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HECKE ALGEBRA REPRESENTATIONS RELATED TO SPHERICAL VARIETIES

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Abstract. Let G be a connected reductive group over the algebraic closure of a finite field and let Y be a spherical variety for G. We consider perverse sheaves on G and on Y which have a weight for the action of a Borel subgroup B and are endowed with an action of Frobenius. This leads to the definition of a "generalized Hecke algebra", attached to G, and of a module over that algebra, attached to Y. The same algebra and the same module can also be defined using constructible sheaves. Comparison of the two definitions gives, in the case of a symmetric variety Y and B-equivariant sheaves, a geometric proof of results which Lusztig and Vogan obtained by representation theoretic means.

1. Introduction

1.1. Let G be a connected reductive group over an algebraically closed field k of characteristic $\neq 2$ and let Y be a symmetric variety for G, i.e. $Y = G/K$, where K is the fixed point group of an involutorial automorphism of G. Choose a Borel subgroup B of G. Then B has only finitely many orbits on Y. Let v be such an orbit and let $\mathcal{L}$ be a B-equivariant local system of rank one on v. Let $j : v \to \bar{v}$ denote the embedding of v into its Zariski closure in Y and consider the intermediate extension $A = j_{!*}\mathcal{L}[\dim v]$ (cf. [BBD, 2.1.7 and 2.2.3]), also called intersection complex. This is a perverse sheaf on $\bar{v}$. The restrictions of the cohomology sheaves $\mathcal{H}^i A$ to a B-orbit u contained in $\bar{v}$ are direct sums of irreducible B-equivariant local systems on u, which have rank one. The problem is how to compute the multiplicities of the irreducible components. By a general principle (see [BBD, section 6]) the question is reduced to the case where k is the algebraic closure of a finite field. The problem was solved by Lusztig and Vogan in [LV] (they considered K-orbits on $B\backslash G$, which is equivalent to considering B-orbits on G/K).

The paper [LV] exploits the purity of the perverse sheaves A, which is a result on the eigenvalues of Frobenius on the stalks of the cohomology sheaves $\mathcal{H}^i A$. It also makes use of representation theory: the numbers we are looking for have an interpretation in the representation theory of real Lie groups and [LV] applies a result of [V]. In the present paper we give a proof of the main results of [LV] without having recourse to representation theory. Our proof gives an algorithm, which is essentially the same as in [LV].

1.2. The problem can be formulated more generally for a spherical variety Y, i.e. a variety with a G-action on which B has only finitely many orbits. One may

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also generalize the problem by considering local systems on the B-orbits which are not necessarily B-equivariant, but have a weight for the B-action (see 2.2.1 for the definition of weights). Actually, in our treatment of the symmetric case we need B_1-equivariant local systems, where B_1 is the Borel subgroup of the simply connected covering of the derived group of G. These local systems have a weight for the B-action satisfying the condition in Theorem 7.1.2.

In the general setting (for a spherical variety Y and weights for B) we define an abelian category $\mathcal{C}_Y$ like $\mathcal{C}'$ in [LV] and an analogous category $\mathcal{A}_Y$ constructed with perverse sheaves instead of local systems. These two categories have isomorphic Grothendieck groups. The local systems on the one hand and the perverse sheaves on the other hand provide two bases for that Grothendieck group and the problem is to express one basis into the other, in other words, determine the coefficients $c_{\eta,u;\xi,v,i}$ introduced in 3.4.1. The construction is such that not only the multiplicities of the components in the cohomology sheaves of the perverse sheaves appear in these coefficients, but also the eigenvalues of Frobenius on the stalks.

Applying Verdier duality we obtain the relations 3.4.2 for the coefficients c, in which appear the coefficients $b_{\eta,u;\xi,v}$ that express the dual of our local systems in the basis elements (they correspond to the $R_{\gamma,\delta}$ of [LV], whereas the coefficients c are related to the $P_{\gamma,\delta}$). When we know that the cohomology sheaves are punctually pure of the correct weight, the coefficients c can be computed from the relations 3.4.2 when the coefficients b are known. This is explained in 3.4.3. Now the module structure which we define in 3.2 on the Grothendieck groups is used in chapter 5 to find relations satisfied by the coefficients b. They are in general not sufficient to determine these coefficients. In the symmetric case and for B_1-equivariant sheaves we use the relations of chapter 5 *together with* 3.4.2 to solve the problem. An important role is played by Lemma 7.4.1, a vanishing result for intersection cohomology.

Although we did not obtain a solution of the problem for general spherical varieties, it seemed worthwhile to present our method in the general setting.

1.3. **Notations and conventions.** All varieties will be defined over an algebraically closed field k. We will deal with $\bar{\mathbb{Q}}_l$-sheaves on varieties over k ($l \neq p = \mathrm{char}(k)$). If $k = \mathbb{C}$, we may also consider sheaves of $\mathbb{C}$-vector spaces for the classical topology. In all cases we denote the coefficient field by E and speak of E-sheaves.

By a local system we mean a smooth E-sheaf. We denote by $\mathcal{D}X$ the bounded derived category of constructible E-sheaves on X. It has a t-structure such that the heart of $\mathcal{D}X$ is the abelian category of perverse sheaves on X. Our main reference for the theory of perverse sheaves is [BBD]. Our notations regarding constructible sheaves and perverse sheaves are those of [BBD].

The cohomology sheaves of a complex K are denoted by $\mathcal{H}^iK$. The notation H^n is used both for cohomology and for hyper-cohomology.

If $\mathcal{L}$ is a local system on a smooth irreducible subvariety U of X, the perverse sheaf $j_{!*}\mathcal{L}[\dim U]$, where j is the embedding $U \to \bar{U}$, will be called the perverse extension of L, as in [MS]. We extend it by zero on $X - \bar{U}$.

For the definition of weights for Frobenius actions and the theorems concerning them we refer to [BBD, section 5]. We will use, in particular, the purity of perverse extensions.

$\mathbb{Z}_{(p)}$ denotes the localisation of $\mathbb{Z}$ at (p), in particular $\mathbb{Z}_{(0)} = \mathbb{Q}$, and $M_{(p)} = \mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} M$ for any $\mathbb{Z}$-module M.

1.4. **A further survey of the contents.** In 2.1 we recall the definition and elementary properties of tame local systems (or Kummer local systems) on tori. See also [MS], where the same notation is used. In 2.1.7 we quote results from [SGA4-1/2] concerning a local system which is related to Jacobi sums.

We establish some results on local systems and perverse sheaves with a weight for B in 2.2.

The purity result in 2.3.3 (not used in [LV]) is of crucial importance for our method, because it allows the argument of 3.4.3.

In chapter 3 we define the categories $\mathcal{C}_Y$ and $\mathcal{A}_Y$ for a spherical variety Y. An algebra structure is defined on the Grothendieck groups $\mathcal{K}(\mathcal{C}_G)$ and $\mathcal{K}(\mathcal{A}_G)$ and a $\mathcal{K}(\mathcal{C}_G)$- (resp. $\mathcal{K}(\mathcal{A}_G)$-) module structure is defined on $\mathcal{K}(\mathcal{C}_Y)$ (resp. $\mathcal{K}(\mathcal{A}_Y)$). The algebra $\mathcal{K}(\mathcal{C}_G)$ is related to the algebra $\mathcal{K}$ of [MS], see the remark at the end of 4.3. Multiplication by t in $\mathcal{K}$ corresponds to a shift in the complexes and multiplication by q in $\mathcal{K}(\mathcal{C}_G)$ corresponds to multiplication of eigenvalues of Frobenius by q. The relation between the two algebras is explained by the fact that in the case under consideration in [MS] all eigenvalues of Frobenius are integral powers of q (up to roots of unity).

As in [LV], Verdier duality is used to define linear endomorphisms of $\mathcal{K}(\mathcal{A}_Y)$ and $\mathcal{K}(\mathcal{C}_Y)$.

The goal of chapter 4 is to make the module structure on $\mathcal{K}(\mathcal{C}_Y)$ explicit by computing the products of the basis elements of $\mathcal{K}(\mathcal{C}_G)$ with those of $\mathcal{K}(\mathcal{C}_Y)$. The results are listed in 4.3.1, which is a non-trivial extension of [LV, Lemma 3.5]. Here Jacobi sums can appear in the coefficients. The classification of B-orbits in 4.1 is well-known, cf. [RS2, 2.2].

In chapter 5 we give explicitly the relations between the coefficients $b_{\eta,u;\xi,v}$ of $D\varepsilon_{\xi,v}$ which result from 4.3.1 by applying D.

So far everything was done in the general situation of a spherical variety and sheaves with a weight for B. In chapters 6 and 7 we deal with symmetric varieties. Chapter 6 reviews material about these. One of the main results is 7.1.2 establishing a parity result for the intersection cohomology complexes (under a restriction on the weights for B). Also the eigenvalues of Frobenius on the stalks are powers of q times a root of unity. Finally, in 7.6 we give an example where Y is still symmetric, but the condition on the weight for B is dropped. In that example Jacobi sums appear as eigenvalues of Frobenius and $A_{\xi,v}$ is not always even.

2. Auxiliary results

2.1. Local systems on tori.

2.1.1. Let T be a torus over an algebraically closed field k of characteristic p and n an integer not divisible by p. We follow Deligne [SGA4-1/2, Sommes trigonométriques, 1.2]. The exact sequence

$$1 \to T_n \to T \xrightarrow{n} T \to 1$$

defines a T_n-torsor $K_n(T)$ on T. If $\rho : T_n \to E^*$ is a character of T_n, there is a sheaf $\rho(K_n(T))$ (a smooth E-sheaf of rank 1) on T with a morphism $\rho : K_n(T) \to \rho(K_n(T))$ which is everywhere $\neq 0$ and such that $\rho(tz) = \rho(t)\rho(z)$ for the action of $z \in T_n$.

If $\chi \in X(T)$, the character group of T, and $\rho(t) = \psi(\chi(t))$ for $t \in T_n$, we denote $\rho(K_n(T))$ by $\mathcal{L}_{\chi,n}$. Here ψ is a fixed embedding of the group $\boldsymbol{\mu}(k)$ of roots of unity of k into E^*. These sheaves are the so-called tame local systems on T. They are called Kummer local systems in [MS]. They have the following properties.

$$\begin{aligned}
\mathcal{L}_{d\chi,dn} &= \mathcal{L}_{\chi,n}.\\
\mathcal{L}_{\chi+\chi',n} &= \mathcal{L}_{\chi,n}\otimes\mathcal{L}_{\chi',n}.\\
&\mathcal{L}_{\chi,n} \text{ is the constant sheaf } E \Leftrightarrow \chi\in nX(T).\\
\mathcal{L}_{-\chi,n} &= \text{dual of } \mathcal{L}_{\chi,n}.
\end{aligned}$$

We put $\widehat{X}(T)=X(T)\otimes_{\mathbb{Z}}(\mathbb{Z}_{(p)}/\mathbb{Z})=X(T)_{(p)}/X(T)$. Then we have an isomorphism of $\widehat{X}(T)$ on the group of tame local systems on T, viz. $\xi\mapsto\mathcal{L}_{\chi,n}$ if ξ is represented by χ/n, $\chi\in X(T)$, $n\in\mathbb{Z}$, $n\not\equiv 0(p)$. We use the notation $\mathcal{L}_\xi$ for $\mathcal{L}_{\chi,n}$.

2.1.2. If k is the algebraic closure of the finite field $\mathbb{F}_q$ and T is defined over $\mathbb{F}_q$, i.e. obtained from a scheme T_0 over $\mathbb{F}_q$ by extension of scalars from $\mathbb{F}_q$ to k, there is a T_n-torsor $K_n(T_0)$ on T_0 defined by the exact sequence $1\to(T_0)_n\to T_0\xrightarrow{n}T_0\to 1$. A character $\rho: T_n\to E^*$ is defined over $\mathbb{F}_q$ if $\rho(Ft)=\rho(t)$ for all $t\in T_n$ (F denotes the Frobenius map). If $\rho=\psi\circ\chi|_{T_n}$, this condition is satisfied if and only if there is $\chi_1\in X(T)$ such that $\chi(t^{-1}Ft)=\chi_1(t)^n$ for all $t\in T$. Then $\rho(K_n(T_0))=\mathcal{L}_\xi$ is a sheaf on T_0.

The endomorphism $\chi\mapsto\chi\circ F$ of $X(T)$ can uniquely be extended to an automorphism of $\widehat{X}(T)$. The invariants of this automorphism correspond to the $\mathcal{L}_\xi$ coming from a sheaf on T_0.

If $a\in T^F=T_0(\mathbb{F}_q)$, the action of F_a^* on $(\mathcal{L}_\xi)_a$ is multiplication by $\psi\circ\chi(b^{-1}Fb)^{-1}$, if $b\in T_0(k)$, $b^n=a$. The factor is an n-th root of unity.

2.1.3. Let T_0 be a torus over $\mathbb{F}_q$ which splits over $\mathbb{F}_{q^N}$. Put $n=q^N-1$. We have a commutative diagram with exact rows

$$\begin{tikzcd}
1 \arrow[r] & (T_0)_n \arrow[r] \arrow[d, "\nu"'] & T_0 \arrow[r, "n"] \arrow[d, "\nu"'] & T_0 \arrow[r] \arrow[d, equal] & 1\\
1 \arrow[r] & T_0(\mathbb{F}_q) \arrow[r] & T_0 \arrow[r, "\mathcal{L}"] & T_0 \arrow[r] & 1
\end{tikzcd}$$

Here $\mathcal{L}$ is the isogeny $t\mapsto t^{-1}Ft$ and the second row defines a $T_0(\mathbb{F}_q)$-torsor L_0. The map ν is defined by $\nu t=t(Ft)\dots(F^{N-1}t)$. Notice that $F^Nt=t^{q^N}$. We have $\mathrm{Ker}\,(\nu)=\{t^{-1}Ft\mid t\in T_n\}$ and ν and $\nu|_{T_n}$ are both surjective.

If $\rho: T_n\to E^*$ is a character defined over $\mathbb{F}_q$, ρ factorizes uniquely via $T_0(\mathbb{F}_q)$. Let $\rho=\eta\circ\nu$, $\eta: T_0(\mathbb{F}_q)\to E^*$. Then $\rho(K_n(T_0))=\eta(L_0)$. Taking $\rho=\psi\circ\chi$, $\chi\in X(T)$, we have $\chi(t^{-1}Ft)=\chi_1(t)^n$ and $\chi=\chi_1\circ\nu$, hence $\mathcal{L}_\xi=\rho(K_n(T_0))=\psi\chi_1(L_0)$.

A diagram as above exists more generally for n divisible by q^N-1.

2.1.4. Corresponding to any homomorphism of tori $\phi: T\to T'$ over k there is a homomorphism $\widehat{\phi}:\widehat{X}(T')\to\widehat{X}(T)$, obtained from $X(T')\to X(T)$. If $\xi\in\widehat{X}(T')$, then $\mathcal{L}_{\widehat{\phi}\xi}=\phi^*\mathcal{L}_\xi$.

2.1.5. Lemma. *Let $\phi: T\to T'$ be a surjective homomorphism of tori. Put $D=$ Ker (ϕ) and let D^0 be the connected component of D (a torus). Then we have an exact sequence*

$$0\to X(D/D^0)\to\widehat{X}(T')\to\widehat{X}(T)\to\widehat{X}(D^0)\to 0.$$

From the exact sequence $0 \to X(T/D) \to X(T) \to X(D) \to 0$ we get an exact sequence

$$0 \to \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_{(p)}/\mathbb{Z}, X(D)) \to \widehat{X}(T/D) \to \widehat{X}(T) \to \mathbb{Z}_{(p)}/\mathbb{Z} \otimes_{\mathbb{Z}} X(D) \to 0.$$

We have isomorphisms $\mathbb{Z}_{(p)}/\mathbb{Z} \otimes_{\mathbb{Z}} X(D) \xrightarrow{\sim} \mathbb{Z}_{(p)}/\mathbb{Z} \otimes_{\mathbb{Z}} X(D^0) = \widehat{X}(D^0)$ and $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_{(p)}/\mathbb{Z}, X(D/D^0)) \xrightarrow{\sim} \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_{(p)}/\mathbb{Z}, X(D))$. For any finite abelian group A whose order is not divisible by p, one has $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_{(p)}/\mathbb{Z}, A) \xrightarrow{\sim} A$. So there is an isomorphism $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_{(p)}/\mathbb{Z}, X(D)) \cong X(D/D^0)$.

To the bijective homomorphism $T/D \to T'$ induced by ϕ there corresponds an injection $X(T') \to X(T/D)$ whose cokernel is a p-group ($= \{0\}$ if $p = 0$). Tensoring with $\mathbb{Z}_{(p)}/\mathbb{Z}$ gives an isomorphism $\widehat{X}(T') \cong \widehat{X}(T/D)$.

Thus we obtain the announced exact sequence.

2.1.6. *If $\xi \in \widehat{X}(T)$, $\xi \neq 0$, then $H_c^i(T, \mathcal{L}_\xi) = H^i(T, \mathcal{L}_\xi) = 0$ for all i.* This is well known.

2.1.7. There is another type of local systems which we shall encounter. It is a special case of the sheaves studied in [SGA4-1/2, Sommes trigonométriques, §4]. We consider the algebra $\mathbb{F}_q^3$ over $\mathbb{F}_q$. Following the notations of [loc. cit.] we define V_0 to be the same algebra considered as a variety over $\mathbb{F}_q$ and W_0 to be the subspace defined by $x_0 + x_1 + x_2 = 0$. Moreover, V_0^* is the multiplicative group of invertible elements of V_0 and $W_0^* = W_0 \cap V_0^*$.

Let n be an integer divisible by $q-1$ and not divisible by p. We have the following commutative diagram with exact rows (it is 2.1.3 with $T_0 = \mathbb{G}_m^3$).

$$\begin{tikzcd}
1 \arrow[r] & (V_0^*)_n \arrow[r] \arrow[d, "\nu"'] & V_0^* \arrow[r, "n"] \arrow[d, "\nu"'] & V_0^* \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & (\mathbb{F}_q^*)^3 \arrow[r] & V_0^* \arrow[r, "q-1"] & V_0^* \arrow[r] & 1
\end{tikzcd}$$

Here $\nu t = t^{\frac{n}{q-1}}$.

Let $\chi : (\mathbb{F}_q^*)^3 \to E^*$ be a character and $\chi \circ \nu = (\chi_0, \chi_1, \chi_2)$, $\chi_i : \mu_n(\mathbb{F}) \to E^*$. Each of the two rows of the diagram defines a torsor on V_0^* and together with the character $\chi \circ \nu$ resp. χ these torsors determine one and the same smooth sheaf on V_0^*, which is $\mathcal{F}(\chi^{-1}) = \mathcal{K}_n(\chi_0^{-1}, \chi_1^{-1}, \chi_2^{-1})$ in the notation of [loc. cit.]. If $\chi_0\chi_1\chi_2 = 1$, $\mathcal{K}_n(\chi_0^{-1}, \chi_1^{-1}, \chi_2^{-1})$ is the inverse image of a sheaf on $V_0^*/\mathbb{G}_m$ (we use the same notation for this sheaf). Now we may identify $V_0^*/\mathbb{G}_m$ with the subset $\{(1, x_1, x_2)\}$ of $\mathbb{F}^3$. Then $W_0^*/\mathbb{G}_m$ is isomorphic to $\mathbb{F} - \{0, -1\}$, by $x \mapsto (1, x, -1-x)$. Obviously, $\mathcal{F}(\chi^{-1})$ on $W^*/\mathbb{G}_m = \mathbb{F} - \{0, -1\}$ is the inverse image of $\mathcal{L}_{\chi_0,n} \boxtimes \mathcal{L}_{\chi_1,n} \boxtimes \mathcal{L}_{\chi_2,n}$ under the map $x \mapsto (1, x, -1-x)$, $\mathbb{F} - \{0, -1\} \to (\mathbb{F}^*)^3$. This is also the inverse image of $\mathcal{L}_{\chi_1,n} \boxtimes \mathcal{L}_{\chi_2,n}$ under the map $\mathbb{F} - \{0, -1\} \to (\mathbb{F}^*)^2$, $x \mapsto (x, -1-x)$.

Proposition 4.16 of [loc. cit.] gives the following.

In the notation above assume that $\chi_0\chi_1\chi_2 = 1$ and that not all $\chi_i = 1$.
Then $H_c^i(W^*/\mathbb{G}_m, \mathcal{F}(\chi^{-1})) = 0$ if $i \neq 1$ and $\dim H_c^1 = 1$.
On $H_c^1(W^*/\mathbb{G}_m, \mathcal{F}(\chi^{-1}))$ the action of F^* is multiplication by

$$J(\chi) = -\sum_{x \in \mathbb{F}_q^*, x \neq -1} \boldsymbol{\chi}_1(x)^{-1} \boldsymbol{\chi}_2(-1-x)^{-1}$$

where $\boldsymbol{\chi}_i$ are the components of χ, i.e. $\chi_i = (\boldsymbol{\chi}_i)^{\frac{n}{q-1}}$.

And by Proposition 4.20 of [SGA4-1/2, Sommes trigonométriques] we have:
If $\chi_1 = \chi_2$ *, the action of the automorphism* $x \mapsto -1-x$ *of* $W^*/\mathbb{G}_m$ *on* H_c^1 *is multiplication by* -1.

2.2. **Weights for solvable group actions.**

2.2.1. Let B be a connected solvable linear algebraic group over the algebraically closed field k. Denote its unipotent radical by U. Fix a maximal torus T of B and let $\pi_T : B \to B/U = T$. If Y is a variety with B-action $a : B \times Y \to Y$, a local system $\mathcal{L}$ on Y is said to have weight $\xi \in \widehat{X}(T)$ for the B-action if there exists an isomorphism $a^*\mathcal{L} \cong \pi_T^*\mathcal{L}_\xi \boxtimes \mathcal{L}$. The same definition applies to perverse sheaves on Y.

2.2.2. Let C be a closed subgroup of B and $Y = B/C$. Put $D = \pi_T C$. Denote by $\phi : T \to T/D$ the canonical map and let $\pi : Y \to T/D$ be the map induced by π_T. If $\xi \in \widehat{X}(T/D)$, then $\pi^*\mathcal{L}_\xi$ is a local system of rank one on Y and it is easily seen that it has weight $\widehat{\phi}\xi$ for the action of B on Y. Conversely we have:

2.2.3. Lemma. *Assume that* $C \cap U$ *is connected. If* $\mathcal{L}$ *is a local system of rank one on* Y *which has a weight for the* B*-action, there is a unique* $\xi \in \widehat{X}(T/D)$ *such that* $\mathcal{L}$ *is isomorphic to* $\pi^*\mathcal{L}_\xi$.

