+$ and some $\omega \in \Omega^K$. Hence $A = (B_u - t)\omega'$ for some $u \in [1,p]$, some $t \in \mathcal{T}^+$ and some $\omega' \in \Omega^K$.

We can find a finite subset F of Ω^K such that any element of Ω^K is the product of an element of F with a translation in $\mathcal{T}^K$. Hence we have $A = (B_u - t)f + t'$ for some $u \in [1,p]$, some $t \in \mathcal{T}^+$, some $f \in F$ and some $t' \in \mathcal{T}^K$. We have

$$A = (B_u - t)f + t' = B_u f - t + t''$$

where $t'' \in \mathcal{T}^K$. Since $B_u f$ runs through a finite subset of X we see that $m \in \tilde{M}^K_{\leftarrow}$. This establishes the first isomorphism (b). The second isomorphism (b) is established in an entirely similar way. From (a) and (b) it follows that

(c) *$M^K_{\leftarrow}, M^K_{\rightarrow}$ are $\mathcal{H}$-submodules of M^K.*

4.9. Let $\epsilon \in \mathbf{S}_{\tilde{\epsilon}}$ (see 2.3). For $A \in D(\epsilon)$ (see 1.1), the following two conditions are equivalent:

(a) $A \subset \Xi$ (that is, $A \in D_\Xi(\epsilon)$),

(b) $A \subset E^+_{\epsilon+i}$ for all $i \in K$.

Let W^I_* be the set of all $w \in W^I$ such that w has maximal length among the elements in the coset wW^K. From the description (b) of $D_\Xi(\epsilon)$ we see that

(c) $$D_\Xi(\epsilon) = \{A_{\epsilon,w} | w \in W^I_*\}.$$

(Notation of 1.1.) Note that $A^!_\epsilon = A_{\epsilon,w_0^K}, A^+_\epsilon = A_{\epsilon,w_0^I}$. We set

$$e_{\epsilon,K} = \sum_{A \in D_\Xi(\epsilon)} v^{-d(A,A^+_\epsilon)} A \in M^K.$$

(For $K = \emptyset$, this specializes to e_ϵ in 3.6.)

Lemma 4.10. *Recall that $\mathcal{T}^K = \sum_{i \in K} \mathbf{Z}\alpha_i = \mathcal{T} \cap \Omega^K$.*

(a) *For any $t \in \mathcal{T}$, the function $A \mapsto d_K(A, A+t)$ on X is constant. Let $\mu_K(t)$ be its value. We have $\mu_K(t) = \sum_{h \in \bar{\mathcal{F}}^K} \check{\alpha}_h(t)$ for all $t \in \mathcal{T}$. The map $t \mapsto \mu_K(t)$ is a group homomorphism $\mathcal{T} \to \mathbf{Z}$.*

(b) *If $\epsilon \in \mathbf{S}_{\tilde{\epsilon}}$, the sum $\sum_{t \in \mathcal{T}^K} v^{\mu_K(t)} e_{\epsilon+t}$ (in M) belongs to $\tilde{M}^K$ and*

$$e_{\epsilon,K} = \mathrm{res}_K\Big(\sum_{t \in \mathcal{T}^K} v^{\mu_K(t)} e_{\epsilon+t}\Big).$$

(c) *For any $t \in \mathcal{T}$ we have $\mu_K(t) + \mu_K(tw_0^K) = 0$ and $\mu_{K^*}(t) + \mu_K(tw_0^I) = 0$.*

(a) follows immediately from 2.14 and 1.3(e).

We prove (b). For $t \in \mathcal{T}^K$ we have $A \in D(\epsilon) \Leftrightarrow A + t \in D(\epsilon + t), A^+_{\epsilon+t} = A^+_\epsilon + t$ and $d(A+t, A^+_\epsilon + t) = d(A, A_\epsilon)$, hence

(d) $$\sum_{t \in \mathcal{T}^K} v^{\mu_K(t)} e_{\epsilon+t} = \sum_{t \in \mathcal{T}^K; A \in D(\epsilon)} v^{\mu_K(t)} v^{-d(A,A^+_\epsilon)}(A+t).$$

Let Ω' be the stabilizer of $\tilde{\epsilon}$ in Ω^K or equivalently, the stabilizer of ϵ in Ω^K. Note that $D_\Xi(\epsilon)$ is a set of representatives for the Ω'-orbits on $D(\epsilon)$. Hence the expression (d) is equal to

$$\sum_{t \in \mathcal{T}^K; A \in D_\Xi(\epsilon); \omega' \in \Omega'} v^{d_K(A,A+t) - d(A\omega', A^+_\epsilon)}(A\omega' + t)$$

We want to show that the exponent of v satisfies:

(e) $$d_K(A, A+t) - d(A\omega', A^+_\epsilon) = d_K(A, A\omega' + t) - d(A, A^+_\epsilon)$$

or equivalently (by the additivity of d, d_K): $d_K(A\omega' + t, A + t) = d(A\omega', A)$. Since $d_K(A\omega' + t, A + t) = d_K(A\omega', A)$, it suffices to show that $d_K(A\omega', A) = d(A\omega', A)$. This follows from the fact that any hyperplane in $\mathfrak{F}$ that separates $A\omega'$ from A must contain ϵ and is automatically in $\mathfrak{F}^K$. (This is the known property which asserts that for an element y in a standard parabolic subgroup of a Coxeter group, the length of y computed in W coincides with the length of y computed in the parabolic subgroup.) Thus (e) is proved and we may therefore rewrite (d) as follows:

$$\sum_{t\in\mathcal{T}^K;A\in D_\Xi(\epsilon);\omega'\in\Omega'} v^{d_K(A,A\omega'+t)-d(A,A_\epsilon^+)}(A\omega'+t).$$

We now observe that Ω^K is the semidirect product of Ω' and $\mathcal{T}^K$ hence the previous expression equals

$$\sum_{A\in D_\Xi(\epsilon);\omega\in\Omega} v^{d_K(A,A\omega)-d(A,A_\epsilon^+)}A\omega.$$

This clearly belongs to $\tilde{M}^K$ and its image under res_K is $\sum_{A\in D_\Xi(\epsilon)} v^{-d(A,A_\epsilon^+)}A = e_{\epsilon,K}$.

We prove the first equality (c). Let $A\in X$ and let $\epsilon\in E$ be a special point. It suffices to show that $d_K(A,A+t)+d_K(A\omega_{\epsilon,K},A\omega_{\epsilon,K}+tw_0^K)=0$. This is a special case of the equality

$$\text{(f)} \qquad d_K(A,B)=-d_K(A\omega_{\epsilon,K},B\omega_{\epsilon,K})$$

valid for any $A,B\in X$. To prove this we note that $H\mapsto H\omega_{\epsilon,K}$ is a bijection between the set of hyperplanes in $\mathfrak{F}^K$ that separate A from B and the set of hyperplanes in $\mathfrak{F}^K$ that separate $A\omega_{\epsilon,K}$ from $B\omega_{\epsilon,K}$). Moreover, the signs attached to corresponding hyperplanes are opposite. The desired equality follows.

We prove the second equality (c). Let $A\in X$ and let $\epsilon\in E$ be a special point. It suffices to show that $d_{K^*}(A,A+t)+d_K(A\omega_\epsilon,A\omega_\epsilon+tw_0^I)=0$. This is a special case of the equality

$$\text{(g)} \qquad d_{K^*}(A,B)=-d_K(A\omega_\epsilon,B\omega_\epsilon)$$

valid for any $A,B\in X$. To prove this we note that $H\mapsto H\omega_\epsilon$ is a bijection between the set of hyperplanes in $\mathfrak{F}^{K^*}$ that separate A from B and the set of hyperplanes in $\mathfrak{F}^K$ that separate $A\omega_\epsilon$ from $B\omega_\epsilon$. Moreover, the signs attached to corresponding hyperplanes are opposite. The desired equality follows.

The lemma is proved.

4.11. Let $\epsilon\in\mathbf{S}_{\tilde\epsilon}$. We set $q=d(A_\epsilon^!,A_\epsilon^+)$. Let $B\in D_\Xi(\epsilon)$; we set $r=d(A_\epsilon^!,B)$. From the description 4.9(c) of $D_\Xi(\epsilon)$ we can deduce that the following holds: $0\le r\le q$ and there exists $s'_1,s'_2,\dots,s'_q$ in S such that $B=s'_rs'_{r-1}\dots s'_1A_\epsilon^!, A_\epsilon^+=s'_qs'_{q-1}\dots s'_1A_\epsilon^!$. Moreover, $s'_n\in\mathcal{L}(s'_{n-1}s'_{n-2}\dots s'_1A_\epsilon^!)$ for $n\in[1,q]$.

Lemma 4.12. *Let $A\in X_\Xi$ and let $\epsilon\in\mathbf{S}_{\tilde\epsilon}$ be such that $A\subset\epsilon+\mathcal{C}^+$. By 2.5 there exist $s_1,s_2,\dots,s_p$ in S such that $p=d(A_\epsilon^+,A)$ and $A=s_ps_{p-1}\dots s_1A_\epsilon^+$. Then in the $\mathcal{H}$-module M^K, both expressions $\tilde{T}_{s_p}\tilde{T}_{s_{p-1}}\dots\tilde{T}_{s_1}(e_{\epsilon,K}),\tilde{T}_{s_p}^{-1}\tilde{T}_{s_{p-1}}^{-1}\dots\tilde{T}_{s_1}^{-1}(e_{\epsilon,K})$ are of the form $A+$ an $\mathcal{A}$-linear combination of elements $A'\in X_\Xi$ with $A'<A$.*

From the definition of $e_{\epsilon,K}$ we see that it is enough to prove statements (a),(b) below.

(a) *Both expressions $\tilde{T}_{s_p}\tilde{T}_{s_{p-1}}\dots\tilde{T}_{s_1}A_\epsilon^+,\tilde{T}_{s_p}^{-1}\tilde{T}_{s_{p-1}}^{-1}\dots\tilde{T}_{s_1}^{-1}A_\epsilon^+$ are of the form A plus an $\mathcal{A}$-linear combination of elements $A'\in X_\Xi$ with $A'<A$.*

(b) *If $B\in D_\Xi(\epsilon),B\ne A_\epsilon^+$, then both expressions $\tilde{T}_{s_p}\tilde{T}_{s_{p-1}}\dots\tilde{T}_{s_1}B,\tilde{T}_{s_p}^{-1}\tilde{T}_{s_{p-1}}^{-1}\dots\tilde{T}_{s_1}^{-1}B$ are $\mathcal{A}$-linear combinations of elements $A'\in X_\Xi$ with $A'<A$.*

We prove (a). Using the formulas for the $\tilde{T}_s$-action (see 4.7) and the equality $\tilde{T}_s^{-1} = \tilde{T}_s + (v^{-1} - v)$, we see that both expressions in (c) are $\mathcal{A}$-linear combinations of elements $A' \in X_\Xi$ such that $A' = s_{j_k} s_{j_{k-1}} \dots s_{j_1} A_\epsilon^+$ for some $p \geq j_k > j_{k-1} > \dots > j_1 \geq 1$. These are all $\leq s_p s_{p-1} \dots s_1 A_\epsilon^+ = A$, by [L1, 3.4]. Moreover, the element $s_p s_{p-1} \dots s_1 A_\epsilon^+ = A$ appears with coefficient 1 in this $\mathcal{A}$-linear combination.

To prove (a) it is then enough to show that, if $k < p$ then $s_{j_k} s_{j_{k-1}} \dots s_{j_1} A_\epsilon^+ \neq A$ (so it is $< A$); this follows from the fact that $d(A_\epsilon^+, s_{j_k} s_{j_{k-1}} \dots s_{j_1} A_\epsilon^+) \leq k$, while $d(A_\epsilon^+, A) = p > k$.

We now prove (b). As above, the two expressions in (b) are $\mathcal{A}$-linear combinations of elements $B' \in X_\Xi$ such that $B' = s_{j_k} s_{j_{k-1}} \dots s_{j_1} B$ for some $p \geq j_k > j_{k-1} > \dots > j_1 \geq 1$.

If $s'_1, s'_2, \dots, s'_q$ in S and r are as in 4.11, then $r < q$ (since $B \neq A_\epsilon^+$) and, by [L1, 3.4], $s_{j_k} \dots s_{j_1} B = s_{j_k} \dots s_{j_1} s'_r s'_{r-1} \dots s'_1 A_\epsilon^! \leq s_p \dots s_1 s'_q s'_{q-1} \dots s'_1 A_\epsilon^! = s_p \dots s_1 A_\epsilon^+ = A$ since

$$d(A_\epsilon^!, s_p s_{p-1} \dots s_1 s'_q s'_{q-1} \dots s'_1 A_\epsilon^!) = d(A_\epsilon^!, A) = d(A_\epsilon^!, A_\epsilon^+) + d(A_\epsilon^+, A) = q + p.$$

It remains to show that $s_{j_k} s_{j_{k-1}} \dots s_{j_1} B \neq A$. But

$$d(A_\epsilon^!, s_{j_k} s_{j_{k-1}} \dots s_{j_1} B) = d(A_\epsilon^!, s_{j_k} s_{j_{k-1}} \dots s_{j_1} s'_r s'_{r-1} \dots s'_1 A_\epsilon^!) \leq k + r$$

while $d(A_\epsilon^!, A) = p + q > k + r$ (since $p \geq k$ and $q > r$). The lemma is proved.

4.13. A subset R of X_Ξ is said to be *bounded above* (resp. *bounded below*) if there exists $A_0 \in X_\Xi$ such that $R \subset \{B \in X_\Xi | B \leq A_0\}$ (resp. $R \subset \{B \in X_\Xi | A_0 \leq B\}$). Let $M^K_\leq$ (resp. $M^K_\geq$) be the set of all elements $m \in M^K$ such that $\mathrm{supp}(m)$ is bounded above (resp. below). Lemma 2.13 can be reformulated as follows:

(a) $$M^K_\leq = M^K_\leftarrow, \quad M^K_\geq = M^K_\rightarrow.$$

Since $M^K_\leftarrow, M^K_\rightarrow$ are $\mathcal{H}$-submodules of M^K (see 4.8(c)), it follows that

(b) *$M^K_\leq, M^K_\geq$ are $\mathcal{H}$-submodules of M^K.*

Let $c : M^K_\dashv \to M^K_{\dashv'}$ be a group homomorphism where $\dashv$ is one of $\leq, \geq$; $\dashv'$ is one of $\leq, \geq$. In analogy with a definition in 3.8, we say that c is *continuous* if the following holds.

Let $(F_\lambda)_{\lambda \in \Lambda}$ be a family of elements of $M^K_\dashv$ which is locally finite (in M^K) and is such that $\bigcup_{\lambda \in \Lambda} \mathrm{supp}(F_\lambda)$ is bounded above (if $\dashv$ is $\leq$) and bounded below (if $\dashv$ is $\geq$). Then the family $(c(F_\lambda))_{\lambda \in \Lambda}$ in $M^K_{\dashv'}$ is locally finite in M^K and $\bigcup_{\lambda \in \Lambda} \mathrm{supp}(c(F_\lambda))$ is bounded above (if $\dashv'$ is $\leq$) and bounded below (if $\dashv'$ is $\geq$). Moreover, $c(\sum_\lambda F_\lambda) = \sum_\lambda c(F_\lambda)$.

We now come to the main result of this section.

Theorem 4.14. (a) *There exists a unique $\mathcal{H}$-antilinear map $b : M^K_\leq \to M^K_\leq$ which is continuous (see 4.13) and satisfies $b(e_{\epsilon',K}) = e_{\epsilon',K}$ for any $\epsilon' \in \bar{\mathbf{S}}_{\tilde{\epsilon}}$.*

(b) *For any $A \in X_\Xi$, $b(A)$ is of the form $A+$ an $\mathcal{A}$-linear combination of elements $A' \in X_\Xi$ with $A' < A$;*

(c) *b^2 is the identity map.*

To begin the proof, we consider the map $a : M^K_\leq \to M$ given by the composition

$$M^K_\leq = M^K_\leftarrow \xrightarrow{a_1} \tilde{M}^K_\leftarrow \xrightarrow{a_2} M(U^-_K) \xrightarrow{a_3} M(U^-_K) \xrightarrow{a_4} M$$

where the equality is given by 4.13(a), a_1 is the inverse of the isomorphism 4.8(b), a_2 is the obvious inclusion, $a_3 = \phi_\epsilon \theta_{H_1, H_2, \dots, H_p}$ is as in 3.18 (in the setup of 3.13) and a_4 is the obvious inclusion. Here, ϵ is chosen in $\mathbf{S}_{\tilde{\epsilon}}$. Note that a is $\mathcal{H}$-antilinear, since a_3 is $\mathcal{H}$-antilinear and a_k for $k \neq 3$ are $\mathcal{H}$-linear.

The proof will be completed in 4.23 after various preparations in 4.15-4.22.

4.15. Assume that for any $A', A \in X_\Xi$ such that $A' \le A$ we are given an element $p_{A',A} \in \mathcal{A}$ such that $p_{A',A} = 1$ whenever $A' = A$. For any $A', A \in X_\Xi$ such that $A' \le A$ we set

$$\tilde{p}_{A',A} = \sum (-1)^r p_{A'_0,A'_1} p_{A'_1,A'_2} \dots p_{A'_{r-1},A'_r}$$

where the sum is taken over all sequences $A'_0, A'_1, A'_2, \dots, A'_r$ in X_Ξ such that $A' = A'_0 < A'_1 < A'_2 < \dots < A'_r = A$.

There are only finitely many such sequences. This is due to the obvious inequality $r \le d(A', A)$ together with the following finiteness property:

(a) *For any $A', A \in X_\Xi$ such that $A' \le A$ the set $\{B \in X_\Xi | A' \le B \le A\}$ is finite.*

This follows from a more general property [L1, 3.5]:

(b) *for any $A', A \in X$ such that $A' \le A$, the set $\{B \in X | A' \le B \le A\}$ is finite.*

Note that $\tilde{p}_{A,A} = 1$ for any A. We have $\sum_{B \in X_\Xi; A' \le B \le A} p_{A',B} \tilde{p}_{B,A} = \delta_{A',A}$. Hence, if $\xi_B = \sum_{A' \in X_\Xi; A' \le B} p_{A',B} A' \in M^K_\le$, then $A = \sum_{B \in X_\Xi; B \le A} \tilde{p}_{B,A} \xi_B$.

(The last sum makes sense in M^K since, for fixed $A \in X_\Xi$, the family $(\tilde{p}_{B,A}\xi_B)$ of elements of M^K (indexed by $\{B \in X_\Xi | B \le A\}$) is locally finite: for any $C \in X_\Xi$, the set of all $B \in X_\Xi$ such that C appears with non-zero coefficient in $\tilde{p}_{B,A}\xi_B$ is contained in the finite set $\{B \in X_\Xi | C \le B \le A\}$.)

4.16. We choose a function $A \mapsto \epsilon_A$ from X_Ξ to $\mathbf{S}_{\tilde{\epsilon}}$ such that $A \subset \epsilon_A + \mathcal{C}^+$ for all A. (Such a function exists by 2.6.) For any A we denote by w_A the element of W such that $w_A(A^+_{\epsilon_A}) = A$. By 4.12, for any $A \in X_\Xi$, we can write

(a) $$\xi_A := \tilde{T}_{w_A} e_{\epsilon_A,K} = \sum_{A' \in X_\Xi; A' \le A} p_{A',A} A',$$

(b) $$\xi'_A := \tilde{T}^{-1}_{w_A^{-1}} e_{\epsilon_A,K} = \sum_{A' \in X_\Xi; A' \le A} q_{A',A} A'$$

where $p_{A',A} \in \mathcal{A}$, $q_{A',A} \in \mathcal{A}$ are 1 whenever $A' = A$ and are zero for all but finitely many A'. Let $\tilde{p}_{A',A} \in \mathcal{A}$ be defined in terms of $p_{A',A}$ as in 4.15 and let $\tilde{q}_{A',A}$ be defined similarly in terms of $q_{A',A}$. As in 4.15, for any $A \in X_\Xi$, we then have

(c) $$A = \sum_{B \in X_\Xi; B \le A} \tilde{p}_{B,A} \xi_B, \quad A = \sum_{B \in X_\Xi; B \le A} \tilde{q}_{B,A} \xi'_B.$$

Lemma 4.17. *For any $A \in X_\Xi$ we have $a(\xi_A) = \mathrm{res}_K^{-1}(\xi'_A)$, $\quad a(\xi'_A) = \mathrm{res}_K^{-1}(\xi_A)$.*

Using the definition of ξ_A, ξ'_A and the fact that a is $\mathcal{H}$-antilinear, we see that it is enough to show that $a(e_{\epsilon_A,K}) = \mathrm{res}_K^{-1}(e_{\epsilon_A,K})$. We show more generally that, for $\epsilon' \in \mathbf{S}_{\tilde{\epsilon}}$ we have

(a) $$a(e_{\epsilon',K}) = \mathrm{res}_K^{-1}(e_{\epsilon',K}).$$

Note that $\epsilon' \omega_{\epsilon,K} = \epsilon'$ since $\omega_{\epsilon,K}$ acts as identity on $\epsilon + T_K$, which contains ϵ'. (Here we use our assumption that $\epsilon \in \mathbf{S}_{\tilde{\epsilon}}$.) Using 4.10(b), 3.18, we have

$$a(e_{\epsilon',K}) = a_3\Big(\sum_{t \in \mathcal{T}^K} v^{\mu_K(t)} e_{\epsilon'+t}\Big) = \sum_{t \in \mathcal{T}^K} v^{-\mu_K(t)} e_{(\epsilon'+t)\omega_{\epsilon,K}} = \sum_{t \in \mathcal{T}^K} v^{-\mu_K(t)} e_{\epsilon'+tw_0^K}.$$

We make the substitution $t' = tw_0^K$ and we obtain $\sum_{t'\in\mathcal{T}^K} v^{-\mu_K(t'w_0^K)}e_{\epsilon'+t'}$. We then use the identity $-\mu_K(t'w_0^K) = \mu_K(t')$ (see 4.10(c)). Thus (a) follows and the lemma is proved.

Lemma 4.18. *Let $A \in X_\Xi$.*

(a) *The family of elements* $\mathrm{res}_K^{-1}(\overline{\tilde{p}_{B,A}}\xi'_B)$ *(where B varies through $\{B \in X_\Xi; B \le A\}$) is locally finite in M and we have $a(A) = \sum_{B\in X_\Xi;B\le A}\mathrm{res}_K^{-1}(\overline{\tilde{p}_{B,A}}\xi'_B)$.*

(b) *We have $a(A) \in \tilde{M}^K$.*

We prove (a). We start with the equality $A = \sum_{B\in X_\Xi;B\le A}\tilde{p}_{B,A}\xi_B$. When B varies through $\{B \in X_\Xi; B \le A\}$, the elements $\tilde{p}_{B,A}\xi_B$ form a locally finite family in M^K and their supports are all contained in $\{C \in X_\Xi | C \le A\}$ and hence are contained in some finite union of sets of the form $\{\gamma_{-t}(A_j)|t \in \mathcal{T}^+\}$ with $A_j \in X_\Xi$. (See 2.13.) Then, by the argument in 4.8, we have that the elements $a_2a_1(\tilde{p}_{B,A}\xi_B)$ (with B as above) form a locally finite family in M and their supports are all contained in some set in $\mathcal{S}(\sum_{i\in I-K}(-\mathbf{N})\alpha_i + \sum_{i\in K}\mathbf{Z}\alpha_i)$, independent of B; moreover, we clearly have $a_2a_1(A) = \sum_{B\in X_\Xi;B\le A}a_2a_1(\tilde{p}_{B,A}\xi_B)$. By the continuity of a_3 (see 3.12), the elements $a_4a_3a_2a_1(\tilde{p}_{B,A}\xi_B)$ (with B as above) form a locally finite family in M and we have

$$a_4a_3a_2a_1(A) = \sum_{B\in X_\Xi;B\le A} a_3a_2a_1(\tilde{p}_{B,A}\xi_B).$$

Using 4.17, we deduce that the elements $\mathrm{res}_K^{-1}(\overline{\tilde{p}_{B,A}}\xi'_B)$ (with B as above) form a locally finite family in M and $a(A) = \sum_{B\in X_\Xi;B\le A}\mathrm{res}_K^{-1}(\overline{\tilde{p}_{B,A}}\xi'_B)$. (a) is proved.

Now (b) follows from (a) using the following statement, whose verification is immediate.

Let $(m^\lambda)_{\lambda\in\Lambda}$ be a locally finite family of elements of M. If $m^\lambda \in \tilde{M}^K$ for any $\lambda \in \Lambda$, then $\sum_{\lambda\in\Lambda} m^\lambda \in \tilde{M}^K$.

The lemma is proved.

Lemma 4.19. *Let $m \in M^K_\le$.*

(a) *The family of elements $\overline{m_A}a(A)$ (where A runs through X_Ξ) is locally finite in M and $a(m) = \sum_A \overline{m_A}a(A)$.*

(b) *We have $a(m) \in \tilde{M}^K$.*

The proof is along the same lines as that of 4.18. We can find $C \in X_\Xi$ such that $\mathrm{supp}(m) \subset \{B \in X_\Xi|B \le C\}$. When A varies through X_Ξ, the elements m_AA form a locally finite family in M^K and their supports are contained in $\{B \in X_\Xi|B \le C\}$ and hence are contained in some finite union of sets of the form $\{\gamma_{-t}(A_j)|t \in \mathcal{T}^+\}$ with $A_j \in X_\Xi$. (See 2.13.) Then, by the argument in 4.8, we have that the elements $a_2a_1(m_AA)$ (with $A \in X_\Xi$) form a locally finite family in M and their support is contained in some set in $\mathcal{S}(\sum_{i\in I-K}(-\mathbf{N})\alpha_i + \sum_{i\in K}\mathbf{Z}\alpha_i)$, independent of A; moreover, we clearly have $a_2a_1(m) = \sum_{A\in X_\Xi}a_2a_1(m_AA)$. By the continuity of a_3 (see 3.12), the elements $a_4a_3a_2a_1(m)$ (with $A \in X_\Xi$) form a locally finite family in M and we have $a_4a_3a_2a_1(m) = \sum_{A\in X_\Xi}a_4a_3a_2a_1(m_AA)$. This proves (a).

Now (b) follows from (a), using 4.18(b), by the same argument as in the proof of 4.18(b). The lemma is proved.

4.20. From 4.19 we see that the image of the homomorphism $a : M^K_\le \to M$ is contained in $\tilde{M}^K$. Using the isomorphism $\mathrm{res}_K : \tilde{M}^K \to M^K$, we deduce that there is a unique map $a' : M^K_{\leq} \to M^K$ such that $\mathrm{res}_K^{-1}(a'(m)) = a(m)$ for all $m \in M^K_{\leq}$. a' is $\mathcal{H}$-antilinear since a is $\mathcal{H}$-antilinear. By 4.17, we have $a'(\xi_A) = \xi'_A, \quad a'(\xi'_A) = \xi_A$.

Lemma 4.21. *Let $A \in X_\Xi$.*

(a) *The family of elements $\overline{\tilde{p}_{B,A}}\xi'_B$ (where B varies through $\{B \in X_\Xi; B \leq A\}$) is locally finite in M^K and we have $a'(A) = \sum_{B\in X_\Xi; B\leq A} \overline{\tilde{p}_{B,A}}\xi'_B$.*

(b) *We have $\mathrm{supp}(a'(A)) \subset \{B \in X_\Xi | B \leq A\}$ and the coefficient of A in $a'(A)$ is 1.*

We prove (a). Using 4.18, we see that it is enough to verify the following statement.

Let $(m_\lambda)_{\lambda\in\Lambda}$ be a family of elements of M^K such that $(\mathrm{res}_K^{-1}(m_\lambda))_{\lambda\in\Lambda}$ is a locally finite family of elements of M. Then $(m_\lambda)_{\lambda\in\Lambda}$ is a locally finite family of elements of M^K and $\mathrm{res}_K^{-1}(\sum_\lambda m_\lambda) = \sum_\lambda \mathrm{res}_K^{-1}(m_\lambda)$.

The verification is immediate.

We prove (b). The coefficient of C in $a'(A)$ is $\sum_{B\in X_\Xi; B\leq A} \overline{\tilde{p}_{B,A}} q_{C,B}$. This is zero unless $C \leq B$ for some B in the sum. Hence it is zero unless $C \leq A$. If now $C = A$, the only contribution to the coefficient is from $B = A$ and it gives 1 since $q_{A,A} = \tilde{p}_{A,A} = 1$. The lemma is proved.

Lemma 4.22. *Let $m \in M^K_{\leq}$.*

(a) *The family of elements $\overline{m_A}a'(A)$ (where A runs through X_Ξ) is locally finite in M^K and $a'(m) = \sum_A \overline{m_A}a'(A)$.*

(b) *We have $a'(m) \in M^K_{\leq}$.*

The proof of (a) is entirely similar to that of 4.21(a). (We use 4.19 instead of 4.18.)

We prove (b). We can find $C \in X_\Xi$ such that $\mathrm{supp}(m) \subset \{B \in X_\Xi | B \leq C\}$. Now let $D \in \mathrm{supp}(a'(m))$. Then there exists A such that $m_A \neq 0$ (hence $A \leq C$) and $D \in \mathrm{supp}(a'(A))$ (hence $D \leq A$ by 4.21(b)). Thus, we have $D \leq C$ and (b) is proved.

4.23. We now prove Theorem 4.14. Lemma 4.22(b) shows that there is a unique homomorphism $b : M^K_{\leq} \to M^K_{\leq}$ such that $b(m) = a'(m)$ for all $m \in M^K_{\leq}$. Now b is $\mathcal{H}$-antilinear since a' is $\mathcal{H}$-antilinear. From 4.17(a) we see that

(a) $b(e_{\epsilon',K}) = e_{\epsilon',K}$ *for any* $\epsilon' \in \mathbf{S}_{\tilde{\epsilon}}$.

From 4.21(b) we see that

(b) *for any $A \in X_\Xi$, $b(A)$ is of the form $A+$ an $\mathcal{A}$-linear combination of elements $A' \in X_\Xi$ with $A' < A$.*

From 4.22(a) we see that

(c) *for any $m \in M^K_{\leq}$, we have* $b(m) = \sum_A \overline{m_A} b(A)$.