The weight of $\mathcal{L}$ is annihilated by an integer m not divisible by the characteristic p. Define $\widetilde{B}$ to be the algebraic group which is a semi-direct product of T and U, the T-action on U now being given by $t^m u t^{-m}$. The map $\rho : \widetilde{B} \to B$, $(t,u) \mapsto t^m u$, is a homomorphism. Put $\widetilde{C} = \rho^{-1}(C)$ and $\widetilde{D} = \widetilde{\pi}_T \widetilde{C}$, where $\widetilde{\pi}_T$ is the projection $\widetilde{B} \to T$. We then have a commutative diagram

$$\begin{array}{ccc} \widetilde{B}/\widetilde{C} & \xrightarrow{\widetilde{\pi}} & T/\widetilde{D} \\ {\scriptstyle\cong}\downarrow & & {\scriptstyle\cong}\downarrow \\ Y = B/C & \xrightarrow{\pi} & T/D \end{array}$$

Here $\widetilde{\pi}$ is induced by $\widetilde{\pi}_T$ and the vertical arrows are isomorphisms induced, respectively, by ρ and by the multiplication by m in T. The sheaf $\widetilde{\mathcal{L}}$ on $\widetilde{B}/\widetilde{C}$ which corresponds to $\mathcal{L}$ under the isomorphism $\widetilde{B}/\widetilde{C} \to B/C$ is a $\widetilde{B}$-equivariant local system and the assertion of the lemma is equivalent to the corresponding one for $\widetilde{B}/\widetilde{C}$ and $\widetilde{\mathcal{L}}$. So the proof is reduced to the case that $\mathcal{L}$ is B-equivariant.

Consider in this case the Cartesian square

$$\begin{array}{ccc} B/C^0 & \longrightarrow & T/D^0 \\ \downarrow & & \downarrow \\ B/C & \xrightarrow[\pi]{} & T/D \end{array}$$

The square is Cartesian, because $C/C^0 \to D/D^0$ is an isomorphism, as a consequence of the assumption that $C \cap U$ is connected. The B-equivariant local systems on B/C are in 1-1-correspondence with the characters of $C/C^0 = D/D^0$, hence by 2.1.5 with the elements of $\mathrm{Ker}\,(\widehat{X}(T/D) \to \widehat{X}(T/D^0))$, the correspondence being $\xi \mapsto \pi^*\mathcal{L}_\xi$. This proves the lemma.

With the same notation we have

2.2.4. Proposition. *Assume that $C \cap U$ is connected. Let $\mathcal{L}$ be a local system on $Y = B/C$ which has a weight $\eta \in \widehat{X}(T)$ for the B-action. Then $\mathcal{L}$ is isomorphic to a direct sum of local systems $\pi^* \mathcal{L}_{\xi_i}$, where the ξ_i are elements of $\widehat{X}(T/D)$ with $\widehat{\phi}\xi_i = \eta$.*

Again, one reduces the proof to the case that $\mathcal{L}$ is B-equivariant. Then the result follows from the fact that $C/C^0 \cong D/D^0$ is abelian.

2.2.5. Now let Y be a variety on which B acts with finitely many orbits. Let V be the set of orbits. If v, $v' \in V$, we write $v' \leq v$ if v' lies in the Zariski closure of v. This defines an order on V. We use the notation $\dot{v}$ for a representative of v. Let $B_{\dot{v}}$ be the isotropy group of $\dot{v}$. Then $T_v = \pi_T B_{\dot{v}}$ is independent of the choice of $\dot{v}$ in v. Assume:

(a) *all orbit maps $B \to v$ $(b \mapsto b\dot{v})$ are separable;*
(b) *all groups $B_{\dot{v}} \cap U$ are connected.*

Assumption (a) implies that an orbit v is isomorphic to $B/B_{\dot{v}}$, as a B-variety. Let $\pi_{\dot{v}} : v \to T/T_v$ be the map $tu\dot{v} \mapsto tT_v$ $(t \in T,\, u \in U)$. If $\xi \in \widehat{X}(T/T_v)$, we write $\mathcal{L}_{\xi,\dot{v}} = \pi_{\dot{v}}^* \mathcal{L}_\xi$ and denote by $A_{\xi,\dot{v}}$ the perverse extension (see 1.3) of $\mathcal{L}_{\xi,\dot{v}}$. The isomorphism class of $\mathcal{L}_{\xi,\dot{v}}$ resp. $A_{\xi,\dot{v}}$ depends only on v. We shall write $\mathcal{L}_{\xi,v}$ resp. $A_{\xi,v}$, when only the isomorphism class is concerned. $\mathcal{L}_{\xi,v}$ is extended by 0 to a sheaf on Y.

We denote by ϕ_v the map $T \to T/T_v$.

2.2.6. Proposition. *Assume* (a) *and* (b). *Let $v \in V$ and $\xi \in \widehat{X}(T/T_v)$.*

(i) *$\mathcal{L}_{\xi,v}$ and $A_{\xi,v}$ have weight $\widehat{\phi}_v \xi$ for the B-action.*
(ii) *If $v' \in V$, $i \in \mathbb{Z}$, then $\mathcal{H}^i A_{\xi,v}|_{v'}$ is a direct sum of local systems $\mathcal{L}_{\xi_j,v'}$, where $\xi_j \in \widehat{X}(T/T_{v'})$ and $\widehat{\phi}_{v'}\xi_j = \widehat{\phi}_v \xi$. It is zero unless $v' \leq v$.*
(iii) *$DA_{\xi,\dot{v}} = A_{-\xi,\dot{v}}(\dim v)$.*

In (iii), D stands for Verdier duality.

2.2.7. Proposition. *Assume* (a) *and* (b) *of 2.2.5. Let A be a semi-simple perverse sheaf on Y which has a weight for the B-action. Then A is isomorphic to a direct sum of perverse sheaves of the form $A_{\xi,v}$.*

This is easily proved. Reduce to the case of a simple perverse sheaf A and use 2.2.3.

2.3. **A purity result.**

2.3.1. Lemma. *Let Z be a variety with a $\mathbb{G}_m$-action $m : \mathbb{G}_m \times Z \to Z$ leaving some $z_0 \in Z$ fixed. Assume that m extends to a morphism $f : \mathbb{A}^1 \times Z \to Z$ such that $f(\{0\} \times Z) = \{z_0\}$. Let $K \in \mathcal{D}Z$ such that there exists an isomorphism $m^* K \cong E \boxtimes K$. Then the canonical maps $H^n(Z,K) \to (\mathcal{H}^n K)_{z_0}$ are isomorphisms.*

It is easy to see that it suffices to prove the assertion when $(\mathcal{H}^n K)_{z_0} = 0$ for all n. Then the restriction of f^*K to $\{0\} \times Z$ is zero and the restriction of f^*K to $\mathbb{G}_m \times Z$ is $m^*K \cong E \boxtimes K$. So we have a morphism $\phi : f^*K \to E \boxtimes K$. It is now obvious that, when we define $f_t : Z \to Z$ by $f_t(z) = f(t,z)$ $(t \in \mathbb{A}^1)$, then f_1 and f_0 are homotopic. So f_1 and f_0 induce the same homomorphism on $H^n(Z,K)$ (see [SGA7, exposé XV, 2.1]). Hence $H^n(Z,K) = 0$ for all n.

Remark. The dual statement is: the canonical maps $H^n_{\{z_0\}}(Z,K) \to H^n_c(Z,K)$ are isomorphisms.

2.3.2. Let G be a connected linear algebraic group, Y an irreducible variety and $a : G \times Y \to Y$ an action of G on Y. Fix $y \in Y$, let $\mathcal{O} = G \cdot y$ be its orbit, and let S be a transverse slice in y with respect to $\mathcal{O}$. By this we mean that:

(a) S is a locally closed subset of Y containing y;
(b) the restriction of a defines a smooth morphism $G \times S \to Y$;
(c) $\dim S = \dim Y - \dim \mathcal{O}$.

Assume moreover that S admits a contraction of the following type. We have a homomorphism $\lambda : \mathbb{G}_m \to G$ such that:

(d) S and y are stable under $\mathrm{Im}(\lambda)$;
(e) the action $\mathbb{G}_m \times S \to S$ extends to a morphism $\mathbb{A}^1 \times S \to S$ sending $\{0\} \times S$ to y.

Let $A \in \mathcal{D}Y$ such that a^*A is isomorphic to $E \boxtimes A$. Using the properties (a), (b), (c) of S we see that $i^*A[-2\dim \mathcal{O}](-\dim \mathcal{O}) \tilde{\to} i^!A$ is an isomorphism. Here $i : S \to Y$ is the embedding.

Now assume that k is the algebraic closure of a finite field and that, in the above situation everything is defined over $\mathbb{F}_q$. For A this means that it comes from a complex in $\mathcal{D}Y_0$, where Y_0 is the $\mathbb{F}_q$-scheme underlying Y. Since $y \in Y_0(\mathbb{F}_q)$ we have a Frobenius action on $(\mathcal{H}^nA)_y$.

2.3.3. Proposition. *Assume that there exists a transverse slice S in y, which satisfies* (a)–(e) *of 2.3.2 and is defined over $\mathbb{F}_q$. If A as in 2.3.2 is pure of weight w, then the weights of $(\mathcal{H}^nA)_y$ are equal to $w + n$.*

From $i^*A[-2\dim \mathcal{O}](-\dim \mathcal{O}) = i^!A$ it follows that i^*A is pure of weight w. So the weights of $H^n(S, A)$ are $\geq w+n$. By Lemma 2.3.1 we have $(\mathcal{H}^nA)_y = H^n(S, A)$. Thus, the weights of $(\mathcal{H}^nA)_y$ are $\geq w + n$, and also $\leq w + n$ by the purity of A.

3. A generalized Hecke algebra and its modules

3.1. Definition of $\mathcal{K}(\mathcal{A}_Y)$ and $\mathcal{K}(\mathcal{C}_Y)$.

3.1.1. Let G be a connected reductive group over the algebraically closed field k and let Y be a spherical variety for G, i.e. a homogeneous space of G such that any Borel subgroup of G has finitely many orbits in Y. We fix a Borel group $B = TU$ of G and use the notations of 2.2.5. In particular V is the set of B-orbits in Y. In the sequel we always assume that the conditions (a) and (b) of 2.2.5 are satisfied. They are, of course, in characteristic 0, and in any characteristic $\neq 2$ they are satisfied if Y is symmetric. We call Y symmetric if $Y = G/K$, where K is the fixed point group of an involution (= automorphism of order 2) of G.

G can be considered as a spherical variety for $G \times G$. The action is $(g_1, g_2) \cdot x = g_1 x g_2^{-1}$ and the set of $B \times B$-orbits is the Weyl group of G. It is symmetric: consider the involution of $G \times G$ which interchanges the two components.

3.1.2. From now on we assume that k is an algebraic closure of a finite field $\mathbb{F}_q$. We take $E = \bar{\mathbb{Q}}_l$, where l is a prime different from the characteristic p of k. We take q big enough, so that G, B, Y are defined over $\mathbb{F}_q$, T is $\mathbb{F}_q$-split, etc. There are Frobenius morphisms $F : G \to G,\ F : Y \to Y$, etc. In 2.2.5 we defined $\mathcal{L}_{\xi,\dot{v}}$ and $A_{\xi,\dot{v}}$. If $F\dot{v} = \dot{v}$ and $F\xi = \xi$, these objects also have an $\mathbb{F}_q$-structure and we have the Frobenius correspondence

$$F^*\mathcal{L}_{\xi,\dot{v}} \tilde{\to} \mathcal{L}_{\xi,\dot{v}} \quad \text{resp.} \quad F^*A_{\xi,\dot{v}} \tilde{\to} A_{\xi,\dot{v}}.$$

We need one more notation. If a is the action of T on Y, a_m is the action such that $a_m(t) = a(t^m)$. If a local system or a perverse sheaf has weight ξ for T and $m\xi = 0$, then it is equivariant for the action a_m (cf. the proof of 2.2.3).

Now we consider perverse sheaves A on Y supplied with an isomorphism $\Phi : F^*A \xrightarrow{\sim} A$. Then there are isomorphism $\Phi^n : (F^n)^*A \xrightarrow{\sim} A$ for all $n \geq 1$. We assume that (A, Φ) satisfies the following conditions.

(1) A is U-equivariant and T-equivariant for the action a_m, where m is a *fixed* integer not divisible by p.

(2) If $y \in Y$, $F^n y = y$, the eigenvalues of Φ_y^n on $(\mathcal{H}^i A)_y$ lie in C_1, where C_1 is a fixed subgroup of $\bar{\mathbb{Q}}_l^*$ containing the roots of unity. Put $C = C_1/\mu$, $\mu =$ group of the roots of unity in $\bar{\mathbb{Q}}_l^*$.

We identify $(A, \Phi) = (A, \Phi')$ if $\Phi^n = {\Phi'}^n$ for some n.

Morphisms between these objects are morphisms of perverse sheaves, compatible with a power of Φ.

In this way an abelian category $\mathcal{A}_Y$ is defined. We denote by $\mathcal{K}(\mathcal{A}_Y)$ the Grothendieck group of $\mathcal{A}_Y$.

If C_1 is taken big enough, $\mathcal{K}(\mathcal{A}_Y)$ is a $\mathbb{Z}[C]$-module with a finite basis corresponding to the $A_{\xi,v}$ with $v \in V$, $\xi \in \widehat{X}(T/T_v)$, $m\widehat{\phi}_v \xi = 0$.

Indeed, if (A, Φ) is an object of $\mathcal{A}_Y$ with A simple, then by Proposition 2.2.7, A is of the form $A_{\xi,v}$. We have $\mathrm{Aut}(A_{\xi,v}) = \mathrm{Aut}(\mathcal{L}_{\xi,v}) = \bar{\mathbb{Q}}_l^*$ and multiplication by the class in C of $c \in \bar{\mathbb{Q}}_l^*$ takes (A, Φ) to (A, Φ') with $\Phi' = c\Phi$. In other words, the eigenvalues of Φ_y^n on $(\mathcal{H}^i A)_y$ are multiplied by c^n.

We need an analogous construction with constructible sheaves instead of perverse sheaves.

We consider constructible $\bar{\mathbb{Q}}_l$-sheaves $\mathcal{S}$ on Y supplied with an isomorphism $\Phi : F^*\mathcal{S} \xrightarrow{\sim} \mathcal{S}$, assuming:

1) $\mathcal{S}$ is U-equivariant and T-equivariant for a_m (m fixed); Φ is equivariant for these actions. This means that we have commutative diagrams

$$\begin{tikzcd} E\boxtimes F^*\mathcal{S} = F^*(E\boxtimes\mathcal{S}) \arrow[r] \arrow[d, "id\times\Phi"] & F^*a^*\mathcal{S} = a^*F^*\mathcal{S} \arrow[d, "a^*\Phi"] \\ E\boxtimes\mathcal{S} \arrow[r] & a^*\mathcal{S} \end{tikzcd}$$

where a stands for the action of U on Y, resp. for the action a_m of T.

2) If $y \in Y$, $F^n y = y$, the eigenvalues of Φ_y^n on $\mathcal{S}_y$ lie in C_1.

We identify $(\mathcal{S}, \Phi) = (\mathcal{S}, \Phi')$ if $\Phi^n = {\Phi'}^n$ for some n.

Morphisms between these objects are morphisms of equivariant sheaves $\mathcal{S} \to \mathcal{S}'$ compatible with a power of Φ and Φ'. This means that we have commutative diagrams

$$\begin{tikzcd} E\boxtimes\mathcal{S} \arrow[r] \arrow[d] & a^*\mathcal{S} \arrow[d] \\ E\boxtimes\mathcal{S}' \arrow[r] & a^*\mathcal{S}' \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} (F^n)^*\mathcal{S} \arrow[r, "\Phi^n"] \arrow[d] & \mathcal{S} \arrow[d] \\ (F^n)^*\mathcal{S}' \arrow[r, "{\Phi'}^n"'] & \mathcal{S}' \end{tikzcd}$$

This defines an abelian category $\mathcal{C}_Y$. Let $\mathcal{K}(\mathcal{C}_Y)$ denote the Grothendieck group of $\mathcal{C}_Y$.

If C_1 is taken big enough, $\mathcal{K}(\mathcal{C}_Y)$ is a $\mathbb{Z}[C]$-module with a finite basis, corresponding to the $\mathcal{L}_{\xi,v}$ with $v \in V$, $\xi \in \widehat{X}(T/T_v)$, $m\widehat{\phi}_v\xi = 0$.

When a constructible sheaf $\mathcal{L}$ has a natural $\mathbb{F}_q$-structure, we denote by $[\mathcal{L}]$ the class in $\mathcal{K}(\mathcal{C}_Y)$ of $\mathcal{L}$ equipped with the Frobenius correspondence. The same convention applies to a perverse sheaf A and its class $[A]$ in $\mathcal{K}(\mathcal{A}_Y)$.

There is a $\mathbb{Z}[C]$-homomorphism $h = h_Y\colon \mathcal{K}(\mathcal{A}_Y) \to \mathcal{K}(\mathcal{C}_Y)$, given by $[A] \mapsto \sum(-1)^i[\mathcal{H}^iA]$ ($\mathcal{H}^iA$ is supplied with the isomorphism $F^*\mathcal{H}^iA \to \mathcal{H}^iA$ coming from $\Phi : F^*A \to A$; we dropped Φ in the notation). Since $h[A_{\xi,v}]$ is the sum of $(-1)^{\dim v}[\mathcal{L}_{\xi,v}]$ and a linear combination with coefficients in $\mathbb{Z}[C]$ of $[\mathcal{L}_{\eta,u}]$, $u < v$, it is clear that h is bijective. The problem is now, how to express one basis into the other.

3.1.3. As a special case of our definitions we have $\mathcal{K}(\mathcal{A}_G)$ and $\mathcal{K}(\mathcal{C}_G)$, where G is considered as a spherical variety for $G \times G$ as in 3.1.1. The $B \times B$-orbits are the sets BwB, $w \in W = N(T)/T$. We denote by $\dot{w}$ a representative in $N(T)$ for w. Let us apply the definitions of 2.2.5 to this case. We have $(T \times T)_w = \{(t_1, t_2) \in T \times T \mid t_1 = w(t_2)\}$, so $T \times T/(T \times T)_w$ can be identified with T by sending the class of (t_1, t_2) to $w^{-1}(t_1)t_2^{-1}$. Then the map $\pi_{\dot{w}} : BwB \to T \times T/(T \times T)_w = T$ is $u\dot{w}tu' \mapsto t$ (u, $u' \in U$, $t \in T$). And $\widehat{X}(T) = \widehat{X}(T \times T/(T \times T)_w) \to \widehat{X}(T \times T) = \widehat{X}(T) \times \widehat{X}(T)$ is the map $\xi \mapsto (w\xi, -\xi)$. So $\mathcal{L}_{\xi,w}$ and $A_{\xi,w}$ have weight $w\xi$ (resp. $-\xi$) for the left (resp. right) B-action.

Remarks. A category like $\mathcal{C}_Y$ with $m = 1$ (B-equivariant sheaves on Y) was defined in [LV]. There K-equivariant sheaves on G/B were considered. In [MS] the same sheaves $\mathcal{L}_{\xi,w}$ and $A_{\xi,w}$ were defined and studied without Frobenius actions.

3.2. **Algebra and module structures.**

3.2.1. Let $w, y \in W$ and $\xi, \eta \in \widehat{X}(T)$. If $\eta = w\xi$, the perverse sheaf $A_{\eta,y} \boxtimes A_{\xi,w}$ on $G \times G$ is equivariant for the action of B given by $b(g_1, g_2) = (g_1b^{-1}, bg_2)$, because $A_{\eta,y}$ has weight $-\eta$ for the right B-action and $A_{\xi,w}$ has weight $w\xi$ for the left B-action (3.1.3). Hence there is a unique perverse sheaf $\widetilde{A}$ on the quotient $G \overset{B}{\times} G$ such that $A_{w\xi,y} \boxtimes A_{\xi,w} = f^*\widetilde{A}[\dim B]$, if f denotes the natural map $G \times G \to G \overset{B}{\times} G$. The product in G defines a proper map $\mu : G \overset{B}{\times} G \to G$ and $\mu_!\widetilde{A}$ is a semi-simple complex on G, which means that $\mu_!\widetilde{A}$ is the direct sum of its shifted perverse cohomology sheaves: $\mu_!\widetilde{A} = \bigoplus {}^pH^i(\mu_!\widetilde{A})[-i]$ and that each ${}^pH^i(\mu_!\widetilde{A})$ is a direct sum of simple perverse sheaves (see [BBD, Théorème 5.4.5 and Théorème 5.3.8]; $\mu_!\widetilde{A}$ is pure, because $\widetilde{A}$ is pure and μ is proper). These simple constituents have weight $yw\xi$ (resp. $-\xi$) for the left (resp. right) B-action, hence are of the form $A_{\xi,x}$, $x \in W$ such that $x\xi = yw\xi$ (Proposition 2.2.7).

3.2.2. Definition.

$$\begin{aligned} [A_{w\xi,y}]\,[A_{\xi,w}] &= \sum(-1)^i[{}^pH^i(\mu_!\widetilde{A})], \\ [A_{\eta,y}]\,[A_{\xi,w}] &= 0 \quad \text{if } \eta \neq w\xi. \end{aligned}$$

Extending this definition by linearity we put a structure of associative $\mathbb{Z}[C]$-algebra on $\mathcal{K}(\mathcal{A}_G)$. The associativity is proved using the map $G \overset{B}{\times} G \overset{B}{\times} G \to G$. We have $[A_{w\xi,e}]\,[A_{\xi,w}] = [A_{\xi,w}]$.

Remark. The convolution of the isomorphism classes of $A_{w\xi,y}$ and $A_{\xi,w}$ (without Frobenius) was defined in [MS].

In an analogous way a product is defined in $\mathcal{K}(\mathcal{C}_G)$.

3.2.3. Definition.

$$[\mathcal{L}_{w\xi,y}]\,[\mathcal{L}_{\xi,w}] = \sum(-1)^i[\mathcal{H}^i(\mu_!\widetilde{\mathcal{L}})],$$
$$[\mathcal{L}_{\eta,y}]\,[\mathcal{L}_{\xi,w}] = 0 \quad \text{if } \eta \neq w\xi.$$

The constructible sheaf $\widetilde{\mathcal{L}}$ is the extension by 0 of the local system on $ByB\overset{B}{\times}BwB$ determined by $\mathcal{L}_{w\xi,y} \boxtimes \mathcal{L}_{\xi,w}$. $\mathcal{K}(\mathcal{C}_G)$ is an associative $\mathbb{Z}[C]$-algebra. We have $[\mathcal{L}_{w\xi,e}]\,[\mathcal{L}_{\xi,w}] = [\mathcal{L}_{\xi,w}]$.

Put $h'_G = (-1)^{\dim B}h_G$ with h_G as in 3.1.2 for $Y = G$.

3.2.4. Proposition. *The map $h'_G : \mathcal{K}(\mathcal{A}_G) \to \mathcal{K}(\mathcal{C}_G)$ is an isomorphism of $\mathbb{Z}[C]$-algebras.*

This can be proved in the same way as Proposition 3.2.8 below.

3.2.5. We shall now define a $\mathcal{K}(\mathcal{A}_G)$-module structure on $\mathcal{K}(\mathcal{A}_Y)$. Let $w \in W$, $v \in V$, $\xi \in \widehat{X}(T/T_v)$, $\eta \in \widehat{X}(T)$. Since $A_{\eta,w}$ has weight $-\eta$ for the right B-action and $A_{\xi,v}$ has weight $\widehat{\phi}_v\xi$ for the B-action (2.2.6 and 3.1.3), the perverse sheaf $A_{\eta,w}\boxtimes A_{\xi,v}$ on $G\times Y$ is equivariant for the action of B given by $b(g,y) = (gb^{-1}, by)$, if $\eta = \widehat{\phi}_v\xi$. So $A_{\widehat{\phi}_v\xi,w}\boxtimes A_{\xi,v}$ is, up to a shift $[\dim B]$, the inverse image of a perverse sheaf $\widetilde{A}$ on the quotient $G\overset{B}{\times}Y$. The action of G on Y defines a proper map $\mu : G\overset{B}{\times}Y \to Y$ and $\mu_!\widetilde{A}$ is a semi-simple complex on Y. The simple constituents of $\mu_!\widetilde{A}$ have weight $w(\widehat{\phi}_v\xi)$ for the B-action, hence are of the form $A_{\xi',v'}$ with $v' \in V$, $\xi' \in \widehat{X}(T/T_{v'})$ such that $\widehat{\phi}_{v'}\xi' = w(\widehat{\phi}_v\xi)$.

3.2.6. Definition.

$$\begin{aligned}[][A_{\widehat{\phi}_v\xi,w}]\,[A_{\xi,v}] &= \sum(-1)^i[{}^pH^i(\mu_!\widetilde{A})],\\ [A_{\eta,w}]\,[A_{\xi,v}] &= 0 \text{ if } \eta \neq \widehat{\phi}_v\xi.\end{aligned}$$

The $\mathbb{Z}[C]$-bilinear map $\mathcal{K}(\mathcal{A}_G)\times\mathcal{K}(\mathcal{A}_Y) \to \mathcal{K}(\mathcal{A}_Y)$ determined by 3.2.6 makes $\mathcal{K}(\mathcal{A}_Y)$ a $\mathcal{K}(\mathcal{A}_G)$-module. That it is a module is proved using the map $G\overset{B}{\times}G\overset{B}{\times}Y \to Y$. We have $[A_{\widehat{\phi}_v\xi,e}][A_{\xi,v}] = [A_{\xi,v}]$.

Notice that the product in $\mathcal{K}(\mathcal{A}_G)$ is not exactly a particular case of 3.2.6; the latter gives a $\mathcal{K}(\mathcal{A}_{G\times G})$-module structure on $\mathcal{K}(\mathcal{A}_G)$.

3.2.7. Definition.

$$\begin{aligned}[][\mathcal{L}_{\widehat{\phi}_v\xi,w}]\,[\mathcal{L}_{\xi,v}] &= \sum(-1)^i[\mathcal{H}^i\mu_!\widetilde{\mathcal{L}}],\\ [\mathcal{L}_{\eta,w}]\,[\mathcal{L}_{\xi,v}] &= 0 \quad \text{if } \eta \neq \widehat{\phi}_v\xi.\end{aligned}$$

Here $\widetilde{\mathcal{L}}$ is the extension by 0 of the local system on $BwB\overset{B}{\times}v$ determined by $\mathcal{L}_{\widehat{\phi}_v\xi,w}\boxtimes\mathcal{L}_{\xi,v}$. The formulas give a $\mathcal{K}(\mathcal{C}_G)$-module structure on $\mathcal{K}(\mathcal{C}_Y)$. Again $[\mathcal{L}_{\widehat{\phi}_v\xi,e}]\,[\mathcal{L}_{\xi,v}] = [\mathcal{L}_{\xi,v}]$.

3.2.8. Proposition. *The bijection $h_Y : \mathcal{K}(\mathcal{A}_Y) \to \mathcal{K}(\mathcal{C}_Y)$ preserves the module structure, in the sense that*

$$h_Y(km) = h'_G(k)h_Y(m) \quad \textit{if} \ \ k \in \mathcal{K}(\mathcal{A}_G),\ m \in \mathcal{K}(\mathcal{A}_Y).$$

It suffices to prove this for $k = [A_{\widehat{\phi}_v\xi,w}]$ and $m = [A_{\xi,v}]$. With $\widetilde{A}$ as above we have

$$\begin{aligned} h_Y(km) &= h_Y(\sum(-1)^i[{}^pH^i(\mu_!\widetilde{A})]) = \sum_{i,j}(-1)^{i+j}[\mathcal{H}^j({}^pH^i(\mu_!\widetilde{A}))] \\ &= \sum(-1)^j[\mathcal{H}^j(\mu_!\widetilde{A})] = \sum_{p,q}(-1)^{p+q}[R^p\mu_!(\mathcal{H}^q\widetilde{A})]. \end{aligned}$$

The latter equality comes from the spectral sequence $R^p\mu_!(\mathcal{H}^q\widetilde{A}) \Rightarrow \mathcal{H}^n(\mu_!\widetilde{A})$. On the other hand $h_G(k)h_Y(m) = \sum_{i,j}(-1)^{i+j}[\mathcal{H}^iA_{\widehat{\phi}_v\xi,w}][\mathcal{H}^jA_{\xi,v}]$.

If p is the projection $G\times Y \to G\overset{B}{\times}Y$, we have $p^*\mathcal{H}^{q+\dim B}\widetilde{A} = \sum_{i+j=q}\mathcal{H}^iA_{\widehat{\phi}_v\xi,w}\boxtimes \mathcal{H}^jA_{\xi,v}$. From this observation and the definitions it follows easily that

$$\sum_p(-1)^p[R^p\mu_!(\mathcal{H}^{q+\dim B}\widetilde{A})] = \sum_{i+j=q}[\mathcal{H}^iA_{\widehat{\phi}_v\xi,w}]\,[\mathcal{H}^jA_{\xi,v}].$$

Now

$$h_G(k)h_Y(m) = \sum_q(-1)^q\sum_p(-1)^p[R^p\mu_!(\mathcal{H}^{q+\dim B}\widetilde{A})] = (-1)^{\dim B}h_Y(km).$$

3.3. **Definition of D.**

3.3.1. Verdier duality induces a $\mathbb{Z}$-linear map $D : \mathcal{K}(\mathcal{A}_Y) \to \mathcal{K}(\mathcal{A}_Y)$. This map is $\mathbb{Z}[C]$-semilinear with respect to the involution of $\mathbb{Z}[C]$ defined by the inverse in C. We also define $D : \mathcal{K}(\mathcal{C}_Y) \to \mathcal{K}(\mathcal{C}_Y)$. This semilinear map is given by $D[\mathcal{S}] = \sum(-1)^i[\mathcal{H}^iD_Y\mathcal{S}]$, where $D_Y\mathcal{S}$ is the Verdier dual of $\mathcal{S}$ as a complex on Y. These definitions apply in particular to $Y = G$.

3.3.2. Proposition. (i) *$h_Y \circ D = D \circ h_Y$.*
(ii) *$D(km) = q^{\dim B}(Dk)(Dm)$ if $k \in \mathcal{K}(\mathcal{A}_G)$, $m \in \mathcal{K}(\mathcal{A}_Y)$ or $\mathcal{K}(\mathcal{A}_G)$.*
(iii) *$D(km) = q^{\dim B}(Dk)(Dm)$ if $k \in \mathcal{K}(\mathcal{C}_G)$, $m \in \mathcal{K}(\mathcal{C}_Y)$ or $\mathcal{K}(\mathcal{C}_G)$.*
(iv) *$D \circ D$ is the identity.*

Proof of (i).

$$\begin{aligned} h_Y(D[A]) = h_Y[D_YA] &= \sum(-1)^i[\mathcal{H}^iD_YA] \\ &= \sum_{p,q}(-1)^{p+q}[\mathcal{H}^pD_Y(\mathcal{H}^qA)] = D(h_Y[A]). \end{aligned}$$

To prove (ii) take $k = [A_{\widehat{\phi}_v\xi,w}]$ and $m = [A_{\xi,v}]$. Then $Dk = q^{-\dim BwB}[A_{-\widehat{\phi}_v\xi,w}]$ and $Dm = q^{-\dim v}[A_{-\xi,v}]$ by 2.2.6(iii). Let $\widetilde{A}$ be as before (3.2.5) and let $\widetilde{\widetilde{A}}$ be the same object with ξ replaced by $-\xi$. Applying Verdier duality to the formula $A_{\widehat{\phi}_v\xi,w}\boxtimes A_{\xi,v} = p^*\widetilde{A}[\dim B]$ we find that $D_Y\widetilde{A} = \widetilde{\widetilde{A}}(l(w)+\dim v)$. Now

$$\begin{aligned} D(km) &= \sum(-1)^iD[{}^pH^i(\mu_!\widetilde{A})] \\ &= \sum(-1)^i[{}^pH^{-i}D_Y(\mu_!\widetilde{A})] = \sum(-1)^i[{}^pH^i(\mu_!D_Y\widetilde{A})] \\ &= q^{-l(w)-\dim v}\sum(-1)^i[{}^pH^i(\mu_!\widetilde{\widetilde{A}})] = q^{\dim B}(Dk)(Dm). \end{aligned}$$

Recall that $\dim BwB = l(w)+\dim B$, where $l(w)$ is the length of w.

(iii) follows from (i) and (ii) and (iv) is obvious.

3.4. **Application of duality.**

3.4.1. We now introduce some notations. If $v \in V$, $\xi \in \widehat{X}(T/T_v)$, we let $\varepsilon_{\xi,v} = [\mathcal{L}_{\xi,v}] \in \mathcal{K}(\mathcal{C}_Y)$, including the case $Y = G$. Expressing $D[\mathcal{L}_{\xi,v}]$ and $[\mathcal{H}^i A_{\xi,v}]$ in the basis elements of $\mathcal{K}(\mathcal{C}_Y)$ we have

$$D\varepsilon_{\xi,v} = q^{-\dim v}\varepsilon_{-\xi,v} + \sum_{u<v} b_{\eta,u;\xi,v}\varepsilon_{\eta,u}$$

and

$$[\mathcal{H}^i A_{\xi,v}] = \delta_{i,-\dim v}\varepsilon_{\xi,v} + \sum_{u<v} c_{\eta,u;\xi,v,i}\varepsilon_{\eta,u}$$

with coefficients $b_{\eta,u;\xi,v}$, $c_{\eta,u;\xi,v,i} \in \mathbb{Z}[C]$. Summation is over (η, u), $u \in V$, $\eta \in \widehat{X}(T/T_u)$. We have $b_{\eta,u;\xi,v} = 0$ unless $\widehat{\phi}_u\eta = -\widehat{\phi}_v\xi$ and $c_{\eta,u;\xi,v,i} = 0$ unless $\widehat{\phi}_u\eta = \widehat{\phi}_v\xi$.

3.4.2. Applying h_Y to the equality $D[A_{\xi,v}] = q^{-\dim v}[A_{-\xi,v}]$ we see that

$$D\sum(-1)^i[\mathcal{H}^i A_{\xi,v}] = q^{-\dim v}\sum(-1)^i[\mathcal{H}^i A_{-\xi,v}],$$

since h_Y commutes with D. Now express both sides of the latter equality in the basis elements, using the notations of 3.4.1. The result is

$$\begin{aligned}\sum_i(-1)^i c_{\eta,u;-\xi,v,i} &- q^{\dim v-\dim u}\sum_i(-1)^i\bar{c}_{-\eta,u;\xi,v,i}\\ &= (-1)^{\dim v}q^{\dim v}b_{\eta,u;\xi,v} + q^{\dim v}\sum_{u<z<v} b_{\eta,u;\zeta,z}\sum_i(-1)^i\bar{c}_{\zeta,z;\xi,v,i} \quad \text{if } u < v.\end{aligned}$$

Here $^-$ denotes the involution of $\mathbb{Z}[C]$ defined by the inverse in C.

3.4.3. Since $\mathcal{L}_{\xi,v}$ has weight zero, we know that $A_{\xi,v}$ is pure of weight $\dim v$. This means that the punctual weights of $\mathcal{H}^i A_{\xi,v}$ are $\leq i + \dim v$. Assume that we are in a situation where it is known that these weights are equal to $i + \dim v$. Then the $c_{\eta,u;\xi,v,i}$ are linear combinations with coefficients in $\mathbb{Z}$ and ≥ 0 of (classes of) algebraic numbers with complex absolute values $q^{\frac{1}{2}(i+\dim v)}$. Since $A_{\xi,v}$ is a perverse extension, $c_{\eta,u;\xi,v,i} \neq 0$ implies $\dim u < -i$, if $u < v$. So the absolute values of the algebraic numbers occurring in $c_{\eta,u;-\xi,v,i}$ are $q^{\frac{1}{2}(i+\dim v)} \leq q^{\frac{1}{2}(\dim v-\dim u-1)}$ and the absolute values of those occurring in $q^{\dim v-\dim u}\bar{c}_{-\eta,u;\xi,v,i}$ are $q^{\frac{1}{2}(\dim v-i)-\dim u} \geq q^{\frac{1}{2}(\dim v-\dim u+1)}$.

It follows that, when ξ, v, η, u are given, the coefficients $c_{\eta,u;-\xi,v,i}$ ($i \in \mathbb{Z}$) are determined by the formula in 3.4.2, if the right hand side is known. In particular, when all the coefficients b are known, the coefficients $c_{\eta,u;\xi,v,i}$ can be computed from the formula by descending induction on $\dim u$.

4. Product of a minimal parabolic and an orbit

One can prove that the algebra $\mathcal{K}(\mathcal{C}_G)$ is generated by the elements $\varepsilon_{\eta,s}$ with s a simple reflection and $\eta \in \widehat{X}(T)$. Hence the $\mathcal{K}(\mathcal{C}_G)$-module structure on $\mathcal{K}(\mathcal{C}_Y)$ is determined by the products $\varepsilon_{\widehat{\phi}_v\xi,s}\varepsilon_{\xi,v}$ with $v \in V$, $\xi \in \widehat{X}(T/T_v)$, s a simple reflection. In this chapter we are going to compute these products.

4.1. **Orbit types.**

4.1.1. We use the notations of chapter 3. The results of 4.1 and 4.2 are valid over any algebraically closed ground field k, from 4.3 on we take up the conventions of 3.1.2.

The choice of B and T determines the root system and the set of simple roots. The Weyl group $W = N(T)/T$ is generated by the set S of simple reflections.

We choose the one parameter subgroups u_α associated to the roots in such a way that $n_\alpha = u_\alpha(1)u_{-\alpha}(-1)u_\alpha(1) \in N(T)$ for every root α. Then n_α is a representative for the reflection s_α corresponding to α. We have $n_\alpha^2 = \check{\alpha}(-1)$ and $n_{-\alpha} = n_\alpha^{-1}$, where $\check{\alpha}$ is the coroot. The following formulas will be used frequently.

$$u_\alpha(x)u_{-\alpha}(-x^{-1})u_\alpha(x) = \check{\alpha}(x)n_\alpha \quad (x \in k^*)$$

$$n_\alpha u_\alpha(x)n_\alpha^{-1} = u_{-\alpha}(-x) \quad (x \in k)$$

$\mathrm{Im}\,(u_\alpha)$ is denoted by U_α.

4.1.2. Fix $s \in S$ and let $P = P_s$ be the parabolic subgroup of G generated by B and s, so $P = \overline{BsB} = B \cup BsB$. Let α be the simple root corresponding to s.

If $v \in V$ and $\dot{v} \in v$, we have an equality of finite sets

$$B\backslash Pv = B\backslash P\dot{v} = B\backslash P/P_{\dot{v}} \cong \mathbb{P}^1/P_{\dot{v}},$$

where $P_{\dot{v}}$ is the isotropy group of $\dot{v}$. We choose the isomorphism $B\backslash P \cong \mathbb{P}^1$ in such a way that $Bn_\alpha u_\alpha(x) \mapsto -x$. The action of P on $B\backslash P$ (given by $p_0 \cdot Bp = Bpp_0^{-1}$) defines a homomorphism $\phi : P \to \mathrm{Aut}\,(\mathbb{P}^1)$. When $\mathrm{Aut}\,(\mathbb{P}^1)$ is identified with PGL_2 in the usual way, we have

$$\phi(u_\alpha(x)) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} k^* \ , \ \phi(n_\alpha) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} k^* \ , \ \phi(\check{\alpha}(x)) = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} k^*$$

and $\mathrm{Ker}\,(\phi) = \mathrm{Ker}\,(\alpha)U_P$, where U_P is the unipotent radical of P.

4.1.3. Let H be a closed subgroup of PGL_2. Modulo conjugation by an element of $\phi(B)$ we have the following possibilities, if H is infinite.

I. $H = PGL_2$.

IIa. $\phi(U_\alpha) \subset H \subset \phi(B)$. IIb. $\phi(U_{-\alpha}) \subset H \subset \phi(n_\alpha B n_\alpha^{-1})$.

IIIa. $H = \phi(T)$. IIIb. H is the conjugate of $\phi(T)$ by $\phi(n_\alpha u_\alpha(-1))$.

IVa., resp. IVb. H is the normalizer of the group of case IIIa, resp. IIIb.

The number of H-orbits in $\mathbb{P}^1$ is 1 in case I, 2 in the cases II and IV, 3 in case III. In case II there is one fix point, in case III there are two fix points and in case IV there is an orbit consisting of two elements.

4.1.4. Let $v \in V$. We can choose $\dot{v} \in v$ such that $H = \phi(P_{\dot{v}})$ is in one of the cases I–IVb of 4.1.3. We call such an element $\dot{v}$ a *special* element of v (for s) and say that v is of type I, IIa, … for s.

The partition of Pv into B-orbits can easily be derived from the partition of $\mathbb{P}^1$ into H-orbits. We give below the situation in the various cases. The open B-orbit in $Pv = P_s v$ is denoted by $m(s)v$ as in [RS1].

I. $Pv = v$.

IIa. $Pv = v \cup m(s)v$, v is closed in Pv, $\dim v = \dim Pv - 1$.
We have $n_\alpha u_\alpha(x)\dot{v} \in m(s)v$ for all $x \in k$.

IIb. $Pv = v \cup v'$, v is open in Pv, $\dim v' = \dim v - 1$.
We have $n_\alpha u_\alpha(x)\dot{v} \in v$ if $x \neq 0$, $n_\alpha \dot{v} \in v'$.

IIIa. $Pv = v \cup v' \cup m(s)v$, v and v' are closed in Pv, $\dim v = \dim v' = \dim Pv - 1$ and $v \neq v'$.
We have $n_\alpha \dot{v} \in v'$, $n_\alpha u_\alpha(x)\dot{v} \in m(s)v$ if $x \neq 0$.

IIIb. $Pv = v \cup v' \cup v''$, v is open, v' and v'' are closed in Pv, $v' \neq v''$, $\dim v' = \dim v'' = \dim v - 1$.
We have $n_\alpha u_\alpha(x)\dot{v} \in v$ if $x \neq 0, -1$, $n_\alpha \dot{v} \in v'$, $n_\alpha u_\alpha(-1)\dot{v} \in v''$.
v' and v'' are of type IIIa with special elements $\dot{v}' = u_\alpha(1)n_\alpha^{-1}\dot{v}$ and $\dot{v}'' = n_\alpha \dot{v}'$.

IVa. $Pv = v \cup m(s)v$, v is closed in Pv, $\dim v = \dim Pv - 1$.
We have $n_\alpha \dot{v} \in v$, $n_\alpha u_\alpha(x)\dot{v} \in m(s)v$ if $x \neq 0$.

IVb. $Pv = v \cup v'$, v is open and v' is closed in Pv, $\dim v' = \dim v - 1$.
We have $n_\alpha u_\alpha(x)\dot{v} \in v$ if $x \neq 0, -1$, $n_\alpha u_\alpha(x)\dot{v} \in v'$ if $x = 0, -1$.
v' is of type IVa with special element $\dot{v}' = u_\alpha(1)n_\alpha^{-1}\dot{v}$.

4.1.5. Lemma. *Let $y \in Y$. Put $H = \phi(P_y)$ (notations of 4.1.2). If $\phi(U_\alpha) \subset H$, then $U_\alpha \subset U_P P_y$. If $\phi(U_{-\alpha}) \subset H$, then $U_{-\alpha} \subset U_P P_y$.*

We prove the first assertion. Assume that $B_y^o = T_y^o U_y^o$ (this can be achieved by replacing y by uy for an appropriate $u \in U$; the assertion to be proved is not changed by that substitution). We have $U_\alpha \subset \operatorname{Ker}(\alpha) U_P P_y$, whence $U_\alpha \subset TU_P B_y^o = TU_P U_y^o$. It follows that $U_\alpha \subset U_P U_y^o \subset U_P P_y$. The second assertion is nothing but the first one applied to $n_\alpha y$.

4.2. **Results involving the group T_v.** In this section notations are as in 4.1. We fix $P = P_s$ and v and choose a special element $\dot{v} \in v$. We now give some results on T_v and $\widehat{X}(T/T_v)$ which are needed in the sequel.

4.2.1. Assume v is of type I for s. By Lemma 4.1.5 the group generated by U_α and $U_{-\alpha}$ is contained in $U_P P_{\dot{v}}$. So $\operatorname{Im}(\check{\alpha}) \subset U_P B_{\dot{v}}$ and by the definition of T_v it follows that $\operatorname{Im}(\check{\alpha}) \subset T_v$. If $\chi \in X(T/T_v)$, then $\langle \chi, \check{\alpha} \rangle = 0$. Obviously $s(T_v) = T_v$.