We show that b is continuous in the sense of 4.13. Let $(F_\lambda)_{\lambda\in\Lambda}$ be a family of elements of $M^K_{\leq}$ which is locally finite (in M^K) and is such that $\bigcup_{\lambda\in\Lambda} \mathrm{supp}(F_\lambda) \subset \{A \in X_\Xi | A \leq A_0\}$ for some $A_0 \in X_\Xi$. From (b),(c) we see that $\bigcup_{\lambda\in\Lambda} \mathrm{supp}(b(F_\lambda)) \subset \{A \in X_\Xi | A \leq A_0\}$. Let $B \in X_\Xi$. From (b),(c) we see that

$$\{\lambda \in \Lambda | B \in \mathrm{supp}(b(F_\lambda))\} \subset \bigcup_{C\in X_\Xi; B\leq C\leq A_0} \{\lambda \in \Lambda | C \in \mathrm{supp}(F_\lambda)\}$$

so that $\{\lambda \in \Lambda | B \in \mathrm{supp}(b(F_\lambda))\}$ is finite. Thus, $(c(F_\lambda))_{\lambda\in\Lambda}$ is locally finite in M^K. Finally, from (c) we see that $c(\sum_\lambda F_\lambda) = \sum_\lambda c(F_\lambda)$. Thus, the continuity of b is established. The existence part of 4.14(a) is proved.

To establish the uniqueness part of 4.14(a), it suffices to verify the following statement.

(d) *Let* $c : M^K_{\leq} \to M^K_{\leq}$ *be a* $\mathcal{H}$*-antilinear continuous map such that* $c(e_{\epsilon',K}) = 0$ *for any* $\epsilon' \in \mathbf{S}_{\tilde{\epsilon}}$*. Then* $c = 0$.

From the definition of ξ_B and the assumptions on c it follows that $c(\xi_B) = 0$ for all $B \in X_\Xi$. For a fixed $B \in X_\Xi$, we consider the family $m_A = \tilde{p}_{A,B}\xi_A$ (see 4.16) in $M^K_{\leq}$ indexed by $\{A \in X_\Xi | A \leq B\}$. This family is locally finite in M^K and the union of supports of the elements in the family is bounded above (by B). By the continuity of c we have

$$c(B) = c(\sum_{A \in X_\Xi; A \leq B} \tilde{p}_{A,B}\xi_A) = \sum_{A \in X_\Xi; A \leq B} c(\tilde{p}_{A,B}\xi_A) = 0.$$

Thus, $c(B) = 0$ for any $B \in X_\Xi$.

Let $m \in M^K_{\leq}$. The family $(m_B B)_{B \in \Xi}$ in $M^K_{\leq}$ is locally finite in M^K and the union of supports of the elements in the family is bounded above. By the continuity of c we have $c(m) = \sum_{B \in X_\Xi} c(m_B B) = 0$. Thus, $c = 0$ and (d) is proved. Hence, 4.14(a) is proved.

Now 4.14(b) is just (b) above. It remains to prove 4.14(c). The $\mathcal{H}$-linear map $c = b^2 - 1 : M^K_{\leq} \to M^K_{\leq}$ satisfies the hypotheses of (d). Hence it is 0. Thus, $b^2 = 1$. Theorem 4.14 is proved.

4.24. For any $A, B \in X_\Xi$ we define $R_{A,B} \in \mathcal{A}$ by $b(B) = \sum_{A \in X_\Xi} R_{A,B} A$.

5. $\mathcal{A}[\mathcal{T}']$-module structure on M^K

5.1. Let $\mathcal{A}[\mathcal{T}']$ (resp. $\mathcal{A}[\mathcal{T}]$) be the group algebra of $\mathcal{T}'$ (resp. $\mathcal{T}$) with coefficients in $\mathcal{A}$. The basis element of $\mathcal{A}[\mathcal{T}']$ corresponding to $t \in \mathcal{T}'$ is denoted by $\lfloor t \rfloor$. The $\mathcal{A}$-module structure on M extends to a $\mathcal{A}[\mathcal{T}']$-module structure by

$$\lfloor t \rfloor m = \sum_{A \in X} m_{A-t} A = \sum_{A \in X} m_A (A + t)$$

for $m \in M, t \in \mathcal{T}'$. It is easy to check that, if $t \in \mathcal{T}$, then the action of t on M is $\mathcal{H}$-linear.

If $i \in I$ and $H \in \mathfrak{F}^i$, then $M_{i,\leq}$ and $M_{i,\geq}$ are $\mathcal{A}[\mathcal{T}']$-submodules of M; using the definitions we see that

(a) $$\theta_H(\lfloor t \rfloor m) = \lfloor t r_i \rfloor \theta_H(m)$$

for all $m \in M_{i,\leq}, t \in \mathcal{T}'$. For any $U \subset \mathcal{T}$, $M(U)$ (see 3.8) is a $\mathcal{A}[\mathcal{T}']$-submodule of M; using (a), we see that, for any $m \in M(U^-_K)$, we have

(b) $$\theta_{H_1,H_2,\dots,H_p}(\lfloor t \rfloor m) = \lfloor t' \rfloor \theta_{H_1,H_2,\dots,H_p}(m)$$

where $t' = t r_{i_1} r_{i_2} \dots r_{i_p} = t w^K_0 w^I_0$. (Notation of 3.14.) If $\epsilon \in E$ is a special point and $t \in \mathcal{T}'$, we see, using the definitions, that, for any $m \in M$ we have

(c) $$\phi_\epsilon(\lfloor t \rfloor m) = \lfloor t w^I_0 \rfloor \phi_\epsilon(m).$$

(Notation of 3.15.) Using (b),(c), we see that, in the setup of 3.18, we have

(d) $$\phi_\epsilon \theta_{H_1,H_2,\dots,H_p}(\lfloor t \rfloor m) = \lfloor t w^K_0 \rfloor \phi_\epsilon \theta_{H_1,H_2,\dots,H_p}(m)$$

for any $t \in \mathcal{T}'$ and any $m \in M(U^-_K)$.

5.2. We show that $\tilde{M}^K$ is a $\mathcal{A}[\mathcal{T}']$-submodule of M. Indeed, let $m \in \tilde{M}^K$ and $t \in \mathcal{T}'$. Then $\lfloor t \rfloor m = \sum_{A \in X} m'_A A$ where $m'_A = m_{A-t}$. Let $\omega \in \Omega^K$, let w be the corresponding element of W^I and let $A \in X$. We have $tw^{-1} = t + t'$ where $t' \in \mathcal{T}^K$. We have

$$\begin{aligned} m'_{A\omega} &= m_{A\omega - t} = m_{(A - tw^{-1})\omega} = m_{(A-t-t')\omega} = v^{d_K(A-t,(A-t-t')\omega)} m_{A-t} \\ &= v^{d_K(A-t,A\omega-t)} m'_A = v^{d_K(A,A\omega)} m'_A \end{aligned}$$

and our assertion follows.

5.3. The $\mathcal{A}$-module structure on M^K extends to a $\mathcal{A}[\mathcal{T}']$-module structure by

$$\lfloor t \rfloor \circ m = \sum_{A \in X_\Xi} m_{\gamma_{-t} A} A = \sum_{A \in X_\Xi} m_A (\gamma_t A).$$

Note that $m \mapsto \lfloor t \rfloor \circ m$ depends only on the coset of t modulo $\mathcal{T}^K$. Hence, there is a unique $\mathcal{A}[\mathcal{T}/\mathcal{T}^K]$-module structure $a[t], m \mapsto a\lfloor \dot{t} \rfloor \circ m$ on M^K where $t \in \mathcal{T}/\mathcal{T}^K$, $\dot{t}$ is a representative of t in $\mathcal{T}$ and $[t]$ denotes the basis element of the group algebra $\mathcal{A}[\mathcal{T}/\mathcal{T}^K]$ defined by t.

Lemma 5.4. *Let $t \in \mathcal{T}$. For $m \in \tilde{M}^K$ we have $\lfloor t \rfloor \circ (\mathrm{res}_K(m)) = v^{\mu_K(t)} \mathrm{res}_K(\lfloor t \rfloor m)$ where res_K is as in 4.6*(a) *and $\mu_K(t)$ is as in 4.10*(a).

We have

$$\lfloor t \rfloor \circ \mathrm{res}_K(m) = \lfloor t \rfloor \circ \left(\sum_{A \in X_\Xi} m_A A \right) = \sum_{A \in X_\Xi} m_{\gamma_{-t} A} A,$$

$$v^{\mu_K(t)} \mathrm{res}_K(\lfloor t \rfloor m) = v^{\mu_K(t)} \mathrm{res}_K \left(\sum_{A \in X} m_{A-t} A \right) = v^{\mu_K(t)} \sum_{A \in X_\Xi} m_{A-t} A.$$

We must only prove that

(a) $$m_{\gamma_{-t} A} = v^{\mu_K(t)} m_{A-t}$$

for any $A \in X_\Xi$. Let $\omega \in \Omega^K$ be such that $(A-t)\omega \in X_\Xi$. We have

(b) $$m_{\gamma_{-t} A} = m_{(A-t)\omega} = v^{d_K(A-t,(A-t)\omega)} m_{A-t}$$

since $m \in \tilde{M}^K$. We show that

(c) $$d_K(A-t,(A-t)\omega) = \mu_K(t).$$

We have

$$d_K(A-t,(A-t)\omega) = d_K(A-t,A) + d_K(A,(A-t)\omega) = \mu_K(t) + d_K(A,(A-t)\omega).$$

Thus, (c) would follow if we show that $d_K(A,(A-t)\omega) = 0$. More generally, we show that $d_K(A,B) = 0$ for any $A, B \in X_\Xi$. If $H \in \mathfrak{F}^K$, then any K-alcove is contained either in E_H^+ or in E_H^-. It follows that A, B lie on the same side of H. From the definition of d_K it therefore follows that $d_K(A,B) = 0$. Thus, (c) is proved.

We introduce (c) into (b). We obtain (a). The lemma is proved.

5.5. Since $\mathrm{res}_K : \tilde{M}^K \to M^K$ is an isomorphism of $\mathcal{H}$-modules, we see from 5.4 and 5.2 that, for $t \in \mathcal{T}$, the map $m \mapsto \lfloor t \rfloor \circ m$ from M^K into itself is $\mathcal{H}$-linear.

Lemma 5.6. (a) *$M^K_{\leq} = M^K_{\leftarrow}$ and $M^K_{\geq} = M^K_{\rightarrow}$ are $\mathcal{A}[\mathcal{T}']$-submodules of M^K.*
(b) *For any subset $U \subset \mathcal{T}$, $M(U)$ (see 3.8) is an $\mathcal{A}[\mathcal{T}']$-submodule of M.*
(c) *$\tilde{M}^K_{\leftarrow}$ and $\tilde{M}^K_{\rightarrow}$ are $\mathcal{A}[\mathcal{T}']$-submodules of M.*

We prove (a). Let $t \in \mathcal{T}'$. If $m \in M^K_{\leftarrow}$ then there exist $A_1, A_2, \dots, A_p$ in X_Ξ with $\mathrm{supp}(m) \subset \{\gamma_{-t'}(A_u) | u \in [1,p], t' \in \mathcal{T}^+\}$. Let $B \in \mathrm{supp}(\gamma_t(m))$. Then $\gamma_{-t}(B) \in \mathrm{supp}(m)$ so that $\gamma_{-t}(B) = \gamma_{-t'}(A_u)$ for some $u \in [1,p], t' \in \mathcal{T}^+$. Then $B = \gamma_{-t'}(\gamma_t A_u)$. This shows that $\lfloor t \rfloor \circ m \in M^K_{\leftarrow}$. Thus, $M^K_{\leftarrow}$ is a $\mathcal{A}[\mathcal{T}']$-submodule of M^K. Similarly, $M^K_{\rightarrow}$ is a $\mathcal{A}[\mathcal{T}']$-submodule of M^K. This proves (a). The proof of (b) is entirely similar.

To prove (c) it remains to note that $\tilde{M}^K_{\leftarrow}$ and $\tilde{M}^K_{\rightarrow}$ are intersections of $\tilde{M}^K$ with a subspace of the form $M(U)$ and both these subspaces are $\mathcal{A}[\mathcal{T}']$-submodules of M.

Proposition 5.7. *For $t \in \mathcal{T}'$, the map $m \mapsto \lfloor t \rfloor \circ m$ from $M^K_{\leq}$ into itself commutes with $b : M^K_{\leq} \to M^K_{\leq}$.*

With the notation in 4.14 we have $a_4a_3a_2a_1 = c_1 \mathrm{res}_K^{-1} c_2 b$ where $c_1 : \tilde{M}^K \to M, c_2 : M^K_{\leq} \to M^K$ are the obvious inclusions. Let $m \in M^K_{\leq}$. We have

$$\begin{aligned}
c_1 \mathrm{res}^{-1} c_2 b(\lfloor t \rfloor \circ m) &= a_4a_3a_2a_1(\lfloor t \rfloor \circ m) = a_4a_3a_2(v^{\mu_K(t)} \lfloor t \rfloor a_1(m)) \\
&= v^{-\mu_K(t)} a_4 a_3 \lfloor t \rfloor a_2 a_1(m) = v^{-\mu_K(t)} a_4 \lfloor t w_0^K \rfloor a_3 a_2 a_1(m) \\
&= v^{-\mu_K(t)} \lfloor t w_0^K \rfloor a_4 a_3 a_2 a_1(m) \\
&= v^{-\mu_K(t)} \lfloor t w_0^K \rfloor c_1 \mathrm{res}_K^{-1} c_2 b(m) = v^{-\mu_K(t)} c_1 \lfloor t w_0^K \rfloor \mathrm{res}_K^{-1} c_2 b(m) \\
&= v^{-\mu_K(t)} v^{-\mu_K(t w_0^K)} c_1 \mathrm{res}_K^{-1}(\lfloor t \rfloor \circ (c_2 b(m))) \\
&= v^{-\mu_K(t)} v^{-\mu_K(t w_0^K)} c_1 \mathrm{res}_K^{-1} c_2(\lfloor t \rfloor \circ b(m)).
\end{aligned}$$

(The second and eighth equality follow from 5.4; the fourth equality follows from 5.1(d); the other equalities follow from the definitions. We have also used the identity $\lfloor t w_0^K \rfloor \circ (c_2 b m) = \lfloor t \rfloor \circ (c_2 b m)$ which follows from the fact that $t w_0^K = t \mod \mathcal{T}^K$.)

Since $c_1 \mathrm{res}_K^{-1} c_2$ is injective, it follows that $b(\lfloor t \rfloor \circ m) = v^{-\mu_K(t)} v^{-\mu_K(t w_0^I)} \lfloor t \rfloor \circ (bm)$. It remains to use the identity $\mu_K(t) + \mu_K(t w_0^K) = 0$ (see 4.10(c)).

6. The isomorphism $\theta_\epsilon : M^K_{\leq} \to M^K_{\geq}$

6.1. In this section we fix $\epsilon \in \mathbf{S}_{\tilde{\epsilon}}$. We define an involution $\kappa_\epsilon : M^K \to M^K$ by

$$\kappa_\epsilon(\sum_{A \in X_\Xi} m_A A) = \sum_{A \in X_\Xi} \overline{m_A} \kappa_\epsilon(A) = \sum_{A \in X_\Xi} \overline{m_{\kappa_\epsilon(A)}} A$$

(notation of 2.3).

Lemma 6.2. *κ_ϵ restricts to a bijection $M^K_{\rightarrow} \xrightarrow{\sim} M^K_{\leftarrow}$, or, equivalently, $M^K_{\geq} \xrightarrow{\sim} M^K_{\leq}$, see 4.13(a), denoted again by κ_ϵ.*

Using the identity $\gamma_t \kappa_\epsilon = \kappa_\epsilon \gamma_{-t} : E \to E$ for $t \in \mathcal{T}$ (see 2.12(d)) we see immediately that κ_ϵ maps $M^K_{\rightarrow}$ isomorphically onto $M^K_{\leftarrow}$. This proves (b).

Lemma 6.3. *$\kappa_\epsilon : M^K_{\geq} \xrightarrow{\sim} M^K_{\leq}$ and its inverse $\kappa_\epsilon^{-1} : M^K_{\leq} \xrightarrow{\sim} M^K_{\geq}$ (which is just the restriction of $\kappa_\epsilon : M^K \to M^K$) are continuous in the sense of 4.13.*

The proof is a refinement of the argument in the proof of 6.2. Namely, let $(F_\lambda)_{\lambda\in\Lambda}$ be a family of elements of $M^K_{\geq}$ which is locally finite (in M^K) and is such that $\bigcup_{\lambda\in\Lambda}\mathrm{supp}(F_\lambda)$ is bounded below. By 2.13(c), there exist $A_1, A_2, \ldots, A_p$ in X_Ξ such that the following holds. If $B \in \mathrm{supp}(F_\lambda)$ for some $\lambda \in \Lambda$, then $B = \gamma_t A_u$ for some $u \in [1,p]$ and some $t \in \mathcal{T}^+$. Using the identity $\gamma_t\kappa_\epsilon = \kappa_\epsilon\gamma_{-t} : E \to E$ for $t \in \mathcal{T}$ (see 2.12(d)) we deduce that the following holds. If $B \in \mathrm{supp}(\kappa_\epsilon F_\lambda)$ (that is, $\kappa_\epsilon B \in \mathrm{supp}(F_\lambda)$) for some $\lambda \in \Lambda$, then $B = \gamma_{-t}\kappa_\epsilon A_u$ for some $u \in [1,p]$ and some $t \in \mathcal{T}^+$. Using now 2.13(b), we deduce that $\bigcup_{\lambda\in\Lambda}\mathrm{supp}(\kappa_\epsilon(F_\lambda))$ is bounded above. The family $(\kappa_\epsilon(F_\lambda))_{\lambda\in\Lambda}$ in $M^K_{\leq}$ is locally finite in M^K since $(F_\lambda)_{\lambda\in\Lambda}$ is locally finite in M^K and $\kappa_\epsilon : X_\Xi \to X_\Xi$ is a bijection. This also shows that $\kappa_\epsilon(\sum_\lambda F_\lambda) = \sum_\lambda \kappa_\epsilon(F_\lambda)$. The continuity of κ_ϵ is proved. The continuity of κ_ϵ^{-1} is proved in an entirely similar way.

6.4. Let $s \mapsto s^*$ be the involution of S defined by the following requirement: if Z is a face of type s then the image of Z under the map $\epsilon + x \mapsto \epsilon - x$ (with $x \in T$) is a face of type s^*. (It follows that $\kappa_\epsilon(Z)$ is a face of type s^*.)

Lemma 6.5. (a) *Let $m \in M^K$ and let $s \in S$. We have $\tilde{T}_s(\kappa_\epsilon(m)) = \kappa_\epsilon(\tilde{T}_{s^*}^{-1}(m))$.*

(b) *For any $\epsilon' \in \mathbf{S}_{\tilde{\epsilon}}$ we have $\kappa_\epsilon(e_{\epsilon',K}) = v^{\nu_K} e_{\epsilon'',K}$ where ν_K is the number of elements of $\bar{\mathcal{F}} - \bar{\mathcal{F}}^K$ and $\epsilon'' = \kappa_\epsilon(\epsilon') \in \mathbf{S}_{\tilde{\epsilon}}$.*

(c) *For any $t \in \mathcal{T}'$ we have $\lfloor t\rfloor \circ (\kappa_\epsilon m') = \kappa_\epsilon(\lfloor -t\rfloor \circ m')$.*

We prove (a). Let $A \in X_\Xi$ and let $H \in \mathfrak{F}$ be the hyperplane separating A, sA. Then $H_1 = \kappa_\epsilon(H)$ is the hyperplane in $\mathfrak{F}$ separating $\kappa_\epsilon(A)$ from $\kappa_\epsilon(sA) = s^*\kappa_\epsilon(A)$. Now H is a wall of Ξ (that is, $H \in \mathfrak{F}^K$) if and only if H_1 is a wall of Ξ (that is, $H_1 \in \mathfrak{F}^K$); see 2.3(b). In this case we have $\kappa_\epsilon(\tilde{T}_{s^*}^{-1}(A)) = \kappa_\epsilon(v^{-1}A) = v\kappa_\epsilon(A) = \tilde{T}_s(\kappa_\epsilon(A))$. Assume now that $H, H_1 \notin \mathfrak{F}^K$. Using 2.3(d), we see that $s^* \in \mathcal{L}(A)$ if and only if $s \notin \mathcal{L}(\kappa_\epsilon(A))$. Hence, $\kappa_\epsilon(\tilde{T}_{s^*}^{-1}(A)) = \kappa_\epsilon(s^*A) = s\kappa_\epsilon(A) = \tilde{T}_s(\kappa_\epsilon(A))$ if $s^* \in \mathcal{L}(A)$;

$$\begin{aligned}\kappa_\epsilon(\tilde{T}_{s^*}^{-1}(A)) &= \kappa_\epsilon(s^*A + (v^{-1} - v)A) = \kappa_\epsilon(s^*A) + (v - v^{-1})\kappa_\epsilon(A)\\ &= s\kappa_\epsilon(A) + (v - v^{-1})\kappa_\epsilon(A) = \tilde{T}_s(\kappa_\epsilon(A)),\end{aligned}$$

if $s^* \notin \mathcal{L}(A)$. Thus we have $\kappa_\epsilon(\tilde{T}_{s^*}^{-1}(A)) = \tilde{T}_s(\kappa_\epsilon(A))$ in all cases. We now use the continuity of κ_ϵ; (a) follows.

We prove (b). Using 2.3(c), we see that it suffices to show that $d(A, A^+_{\epsilon'}) = \nu_K - d(\kappa_\epsilon(A), A^+_{\epsilon''})$ for any $A \in D_\Xi(\epsilon')$. We have $\nu_K = d(A^!_{\epsilon''}, A^+_{\epsilon''})$ hence $\nu_K - d(\kappa_\epsilon(A), A^+_{\epsilon''}) = d(A^!_{\epsilon''}, \kappa_\epsilon(A)) = d(\kappa_\epsilon(A^+_{\epsilon''}), \kappa_\epsilon(A))$. Hence it is enough to show that

$d(A, B) = -d(\kappa_\epsilon(A), \kappa_\epsilon(B))$ for any $A, B \in X_\Xi$. The proof is similar to that of 4.10(f). We note that $H \mapsto \kappa_\epsilon(H)$ is a bijection between the set of hyperplanes in $\mathfrak{F}$ that separate A from B and the set of hyperplanes in $\mathfrak{F}$ that separate $\kappa_\epsilon(A)$ from $\kappa_\epsilon(B)$ (these hyperplanes are automatically in $\mathfrak{F}^K$). Moreover, the signs attached to corresponding hyperplanes are opposite. The desired equality follows; (b) is proved.

(c) follows from 2.12(d).

6.6. Let $\theta_\epsilon : M^K_\leq \xrightarrow{\sim} M^K_\geq$ be the bijection defined by the equality

$$b = v^{-\nu_K}\kappa_\epsilon\theta_\epsilon : M^K_\leq \xrightarrow{\sim} M^K_\leq$$

where $\kappa_\epsilon : M^K_\geq \xrightarrow{\sim} M^K_\leq$ is as in 6.2.

Proposition 6.7. (a) *For any $s \in S$ and $m \in M^K_\leq$, we have $\theta_\epsilon(\tilde{T}_{s^*}(m)) = \tilde{T}_s(\theta_\epsilon(m))$.*

(b) *For any $\epsilon' \in \mathbf{S}_{\tilde{\epsilon}}$ we have $\theta_\epsilon(e_{\epsilon',K}) = e_{\epsilon'',K}$ where $\epsilon'' = \kappa_\epsilon(\epsilon')$.*

(c) *For any $t \in \mathcal{T}'$ and any $m' \in M^K_\leq$ we have $\lfloor t\rfloor \circ (\theta_\epsilon m') = \theta_\epsilon(\lfloor -t\rfloor \circ m')$.*

(a) follows from the $\mathcal{H}$-antilinearity of (b) and 6.5(a); (b) follows from the definition of b (see 4.14(a)) and 6.5(b); (c) follows from 5.7 and 6.5(c).

Proposition 6.8. (a) *The map $\theta_\epsilon : M^K_\leq \xrightarrow{\sim} M^K_\geq$ is continuous in the sense of 4.13.*

(b) *The map $\theta_\epsilon^{-1} : M^K_\geq \xrightarrow{\sim} M^K_\leq$ is continuous in the sense of 4.13.*

This follows from the continuity of $\kappa_\epsilon, \kappa_\epsilon^{-1}$ (see 6.3) and the continuity of $b = b^{-1}$.

The following characterization of θ_ϵ (similar to that of b) follows from 4.14(a).

Proposition 6.9. *θ_ϵ is the unique continuous map $M^K_\leq \to M^K_\geq$ that satisfies 6.7*(a),(b).

6.10. Let M^K_c be the $\mathcal{A}$-submodule of M^K consisting of all elements $m \in M^K$ such that $\mathrm{supp}(m)$ is a finite set. Then M^K_c is a $\mathcal{A}[\mathcal{T}]$-submodule and a $\mathcal{H}$-submodule of M^K.

Lemma 6.11. *Let $\varrho = \prod_{h\in\bar{\mathcal{F}}-\bar{\mathcal{F}}^K}(1 - v^{-2-\mu_K(\alpha_h)}\lfloor\alpha_h\rfloor) \in \mathcal{A}[\mathcal{T}]$. For any $B \in X_\Xi$ we have $\varrho \circ \theta_\epsilon(B) \in M^K_c$.*

For $i \in I$ and $H \in \mathfrak{F}^i$, we define a $\mathcal{A}$-linear map $\hat{\theta}_H : M \to M$ by

(a) $\hat{\theta}_H(\sum_{A\in X} m_A A) = \sum_{A\in X} m_A((v^{-1}A^0 + A^1) - v^{-1}(v^{-1}A^1 + A^2))$

where A^z are as in 3.3. From 3.3(a) we see that $\theta_H(A) - v^{-2}\lfloor\alpha_i\rfloor\theta_H(A) = \hat{\theta}_H(A)$ for any $A \in X$. Hence $\theta_H(m) - v^{-2}\lfloor\alpha_i\rfloor\theta_H(m) = \hat{\theta}_H(m)$ for any $m \in M_{i,\leq}$.

Let $P_B = \sum_{\omega\in\Omega^K} v^{d_K(B,B\omega)}B\omega \in \tilde{M}^K$ (see 4.7); then $\mathrm{res}_K(P_B) = B$. The Ω^K-orbit of B decomposes into finitely many $\mathcal{T}^K$-orbits and we fix a set of representatives $B_1, B_2, \dots, B_N$ for these $\mathcal{T}^K$-orbits. We have

$$P_B = \sum_{j=1}^N v^{d_K(A,B_j)} \sum_{t\in\mathcal{T}^K} v^{\mu_K(t)}(B_j + t).$$

In the setup of 3.13, we have $b(B) = \mathrm{res}_K(\phi_\epsilon\theta_{\epsilon+i_p}\dots\theta_{\epsilon+i_2}\theta_{\epsilon+i_1}(P_B)$. Let $\varrho'_1 = \prod_{h\in\bar{\mathcal{F}}-\bar{\mathcal{F}}^{K*}}(1 - v^{-2}\lfloor\alpha_h\rfloor)$ and let $\varrho'_2 = \prod_{h\in\bar{\mathcal{F}}-\bar{\mathcal{F}}^{K*}}(1 - v^2\lfloor-\alpha_h\rfloor)$.

Using several times 5.1(a), we have

$$\begin{aligned}
&\varrho'_1\theta_{\epsilon+i_p}\dots\theta_{\epsilon+i_1}P_B\\
&= (1 - v^{-2}\lfloor\alpha_{i_p}\rfloor)(1 - v^{-2}\lfloor\alpha_{i_{p-1}}r_{i_p}\rfloor)\dots(1 - v^{-2}\lfloor\alpha_{i_1}r_{i_2}r_{i_3}\dots r_{i_p}\rfloor)\theta_{\epsilon+i_p}\dots\theta_{\epsilon+i_1}P_B\\
&= (1 - v^{-2}\lfloor\alpha_{i_p}\rfloor)\theta_{\epsilon+i_p}(1 - v^{-2}\lfloor\alpha_{i_2}\rfloor)\theta_{\epsilon+i_2}(1 - v^{-2}\lfloor\alpha_{i_1}\rfloor)\theta_{\epsilon+i_1}P_B\\
&= \hat{\theta}_{\epsilon+i_p}\dots\hat{\theta}_{\epsilon+i_2}\hat{\theta}_{\epsilon+i_1}P_B = \sum_{j=1}^N v^{d_K(A,B_j)} \sum_{t\in\mathcal{T}^K} v^{\mu_K(t)}\hat{\theta}_{\epsilon+i_p}\dots\hat{\theta}_{\epsilon+i_2}\hat{\theta}_{\epsilon+i_1}(B_j + t),
\end{aligned}$$

hence, using 5.1(c):

$$\varrho_2'\phi_\epsilon\theta_{\epsilon+i_p}\dots\theta_{\epsilon+i_1}P_B = \phi_\epsilon\varrho_1'\theta_{\epsilon+i_p}\dots\theta_{\epsilon+i_1}P_B$$
$$= \phi_\epsilon\sum_{j=1}^{N} v^{d_K(A,B_j)}\sum_{t\in\mathcal{T}^K} v^{\mu_K(t)}\hat\theta_{\epsilon+i_p}\dots\hat\theta_{\epsilon+i_2}\hat\theta_{\epsilon+i_1}(B_j+t).$$

Using (a) and the definition of ϕ_ϵ, we see that there exist $B_{j,j'}\in X$ and $a_{j,j'}\in\mathcal{A}$ for $j\in[1,N], j'\in[1,N']$ such that $\phi_\epsilon\hat\theta_{\epsilon+i_p}\dots\hat\theta_{\epsilon+i_2}\hat\theta_{\epsilon+i_1}(B_j)=\sum_{j'\in[1,N']}a_{j,j'}B_{j,j'}$. It follows that

$$\phi_\epsilon\hat\theta_{\epsilon+i_p}\dots\hat\theta_{\epsilon+i_2}\hat\theta_{\epsilon+i_1}(B_j+t)=\sum_{j'\in[1,N']}a_{j,j'}(B_{j,j'}+tw_0^K)$$

for any $t\in\mathcal{T}^K$ (see 5.1). Thus we have

$$\varrho_2'\phi_\epsilon\theta_{\epsilon+i_p}\dots\theta_{\epsilon+i_1}P_B=\sum_{\substack{j\in[1,N]\\ j'\in[1,N']}} v^{-d_K(A,B_j)}\sum_{t\in\mathcal{T}^K}v^{-\mu_K(t)}\sum_{j'\in[1,N']}a_{j,j'}(B_{j,j'}+tw_0^K).$$

If we apply res_K to the right hand side of the last equality, we obtain an $\mathcal{A}$-linear combination of at most NN' alcoves in X_Ξ since for any $C\in X_\Xi$ and any j,j' there is at most one $t\in\mathcal{T}^K$ such that $C=B_{j,j'}+tw_0^K$. It follows that $\mathrm{res}_K(\varrho_2'\phi_\epsilon\theta_{\epsilon+i_p}\dots\theta_{\epsilon+i_1}P_B)\in M_c^K$. Let $\varrho_3'=\prod_{h\in\bar{\mathcal{F}}-\bar{\mathcal{F}}^{K*}}(1-v^2v^{\mu_K(\alpha_h)}\lfloor-\alpha_h\rfloor)$. Using 5.4, we have

$$\varrho_3'\circ b(B)=\varrho_3'\,\mathrm{res}_K(\phi_\epsilon\theta_{\epsilon+i_p}\dots\theta_{\epsilon+i_1}P_B)=\mathrm{res}_K(\varrho_2'\phi_\epsilon\theta_{\epsilon+i_p}\dots\theta_{\epsilon+i_1}P_B)\in M_c^K.$$

Applying κ_ϵ^{-1} to $\varrho_3'\circ b(B)\in M_c^K$, we deduce that $\rho\circ\theta_\epsilon(B)\in M_c^K$. The lemma is proved.