4.2.2. Assume v of type IIa. We have $P_{\dot{v}} \subset B$ and, by Lemma 4.1.5, $U_\alpha \subset U_P U_{\dot{v}}$. It is easily seen that $s(T_v) = T_{m(s)v}$.

4.2.3. Assume v of type IIIa. Then

(i) $T_{m(s)v} = T_v \cap \operatorname{Ker}(\alpha)$.
(ii) $T = T_v \operatorname{Ker}(\alpha)$.
(iii) $s(T_v) = T_{v'}$.

We have $P_{\dot{v}} \subset TU_P$ and $\operatorname{Im}(\check{\alpha}) \subset \operatorname{Ker}(\alpha) U_P P_{\dot{v}}$. So $\operatorname{Im}(\check{\alpha}) \subset \operatorname{Ker}(\alpha) T_v$, which implies (ii). Also (iii) is easy. We prove (i).

Let $t \in T_{m(s)v}$. There is $u \in U$ such that $tun_\alpha u_\alpha(1)\dot{v} = n_\alpha u_\alpha(1)\dot{v}$ (recall that $n_\alpha u_\alpha(1)\dot{v} \in m(s)v$). Then $tun_\alpha u_\alpha(1) = n_\alpha u_\alpha(1)t'u'$ with $t' \in T$, $u' \in U_P$, $t'u' \in P_{\dot{v}}$. It follows that $tun_\alpha u_\alpha(1) \in U_P s(t') n_\alpha u_\alpha(\alpha(t')^{-1})$, whence $\alpha(t') = 1$, $t = s(t') = t'$. So we have $t \in \operatorname{Ker}(\alpha)$ and from $t'u' \in B_{\dot{v}}$ it follows that $t \in T_v$.

Conversely, let $t \in T_v \cap \operatorname{Ker}(\alpha)$. Choose $u \in U$ such that $ut \in B_{\dot{v}}$. Then $u \in U_P$ and $tn_\alpha u_\alpha(1)\dot{v} = n_\alpha u_\alpha(1)t\dot{v} \in U_P n_\alpha u_\alpha(1)ut\dot{v} = U_P n_\alpha u_\alpha(1)\dot{v}$, so $t \in T_{m(s)v}$.

4.2.4. Assume v of type IVa and assume that the characteristic of k is not 2. Then

(i) $\mathrm{Im}\,(\check{\alpha}) \subset T_v$.
(ii) $\alpha(T_{m(s)v}) = \{\pm 1\}$.
(iii) $T_v \cap \mathrm{Ker}\,(\alpha) = T_{m(s)v} \cap \mathrm{Ker}\,(\alpha)$.
(iv) $s(T_v) = T_v$ and $s(T_{m(s)v}) = T_{m(s)v}$.

We have $P_{\dot{v}} \subset TU_P \cup n_\alpha TU_P$ and $\langle n_\alpha, \mathrm{Im}\,(\check{\alpha})\rangle \subset \mathrm{Ker}\,(\alpha)U_P P_{\dot{v}}$. That $s(T_v) = T_v$ follows from the facts that $B_{\dot{v}} \subset TU_P$ and $n_\alpha \dot{v} \in v$. We have obviously $\mathrm{Im}\,(\check{\alpha}) \subset \mathrm{Ker}\,(\alpha)T_v$. Choose $\tau \in k^*$ and let $t \in \mathrm{Ker}\,(\alpha)$ be such that $t\check{\alpha}(\tau) \in T_v$. Then $t\check{\alpha}(\tau^{-1}) = s(t\check{\alpha}(\tau)) \in s(T_v) = T_v$. Hence $\check{\alpha}(\tau^2) \in T_v$. This proves (i).

One proves that $T_v \cap \mathrm{Ker}\,(\alpha) \subset T_{m(s)v}$ as in case IIIa.

Now we first prove the following lemma, which we shall also need later.

4.2.5. Lemma. *Let x_1, $x_2 \in k$, t_1, $t_2 \in T$, u_1, $u_2 \in U_P$ be such that $u_\alpha(x_1)u_1t_1\dot{v}' = u_\alpha(x_2)u_2t_2\dot{v}'$, where $\dot{v}' = n_\alpha u_\alpha(-1)\dot{v} \in m(s)v$. Then we have either $x_1 = x_2$, $\alpha(t_1) = \alpha(t_2)$ or $x_2 - x_1 = \alpha(t_1) = -\alpha(t_2)$.*

For the proof we may take $x_2 = 0$, $u_2 = t_2 = 1$ and drop the indices 1. The element $g = u_\alpha(1)n_\alpha^{-1}u_\alpha(x)utn_\alpha u_\alpha(-1)$ stabilizes $\dot{v}$. There are two possibilities: $g \in TU_P$ or $g \in n_\alpha TU_P$. If $g \in TU_P$, we must have $x = 0$, $\alpha(t) = 1$, and it follows also that $t \in T_v$. Assume now that $g \in n_\alpha TU_P$. Then $x \neq 0$ and

$$g = u_\alpha(1 - x^{-1})n_\alpha\check{\alpha}(x)u_\alpha(-x^{-1}).n_\alpha^{-1}un_\alpha.s(t)u_\alpha(-1).$$

The condition $g \in n_\alpha TU_P$ gives $x = 1$, $\alpha(t) = -1$.

We now finish the proof of 4.2.4. If $t \in T_{m(s)v}$, there are $x \in k$ and $u \in U_P$ such that $u_\alpha(x)ut\dot{v}' = \dot{v}'$, with $\dot{v}'$ as in Lemma 4.2.5. By that lemma and its proof we have then $\alpha(t) = 1$ and $t \in T_v$ or $\alpha(t) = -1$. Assertion (iii) of 4.2.4 is now proved. To complete the proof of (ii), choose $i \in k$ such that $i^2 = -1$ and observe that $\check{\alpha}(i)n_\alpha \in \mathrm{Ker}\,(\alpha)U_P P_{\dot{v}}$. Conjugation by $n_\alpha u_\alpha(-1)$ gives $u_\alpha(1)\check{\alpha}(i) \in \mathrm{Ker}\,(\alpha)U_P P_{\dot{v}'}$. Hence there is $t \in T_{m(s)v}$ with $\alpha(t) = -1$.

Finally, we have $\check{\alpha}(-1) \in T_{m(s)v}$ (by (i) and (iii)). Together with (ii) this gives the second assertion of (iv).

Remark. It has been shown by Knop [K] that there is an action of the Weyl group on V such that the simple reflections act as follows. In case II the two B-orbits are interchanged, in case III the two small orbits are interchanged, all other orbits are fixed. If this action is denoted by $(w, v) \mapsto w \cdot v$, 4.2.1–4.2.4 show that $s(T_v) = T_{s \cdot v}$, hence $w(T_v) = T_{w \cdot v}$ ($w \in W$, $v \in V$).

4.2.6. Let v be of type IIIa for s. Consider the natural homomorphism $T/T_{m(s)v} \to T/T_v$. Its kernel is $T_v/T_{m(s)v}$ and from 4.2.3 we see that there are bijective homomorphisms $T_v/T_{m(s)v} \to T/\mathrm{Ker}\,(\alpha) \xrightarrow{\alpha} \mathbb{G}_m$. So Lemma 2.1.5 gives an exact sequence

$$0 \to \widehat{X}(T/T_v) \to \widehat{X}(T/T_{m(s)v}) \to \widehat{X}(\mathbb{G}_m) \to 0.$$

If v is of type IVa for s, we find in the same way, using 4.2.4(i), an exact sequence

$$0 \to \widehat{X}(T/T_v) \to \widehat{X}(T/T_v \cap \mathrm{Ker}\,(\alpha)) \to \widehat{X}(\mathbb{G}_m) \to 0.$$

Moreover, in this case we have a homomorphism $\widehat{X}(T/T_{m(s)v}) \to \widehat{X}(T/T_v \cap \mathrm{Ker}(\alpha))$, which is surjective and has a kernel of two elements. This comes from the homomorphism $T/T_{m(s)v} \cap \mathrm{Ker}\,(\alpha) \to T/T_{m(s)v}$, using 4.2.4(ii) and (iii).

4.2.7. It is convenient to formulate the above results also starting with an orbit of type IIIb or IVb. We take these cases together. So let v be of type IIIb or IVb for s and let $v' \subset Pv$, $v' \neq v$. We assume that the characteristic of k is not 2. Then

$$T_v \cap \mathrm{Ker}\,(\alpha) = T_{v'} \cap \mathrm{Ker}\,(\alpha)\ ,\ T = T_{v'}\,\mathrm{Ker}\,(\alpha),$$
$$\alpha(T_v) = \{\pm 1\} \text{ in case IV},\ T_v \subset \mathrm{Ker}\,(\alpha) \text{ in case III}.$$
$$\text{In case IV we have } \mathrm{Im}\,(\check{\alpha}) \subset T_{v'}.$$

In the following diagram the row is exact. The map a is defined such that the triangle is commutative.

$$\begin{tikzcd}
 & & \widehat{X}(T/T_v) \arrow[d] \arrow[dr, "a"] & & \\
0 \arrow[r] & \widehat{X}(T/T_{v'}) \arrow[r] & \widehat{X}(T/T_v \cap \mathrm{Ker}\,(\alpha)) \arrow[r] & \widehat{X}(\mathbb{G}_m) \arrow[r] & 0
\end{tikzcd}$$

The homomorphism $\widehat{X}(T/T_v) \to \widehat{X}(T/T_v \cap \mathrm{Ker}\,(\alpha))$ is surjective and has a kernel of order two in case IV, it is the identity in case III.

An easy computation shows that $a(\xi) - a(s\xi) = \langle \widehat{\phi}_v\xi, \check{\alpha}\rangle$.

Notice that s induces an automorphism of $\widehat{X}(T/T_v)$, since $s(T_v) = T_v$. The coroot $\check{\alpha} : X(T) \to \mathbb{Z}$ is extended to a map $\widehat{X}(T) \to \mathbb{Z}_{(p)}/\mathbb{Z}$, which explains the notation $\langle \widehat{\phi}_v\xi, \check{\alpha}\rangle$ (ϕ_v as in 2.2.5).

If $\mathrm{Im}\,(\check{\alpha}) \subset T_{v'}$, we have $\langle \widehat{\phi}_v\xi, \check{\alpha}\rangle = 2a(\xi)$.

In case III the map a corresponding to the other small orbit in Pv is $\xi \mapsto -a(s\xi)$, so the sum of the two is $\langle \widehat{\phi}_v\xi, \check{\alpha}\rangle$. We shall denote these two maps $\widehat{X}(T/T_v) \to \widehat{X}(\mathbb{G}_m)$ by $a_{v'}$ and $a_{v''}$ (when s and v are fixed).

Remark. If Y is symmetric, one has $\mathrm{Im}\,(\check{\alpha}) \subset T_{v'}$ in both cases III and IV (see 6.7).

4.3. **The product** $\varepsilon_{\widehat{\phi}_v\xi,s}\varepsilon_{\xi,v}$.

4.3.1. For $s \in S$, $v \in V$ and $\xi \in \widehat{X}(T/T_v)$, the product $\varepsilon_{\widehat{\phi}_v\xi,s}\varepsilon_{\xi,v}$ is given in the following table according to the type of v for s. In the case of an orbit of type IV we assume that the characteristic of k is different from 2. It is assumed that all B-orbits contain points over $\mathbb{F}_q$ and that $F\xi = \xi$. The notation of the orbits is as in 4.1.4, notations for elements of $\widehat{X}(T/T_{v'})$ etc. are explained below. The proofs will be given in 4.3.3–4.3.11.

I. $q\varepsilon_{\xi,v}$

IIa. $\varepsilon_{s\xi,m(s)v}$

IIb. $(q-1)\varepsilon_{\xi,v} + q\varepsilon_{s\xi,v'}$ if $\langle \widehat{\phi}_v\xi, \check{\alpha}\rangle = 0$,
$q\varepsilon_{s\xi,v'}$ if $\langle \widehat{\phi}_v\xi, \check{\alpha}\rangle \neq 0$.

IIIa. $\varepsilon_{s\xi,v'} + \varepsilon_{s\xi,m(s)v}$

IIIb. $(q-2)\varepsilon_{\xi,v} + (q-1)(\varepsilon_{\xi,v'} + \varepsilon_{\xi,v''})$ if $a_{v'}(\xi) = a_{v''}(\xi) = 0$,
$-\varepsilon_{\xi,v}$ if $\langle \widehat{\phi}_v\xi, \check{\alpha}\rangle = 0$, $a_{v'}(\xi) \neq 0$,
$-\varepsilon_{s\xi,v} + (q-1)\varepsilon_{s\xi,v'}$ if $a_{v'}(\xi) \neq 0$, $a_{v''}(\xi) = 0$,
$-\varepsilon_{s\xi,v} + (q-1)\varepsilon_{s\xi,v''}$ if $a_{v'}(\xi) = 0$, $a_{v''}(\xi) \neq 0$,
$-\mathcal{J}_v(\xi)\varepsilon_{s\xi,v}$ if $a_{v'}(\xi) \neq 0$, $a_{v''}(\xi) \neq 0$, $\langle \widehat{\phi}_v\xi, \check{\alpha}\rangle \neq 0$.

IVa. $\varepsilon_{\xi,v} + \varepsilon_{\xi_1,m(s)v} + \varepsilon_{\xi_2,m(s)v}$

IVb. $(q-1)\varepsilon_{\xi,v} - \varepsilon_{\xi',v} + (q-1)\varepsilon_{\bar{\xi},v'}$ if $a(\xi) = 0$,
$-\varepsilon_{\xi,v}$ if $a(\xi) \neq 0$, $2a(\xi) = 0$,
$-\mathcal{J}_v(\xi)\varepsilon_{s\xi',v}$ if $2a(\xi) \neq 0$.

If $\xi \in \widehat{X}(T/T_v)$, then $s\xi \in \widehat{X}(T/s(T_v))$. In case IIIa we have $s\xi \in \widehat{X}(T/T'_v) \subset \widehat{X}(T/T_{m(s)v})$ (see 4.2.6). If v is of type IIIb and $a_{v'}(\xi) = 0$, then ξ lies in the subgroup $\widehat{X}(T/T_{v'})$ of $\widehat{X}(T/T_v)$ and $s\xi$ in the subgroup $\widehat{X}(T/T_{v''})$ of $\widehat{X}(T/T_v)$ (see 4.2.7). In case IVa, ξ_1 and ξ_2 are the two elements of $\widehat{X}(T/T_{m(s)v})$ which are mapped onto the image of ξ in $\widehat{X}(T/T_v \cap \mathrm{Ker}\,(\alpha))$ (see 4.2.6). In case IVb, ξ' is the element $\neq \xi$ of $\widehat{X}(T/T_v)$ which has the same image as ξ in $\widehat{X}(T/T_v \cap \mathrm{Ker}\,(\alpha))$. When $a(\xi) = 0$, this image can be considered as an element $\bar{\xi}$ of $\widehat{X}(T/T_{v'})$ (see 4.2.7).

The definition of $\mathcal{J}_v(\xi)$ in the cases IIIb and IVb is as follows. Since we assume that T is defined and split over $\mathbb{F}_q$ and that $F\xi = \xi$, i.e. $(q-1)\xi = 0$, we have also $(q-1)a_{v'}(\xi) = (q-1)a_{v''}(\xi) = 0$, in case IIIb. Therefore $a_{v'}(\xi)$ and $a_{v''}(\xi)$ determine two characters $\chi_1, \chi_2 : \mathbb{F}_q^* \to \bar{\mathbb{Q}}_l^*$ (see [MS, 2.3.1]; in our particular case, if $a_{v'}(\xi) = \frac{m}{q-1} + \mathbb{Z}$, then $\chi_1 = \psi \circ m$, ψ as in 2.1.1). Now $\mathcal{J}_v(\xi)$ is the class in C (cf. 3.1.2) of the Jacobi sum $J(\chi^{-1})$ defined in 2.1.7 for χ with components $\chi_0 = \chi_1^{-1}\chi_2^{-1}$, χ_1, χ_2:

$$J(\chi^{-1}) = -\sum_{x \in \mathbb{F}_q^*, x \neq -1} \chi_1(x)\chi_2(-1-x).$$

If v is of type IVb, the definition is the same, just take $\chi_1 = \chi_2$ equal to the character of $\mathbb{F}_q^*$ determined by $a(\xi)$.

4.3.2. The product in $\mathcal{K}(\mathcal{C}_G)$ is described by the following formulas, which can be derived from 4.3.1, IIa and IIb, or proved directly. The notation is as in 3.1.3. Let $s \in S$, $w \in W$, $\xi \in \widehat{X}(T)$.

$$\begin{aligned}
\varepsilon_{w\xi,s}\varepsilon_{\xi,w} &= \varepsilon_{\xi,sw} && \text{if } sw > w,\\
&= (q-1)\varepsilon_{\xi,w} + q\varepsilon_{\xi,sw} && \text{if } sw < w \text{ and } \langle w\xi, \check{\alpha}\rangle = 0,\\
&= q\varepsilon_{\xi,sw} && \text{if } sw < w \text{ and } \langle w\xi, \check{\alpha}\rangle \neq 0,
\end{aligned}$$

4.3.3. Let s, v and ξ be as in 4.3.1. The local system $\mathcal{L}_{\widehat{\phi}_v\xi,s} \boxtimes \mathcal{L}_{\xi,v}$ on $BsB \times v$ is the pull-back of a local system $\widetilde{\mathcal{L}}$ on $BsB\overset{B}{\times}v$ and we have to compute $\mu_!\widetilde{\mathcal{L}}$, where μ is the map $BsB\overset{B}{\times}v \to Y$ induced by the action of G on Y. The image of μ is $m(s)v$ if v is of type IIa for s, it is $v' \cup m(s)v$ if v is of type IIIa and it is Pv in all other cases. This can be read off from 4.1.4. We shall now treat the different cases. Elements of $G\overset{B}{\times}Y$ are denoted by $g * y$ and $\dot{v}$ is a special element of v (see 4.1.4).

4.3.4. Assume v is of type IIa for s. Then μ is an isomorphism from $BsB\overset{B}{\times}v$ onto $m(s)v$, because $P_{\dot{v}} = B_{\dot{v}}$ (4.2.2). If $p \in Un_\alpha t_1 U$ and $y \in t_2 U\dot{v}$ with t_1, $t_2 \in T$, then $py \in Us(t_1t_2)n_\alpha\dot{v}$, because $U = U_P U_{\dot{v}}$ (4.2.2). Since $n_\alpha\dot{v} \in m(s)v$, it follows immediately from the definitions that $\mu_!\widetilde{\mathcal{L}} = \mathcal{L}_{s\xi,m(s)v}$ and so $\varepsilon_{\widehat{\phi}_v\xi,s}\varepsilon_{\xi,v} = \varepsilon_{s\xi,m(s)v}$.

4.3.5. Assume v of type IIIa. Now μ is an isomorphism from $BsB\overset{B}{\times}v$ onto its image $v' \cup m(s)v$, since $P_{\dot{v}} = B_{\dot{v}}$. If $p = bn_\alpha b_1 \in Bn_\alpha B$ and $y = b_2\dot{v} \in v$, then $py \in v'$ if $b_1b_2 \in TU_P$, $py \in m(s)v$ if $b_1b_2 \notin TU_P$.

We shall make use of the following observation. The identity $\check{\alpha}(\tau)n_\alpha u_\alpha(-1)\check{\alpha}(\tau) = n_\alpha u_\alpha(-\tau^{-2})$ implies, since $\mathrm{Im}\,(\check{\alpha}) \subset \mathrm{Ker}\,(\alpha)U_P P_{\dot{v}}$, that $n_\alpha u_\alpha(-\tau^{-2})\dot{v} \in U_P t_0 \check{\alpha}(\tau)\dot{v}''$ with $\dot{v}'' = n_\alpha u_\alpha(-1)\dot{v} \in m(s)v$, if $\check{\alpha}(\tau) \in t_0 T_v$, $t_0 \in \mathrm{Ker}\,(\alpha)$.

The local system $\mathcal{L}_{\widehat{\phi}_v \xi, s} \boxtimes \mathcal{L}_{\xi,v}$ is the inverse image of $\mathcal{L}_\xi$ under the map $f : BsB \times v \to T/T_v$, where $Un_\alpha t_1 U \times U t_2 \dot{v} \to t_1 t_2 T_v$. Let $(BsB \times v)'$, resp. $(BsB \times v)''$, denote the subset of $BsB \times v$ which is mapped onto v', resp. $v'' = m(s)v$, by the product map. The restrictions f' and f'' of f to these subsets factorize as follows:

$$f' : (BsB \times v)' \to v' \xrightarrow{\pi_{\dot{v}'}} T/T_{v'} \xrightarrow{s} T/T_v,$$

$$f'' : (BsB \times v)'' \to v'' \xrightarrow{\pi_{\dot{v}''}} T/T_{v''} \xrightarrow{s} T/T_v.$$

Indeed, if $p = un_\alpha b_1$, $y = b_2\dot{v}$ and $b_1 b_2 \in U_P u_\alpha(x)t$ with $u \in U$, b_1, $b_2 \in B$, $t \in T$, then we have $py \in Un_\alpha t\dot{v} = Us(t)\dot{v}'$ if $x = 0$ and, applying the observation above, we see that $py \in Us(t)n_\alpha u_\alpha(\alpha(t)^{-1}x)\dot{v} = Us(t)t_0\check{\alpha}(\tau)\dot{v}''$ if $x \neq 0$ and $\tau^2 = -\alpha(t)x^{-1}$, $\check{\alpha}(\tau) \in t_0 T_v$, $t_0 \in \mathrm{Ker}\,(\alpha)$. Notice that $s(t_0\check{\alpha}(\tau)) = t_0\check{\alpha}(\tau)^{-1} \in T_v$.

It follows now immediately that $\mu_! \widetilde{\mathcal{L}}|_{v'} = \mathcal{L}_{s\xi,v'}$ and $\mu_! \widetilde{\mathcal{L}}|_{m(s)v} = \mathcal{L}_{s\xi,m(s)v}$. This proves 4.3.1 in case IIIa.