7. M_c^K as an induced $\mathcal{H}$-module

7.1. Let $\tilde M_c^K$ be the $\mathcal{A}$-submodule of $\tilde M^K$ which corresponds to M_c^K under the isomorphism $\mathrm{res}_K:\tilde M^K\xrightarrow{\sim}M^K$. This is then an $\mathcal{H}$-submodule of $\tilde M^K$. The elements P_A (see 4.3), where A runs over a set of representatives for the Ω^K-orbits in X form an $\mathcal{A}$-basis of $\tilde M_c^K$. (This follows from 4.3(d).)

7.2. In this section we fix $\epsilon\in\mathbf{S}_{\tilde\epsilon}$. This allows us to identify I with a subset of S: to $i\in I$ corresponds the element $s_i\in S$ such that the face $\delta_{s_i}(A_\epsilon^+)$ is supported by $\epsilon+i$.

Let $W^{(I)}$ be the (parabolic) subgroup of W generated by $\{s_i|i\in I\}$ and let $W^{(K)}$ be the (parabolic) subgroup of W generated by $\{s_i|i\in K\}$. We have $W^{(I)}=W_\epsilon$ with the notation of 1.1. We denote by W_K the set of elements $w\in W$ which have minimal length in the coset $W^{(K)}w$.

Lemma 7.3. *The alcoves $w^{-1}A_{\epsilon+x}^+=w^{-1}(A_\epsilon^++x)$ (for various $w\in W_K$ and various $x\in\mathcal{T}^{I-K}$) form a set of representatives for the Ω^K-orbits on X. Hence the elements $P_{w^{-1}A_{\epsilon+x}^+}$ (with w,x as above) form an $\mathcal{A}$-basis of $\tilde M_c^K$.*

For A,B in X we write $A\sim B$ instead of "A,B are in the same Ω^K-orbit".

If $A \in X$, then $A = w_1^{-1} A^+_{\epsilon+x_1}$ for a unique $w_1 \in W^{(I)}$ and a unique $x_1 \in \mathcal{T}$. We can write uniquely $w_1 = w'w$ where $w \in W_K, w' \in W^{(K)}$. Let $\omega' \in \Omega$ be the element corresponding to w' under j_ϵ; then $\omega' \in \Omega^K$. We have

$$\begin{aligned} A &= w^{-1}w'^{-1}(A^+_\epsilon + x) = w^{-1}((w'^{-1}A^+_\epsilon) + x) \\ &= w^{-1}(A^+_\epsilon \omega^{-1} + x) = w^{-1}(A^+_\epsilon + x_1)\omega^{-1} \end{aligned}$$

for some $x_1 \in \mathcal{T}$. Writing $x_1 = x' + x''$ where $x' \in \mathcal{T}^{I-K}, x'' \in \mathcal{T}^K$ we see that

$$A \sim w^{-1}(A^+_\epsilon + x_1) = w^{-1}A^+_\epsilon + x' + x'' \sim w^{-1}A^+_\epsilon + x'.$$

Thus, $A \sim w^{-1}A^+_\epsilon + x'$. It remains to verify the following statement:

Let $w, w' \in W_K$ and $x, x' \in \mathcal{T}^{I-K}$ be such that $w^{-1}(A^+_\epsilon + x) \sim w'^{-1}(A^+_\epsilon + x')$. Then $w = w'$ and $x = x'$. Our assumption implies $A^+_\epsilon + x \sim ww'^{-1}(A^+_\epsilon + x')$ and $A^+_\epsilon \sim ww'^{-1}(A^+_\epsilon + x') - x$. But the Ω^K-orbit of A^+_ϵ consists of the alcoves $w_1 A^+_\epsilon + x_1$ with $w_1 \in W^{(K)}, x_1 \in \mathcal{T}^K$. Thus, there exist $w_1 \in W^{(K)}, x_1 \in \mathcal{T}^K$ such that $ww'^{-1}(A^+_\epsilon + x') - x = w_1 A^+_\epsilon + x_1$. It follows that $ww'^{-1} = w_1$ and $x' - x = x_1$. This forces $w = w', x' = x$. The lemma is proved.

7.4. The restriction of $j_\epsilon : \Omega \xrightarrow{\sim} W$ (see 1.1) gives an isomorphism $x \to a^x$ of $\mathcal{T}$ onto a normal subgroup of W. For $x \in \mathcal{T}$ we set $\vartheta_x = \tilde{T}^{-1}_{a^{x_2}} \tilde{T}_{a^{x_1}} \in \mathcal{H}$ where $x_1, x_2 \in \mathcal{T}_{\mathrm{dom}}$ are such that $x = x_1 - x_2$. Then ϑ_x is a well defined element of $\mathcal{H}$. (See [L2, 2.6].)

Lemma 7.5. *Let $\epsilon' \in \epsilon + \mathcal{T}$. For $x \in \mathcal{T}$ we have $\vartheta_x(P_{A^+_{\epsilon'}}) = v^{2\mu_K(x)} P_{A^+_{\epsilon'+x}} = v^{\mu_K(x)} P_{\gamma_x A^+_{\epsilon'}}$.*

We prove the first equality in the lemma. Using the definition of ϑ_x and the linearity of the function $x \mapsto \mu_K(x)$ we see that it is enough to do that under the additional assumption that $x \in \mathcal{T}_{\mathrm{dom}}$. From [L1, 3.6] we see that $d(A^+_{\epsilon'}, a^x(A^+_{\epsilon'}) = l(a^x)$. Hence there exists a sequence $s_{(1)}, s_{(2)}, \dots, s_{(n)}$ in S such that $a^x = s_{(n)} s_{(n-1)} \dots s_{(1)}$, $n = l(a^x)$ and

$$A_{\epsilon'} < s_{(1)} A_{\epsilon'} < s_{(2)} s_{(1)} A_{\epsilon'} < \dots < s_{(n)} s_{(n-1)} \dots s_{(1)} A_{\epsilon'} = a^x A_{\epsilon'}.$$

For $A \in X$ and $s \in S$ such that $s \notin \mathcal{L}(A)$ we have

(a) $\tilde{T}_s P_A = P_{sA}$ if $L \notin \mathfrak{F}^K$; $\tilde{T}_s P_A = v^2 P_{sA}$ if $L \in \mathfrak{F}^K$.

Here L is the hyperplane separating A, sA. (The first equality is just 4.3(a); the second equality follows from 4.3(c), since we now have $P_A = v P_{sA}$ by 4.3(d).) Using this repeatedly we see that

$$\begin{aligned} \tilde{T}_{a^x} P_{A^+_{\epsilon'}} &= \tilde{T}_{s_{(n)}} \dots \tilde{T}_{s_{(1)}} P_{A^+_{\epsilon'}} = v^{2N} P_{s_{(n)} \dots s_{(1)} A^+_{\epsilon'}} = v^{2N} P_{a^x A^+_{\epsilon'}} \\ &= v^{2N} P_{A^+_{\epsilon'} + x} = v^{2N} P_{A^+_{\epsilon'+x}} \end{aligned}$$

where N is the number of consecutive pairs of alcoves in the sequence

$$A^+_{\epsilon'}, s_{(1)} A^+_{\epsilon'}, s_{(2)} s_{(1)} A^+_{\epsilon'}, \dots, s_{(n)} s_{(n-1)} \dots s_{(1)} A^+_{\epsilon'}$$

such that the hyperplane separating them is in $\mathfrak{F}^K$. In other words, we have

$$N = d_K(A^+_{\epsilon'}, s_{(n)} s_{(n-1)} \dots s_{(1)} A^+_{\epsilon'}) = d_K(A^+_{\epsilon'}, A^+_{\epsilon'+x}) = \mu_K(x).$$

The first equality in the lemma follows. We prove the second equality in the lemma. Let $A = A^+_{\epsilon'}$. Let $\omega \in \Omega^K$ be such that $(A + x)\omega \in X_\Xi$; then

$$P_{\gamma_x A} = P_{(A+x)\omega} = v^{-d_K(A+x, (A+x)\omega)} P_{A+x}$$

and it remains to use $-d_K(A+x,(A+x)\omega)=\mu_K(x)$ which follows from 5.4(c).

Lemma 7.6. *Let $\epsilon'\in\epsilon+\mathcal{T}$.*

(a) *If $x\in\mathcal{T}^K$, then $\vartheta_x P_{A^+_{\epsilon'}}=v^{\mu_K(x)}P_{A^+_{\epsilon'}}$.*

(b) *If $w\in W^{(K)}$, then $T_w^{-1}P_{A^+_{\epsilon'}}=v^{-l(w)}P_{A^+_{\epsilon'}}$.*

(c) *If $w\in W^{(I)}$, then $T_w^{-1}P_{A^+_{\epsilon'}}=v^{2n}P_{w^{-1}A^+_{\epsilon'}}$ for some $n\in\mathbf{Z}$.*

(a) follows from 7.5; (b) follows by repeated application of 4.3(c); finally, (c) follows by repeated application of 7.5(a), using the equality $d(w^{-1}A^+_{\epsilon'},A^+_{\epsilon'})=l(w)$. The lemma is proved.

Lemma 7.7. $\tilde{T}_{s_i}\vartheta_x-\vartheta_{xr_i}\tilde{T}_{s_i}=(v-v^{-1})\frac{\vartheta_x-\vartheta_{xr_i}}{1-\vartheta_{-\alpha_i}}$ *for $i\in I$ and $x\in\mathcal{T}$.*

(The last fraction is a well defined element of the $\mathcal{A}$-submodule of $\mathcal{H}$ spanned by $\{\vartheta_y|y\in\mathcal{T}\}$.) See [L2]. There is a slight difference between the equality in the lemma and that in [L2]; this is due to the fact that what was a left action in [L2] is now a right action, hence we have to adjust formulas accordingly.

Lemma 7.8. *Let $w'\in W^{(K)}$ and let $x\in\mathcal{T}^{I-K}$. We have $\tilde{T}_{w'}^{-1}\vartheta_x=\vartheta_x\tilde{T}_{w'}^{-1}\vartheta_{x-x\tilde{w}'^{-1}}$ plus an $\mathcal{A}$-linear combination of elements of the form $\vartheta_z\tilde{T}_{w''}\vartheta_u$ with $z\in\mathcal{T}^{I-K},u\in\mathcal{T}^K$ and $w''\in W^{(K)},l(w'')<l(w)$. Here $\tilde{w}'$ is the image of $\omega=j_\epsilon^{-1}(w')\in\Omega$ in $W^{(I)}$. (Note also that $x-x\tilde{w}'^{-1}\in\mathcal{T}^K$.)*

The proof (by induction on $l(w')$) is based on [L2]. We omit the details.

Lemma 7.9. *The elements $\tilde{T}_w^{-1}\vartheta_x$ (with $w\in W^{(I)}$ and $x\in\mathcal{T}$) form an $\mathcal{A}$-basis of $\mathcal{H}$.*

The arguments in [L2] show that the elements $\tilde{T}_{w^{-1}}\vartheta_x$ (with $w\in W^{(I)}$ and $x\in\mathcal{T}$) form an $\mathcal{A}$-basis of $\mathcal{H}$. But this basis is related to the family of elements in the lemma by an "upper triangular matrix with 1 on diagonal". The lemma follows.

Lemma 7.10. (a) *The elements $\tilde{T}_w^{-1}\vartheta_x\tilde{T}_{w'}^{-1}\vartheta_{x'}$, $w\in W_K,w'\in W^{(K)},x\in\mathcal{T}^{I-K}$, $x'\in\mathcal{T}^K$, form an $\mathcal{A}$-basis of $\mathcal{H}$.*

(b) *The elements*

(∗) $\tilde{T}_w^{-1}\vartheta_x(\tilde{T}_{w'}^{-1}\vartheta_{x'}-v^{-l(w')+\mu_K(x')})$, $w\in W_K,x\in\mathcal{T}^{I-K},w'\in W^{(K)},x'\in\mathcal{T}^K,(w',x)\neq(1,0)$, *and*

(∗∗) $\tilde{T}_w^{-1}\vartheta_x$, $w\in W_K,x\in\mathcal{T}^{I-K}$ *form an $\mathcal{A}$-basis of $\mathcal{H}$.*

From the definition of $W_K,W^{(K)}$ we have $\tilde{T}_{w'}\tilde{T}_w=\tilde{T}_{w'w}$ for $w'\in W^{(K)},w\in W_K$. We see that 7.9 can be restated as follows.

The elements $\tilde{T}_w^{-1}\tilde{T}_{w'}^{-1}\vartheta_x\vartheta_{x'}$, $w\in W_K,w'\in W^{(K)},x\in\mathcal{T}^{I-K},x'\in\mathcal{T}^K$ form an $\mathcal{A}$-basis of $\mathcal{H}$. Using 7.8, we see that this last basis is related to the family $\tilde{T}_w^{-1}\vartheta_x\tilde{T}_{w'}^{-1}\vartheta_{x'+x-x\tilde{w}'^{-1}}$, $w\in W_K,w'\in W^{(K)},x\in\mathcal{T}^{I-K},x'\in\mathcal{T}^K,\tilde{w}'$ as in 7.8 by an "upper triangular matrix with 1 on diagonal". Hence the last family is an $\mathcal{A}$-basis of $\mathcal{H}$. By a change of indexing, we see that the elements in (a) form an $\mathcal{A}$-basis of $\mathcal{H}$. Now the family of elements in (b) is related to the previous basis by an "upper triangular matrix with 1 on diagonal". The lemma follows.

7.11. Let $\mathcal{H}^K$ be the $\mathcal{A}$-submodule of $\mathcal{H}$ spanned by the elements

$$\tilde{T}_{w'}^{-1}\vartheta_{x'}\ (w'\in W^{(K)},x'\in\mathcal{T}^K);$$

these elements form a $\mathcal{A}$-basis of $\mathcal{H}^K$. Let $\mathcal{H}'^K$ be the $\mathcal{A}$-submodule of $\mathcal{H}$ spanned by the elements $\tilde{T}_{w'}^{-1}\vartheta_{x'}$ $(w' \in W^{(K)}, x' \in \mathcal{T})$; these elements form a $\mathcal{A}$-basis of $\mathcal{H}'^K$. Note that $\mathcal{H}^K \subset \mathcal{H}'^K$. From 7.7 we see that $\mathcal{H}^K, \mathcal{H}'^K$ are $\mathcal{A}$-subalgebras of $\mathcal{H}$.

Lemma 7.12. (a) *The elements $\vartheta_x\tilde{T}_{w'}^{-1}\vartheta_{x'}$, $w' \in W^{(K)}, x \in \mathcal{T}^{I-K}, x' \in \mathcal{T}^K$, form an $\mathcal{A}$-basis of $\mathcal{H}'^K$. The elements $\tilde{T}_{w'}^{-1}\vartheta_{x'}$, $w' \in W^{(K)}, x' \in \mathcal{T}^K$, form an $\mathcal{A}$-basis of $\mathcal{H}^K$.*

(b) *The elements*

(∗) $\vartheta_x(\tilde{T}_{w'}^{-1}\vartheta_{x'} - v^{-l(w')+\mu_K(x')})$, $x \in \mathcal{T}^{I-K}, w' \in W^{(K)}, x' \in \mathcal{T}^K, (w', x) \neq (1, 0)$, *and*

(∗∗) ϑ_x, $x \in \mathcal{T}^{I-K}$ *form an $\mathcal{A}$-basis of $\mathcal{H}$.*

The proof is the same as that of 7.10.

7.13. From 7.6(a),(b), we see that there is a well defined $\mathcal{A}$-algebra homomorphism $\chi : \mathcal{H}^K \to \mathcal{A}$ such that $HP_{A_\epsilon^+} = \chi(H)P_{A_\epsilon^+}$ for all $H \in \mathcal{H}^K$. In fact, we have $\chi(\tilde{T}_{w'}^{-1}\vartheta_{x'}) = v^{-l(w')+\mu_K(x')}$ for $w' \in W^{(K)}, x' \in \mathcal{T}^K$. Consider the left ideal $\mathcal{J}$ of $\mathcal{H}$ generated by the elements $H - \chi(H)$ for various $H \in \mathcal{H}^K$. From 7.10, it follows easily that $\mathcal{J}$ is exactly the $\mathcal{A}$-submodule of $\mathcal{H}$ spanned by the elements (∗) in that lemma. Hence the images of the elements (∗∗) in 7.10 in $\mathcal{H}/\mathcal{J}$ form an $\mathcal{A}$-basis of $\mathcal{H}/\mathcal{J}$.

Similarly, let $\mathcal{J}'$ be the left ideal of $\mathcal{H}'^K$ generated by the elements $H - \chi(H)$ for various $H \in \mathcal{H}^K$. From 7.12, it follows easily that $\mathcal{J}'$ is exactly the $\mathcal{A}$-submodule of $\mathcal{H}'^K$ spanned by the elements (∗) in that lemma. Hence the images in $\mathcal{H}'^K/\mathcal{J}'$ of the elements (∗∗) in 7.12 form an $\mathcal{A}$-basis of $\mathcal{H}'^K/\mathcal{J}'$. Thus we have an isomorphism of $\mathcal{A}$-modules $\mathcal{A}[\mathcal{T}^{I-K}] \xrightarrow{\sim} \mathcal{H}'^K/\mathcal{J}'$ (the first of which is a group algebra) given by $x \mapsto v^{-\mu_K(x)}\vartheta_x$; we use this to identify these two $\mathcal{A}$-modules. In particular, $\mathcal{A}[\mathcal{T}^{I-K}]$ becomes an $\mathcal{H}'^K$-module (a quotient of $\mathcal{H}'^K$).

Let us consider the tensor products $\mathcal{H} \otimes_{\mathcal{H}^K} \mathcal{A}$, $\mathcal{H}'^K \otimes_{\mathcal{H}^K} \mathcal{A}$, where $\mathcal{H}, \mathcal{H}'K$ are regarded as a right $\mathcal{H}^K$-module using the algebra imbeddings $\mathcal{H}^K \subset \mathcal{H}'^K \subset \mathcal{H}$ and $\mathcal{A}$ is regarded as a left $\mathcal{H}^K$-module via χ. Then $\mathcal{H} \otimes_{\mathcal{H}^K} \mathcal{A}$ is naturally a $\mathcal{H}$-module and $\mathcal{H}'^K \otimes_{\mathcal{H}^K} \mathcal{A}$ is naturally a $\mathcal{H}'^K$-module. From the definition of tensor product, we have

(a) $\mathcal{H}/\mathcal{J} = \mathcal{H} \otimes_{\mathcal{H}^K} \mathcal{A} = \mathcal{H} \otimes_{\mathcal{H}'^K} (\mathcal{H}'^K \otimes_{\mathcal{H}^K} \mathcal{A}) = \mathcal{H} \otimes_{\mathcal{H}'^K} (\mathcal{H}'^K/\mathcal{J}') = \mathcal{H} \otimes_{\mathcal{H}'^K} \mathcal{A}[\mathcal{T}^{I-K}]$.

Proposition 7.14. (a) *The $\mathcal{A}$-linear map $\mathcal{H} \otimes_{\mathcal{H}^K} \mathcal{A} \to \tilde{M}_c^K$ given by $H \otimes 1 \mapsto HP_{A_\epsilon^+}$ is a well defined isomorphism of $\mathcal{H}$-modules.*

(b) *The $\mathcal{A}$-linear map $\mathcal{H} \otimes_{\mathcal{H}^K} \mathcal{A} \to M_c^K$ given by $H \otimes 1 \mapsto HA_\epsilon^+$ is a well defined isomorphism of $\mathcal{H}$-modules.*

(c) *The $\mathcal{A}$-linear map $\mathcal{H} \otimes_{\mathcal{H}'^K} \mathcal{A}[\mathcal{T}^{I-K}] \to M_c^K$ given by $H \otimes x \mapsto v^{-\mu_K(x)}H\vartheta_x A_\epsilon^+ = H(\gamma_x A_\epsilon^+) = \gamma_x(HA_\epsilon^+)$ (with $x \in \mathcal{T}^{I-K}$) is a well defined isomorphism of $\mathcal{H}$-modules.*

We prove (a). The map $H \to \tilde{M}_c^K$ given by $H \mapsto HP_{A_\epsilon^+}$ is clearly zero on $\mathcal{J}$ (as in 7.13) hence it induces an $\mathcal{A}$-linear map $\mathcal{H}/\mathcal{J} \to \tilde{M}_c^K$. The image under this map of the basis element $T_w^{-1}\vartheta_x$ (with $w \in W_K, x \in \mathcal{T}^{I-K}$) is, by 7.5 and 7.6(c), the element $P_{w^{-1}A_{\epsilon+x}^+}$ times a power of v; and these elements form an $\mathcal{A}$-basis of $\tilde{M}_c^K$

(by 7.3). Thus our map takes an $\mathcal{A}$-basis of $\mathcal{H}/\mathcal{J}$ onto an $\mathcal{A}$-basis of $\tilde{M}_c^K$; hence it is an isomorphism. This proves (a), since $\mathcal{H}/\mathcal{J} = \mathcal{H} \otimes_{\mathcal{H}^K} \mathcal{A}$.

Since $\mathrm{res}_K : \tilde{M}_c^K \to M_c^K$ is an isomorphism of $\mathcal{H}$-modules carrying $P_{A_\epsilon^+}$ to A_ϵ^+, we see that (b) follows from (a). Finally, (c) follows from (b) via the isomorphism 7.13(a). The proposition is proved.

8. The element $\Delta \in \mathcal{A}[\mathcal{T}/\mathcal{T}^K]$

8.1. Throughout this section we use the following notation. An underlined symbol denotes $\mathbf{C}\otimes_{\mathbf{Z}}$(that symbol). For example, $\underline{\mathcal{A}} = \mathbf{C} \otimes_{\mathbf{Z}} \mathcal{A}, \underline{\mathcal{H}} = \mathbf{C} \otimes_{\mathbf{Z}} \mathcal{H}$.

8.2. Let $\mathbf{G}$ be a reductive connected adjoint algebraic group over $\mathbf{C}$ with a fixed maximal torus $\mathbf{T}$ such that the group of characters $\mathbf{T} \to \mathbf{C}^*$ is the group $\mathcal{T}$ in 1.1 and such that $\{\alpha_i | i \in I\}$ (in 1.3) are the simple roots of $\mathbf{G}$ with respect to $\mathbf{T}$ and to a Borel subgroup containing $\mathbf{T}$. Let $\mathfrak{g}, \mathfrak{t}$ be the Lie algebras of $\mathbf{G}$, $\mathbf{T}$. We fix standard Chevalley generators $\mathbf{e}_i, \mathbf{f}_i, \mathbf{h}_i (i \in I)$ for $\mathfrak{g}$ so that $\mathbf{h}_i$ span $\mathbf{T}$ and $\mathbf{e}_i, \mathbf{f}_i$ corresponds to $\alpha_i, -\alpha_i$ in the usual way. Let $\varpi : \mathbf{G} \to \mathbf{G}$ be the involutive automorphism of $\mathbf{G}$ whose tangent map satisfies $\mathbf{e}_i \mapsto \mathbf{f}_i, \mathbf{f}_i \mapsto \mathbf{e}_i, \mathbf{h}_i \mapsto -\mathbf{h}_i$ for all i. Note that $\varpi(t) = t^{-1}$ for all $t \in \mathbf{T}$.

8.3. Let $\mathbf{T}_K = \{t \in \mathbf{T} | \alpha_i(t) = 1 \quad \forall i \in K\}$; this is a torus in $\mathbf{T}$. Let $\mathbf{G}_K$ be the centralizer of $\mathbf{T}_K$ in $\mathbf{G}$. Let $\mathfrak{g}_K$ be the Lie algebra of $\mathbf{G}_K$. Clearly, $\varpi(\mathbf{T}_K) = \mathbf{T}_K, \varpi(\mathbf{G}_K) = \mathbf{G}_K$. We can find uniquely strictly positive real numbers $c_i (i \in K)$ such that $\mathbf{e} = \sum_{i \in K} c_i \mathbf{e}_i, \mathbf{f} = \sum_{i \in K} c_i \mathbf{f}_i, \mathbf{h} = \sum_{i \in K} c_i^2 \mathbf{h}_i$ satisfy the $\mathfrak{sl}_2$-relations $[\mathbf{e}, \mathbf{f}] = \mathbf{h}, [\mathbf{h}, \mathbf{e}] = 2\mathbf{e}, [\mathbf{h}, \mathbf{f}] = -2\mathbf{f}$. (In fact, $c_i^2 \in \mathbf{N}$.) Then $\tilde{\mathbf{e}} = (\mathbf{e}-\mathbf{f}+\mathbf{h})/2, \tilde{\mathbf{f}} = (-\mathbf{e}+\mathbf{f}+\mathbf{h})/2, \tilde{\mathbf{h}} = -\mathbf{e}-\mathbf{f}$ again satisfy the $\mathfrak{sl}_2$-relations and in addition, the tangent map of ϖ takes $\tilde{\mathbf{e}}, \tilde{\mathbf{f}}, \tilde{\mathbf{h}}$ to $-\tilde{\mathbf{e}}, -\tilde{\mathbf{f}}, \tilde{\mathbf{h}}$ respectively. There is a unique homomorphism of algebraic groups $p : SL_2(\mathbf{C}) \to \mathbf{G}_K$ whose tangent map satisfies $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \mapsto \tilde{\mathbf{e}}, \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \mapsto \tilde{\mathbf{f}}, \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \mapsto \tilde{\mathbf{h}}$. Then $u = p\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ is a regular unipotent element of $\mathbf{G}_K$ such that $\varpi(u) = u^{-1}$. For any $\lambda \in \mathbf{C}^*$ we set $p'(\lambda) = p\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right) \in \mathbf{G}_K$. Then $p'(\lambda) u p'(\lambda)^{-1} = u^{\lambda^2}$ and $\varpi(p'(\lambda)) = p'(\lambda)$ for all $\lambda \in \mathbf{C}^*$.

Let $\mathcal{B}_u$ be the variety of Borel subgroups of $\mathbf{G}$ that contain u. The torus $\mathbf{T}_K \times \mathbf{C}^*$ acts on $\mathcal{B}_u$ by $(g, \lambda) : B \to {}^{(g,\lambda)}B = gp'(\lambda) B (gp'(\lambda))^{-1}$.

8.4. In this section we will use for $\mathbf{G}$ some results which appear in [KL2] in the case of simply connected groups; however, the results we need can be deduced from the corresponding results for the simply connected covering of $\mathbf{G}$.

We consider the (complexified) equivariant K-homology space $\mathbf{K}_0^{\mathbf{T}_K \times \mathbf{C}^*}(\mathcal{B}_u)$. We regard this space as a $\underline{\mathcal{H}}$-module by composing the $\underline{\mathcal{H}}$-module structure in [KL2] with the algebra involution ${}^\bullet : \underline{\mathcal{H}} \to \underline{\mathcal{H}}$ given by $\tilde{T}_s \mapsto -\tilde{T}_s^{-1}$ for all $s \in S$ or, equivalently, by $\tilde{T}_{s_i} \mapsto -\tilde{T}_{s_i}^{-1}$ for all $i \in I$ and $\vartheta_x \mapsto \vartheta_{-x}$ for all $x \in \mathcal{T}$.

This K-homology space is naturally a module over the (complexified) representation ring $\mathfrak{R}_{\mathbf{T}_K \times \mathbf{C}^*}$ which is just the algebra of all regular functions $\mathbf{T}_K \times \mathbf{C}^* \to \mathbf{C}$ and this module structure commutes with the $\underline{\mathcal{H}}$-module structure. In fact, $\mathbf{K}_0^{\mathbf{T}_K \times \mathbf{C}^*}(\mathcal{B}_u)$ may be regarded as a space of sections of an algebraic vector bundle $\mathbf{V} \to \mathbf{T}_K \times \mathbf{C}^*$ and the action of $\underline{\mathcal{H}}$ may be regarded as an algebraic family of finite dimensional representations of $\underline{\mathcal{H}}$ on the various fibres of $\mathbf{V}$; moreover, these finite dimensional representations of $\underline{\mathcal{H}}$ are irreducible for almost all fibres.

Since $\varpi(u) = u^{-1}$, we have $B \in \mathcal{B}_u \implies \varpi(B) \in \mathcal{B}_u$; thus, we obtain an involution $\varpi : \mathcal{B}_u \to \mathcal{B}_u$. Note also that this involution is related to the action of

$\mathbf{T}_K \times \mathbf{C}^*$ on $\mathcal{B}_u$ by $\varpi({}^{(g,\lambda)}B) = {}^{\pi(g,\lambda)}(\varpi(B))$ for all $B \in \mathcal{B}_u$ and $(g,\lambda) \in \mathbf{T}_K \times \mathbf{C}^*$; here π is the involution $(g,\lambda) \mapsto (g^{-1},\lambda)$ of $\mathbf{T}_K \times \mathbf{C}^*$. Then ϖ induces, as in [KL2, 1.3(j)], an involution $\varpi^\dagger : \mathbf{K}_0^{\mathbf{T}_K\times\mathbf{C}^*}(\mathcal{B}_u) \to \mathbf{K}_0^{\mathbf{T}_K\times\mathbf{C}^*}(\mathcal{B}_u)$. From the definitions we see that $\varpi^\dagger$ is a semilinear automorphism of the $\underline{\mathcal{H}}$-module $\mathbf{K}_0^{\mathbf{T}_K\times\mathbf{C}^*}(\mathcal{B}_u)$ with respect to the $\underline{\mathcal{A}}$-algebra involution of $\underline{\mathcal{H}}$ such that $\tilde{T}_s \mapsto \tilde{T}_{s^*}$ (where $s \in S$ and s^* is as in 6.4). Also from the definitions we see that $\varpi^\dagger$ is a semilinear automorphism of the $\mathfrak{R}_{\mathbf{T}_K\times\mathbf{C}^*}$-module $\mathbf{K}_0^{\mathbf{T}_K\times\mathbf{C}^*}(\mathcal{B}_u)$ with respect to the $\mathbf{C}$-algebra involution of $\mathfrak{R}_{\mathbf{T}_K\times\mathbf{C}^*}$ induced by π. Equivalently, there exists an involution $\varpi^\dagger : \mathbf{V} \to \mathbf{V}$ (as a vector bundle) inducing π on the base $\mathbf{T}_K \times \mathbf{C}^*$.

8.5. There exists a unique parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ with Lie algebra $\mathfrak{p}$ such that

(a) $\mathbf{G}_K$ is a Levi subgroup of $\mathbf{P}$;
(b) the $\mathbf{T}_K$-module $\mathbf{G}/\mathbf{P}$ is a direct sum of one dimensional modules corresponding to characters of the form $\alpha_h : \mathbf{T}_K \to \mathbf{C}^*$ where $h \in \bar{\mathcal{F}} - \bar{\mathcal{F}}_K$.

(Note that $\mathbf{P}$ is not stable under ϖ, in general.) Let $(\mathfrak{g}/\mathfrak{p})_u = \ker(1 - \mathrm{Ad}(u) : \mathfrak{g}/\mathfrak{p} \to \mathfrak{g}/\mathfrak{p})$.

8.6. Let $\hat{\mathcal{B}}_u$ be the variety of Borel subgroups of $\mathbf{G}_K$ that contain u. We may identify $\hat{\mathcal{B}}_u$ with a point of $\mathcal{B}_u$, namely the unique Borel subgroup of $\mathbf{G}$ that is contained in $\mathbf{P}$ and contains u.

We regard the (complexified) equivariant K-homology space $\mathbf{K}_0^{\mathbf{T}_K\times\mathbf{C}^*}(\hat{\mathcal{B}}_u)$ as a $\underline{\mathcal{H}}'^K$-module obtained by composing the $\underline{\mathcal{H}}'^K$-module structure in [KL2] with the algebra involution of $\underline{\mathcal{H}}$ obtained by restricting ${}^\bullet : \underline{\mathcal{H}} \to \underline{\mathcal{H}}$. The K-homology space above may be naturally identified with $\mathfrak{R}_{\mathbf{T}_K\times\mathbf{C}^*}$ and the obvious $\mathfrak{R}_{\mathbf{T}_K\times\mathbf{C}^*}$-module structure commutes with the $\underline{\mathcal{H}}'^K$-module structure. Let $\mathfrak{R}'$ be the ring obtained by adjoining Δ'^{-1} to $\mathfrak{R}_{\mathbf{T}_K\times\mathbf{C}^*}$ where $\Delta' : \mathbf{T}_K \times \mathbf{C}^* \to \mathbf{C}$ is given by

$$\Delta'(g,\lambda) = \det(1 - \lambda^2\,\mathrm{Ad}(gp'(\lambda)), (\mathfrak{g}/\mathfrak{p})_u).$$

By [KL2, 6.2] the natural $\underline{\mathcal{H}}$-module homomorphism

(a) $\underline{\mathcal{H}} \otimes_{\underline{\mathcal{H}}'^K} \mathbf{K}_0^{\mathbf{T}_K\times\mathbf{C}^*}(\hat{\mathcal{B}}_u) \to \mathbf{K}_0^{\mathbf{T}_K\times\mathbf{C}^*}(\mathcal{B}_u)$

induced by the inclusion $\hat{\mathcal{B}}_u \subset \mathcal{B}_u$ is an isomorphism after tensoring over $\mathfrak{R}_{\mathbf{T}_K\times\mathbf{C}^*}$ by the ring of fractions $\mathfrak{R}'$.

8.7. We may identify

(a) $\underline{\mathcal{H}} \otimes_{\underline{\mathcal{H}}'^K} \mathbf{K}_0^{\mathbf{T}_K\times\mathbf{C}^*}(\hat{\mathcal{B}}_u) = \underline{\mathcal{H}} \otimes_{\underline{\mathcal{H}}'^K} \mathfrak{R}_{\mathbf{T}_K\times\mathbf{C}^*} = \underline{\mathcal{H}} \otimes_{\underline{\mathcal{H}}'^K} \underline{\mathcal{A}}[\mathcal{T}^{I-K}] = \underline{M}_c^K$

where the third equality is deduced from 7.14(c) by tensoring with $\mathbf{C}$. Hence we may regard 8.6(a) as a homomorphism of $(\underline{\mathcal{H}}, \mathfrak{R}_{\mathbf{T}_K\times\mathbf{C}^*})$-bimodules

(b) $\underline{M}_c^K \to \mathbf{K}_0^{\mathbf{T}_K\times\mathbf{C}^*}(\mathcal{B}_u)$

which becomes an isomorphism after tensoring over $\mathfrak{R}_{\mathbf{T}_K\times\mathbf{C}^*}$ by the ring of fractions $\mathfrak{R}'$. (We regard $\underline{M}_c^K$ as an $\underline{\mathcal{A}}[\mathcal{T}/\mathcal{T}^K]$-submodule of $\underline{M}^K$, see 5.3, and we identify

(c) $\underline{\mathcal{A}}[\mathcal{T}/\mathcal{T}^K] = \mathfrak{R}_{\mathbf{T}_K\times\mathbf{C}^*}$

by attaching to $v^n x$, where $n \in \mathbf{Z}, x \in \mathcal{T}/\mathcal{T}^K$, the function $(g,\lambda) \mapsto \lambda^n \dot{x}(g^{-1})$ where $\dot{x} \in \mathcal{T}$ is a representative of x.) We may regard $\underline{M}_c^K$ as the space of sections of an algebraic vector bundle $\mathbf{V}' \to \mathbf{T}_K \times \mathbf{C}^*$ and the action of $\underline{\mathcal{H}}$ on $\underline{M}_c^K$ may be regarded as an algebraic family of finite dimensional representations of $\underline{\mathcal{H}}$ on the various fibres of $\mathbf{V}'$. The map (b) can then be interpreted as a morphism of vector bundles

(d) $\zeta : \mathbf{V}' \to \mathbf{V}$
(inducing the identity on the base $\mathbf{T}_K \times \mathbf{C}^*$) such that this morphism is an isomorphism over the open dense subset $\mathcal{U}$ of $\mathbf{T}_K \times \mathbf{C}^*$ defined by the condition that $\Delta' \neq 0$. This morphism is compatible with the $\underline{\mathcal{H}}$-actions. In particular, the representations of $\underline{\mathcal{H}}$ on the fibres of $\mathcal{V}'$ are irreducible for almost all fibres.

8.8. Now from 6.11 and 6.7(c) we see that θ_ϵ defines a meromorphic map of vector bundles $\mathbf{V}' \to \mathbf{V}'$ inducing π on the base $\mathbf{T}_K \times \mathbf{C}^*$.

Proposition 8.9. *There exists an integer $N \geq 1$ such that, for any (regular) section σ of $\mathbf{V}'$, ${\Delta'}^N \theta_\epsilon(\sigma)$ is a (regular) section of $\mathbf{V}'$.*

The square of θ_ϵ induces the identity on the base and (by 6.7(a)) it commutes with the $\underline{\mathcal{H}}$-action. Since $\underline{\mathcal{H}}$ acts irreducibly on almost all fibres, it follows that this square is given by multiplication by some (meromorphic) function $\mathbf{T}_K \times \mathbf{C}^* \to \mathbf{C}$. On the other hand, from 6.7(b) we see that, if $\epsilon' \in \mathbf{S}_{\tilde{\epsilon}}$, then $e_{\epsilon',K}$ (regarded as a section of $\mathbf{V}'$) is mapped to itself by the square above. Hence the meromorphic function just considered must be identically 1 so that the square of θ_ϵ is the identity map of $\mathbf{V}'$.

Consider the meromorphic vector bundle map $\theta_\epsilon \zeta^{-1} \varpi^\dagger \zeta$ of $\mathbf{V}'$ into itself (recall that $\zeta : \mathbf{V}' \to \mathbf{V}$ is regular map but its inverse ζ^{-1} is only a meromorphic map). This induces the identity on the base $\mathbf{T}_K \times \mathbf{C}^*$. It commutes with the action of $\underline{\mathcal{H}}$ (since both $\varpi^\dagger$ and θ_ϵ are $\mathcal{H}$-semilinear with respect to $\tilde{T}_s \mapsto \tilde{T}_{s^*}$, see 6.4, 6.7(a), and ζ is $\mathcal{H}$-linear). Since $\underline{\mathcal{H}}$ acts irreducibly on almost all fibres of $\mathbf{V}'$, it follows that $\theta_\epsilon \zeta^{-1} \varpi^\dagger \zeta$ is multiplication by some (meromorphic) function $f : \mathbf{T}_K \times \mathbf{C}^* \to \mathbf{C}$. Hence $\theta_\epsilon = f \zeta^{-1} \varpi^\dagger \zeta$. Since both θ_ϵ and $\zeta^{-1} \varpi^\dagger \zeta$ have square 1, it follows that $f^2 = 1$ so that $\theta_\epsilon = \pm \zeta^{-1} \varpi^\dagger \zeta$.

Let σ be a (regular) section of $\mathbf{V}'$ and let $\sigma_1 = \theta_\epsilon(\sigma) = \pm \zeta^{-1} \varpi^\dagger \zeta(\sigma)$ (a meromorphic section on $\mathbf{V}'$). Since ζ^{-1} is regular over the open set $\mathcal{U}$ and $\varpi^\dagger, \zeta$ are regular everywhere, it follows that σ_1 is regular over the open set $\mathcal{U}$. Hence there exists a regular section σ_2 of $\mathbf{V}'$ and an integer $N \geq 1$ such that ${\Delta'}^N \sigma_1 = \sigma_2$. In other words, ${\Delta'}^N \theta_\epsilon(\sigma)$ is a regular section of $\mathbf{V}'$. Note that the integer $N \geq 1$ can be chosen independently of σ since the space of sections of $\mathbf{V}'$ is a finitely generated module over the algebra of regular functions on the base. The proposition is proved.

8.10. The function $(g, \lambda) \mapsto \det(1 - \lambda^{-2} \operatorname{Ad}(g^{-1} p'(\lambda^{-1})), (\mathfrak{g}/\mathfrak{p})_u)$ on $\mathbf{T}_K \times \mathbf{C}^*$ corresponds, under the identification 8.7(c), to an element $\Delta \in \underline{\mathcal{A}}[\mathcal{T}/\mathcal{T}^K]$.

Lemma 8.11. *There exist homomorphisms $\xi_1, \xi_2, \dots, \xi_n : \mathbf{T}_K \to \mathbf{C}^*$ that are restrictions of roots $\alpha_h : \mathbf{T} \to \mathbf{C}^*$ with $h \in \bar{\mathcal{F}} - \bar{\mathcal{F}}^K$, and $k_1, k_2, \dots, k_n$ in $\{2, 3, 4, \dots\}$ such that*

(a) $\Delta'(g, \lambda) = \prod_{j=1}^n (1 - \lambda^{k_j} \xi_j(g))$ *for all* $(g, \lambda) \in \mathbf{T}_K \times \mathbf{C}^*$;

(b) $\Delta = \prod_{j=1}^n (1 - v^{-k_j} [\xi_j])$.

We prove (a). Let V be a finite dimensional $\mathbf{C}$-vector space with a given algebraic representation $\rho : \mathbf{T}_K \times SL_2(\mathbf{C}) \to GL(V)$ such that all characters of $\mathbf{T}_K$ appearing in the $\mathbf{T}_K$-module V are restrictions of roots $\alpha_h : \mathbf{T} \to \mathbf{C}^*$ with $h \in \bar{\mathcal{F}} - \bar{\mathcal{F}}^K$.

Then we can define $\Delta'_V : \mathbf{T}_K \times \mathbf{C}^* \to \mathbf{C}$ as the determinant of the linear transformation $1 - \lambda^2 \rho(g, \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right))$ on $\ker(1 - \rho(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)) : V \to V)$.

We will show that Δ'_V has a factorization of the form stated in the lemma for Δ'. (This will imply (a) since $\Delta' = \Delta'_V$ for $V = \mathfrak{g}/\mathfrak{p}$.) To do this, we may assume that V is an irreducible $\mathbf{T}_K \times SL_2(\mathbf{C})$-module. Then $\mathbf{T}_K$ acts on V via a one-dimensional character ξ and $SL_2(\mathbf{C})$ acts on V irreducibly. But then it is clear that $\Delta'_V(g,\lambda) = 1 - \lambda^{\dim V+1}\xi(g)$. This proves (a). Now (b) follows immediately from (a), using the definitions. The lemma is proved.

Proposition 8.12. (a) *We have* $\Delta \in \mathcal{A}[\mathcal{T}/\mathcal{T}^K]$.

(b) *There exists an integer* $N \geq 1$ *such that, for any* $m \in M_c^K$, *the element* $\Delta^N \theta_\epsilon(m)$ *(which belongs to* $M_{\geq}^K$, *by* (a)*) actually belongs to* M_c^K.

(a) is clear from 8.11(b). To prove (b), it suffices to show that our element in (b) belongs to $\underline{M}_c^K$. (An element of $M_{\geq}^K$ which is contained in $\underline{M}_c^K$ is also contained in M_c^K.) Let $\Delta_1 = \prod_{j=1}^n (1 - v^{k_j}[-\xi_j]) \in \mathcal{A}[\mathcal{T}/\mathcal{T}^K]$. Since $\Delta = (-1)^n v^{-(k_1+\cdots+k_n)}\lfloor \xi_1 + \cdots + \xi_n \rfloor \Delta_1$, in order to show that $\Delta^N \circ (\theta_\epsilon m) \in \underline{M}_c^K$ it is enough to show that $\Delta_1^N \circ (\theta_\epsilon m) \in \underline{M}_c^K$. But this follows from 8.9. The proposition is proved.

8.13. It is likely that one can take $N = 1$ in 8.12(b). This is true if $K = \emptyset$, as well as in the following examples (in which $I = \{1,2\}, \alpha_2(\check{\alpha}_1) = -1$):
(a) type A_2, $K = \{1\}$; $\Delta = 1 - v^{-3}[\alpha_2]$;
(b) type B_2, $K = \{1\}$; $\Delta = (1 - v^{-3}[\alpha_2])(1 - v^{-2}[2\alpha_2])$;
(c) type B_2, $K = \{2\}$; $\Delta = 1 - v^{-4}[\alpha_1]$.

8.14. In general, the number n in 8.11 (number of factors of Δ) is equal to $\dim \mathcal{B}_u$. Indeed, by definition, $n = \dim\ker(1 - \mathrm{Ad}(u) : \mathfrak{g}/\mathfrak{p} \to \mathfrak{g}/\mathfrak{p})$. Moreover, $\dim\ker(1 - \mathrm{Ad}(u) : \mathfrak{g}_K \to \mathfrak{g}_K) = \mathrm{rank}\,\mathfrak{g}_K = \mathrm{rank}\,\mathfrak{g}$ since u is regular unipotent in $\mathbf{G}_K$. Hence $\dim\ker(1 - \mathrm{Ad}(u) : \mathfrak{g} \to \mathfrak{g}) = 2n + \mathrm{rank}\,\mathfrak{g}$. On the other hand, $\dim\ker(1 - \mathrm{Ad}(u) : \mathfrak{g} \to \mathfrak{g})$ is the dimension of the centralizer of u in $\mathbf{G}$ and this is known to be equal to $2\dim \mathcal{B}_u + \mathrm{rank}\,\mathfrak{g}$; our assertion follows.

9. Study of a pairing $(,) : M_{\leq}^K \times M_{\geq}^K \to \mathcal{A}$

9.1. We define an $\mathcal{A}$-bilinear pairing $(,) : M_{\leq}^K \times M_{\geq}^K \to \mathcal{A}$ by $(m,m') = \sum_{A \in X_\Xi} m_A m'_A$. The sum makes sense since $\mathrm{supp}(m) \cap \mathrm{supp}(m')$ is a finite set whenever $m \in M_{\leq}^K, m' \in M_{\geq}^K$.

We shall make the following conventions (for fixed $s \in S$). For $A \in X_\Xi$ we write $A : \heartsuit$ if (s,A) is as in 4.7(a); we write $A : \clubsuit$ if (s,A) is as in 4.7(b); we write $A : \spadesuit$ if (s,A) is as in 4.7(c). Note that for any A, exactly one of the statements $A : \heartsuit, A : \clubsuit, A : \spadesuit$ holds. Note that $A : \heartsuit$ if and only if $sA : \clubsuit$. We write $A : \clubsuit\spadesuit$ if either $A : \clubsuit$ or $A : \spadesuit$ holds.

Lemma 9.2. *For* $m \in M_{\leq}^K, m' \in M_{\geq}^K, w \in W$ *we have* $(\tilde{T}_w(m), m') = (m, \tilde{T}_{w^{-1}}(m'))$.

We may assume that $m = A, m' = A'$ where $A, A' \in X_\Xi$ and that $w = s \in S$. We use the conventions of 9.1 relative to s.

(i) Assume that $A : \heartsuit$ and $A' : \heartsuit$. Then $sA : \clubsuit$ and $sA' : \clubsuit$, hence $A \neq sA'$ and $A' \neq sA$. We have

$$(\tilde{T}_s A, A') = (sA, A') = \delta_{sA,A'} = 0, \ (A, \tilde{T}_s A') = (A, sA') = \delta_{A,sA'} = 0.$$

(ii) Assume that $A : \clubsuit$ and $A' : \clubsuit$. Then $sA : \heartsuit$ and $sA' : \heartsuit$ hence $A \neq sA'$ and $A' \neq sA$. We have

$$(\tilde{T}_s A, A') = (sA + (v - v^{-1})A, A') = \delta_{sA,A'} + (v - v^{-1})\delta_{A,A'} = (v - v^{-1})\delta_{A,A'},$$
$$(A, \tilde{T}_s A') = (A, sA' + (v - v^{-1})A') = \delta_{A,sA'} + (v - v^{-1})\delta_{A,A'} = (v - v^{-1})\delta_{A,A'}.$$

(iii) Assume that $A : \spadesuit$ and $A' : \spadesuit$. We have

$$(\tilde{T}_s A, A') = (vA, A') = v\delta_{A,A'},\ (A, \tilde{T}_s A') = (A, vA') = v\delta_{A,A'}.$$

(iv) Assume that $A : \heartsuit$ and $A' : \clubsuit$. We have

$$(\tilde{T}_s A, A') = (sA, A') = \delta_{sA,A'},$$
$$(A, \tilde{T}_s A') = (A, sA' + (v - v^{-1})A') = \delta_{A,sA'} + (v - v^{-1})\delta_{A,A'}.$$

It remains to use $\delta_{A,sA'} = \delta_{sA,A'}$ and $\delta_{A,A'} = 0$. The case where $A : \clubsuit$ and $A' : \heartsuit$ is entirely similar.

(v) Assume that $A : \heartsuit$ and $A' : \spadesuit$. Then $sA : \clubsuit$ hence $sA \neq A'$. We have

$$(\tilde{T}_s A, A') = (sA, A') = \delta_{sA,A'} = 0,\ (A, \tilde{T}_s A') = (A, vA') = v\delta_{A,A'} = 0.$$

The case where $A : \spadesuit$ and $A' : \heartsuit$ is entirely similar.

(vi) Assume that $A : \clubsuit$ and $A' : \spadesuit$. Then $sA : \heartsuit$ hence $sA \neq A'$. We have

$$(\tilde{T}_s A, A') = (sA + (v - v^{-1})A, A') = \delta_{sA,A'} + (v - v^{-1})\delta_{A,A'} = 0,$$
$$(A, \tilde{T}_s A') = (A, vA') = v\delta_{A,A'} = 0.$$

The case where $A : \spadesuit$ and $A' : \clubsuit$ is entirely similar. The lemma is proved.

Lemma 9.3. *Let $\epsilon \in \mathbf{S}_{\tilde{\epsilon}}$ and let $m \in \tilde{M}^K$. Let $N = \sum_{w \in W^K} v^{-2l(w)}$, where l denotes the length function of W^K. We have*

$$\sum_{A \in D(\epsilon)} m_A v^{-d(A, A^+_\epsilon)} = N \sum_{A \in D_\Xi(\epsilon)} m_A v^{-d(A, A^+_\epsilon)}.$$

With notation as in the proof of 4.10(b) we have

$$\begin{aligned}\sum_{A \in D(\epsilon)} m_A v^{-d(A, A^+_\epsilon)} &= \sum_{\substack{A \in D_\Xi(\epsilon) \\ \omega' \in \Omega'}} m_{A\omega'} v^{-d(A\omega', A^+_\epsilon)} \\ &= \sum_{\substack{A \in D_\Xi(\epsilon) \\ \omega' \in \Omega'}} m_A v^{d_K(A, A\omega')} v^{-d(A\omega', A^+_\epsilon)}.\end{aligned}$$

Here we replace $d(A\omega', A^+_\epsilon)$ by $d(A, A^+_\epsilon) - d(A, A\omega')$ and (using the definitions and an argument in the proof of 4.10(b)) we replace $d(A, A\omega') = d_K(A, A\omega')$ by $-l(w)$ where $w \in W^K$ corresponds to ω' under the canonical isomorphism $\Omega' \xrightarrow{\sim} W^K$. Our sum becomes

$$\sum_{A \in D_\Xi(\epsilon); w' \in W^K} m_A v^{-2l(w)} v^{-d(A, A^+_\epsilon)} = N \sum_{A \in D_\Xi(\epsilon)} m_A v^{-d(A, A^+_\epsilon)}.$$

The lemma is proved.

Lemma 9.4. *Let $i \in I$ and let $H \in \mathfrak{F}^i$. Let $m \in M_{i,\le}$ and let $m' = \theta_H(m) \in M_{i,\ge}$. Let ϵ be a special point in E and let $\epsilon' = \epsilon\sigma_H$. We have*

$$\sum_{A \in D(\epsilon)} m'_A v^{-d(A, A^+_\epsilon)} = \sum_{A \in D(\epsilon')} m_A v^{-d(A, A^+_{\epsilon'})}.$$

Let $H_1, H'_1, Y(\epsilon), Y(\epsilon'), x \in \mathcal{T}$ be as in the proof of 3.7. By an argument in that proof we have

$$\sum_{A \in D(\epsilon)} m'_A v^{-d(A,A^+_\epsilon)} = \sum_{A \in Y(\epsilon)} v^{-d(A,A^+_\epsilon)}(m'_A + m'_{A\sigma_{H_1}} v^{-1}),$$

$$\begin{aligned}
&\sum_{A \in D(\epsilon')} m_A v^{-d(A,A^+_{\epsilon'})} \\
&\quad = \sum_{A \in Y(\epsilon')} v^{-d(A,A^+_{\epsilon'})}(m_A + m_{A\sigma_{H'_1}} v^{-1}) \\
&\quad = \sum_{A \in Y(\epsilon)} v^{-d(A+x,A^+_{\epsilon'})}(m_{A+x} + m_{(A+x)\sigma_{H'_1}} v^{-1}) \\
&\quad = \sum_{A \in Y(\epsilon)} v^{-d(A,A^+_{\epsilon})}(m_{A+x} + m_{(A+x)\sigma_{H'_1}} v^{-1}) \\
&\quad = \sum_{A \in Y(\epsilon)} v^{-d(A,A^+_{\epsilon})}(m_{A\sigma_{H_1}\sigma_H} + m_{A\sigma_H} v^{-1}).
\end{aligned}$$

We see that it is enough to verify the equality $m'_A + m'_{A\sigma_{H_1}} v^{-1} = m_{A\sigma_{H_1}\sigma_H} + m_{A\sigma_H} v^{-1}$ for any $A \in Y(\epsilon)$. For any $B \in X$, let $(B^z)_{z \in \mathbf{Z}}$ be the sequence of alcoves defined as in 3.3 in terms of H and B (instead of A). Note that for $A, B \in X$ and $z \in \mathbf{Z}$, the conditions $B^z = A$ and $A^z = B$ are equivalent.

Let $A \in Y(\epsilon)$. We set $\tilde{A} = A\sigma_{H_1}$. For $z \in \mathbf{N}$ we define

$$c_z = (-1)^{z-1}(v^{-z+1} - v^{-z-1})$$

if $z > 0$ and $c_0 = v^{-1}$. From the definitions we have

$$\begin{aligned}
m'_A &= \sum_{z \geq 0; B \in X; B^z = A} c_z m_B = \sum_{z \geq 0} c_z m_{A^z}, \\
m'_{\tilde{A}} &= \sum_{z \geq 0; B \in X; B^z = \tilde{A}} c_z m_B = \sum_{z \geq 0; B \in X; B^{z+1} = A} c_z m_B = \sum_{z \geq 0} c_z m_{A^{z+1}}.
\end{aligned}$$

Hence

$$\begin{aligned}
m'_A + m'_{A\sigma_{H_1}} v^{-1} &= \sum_{z \geq 0} c_z m_{A^z} + \sum_{z \geq 0} v^{-1} c_z m_{A^{z+1}} = v^{-1} m_{A^0} \\
&+ \sum_{z \geq 1} (c_z + v^{-1} c_{z-1}) m_{A^z} = v^{-1} m_{A^0} + m_{A^1} = m_{A\sigma_{H_1}\sigma_H} + m_{A\sigma_H} v^{-1}.
\end{aligned}$$

The lemma is proved.

Lemma 9.5. *Let $\epsilon, \epsilon' \in \mathbf{S}_{\tilde{\epsilon}}$ and let $\epsilon'' = \kappa_\epsilon(\epsilon' \omega_{\epsilon,K}) \in \mathbf{S}_{\tilde{\epsilon}}$. Let $m \in M^K_{\leq}$. We have*

$$\sum_{A \in D_\Xi(\epsilon'')} (\theta_\epsilon m)_A v^{-d(A,A^+_{\epsilon''})} = \sum_{A \in D_\Xi(\epsilon')} m_A v^{-d(A,A^+_{\epsilon'})}$$

or, equivalently, $(m, e_{\epsilon',K}) = (e_{\epsilon'',K}, \theta_\epsilon m)$.

First note that, for any $m', m'_1 \in M^K_{\geq}$ we have (from the definitions)

(a) $(\kappa m', m'_1) = \overline{(\kappa m'_1, m')}$. Taking $m'_1 = e_{\epsilon_1,K}$ where $\epsilon_1 \in \mathbf{S}_{\tilde{\epsilon}}$ and using 6.5(b), we obtain

(b) $\sum_{A \in D_\Xi(\epsilon_1)} (\kappa m')_A v^{-d(A,A^+_{\epsilon_1})} = \sum_{A \in D_\Xi(\kappa_\epsilon \epsilon_1)} \overline{m'_A} v^{-\nu_K + d(A,A^+_{\kappa_\epsilon \epsilon_1})}$.

A similar argument (using 3.16 instead of 6.5(b)) shows that for any $m'' \in M$ and any special point ϵ_2 we have

(c) $\sum_{A\in D(\epsilon_2)}(\phi_\epsilon m'')_A v^{-d(A,A^+_{\epsilon_2})} = \sum_{A\in D(\epsilon_2\omega_\epsilon)} \overline{m''_A} v^{-2\nu+d(A,A^+_{\epsilon_2\omega_\epsilon})}$.

Now let $\tilde m \in \tilde M^K_{\leftarrow}$ be defined by $\mathrm{res}_K(\tilde m) = m$. In the setup of 3.13, let $\tilde m' = \theta_{\epsilon+i_p}\dots\theta_{\epsilon+i_2}\theta_{\epsilon+i_1}\tilde m \in M$. Let ϵ'_1 be defined by $\epsilon' = \epsilon'_1\omega_\epsilon\omega_{\epsilon,K} = \epsilon'_1\sigma_{\epsilon+i_p}\dots\sigma_{\epsilon+i_2}\sigma_{\epsilon+i_1}$. Applying p times 9.4, we have

$$\begin{aligned}\sum_{A\in D(\epsilon'_1)}\tilde m'_A v^{-d(A,A^+_{\epsilon'_1})} &= \sum_{A\in D(\epsilon'_1)}(\theta_{\epsilon+i_p}\dots\theta_{\epsilon+i_2}\theta_{\epsilon+i_1}\tilde m)_A v^{-d(A,A^+_{\epsilon'_1})}\\ &= \sum_{A\in D(\epsilon'_1\sigma_{\epsilon+i_p})}(\theta_{\epsilon+i_{p-1}}\dots\theta_{\epsilon+i_2}\theta_{\epsilon+i_1}\tilde m)_A v^{-d(A,A^+_{\epsilon'_1\sigma_{\epsilon+i_p}})}\\ &= \dots\\ &= \sum_{A\in D(\epsilon'_1\sigma_{\epsilon+i_p}\dots\sigma_{\epsilon+i_2}\sigma_{\epsilon+i_1})}\tilde m_A v^{-d(A,A^+_{\epsilon'_1\sigma_{\epsilon+i_p}\dots\sigma_{\epsilon+i_2}\sigma_{\epsilon+i_1}})}\\ &= \sum_{A\in D(\epsilon')}\tilde m_A v^{-d(A,A^+_{\epsilon'})} = N\sum_{A\in D_\Xi(\epsilon')} m_A v^{-d(A,A^+_{\epsilon'})}.\end{aligned}$$

(We used 9.3.) The first term of the last sequence of equalities equals (by (c)):

$$\overline{\sum_{A\in D(\epsilon'_1\omega_\epsilon)}(\phi_\epsilon\tilde m')_A v^{2\nu-d(A,A^+_{\epsilon'_1\omega_\epsilon})}} = \overline{N\sum_{A\in D_\Xi(\epsilon'_1\omega_\epsilon)}(bm)_A v^{2\nu-d(A,A^+_{\epsilon'_1\omega_\epsilon})}}.$$

(We have used 9.3 and $bm = \mathrm{res}_K\,\phi_\epsilon\tilde m'$.) Hence we have

(d) $N\sum_{A\in D_\Xi(\epsilon')} m_A v^{-d(A,A^+_{\epsilon'})} = \bar N v^{-2\nu}\overline{\sum_{A\in D_\Xi(\epsilon'_1\omega_\epsilon)}(bm)_A v^{-d(A,A^+_{\epsilon'_1\omega_\epsilon})}}$. Using (b) and $bm = v^{-\nu_K}\kappa_\epsilon\theta_\epsilon m$, we see that

$$\begin{aligned}\sum_{A\in D_\Xi(\epsilon'_1\omega_\epsilon)}(bm)_A v^{-d(A,A^+_{\epsilon'_1\omega_\epsilon})} &= v^{-\nu_K}\sum_{A\in D_\Xi(\epsilon'_1\omega_\epsilon)}(\kappa_\epsilon\theta_\epsilon m)_A v^{-d(A,A^+_{\epsilon'_1\omega_\epsilon})}\\ &= v^{-\nu_K}\sum_{A\in D_\Xi(\epsilon'')}\overline{(\theta_\epsilon m)_A} v^{-\nu_K+d(A,A^+_{\epsilon''})}.\end{aligned}$$

Introducing this in (d) we obtain

$$N\sum_{A\in D_\Xi(\epsilon')} m_A v^{-d(A,A^+_{\epsilon'})} = \bar N v^{-2\nu}v^{2\nu_K}\sum_{A\in D_\Xi(\epsilon'')}(\theta_\epsilon m)_A v^{-d(A,A^+_{\epsilon''})}.$$

It remains to use the equality $N = \bar N v^{-2\nu+2\nu_K}$. The lemma is proved.