4.3.6. Assume v of type IVa. The image of μ is $Pv = v \cup m(s)v$. If $p = bn_\alpha b_1 \in Bn_\alpha B$ and $y = b_2\dot{v} \in v$, then $py \in v$ if $b_1 b_2 \in TU_P$, $py \in m(s)v$ if $b_1 b_2 \notin TU_P$. Since $\mathrm{Im}\,(\check{\alpha}) \subset T_v$ (4.2.4), we have in this case $n_\alpha u_\alpha(-\tau^{-2})\dot{v} \in U_P\check{\alpha}(\tau)\dot{v}''$ with $\dot{v}'' = n_\alpha u_\alpha(-1)\dot{v} \in m(s)v$ (cf. 4.3.5). Consider, as in 4.3.5, the map $f : BsB \times v \to T/T_v$. Let $(BsB \times v)'$, resp. $(BsB \times v)''$, denote the subset of $BsB \times V$ which is mapped onto v, resp. $m(s)v$, by the product map. The restriction f' of f to $(BsB \times v)'$ factorizes in

$$f' : (BsB \times v)' \to v \xrightarrow{\pi_{n_\alpha \dot{v}}} T/T_v,$$

as is easily checked. Since $(BsB \overset{B}{\times} v)' \to v$ is an isomorphism, it follows that $\mu_! \widetilde{\mathcal{L}}|_v = \mathcal{L}_{\xi,v}$. The restriction of $\mu_! \widetilde{\mathcal{L}}$ to $m(s)v$ can be determined from the following diagram.

$$\begin{tikzcd}
(BsB \times v)'' \arrow[r] & (BsB \overset{B}{\times} v)'' \arrow[r, "\rho"] \arrow[d] & T/T_v \cap \mathrm{Ker}\,(\alpha) \arrow[r] \arrow[d] & T/T_v \\
 & m(s)v \arrow[r, "\pi_{\dot{v}''}"] & T/T_{m(s)v} &
\end{tikzcd}$$

The first row is a factorization of the restriction of f to $(BsB \times v)''$. It is defined as follows. Let $p = un_\alpha b_1 \in Un_\alpha B$, $y = b_2\dot{v} \in v$, $b_1 b_2 \in U_P u_\alpha(x)t$ with $x \neq 0$. Then $py \in Us(t)\check{\alpha}(\tau)\dot{v}''$, if $\tau^2 = -\alpha(t)x^{-1}$. The morphism ρ is defined by $\rho(un_\alpha b_1 * b_2\dot{v}) = s(t)\check{\alpha}(\tau)(T_v \cap \mathrm{Ker}\,\alpha))$ and the other maps are the obvious ones. The vertical arrows are 2-fold Galois coverings (see 4.2.4), the square is Cartesian. We see from the diagram that $\mu_! \widetilde{\mathcal{L}}|_{m(s)v} = \mathcal{L}_{\xi_1,m(s)v} \oplus \mathcal{L}_{\xi_2,m(s)v}$, where ξ_1, $\xi_2 \in \widehat{X}(T/T_{m(s)v})$ are as explained in 4.3.1. This completes the proof of 4.3.1 in case IVa.

4.3.7. In the cases where v is open in Pv we use the isomorphisms

$$\begin{aligned} BsB \overset{B}{\times} v &\cong \{(pB, y) \in BsB/B \times Pv \mid p^{-1}y \in v\} \\ &\cong \{(x, y) \in k \times Pv \mid n_\alpha^{-1} u_\alpha(-x)y \in v\}. \end{aligned}$$

The first isomorphism is $p * y \leftrightarrow (pB, py)$, for the second one, write $p = u_\alpha(x)n_\alpha$. The morphism $\mu : BsB \overset{B}{\times} v \to Y$ corresponds to the projection $(x, y) \mapsto y$. Recall that the image of μ is Pv. Now $\mu^{-1}(v) \cong \{(x,y) \in k \times v \mid n_\alpha^{-1}u_\alpha(-x)y \in v\}$ and $\mu^{-1}(v') \cong \{(x,y) \in k \times v' \mid n_\alpha^{-1}u_\alpha(-x)y \in v\}$ in the cases IIb, IIIb, IVb.

From 4.1.4 we find the following necessary and sufficient conditions on (x,y) in order that $n_\alpha^{-1}u_\alpha(-x)y \in v$.

If v is of type IIb and $y \in u_\alpha(x')U_P T\dot{v} \; : \; x \neq x'$.

If v is of type IIIb or IVb and $y \in u_\alpha(x')U_P t\dot{v} \; : \; x \neq x'$ and $x - x' \neq \alpha(t)$.

If v is of type IIb and $y \in v'$: no condition.

If v is of type IIIb or IVb and $y \in u_\alpha(x')U_P T\dot{v}' \; : \; x \neq x'$.

In each of these cases we can write $n_\alpha^{-1}u_\alpha(-x)y = b\dot{v}$ with $b \in B$ and the local system $\widetilde{\mathcal{L}}$ corresponds to the inverse image of $\mathcal{L}_\xi$ under the map $(x,y) \mapsto \pi_T(b)T_v \in T/T_v$. We shall now consider the different cases separately.

4.3.8. Assume v is of type I. Since $Pv = v$, 4.3.7 gives an isomorphism $BsB\overset{B}{\times}v \cong k \times v$. Let $x \in k$, $y = tu\dot{v} \in v$ $(t \in T, u \in U)$. Then $n_\alpha^{-1}u_\alpha(-x)y \in s(t)n_\alpha^{-1}U\dot{v} \subset U_P s(t)\dot{v}$, since the group generated by U_α and $U_{-\alpha}$ is contained in $U_P P_{\dot{v}}$ (4.2.1). We have also $s(t) \in tT_v$, since $\mathrm{Im}\,(\check{\alpha}) \subset T_v$. So the image of (x,y) in T/T_v is tT_v and $\mu_!\widetilde{\mathcal{L}} = (pr_2)_!(\bar{\mathbb{Q}}_l \boxtimes \mathcal{L}_{\xi,v}) = \mathcal{L}_{\xi,v}[-2](-1)$. Hence $\varepsilon_{\widehat{\phi}_v\xi,s}\varepsilon_{\xi,v} = q\varepsilon_{\xi,v}$.

4.3.9. Assume v of type IIb. We apply 4.3.7. Let $x \in k$ and $y \in u_\alpha(x')U_P t\dot{v}$ with $x \neq x'$. With $z = x - x'$ we have $n_\alpha^{-1}u_\alpha(-x)y \in U_P n_\alpha^{-1}u_\alpha(-z)t\dot{v} = U_P u_\alpha(z^{-1})\check{\alpha}(-z^{-1})u_{-\alpha}(-z^{-1})t\dot{v} \subset U\check{\alpha}(-z^{-1})t\dot{v}$, since $U_{-\alpha} \subset U_P P_{\dot{v}}$ by Lemma 4.1.5. So the image of (x,y) in T/T_v is $\check{\alpha}(-z^{-1})tT_v$. Since $B_{\dot{v}} \subset B \cap n_\alpha B n_\alpha^{-1} = TU_P$, the map $(x',y') \mapsto u_\alpha(x')y'$ gives an isomorphism $k \times TU_P\dot{v} \xrightarrow{\sim} V$. Take $z = x - x'$ as a coordinate in the place of x. Then we see that

$$\mu_!\widetilde{\mathcal{L}}\mid_v \cong (pr_2)_!(\mathcal{L}_{-\langle\xi,\check{\alpha}\rangle} \boxtimes \mathcal{L}_{\xi,v}),$$

where pr_2 is the projection of $k^* \times v$ on v. If $\langle\xi,\check{\alpha}\rangle = 0$, we have $\mathcal{H}^1(\mu_!\widetilde{\mathcal{L}})\mid_v = \mathcal{L}_{\xi,v}$ and $\mathcal{H}^2(\mu_!\widetilde{\mathcal{L}})\mid_v = \mathcal{L}_{\xi,v}(-1)$. If $\langle\xi,\check{\alpha}\rangle \neq 0$, then $\mu_!\widetilde{\mathcal{L}}\mid_v = 0$ by 2.1.6. This gives the contribution $(q-1)\varepsilon_{\xi,v}$ to $\varepsilon_{\widehat{\phi}_v\xi,s}\varepsilon_{\xi,v}$ in case $\langle\xi,\check{\alpha}\rangle = 0$. Notice that we may neglect roots of unity as factors in Frobenius eigenvalues.

Next, let $x \in k$ and $y \in tU\dot{v}'$, where $\dot{v}' = n_\alpha\dot{v}$. Then $n_\alpha^{-1}u_\alpha(-x)y \in U_P s(t)\dot{v}$, the image of (x,y) in T/T_v is $s(t)T_v$ and $\mu_!\widetilde{\mathcal{L}}\mid_{v'} = (pr_2)_!(\bar{\mathbb{Q}}_l \boxtimes \mathcal{L}_{s\xi,v'}) = \mathcal{L}_{s\xi,v'}[-2](-1)$. This gives the contribution $q\varepsilon_{s\xi,v'}$ to $\varepsilon_{\widehat{\phi}_v\xi,s}\varepsilon_{\xi,v}$.

4.3.10. Assume v of type IIIb. We apply again 4.3.7. Let $x \in k$ and $y \in U_P u_\alpha(x')t\dot{v}$ with $x \neq x'$ and $x - x' \neq \alpha(t)$. Then $n_\alpha^{-1}u_\alpha(-x)y \in U_P s(t)n_\alpha^{-1}u_\alpha(z)\dot{v}$, where $z = \alpha(t)^{-1}(x' - x)$, so that $z \neq 0, -1$. Now $\dot{v}' = u_\alpha(1)n_\alpha^{-1}\dot{v}$ is a special element of the orbit v', which is of type IIIa (see 4.1.4). We have $n_\alpha^{-1}u_\alpha(z)\dot{v} = n_\alpha^{-1}u_\alpha(z)n_\alpha u_\alpha(-1)\dot{v}' = u_{-\alpha}(-z)u_\alpha(-1)\dot{v}' \in U\check{\alpha}(z^{-1})n_\alpha u_\alpha(-z^{-1}-1)\dot{v}'$. By the observation made in the beginning of 4.3.5 we have $n_\alpha u_\alpha(-z^{-1}-1)\dot{v}' \in U_P t_0\check{\alpha}(\tau)\dot{v}$, if $\tau^2 = z(z+1)^{-1}$, $\check{\alpha}(\tau) \in t_0 T_{v'}$, $t_0 \in \mathrm{Ker}\,(\alpha)$. So finally $n_\alpha^{-1}u_\alpha(-x)y \in Us(t)\check{\alpha}(z^{-1})t_0\check{\alpha}(\tau)\dot{v}$ and the image of (x,y) in T/T_v is

$$s(t)\check{\alpha}(z^{-1})t_0\check{\alpha}(\tau)T_v.$$

As in case IIb the map $(x',y') \mapsto u_\alpha(x')y'$ gives an isomorphism $k \times TU_P\dot{v} \xrightarrow{\sim} v$. In $y \in U_P u_\alpha(x')t\dot{v}$, the element t is determined modulo T_v, so $\alpha(t)$ is determined by y, since $\alpha(T_v) = 1$ (4.2.7). It follows that we may take $z = \alpha(t)^{-1}(x' - x)$ as a coordinate in the place of x. Let k^{**} denote $k - \{0, -1\}$. Then $\widetilde{\mathcal{L}}$ gives a local system on $k^{**} \times v$ and we have to compute the direct image with proper support of that local system for the second projection.

Suppose $\xi \in \widetilde{X}(T/T_v)$ is represented by χ/n with $\chi \in X(T/T_v)$, n an integer prime to p. In 4.2.7 we defined $a(\xi) \in \widehat{X}(\mathbb{G}_m) = \mathbb{Z}_{(p)}/\mathbb{Z}$. It is computed as follows. On $T_{v'}$ we have $\chi = a(\chi)\alpha$ with an integer $a(\chi)$. Then $a(\chi)/n$ is a representative for $a(\xi)$. Let us compute $\chi(s(t)\check{\alpha}(z^{-1})t_0\check{\alpha}(\tau))$. Since $t_0\check{\alpha}(\tau)^{-1} \in T_{v'}$ and $\alpha(t_0) = 1$, we have $\chi(t_0)\tau^{-\langle\chi,\check{\alpha}\rangle} = \tau^{-2a(\chi)}$, so that $\chi(\check{\alpha}(z^{-1})t_0\check{\alpha}(\tau)) = z^{-\langle\chi,\check{\alpha}\rangle}\tau^{2\langle\chi,\check{\alpha}\rangle-2a(\chi)} = z^{-\langle\chi,\check{\alpha}\rangle}(z(z+1)^{-1})^{\langle\chi,\check{\alpha}\rangle-a(\chi)} = z^{-a(\chi)}(z+1)^{a(s\chi)}$, as $a(\chi) - a(s\chi) = \langle\chi,\check{\alpha}\rangle$.

It follows that $\mu_!\widetilde{\mathcal{L}} \mid_v \cong (pr_2)_!(\mathcal{J} \boxtimes \mathcal{L}_{s\xi,v})$ where $\mathcal{J}$ is the inverse image of $\mathcal{L}_{-a(\xi)} \boxtimes \mathcal{L}_{a(s\xi)}$ under the map $k^{**} \to k^* \times k^*$, $z \mapsto (z, -1-z)$. If $a(\xi) = a(s\xi) = 0$, then $\mathcal{J}$ is constant and $\mathcal{H}^1(\mu_!\widetilde{\mathcal{L}}) \mid_v = \mathcal{L}_{\xi,v} \oplus \mathcal{L}_{\xi,v}$, $\mathcal{H}^2(\mu_!\widetilde{\mathcal{L}}) \mid_v = \mathcal{L}_{\xi,v}(-1)$, the other $\mathcal{H}^i$ are zero (in this case $s\xi = \xi$). If $a(\xi)$ and $a(s\xi)$ are not both zero, we can apply Proposition 4.16 of [SGA4-1/2, Sommes trigonométriques] (see 2.1.7). The Jacobi sum in 2.1.7 equals ± 1 if exactly one of $a(\xi)$, $a(s\xi)$, $\langle\widehat{\phi}_v\xi, \check{\alpha}\rangle$ is zero, but it has complex absolute value $q^{1/2}$ if all three are non-zero. So we find the contribution of v to $\varepsilon_{\widehat{\phi}_v\xi,s}\varepsilon_{\xi,v}$ as stated in 4.3.1.

Next, let $x \in k$ and $y \in U_P u_\alpha(x')t\dot{v}'$ with $x \neq x'$. Then $n_\alpha^{-1}u_\alpha(-x)y \in U_P\check{\alpha}(-1)s(t)n_\alpha u_\alpha(-\alpha(t)^{-1}z)\dot{v}'$, where $z = x - x' \neq 0$. In the same way as before we find $n_\alpha u_\alpha(-\alpha(t)^{-1}z)\dot{v}' \in U_P t_0\check{\alpha}(\tau)\dot{v}$ if $\tau^2 = \alpha(t)z^{-1}$, $\check{\alpha}(\tau) \in t_0T_{v'}$, $t_0 \in \operatorname{Ker}(\alpha)$. So $n_\alpha^{-1}u_\alpha(-x)y \in U_P\check{\alpha}(-1)s(t)t_0\check{\alpha}(\tau)\dot{v}$ and the image of (x,y) in T/T_v is $\check{\alpha}(-1)s(t)t_0\check{\alpha}(\tau)T_v$. Let χ and n be as above and put $\chi' = \chi - a(\chi)\alpha \in X(T/T_{v'})$, $\xi' = \chi'/n + X(T/T_{v'}) \in \widehat{X}(T/T_{v'})$. Then $\chi(t_0\check{\alpha}(\tau)) = \tau^{2\langle\chi,\check{\alpha}\rangle-2a(\chi)} = (\alpha(t)z^{-1})^{\langle\chi,\check{\alpha}\rangle-a(\chi)}$ and $s\chi(t) = \chi(t)\alpha(t)^{-\langle\chi,\check{\alpha}\rangle}$, so $\chi(s(t)t_0\check{\alpha}(\tau)) = \chi'(t)z^{a(s\chi)}$. It follows that $\mu_!\widetilde{\mathcal{L}} \mid_{v'} = (pr_2)_!(\mathcal{L}_{a(s\xi)} \boxtimes \mathcal{L}_{\xi',v'})$, where pr_2 is the projection $k^* \times v' \to v'$. If $a(s\xi) \neq 0$, then $\mu_!\widetilde{\mathcal{L}}|_{v'} = 0$ by 2.1.6. If $a(s\xi) = 0$, then $s\xi = \xi' \in \widehat{X}(T/T_{v'})$ and we have $\mathcal{H}^1(\mu_!\widetilde{\mathcal{L}}) \mid_{v'} = \mathcal{L}_{s\xi,v'}$, $\mathcal{H}^2(\mu_!\widetilde{\mathcal{L}}) \mid_{v'} = \mathcal{L}_{s\xi,v'}(-1)$. This gives the contribution $(q-1)\varepsilon_{s\xi,v'}$ to $\varepsilon_{\widehat{\phi}_v\xi,s}\varepsilon_{\xi,v}$, when $a_{v''}(\xi) = -a(s\xi) = 0$. The contribution of v'' is determined by interchanging v' and v''. Now 4.3.1 is proved in case IIIb.

4.3.11. Assume v of type IVb. Let $x \in k$ and $y \in U_P u_\alpha(x')t\dot{v}$ with $x \neq x'$ and $x - x' \neq \alpha(t)$. Put $z = \alpha(t)^{-1}(x' - x)$. Proceeding as in 4.3.10 we find that $n_\alpha^{-1}u_\alpha(-x)y \in Us(t)\check{\alpha}(z^{-1}\tau)\dot{v}$, if $\tau^2 = z(z+1)^{-1}$ (the factor $t_0 \in \operatorname{Ker}(\alpha)$ is absent, because in this case $\operatorname{Im}(\check{\alpha}) \subset T_{v'}$). The image of (x,y) in T/T_v is $s(t)\check{\alpha}(z^{-1}\tau)T_v$. Suppose $\xi \in \widehat{X}(T/T_v)$ is represented by χ/n with $\chi \in X(T/T_v)$ and $n \in \mathbb{Z}$, prime to p. In this case we have $\langle\chi,\check{\alpha}\rangle = 2a(\chi)$, where $a(\chi) \in \mathbb{Z}$ is determined by the condition that $\chi = a(\chi)\alpha$ on $T_{v'}$, so $a(\xi) = a(\chi)/n + \mathbb{Z}$ (see 4.2.7). Now $\chi(\check{\alpha}(z^{-1}\tau)) = (z^{-1}\tau)^{2a(\chi)} = (z(z+1))^{-a(\chi)}$.

Let Z denote the variety $\{(x,y) \in k \times v \mid n_\alpha^{-1}u_\alpha(-x)y \in v\}$ (which is isomorphic to $\mu^{-1}(v)$) and $\widetilde{Z} = k^{**} \times k \times TU_P\dot{v}$, where $k^{**} = k - \{0,-1\}$. By Lemma 4.2.5 we have a 2-fold Galois covering $\gamma : \widetilde{Z} \to Z$ defined by $\gamma(z,x',y') = (x' - \alpha(t)z, u_\alpha(x')y')$ if $y' \in U_P t\dot{v}$. Notice that $\alpha(t)$ is determined by y', as a special case of Lemma 4.2.5. The group of order two acts on $\widetilde{Z}$ by $(z,x',y') \mapsto (-z-1, x'+\alpha(t), u_\alpha(-\alpha(t))y')$, if $y' \in U_P t\dot{v}$. We have a Cartesian diagram

$$\begin{tikzcd}
\widetilde{Z} \arrow[r, "\gamma"] \arrow[d, "\beta"'] & Z \arrow[d, "\alpha"] \\
k \times TU_P\dot{v} \arrow[r, "\pi"'] & v
\end{tikzcd}$$

where $\beta(z,x',y') = (x',y')$, $\alpha(x,y) = y$ and $\pi(x',y') = u_\alpha(x')y'$.

Define $f : Z \to k^*$ by $f \circ \gamma(z, x', y') = z(z+1)$. It follows from the computation above that restriction of $\widetilde{\mathcal{L}}$ gives on Z the sheaf $f^*\mathcal{L}_{-a(\xi)} \otimes \alpha^*\mathcal{L}_{s\xi,v}$. Hence $\mu_!\widetilde{\mathcal{L}} \mid_v \cong \alpha_!(f^*\mathcal{L}_{-a(\xi)} \otimes \alpha^*\mathcal{L}_{s\xi,v}) = \alpha_! f^*\mathcal{L}_{-a(\xi)} \otimes \mathcal{L}_{s\xi,v}$.

Denote the complex $\alpha_! f^*\mathcal{L}_{-a(\xi)}$ by K. We are going to compute its cohomology sheaves. On $\widetilde{Z}$ we have $\gamma^* f^*\mathcal{L}_{-a(\xi)} = \mathcal{J} \boxtimes \bar{\mathbb{Q}}_l$, where $\mathcal{J}$ is the inverse image of $\mathcal{L}_{-a(\xi)} \boxtimes \mathcal{L}_{-a(\xi)}$ under the map $k^{**} \to k^* \times k^*$, $z \mapsto (z, -1-z)$, and $\bar{\mathbb{Q}}_l$ is the constant sheaf on $k \times TU_P\dot{v}$. Now $\pi^*K = \pi^*\alpha_! f^*\mathcal{L}_{-a(\xi)} = \beta_!\gamma^* f^*\mathcal{L}_{-a(\xi)} = \beta_!(\mathcal{J} \boxtimes \bar{\mathbb{Q}}_l) = p_!\mathcal{J} \otimes \bar{\mathbb{Q}}_l$, where p denotes projection on a point. So $\pi^*\mathcal{H}^iK = H^i_c(k^{**}, \mathcal{J}) \otimes \bar{\mathbb{Q}}_l$, a constant sheaf, and $\mathcal{H}^iK$ is a direct summand of $\pi_*\pi^*\mathcal{H}^iK = H^i_c(k^{**}, \mathcal{J}) \otimes \pi_*\bar{\mathbb{Q}}_l$. Let $\pi_*\bar{\mathbb{Q}}_l = \bar{\mathbb{Q}}_l \oplus \mathcal{E}$, where $\mathcal{E}$ corresponds to the non-trivial character of the Galois group of π. The action of the Galois group of π on the stalks $(\mathcal{H}^iK)_y \cong H^i_c(k^{**}, \mathcal{J})$ of $\mathcal{H}^iK$ comes from the action of the automorphism $\sigma : z \mapsto -1-z$ of k^{**} on $H^i_c(k^{**}, \mathcal{J})$.

Assume first that $a(\xi) = 0$. Then $\mathcal{J} = \bar{\mathbb{Q}}_l$ and the non-zero cohomology spaces are:

$$H^1_c(k^{**}, \bar{\mathbb{Q}}_l) = \bar{\mathbb{Q}}_l^2,\ \sigma \text{ acts with eigenvalues 1 and } -1;$$
$$H^2_c(k^{**}, \bar{\mathbb{Q}}_l) = \bar{\mathbb{Q}}_l(-1),\ \sigma \text{ acts trivially.}$$

We conclude that $\mathcal{H}^1K = \bar{\mathbb{Q}}_l \oplus \mathcal{E}$ and $\mathcal{H}^2K = \bar{\mathbb{Q}}_l(-1)$. Hence $\mathcal{H}^1(\mu_!\widetilde{\mathcal{L}})|_v = \mathcal{H}^1K \otimes \mathcal{L}_{\xi,v} = \mathcal{L}_{\xi,v} \oplus \mathcal{L}_{\xi',v}$ and $\mathcal{H}^2(\mu_!\widetilde{\mathcal{L}})|_v = \mathcal{L}_{\xi,v}(-1)$. Here ξ' is as in 4.3.1 and $s\xi = \xi$, because $a(\xi) = 0$.

This gives $(q-1)\varepsilon_{\xi,v} - \varepsilon_{\xi',v}$ as a contribution to $\varepsilon_{\widehat{\phi}_v\xi,s}\varepsilon_{\xi,v}$.

Consider now the case $a(\xi) \neq 0$. The only non-zero cohomology space is $H^1_c(k^{**}, \mathcal{J})$. This space is one-dimensional and the action of σ is multiplication by -1, by Proposition 4.20 of [SGA4-1/2, Sommes trigonométriques] (see 2.1.7).

Now $\mathcal{H}^1K = H^1_c(k^{**}, \mathcal{J}) \otimes \mathcal{E}$ and $\mathcal{H}^1(\mu_!\widetilde{\mathcal{L}}) \mid_v = H^1_c(k^{**}, \mathcal{J}) \otimes \mathcal{L}_{s\xi',v}$. The contribution to $\varepsilon_{\widehat{\phi}_v\xi,s}\varepsilon_{\xi,v}$ is $-\varepsilon_{\xi,v}$ if $2a(\xi) = 0$ and $-\mathcal{J}_v(\xi)\varepsilon_{s\xi',v}$ if $2a(\xi) \neq 0$ (notice that $s\xi' = \xi$ if $a(\xi) \neq 0$, $2a(\xi) = 0$).

Finally, we have to determine the contribution of v'. Let $x \in k$ and $y \in U_P u_\alpha(x')t\dot{v}'$ with $x \neq x'$. Put $z = x - x'$. In the same way as in case IIIb (4.3.10) we find that $n_\alpha^{-1}u_\alpha(-x)y \in U_P\check{\alpha}(-1)s(t)\check{\alpha}(\tau)\dot{v}$, if $\tau^2 = \alpha(t)z^{-1}$. The image of (x, y) in T/T_v is $\check{\alpha}(-1)s(t)\check{\alpha}(\tau)T_v$. Let χ and n be as above and put $\chi_1 = \chi - a(\chi)\alpha \in X(T/T_{v'})$ and $\xi_1 = \chi_1/n + X(T/T_{v'}) \in \widehat{X}(T/T_{v'})$. Then $\chi(s(t)\check{\alpha}(\tau)) = s\chi(t)\tau^{2a(\chi)} = \chi_1(t)z^{-a(\chi)}$.

It follows that $\mu_!\widetilde{\mathcal{L}} \mid_{v'} = (pr_2)_!(\mathcal{L}_{-a(\xi)} \boxtimes \mathcal{L}_{\xi_1,v'})$, where pr_2 is the projection $k^* \times v' \to v'$. If $a(\xi) \neq 0$, then $\mu_!\widetilde{\mathcal{L}} \mid_{v'} = 0$ by 2.1.6. If $a(\xi) = 0$, then ξ_1 is equal to the element $\bar{\xi}$ defined in 4.3.1 and $\mathcal{H}^1(\mu_!\widetilde{\mathcal{L}}) \mid_{v'} = \mathcal{L}_{\bar{\xi},v'}$, $\mathcal{H}^2(\mu_!\widetilde{\mathcal{L}}) \mid_{v'} = \mathcal{L}_{\bar{\xi},v'}(-1)$. So the contribution of v' is $(q-1)\varepsilon_{\bar{\xi},v'}$, if $a(\xi) = 0$.

The proof of 4.3.1 is now complete.

Remark. The relation between the algebra $\mathcal{K}(\mathcal{C}_G)$ and the algebra $\mathcal{K}$ of [MS] is the following. In $\mathcal{K}(\mathcal{C}_G)$ we have the formulas 4.3.2. Comparison of these formulas with [MS, 3.3.1] shows that the $\mathbb{Z}[q]$-subalgebra of $\mathcal{K}(\mathcal{C}_G)$ generated by the elements $\varepsilon_{\xi,w}$ is isomorphic to a $\mathbb{Z}[t]$-subalgebra of the algebra $\mathcal{K}$: to $\varepsilon_{\xi,w}$ corresponds $t^{l(w)-l_\xi(w)}e_{\xi,w}$ and to multiplication by q corresponds multiplication by t^2.

4.4. **The restriction of $A_{\xi,v}$ to Pv.**

4.4.1. Let $v \in V$, $s \in S$ and $P = P_s$ as before. If v is of a type b for s, then $Pv \subset \bar{v}$ and v is open in Pv. The restriction of $A_{\xi,v}$ $(\xi \in \widehat{X}(T/T_v))$ to Pv is the perverse extension of $\mathcal{L}_{\xi,v}$ to Pv. It is given by one sheaf, in degree $-\dim v$, which we shall now compute. The result will be needed later.

4.4.2. Assume v is of type IIb for s. We have $Pv = v \cup v'$. Let $\dot{v}$ be a special element of v, then $\dot{v}' = n_\alpha^{-1}\dot{v}$ is one of v'. The set $U_{-\alpha}v' = U_{-\alpha}B\dot{v}'$ is an open neighbourhood of v' in Pv. The map $k \times v' \to U_{-\alpha}v'$, $(x, y) \mapsto u_{-\alpha}(x)y$, is an isomorphism, since $P_{\dot{v}'} \subset B$. If $t \in T$, $u \in U$ and $x \in k^*$, then $u_{-\alpha}(x)tu\dot{v}' = u_\alpha(x^{-1})\check{\alpha}(-x^{-1})n_\alpha u_\alpha(x^{-1})tu\dot{v}' \in U\check{\alpha}(-x^{-1})s(t)\dot{v}$, since $U \subset U_P P_{\dot{v}'}$, so that $\pi_{\dot{v}}(u_{-\alpha}(x)tu\dot{v}') = \check{\alpha}(-x^{-1})s(t)T_v$ (notation of 2.2.5). Let $\xi = \chi/n + X(T/T_v) \in \widehat{X}(T/T_v)$. From $\chi(\check{\alpha}(-x^{-1})s(t)) = (-x)^{-\langle \chi, \check{\alpha}\rangle} s\chi(t)$ we see that $\mathcal{L}_{\xi,v}$ has trivial local monodromy around v' if $\langle \widehat{\phi}_v\xi, \check{\alpha}\rangle = 0$. In that case $\mathcal{L}_{\xi,v}$ can be extended to a smooth sheaf $\mathcal{L}$ on Pv, whose restriction to v' is $\mathcal{L}_{s\xi,v'}$. Then $A_{\xi,v}\mid_{Pv} = \mathcal{L}[\dim v]$. If $\langle \widehat{\phi}_v\xi, \check{\alpha}\rangle \neq 0$, $A_{\xi,v}\mid_{Pv}$ is the extension by zero of $\mathcal{L}_{\xi,v}[\dim v]$.

4.4.3. We assume now that v is of type IIIb and use the notations of 4.3.10. We have $Pv = v \cup v' \cup v''$. The set $v \cup v''$ is an open neighbourhood of v'' in Pv. It is isomorphic to $k \times k \times TU_P\dot{v}'$ by the map $(x_1, x, y) \mapsto u_\alpha(x_1)n_\alpha u_\alpha(x)y$ ($\in v$ if $x \neq 0$, $\in v''$ if $x = 0$). By the computations in the last part of 4.3.10 we have $u_\alpha(x_1)n_\alpha u_\alpha(x)y \in Us(t)t_0\check{\alpha}(\tau)\dot{v}$, if $x \neq 0$, $y \in U_P t\dot{v}'$, $\tau^2 = -\alpha(t)x^{-1}$, $\check{\alpha}(\tau) \in t_0 T_{v'}$, $t_0 \in \mathrm{Ker}\,(\alpha)$, and $\chi(s(t)t_0\check{\alpha}(\tau)) = \chi'(t)(-x)^{a(s\chi)}$ with $\chi' = \chi - a(\chi)\alpha \in X(T/T_{v'})$. So $\mathcal{L}_{\xi,v}$ has trivial local monodromy around v'' if $a(s\xi) = 0$, i.e. $a_{v''}(\xi) = 0$. In that case $\mathcal{L}_{\xi,v}$ can be extended to a smooth sheaf on $v \cup v''$, whose restriction on v'' is $\mathcal{L}_{\xi,v''}$. Interchanging v' and v'' we have: $\mathcal{L}_{\xi,v}$ extends to a smooth sheaf on $v \cup v'$, whose restriction to v' is $\mathcal{L}_{\xi,v'}$, if $a_{v'}(\xi) = 0$.

4.4.4. We assume that v is of type IVb and use the notations of 4.3.11. We have $Pv = v \cup v'$. The action of P on Y induces a 2-fold Galois covering $\mu : P \overset{B}{\times} v' \to Pv$, which is trivial over v' (cf. 4.3.6). The open subset $BsB \overset{B}{\times} v'$ of $P \overset{B}{\times} v'$ is isomorphic to $k \times k \times TU_P\dot{v}'$ by the map $(x_1, x, y) \mapsto u_\alpha(x_1)n_\alpha \star u_\alpha(x)y$. By the computations in 4.3.11 we have $u_\alpha(x_1)n_\alpha u_\alpha(x)y \in Us(t)\check{\alpha}(\tau)\dot{v}$, if $x \neq 0$, $y \in U_P t\dot{v}'$, $\tau^2 = -\alpha(t)x^{-1}$, and $\chi(s(t)\check{\alpha}(\tau)) = \chi_1(t)(-x)^{-a(\chi)}$ with $\chi_1 = \chi - a(\chi)\alpha \in X(T/T_{v'})$. Let γ denote the restriction of μ to $BsB \overset{B}{\times} v'$. We see that the inverse image of $\mathcal{L}_{\xi,v}$ on $\gamma^{-1}(v)$ has trivial local monodromy around $\gamma^{-1}(v')$ if $a(\xi) = 0$. In that case $\mathcal{L}_{\xi,v}$ extends to a smooth sheaf on Pv, whose restriction to v' is $\mathcal{L}_{\bar{\xi},v'}$.

In the notations of 3.4.1 these results can be expressed as follows.

4.4.5. Lemma. *Let $u \subset Pv$, $u < v$. Then $c_{\eta,u;\xi,v,i} = 0$ if $i \neq -\dim v$ and $c_{\eta,u;\xi,v,-\dim v} = 0$ except for the following cases, in which it equals 1:*

$(\eta, u) = (s\xi, v')$ if v is of type IIb and $\langle \widehat{\phi}_v\xi, \check{\alpha}\rangle = 0$,
$(\eta, u) = (\xi, v')$ (resp. (ξ, v'')) if v is of type IIIb and $a_{v'}(\xi) = 0$ (resp. $a_{v''}(\xi) = 0$),
$(\eta, u) = (\bar{\xi}, v')$ if v is of type IVb and $a(\xi) = 0$.

The formula in 3.4.2 can now be used to derive the following result.

4.4.6. Lemma. *Let $u \subset Pv$, $u < v$. Then $b_{\eta,u;\xi,v} = 0$ except for the following cases, in which it equals $q^{-\dim v}(1 - q)$:*

$(\eta, u) = (-s\xi, v')$ if v is of type IIb and $\langle \widehat{\phi}_v\xi, \check{\alpha}\rangle = 0$,
$(\eta, u) = (-\xi, v')$ (resp. $(-\xi, v'')$) if v is of type IIIb and $a_{v'}(\xi) = 0$ (resp. $a_{v''}(\xi) = 0$),
$(\eta, u) = (-\bar{\xi}, v')$ if v is of type IVb and $a(\xi) = 0$.

This can, of course, also be proved directly by determining the Verdier dual on Pv of $j_!\mathcal{L}_{\xi,v}$, where j is the embedding $v \to Pv$. In particular we have

4.4.7. Lemma. *In $\mathcal{K}(\mathcal{C}_G)$ one has*

$$\begin{aligned} D\varepsilon_{\xi,s} &= q^{-\dim B-1}(\varepsilon_{-\xi,s} + (1-q)e_{-\xi,e}) \quad \text{if } \langle \xi, \check{\alpha} \rangle = 0, \\ &= q^{-\dim B-1}\varepsilon_{-\xi,s} \quad \text{if } \langle \xi, \check{\alpha} \rangle \neq 0. \end{aligned}$$

Here $s \in S$, $\xi \in \widehat{X}(T)$, α is the simple root corresponding to s.

5. Recurrence relations for the coefficients b

5.1. Let $v \in V$, $\xi \in \widehat{X}(T/T_v)$ and $s \in S$. Let α denote the simple root corresponding to s. By 3.3.2 we have

$$D(\varepsilon_{\widehat{\phi}_v\xi,s}\varepsilon_{\xi,v}) = q^{\dim B}(D\varepsilon_{\widehat{\phi}_v\xi,s})(D\varepsilon_{\xi,v})$$

and 4.4.7 gives a formula for $D\varepsilon_{\widehat{\phi}_v\xi,s}$, so that we have

$$\begin{aligned} D(\varepsilon_{\widehat{\phi}_v\xi,s}\varepsilon_{\xi,v}) &= (q^{-1}\varepsilon_{-\widehat{\phi}_v\xi,s} + (q^{-1}-1)\varepsilon_{-\widehat{\phi}_v\xi,e})D\varepsilon_{\xi,v} \quad \text{if } \langle \widehat{\phi}_v\xi, \check{\alpha} \rangle = 0, \\ &= q^{-1}\varepsilon_{-\widehat{\phi}_v\xi,s}D\varepsilon_{\xi,v} \quad \text{if } \langle \widehat{\phi}_v\xi, \check{\alpha} \rangle \neq 0. \end{aligned}$$

Now $\varepsilon_{\widehat{\phi}_v\xi,s}\varepsilon_{\xi,v}$ is a linear combination of $\varepsilon_{\eta,u}$ with $u \subset Pv$, which is given by 4.3.1. Thus we get relations between the coefficients b defined in 3.4.1. In general these relations are not sufficient to determine the coefficients b, even when Y is symmetric.

5.2. We now give the relations more explicitly in the different cases. If v is of type IIa for s, we have

$$\begin{aligned} D\varepsilon_{s\xi,m(s)v} &= (q^{-1}\varepsilon_{-\widehat{\phi}_v\xi,s} + (q^{-1}-1)\varepsilon_{-\widehat{\phi}_v\xi,e})D\varepsilon_{\xi,v} \quad \text{if } \langle \widehat{\phi}_v\xi, \check{\alpha} \rangle = 0, \\ &= q^{-1}\varepsilon_{-\widehat{\phi}_v\xi,s}D\varepsilon_{\xi,v} \quad \text{if } \langle \widehat{\phi}_v\xi, \check{\alpha} \rangle \neq 0. \end{aligned}$$

If v is of type IIIa, we have the same expression for $D\varepsilon_{s\xi,m(s)v} + D\varepsilon_{s\xi,v'}$.

Now assume v is of type IVa. The element $\xi \in \widehat{X}(T/T_v)$ defines two elements $\xi_1, \xi_2 \in \widehat{X}(T/T_{m(s)v})$, as in 4.3.1. Obviously $\widehat{\phi}_{m(s)v}\xi_1 = \widehat{\phi}_{m(s)v}\xi_2 = \widehat{\phi}_v\xi$ (see the diagram in 4.2.7). Also $\langle \widehat{\phi}_v\xi, \check{\alpha} \rangle = 0$, since $\mathrm{Im}\,(\check{\alpha}) \subset T_v$.

We apply the formula of 5.1 to ξ_1 and $m(s)v$. By 4.3.1 we have $\varepsilon_{\widehat{\phi}_v\xi,s}\varepsilon_{\xi_1,m(s)v} = (q-1)\varepsilon_{\xi_1,m(s)v} - \varepsilon_{\xi_2,m(s)v} + (q-1)\varepsilon_{\xi,v}$ and the result is

$$q^{-1}\varepsilon_{-\widehat{\phi}_v\xi,s}D\varepsilon_{\xi_1,m(s)v} + D\varepsilon_{\xi_2,m(s)v} = (q^{-1}-1)D\varepsilon_{\xi,v}.$$

The same is valid with ξ_1 and ξ_2 interchanged. From these two identities we obtain by adding them together and multiplying by $q^{-1}\varepsilon_{-\widehat{\phi}_v\xi,s} + (q^{-1}-2)\varepsilon_{-\widehat{\phi}_v\xi,e}$, resp. by subtracting, the following identities.

(i) $D\varepsilon_{\xi_1,m(s)v} + D\varepsilon_{\xi_2,m(s)v} = (q^{-1}\varepsilon_{-\widehat{\phi}_v\xi,s} + (q^{-1}-2)\varepsilon_{-\widehat{\phi}_v\xi,e})D\varepsilon_{\xi,v}$.

(ii) $(q^{-1}\varepsilon_{-\widehat{\phi}_v\xi,s} - \varepsilon_{-\widehat{\phi}_v\xi,e})(D\varepsilon_{\xi_1,m(s)v} - D\varepsilon_{\xi_2,m(s)v}) = 0$.

(To derive (i), use that in $\mathcal{K}(\mathcal{C}_G)$ one has $\varepsilon_{\eta,s}^2 = (q-1)\varepsilon_{\eta,s} + q\varepsilon_{\eta,e}$ if $\langle \eta, \check{\alpha} \rangle = 0$ and that in $\mathcal{K}(\mathcal{C}_Y)$ the relation $(q^{-1}-1)m = 0$ implies $m = 0$.)

A consequence of (i) and (ii) is

(iii) $(q^{-1}\varepsilon_{-\widehat{\phi}_v\xi,s} - \varepsilon_{-\widehat{\phi}_v\xi,e})(D\varepsilon_{\xi_1,m(s)v} + D\varepsilon_{\xi,v}) = 0$.

Next, let v be of type IIIb or IVb and assume $\langle\widehat{\phi}_v\xi,\check{\alpha}\rangle = 0$, $a(\xi)\neq 0$. Then we find immediately that

$$(\varepsilon_{-\widehat{\phi}_v\xi,s}+\varepsilon_{-\widehat{\phi}_v\xi,e})D\varepsilon_{\xi,v}=0.$$

If v is of type IIIb and $\langle\widehat{\phi}_v\xi,\check{\alpha}\rangle\neq 0$, $a_{v'}(\xi)\neq 0$, $a_{v''}(\xi)\neq 0$, then

$$\mathcal{J}_v(\xi)^{-1}D\varepsilon_{s\xi,v}+q^{-1}\varepsilon_{-\widehat{\phi}_v\xi,s}D\varepsilon_{\xi,v}=0.$$

If v is of type IVb and $\langle\widehat{\phi}_v\xi,\check{\alpha}\rangle\neq 0$, then

$$\mathcal{J}_v(\xi)^{-1}D\varepsilon_{s\xi',v}+q^{-1}\varepsilon_{-\widehat{\phi}_v\xi,s}D\varepsilon_{\xi,v}=0,$$

where ξ' is as in 4.3.1.

These are all the relations which can be obtained by the method of 5.1. In fact, when applied to an orbit of type IIb, the method gives the same result as when applied to an orbit of type IIa, etc.

5.3. The equations (ii) and (iii) in 5.2 are of the type

$$(q^{-1}\varepsilon_{\zeta,s}-\varepsilon_{\zeta,e})\sum a_{\eta,u}\varepsilon_{\eta,u}=0,$$

where $\zeta\in\widehat{X}(T)$ such that $\langle\zeta,\check{\alpha}\rangle=0$ and $a_{\eta,u}\in\mathbb{Z}[C]$ such that $a_{\eta,u}\neq 0\Rightarrow\widehat{\phi}_u\eta=\zeta$. This system of equations can easily be solved, using 4.3.1. The solution is

(iv)

$a_{\eta,u}=a_{s\eta,m(s)u}$	if u is of type IIa,
$a_{\eta,u}=a_{s\eta,u'}=a_{\eta,m(s)u}$	if u is of type IIIa,
$a_{\eta,u}=a_{\eta_1,m(s)u}+a_{\eta_2,m(s)u}$	if u is of type IVa,
$a_{\eta,u}=0$	if u is of type IIIb or IVb and $a(\eta)\neq 0$.

The system

$$(\varepsilon_{\zeta,s}+\varepsilon_{\zeta,e})\sum b_{\eta,u}\varepsilon_{\eta,u}=0,$$

which also occurs in 5.2, can be solved in the same way. Here again $\langle\zeta,\check{\alpha}\rangle=0$ and it is assumed that $b_{\eta,u}\neq 0\Rightarrow\widehat{\phi}_u\eta=\zeta$. The solution is

(v)

$b_{\eta,u}=0$	if u is of type I,
$b_{\eta,u}+qb_{s\eta,m(s)u}=0$	if u is of type IIa,
$b_{\eta,u}+b_{s\eta,u'}+(q-1)b_{\eta,m(s)u}=0$	if u is of type IIIa,
$b_{\eta_1,m(s)u}=b_{\eta_2,m(s)u},\ b_{\eta,u}=(1-q)b_{\eta_1,m(s)u}$	if u is of type IVa.

6. Symmetric varieties

6.1. We collect here some facts concerning symmetric varieties. References are [S2] and [RS1].

Let G be a connected reductive group over an algebraically closed field k of characteristic $\neq 2$. Let θ be an involution of G, i.e. an automorphism of the algebraic group G of order 2, and K the group of elements of G which are fixed by θ. Then K is a reductive group, not necessarily connected. The homogeneous space $Y=G/K$ is called a symmetric variety for G.

Choose in G a θ-stable maximal torus T and a θ-stable Borel group B such that $B\supset T$ (such a pair T, B always exists). We denote the normalizer of T in G by N and we put $N/T=W$. Then N is θ-stable, θ acts on W and on the root system and permutes the simple roots.

Let $\mathcal{V}=\{x\in G\mid x(\theta x)^{-1}\in N\}$. Then $\mathcal{V}$ is left T-invariant and right K-invariant and the map $T\backslash\mathcal{V}/K\to B\backslash G/K$, $TxK\mapsto BxK$, is a bijection of finite sets. In particular Y is a spherical variety and the set V of B-orbits in Y can be identified with the set $T\backslash\mathcal{V}/K$.