Lemma 9.6. (a) *There exists a unique $\mathcal{H}$-antilinear map $\tilde b : M^K_{\geq} \to M^K_{\geq}$ such that $\overline{(b(m), m')} = (m, \tilde b(m'))$ for all $m \in M^K_{\leq}, m' \in M^K_{\geq}$.*

(b) *We have $\tilde b^2 = 1$.*

(c) *For any $A \in X_\Xi$ we have $\tilde b(A) = \sum_{B\in X_\Xi}\overline{R_{A,B}}B$ where $R_{A,B} \in \mathcal{A}$ are as in 4.24.*

(d) *$\tilde b : M^K_{\geq} \to M^K_{\geq}$ is continuous in the sense of 4.13.*

We prove (a). For $m' \in M^K_{\geq}$, we set

(e) $\tilde b(m') = \sum_{A\in X_\Xi}\overline{(b(A), m')}A$.

Using 4.14(b), we see that the last sum belongs to $M^K_{\geq}$. Thus, $m' \mapsto \tilde b(m')$ is a well defined group homomorphism $\tilde{b} : M_{\geq}^{K} \to M_{\geq}^{K}$. Clearly, $\tilde{b}$ satisfies the equation in (a). For all $p \in \mathcal{A}$ and $m' \in M_{\geq}^{K}$ we have $\tilde{b}(pm') = \bar{p}\tilde{b}(m')$. Let $w \in W, m \in M_{\leq}^{K}, m' \in M_{\geq}^{K}$. We have

$$\begin{aligned}(m, \tilde{T}_w \tilde{b} m') &= (\tilde{T}_{w^{-1}} m, \tilde{b} m') = \overline{(b\tilde{T}_{w^{-1}} m, m')} = \overline{(\tilde{T}_w bm, m')} = \overline{(bm, \tilde{T}_{w^{-1}} m')} \\ &= (m, \tilde{b}\tilde{T}_{w^{-1}} m').\end{aligned}$$

Thus, $(m, \tilde{T}_w \tilde{b} m') = (m, \tilde{b}\tilde{T}_{w^{-1}} m')$. Since this holds for any m, it follows that $\tilde{b}$ is $\mathcal{H}$-antilinear; the existence part of (a) is proved. The uniqueness part of (a) is obvious.

We prove (b). For any $m \in M_{\leq}^{K}, m' \in M_{\geq}^{K}$ we have $(m, \tilde{b}^2 m') = \overline{(bm, \tilde{b} m')} = (b^2 m, m') = (m, m')$ so that $(m, \tilde{b}^2 m') = (m, m')$. Since this holds for any m, it follows that $\tilde{b}^2 m' = m'$ and (b) is proved; (c) follows from (a), taking $m = B, m' = A$; (d) follows from (e), using 4.14(b). The lemma is proved.

Theorem 9.7. (a) *For $m, m' \in M_{\leq}^{K}$ we have $(m, \theta_\epsilon(m')) = (m', \theta_\epsilon(m))$.*

(b) *For $m'' \in M_{\geq}^{K}$ we have $\tilde{b}(m'') = v^{\nu_K} \theta_\epsilon \kappa_\epsilon (m'') = \kappa_\epsilon^{-1} b \kappa_\epsilon (m'') = v^{2\nu_K} \theta_\epsilon b \theta_\epsilon^{-1}(m'')$.*

(c) *We have $\tilde{b}(v^{\nu_K} e_{\epsilon', K}) = v^{\nu_K} e_{\epsilon', K}$ for any $\epsilon' \in \mathbf{S}_{\tilde{\epsilon}}$.*

We prove (a). By 9.6, we have $\overline{(b(m'), m'')} = (m', \tilde{b}(m''))$ for any $m' \in M_{\leq}^{K}$, $m'' \in M_{\geq}^{K}$ or, equivalently: $\overline{(v^{-\nu_K} \kappa_\epsilon \theta_\epsilon(m'), m'')} = (m', \tilde{b}(m''))$. Using 9.5(a), we see that the left hand side of the last equality equals $v^{\nu_K}(\kappa_\epsilon(m''), \theta_\epsilon(m'))$. Thus, we have $v^{\nu_K}(\kappa_\epsilon(m''), \theta_\epsilon(m')) = (m', \tilde{b}(m''))$. We make the substitution $m = \kappa_\epsilon(m'') \in M_{\leq}^{K}$. We obtain the equality $(m, \theta_\epsilon(m')) = v^{-\nu_K}(m', \tilde{b}\kappa_\epsilon^{-1}(m))_K$ for any m, m' as in (a). Let τ be the composition $M_{\leq}^{K} \xrightarrow{v^{-\nu_K}\kappa_\epsilon^{-1}} M_{\geq}^{K} \xrightarrow{\tilde{b}} M_{\geq}^{K}$. Then we have $(m, \theta_\epsilon(m')) = (m', \tau(m))$ for any m, m' as in (a). Now, for any $s \in S$ and $m \in M_{\leq}^{K}$, we have $\tau(\tilde{T}_{s^*}(m)) = \tau(\theta_\epsilon(m))$ (by 9.6(a) and 6.5(a)). Moreover, τ is continuous in the sense of 4.13 since so are $\kappa_\epsilon^{-1}, \tilde{b}$.

Let $\epsilon' \in \mathbf{S}_{\tilde{\epsilon}}$ and let $\epsilon'' = \kappa_\epsilon(\epsilon' \omega_{\epsilon, K})$. For any $m' \in M_{\leq}^{K}$ we have

$$(m', \tau(e_{\epsilon'', K})) = (e_{\epsilon'', K}, \theta_\epsilon(m')) = (m', e_{\epsilon', K})$$

where the second equality follows from 9.5. Thus, $(m', \tau(e_{\epsilon'', K})) = (m', e_{\epsilon', K})$. Since this holds for any $m' \in M_{\leq}^{K}$, it follows that $\tau(e_{\epsilon'', K}) = e_{\epsilon', K}$. We now see that τ satisfies the properties that characterize θ_ϵ (see 6.9). Hence we have $\tau = \theta_\epsilon$. This proves (a).

We prove (b). Let $m' \in M_{\leq}^{K}, m'' \in M_{\geq}^{K}$. By the arguments in the proof of (a) we have $(m', \tilde{b}(m'')) = \overline{(b(m'), m'')} = v^{\nu_K}(\kappa_\epsilon(m''), \theta_\epsilon(m'))$ and, by (a), the last expression is equal to $v^{\nu_K}(m', \theta_\epsilon \kappa_\epsilon(m''))$. Thus, we have $(m', \tilde{b}(m'')) = v^{\nu_K}(m', \theta_\epsilon \kappa_\epsilon(m''))$. Since this holds for all $m' \in M_{\leq}^{K}$, it follows that $\tilde{b}(m'') = v^{\nu_K} \theta_\epsilon \kappa_\epsilon(m'')$ and the first equality in (b) is proved. The second and third equalities in (b) follow from the definition 6.6 of θ_ϵ.

Now (c) follows from (b) using 6.5(b) and 4.14(a). The theorem is proved.

Corollary 9.8. *For any $A, B \in X_\Xi$ we have $R_{A,B} = R_{\kappa_\epsilon(B), \kappa_\epsilon(A)}$. (Notation of 4.24.)*

Using 9.7(b) and the formula $b(C)=\sum_{B\in X_\Xi} R_{B,C}B$ (see 4.24), we see that

$$\tilde{b}(A)=\kappa_\epsilon^{-1}b\kappa_\epsilon(A)=\kappa_\epsilon^{-1}\left(\sum_{B\in X_\Xi} R_{B,\kappa_\epsilon(A)}B\right)=\sum_{B\in X_\Xi}\overline{R_{\kappa_\epsilon(B),\kappa_\epsilon(A)}}B$$

for any $A\in X_\Xi$. We compare this with the formula in 9.6(c). The corollary follows.

10. Integrability of $\theta_\epsilon(A)$

10.1. Let $\mathfrak{A}=\mathbf{Z}[\mathcal{T}/\mathcal{T}^K]((v^{-1}))$ be the ring of all power series in v^{-1} with coefficients in the group algebra of $\mathcal{T}/\mathcal{T}^K$ over $\mathbf{Z}$; a typical element of $\mathfrak{A}$ is $\sum_{t\in\mathcal{T}/\mathcal{T}^K;n\in\mathbf{Z}} a_{n,t}[t]v^n$ where $a_{n,t}\in\mathbf{Z}$ and there exists $n_0\in\mathbf{Z}$ such that $a_{n,t}=0$ for $n>n_0$; moreover, for any given $n\in\mathbf{Z}$, $a_{n,t}$ is zero for all but finitely many $t\in\mathcal{T}/\mathcal{T}^K$. Note that $\mathfrak{A}$ is an integral domain since $\mathcal{T}/\mathcal{T}^K$ is a finitely generated free abelian group. It contains as a subring the group algebra $\mathcal{A}[\mathcal{T}/\mathcal{T}^K]$ of $\mathcal{T}/\mathcal{T}^K$ over $\mathcal{A}$. Let $a\mapsto a^\ddagger$ be the involution of the ring $\mathfrak{A}$ given by

$$\sum_{t\in\mathcal{T}/\mathcal{T}^K;n\in\mathbf{Z}} a_{n,t}[t]v^n\mapsto\sum_{t\in\mathcal{T}/\mathcal{T}^K;n\in\mathbf{Z}} a_{n,t}[-t]v^n.$$

Let M^K_{int} be the set of formal expressions $m=\sum_{A\in X_\Xi;n\in\mathbf{Z}} m_{A,n}v^nA$ (where $m_{A,n}\in\mathbf{Z}$) such that there exists $n_0\in\mathbf{Z}$ with $m_{A,n}=0$ whenever $n>n_0$ and such that, for any $n\in\mathbf{Z}$, the set $\{A|m_{A,n}\neq 0\}$ is finite.

For any $m\in M^K$ and $A\in X_\Xi$, we write $m_A=\sum_{n\in\mathbf{Z}} m_{A,n}v^n$ where $m_{A,n}\in\mathbf{Z}$; we say that m is *integrable* if $\sum_{A\in X_\Xi;n\in\mathbf{Z}} m_{A,n}v^nA\in M^K_{\text{int}}$. Note that $M^K_c\subset M^K_{\text{int}}$.

We regard M^K as a $\mathcal{A}[\mathcal{T}/\mathcal{T}^K]$-module in which $[t]m=\lfloor t\rfloor\circ m$ for $m\in M^K$; in the right hand side, t is an element of $\mathcal{T}$ while in the left hand side $[t]$ is determined by the image of $t\in\mathcal{T}$ in $\mathcal{T}/\mathcal{T}^K$. We regard M^K_{int} as an $\mathfrak{A}$-module by $(a,m)\mapsto am=m'$ where $m'_{A,n}=\sum_{t,B,p,q;\gamma_t(B)=A;p+q=n} a_{p,t}m_{B,q}$. By restriction, we obtain on M^K_c (which is contained both in M^K_c and in M^K_{int}) two $\mathcal{A}[\mathcal{T}/\mathcal{T}^K]$-module structures, which coincide.

10.2. We define a pairing $(|):M^K_{\text{int}}\times M^K_{\text{int}}\to\mathfrak{A}$ by

(a) $$(m|m')=\sum_{t\in\mathcal{T}/\mathcal{T}^K,n\in\mathbf{Z}} a_{n,t}[t]v^n$$

where $a_{n,t}=\sum_{A\in X_\Xi}\sum_{p,q\in\mathbf{Z};p+q=n} m_{A,p}m'_{\gamma_tA,q}$. We show that the right hand side of (a) is a well defined element of $\mathfrak{A}$. Let n_0,n'_0 be integers such that, for all A, $m_{A,p}=0$ whenever $p>n_0$ and $m'_{A,q}=0$ whenever $q>n_0$. Then $a_{n,t}=0$ whenever $n>p_0+q_0$. For fixed n, we may restrict the sum over p,q to those p,q such that $p\le n_0,q\le n'_0,p+q=n$. There are only finitely many such p,q. Hence it is enough to show that for fixed p,q such that $p+q=n$, the set

$$\{(A,t)\in X_\Xi\times(\mathcal{T}/\mathcal{T}^K)|m_{A,p}m'_{\gamma_tA,q}\neq 0\}$$

is finite. If (A,t) is in this set then, since $m_{A,p}\neq 0$, we see that A runs over a fixed finite set F and γ_tA runs over a fixed finite set F'. But then t runs over the set of all $t'\in\mathcal{T}/\mathcal{T}^K$ such that $\gamma_{t'}A=B$ for some $A\in F$ and some $B\in F'$. This set is finite by 2.12(f). Our assertion is verified.

10.3. From the definitions we see that, for any $m, m' \in M^K_{\rm int}$ and $a \in \mathfrak{A}$, $(am|m') = (m|a^\ddagger m') = a(m|m')$, $\quad (m|m') = (m'|m)^\ddagger$.

If m, m' belong to M^K_c, then clearly, $(m, [t]m')$ (notation of 9.1) is zero for all but finitely many $t \in \mathcal{T}/\mathcal{T}^K$, so that

$$(m|m') = \sum_{t\in\mathcal{T}/\mathcal{T}^K;A\in X_\Xi} m_A m'_{\gamma_t A}[t] = \sum_{t\in\mathcal{T}/\mathcal{T}^K} (m, [t]m')[t] \in \mathcal{A}[\mathcal{T}/\mathcal{T}^K].$$

We now fix $\epsilon \in \mathbf{S}_{\tilde{\epsilon}}$. Let $N \geq 1$ be as in 8.12(b), so that $m \mapsto \Delta^N\theta_\epsilon(m)$ is a well defined $\mathcal{A}$-linear map $\Theta_\epsilon : M^K_c \to M^K_c$.

Lemma 10.4. *If $m, m' \in M^K_c$, then $\Theta_\epsilon m, \Theta_\epsilon m' \in M^K_c$ satisfy $(m|\Theta_\epsilon m') = (m'|\Theta_\epsilon m)$.*

By 10.3, it suffices to prove that $(m, [t]\Theta_\epsilon(m')) = (m', [t]\Theta_\epsilon(m))$ for $t \in \mathcal{T}/\mathcal{T}^K$. We have

$$\begin{aligned}(m, [t]\Theta_\epsilon(m')) &= (m, [t]\Delta^N\theta_\epsilon(m')) = ([-t](\Delta^N)^\ddagger m, \theta_\epsilon(m'))\\ &= (m', \theta_\epsilon([-t](\Delta^N)^\ddagger m)) = (m', [t]\Delta^N\theta_\epsilon(m)) = (m', [t]\Theta_\epsilon(m)).\end{aligned}$$

(The third equality follows from 9.7(a).) The lemma is proved.

Lemma 10.5. *For any $m \in M^K_c$ we have $\Theta^2_\epsilon(m) = (\Delta\Delta^\ddagger)^N m$.*

With notation in the proof of 8.9, the (meromorphic) map $\theta_\epsilon : \mathbf{V}' \to \mathbf{V}'$ has square 1. This clearly implies the lemma.

Theorem 10.6. (a) *For any $m \in M^K_c$ we have $\theta_\epsilon(m) \in M^K_{\rm int}$.*
(b) *For any $m_1, m_2 \in M^K_c$ we have $(\theta_\epsilon(m_1)|\theta_\epsilon(m_2)) = (m_2|m_1)$.*

We prove (a). Let $m' = \theta_\epsilon(m) \in M^K_{\geq}$. By 8.12(b) we have $\Delta^N m' \in M^K_c$ and in particular, $\Delta^N m' \in M^K_{\rm int}$ for some $N \geq 1$. Multiplying with $\Delta^{-N} \in \mathfrak{A}$ we obtain $m' \in M^K_{\rm int}$. This proves (a).

We prove (b). Using 10.4 with $m = \Theta_\epsilon(m_1), m' = m_2$, as well as 10.5, we have

$$(\Theta_\epsilon(m_1)|\Theta_\epsilon(m_2)) = (m_2|\Theta^2_\epsilon(m_1)) = (m_2|(\Delta\Delta^\ddagger)^N m_1) = (\Delta\Delta^\ddagger)^N(m_2|m_1).$$

On the other hand,

$$(\Delta\Delta^\ddagger)^N(\theta_\epsilon(m_1)|\theta_\epsilon(m_2)) = (\Delta^N\theta_\epsilon(m_1)|\Delta^N\theta_\epsilon(m_2)) = (\Theta_\epsilon(m_1)|\Theta_\epsilon(m_2)).$$

It follows that $(\Delta\Delta^\ddagger)^N(\theta_\epsilon(m_1)|\theta_\epsilon(m_2)) = (\Delta\Delta^\ddagger)^N(m_2|m_1)$. This implies (b), since $(\Delta\Delta^\ddagger)^N \neq 0$ and $\mathfrak{A}$ is an integral domain. The theorem is proved.

Corollary 10.7. *For $A \in X_\Xi$, we write $\theta_\epsilon(A) = \sum_{C\in X_\Xi} f^\epsilon_{C,A} C$ where $f^\epsilon_{C,A} \in \mathcal{A}$. We also write $f^\epsilon_{C,A} = \sum_{n\in\mathbf{Z}} f^\epsilon_{C,A;n} v^n$ where $f^\epsilon_{C,A;n} \in \mathbf{Z}$. Then, for any $A, B \in X_\Xi$ we have*

(a) $$\sum_{n\in\mathbf{Z}}(\sum_{C\in X_\Xi;p,q\in\mathbf{Z};p+q=n} f^\epsilon_{C,A;p} f^\epsilon_{C,B;q})v^n = \delta_{A,B}.$$

(By the integrability of $\theta_\epsilon(A)$ and $\theta_\epsilon(B)$, the coefficient of v^n is well defined for any n, since it is given by a finite sum.)

We consider the map from $\mathfrak{A}$ to the set $\mathbf{Z}((v^{-1}))$ of power series in v^{-1} with integer coefficients, given (with the notation of 10.1) by

$$\sum_{t\in\mathcal{T}/\mathcal{T}^K;n\in\mathbf{Z}} a_{n,t}[t]v^n \mapsto \sum_{n\in\mathbf{Z}} a_{n,0}v^n.$$

We apply this map to both sides of the identity in 10.6(b) with $m_1 = A, m_2 = B$. The corollary follows.

Corollary 10.8. (a) *For any $A, C \in X_\Xi$ we have $f^\epsilon_{C,A} \in \mathbf{Z}[v^{-1}]$.*

(b) *For any $A \in X_\Xi$ there exists a unique alcove $l_\epsilon(A) \in X_\Xi$ such that $f^\epsilon_{C',A} \in v^{-1}\mathbf{Z}[v^{-1}]$ for all $C' \neq l_\epsilon(A)$ and $f^\epsilon_{l_\epsilon(A),A} \notin v^{-1}\mathbf{Z}[v^{-1}]$. In fact, we have $f^\epsilon_{l_\epsilon(A),A} \in \mathrm{sgn}_A + v^{-1}\mathbf{Z}[v^{-1}]$ where $\mathrm{sgn}_A = \pm 1$.*

(c) *If $m \in M^K_{\mathrm{int}}$, then $m' = \sum_{C,n} m'_{C,n} C v^n$, where*

$$m'_{C,n} = \sum_{A,p,q;p+q=n} m_{A,p} f^\epsilon_{C,A;q},$$

is a well defined element of M^K_{int}, denoted by $\theta_\epsilon(m)$. (This agrees with the earlier meaning of $\theta_\epsilon(m)$, when $m \in M^K_c$.) Thus we have a map $\theta_\epsilon : M^K_{\mathrm{int}} \to M^K_{\mathrm{int}}$. For $a \in \mathfrak{A}, m \in M^K_{\mathrm{int}}$, we have $\theta_\epsilon(am) = a^\ddagger \theta_\epsilon(m)$.

(d) *The map $m \to \theta_\epsilon(m)$ of M^K_{int} into itself has square 1.*

(e) *The map $A \mapsto l_\epsilon(A)$ of X_Ξ into itself is an involution and $\mathrm{sgn}_A = \mathrm{sgn}_{l_\epsilon(A)}$.*

The equality 10.7(a), with $A = B$, can be written in the form $\sum_{C \in X_\Xi} (f^\epsilon_{C,A})^2 = 1$, where the sum in the left hand side is convergent in the power series topology of $\mathbf{Z}((v^{-1}))$. This implies immediately (a) and (b). (We use the fact that $\sum_{j=1}^N n_j^2 = 0$ with $n_j \in \mathbf{Z}$ implies $n_j = 0$ for all j and that $\sum_{j=1}^N n_j^2 = 1$ with $n_j \in \mathbf{Z}$ implies $n_j = 0$ for all but one j, for which $n_j = \pm 1$.)

We prove (c). We can find n_0 such that $m_{A,n} = 0$ whenever $n > n_0$. Using (a), we see that $m'_{C,n} = \sum_{A,p,q;p \le n_0; q \le 0; p+q=n} m_{A,p} f^\epsilon_{C,A;q}$. Here (p,q) runs over a finite set and then $m_{A,p} = 0$ unless A is in some finite set; thus, $m'_{C,n}$ is a well defined integer. Clearly, $m'_{C,n} = 0$ if $n > n_0$. We consider, for fixed n, the set $\{C | m'_{C,n} \neq 0\}$. If C is in this set, then there exist (A,p,q) running through some finite set (depending on n) so that $f^\epsilon_{C,A,q} \neq 0$. But then C can take only finitely many values, since $\sum_{C,q} f^\epsilon_{C,A;q} = \theta_\epsilon(A)$ is integrable, by 10.6(a). Thus, m' is integrable. The last assertion of (c) follows from the definitions, using 6.7(c).

We prove (d). The identity in (d) is equivalent to the collection of identities

(f) $\sum_{C,p,q;p+q=n} f^\epsilon_{B,C;p} f^\epsilon_{C,A;q} = \delta_{A,B}\delta_{n,0}$ for $A, B \in X_\Xi$ and $n \in \mathbf{Z}$.

From 10.5 we have, for any $A \in X_\Xi$: $(\Delta\Delta^\ddagger)^N(\theta^2_\epsilon(A)) = \Delta^N \theta_\epsilon(\Delta^N \theta_\epsilon(A)) = \Theta^2_\epsilon(A) = (\Delta\Delta^\ddagger)^N A$. Multiplying with $(\Delta\Delta^\ddagger)^{-N} \in \mathfrak{A}$ gives $\theta^2_\epsilon(A) = A$. This implies (f); thus, (d) is proved.

We prove (e). We write (f) for $A = B$ and $n = 0$, taking (a) into account:

$$\sum_{C,p,q;p \le 0; q \le 0; p+q=0} f^\epsilon_{A,C;p} f^\epsilon_{C,A;q} = 1$$

or, equivalently, $\sum_C f^\epsilon_{A,C;0} f^\epsilon_{C,A;0} = 1$. Now using (b), we can rewrite this in the form $f^\epsilon_{A,l_\epsilon(A);0} = \mathrm{sgn}_A = \pm 1$. Using (b) once more, we deduce that $l_\epsilon(l_\epsilon(A)) = A$ and $\mathrm{sgn}_A = \mathrm{sgn}_{l_\epsilon(A)}$. The corollary is proved.

Corollary 10.9. *For any $A, C \in X_\Xi$, we have $R_{A,C} \in v^{-\nu_K}\mathbf{Z}[v]$. More precisely, $R_{A,C} \in v^{-\nu_K+1}\mathbf{Z}[v]$ if $\kappa_\epsilon(A) \neq l_\epsilon(C)$ and $R_{A,C} \in \mathrm{sgn}_C\, v^{-\nu_K} + v^{-\nu_K+1}\mathbf{Z}[v]$ if $\kappa_\epsilon(A) = l_\epsilon(C)$. The coefficient of $v^{-\nu_K+1}$ in $R_{A,C}$ is equal to $f^\epsilon_{\kappa_\epsilon(A),C;-1}$.*

By the definition of $R_{A,C}$ (see 4.24) we have

$$\sum_{A\in X_\Xi} R_{A,C}A = b(C) = v^{-\nu_K}\kappa_\epsilon\theta_\epsilon(C) = v^{-\nu_K}\kappa_\epsilon(\sum_{B\in X_\Xi} f^\epsilon_{B,C}B)$$
$$= v^{-\nu_K}\sum_{B\in X_\Xi}\overline{f^\epsilon_{B,C}}\kappa_\epsilon(B) = v^{-\nu_K}\sum_{A\in X_\Xi}\overline{f^\epsilon_{\kappa_\epsilon(A),C}}A.$$

Hence $R_{A,C} = v^{-\nu_K}\overline{f^\epsilon_{\kappa_\epsilon(A),C}}$ and the conclusion follows from 10.8.

Conjecture 10.10. $\mathrm{sgn}_A = 1$ *for any* A.

This holds for $K=\emptyset$, by [L1].

10.11. By extending the scalars of the $\mathcal{A}[\mathcal{T}/\mathcal{T}^K]$-module M^K_c to the field of fractions of $\mathcal{A}[\mathcal{T}/\mathcal{T}^K]$, we obtain a vector space $\hat{M}^K_c$ over that field. We can regard $\theta_\epsilon,\kappa_\epsilon$ as isomorphisms $\hat{M}^K_c \xrightarrow{\sim} \hat{M}^K_c$ which are semilinear in a suitable sense. We then define $b,b' : \hat{M}^K_c \xrightarrow{\sim} \hat{M}^K_c$ by $b = v^{-\nu_K}\kappa_\epsilon\theta_\epsilon$ and $b' = \theta_\epsilon b\theta_\epsilon^{-1} = v^{-2\nu_K}\kappa_\epsilon^{-1}b\kappa_\epsilon = v^{-\nu_K}\theta_\epsilon\kappa_\epsilon$.

Proposition 10.12. *The operators* b,b' *from* $\hat{M}^K_c$ *to* $\hat{M}^K_c$ *coincide.*

Since $b^2=1$, it suffices to show that $bb'=1$. We have $bb' = \kappa_\epsilon\theta_\epsilon\theta_\epsilon\kappa_\epsilon$. As we have seen in the proof of 10.5, we have $\theta_\epsilon^2 = 1$, hence $bb' = \kappa_\epsilon\kappa_\epsilon = 1$. The proposition is proved.

10.13. The operator $\theta_\epsilon : \hat{M}^K_c \to M^K_c$ can be characterized (up to multiplication by ± 1) by the following three properties: it is linear over the field of fractions of $\mathcal{A}[\mathcal{T}/\mathcal{T}^K]$; it is $\mathcal{H}$-linear; it has square 1. (This follows from the argument in the proof of 8.9.)

11. The elements $B^\flat$

11.1. We consider the $\mathbf{Z}[v^{-1}]$-submodule $\mathfrak{M}^K_\le = \{m \in M^K_\le | m_A \in \mathbf{Z}[v^{-1}] \quad \forall A\}$ of $M^K_\le$.

Theorem 11.2. *Let* $B \in X_\Xi$.

(a) *There exists a unique element* $B^\flat \in \mathfrak{M}^K_\le$ *such that* $B^\flat - B \in v^{-1}\mathfrak{M}^K_\le$ *and such that* $b(B^\flat) = B^\flat$.

(b) *We have* $B^\flat = \sum_{A\in X_\Xi; A\le B}\Pi_{A,B}A$ *where* $\Pi_{A,B}\in v^{-1}\mathbf{Z}[v^{-1}]$ *for all* $A<B$ *and* $\Pi_{B,B}=1$.

The elements $R_{A,B}\in\mathcal{A}$ (see 4.24) for $A,B\in X_\Xi$ satisfy $R_{A,B}=0$ unless $A\le B$. The equation $b^2=1$ can be written as

(c) $$\sum_{A\in X_\Xi; C\le A\le B}\bar{R}_{A,B}R_{C,A} = \delta_{B,C}$$

for all $C\le B$. We have

(d) $R_{B,B}=1$ for all B.

We apply [L3, 24.2.1] with $(H,\le) = (X_\Xi,\ge)$ and $r_{B,A} = R_{A,B}$ for $A\le B$ in X_Ξ (that is $B\le A$ in H). The assumptions of that lemma are satisfied by (c),(d). Replacing $p_{B,A}$ in the conclusion of that lemma by $\Pi_{A,B}$ we see that there exist elements $\Pi_{A,B}\in\mathbf{Z}[v^{-1}]$ defined for $A\in X_\Xi$ such that $A\le B$ so that

(e) $\Pi_{B,B}=1$;

(f) $\Pi_{A,B}\in v^{-1}\mathbf{Z}[v^{-1}]$ if $A<B$;

(g) $\Pi_{C,B} = \sum_{A \in X_\Xi; C \le A \le B} \bar{\Pi}_{A,B} R_{C,A}$ for all $C \le B$.

Setting $B^\flat = \sum_{A \in X_\Xi; A \le B} \Pi_{A,B} A$, we see that (g) implies $b(B^\flat) = B^\flat$. Thus, the existence part of (a) is established (in the stronger form (b)).

We now prove the uniqueness part of (a). It suffices to prove the following statement.

(h) *Let $G \in v^{-1}\mathfrak{M}^K_{\le}$ be such that $b(G) = G$. Then $G = 0$.*

Assume that $G \ne 0$. Since $\mathrm{supp}(G)$ is bounded above, we can find $B \in \mathrm{supp}(G)$ such that $A \notin \mathrm{supp}(G)$ for any $A > B$.