6.2. Let $v \in V$ and choose a representative $x \in \mathcal{V}$ for v. The image w of $n = x(\theta x)^{-1}$ in W depends only on v. Put $\psi = \mathrm{Int}\,(n) \circ \theta$. This is an involution of G (because $\theta n = n^{-1}$) and is determined by v up to conjugation by $\mathrm{Int}\,(t)$, $t \in T$. The torus T is ψ-stable, ψ acts on it as $w \circ \theta$. The isotropy group of $\dot{v} = xK$ is the fixed point group G^ψ of ψ.

Now $B \cap G^\psi$ is the fixed point group of the restriction of ψ to $B \cap nBn^{-1}$. Since $B \cap nBn^{-1} = T(U \cap nUn^{-1})$, we have $B \cap G^\psi = T^\psi(U \cap nUn^{-1})^\psi$, in particular $T_v = T^\psi$. It also follows that $U \cap G^\psi$ is connected, as fixed point group of a semi-simple automorphism of the connected unipotent group $U \cap nUn^{-1}$. So condition (b) of 2.2.5 is satisfied.

6.3. Put $\tau g = g(\psi g)^{-1}$. Then τG is closed in G and $\tau : G \to \tau G$ is separable (this is true for any semi-simple automorphism ψ of a connected linear algebraic group G, see [S1, 4.4.4]). So τ induces an isomorphism of $Y = G/G^\psi$ on τG.

The morphism $B \to \tau B$ induced by τ is also separable. This is a slight extension of [loc. cit.] and can be proved in the same way. It follows that $B \to v$, $b \mapsto b \cdot \dot{v}$ is separable, i.e. condition (a) of 2.2.5 is satisfied.

6.4. Let $U^- = \prod_{\alpha<0} U_\alpha$ and put $\mathcal{S} = (U^- \cap \psi U^-)\dot{v}$ (notations as in 6.2). Then $\mathcal{S}$ is a transverse slice in $\dot{v}$ with respect to v, as defined in 2.3.2, where G is replaced by B. The properties (a), (b), (c) are easily checked. For (b), observe that the morphism $B \times \mathcal{S} \to Y$ is the map $B \times (U^- \cap \psi U^-)/(U^-)^\psi \to G/G^\psi$, induced by the product in G and it suffices to check that this map is submersive.

If v is not the open orbit, we have $\dim \mathcal{S} > 0$ and there is a contraction of $\mathcal{S}$ such that (d) and (e) of 2.3.2 are satisfied. Choose a homomorphism $\lambda : \mathbb{G}_m \to T^\psi$ such that $\langle \lambda, \alpha \rangle < 0$ for all positive roots α with $\psi\alpha > 0$. The same $\mathcal{S}$ is also a slice with contraction for the action of $\tilde{B} = T \ltimes U$, defined in 2.2.3.

6.5. Let $v \in V$ be fixed. Fix also a simple root α and denote by s the simple reflection corresponding to α. We use the notations of 6.2. The orbit type of v for s is related to certain properties of ψ with respect to α. Namely, we have

v is of type I for s if and only if $\psi\alpha = \alpha$ and ψ is the identity on U_α and $U_{-\alpha}$ (α is “compact imaginary” for v).

v is of type II for s if and only if $\psi\alpha \neq \pm\alpha$ (α is “complex” for v).

v is of type IIIa or IVa for s if and only if $\psi\alpha = \alpha$ and $\psi(g) = g^{-1}$ on $U_{\pm\alpha}$ (α is “non-compact imaginary” for v).

v is of type IIIb or IVb for s if and only if $\psi\alpha = -\alpha$ (α is “real” for v).

If v is of type I, IIa, IIIa or IVa, then $sw > w$ ($\Leftrightarrow w^{-1}\alpha > 0 \Leftrightarrow \psi\alpha > 0$). If v is of type IIb, IIIb or IVb, then $sw < w$ ($\Leftrightarrow w^{-1}\alpha < 0 \Leftrightarrow \psi\alpha < 0$). The types III and IV are distinguished from each other by the number of B-orbits in $P_s v$.

6.6. Symmetric varieties have the following important property. If $v \in V$ is not a minimal B-orbit, then there exists a simple reflection s such that v is of a type b for s.

For an arbitrary spherical variety this is not always true.

6.7. We use the notations of 6.5. If v is of type IIIa or IVa for s, then $\psi\check{\alpha} = \check{\alpha}$, hence $\mathrm{Im}\,(\check{\alpha}) \subset T^{\psi} = T_v$.

Now let v be of type IIIb for s and $P_s v = v \cup v' \cup v''$ as usual.

If $\xi \in \widehat{X}(T/T_v)$, then $\langle \widehat{\phi}_v \xi, \check{\alpha} \rangle = 2a_{v'}(\xi)$, since $\mathrm{Im}\,(\check{\alpha}) \subset T_{v'}$ (cf. 4.2.7). Since $\langle \widehat{\phi}_v \xi, \check{\alpha} \rangle = a_{v'}(\xi) + a_{v''}(\xi)$, we have $a_{v'}(\xi) = a_{v''}(\xi)$.

6.8. Now let T be a θ-stable maximal torus in G and let B be any Borel group containing T. When $\mathcal{V}$ is defined as in 6.1 we still have a bijection $T\backslash\mathcal{V}/K \to B\backslash G/K$. This follows from the result mentioned in 6.1. Choose $g \in G$ such that ${}^gT$ and ${}^gB$ are θ-stable and consider the map $x \mapsto gx$ of G onto itself.

For $x \in \mathcal{V}$ we can define $n = x(\theta x)^{-1} \in N$ and $\psi = \mathrm{Int}\,(n) \circ \theta$ as in 6.2. The classification of orbits in 6.5 is also valid in this case. The condition for v to be of type I, IIa, IIIa or IVa is $\psi\alpha > 0$, which is now equivalent to $\theta(w^{-1}\alpha) > 0$ or to $sww_0 > ww_0$, where $w_0 \in W$ is the image of $(\theta g)^{-1} g \in N$, g as above.

7. Proof of the main result

7.1. **Statement of result.**

7.1.1. Assumptions are as in chapters 3 and 4. In particular k is the algebraic closure of $\mathbb{F}_q$, where q is large. All B-orbits in Y contain elements over $\mathbb{F}_q$ and $F\xi = \xi$ for all ξ in consideration (they are finite in number). Moreover, we assume now that Y is symmetric, so $Y = G/G^{\theta}$, where θ is an involution of G, and the characteristic of k is odd.

7.1.2. Theorem. *Let $v \in V$, $\xi \in \widehat{X}(T/T_v)$. Assume that $\langle \hat{\phi}_v \xi, \check{\alpha} \rangle = 0$ for all roots α. Then*

(i) $q^{\dim v} b_{\eta,u;\xi,v}$ *is a polynomial in q with coefficients in $\mathbb{Z}$, if $u \in V$, $\eta \in \hat{X}(T/T_u)$.*

(ii) $c_{\eta,u;\xi,v,i} \in \mathbb{N} q^{\frac{1}{2}(i+\dim v)}$, *if $u \in V$, $\eta \in \hat{X}(T/T_u)$, $i \in \mathbb{Z}$. Moreover, if $c_{\eta,u;\xi,v,i} \neq 0$, then $i + \dim v$ is even.*

Assertion (ii) means that the eigenvalues of Frobenius on the stalks of $\mathcal{H}^i A_{\xi,v}$ are $q^{\frac{1}{2}(i+\dim v)}$ times a root of unity and $\mathcal{H}^i A_{\xi,v} = 0$ unless $i + \dim v$ is even.

We give an algorithm to compute the coefficients b and c, which will prove the theorem. After 3.4.3 part (ii) is a consequence of part (i), since we have slices with contraction by 6.4, the condition on weights in 3.4.3 is satisfied by Proposition 2.3.3. However, in general we cannot prove (i) without making use of the relation between the coefficients b and c. In a few special cases assertion (i) does follow from the recurrence relations for the coefficients b alone. This is the case, for instance, when the only orbit type occurring is type II. The space $Y = G$ with the action of $G \times G$ is an example.

7.2. **First reduction.**

7.2.1. We prove Theorem 7.1.2(i) by an inductive procedure. The assertion is obvious for a minimal orbit v. Suppose that v is not minimal and that 7.1.2(i) is known to hold for all orbits of lower dimension. If there is $s \in S$ such that v is of type IIb for s, then $b_{\eta,u;\xi,v}$ can be computed from

$$D\varepsilon_{\xi,v} = (q^{-1}\varepsilon_{-\hat{\phi}_v\xi,s} + (q^{-1} - 1)\varepsilon_{-\hat{\phi}_v\xi,e}) D\varepsilon_{s\xi,v'}$$

(see 5.2), using the formulas of 4.3.1, and they satisfy 7.1.2(i).

Notice that $\hat{\phi}_{v'}(s\xi) = s\hat{\phi}_v\xi = \hat{\phi}_v\xi$ and $\langle\hat{\phi}_{v'}(s\xi), \check{\beta}\rangle = 0$ for all roots β. Now assume that we have $v \in V$ such that there is no $s \in S$ which is of type IIb for v. We may assume that B and T are θ-stable. Choose $x \in \mathcal{V}$ representing v and define w and ψ as in 6.2. We denote the set of simple roots by Δ. Let I be the set of simple roots α such that s_α is of type IIIb or IVb for v. Then $\psi\alpha = -\alpha$ if $\alpha \in I$ and $\psi\alpha > 0$ if $\alpha \in \Delta - I$. This follows from 6.5, since the case IIb does not occur here. Notice that $I \neq \emptyset$ if v is not minimal, which we assume. Since $\psi\alpha = w(\theta\alpha)$ and $\theta\Delta = \Delta$, we have $w(\Delta) \subset -\Delta \cup R^+$, where R^+ is the set of positive roots. It is known that there is a subset J of Δ such that w is the longest element of the subgroup W_J of W generated by J (cf. [B, ch.VI, §1, exercise 17]). It is easily seen that in our case $J = \theta I = I$.

Let $P = P_I$ be the parabolic subgroup of G containing B and corresponding to the subset I of Δ. Then P is ψ-stable. Let τ be defined as in 6.3. Since τP is closed in P, it is closed in G, hence in τG. Moreover, τ induces isomorphisms $P/P^\psi \xrightarrow{\sim} \tau P$ and $G/G^\psi \xrightarrow{\sim} \tau G$. It follows that Pv is closed in Y (see also [S2, 6.3]). Now $\bar{v}$ is invariant under $P = P_I$, since all $\alpha \in I$ are of some type b for v. Hence $Pv \subset \bar{v}$ and since Pv is closed we have $Pv = \bar{v}$.

7.2.2. Let L be the Levi subgroup of P containing T. It is ψ-stable. The projection of $P = LU_P$ on L is denoted by π_L.

We define a symmetric variety $Y^{(L)}$ for L by $Y^{(L)} = L\dot{v} \subset Y$, where $\dot{v} = xK$, x as above. By an argument as before we see that $L\dot{v}$ is closed in Y and isomorphic to L/L^ψ.

In L we have the Borel group $\pi_L(B) = B \cap L$ containing T. Notice that T is ψ-stable, but $B \cap L$ is not.

There is a surjective morphism $\pi : \bar{v} \to Y^{(L)}$ defined by $\pi(p\dot{v}) = \pi_L(p)\dot{v}$ for $p \in P$. The fibers of π are isomorphic to U_P/U_P^ψ. The B-orbits contained in $\bar{v}$ are in 1-1-correspondence with the $(B \cap L)$-orbits in $Y^{(L)}$ by $u \mapsto \pi u$. The image πv of v is the open orbit in $Y^{(L)}$.

It is easily checked that, for any B-orbit u in $\bar{v}$, we have $T_u = T_{\pi u}$ and if $\eta \in \hat{X}(T/T_u) = \hat{X}(T/T_{\pi u})$, then $\mathcal{L}_{\eta,u} = (\pi|_u)^*\mathcal{L}_{\eta,\pi u}$. Denote by $\hat{\pi}$ the $\mathbb{Z}[\mathcal{C}]$-linear map $\mathcal{K}(\mathcal{C}_{Y^{(L)}}) \to \mathcal{K}(C_Y)$ such that $\hat{\pi}(\varepsilon_{\eta,\pi u}) = \varepsilon_{\eta,u}$ if $u \in V$, $u \leq v$ and $\eta \in \hat{X}(T/T_u)$.

7.2.3. Lemma. *Let $\xi \in \hat{X}(T/T_v)$. With $d = \dim(U_P/U_P^\psi)$ we have:*

(i) $D\varepsilon_{\xi,v} = q^{-d}\hat{\pi}(D\varepsilon_{\xi,\pi v})$,
(ii) $[\mathcal{H}^i A_{\xi,v}] = \hat{\pi}[\mathcal{H}^{i+d} A_{\xi,\pi v}]$.

Consider the Cartesian square

$$\begin{array}{ccc} v & \xrightarrow{j} & \bar{v} \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ \pi v & \xrightarrow[j^{(L)}]{} & Y^{(L)} \end{array}$$

We have $Dj_!\mathcal{L}_{\xi,v} = D\pi^* j_!^{(L)}\mathcal{L}_{\xi,\pi v} = \pi^*(Dj_!^{(L)}\mathcal{L}_{\xi,\pi v})[2d](d)$, since π is smooth with relative dimension d. This proves (i). Also $A_{\xi,v} = \pi^* A_{\xi,\pi v}[d]$, which gives (ii).

7.2.4. By 7.2.3(i) assertion (i) of 7.1.2 for ξ, v will follow from the analogous assertion for ξ, πv. This leads us to consider the particular case of a group G with involution θ with the property that there exists a θ-stable maximal torus T in G such that $\theta\alpha = -\alpha$ for all roots of T. The open orbit only is of interest. The condition on θ is equivalent to the condition that $t\theta t$ lies in the center Z of G for all $t \in T$.

7.3. **Further reduction.**

7.3.1. We take up the notation of 7.1.1. In addition, let C be a finite θ-stable subgroup of the center of G. Let $G' = G/C$. Then θ induces an involution θ' of G'. Put $K = G^\theta$, $K' = (G')^{\theta'}$, $Y = G/K$ and $Y' = G'/K'$. Let f denote the natural homomorphism $G \to G'$ and π the morphism $Y \to Y'$ induced by f.

Let $K^* = \{g \in G \mid g^{-1}\theta g \in C\}$. Then $K^* = f^{-1}(K')$ and $g \mapsto g^{-1}\theta g$ defines a homomorphism $K^* \to C$ with kernel K. The finite abelian group K^*/K acts on Y by $xK \mapsto xgK$, if $g \in K^*$, and Y' is the quotient of Y for this action.

Define $T' = f(T)$, $B' = f(B)$, where T and B are as usual (they need not be θ-stable). Let V' be the set of B'-orbits in Y'.

Obviously, the image under π of a B-orbit is a B'-orbit and the inverse image of a B'-orbit is a union of B-orbits of the same dimension. We have $v_1 \leq v_2 \Rightarrow \pi v_1 \leq \pi v_2$ if v_1, $v_2 \in V$. If $v \in V$, then $\pi\bar{v}$ is closed in Y', hence $\pi\bar{v} = \overline{\pi v}$.

7.3.2. If $v \in V$, then $f(T_v) \subset T'_{\pi v}$ and f induces a surjective homomorphism $\psi_v : T/T_v \to T'/T'_{\pi v}$ with finite kernel D_v. So by 2.1.5 we have an exact sequence

$$0 \to X(D_v) \to \hat{X}(T'/T'_{\pi v}) \xrightarrow{\hat{\psi}_v} \hat{X}(T/T_v) \to 0.$$

There is a bijection between the fiber of $v \to \pi v$ and D_v.

Indeed, we may identify v (resp. πv) with $B/B_{\dot{v}}$ (resp. $B'/B'_{\pi\dot{v}}$), so that the fiber is $f^{-1}(B'_{\pi\dot{v}})/B_{\dot{v}}$. Now π_T induces a surjective map $\beta : f^{-1}(B'_{\pi\dot{v}})/B_{\dot{v}} \to f^{-1}(T'_{\pi v})/T_v = D_v$ and $\beta^{-1}(1) = U \cap f^{-1}(U'_{\pi\dot{v}})/U_{\dot{v}}$ is connected, because $U \cap f^{-1}(U'_{\pi\dot{v}}) \cong U'_{\pi\dot{v}}$ is connected (6.2), and it is finite, so it is $\{1\}$.

Let $\xi' \in \hat{X}(T'/T'_{\pi v})$ and $\xi = \hat{\psi}_v\xi'$. Then the restriction of $\pi^*\mathcal{L}_{\xi',\pi v}$ to v is $\mathcal{L}_{\xi,v}$. In case $\bar{v} \to \overline{\pi v}$ is smooth, we have

$$A_{\xi,v} = \pi^* A_{\xi',\pi v},$$
$$Dj_!\mathcal{L}_{\xi,v} = \pi^*(Dj'_!\mathcal{L}_{\xi',\pi v}),$$

as is seen from the diagram

$$\begin{tikzcd} v \arrow[r, "j"] \arrow[d] & \bar{v} \arrow[d, "\pi"] \\ \pi v \arrow[r, "j'"'] & \overline{\pi v} \end{tikzcd}$$

Let $\hat{\pi}_v$ denote the $\mathbb{Z}[C]$-linear map $\mathcal{K}(\mathcal{C}'_Y) \to \mathcal{K}(\mathcal{C}_Y)$ determined by $\hat{\pi}_v(\varepsilon_{\eta',u'}) = \sum_{u \subset \pi^{-1}(u') \cap \bar{v}} \varepsilon_{\hat{\psi}_u\eta',u}$ if $u' \in V'$, $\eta' \in \hat{X}(T'/T'_{u'})$. Summation is over all B-orbits u contained in $\pi^{-1}(u') \cap \bar{v}$. The following lemma is now immediate.

7.3.3. Lemma. *Let $v \in V$ be such that $\bar{v} \to \overline{\pi v}$ is smooth. Let $\xi \in \hat{X}(T/T_v)$ and choose $\xi' \in \hat{X}(T'/T'_{\pi v})$ such that $\hat{\psi}_v\xi' = \xi$. Then*

(i) $D\varepsilon_{\xi,v} = \hat{\pi}_v(D\varepsilon_{\xi',\pi v})$,
(ii) $[\mathcal{H}^i A_{\xi,v}] = \hat{\pi}_v[\mathcal{H}^i A_{\xi',\pi v}]$.

7.3.4. Now consider the case where G has a maximal torus T such that $t\theta t \in Z$ for all $t \in T$. Take for the group C in 7.3.1 the center of the derived group of G. The image πv of the open orbit v in Y is the open orbit in Y' and Lemma 7.3.3 can be applied. Notice that $\langle \hat{\phi}_v \xi, \check{\alpha} \rangle = \langle \hat{\phi}_{\pi v} \xi', \check{\alpha} \rangle$. In $G' = G/C$ we have again $t'\theta' t' \in Z'$ for all $t' \in T'$, where Z' is the center of G'. This leads us to consider the case where G satisfies the extra condition that its derived group has trivial center. Then G is the direct product of a finite number of simple groups and a torus. All factors are θ-stable, as a consequence of the condition on θ. The sheaves on G we consider are the exterior tensor products of the corresponding sheaves on the factors of G. So it is enough to consider the factors separately. For the central torus there is nothing to prove.

On each simple factor the involution is split, i.e. there exists a maximal torus on which $\theta t = t^{-1}$.

7.3.5. The arguments of 7.2 and 7.3.4 show that Theorem 7.1.2 will be proved in general when it has been proved for all simple groups with split involution.

7.4. **The case of a simple group with split involution.** In this section we shall mainly be occupied with the case of a simple group with split involution, but first we formulate a result in a more general setting.

7.4.1. Lemma. *In the situation of 7.1.1 let $v \in V$ be such that there is no $s \in S$ of type IIb for v. Let $\xi \in \hat{X}(T/T_v)$ such that for every $s = s_\alpha \in S$ of type IIIb or IVb for v we have $\langle \hat{\phi}_v \xi, \check{\alpha} \rangle = 0$, $a(\xi) \neq 0$ (notation of 4.2.7). Then $A_{\xi,v} = \mathcal{L}_{\xi,v}[\dim v]$ and $D_Y \mathcal{L}_{\xi,v} = \mathcal{L}_{-\xi,v}[2\dim v](\dim v)$.*

The method of 7.2 reduces the proof to the case where all $s \in S$ are of type IIIb or IVb for v (which implies that v is the open orbit). We prove that $b_{\eta,u;\xi,v} = c_{\eta,u;\xi,v,i} = 0$ for all $\eta,\ u,\ i$ with $u < v$. This is done by descending induction on $\dim u$. If $\dim u = \dim v - 1$, there is $s \in S$ such that $v = m(s)u$, i.e. $u \subset P_s v$, and our assertion follows from Lemmas 4.4.5 and 4.4.6. Now assume $\dim u < \dim v - 1$. Choose s of some type a for u. By 5.2 we have $(\varepsilon_{-\hat{\phi}_v \xi, s} + \varepsilon_{-\hat{\phi}_v \xi, e}) D \varepsilon_{\xi,v} = 0$ and this implies by 5.3 that

$$\begin{aligned} &b_{\eta,u} = -q b_{s\eta, m(s)u} && \text{if } s \text{ is of type IIa for } u,\\ &b_{\eta,u} + b_{s\eta,u'} = (1-q) b_{\eta,m(s)u} && \text{if } s \text{ is of type IIIa for } u,\\ &b_{\eta,u} = (1-q) b_{\eta_1, m(s)u} && \text{if } s \text{ is of type IVa for } u,\end{aligned}$$

where we wrote $b_{\eta,u}$ for $b_{\eta,u;\xi,v}$. Using the induction hypothesis we find immediately that $b_{\eta,u} = 0$ in case IIa or IVa. In case IIIa we have $b_{\eta,u} + b_{s\eta,u'} = 0$. Apply the relation 3.4.2 between the coefficients b and c to (η, u) and $(s\eta, u')$ and add the two equalities thus obtained together. Then the right hand side is zero. This gives, by the argument in 3.4.3, that $c_{-\eta,u;\xi,v,i} + c_{-s\eta,u';\xi,v,i} = 0$. Since $c_{-\eta,u;\xi,v,i}$ and $c_{-s\eta,u';\xi,v,i}$ have coefficients ≥ 0 in $\mathbb{Z}$, both must be zero. Also $c_{\eta,u;-\xi,v,i} = 0$ and finally $b_{\eta,u;\xi,v} = 0$ by 3.4.2.