Assume that B appears with non-zero coefficient in $\overline{G_A} b(A)$. Then $B \in \mathrm{supp}(b(A))$, so that $B \le A$ and $G_A \ne 0$, so that $A \in \mathrm{supp}(G)$. Hence the inequality $B \le A$ cannot be strict. Thus B appears with coefficient 0 in $\overline{G_A} b(A)$ except when $A = B$ in which case the coefficient is $\overline{G_B}$. It follows that B appears in $b(G) = \sum_{A \in X_\Xi} \overline{G_A} b(A)$ with coefficient $\overline{G_B}$. Since $b(G) = G$, the coefficient of B in G is equal to the coefficient of B in $b(G)$. Hence $G_B = \overline{G_B}$. Since $G_B \in v^{-1}\mathbf{Z}[v^{-1}]$, it follows that $G_B = 0$. This contradicts the assumption $B \in \mathrm{supp}(G)$. The proof is complete.

11.3. We make the convention that $\Pi_{A,B} = 0$ if $A \not\le B$. Thus, we have

$$B^\flat = \sum_{A \in X_\Xi} \Pi_{A,B} A.$$

For any A, B in X_Ξ we define $\mu_{A,B} \in \mathbf{Z}$ to be the coefficient of v^{-1} in $\Pi_{A,B} \in \mathbf{Z}[v^{-1}]$. Note that $\mu_{A,B} = 0$ unless $A < B$.

Example 11.4. Let $\epsilon \in \mathbf{S}_{\tilde{\epsilon}}$. We have $(A_\epsilon^+)^\flat = e_{\epsilon,K}$.

For $A \in D_\Xi(\epsilon)$ we have $A \le A_\epsilon^+$ hence $-d(A, A_\epsilon^+) < 0$ if $A \ne A_\epsilon^+$ and $-d(A, A_\epsilon^+) = 0$ if $A = A_\epsilon^+$. Thus, $e_{\epsilon,K} - A_\epsilon^+ \in v^{-1}\mathbf{Z}[v^{-1}]$. On the other hand, we have $b(e_{\epsilon,K}) = e_{\epsilon,K}$ by the definition of b. We see that $e_{\epsilon,K}$ satisfies the defining properties 5.2(a) of $(A_\epsilon^+)^\flat$. Our assertion follows.

11.5. In the remainder of this section we use the conventions of 10.1 relative to $s \in S$.

Proposition 11.6. *Let $s \in S$ and let $B \in X_\Xi$. If $B : \heartsuit$, then*

$$(\tilde{T}_s + v^{-1})B^\flat = (sB)^\flat + \sum_{A : \clubsuit\spadesuit; A < B} \mu_{A,B} A^\flat.$$

Let

$$G = (\tilde{T}_s + v^{-1})B^\flat - \sum_{A : \clubsuit\spadesuit; A < B} \mu_{A,B} A^\flat - (sB)^\flat.$$

We show that G satisfies the assumptions of 11.2(h). We have

$$G = \sum_C \Pi_{C,B}\tilde{T}_s C + \sum_C v^{-1}\Pi_{C,B}C - \sum_{A:\clubsuit\spadesuit;C;A<B} \mu_{A,B}\Pi_{C,A}C - \sum_C \Pi_{C,sB}C$$
$$= \sum_{C:\heartsuit} \Pi_{C,B}sC + \sum_{C:\clubsuit} \Pi_{C,B}(sC + (v - v^{-1})C) + \sum_{C:\spadesuit} v\Pi_{C,B}C$$
$$+ \sum_C v^{-1}\Pi_{C,B}C - \sum_{A:\clubsuit\spadesuit;C;A<B} \mu_{A,B}\Pi_{C,A}C - \sum_C \Pi_{C,sB}C$$
$$= \sum_{C:\clubsuit} \Pi_{sC,B}C + \sum_{C:\heartsuit} \Pi_{sC,B}C + \sum_{C:\clubsuit} \Pi_{C,B}(v - v^{-1})C + \sum_{C:\spadesuit} v\Pi_{C,B}C$$
$$+ \sum_C v^{-1}\Pi_{C,B}C - \sum_{A:\clubsuit\spadesuit;C;A<B} \mu_{A,B}\Pi_{C,A}C - \sum_C \Pi_{C,sB}C.$$

Thus,

$$G_C = \Pi_{sC,B} + v^{-1}\Pi_{C,B} - \Pi_{C,sB} - \sum_{A:\clubsuit\spadesuit;A<B} \mu_{A,B}\Pi_{C,A}$$

if $C : \heartsuit$;

$$G_C = \Pi_{sC,B} + \Pi_{C,B}(v - v^{-1}) + v^{-1}\Pi_{C,B} - \Pi_{C,sB} - \sum_{A:\clubsuit\spadesuit;A<B} \mu_{A,B}\Pi_{C,A}$$

if $C : \clubsuit$;

$$G_C = (v + v^{-1})\Pi_{C,B} - \Pi_{C,sB} - \sum_{A:\clubsuit\spadesuit;A<B} \mu_{A,B}\Pi_{C,A}$$

if $C : \spadesuit$. Modulo $v^{-1}\mathbf{Z}[v^{-1}]$, we have

$$\sum_{A:\clubsuit\spadesuit;A<B} \mu_{A,B}\Pi_{C,A} = \sum_{A:\clubsuit\spadesuit;A<B} \mu_{A,B}\delta_{C,A}$$

and this is $\mu_{C,B}$ if $C : \clubsuit\spadesuit$ and is 0 otherwise. Hence modulo $v^{-1}\mathbf{Z}[v^{-1}]$, we have
$G_C = \delta_{sC,B} - \delta_{C,sB} = 0$ if $C : \heartsuit$;
$G_C = \delta_{sC,B} - \delta_{C,sB} + \mu_{C,B} - \mu_{C,B} = 0$ if $C : \clubsuit$;
$G_C = \mu_{C,B} - \delta_{C,sB} - \mu_{C,B} = 0$ if $C : \spadesuit$.

(If $C : \clubsuit\spadesuit$, we must have $C \neq B$ since $B : \heartsuit$. If $C : \spadesuit$, we must have $C \neq sB$ since $sB : \clubsuit$.) Thus, $G_C = 0 \mod v^{-1}\mathbf{Z}[v^{-1}]$ for all C so that $G \in v^{-1}\mathfrak{M}^K_{\leq}$. On the other hand, using the continuity and the $\mathcal{H}$-antilinearity of b we have

$$b(G) = \overline{(\tilde{T}_s + v^{-1})}b(B^\flat) - \sum_{A:\clubsuit\spadesuit;A<B} \mu_{A,B}b(A^\flat) - b((sB)^\flat)$$
$$= (\tilde{T}_s + v^{-1})B^\flat - \sum_{A:\clubsuit\spadesuit;A<B} \mu_{A,B}A^\flat - (sB)^\flat = G.$$

Thus, $b(G) = G$. Using 11.2(h), we see that $G = 0$. The proposition is proved.

Corollary 11.7. *In the setup of 11.6, we have* $\Pi_{B,sB} = v^{-1}$. *In particular,* $\mu_{B,sB} = 1$.

By 11.6 and its proof we have $G_C = 0$ for all C. In particular $G_B = 0$ so that

(a) $$\Pi_{sB,B} + v^{-1}\Pi_{B,B} - \Pi_{B,sB} - \sum_{A:\clubsuit\spadesuit;A<B} \mu_{A,B}\Pi_{B,A} = 0.$$

Since $B < sB$ we have $\Pi_{sB,B} = 0$. For any A in the sum we have $\Pi_{A,B} = 0$. Hence (a) becomes $v^{-1}\Pi_{B,B} - \Pi_{B,sB} = 0$. The corollary follows.

Lemma 11.8. *Let $s \in S$ and let $B \in X_\Xi$. If $B : \clubsuit\spadesuit$, then*

$$\tilde{T}_s B^\flat = vB^\flat - \sum_{A:\heartsuit; A<B; sA\neq B} \mu_{A,B} A^\flat.$$

Note that the condition $sA \neq B$ in the sum is automatic if $B : \spadesuit$ (since $sA : \clubsuit$). Let

$$G = \tilde{T}_s B^\flat - vB^\flat + \sum_{A:\heartsuit; A<B; sA\neq B} \mu_{A,B} A^\flat.$$

We show that G satisfies the assumptions of 11.2(h). We have

$$\begin{aligned}
G &= \sum_C \Pi_{C,B}\tilde{T}_s C - \sum_C v\Pi_{C,B} C + \sum_{A:\heartsuit; C; A<B; sA\neq B} \mu_{A,B}\Pi_{C,A} C \\
&= \sum_{C:\heartsuit} \Pi_{C,B} sC + \sum_{C:\clubsuit} \Pi_{C,B}(sC + (v - v^{-1})C) + \sum_{C:\spadesuit} v\Pi_{C,B} C \\
&\quad - \sum_C v\Pi_{C,B} C + \sum_{A:\heartsuit; C; A<B; sA\neq B} \mu_{A,B}\Pi_{C,A} C \\
&= \sum_{C:\clubsuit} \Pi_{sC,B} C + \sum_{C; C:\heartsuit} \Pi_{sC,B} C + \sum_{C:\clubsuit} \Pi_{C,B}(v - v^{-1})C + \sum_{C:\spadesuit} v\Pi_{C,B} C \\
&\quad - \sum_C v\Pi_{C,B} C + \sum_{A:\heartsuit; C; A<B; sA\neq B} \mu_{A,B}\Pi_{C,A} C.
\end{aligned}$$

Thus,

$G_C = \Pi_{sC,B} - v\Pi_{C,B} + \sum_{A:\heartsuit; A<B; sA\neq B} \mu_{A,B}\Pi_{C,A}$ if $C : \heartsuit$;

$G_C = \Pi_{sC,B} - v^{-1}\Pi_{C,B} + \sum_{A:\heartsuit; A<B; sA\neq B} \mu_{A,B}\Pi_{C,A}$ if $C : \clubsuit$;

$G_C = \sum_{A:\heartsuit; A<B; sA\neq B} \mu_{A,B}\Pi_{C,A}$ if $C : \spadesuit$.

The sum $\sum_{A:\heartsuit; A<B; sA\neq B} \mu_{A,B}\Pi_{C,A}$ is equal modulo $v^{-1}\mathbf{Z}[v^{-1}]$ to $\mu_{C,B}$ if $C : \heartsuit, sC \neq B$ and to 0 otherwise. If $C : \heartsuit$ we have $C \neq B$. Thus, if $C : \heartsuit, sC \neq B$ we have (modulo $v^{-1}\mathbf{Z}[v^{-1}]$) $G_C = -\mu_{C,B} + \mu_{C,B} = 0$; if $C : \heartsuit, sC = B$ (so that $B : \clubsuit$) then we have (modulo $v^{-1}\mathbf{Z}[v^{-1}]$) $G_C = 1 - \mu_{sB,B}$ and this is 0 by 11.7 applied to sB instead of B. If $C : \clubsuit$ we have $sC \neq B$ since $sC : \heartsuit$ and $B : \clubsuit\spadesuit$. Thus, in this case, we have $G_C = 0$ (modulo $v^{-1}\mathbf{Z}[v^{-1}]$). If $C : \spadesuit$ we again have $G_C = 0$ (modulo $v^{-1}\mathbf{Z}[v^{-1}]$).

Thus, $G_C = 0 \mod v^{-1}\mathbf{Z}[v^{-1}]$ for all C so that $G \in v^{-1}\mathfrak{M}^K_\leq$. On the other hand, using the continuity and the $\mathcal{H}$-antilinearity of b we have

$$b(G) = \overline{(\tilde{T}_s - v)} b(B^\flat) + \sum_{\substack{A:\heartsuit\\ A<B\\ sA\neq B}} \mu_{A,B} b(A^\flat) = (\tilde{T}_s - v)B^\flat + \sum_{\substack{A:\heartsuit\\ A<B\\ sA\neq B}} \mu_{A,B} A^\flat.$$

Thus, $b(G) = G$. We may now use 11.2(h) and we see that $G = 0$. The lemma is proved.

Lemma 11.9. *We preserve the assumptions of 11.8. Let $A \in X_\Xi$ be such that $A : \heartsuit, sA \neq B$. We have $\mu_{A,B} = 0$.*

We may assume that $A \leq B$. We argue by induction on $d(A, B) \geq 0$. If $d(A, B) = 0$ we have $A = B$ so that $\mu_{A,B} = 0$. Assume now that $d(A, B) \geq 1$.

With the notation in the proof of 11.8 we have $G_C = 0$ for any C. In particular, $G_{sA} = 0$. Note that $sA : \clubsuit$ so that the equation $G_{sA} = 0$ reads

$$\Pi_{A,B} - v^{-1}\Pi_{sA,B} + \sum_{A':\heartsuit; A'<B; sA'\neq B} \mu_{A',B}\Pi_{sA,A'} = 0.$$

Taking the coefficient of v^{-1} we obtain

(a) $$\mu_{A,B} + \sum_{A':\heartsuit; sA<A'<B; sA'\neq B} \mu_{A',B}\mu_{sA,A'} = 0.$$

since $sA \neq B$. Since $A : \heartsuit$, we have $A < sA$ hence for A' in the sum we have $A < sA < A' < B$ so that $d(A', B) < d(A, B)$. The induction hypothesis is applicable to A' and gives $\mu_{A',B} = 0$. Introducing this in (a), we obtain $\mu_{A,B} = 0$. The lemma is proved.

Proposition 11.10. *Let $s \in S$ and let $B \in X_\Xi$. If $B : \clubsuit\spadesuit$, then $\tilde{T}_s B^\flat = vB^\flat$.*

This follows immediately from 11.7, 11.8.

11.11. Let $p \mapsto p^\dagger$ be the ring homomorphism $\mathcal{A} \to \mathcal{A}$ which takes v^n to $(-v)^n$ for any n. Let $\chi \to \chi^\dagger$ be the group homomorphism $\mathcal{H} \to \mathcal{H}$ which takes pT_w to $(-1)^{l(w)}p^\dagger T_w$ for any $p \in \mathcal{A}$ and any $w \in W$. It is easy to check that this is a ring homomorphism with square 1, which commutes with $^-: \mathcal{H} \to \mathcal{H}$.

Let $m \to m^\dagger$ be the group homomorphism $M_{\leq}^K \to M_{\leq}^K$ defined by $\sum_{A\in X_\Xi} m_A A \mapsto \sum_{A\in X_\Xi}(m_A)^\dagger \iota_A A$ where $A \mapsto \iota_A$ is a fixed function $X \to \{1, -1\}$ such that $\iota(A)\iota(B)^{-1} = (-1)^{d(A,B)}$ for all $A, B \in X$. (Such a function exists by the additivity property of d.) It is clear that $m \mapsto m^\dagger$ has square 1. One checks easily that $(\chi m)^\dagger = \chi^\dagger m^\dagger$ for all $\chi \in \mathcal{H}, m \in M_{\leq}^K$.

Let $b^\dagger : M_{\leq}^K \to M_{\leq}^K$ be defined by $b^\dagger(m) = (b(m^\dagger))^\dagger$. We show that $b^\dagger = b$. First, $b^\dagger$ is continuous (in the sense of 4.13) since $m \to m^\dagger$ is obviously continuous and b is continuous, by definition. Next, if $\chi \in \mathcal{H}, m \in M_{\leq}^K$, we have

$$b^\dagger(hm) = (b((hm)^\dagger))^\dagger = (b(h^\dagger m^\dagger))^\dagger = (\overline{h^\dagger} b(m^\dagger))^\dagger = \bar{h} b^\dagger(m)$$

so that $b^\dagger$ is $\mathcal{H}$-antilinear. Finally, if $\epsilon \in \mathbf{S}_{\tilde{\epsilon}}$, we have

$$(e_{\epsilon,K})^\dagger = \sum_{A\in D_\Xi(\epsilon)} v^{-d(A,A_\epsilon^+)}(-1)^{-d(A,A_\epsilon^+)}\iota_A A = \iota_{A_\epsilon^+} e_{\epsilon_K}$$

hence

$$b^\dagger(e_{\epsilon,K}) = (b((e_{\epsilon,K})^\dagger))^\dagger = \iota_{A_\epsilon^+}(b(e_{\epsilon,K}))^\dagger = \iota_{A_\epsilon^+}(e_{\epsilon,K})^\dagger = \iota_{A_\epsilon^+}\iota_{A_\epsilon^+} e_{\epsilon,K} = e_{\epsilon,K}.$$

Thus, $b^\dagger$ satisfies the defining properties of b hence $b^\dagger = b$.

Proposition 11.12. *For any A, B in X_Ξ we have $(\Pi_{A,B})^\dagger = (-1)^{d(A,B)}\Pi_{A,B}$. In particular, $\mu_{A,B} = 0$ if $d(A, B)$ is even.*

An equivalent statement is that

(a) $$\iota_B(B^\flat)^\dagger = B^\flat$$

for any $B \in X_\Xi$. It is therefore enough to show that $\iota_B(B^\flat)^\dagger$ satisfies the defining properties 11.2(a) of $B^\flat$. First, we have $\iota_B(B^\flat)^\dagger = \iota_B \sum_{A\in X_\Xi} \Pi^\dagger_{A,B}\iota_A A$ and this is clearly equal to B modulo $v^{-1}\mathfrak{M}^K_\le$. Next, we have

$$b(\iota_B(B^\flat)^\dagger) = \iota_B(b^\dagger(B^\flat))^\dagger = \iota_B(b(B^\flat))^\dagger = \iota_B(B^\flat)^\dagger$$

(we have used $b^\dagger = b$, see 11.11). Thus, (a) is verified. The proposition is proved.

11.13. For any $A \in X_\Xi$ let $\mathfrak{I}_A$ be the set of all $s \in S$ such that either $sA \notin \Xi$ or $sA \in \Xi, s \in \mathcal{L}(A)$ (or equivalently, $A : \clubsuit\spadesuit$, relative to s, in the notation of 11.5).

For $A, B \in X_\Xi$ such that $\mathfrak{I}_A \not\subset \mathfrak{I}_B$, we set

$$\mu'_{A,B} = \begin{cases} \mu_{A,B} & \text{if } A \le B, \\ 1 & \text{if } B < A = sB \text{ for some } s \in S, \\ 0 & \text{otherwise .} \end{cases}$$

We prove the following statement.

(a) *Let $n \ge 1$ and let $(A_0, A_1, \dots, A_n) \in X_\Xi^{n+1}$ be such that $\mu'_{A_0,A_1}\mu'_{A_1,A_2}\cdots\mu'_{A_{n-1},A_n}$ is defined and non-zero. Then there exist $w_1, w_2, \dots, w_n$ in W such that $l(w_1) \le 1, l(w_2) \le 2, \dots, l(w_n) \le n$ and such that $A_0 \le w_1A_1 \le w_2A_2 \le \cdots \le w_nA_n$.*

We argue by induction on n. For $n = 1$ this follows directly from the definition of μ'. Assume now that $n \ge 2$. By the induction hypothesis there exist $w_1, w_2, \dots, w_{n-1}$ in W such that $l(w_1) \le 1, l(w_2) \le 2, \dots, l(w_{n-1}) \le n-1$ and such that $A_0 \le w_1A_1 \le w_2A_2 \le \cdots \le w_{n-1}A_{n-1}$. Since $\mu'_{A_{n-1},A_n} \ne 0$ we have either $A_{n-1} \le A_n$ or $A_{n-1} = sA_n$ for some $s \in S$. In the second case, we set $w_n = w_{n-1}s$ and we have $l(w_n) \le n$ and $w_{n-1}A_{n-1} = w_nA_n$.

In the first case we apply the following statement with $A = A_{n-1}, B = B_n, w = w_{n-1}$ and we set $w_n = w'$.

Let $A, B \in \Xi$ be such that $A \le B$ and let $w \in W$. Then there exists $w' \in W$ such that $l(w') \le l(w)$ and $wA \le w'B$. We argue by induction on $l(w)$. The result is trivial in the case where $l(w) = 0$. Assume now that $l(w) \ge 1$. Then $w = sw_1$ where $s \in S, l(w_1) = l(w)-1$. By the induction hypothesis, there exists $w'_1 \in W$ such that $l(w'_1) \le l(w_1)$ and $w_1A \le w'_1B$. Let $A' = w_1A, B' = w'_1B$ so that $A' \le B'$. From [L1, 3.2] it follows that we have either $sA' \le B'$ or $sA' \le sB'$. Hence $wA \le w'_1B$ or $wA \le sw'_1B$. This completes the proof.

Theorem 11.14. *The triple $(X_\Xi, (\mathfrak{I}_A)_{A\in X_\Xi}, \mu')$ is a W-graph (see A.2).*

We show that the finiteness condition A.1(a) is satisfied in our case. Let $A, B \in X_\Xi$ and let $n \ge 1$. Let $X_{\Xi,n}(A, B)$ be the set of all $(A_0, A_1, \dots, A_n) \in X_\Xi^{n+1}$ such that $A_0 = A, A_n = B$ and $\mu'_{A_0,A_1}\mu'_{A_1,A_2}\cdots\mu'_{A_{n-1},A_n}$ is defined and non-zero. It is enough to show that $X_{\Xi,n}(A, B)$ is finite. Let $(A_0, A_1, \dots, A_n) \in X_{\Xi,n}(A, B)$. By 11.13(a) there exist $w_1, w_2, \dots, w_n$ in W such that $l(w_1) \le 1, l(w_2) \le 2, \dots, l(w_n) \le n$ and such that

(a) $A_0 \le w_1A_1 \le w_2A_2 \le \cdots \le w_nA_n$.

Then each of $w_1, w_2, \dots, w_n$ can only take finitely many values. In particular, $w_nA_n = w_nB$ takes only finitely many values. Since $A_0 = A$, we deduce from (a) and 4.15(b) that each of $w_1A_1, w_2A_2, \dots, w_{n-1}A_{n-1}$ takes only finitely many values. Hence each of $A_1, A_2, \dots, A_{n-1}$ takes only finitely many values. Thus, the finiteness condition A.1(a) is verified in our case.

Next we note that any subset of Ξ that is of finite type in the sense of A.3 is necessarily bounded above. It is enough to check that for any $B \in X_\Xi$ and any $n \geq 1$, the set $\{A \in X_\Xi | X_{\Xi,n}(A, B) \neq \emptyset\}$ is bounded above. Let A be an element of this set. Using again 11.13(a), we see that $A \leq wB$ for some $w \in W$ such that $l(w) \leq n$. Since w takes only finitely many values, we see that our set is indeed bounded above.

As in A.3, we consider the set $\mathcal{E}$ consisting of all formal sums $\sum_{A\in X_\Xi} c_A A$ with $c_A \in \mathcal{A}$ such that $\{A \in X_\Xi | c_A \neq 0\}$ is of finite type (and in particular, bounded above). For any $\sum_{A\in X_\Xi} c_A A \in \mathcal{E}$, the sum $\sum_{A\in X_\Xi} c_A A^\flat$ is a well defined element of $M^K_\leq$. (The family of elements $(c_A A^\flat)_{A\in X_\Xi}$ is locally finite in M^K.) The correspondence $\sum_{A\in X_\Xi} c_A A \mapsto \sum_{A\in X_\Xi} c_A A^\flat \in M^K_\leq$ identifies $\mathcal{E}$ with an $\mathcal{A}$-submodule of $M^K_\leq$. From 11.6 and 11.10 we see that $\mathcal{E}$ is a $\mathcal{H}$-submodule of $M^K_\leq$ and that the operator τ'_s on $\mathcal{E}$ coincides with the operator $\tilde{T}_s$ in this $\mathcal{H}$-module structure. It follows that the operators τ'_s satisfy the identity defining a W-graph (see A.4(c)). The theorem is proved.

Proposition 11.15. *Let $B \in X_\Xi$ and let $t \in \mathcal{T}'$. We have $\lfloor t \rfloor \circ B^\flat = (\gamma_t B)^\flat$. Equivalently, $\Pi_{\gamma_t A, \gamma_t B} = \Pi_{A,B}$ for all $A, B \in X_\Xi$. In particular, $\mu_{\gamma_t A, \gamma_t B} = \mu_{A,B}$ for all $A, B \in X_\Xi$.*

We write $B^\flat = \sum_{A\in X_\Xi} \Pi_{A,B} A$. We have $\lfloor t \rfloor \circ B^\flat = \sum_{A\in X_\Xi} \Pi_{A,B} \gamma_t A$. Thus, $\lfloor t \rfloor \circ B^\flat - \gamma_t B = \sum_{A\in X_\Xi; A\neq B} \Pi_{A,B} \gamma_t A \in v^{-1}\mathfrak{M}^K_\leq$. Using the definition 11.2(a) of $(\gamma_t B)^\flat$ we see that it remains to verify the equality $b(\lfloor t \rfloor \circ B^\flat) = \lfloor t \rfloor \circ B^\flat$. But this follows from 6.7 and $b(B^\flat) = B^\flat$. The proposition is proved.

Proposition 11.16. *Let $\epsilon \in \mathbf{S}$. (See 2.3.)*

(a) *There is a unique alcove $B \in D_\Xi(\epsilon)$ such that $d(C, B) > 0$ for all $C \in D_\Xi(\epsilon), C \neq B$.*
(b) *We have $B^\flat = \sum_{A\in D_\Xi(\epsilon)} v^{-d(A,B)} A$.*

By 2.12(b), we can choose $t \in \mathcal{T}'$ so that $\epsilon + t \in \mathbf{S}_{\tilde{\epsilon}}$. Since γ_{-t} is a homeomorphism $\Xi \to \Xi$ carrying $\epsilon + t$ to ϵ, an alcove $A \subset \Xi$ contains $\epsilon + t$ in its closure if and only if $\gamma_{-t} A$ contains ϵ in its closure. It follows that $A \mapsto \gamma_{-t} A$ is a bijection $D_\Xi(\epsilon + t) \to D_\Xi(\epsilon)$. In particular, $B = \gamma_{-t} A^+_{\epsilon+t} \in D_\Xi(\epsilon)$. From 11.4 and 11.15 we see that

$$B^\flat = \lfloor -t \rfloor \circ e_{\epsilon+t,K} = \sum_{A\in D_\Xi(\epsilon+t)} v^{-d(A,A^+_{\epsilon+t})} \gamma_{-t} A.$$

We make the substitution $\gamma_{-t} A = C$ and use 2.12(c); we obtain

$$B^\flat = \sum_{C\in D_\Xi(\epsilon)} v^{-d(C,B)} C.$$

This, together with 11.2(b) shows that $d(C, B) > 0$ for all $C \in D_\Xi(\epsilon), C \neq B$. The proposition follows.

Theorem 11.17. *Let $\epsilon \in \mathbf{S}_{\tilde{\epsilon}}$. For any $A, C \in X_\Xi$, we have $\Pi_{A,C} \in v^{-\nu_K}\mathbf{Z}[v]$. More precisely, $\Pi_{A,C} \in v^{-\nu_K+1}\mathbf{Z}[v]$ if $\kappa_\epsilon(A) \neq l_\epsilon(C)$ and $\Pi_{A,C} \in \mathrm{sgn}_C v^{-\nu_K} + v^{-\nu_K+1}\mathbf{Z}[v]$ if $\kappa_\epsilon(A) = l_\epsilon(C)$. (Notation of 10.8.)*

We have

(a) $\Pi_{A,C} = R_{A,C} + \sum_{B\in X_\Xi; B<C; A\leq B} \bar{\Pi}_{B,C} R_{A,B}$.

(For $A \le C$ this follows from 11.2(g); if $A \not\le C$, both sides of (a) are 0.) For any B in the sum we have $\Pi_{B,C} \in v^{-1}\mathbf{Z}[v^{-1}]$ hence $\bar{\Pi}_{B,C} \in v\mathbf{Z}[v]$; on the other hand, by 10.9, we have $R_{A,B} \in v^{-\nu_K}\mathbf{Z}[v]$. Hence $\bar{\Pi}_{B,C}R_{A,B} \in v^{-\nu_K+1}\mathbf{Z}[v]$. Hence our sum over B belongs to $v^{-\nu_K+1}\mathbf{Z}[v]$. Combining this with the conclusion of 10.9, we see from (a) that the theorem holds.

11.18. Let us write $\Pi_{A,B} = \sum_{n\in\mathbf{Z}} \Pi_{A,B;n}v^n$ with $\Pi_{A,B;n} \in \mathbf{Z}$. Picking up the coefficients of $v^{-\nu_K+1}$ in the two sides of 11.17(a) and using 10.9, 10.8, we obtain

(a) $\Pi_{A,B;-\nu_K+1} - \Pi_{l_\epsilon\kappa_\epsilon(A),B;-1}\operatorname{sgn}_{\kappa_\epsilon(A)} = f^\epsilon_{\kappa_\epsilon(A),B;-1}$ where $f^\epsilon_{\kappa_\epsilon(A),B;-1}$ is as in 10.7. We have used that $A \le l_\epsilon\kappa_\epsilon(A)$, which follows from 10.9).

11.19. Let $\epsilon \in \mathbf{S}_{\tilde{\epsilon}}$. Let $A, B \in D_\Xi(\epsilon)$. Write $A = A_{\epsilon,y}, B = A_{\epsilon,w}$ where $y, w \in W^I_*$ (see 4.9). Then the polynomial $\Pi_{A,B}$ is (up to a power of v) the same as the polynomial attached to $y, w \in W^I$ in [KL1] (with q replaced by v^2). This is proved in the same way as in the case $K = \emptyset$ [L1, 11.15].

12. The elements $B^\sharp$

12.1. We consider the $\mathbf{Z}[v^{-1}]$-submodule $\mathfrak{M}^K_\ge = \{m \in M^K_\ge | m_A \in \mathbf{Z}[v^{-1}] \quad \forall A\}$ of $M^K_\ge$.

Theorem 12.2. *Let $B \in X_\Xi$.*

(a) *There exists a unique element $B^\sharp \in \mathfrak{M}^K_\ge$ such that $B^\sharp - B \in v^{-1}\mathfrak{M}^K_\ge$ and such that $\tilde{b}(B^\sharp) = B^\sharp$.*

(b) *We have $B^\sharp = \sum_{A\in X_\Xi;B\le A}\Pi'_{B,A}A$ where $\Pi'_{B,A} \in v^{-1}\mathbf{Z}[v^{-1}]$ for all $B < A$ and $\Pi'_{B,B} = 1$.*

(c) *For any $C \in X_\Xi$ such that $B \le C$, we have $\sum_{D\in X_\Xi;B\le D\le C}\Pi'_{B,D}\Pi_{D,C} = \delta_{B,C}$.*

(d) *For any $B, C \in X_\Xi$ we have $(C^\flat, B^\sharp) = \delta_{C,B}$.*

(e) *For any $C \in X_\Xi$ such that $B \le C$, we have $\sum_{D\in X_\Xi;B\le D\le C}\Pi_{B,D}\Pi'_{D,C} = \delta_{B,C}$.*

For any C such that $B \le C$ we define $\Pi'_{B,C} \in \mathbf{Z}[v^{-1}]$ by induction on $d(B,C)$ from the formula $\sum_{D\in X_\Xi;B\le D\le C}\Pi'_{B,D}\Pi_{D,C} = \delta_{B,C}$ together with $\Pi'_{B,B} = 1$. (If D in the sum satisfies $B \le D < C$, then $d(B,D) < d(B,C)$; the term corresponding to $D = C$ is $\Pi'_{B,D}$.) From the inductive formula and 11.2(e),(f), we see that $\Pi'_{B,C} \in v^{-1}\mathbf{Z}[v^{-1}]$ if $B < C$. We show that $\Pi''_{B,C} = \sum_{A\in X_\Xi;B\le A\le C}\overline{\Pi}'_{B,A}\overline{R}_{A,C}$ satisfies the same inductive formula as $\Pi'_{B,C}$. Indeed, we have $\Pi''_{B,B} = \Pi'_{B,B} = 1$ and

$$\begin{aligned}\sum_{D;B\le D\le C}&\Pi''_{B,D}\Pi_{D,C}\\ &= \sum_{A,D;B\le A\le D\le C}\overline{\Pi'}_{B,A}\overline{R}_{A,D}\Pi_{D,C} = \sum_{A;B\le A\le C}\overline{\Pi'}_{B,A}\overline{P}_{A,C} = \delta_{B,C}\end{aligned}$$

(the second equality follows from 11.2(g)). Our assertion is verified. It follows that $\Pi''_{B,C} = \Pi'_{B,C}$, that is,

(f) $$\sum_{A\in X_\Xi;B\le A\le C}\overline{\Pi'}_{B,A}\overline{R}_{A,C} = \Pi'_{B,C}$$

for all C such that $B \le C$. We set $B^\sharp = \sum_{A\in X_\Xi; B\le A} \Pi'_{B,A} A$. We have

$$\tilde{b}(B^\sharp) = \sum_{A; B\le A} \overline{\Pi'}_{B,A}\tilde{b}(A) = \sum_{A,C; B\le A\le C} \overline{\Pi'}_{B,A}\overline{R}_{A,C}C = \sum_{C; B\le C} \Pi'_{B,C}C = B^\sharp$$

where the second equality uses 9.6(c) and the third equality uses (f). Thus the existence part of (a) is proved. The proof of the uniqueness part of (a) is entirely analogous to the proof of the corresponding part of 11.2(a). Thus (a) is proved. The arguments above yield at the same time (b) and (c).

We prove (d). We have

$$\begin{aligned}(C^\flat, B^\sharp) &= (\sum_{D\in X_\Xi; D\le C} \Pi_{D,C}D, \sum_{D\in X_\Xi; B\le D} \Pi'_{B',D}D)\\ &= \sum_{D\in X_\Xi; B\le D\le C} \Pi_{D,C}\Pi'_{B',D} = \delta_{B,C}\end{aligned}$$

where the last equality uses (c). This proves (d).

We prove (e). For $B \le C$ we set $x_{B,C} = \sum_{D\in X_\Xi; B\le D\le C} \Pi_{B,D}\Pi'_{D,C}$. Clearly, $x_{B,B} = 1$. We now assume that $B < C$. Using (c) we have

$$\begin{aligned}x_{B,C} &= \sum_{D\in X_\Xi; B\le D\le C} \Pi_{B,D}\Pi'_{D,C} = \sum_{A,D\in X_\Xi; B\le A\le D\le C} \Pi_{B,A}\delta_{A,D}\Pi'_{D,C}\\ &= \sum_{A,D,F\in X_\Xi; B\le A\le F\le D\le C} \Pi_{B,A}\Pi'_{A,F}\Pi_{F,D}\Pi'_{D,C}\\ &= \sum_{F\in X_\Xi; B\le F\le C} x_{B,F}x_{F,C} = 2x_{B,C} + \sum_{F\in X_\Xi; B<F<C} x_{B,F}x_{F,C}.\end{aligned}$$

Thus, $x_{B,C} = -\sum_{F\in X_\Xi; B<F<C} x_{B,F}x_{F,C}$. This shows by induction on $d(B,C)$ that $x_{B,C} = 0$. The theorem is proved.

12.3. The W-graph complementary (see A.6) to the W-graph $(X_\Xi, (\mathfrak{I}_A)_{A\in X_\Xi}, \mu')$ in 11.14 is $(X_\Xi, (\tilde{\mathfrak{I}}_A)_{A\in X_\Xi}, \tilde{\mu}')$ where, for any $A \in X_\Xi$, $\tilde{\mathfrak{I}}_A$ is the set of all $s \in S$ such that $sA \in \Xi, s \notin \mathcal{L}(A)$ (or equivalently, $A : \heartsuit$ relative to s with the convention of 9.1) and, for $A, B \in X_\Xi$ such that $\tilde{\mathfrak{I}}_A \not\subset \tilde{\mathfrak{I}}_B$, we set

$$\tilde{\mu}'_{A,B} = \begin{cases}\mu_{B,A} & \text{if } B \le A,\\ 1 & \text{if } B < A = sB \text{ for some } s \in S,\\ 0 & \text{otherwise .}\end{cases}$$

Hence the following result holds.

Theorem 12.4. *The triple $(X_\Xi, (\tilde{\mathfrak{I}}_A)_{A\in X_\Xi}, \tilde{\mu}')$ is a W-graph in the sense of A.2.*

The theorem can also be deduced from the following result.

Proposition 12.5. *Let $s \in S$ and let $B \in X_\Xi$. We use the convention of 9.1 relative to s.*

(a) *If $B : \heartsuit$, then $\tilde{T}_s B^\sharp = -v^{-1}B^\sharp$.*

(b) *If $B : \clubsuit$, then $(\tilde{T}_s - v)B^\sharp = (sB)^\sharp + \sum_{A:\heartsuit; B<A} \mu_{B,A}A^\sharp$.*

(c) *If $B : \spadesuit$, then $(\tilde{T}_s - v)B^\sharp = \sum_{A:\heartsuit; B<A} \mu_{B,A}A^\sharp$.*

In the proof we shall use the following fact: if $m \in M^K_\ge$ satisfies $(A^\flat, m) = 0$ for any $A \in X_\Xi$, then $m = 0$. (Indeed, if we assume that $m \ne 0$, we can find

$A \in X_\Xi$ such that $m_A \neq 0$ and such that $m_{A'} = 0$ for any $A' < A$. But then $(A^\flat, m) = m_A \neq 0$, contradiction.)

A statement equivalent with that of the proposition is that

(d) $$\tilde{T}_s B^\sharp = -v^{-1} B^\sharp \delta_{s \notin \mathfrak{I}_B} + v B^\sharp \delta_{s \in \mathfrak{I}_B} + \sum_{C; s \notin \mathfrak{I}_C} \mu'_{B,C} \delta_{s \in \mathfrak{I}_B} C^\sharp$$

where $\delta_{s \in \mathfrak{I}_B}$ is 1 if $s \in \mathfrak{I}_B$ and 0 if $s \in \mathfrak{I}_B$ while $\delta_{s \notin \mathfrak{I}_B}$ is 0 if $s \in \mathfrak{I}_B$ and 1 if $s \in \mathfrak{I}_B$. Let $G(B)$ be the left hand side of (d) minus the right hand side of (d). We can rewrite 11.6 and 11.10 in the following form:

(e) $$\tilde{T}_s A^\flat = v A^\flat \delta_{s \in \mathfrak{I}_A} - v^{-1} A^\flat \delta_{s \notin \mathfrak{I}_A} + \sum_{C; s \in \mathfrak{I}_C} \mu'_{C,A} \delta_{s \notin \mathfrak{I}_A} C^\flat$$

for any $A \in X_\Xi$. Let $G'(A)$ be the left hand side of (e) minus the right hand side of (e). Thus we have $G'(A) = 0$. Using 12.2(d), we compute:

$$(A^\flat, G(B)) = (A^\flat, \tilde{T}_s B^\sharp) + v^{-1} \delta_{A,B} \delta_{s \notin \mathfrak{I}_B} - v \delta_{A,B} \delta_{s \in \mathfrak{I}_B} - \sum_{C; s \notin \mathfrak{I}_C} \mu'_{B,C} \delta_{s \in \mathfrak{I}_B} \delta_{A,C},$$

$$0 = (G'(A), B^\sharp) = (\tilde{T}_s A^\flat, B^\sharp) - v \delta_{A,B} \delta_{s \in \mathfrak{I}_A} + v^{-1} \delta_{A,B} \delta_{s \notin \mathfrak{I}_A} - \sum_{C; s \in \mathfrak{I}_C} \mu'_{C,A} \delta_{s \notin \mathfrak{I}_A} \delta_{C,B}.$$

Subtracting, we find $(A^\flat, G(B)) = (A^\flat, \tilde{T}_s B^\sharp) - (\tilde{T}_s A^\flat, B^\sharp) = 0$ where the last equality follows from 9.2. Thus we have $(A^\flat, G(B)) = 0$ for all A. It follows that $G(B) = 0$. The proposition is proved.

13. Conjectures, comments

13.1. To simplify the statements in this section we will assume that Conjecture 10.10 holds; without this assumption, we would have to insert appropriate signs in the various statements that follow. We fix $\epsilon \in \mathbf{S}_{\tilde{\epsilon}}$. For $B \in X_\Xi$ we set

(a) $$B^\natural = \theta_\epsilon(l_\epsilon(B)^\flat) \in M_{\geq}^K = v^{-\nu_K} \kappa_\epsilon^{-1}(l_\epsilon(B)^\flat).$$

(The second equality holds since $l_\epsilon(B)^\flat)$ is fixed by b.) We have

(b) $B^\natural - B \in v^{-1} \mathfrak{M}_{\geq}^K$,

(c) $\theta_\epsilon b \theta_\epsilon^{-1}(B^\natural) = B^\natural$,

where $\mathfrak{M}_{\geq}^K$ is as in 12.1. Indeed, from 10.8(a) and the continuity of θ_ϵ we see that θ_ϵ maps $\mathfrak{M}_{\leq}^K$ into $\mathfrak{M}_{\geq}^K$. Since $l_\epsilon(B)^\flat = l_\epsilon(B) \mod v^{-1}\mathfrak{M}_{\leq}^K$, it follows that $B^\natural = \theta_\epsilon(l_\epsilon(B)) \mod v^{-1}\mathfrak{M}_{\geq}^K$. But $\theta_\epsilon(l_\epsilon(B)) = B \mod v^{-1}\mathfrak{M}_{\geq}^K$, by the definition of $l_\epsilon(B)$. Hence (b) holds. (c) is obvious.

In the remainder of this section we will assume that the following conjecture holds and we will derive some consequences of it.

Conjecture 13.2. *For any $B \in X_\Xi$ we have $B^\flat \in M_c^K$.*

This holds when $K = \emptyset$, by [L1], and in the examples of §14,§15.

Consequence 13.3. *For any $B \in X_\Xi$, we have $B^\flat = B^\natural$.*

Since, by assumption, $l_\epsilon(B)^\flat \in M_c^K$, and κ_ϵ^{-1} maps M_c^K into M_c^K we see (from 13.1(a)) that $B^\natural \in M_c^K$. Hence from 13.1(b) we have that

(a) $B^\natural - B \in v^{-1} \mathfrak{M}_c^K$

where $\mathfrak{M}_c^K = \sum_A \mathbf{Z}[v^{-1}]A \subset M_c^K$. By 10.12, we have $b'(B^\natural) = b(B^\natural)$ (equality in

$\hat{M}_c^K$); this is in fact an equality in M_c^K since $b'(B^\natural) = \theta_\epsilon b \theta_\epsilon^{-1} b(B^\natural) = B^\natural \in M_c^K$. Thus,

(b) $b(B^\natural) = B^\natural$.

From (a),(b), we see that $B^\natural$ satisfies the defining properties 11.2(a) of $B^\flat$ hence $B^\natural = B^\flat$.

Remark 13.4. Conversely, if the conclusion of 13.3 holds, then 13.2 holds as well. Indeed, in this case we would have $B^\flat \in M_{\leq}^K, B^\flat = B^\natural \in M_{\geq}^K$ hence $B^\flat \in M_{\leq}^K \cap M_{\geq}^K = M_c^K$.

Consequence 13.5. *We have $\theta_\epsilon(C^\flat) = l_\epsilon(C)^\flat$ for all $C \in X_\Xi$. In particular, θ_ϵ defines a bijection of $\{C^\flat | C \in X_\Xi\}$ onto itself.*

This follows from 13.3 using 13.1(a).

Consequence 13.6. *We have $\Pi_{A,B} = v^{-\nu_K} \overline{\Pi}_{\kappa_\epsilon(A), l_\epsilon(B)}$ for any $A, B \in X_\Xi$.*

This follows from 13.3, using 13.1(a).

We take the coefficient of $v^{-\nu_K+1}$ in the two sides of the previous equality and introduce it in 11.18(a); we then replace $\kappa_\epsilon(A)$ by A and we obtain

(a) $\mu_{A,l_\epsilon(B)} - \mu_{l_\epsilon(A),B} = f^\epsilon_{A,B;-1}$.

Consequence 13.7. *We use notation of 11.13.*

(a) *For any $B \in X_\Xi$, the set $\mathfrak{I}_{l_\epsilon(B)}$ is the image of $\mathfrak{I}_B$ under the involution $s \mapsto s^*$ of S.*

(b) *Let $A, B \in X_\Xi$ be such that $\mathfrak{I}_A \not\subset \mathfrak{I}_B$. Then $\mu'_{A,B} = \mu'_{l_\epsilon(A), l_\epsilon(B)}$.*

We prove (a). If $s \notin \mathfrak{I}_B$, then by 11.6 and 13.5, we have

(c) $$\theta_\epsilon(\tilde{T}_s B^\flat) = \theta_\epsilon(-v^{-1}B^\flat + \sum_{C; s \in \mathfrak{I}_C} \mu'_{C,B} C^\flat) = -v^{-1} l_\epsilon(B)^\flat + \sum_{C; s \in \mathfrak{I}_C} \mu'_{C,B} l_\epsilon(C)^\flat.$$

In particular,

(d) $\theta_\epsilon(\tilde{T}_s B^\flat) \neq v l_\epsilon(B)^\flat$.

We show that

(e) $s^* \notin \mathfrak{I}_{l_\epsilon(B)}$.

Assume that $s^* \in \mathfrak{I}_{l_\epsilon(B)}$. Then, by 11.10, $\tilde{T}_{s^*} l_\epsilon(B)^\flat = v l_\epsilon(B)^\flat$. Applying θ_ϵ, we obtain $v\theta_\epsilon l_\epsilon(B)^\flat = \theta_\epsilon \tilde{T}_{s^*} l_\epsilon(B)^\flat = \tilde{T}_s \theta_\epsilon l_\epsilon(B)^\flat$ hence (using 13.5) $\tilde{T}_s B^\flat = v B^\flat$. This contradicts (d); (e) is proved. Thus, $\mathfrak{I}_{l_\epsilon(B)}$ is contained in the image of $\mathfrak{I}_B$ under $s \mapsto s^*$. The same argument applied to $l_\epsilon(B)$ instead of B shows that $\mathfrak{I}_B$ is contained in the image of $\mathfrak{I}_{l_\epsilon(B)}$ under $s \mapsto s^*$. This proves (a).

We prove (b). Choose $s \in S$ such that $s \in \mathfrak{I}_A, s \notin \mathfrak{I}_B$. Then (c) above holds. Using (a), 13.5 and 11.6 we have

$$\begin{aligned}\theta_\epsilon(\tilde{T}_s B^\flat) = \tilde{T}_{s^*}\theta_\epsilon B^\flat = \tilde{T}_{s^*} l_\epsilon(B)^\flat &= -v^{-1} l_\epsilon(B)^\flat + \sum_{C; s^* \in \mathfrak{I}_C} \mu'_{C, l_\epsilon(B)} C^\flat \\ &= -v^{-1} l_\epsilon(B)^\flat + \sum_{C'; s \in \mathfrak{I}_{C'}} \mu'_{l_\epsilon(C'), l_\epsilon(B)} l_\epsilon(C')^\flat\end{aligned}$$

where the last equality is obtained by the substitution $C = l_\epsilon(C')$, using (a). Comparing with (c), we deduce

$$-v^{-1} l_\epsilon(B)^\flat + \sum_{C; s \in \mathfrak{I}_C} \mu'_{C,B} l_\epsilon(C)^\flat = -v^{-1} l_\epsilon(B)^\flat + \sum_{C; s \in \mathfrak{I}_C} \mu'_{l_\epsilon(C), l_\epsilon(B)} l_\epsilon(C)^\flat.$$

We now compare the coefficients of $l_\epsilon(A)^\flat$ in the two sides of the last equality; (b) follows.

Consequence 13.8. *Let $A \in X_\Xi$. Then the set $\{B \in X_\Xi | \Pi_{A,B} \neq 0\}$ is finite.*

Let $C_1, C_2, \dots, C_n$ be a set of representatives for the orbits of the γ_t-action of $\mathcal{T}/\mathcal{T}^K$-orbits on X_Ξ. For $j \in [1,n]$, let $F_j = \{D \in X_\Xi | \Pi_{D,C_j} \neq 0\}$ (a finite set, by 13.2). If $\Pi_{A,B} \neq 0$, then, by 11.15, we have $\Pi_{\gamma_t A, C_j} \neq 0$ where $t \in \mathcal{T}/\mathcal{T}^K$ and $j \in [1,n]$ are such that $\gamma_t B = C_j$. Note that t, j are uniquely determined by B. Thus, it is enough to show that the set of all pairs (t,j) in $\mathcal{T}/\mathcal{T}^K \times [1,n]$ such that $\gamma_t A \in F_j$ is finite. But for each $j \in [1,n]$ there are only finitely many t such that $\gamma_t A \in F_j$ (since F_j is finite).

Consequence 13.9. *Let $M^K_{d'}$ be the $\mathcal{A}$-submodule of $M^K_\leq$ spanned by $\{B^\flat | B \in X_\Xi\}$ (which is then an $\mathcal{A}$-basis of M^K_d). Then $M^K_{d'}$ is an $\mathcal{H}$-submodule of M^K_c.*

The fact that $M^K_{d'} \subset M^K_c$ follows directly from 13.2. The fact that $M^K_{d'}$ is an $\mathcal{H}$-submodule of M^K follows from 11.6, 11.10. (The sum appearing in 11.6 is finite, by 13.2.)

Consequence 13.10. *Let M^K_d be the $\mathcal{A}$-submodule of $M^K_\geq$ spanned by $\{B^\sharp | B \in X_\Xi\}$ (which is then an $\mathcal{A}$-basis of M^K_d).*

(a) *M^K_d is an $\mathcal{H}$-submodule of M^K.*

(b) *$M^K_c \subset M^K_d$.*

From 13.8 we see that, for fixed $B \in X_\Xi$, the set $\{A \in X_\Xi | \mu_{B,A} \neq 0\}$ is finite. Hence (a) follows from 12.5. From 12.2(e) we see that, for any $C \in X_\Xi$, we have $C = \sum_{B \in X_\Xi} \Pi_{C,B} B^\sharp$. By 13.8, only finitely many of the coefficients $\Pi_{C,B}$ in the sum are non-zero. It follows that $C \in M^K_d$. This proves (b).

Conjecture 13.11. (a) *We have $\Delta^\ddagger M^K_c \subset M^K_{d'}$.*

(b) *We have $\Delta M^K_d \subset M_c$.*

Using 12.2(d), we see that (a) and (b) are equivalent. Note that (b) implies that $B^\sharp \in M^K_{\mathrm{int}}$. In the case $K = \emptyset$, (b) holds by [Kt].

13.12. We want to give a conjectural relationship between the W-graph in 12.4 and the W-graphs in [KL1].

For any $w \in W$ we set $A_w = wA^+_\epsilon$ so that $w \to A_w$ is a bijection $W \xrightarrow{\sim} X$. Let W_Ξ be the subset of W corresponding to $X_\Xi \subset X$ under this bijection. For $t \in \mathcal{T}$, the map $\gamma_t : X_\Xi \to X_\Xi$ corresponds under the bijection above to a map $W_\Xi \to W_\Xi$, denoted again by γ_t. An element $t \in \mathcal{T}$ is said to be large if $\check{\alpha}_i(t) \gg 0$ for all $i \in I$.

For $w \in W_\Xi$, let $\mathfrak{I}'_w$ be the set of all $s \in S$ such that $l(s\gamma_t(w)) = l(\gamma_t(w)) - 1$ where t is a large element of $\mathcal{T}$. For w, w' in W_Ξ such that $\mathfrak{I}'_w \not\subset \mathfrak{I}'_{w'}$, let $\mu'(w,w')$ be defined as $\mu(\gamma_t w, \gamma_t w')$ where $t \in \mathcal{T}$ is large and μ is as in [KL1].

Conjecture 13.13. (a) $(W_\Xi, (\mathfrak{I}'_w)_{w \in W_\Xi}, \mu')$ *(as in 13.12) are well defined and they form a W-graph isomorphic to the W-graph $(X_\Xi, (\tilde{\mathfrak{I}}_A)_{A \in X_\Xi}, \tilde{\mu}')$ in 12.4.*

(b) *There is a unique left cell Γ of W such that, for any $w \in W_\Xi$, we have $\gamma_t w \in \Gamma$ for large $t \in \mathcal{T}$.*

This holds for $K = \emptyset$, by [L1].

13.14. In the setup of §8, the homomorphism 8.7(b) comes by complexification from a natural homomorphism of $\mathcal{A}[\mathcal{T}/\mathcal{T}^K]$-modules and $\mathcal{H}$-modules $z : M_c^K \to K_0^{\mathbf{T}_K \times \mathbf{C}^*}(\mathcal{B}_u)$ where the second space is (uncomplexified) equivariant K-homology. Here equivariant K-homology is taken in the topological sense, but in the present case it coincides with the K-group based on coherent sheaves on $\mathcal{B}_u$, equivariant with respect to the action of $\mathbf{T}_K \times \mathbf{C}^*$. (This coincidence can be proved using techniques of [DLP].) Note also that $K_0^{\mathbf{T}_K \times \mathbf{C}^*}(\mathcal{B}_u)$ is a free $\mathcal{A}[\mathcal{T}/\mathcal{T}^K]$-module of rank $|W^I/W^K|$. (Again, this can be proved using techniques of [DLP].)

Thus, z is a homomorphism between free $\mathcal{A}[\mathcal{T}/\mathcal{T}^K]$-modules of the same rank.

Conjecture 13.15. *There is a unique isomorphism* $K_0^{\mathbf{T}_K \times \mathbf{C}^*}(\mathcal{B}_u) \xrightarrow{\sim} M_d^K$ *of* $\mathcal{A}[\mathcal{T}/\mathcal{T}^K]$*-modules such that the diagram*

$$\begin{array}{ccc} M_c^K & \xrightarrow{=} & M_c^K \\ {\scriptstyle z}\downarrow & & \downarrow \\ K_0^{\mathbf{T}_K \times \mathbf{C}^*}(\mathcal{B}_u) & \xrightarrow{\sim} & M_d^K \end{array}$$

with the right vertical map as in 13.10(b)*, is commutative.*

This isomorphism is then automatically compatible with the $\mathcal{H}$-module structures. Under this isomorphism, the $\mathcal{A}$-basis $\{B^\sharp | B \in X_\Xi\}$ corresponds to an $\mathcal{A}$-basis $\mathbf{B}$ of $K_0^{\mathbf{T}_K \times \mathbf{C}^*}(\mathcal{B}_u)$. It would be very interesting to characterize this basis geometrically, in terms of equivariant coherent sheaves on $\mathcal{B}_u$. Note that the elements of this basis are fixed by the composition of two maps (corresponding to θ_ϵ and κ_ϵ); the first of these maps is $\varpi^\dagger$ as in 8.4 and the second one is closely related to the Serre-Grothendieck duality for coherent sheaves.

Conjecture 13.16. *For any $A, B \in X_\Xi$, the coefficients of the polynomial* $\Pi_{A,B} \in \mathbf{Z}[v^{-1}]$ *are* ≥ 0.

13.17. Consider the semisimple Lie algebra $\mathfrak{g}'$ over an algebraically closed field $\mathbf{k}$ of sufficiently large characteristic $p > 0$, of the same type as $\mathfrak{g}$ in 8.2. According to [KW] the set $\mathcal{E}$ of isomorphism classes of finite dimensional irreducible representations of $\mathfrak{g}'$ has a natural partition $\mathcal{E} = \bigsqcup_\lambda \mathcal{E}_\lambda$ where λ runs over the linear forms $\mathfrak{g}' \to \mathbf{k}$. Let us fix λ such that the element of $\mathfrak{g}'$ corresponding to λ via the Killing form is a nilpotent element that is regular inside a Levi subalgebra of a parabolic subalgebra of type K. One can hope that the integers $\Pi_{A,B}(1)$, for $A, B \in X_\Xi$, play the same role in computing the characters of the representations in $\mathcal{E}_\lambda$ as they play in the case where $K = \emptyset$ (which corresponds to $\lambda = 0$, that is, to restricted representations).

13.18. For any integer $n \geq 0$ we define the notion of alcove of *level* n by induction on n. We say that $B \in X_\Xi$ has level 0 if for any $A \in X_\Xi$ we have $\Pi_{A,B} = 0$ or $\Pi_{A,B} = v^{-k}$ for some $k \in \mathbf{N}$ and if $\sum_A \Pi_{A,B} = \sum_w v^{-l(w)}$ where w runs over the set of elements of W^I which have minimal length in their left W^K-coset. (In particular, $\mathrm{supp}(B^\flat)$ has $|W^I/W^K|$ elements.) For example, if $\epsilon \in \mathbf{S}_{\tilde{\epsilon}}$, then A_ϵ^+ has level 0 (see 11.4); more generally, the alcoves B in 11.16 have level 0, but there may exist alcoves of level 0 other than those just described. (See §15, Figure 2 and 4.) Assume now that $n \geq 1$. We say that $B \in X_\Xi$ has level n if it does not have level n' with $n' < n$ and if there exists $A \in X_\Xi$ of level $n-1$ and $s \in S$ such that

$A < sA = B$ and $(\tilde{T}_s + v^{-1})A^\flat - B^\flat$ is a finite $\mathcal{A}$-linear combination of alcoves $A' \in X_\Xi$ of level $< n-1$.

It seems likely that any alcove in X_Ξ has a level and that the levels of the various alcoves are bounded above. This property, which would imply 13.2, holds for $K = \emptyset$ (see [L1, §10]) and for the examples in §14,§15. It would also follow that $M_{d'}^K$ (see 13.9) would be generated as an $\mathcal{H}$-module by the elements $B^\flat$ with B of level 0.

13.19. In the case where $K = \emptyset$ the W-graph 11.14 admits a large group of automorphisms (see [L1, 8.10]) implementing the various intertwining operators of the principal series.

The same holds in general, for the union of the W-graphs 11.14 corresponding to the various subsets L of I that are of the form $w(K)$ for some $w \in W^I$. Indeed, in this case, the intertwining operators can be expressed as compositions of elementary ones and for these elementary ones, the arguments in 13.5 can be used.

14. Examples in type A

14.1. We fix an integer $n \geq 1$. Let (W, S) be the affine Weyl group of type A with a set of simple reflections s_p indexed by $p \in \mathbf{Z}/(n+2)\mathbf{Z}$ and such that $s_p s_{p+1}$ has order 3 for all p. Let $\mathcal{H}$ be the corresponding affine Hecke algebra over $\mathcal{A}$.

14.2. Let $Y = \mathbf{Z}$. Let $\mathcal{A}[Y]$ be the free $\mathcal{A}$-module with basis $(A_a)_{a \in \mathbf{Z}}$. For any $p \in \mathbf{Z}/(n+2)\mathbf{Z}$ we define an $\mathcal{A}$-linear map $\tilde{T}_{s_p} : \mathcal{A}[Y] \to \mathcal{A}[Y]$ by

$$\tilde{T}_{s_p}(A_a) = \begin{cases} A_{a+1}, & \text{if } p = -a \mod n+2, \\ A_{a-1} + (v - v^{-1})A_a, & \text{if } p = -a+1 \mod n+2, \\ vA_{a,b}, & \text{otherwise.} \end{cases}$$

One checks that these formulas define a $\mathcal{H}$-module structure on $\mathcal{A}[Y]$.

We can identify the $\mathcal{H}$-module $\mathcal{A}[Y]$ with its $\mathcal{A}$-basis (A_a), with the $\mathcal{H}$-module M_c^K with its $\mathcal{A}$-basis $\{A | A \in X_\Xi\}$ where K corresponds to a maximal parabolic subgroup of SL_{n+2} with Levi subgroup GL_{n+1}, in such a way that the following hold.

The polynomials $\Pi_{A_{a'}, A_a}$ of §11 are given by

$$\Pi_{A_{a'}, A_a} = \begin{cases} v^{a'-a}, & \text{if } -n-1 \leq a' - a \leq 0, \\ 0, & \text{otherwise .} \end{cases}$$

The polynomials $\Pi'_{A_{a'}, A_a}$ of §12 are given by

$$\Pi_{A_{a'}, A_a} = \begin{cases} v^{a'-a}, & \text{if } a' \leq a, a - a' = 0 \mod n+2, \\ -v^{a'-a}, & \text{if } a' \leq a, a - a' = 1 \mod n+2, \\ 0, & \text{otherwise.} \end{cases}$$

The W-graph structure on Y (of 11.14) is given by

$$\mathfrak{I}_a = \{p; p \neq -a \mod n+2\}, \quad \mu_{a',a} = \begin{cases} 1, & \text{if } a - a' = \pm 1, \\ 0, & \text{otherwise .} \end{cases}$$

14.3. In the remainder of this section, we take $Y = \{(a,b) \in \mathbf{Z}^2 | 0 \le b-a \le n\}$. Let $\mathcal{A}[Y]$ be the free $\mathcal{A}$-module with basis $(A_{a,b})$ indexed by Y.

For any $p \in \mathbf{Z}/(n+2)\mathbf{Z}$ we define an $\mathcal{A}$-linear map $\tilde{T}_{s_p} : \mathcal{A}[Y] \to \mathcal{A}[Y]$ by

$$\tilde{T}_{s_p}(A_{a,b}) = \begin{cases} A_{a+1,b}, & \text{if } p = -a \mod n+2 \text{ and } b-a > 0, \\ A_{a,b+1}, & \text{if } p = -b-1 \mod n+2 \text{ and } b-a < n, \\ A_{a-1,b} + (v - v^{-1})A_{a,b}, & \text{if } p = -a+1 \mod n+2 \text{ and } b-a < n, \\ A_{a,b-1} + (v - v^{-1})A_{a,b}, & \text{if } p = -b \mod n+2 \text{ and } b-a > 0, \\ vA_{a,b}, & \text{otherwise.} \end{cases}$$

It is easy to check that this is a well defined map (that is, the five cases above are disjoint). One checks that these formulas define an $\mathcal{H}$-module structure on $\mathcal{A}[Y]$.

We can identify the $\mathcal{H}$-module $\mathcal{A}[Y]$ with its $\mathcal{A}$-basis $(A_{a,b})$, with the $\mathcal{H}$-module M_c^K with its $\mathcal{A}$-basis $\{A | A \in X_\Xi\}$ where K corresponds to a maximal parabolic subgroup of SL_{n+2} with Levi subgroup $S(GL_2 \times GL_n)$, in such a way that the following hold.

The polynomials $\Pi_{A_{a',b'},A_{a,b}}$ of §11 are given by

(a)
$$\Pi_{A_{a',b'},A_{a,b}} = v^{a'+b'-a-b}, \text{ if } b-a \in \{0,n\}, \quad -n \le a'-a \le 0, \quad -n \le b'-b \le 0;$$

$$\Pi_{A_{a',b'},A_{a,b}} = v^{a'+b'-a-b}\zeta_{(a',b'),(a,b)} + v^{a'+b'-a-b+2}\tilde{\zeta}_{(a',b'),(a,b)},$$

if

(b) $$0 < b-a < n, \quad -n-1 \le a'-a \le 0, \quad -n-1 \le b'-b \le 0,$$

where

$$\zeta_{(a',b'),(a,b)} = \begin{cases} 1, & \text{if } b-a' \le n \text{ or } 0 \le b'-a, \\ 0, & \text{otherwise ,} \end{cases}$$

$$\tilde{\zeta}_{(a',b'),(a,b)} = \begin{cases} 1, & \text{if } n+1 \le b-a' \quad \text{or } b'-a \le -1, \\ 0, & \text{otherwise .} \end{cases}$$

Let us verify directly that in the case (b) we have

(c) $P_{(a',b'),(a,b)} \in v^{-1}\mathbf{Z}^{-1}$ if $(a',b') \ne (a,b)$. Since $a' \le a, b' \le b$ it follows that $a'+b'-a-b < 0$ hence $v^{a'+b'-a-b} \in v^{-1}\mathbf{Z}[v^{-1}]$.

Assume that $b'-a \le -1$. We have $a'-b \le a-b \le -1$ (since $0 < b-a$) and $b'-a \le -1$; adding, we obtain $a'+b'-a-b \le -2$. If this is an equality, then we must have $a'-b = a-b = -1, b'-a = -1$. Hence $a = a'$ and $b'-a' = -1$, contradicting $0 \le b'-a'$. We see that $a'+b'-a-b \le -2$ must be a strict inequality hence $v^{a'+b'-a-b+2} \in v^{-1}\mathbf{Z}[v^{-1}]$.

Assume that $n+1 \le b-a'$. We have $b'-a \le b-a \le n-1$ (since $b-a < n$) and $a'-b \le -n-1$; adding, we obtain $a'+b'-a-b \le -2$. If this is an equality, then we must have $b'-a = b-a = n-1$ and $a'-b = -n-1$. Hence $b = b'$ and $a'-b' = -n-1$, contradicting $b'-a' \le n$. We see that $a'+b'-a-b \le -2$ must be a strict inequality hence $v^{a'+b'-a-b+2} \in v^{-1}\mathbf{Z}[v^{-1}]$. Thus, (c) is verified.

The W-graph structure on Y (of 11.14) is given by:

$$\mathfrak{I}_{(a,b)} = \begin{cases} \{p; p \neq -a+1 \mod n+2, p \neq -b \mod n+2\}, & \text{if } 0 < b-a < n, \\ \{p; p \neq -a+1 \mod n+2\}, & \text{if } b-a=0, \\ \{p; p \neq -b \mod n+2\}, & \text{if } b-a=n, \end{cases}$$

$$\mu_{(a',b'),(a,b)} = \begin{cases} 1, & \text{if } a'=a+1, b'=b, \\ 1, & \text{if } a'=a, b'=b+1, \\ 1, & \text{if } a'=a-1, b'=b, b-a \neq 0, \\ 1, & \text{if } a'=a, b'=b-1, b-a \neq n, \\ 1, & \text{if } a'=a-2, b'=b-1, b-a=n-1, \\ 1, & \text{if } a'=a-1, b'=b-2, b-a=1, \\ 0, & \text{otherwise }. \end{cases}$$

15. Examples in rank 2

The five figures below describe the elements $B^\flat$ in type A_2, B_2, G_2 with $|K|=1$. In each picture, we specify the alcove B by inserting 1 in it; in other alcoves $A \in X_\Xi$, we insert the value of $\Pi_{A,B}$ whenever it is non-zero. We only have to describe the situation for one B in each $\mathcal{T}$-orbit on X_Ξ.

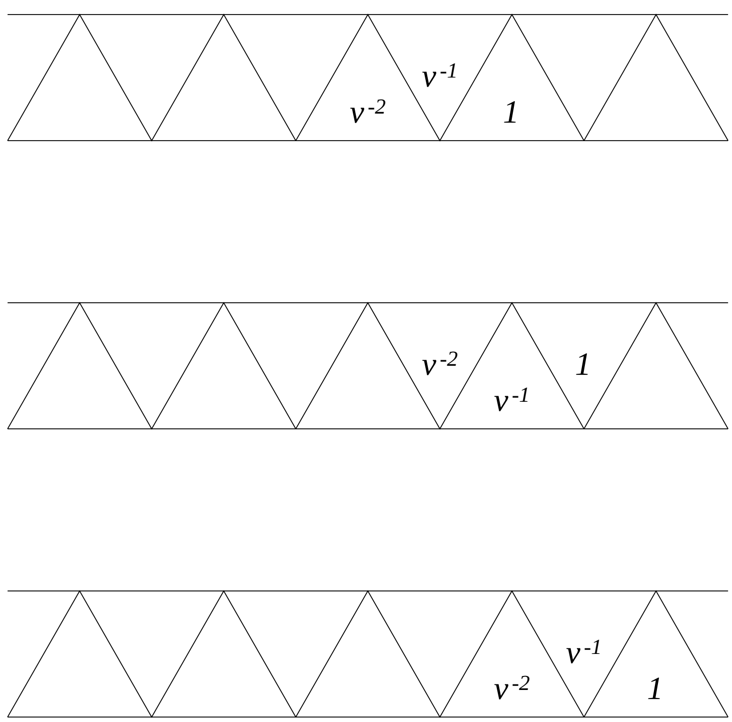

Figure 1. Type $A_2, K=\{i\}$

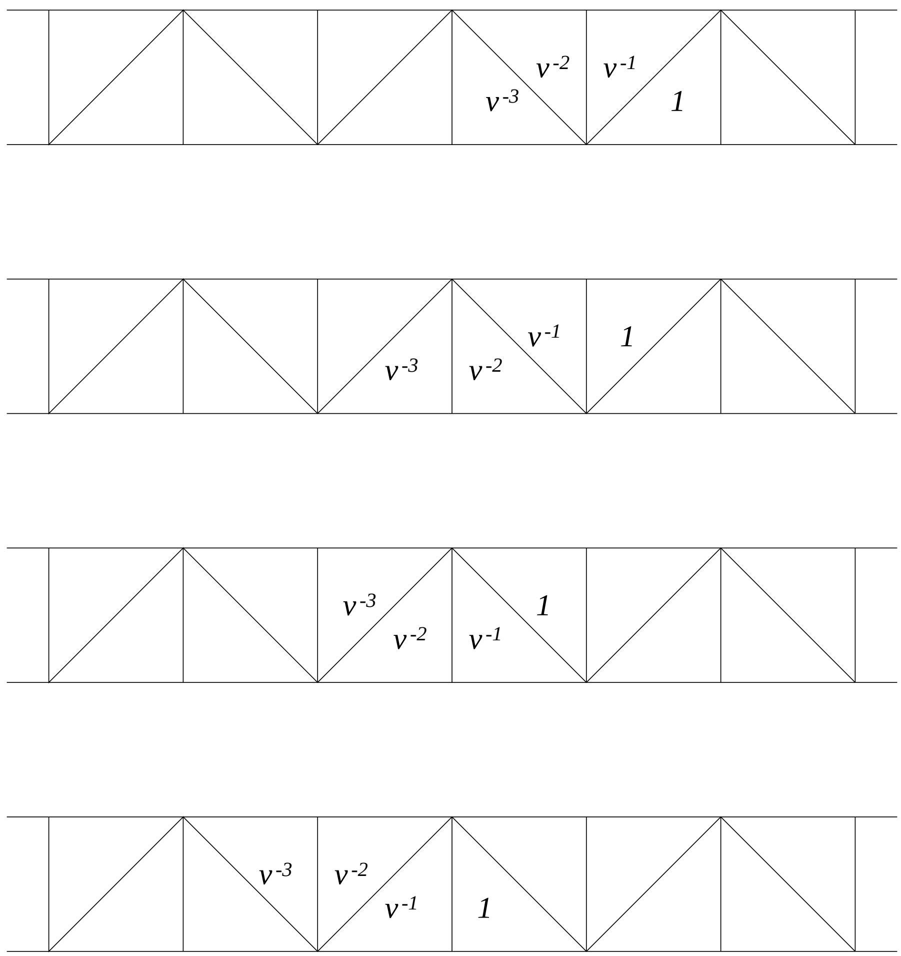

FIGURE 2. Type $B_2, K = \{i_1\}$

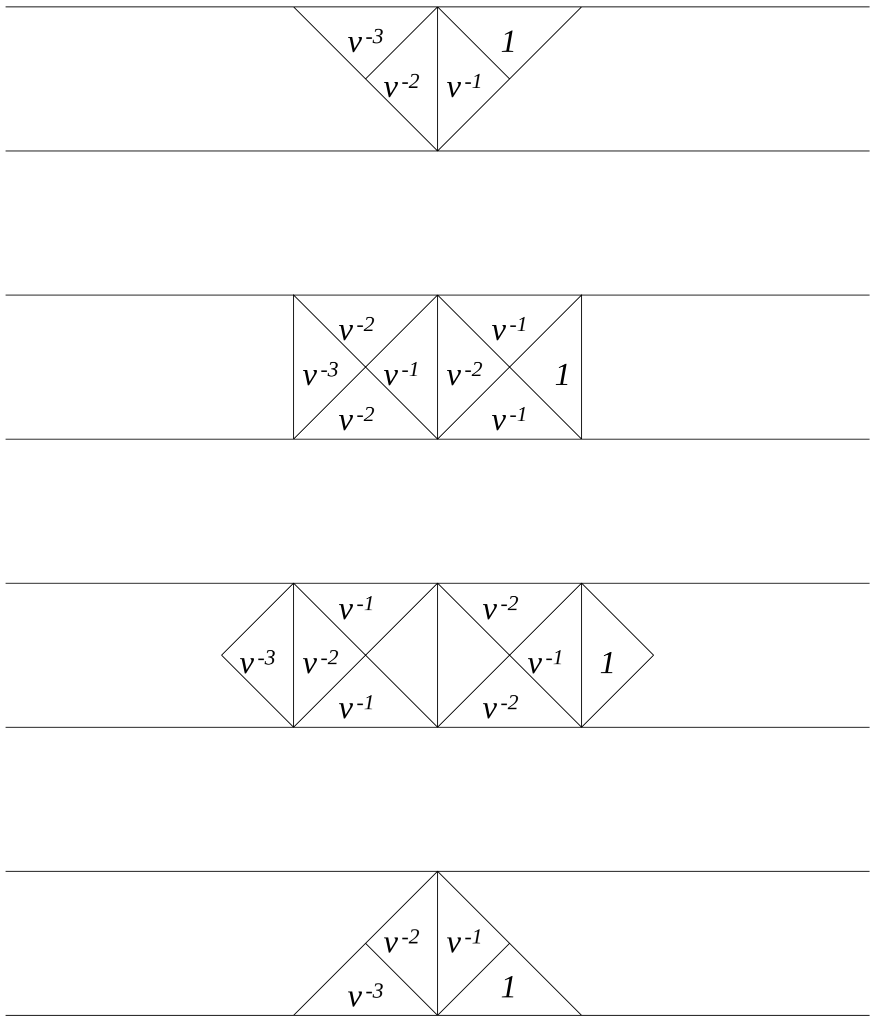

FIGURE 3. Type $B_2, K = \{i_2\}$

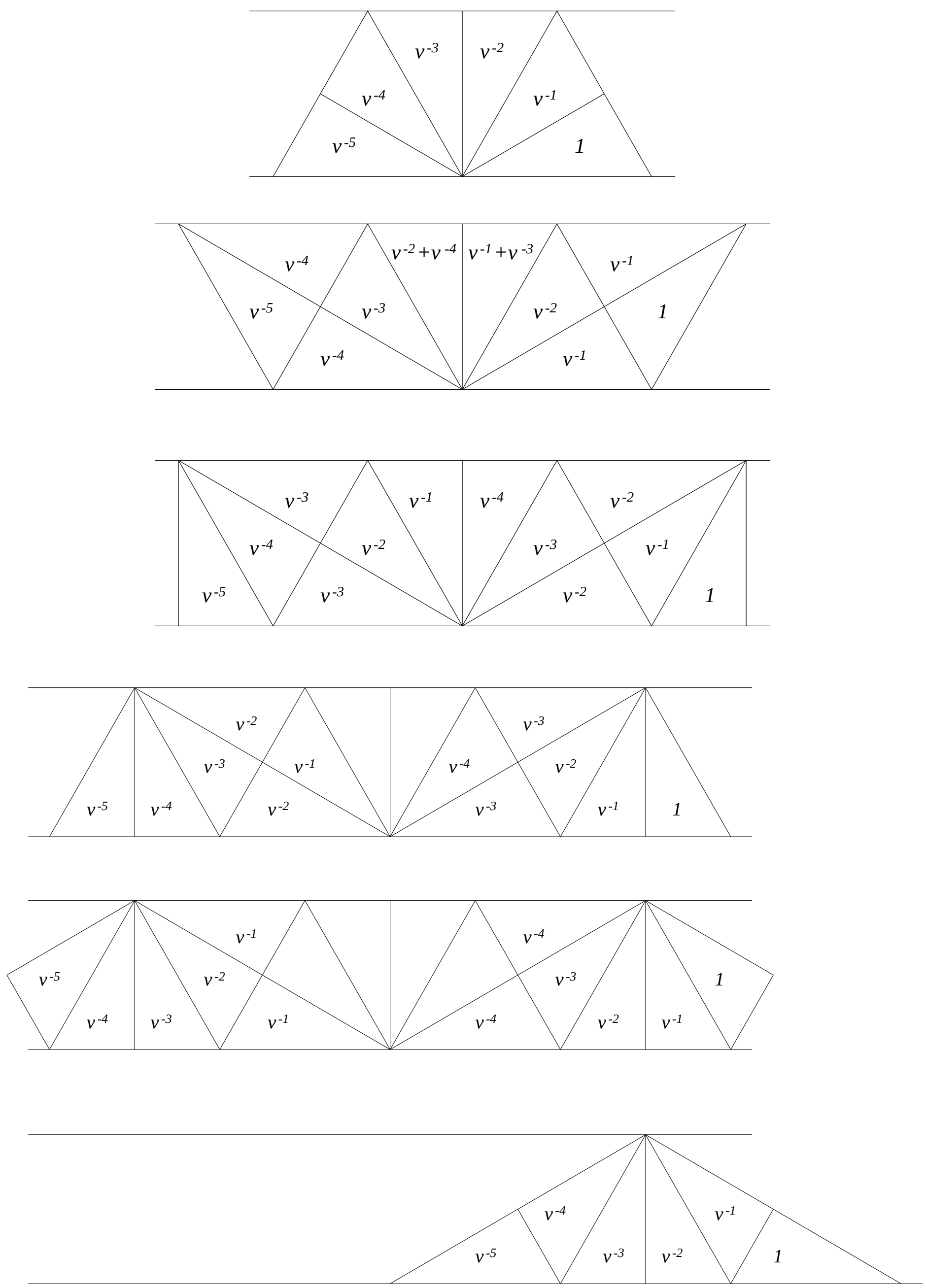

FIGURE 4. Type $G_2, K = \{i_1\}$

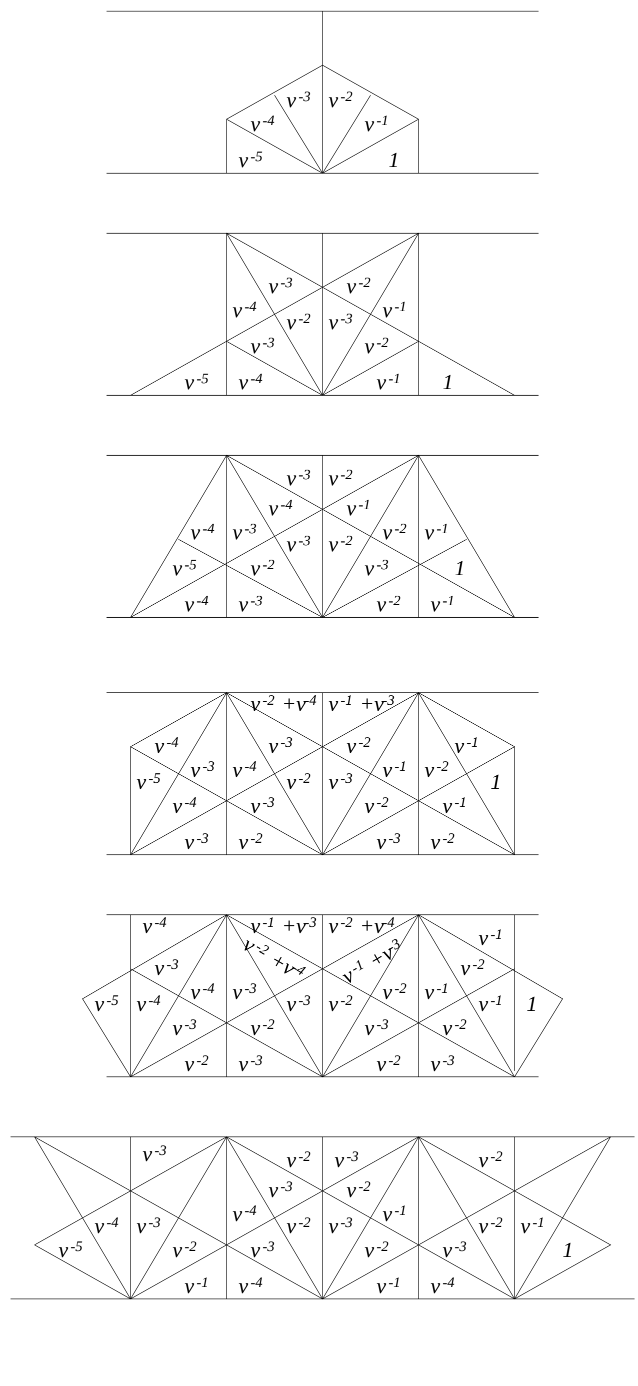

FIGURE 5. Type $G_2, K = \{i_2\}$

Appendix

A.1. We want to give a definition of a W-graph which is slightly more general than the one given in [KL1].

Let (W,S) be a Coxeter group; here S, the set of simple reflections, is assumed to be finite. Let Y be a set. Assume that for each $y \in Y$ we are given a subset $\mathfrak{I}_y$ of S and that for any y, y' in Y such that $\mathfrak{I}_y \not\subset \mathfrak{I}_{y'}$ we are given an integer $\mu'_{y,y'}$, so that

(a) *for any $y, z \in Y$ and any integer $n \geq 1$, the set*

$$Y_n(y,z)\{(y_0, y_1, \dots, y_n) \in Y^{n+1} | y_0 = y, y_n = z,$$
$$\prod_{t=0}^{n-1} \mu'_{y_t,y_{t+1}} \neq 0, \mathfrak{I}_{y_j} \not\subset \mathfrak{I}_{y_{j+1}} \text{for } j \in [0, n-1]\}$$

is finite.

A.2. Let $s \neq s'$ in S. For any integer $n \geq 1$ and any $a, b \in Y$ such that $s' \in \mathfrak{I}_a$ and $s' \notin \mathfrak{I}_b$ (if n is odd), $s \notin \mathfrak{I}_b$ (if n is even), we set $f_{n;s,s'}(a,b) = \sum \prod_{t=0}^{n-1} \mu'_{y_t,y_{t+1}}$ where the sum runs over all $(y_0, y_1, \dots, y_n) \in Y^{n+1}$ such that $y_0 = a, \mathfrak{I}_{y_t} \cap \{s, s'\} = \{s\}$ for t odd in $[1, n-1]$, $\mathfrak{I}_{y_t} \cap \{s, s'\} = \{s'\}$ for t even in $[1, n-1]$ and $y_n = b$. The sum is well defined by condition (a) above. We also set $f_{0;s,s'} = \delta_{a,b}$ for any $a, b \in W$.

For an integer $m \geq 2$ we define integers $p_{0,m}, p_{1,m}, p_{2,m} \dots, p_{m-1,m}$ by

$$\prod_{k=1}^{m-1} (v - 2\cos\frac{k\pi}{m}) = p_{0,m} + p_{1,m}v + p_{2,m}v^2 + \dots + p_{m-1,m}v^{m-1}.$$

We say that Y (with the additional data $(\mathfrak{I}_y)_{y \in Y}, \mu'$) is a W-*graph* if for any $s \neq s'$ in S such that ss' has finite order m, we have

(b) $f_{n;s,s'}(a,b) = f_{n;s',s}(a,b)$ for all $a, b \in Y$ such that $\{s, s'\} \subset \mathfrak{I}_a$, $\mathfrak{I}_b \cap \{s, s'\} = \emptyset$ and any $n \in [2, m]$; and

(c) $\sum_{n=0}^{m-1} p_{n,m} f_{n;s,s'}(a,b) = 0$ for all $a \in Y$ such that $\{s, s'\} \cap \mathfrak{I}_a = \{s'\}$ and all $b \in Y$ such that $\mathfrak{I}_b \cap \{s, s'\} = \{s'\}$ (if n is even), $\mathfrak{I}_b \cap \{s, s'\} = \{s\}$ (if n is odd).

A.3. We return to the setup of A.1. For any $z \in Y$ and $n \geq 1$ we set $P_n(z) = \{y \in Y | Y_n(y,z) \neq \emptyset\}$. For $n = 0$ we set $P_0(z) = \{z\}$. A subset $P \subset Y$ is said to be *of finite type* if it is contained in a union of finitely many sets of the form $P_n(z)$ (for various $z \in Y$ and $n \geq 0$). For example, any finite subset of Y is of finite type. Let $\mathcal{E}$ be the set consisting of all formal sums $\sum_{y \in Y} c_y y$ with $c_y \in \mathcal{A}$ such that $\{y \in Y | c_y \neq 0\}$ is of finite type. Then $\mathcal{E}$ is an $\mathcal{A}$-module in an obvious way. From A.1(a) it follows that

$$\tau_s : \sum_y c_y y \mapsto -\sum_{\substack{y \in Y \\ s \in \mathfrak{I}_y}} v^{-1} c_y y + \sum_{\substack{y \in Y \\ s \notin \mathfrak{I}_y}} v c_y y + \sum_{\substack{y \in Y \\ s \in \mathfrak{I}_y}} (\sum_{\substack{y' \in Y \\ s \notin \mathfrak{I}_{y'}}} \mu'_{y,y'} c_{y'}) y,$$

$$\tau'_s : \sum_y c_y y \mapsto \sum_{\substack{y \in Y \\ s \in \mathfrak{I}_y}} v c_y y - \sum_{\substack{y \in Y \\ s \notin \mathfrak{I}_y}} v^{-1} c_y y + \sum_{\substack{y \in Y \\ s \in \mathfrak{I}_y}} (\sum_{\substack{y' \in Y \\ s \notin \mathfrak{I}_{y'}}} \mu'_{y,y'} c_{y'}) y$$

are well defined $\mathcal{A}$-linear maps $\mathcal{E} \to \mathcal{E}$.

Proposition A.4. *The following three conditions for $Y, (\mathfrak{I}_y)_{y \in Y}, \mu'$ are equivalent:*

(a) *$Y, (\mathfrak{I}_y)_{y \in Y}, \mu'$ is a W-graph in the sense of A.2.*

(b) *For any $s \neq s'$ in S such that ss' has finite order m, we have*

$$\underbrace{\tau_s \tau_{s'} \tau_s \dots}_{m \text{ factors}} = \underbrace{\tau_{s'} \tau_s \tau_{s'} \dots}_{m \text{ factors}}.$$

(c) *For any $s \neq s'$ in S such that ss' has finite order m, we have*

$$\underbrace{\tau'_s \tau'_{s'} \tau'_s \dots}_{m \text{ factors}} = \underbrace{\tau'_{s'} \tau'_s \tau'_{s'} \dots}_{m \text{ factors}}.$$

The equivalence of (b),(c) holds since $\tau'_s = \zeta \tau_s \zeta$ where ζ is the involution of $\mathcal{E}$ given by $\sum_{y \in Y} c_y y \mapsto \sum_{y \in Y} c'_y y$ and c'_y is the image of c_y under the ring involution of A which takes v to $-v^{-1}$. The equivalence of (a),(b) is proved by computation.

A.5. The definition of a W-graph given above (which is adopted in this paper) is slightly more general than that given in [KL1] in which the finiteness property A.1(a) is replaced by the stronger property that $P_1(z)$ is finite for any $z \in Y$ (this implies that $P_n(z)$ is finite for any $z \in Y$ and any n, so that $\mathcal{E}$ is just the free $\mathcal{A}$-module with basis Y.)

A.6. Assume that $Y, (\mathfrak{I}_y)_{y \in Y}, \mu'$ is a W-graph in the sense of A.2. Let $\tilde{\mathfrak{I}}_y = S - \mathfrak{I}_y$ for $y \in Y$. For y, y' such that $\tilde{\mathfrak{I}}_y \not\subset \tilde{\mathfrak{I}}_{y'}$ (that is, $\mathfrak{I}_{y'} \not\subset \mathfrak{I}_y$) we set $\tilde{\mu}'(y, y') = \mu'(y', y)$. Then $Y, (\tilde{\mathfrak{I}}_y)_{y \in Y}, \tilde{\mu}'$ is again a W-graph. (This is clear from the definition in A.2.)

The W-graph $Y, (\tilde{\mathfrak{I}}_y)_{y \in Y}, \tilde{\mu}'$ is said to be *complementary* to the W-graph $Y, (\mathfrak{I}_y)_{y \in Y}, \mu'$.

Index of notation

1.1. $E, \mathfrak{F}, T, \sigma_H, \Omega, \mathcal{T}, \mathcal{T}', \bar{\mathcal{F}}, \mathcal{C}^+, \mathcal{C}^-, r_h, W^I, W^K, w_0^K, X, S, cl(), W, A_\epsilon^+, A_\epsilon^-, \Omega_\epsilon,$ $W_\epsilon, \omega_{\epsilon,K}, \omega_\epsilon, D(\epsilon)$
1.2. $E_H^+, E_H^-, \mathcal{L}(A), d(A,B), d_h(A,B)$
1.3. $\alpha_i, \mathcal{T}^+, \mathcal{T}_{\mathrm{dom}}$
1.4. $\leq$
1.5. $\mathcal{T}^K$
2.1. $T_K, \mathfrak{F}^K, \Omega^K$
2.2. $\tilde{\epsilon}, \Xi, X_\Xi$
2.3. $\mathbf{S}, \mathbf{S}_{\tilde{\epsilon}}, \kappa_\epsilon$
2.4. $D_\Xi(\epsilon), A_\epsilon^!$
2.12. γ_t
2.14. $d_K(A,B)$
3.1. $\mathcal{A}, \mathcal{H}$
3.2. $M, M_{i,\leq}, M_{i,\geq}$
3.3. θ_H
3.6. e_ϵ
3.8. $M(U)$
3.13. U_K^+, U_K^-
3.15. $\bar{\ }, \phi_\epsilon$
4.2. $\tilde{M}^K$
4.6. M^K, res_K
4.8. $M^K_{\leftarrow}, M^K_{\rightarrow}$.
4.9. $e_{\epsilon,K}$
4.10. $\mu_K(t)$

References

[B] N. Bourbaki, *Groupes et algèbres de Lie, Ch. 4,5,6*, Hermann, Paris, 1968. MR **39:**1590

[DLP] C. De Concini, G. Lusztig and C. Procesi, *Homology of the zero set of a nilpotent vector field on a flag manifold*, Jour. Amer. Math. Soc. **1** (1988), 15-34. MR **89f:**14052

[KW] V. G. Kac and B. Weisfeiler, *On irreducible representations of p-Lie algebras*, Funkt. Analiz **5** (1971), 28-36. MR **44:**2793

[Kt] S.I. Kato, *On the Kazhdan-Lusztig polynomials for affine Weyl groups*, Adv. in Math. **55** (1985), 103-130. MR **86d:**20048

[KL1] D. Kazhdan and G. Lusztig, *Representations of Coxeter groups and Hecke algebras*, Invent. Math. **53** (1979), 165-184. MR **81j:**20066

[KL2] D. Kazhdan and G. Lusztig, *Proof of the Deligne-Langlands conjecture for Hecke algebras*, Invent. Math. **87** (1987), 153-215. MR **88d:**11121

[L1] G. Lusztig, *Hecke algebras and Jantzen's generic decomposition patterns*, Adv. in Math. **37** (1980), 121-164. MR **82b:**20059

[L2] G.Lusztig, *Affine Hecke algebras and their graded version*, Jour. Amer.Math.Soc. **2** (1989), 599-635. MR **90e:**16049

[L3] G. Lusztig, *Introduction to quantum groups*, Progr. in Math. 110, Birkhäuser, Boston, 1993. MR **94m:**17016

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

E-mail address: `gyuri@math.mit.edu`

ERRATUM: PERIODIC W-GRAPHS

G. LUSZTIG

I thank Jens C. Jantzen for pointing out the following misprints.

In the second line of the display in 3.2, replace $s \notin \mathcal{L}$ by $s \in \mathcal{L}$.

In 3.8, replace the last $\mathcal{H}$ by M in the line: "It is clear that $M(U)$ is an $\mathcal{H}$-submodule of $\mathcal{H}$".

In 14.2, in the third line of the first display, replace $vA_{a,b}$ by vA_a.

In the line 14.3(c) replace $v^{-1}\mathbf{Z}^{-1}$ by $v^{-1}\mathbf{Z}[v^{-1}]$.