7.4.2. We assume from now on that G is simple and θ split.

We fix a maximal torus T in G such that $\theta t = t^{-1}$ for all $t \in T$. For any Borel group B containing T we have $\theta B = B^-$, the opposite Borel group, and BK is open in G. The open B-orbit v is the image of BK in $Y = G/K$. We have $T_v = T^\theta = B \cap K = \{t \in T \mid t^2 = 1\}$ and $X(T/T_v) = 2X(T)$.

Let R denote the root system of T and $P(R)$ resp. $Q(R)$ the weight lattice and the root lattice. If G is adjoint, then $X(T) = Q(R)$ by definition, and otherwise G is of type A_l and the characteristic p divides $l+1$ or G is of type E_6 and $p = 3$.

All $s \in S$ are of type IVb for v. We know already that they are of type IIIb or IVb. If s_α were of type IIIb for v, we would have $\alpha(T_v) = 1$ (see 4.2.7), so $\alpha \in 2X(T)$. This is impossible when G is adjoint and for A_l and E_6 there is no simple root in $2P(R)$.

We have $\hat{X}(T/T_v) = (\mathbb{Z}_{(p)}/\mathbb{Z}) \otimes_{\mathbb{Z}} 2X(T) = X(T)_{(p)}/2X(T)$ and $\hat{\phi}_v$ is the obvious map $X(T)_{(p)}/2X(T) \to X(T)_{(p)}/X(T)$. The condition $\langle \hat{\phi}_v \xi, \check{\alpha} \rangle = 0$ for all α means that ξ lies in the subgroup $P(R) \cap X(T)_{(p)}/2X(T)$ of $\hat{X}(T/T_v)$.

We shall write $P(R)'$ for $P(R) \cap X(T)_{(p)}$. So we consider only ξ in $P(R)'/2X(T)$.

For $s \in S$ we denote by a_s the map $\hat{X}(T/T_v) \to \hat{X}(\mathbb{G}_m)$ which was called a in 4.2.7. If $\xi = \omega + 2X(T)$ with $\omega \in P(R)'$ and s is the reflection corresponding to the simple root α, then $a_s(\xi) = \frac{1}{2}\langle \omega, \check{\alpha} \rangle + \mathbb{Z} \in \frac{1}{2}\mathbb{Z}/\mathbb{Z}$. The other element $\xi' \in \hat{X}(T/T_v)$ with the same image as ξ in $\hat{X}(T/T_v \cap \operatorname{Ker}(\alpha))$ is $\xi + \alpha$, since $2\alpha \in X(T/T_v)$ and $\alpha \notin X(T/T_v)$.

7.4.3. Let G be simple and θ split. For the proof of Theorem 7.1.2(i) for G and θ we follow again the procedure of 7.2 and 7.3.4. Suppose the theorem has been proved for all simple groups of lower dimension with a split involution. Then induction on $\dim v$ ($v \in V$) proves 7.1.2(i) for G and all orbits other than the open orbit. So, Theorem 7.1.2 will be completely proved when we have proved:

7.4.4. Lemma. *Assume G is simple and θ split. If 7.1.2*(i) *holds for all orbits other than the open orbit, then it holds for the open orbit also.*

7.4.5. We use the notations of 7.4.2. In particular v is the open B-orbit in Y. We assume that all $b_{\eta,u;\xi_1,v_1}$ with $v_1 < v$ are known and satisfy 7.1.2(i).

Let $s = s_\alpha \in S$ and $\xi \in P(R)'/2X(T)$. We know that s is of type IVb for v. If $a_s(\xi) = 0$, we have by 5.2(i) applied to $\bar{\xi}$ (defined as in 4.3.1) and v':

$$D\varepsilon_{\xi,v} + D\varepsilon_{\xi+\alpha,v} = (q^{-1}\varepsilon_{-\hat{\phi}_v\xi,s} + (q^{-1} - 2)\varepsilon_{-\hat{\phi}_v\xi,e} D\varepsilon_{\bar{\xi},v'}.$$

So $b_{\eta,u;\xi,v} + b_{\eta,u;\xi+\alpha,v}$ is known for all η, u.

By 5.2(iii) we have

$$(q^{-1}\varepsilon_{-\hat{\phi}_v\xi,s} - \varepsilon_{-\hat{\phi}_v\xi,e})(D\varepsilon_{\xi,v} + D\varepsilon_{\bar{\xi},v'}) = 0.$$

The solution of this system of equations is given by 5.3(iv) with $a_{\eta,u} = b_{\eta,u;\xi,v} + b_{\eta,u;\bar{\xi},v'}$ and $\zeta = -\hat{\phi}_v\xi$. In particular we see that

$$\begin{array}{ll} b_{\eta,u;\xi,v} - b_{s\eta,m(s)u;\xi,v} & \text{is known if } s \text{ is of type IIa for } u, \\ b_{\eta,u;\xi,v} - b_{\eta,m(s)u;\xi,v} & \text{is known if } s \text{ is of type IIIa for } u, \\ b_{\eta,u;\xi,v} - b_{\eta_1,m(s)u;\xi,v} - b_{\eta_2,m(s)u;\xi,v} & \text{is known if } s \text{ is of type IVa for } u. \end{array}$$

If $a_s(\xi) \neq 0$, then by 5.2 we have $(\varepsilon_{-\hat{\phi}_v\xi,s} + \varepsilon_{-\hat{\phi}_v\xi,e})D\varepsilon_{\xi,v} = 0$ and 5.3(v) gives:

$$\begin{array}{ll} b_{\eta,u;\xi,v} + qb_{s\eta,m(s)u;\xi,v} = 0 & \text{if } s \text{ is of type IIa for } u, \\ b_{\eta,u;\xi,v} + b_{s\eta,u';\xi,v} + (q-1)b_{\eta,m(s)u;\xi,v} = 0 & \text{if } s \text{ is of type IIIa for } u, \\ b_{\eta,u;\xi,v} + (q-1)b_{\eta_1,m(s)u;\xi,v} = 0 & \text{if } s \text{ is of type IVa for } u. \end{array}$$

7.4.6. Fix $u \in V$, $u < v$, and $\eta \in \hat{X}(T/T_u)$. Assume that $b_{\eta_1,u_1;\xi,v}$ is known and satisfies 7.1.2(i) for all ξ and all (η_1, u_1) with $\dim u_1 > \dim u$. From 7.4.5 we conclude that

(1) $b_{\eta,u;\xi,v} + b_{\eta,u;\xi+\alpha,v}$ is known if $a_{s_\alpha}(\xi) = 0$;

(2) $b_{\eta,u;\xi,v}$ is known if there is $s \in S$ of type a for u such that $a_s(\xi) = 0$;

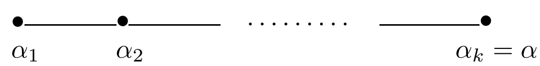
(3) $b_{\eta,u;\xi,v}$ is known if there is $s \in S$ of type IIa or IVa for u such that $a_s(\xi) \neq 0$.

There are always $s \in S$ of type a for u. If there is one of type IIa or IVa we are done by (2) and (3).

We shall write b_ω for $b_{\eta,u;\xi,v}$ if $\xi = \omega + 2X(T)$, $\omega \in P(R)'$. We have $a_{s_\alpha}(\xi) = 0$ if and only if $\langle \omega, \check{\alpha} \rangle$ is even (see 7.4.2).

Let $s = s_\alpha$ be of type IIIa for u. By (2), b_ω is known if $\langle \omega, \check{\alpha} \rangle$ is even and, by (1), $b_\omega + b_{\omega+\beta}$ is known if β is a simple root such that $\langle \omega, \check{\beta} \rangle$ is even. Assume there are simple roots $\alpha_1, \dots, \alpha_k$ such that $\alpha_k = \alpha$ and $\langle \omega, \check{\alpha}_1 \rangle$, $\langle \omega + \alpha_1, \check{\alpha}_2 \rangle$, $\langle \omega + \alpha_1 + \alpha_2, \check{\alpha}_3 \rangle$, $\dots$, $\langle \omega + \alpha_1 + \dots + \alpha_{k-1}, \check{\alpha}_k \rangle$ are all even. Then by the above $b_{\omega+\alpha_1+\dots+\alpha_{i-1}} + b_{\omega+\alpha_1+\dots+\alpha_i}$ $(1 \leq i \leq k-1)$ and $b_{\omega+\alpha_1+\dots+\alpha_{k-1}}$ are known. Hence b_ω is known.

If the Dynkin diagram contains a subset of the type



with simple laces, then $\alpha_1, \dots, \alpha_k$ satisfy the above condition if $\langle \omega, \check{\alpha}_1 \rangle$ is even and $\langle \omega, \check{\alpha}_i \rangle$ are odd for $2 \leq i \leq k$.

This means that for G of type A_l, D_l or E_l we find b_ω except for the case where $\langle \omega, \check{\alpha} \rangle$ is odd for all simple α. But then $b_\omega = 0$ by 7.4.1. The same argument works for G_2 since $\langle \alpha_1, \check{\alpha}_2 \rangle$ and $\langle \alpha_2, \check{\alpha}_1 \rangle$ are odd, and also for type B_l, C_l, F_4 if α is a long root.

If G is of type B_l, it is adjoint and there is no orbit of type III for s_α when α is the short root α_l. Indeed, suppose s_α were of type IIIb for some orbit. With ψ as in 6.8 we would have $\psi\alpha = -\alpha$ and $\alpha(T^\psi) = 1$. Since $\alpha(T^\psi) = 1$, there exists $\chi \in X(T)$ such that $\alpha = \psi\chi - \chi$, and then $\langle \chi, \check{\alpha} \rangle = -1$. This is impossible, since for the adjoint group of type B_l and $\alpha = \alpha_l$ we have $\langle \chi, \check{\alpha} \rangle \in 2\mathbb{Z}$ for all $\chi \in X(T)$.

The two remaining cases, C_l and F_4, are settled by the following lemma.

7.4.7. Lemma. *Let G be of type C_l or F_4 (and θ split) and let u be a B-orbit in Y which is not the open orbit. At least one of the three holds:*

(i) *there is $s \in S$ of type IIa for u,*
(ii) *there is $s \in S$ of type IVa for u,*
(iii) *there is $s \in S$ corresponding to a long root such that s is of type IIIa for u.*

In any of the three cases $b_{\eta,u;\xi,v}$ can be computed. The proof of Lemma 7.4.7 will be given in 7.5. Then the proof of Lemma 7.4.4 is complete.

Remark. In 7.4.6 we have besides (1), (2), (3) also

(4) $b_{\eta,u;\xi,v} + b_{s\eta,u';\xi,v}$ is known if s is of type IIIa for u,

and this expression is $q^{-\dim v}$ times a polynomial in q with coefficients in $\mathbb{Z}$. It is possible to deduce from this that $q^{\dim v} b_{\eta,u;\xi,v}$ is a polynomial in q with coefficients in $\mathbb{Z}$. Take, as in the proof of 7.4.1, the sum of the equality 3.4.2 and the same with (η, u) replaced by $(s\eta, u')$. Then the right hand side is a polynomial in q with coefficients in $\mathbb{Z}$ (this follows from the assumptions on the coefficients b made in 7.4.5 and 7.4.6 and application of 3.4.2). By the argument of 3.4.3 we conclude that $c_{-\eta,u;\xi,v,i} + c_{-s\eta,u';\xi,v,i}$ is a polynomial in q with coefficients in $\mathbb{Z}$, hence is in $\mathbb{Z}q^{\frac{1}{2}(i+\dim v)}$. Since $c_{-\eta,u;\xi,v,i}$ and $c_{-s\eta,u';\xi,v,i}$ have coefficients in $\mathbb{N}$, both must be in $\mathbb{N}q^{\frac{1}{2}(i+\dim v)}$. Applying 3.4.2 once more we see that $q^{\dim v}b_{\eta,u;\xi,v}$ is a polynomial in q with coefficients in $\mathbb{Z}$. This argument however does not yield a way to compute $b_{\eta,u;\xi,v}$.

7.5. **Proof of Lemma 7.4.7.**

7.5.1. In the notations of 7.4.2 assume that G is of type C_l or F_4. The involution θ then is an inner automorphism $\mathrm{Int}\,(n_0)$ with $n_0 \in N$, $n_0^2 \in Z$ and the image of n_0 in W is $w_0 = -1$. We use the notations of Chapter 6. If $x \in \mathcal{V}$, then $\tau x = x(\theta x)^{-1} = xn_0x^{-1}n_0^{-1} \in N$, so $\tau G = C(n_0)n_0^{-1}$, where $C(n_0)$ is the conjugacy class of n_0. The fixed point group K of θ is the centralizer of n_0 in G. The map $x \mapsto xn_0x^{-1}$ induces a bijection

$$V = T\backslash\mathcal{V}/K \to T\backslash N \cap C(n_0),$$

where the right hand side is the quotient of $N \cap C(n_0)$ for the action of T by conjugation.

Let $v \in V$ and let x be a representative for v in $\mathcal{V}$. Put $n = xn_0x^{-1}$. There is a map $\phi : V \to W$ defined by $\phi(v) = w =$ image of n in W. We have $w^2 = e$, since $n_0^2 \in Z$. The involution ψ introduced in Chapter 6 is now $\psi = \mathrm{Int}\,(nn_0^{-1}) \circ \theta = \mathrm{Int}\,(n)$.

If $s \in S$ corresponds to the simple root α, then $\psi\alpha = w\alpha$ and we see from 6.5 that

v is of type I for s if $w\alpha = \alpha$, $\mathrm{Int}\,(n) = 1$ on $U_{\pm\alpha}$,
v is of type IIIa/IVa for s if $w\alpha = \alpha$, $\mathrm{Int}\,(n) = -1$ on $U_{\pm\alpha}$,
v is of type IIIb/IVb for s if $w\alpha = -\alpha$,
v is of type II for s if $w\alpha \neq \pm\alpha$,
v is of type I or a $\Leftrightarrow w\alpha > 0$.

If v is of type IIIa or IVa, then $\phi(m(s)v) = s\phi(v)$. If v is of type IIa, then $\phi(m(s)v) = s\phi(v)w_0sw_0$. We have $T_v = T^\psi = \{t \in T \mid w(t) = t\}$. If $w\alpha = -\alpha$, then v is of type IIIb if and only if $\alpha(T_v) = 1$, i.e. if $\alpha \in (w-1)X(T)$.

7.5.2. Let $u \in V$ be such that there is no $s \in S$ of type IIa for u. With $w = \phi(u)$ we have for $\alpha \in \Delta$ either $w\alpha = \alpha$ or $w\alpha < 0$. Then there is a subset I of Δ such that $ww_0 = w_I$, the longest element of W_I, and that $w_I = -1$ on I (recall that w_0 is the longest element of W). If u is not the open orbit, then $I \neq \emptyset$.

7.5.3. We first consider the case where G is of type C_l. We use the notations for the roots as in [B]. So the long simple root is α_l. Let u and I be as in 7.5.2. Since $w_I = -1$ on I, I may have an irreducible component of the form $\{\alpha_j \mid j \geq k\}$ and all other irreducible components consist of one element.

Assume α_i is of type IIIa for u, for some $i < l$. Then $w_I\alpha_i = -\alpha_i$ and $\alpha_i \in (s_iw_I + 1)X(T)$. Let $\alpha_i = (s_iw_I + 1)\chi$. Then $2 = \langle\alpha_i, \check{\alpha}_i\rangle = \langle(s_iw_I + 1)\chi, \check{\alpha}_i\rangle = 2\langle\chi, \check{\alpha}_i\rangle$. So we have $\chi \in X(T)$ such that $\langle\chi, \check{\alpha}_i\rangle = 1$ and $w_I\chi = -\chi$. Then χ is a linear combination of the roots in I and α_i cannot be isolated in I, so I has an irreducible component consisting of more than one element (and α_i lies in this component). This proves Lemma 7.4.7 when all components of I have only one element.

Assume now that I has an irreducible component $\{\alpha_i \mid i > k\}$ with $0 \leq k \leq l-2$. Let $J = \{j \leq k,\ \alpha_j \in I\}$. Then all indices in J are strictly smaller than k and the difference between two of them is at least 2. Let β_i denote the long root $2\varepsilon_i$ $(1 \le i \le l)$.

We have $w_I = \prod_{j\in J} s_{\alpha_j} \prod_{i>k} s_{\beta_i}$ and $\phi(u) = w_I w_0 = \prod_{j\in J} s_{\alpha_j} \prod_{i\le k} s_{\beta_i}$. The element $n = \prod_{j\in J} n_{\alpha_j} \prod_{i\le k} n_{\beta_i}$ of N is a representative for $\phi(u)$. Let $tn \in N \cap C(n_0)$ correspond to u in the bijection $T\backslash N \cap C(n_0) \to V$. The involution ψ is then $\mathrm{Int}\,(tn)$. We know that α_l is of type I, IIIa or IVa for u, since $\phi(u)\alpha_l = \alpha_l$, and we want to show that it is of type IIIa or IVa. Then Lemma 7.4.7 will be proved.

Obviously all factors in the definition of n centralize U_{α_l}, so $\mathrm{Int}\,(n)$ is the identity on U_{α_l} and it suffices to prove that $\alpha_l(t) = -1$. This can be done by a computation in $Sp\,(2l)$. The class $C(n_0)$ in G is the projection of the analogous class in $Sp\,(2l)$, which is $\{x \in Sp\,(2l), x^2 = -1\}$. Working with representatives in $Sp\,(2l)$ we have $tw_I w_0(t) = tntn^{-1} = (tn)^2 n^{-2} = -n^{-2}$ and $\alpha_l(t) = \varepsilon_l(t)^2 = \varepsilon_l(tw_I w_0(t)) = -\varepsilon_l(n^{-2})$. Now $n^2 \in T$ and by the definition of n it lies in $Sp\,(2k)$ embedded in an obvious way in $Sp(2l)$. Hence $\varepsilon_l(n^2) = 1$. Thus $\alpha_l(t) = -1$.

7.5.4. We now consider G of type F_4. The notations concerning the root system are again those of [B].

Let u and I be as in 7.5.2. So $\phi(u) = w_I w_0$ and $w_I = -1$ on I.

Let α be one of the short roots α_3, α_4 and suppose α is of type IIIa for u. As in the case C_l there must be $\chi \in X(T)$ such that $w_I\chi = -\chi$ and $\langle \chi, \check{\alpha} \rangle = 1$. Then $\langle \beta, \check{\alpha} \rangle$ must be odd for at least one $\beta \in I$. In combination with the fact that $w_I = -1$ on I, this implies that $I = \{\alpha_2, \alpha_3, \alpha_4\}$ or $I = \Delta$. So Lemma 7.4.7 holds when I is not one of these two sets.

Assume now $I = \{\alpha_2, \alpha_3, \alpha_4\}$. It is easily checked that $w_I w_0 = s_\beta$, where $\beta = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$ is the highest root. So n_β is a representative for $\phi(u)$. Let tn_β be an element of $N \cap C(n_0)$ corresponding to u. Since $\beta \pm \alpha_2$ are not roots, U_β and $U_{-\beta}$ centralize U_{α_2} and so does n_β. When we show that $\alpha_2(t) = -1$, it will follow that $\psi = \mathrm{Int}\,(tn_\beta)$ is -1 on U_{α_2}, so that α_2 is of type IIIa or IVa for u and Lemma 7.4.7 is proved in this case.

We may replace tn_β by a conjugate under T. This means that t may be multiplied by an element of $\mathrm{Im}(\check{\beta})$, so take $t \in \mathrm{Ker}\,(\beta)$. Since $tn_\beta \in C(n_0)$, we have $(tn_\beta)^2 = 1$, that is $t^2 = \check{\beta}(-1)$. Then $\alpha(t)^2 = (-1)^{\langle \alpha, \check{\beta} \rangle}$ for any α, which gives $\alpha_1(t)^2 = -1$, $\alpha_2(t)^2 = \alpha_3(t)^2 = \alpha_4(t)^2 = 1$ and $\beta(t) = -\alpha_2(t)$. Also $\beta(t) = 1$, hence $\alpha_2(t) = -1$.

It remains to consider the case $I = \Delta$. Now $\phi(u) = e$ and $\psi = \mathrm{Int}\,(t)$ with some $t \in C(n_0)$, in particular $t^2 = 1$.

Let t_0 be the element of T with $\alpha_i(t_0) = -1$ for $1 \le i \le 4$. By [S3, 6.1], the involution $\mathrm{Int}\,(t_0)$ is split. Hence n_0 is conjugate to t_0 and $T \cap C(n_0)$ is the orbit of t_0 under the Weyl group. Using the description of the Weyl group of F_4 in [B, p. 213], it is easy to check that for any of the twelve elements t of $W \cdot t_0$ one has $\alpha_1(t) = -1$ or $\alpha_2(t) = -1$. So for each of the corresponding B-orbits there is a long root $\alpha \in \Delta$ which is of type IIIa or IVa for that orbit (it is in fact IIIa). This finishes the proof of Lemma 7.4.7.

7.6. **An example.** Let G be a semi-simple group of type C_2 and θ a split involution of G. Let T be a maximal torus in G such that $\theta t = t^{-1}$ for all $t \in T$ and choose $B \supset T$. We recall some facts already mentioned in 7.5.1. We have $\theta = \mathrm{Int}\,(n_0)$ with $n_0 \in N$, $n_0^2 \in Z$ and n_0 represents the longest element w_0 of W. The B-orbits in G/G^θ are in 1-1-correspondence with the T-conjugacy classes in $N \cap C(n_0)$, where $C(n_0)$ is the G-conjugacy class of n_0. If the B-orbit v corresponds to $n \in N \cap C(n_0)$, then $\phi(v)$ is the image of n in W. We have $\phi(v)^2 = e$. If $G = Sp_4$, then $C(n_0) = \{x \in Sp_4 \mid x^2 = -1\}$.

Let the simple roots be α_1 (short) and α_2 (long), let s_1 and s_2 be the simple reflections and P_1 and P_2 the corresponding parabolic subgroups. It is easy to determine the ordered set V of B-orbits in G/G^θ by the method of [RS1]. Doing this one finds at the same time the P_1-orbits and the P_2-orbits. We describe the result.

We let π denote the natural morphism from the symmetric space for Sp_4 onto the symmetric space for PSp_4 (see 7.3.1).

In the case of Sp_4 we denote the B-orbits by $v_1, v_2, \ldots$ and number them in such a way that $\phi(v_i) = e$ for $i = 1, 2, 3, 4$, $\phi(v_5) = \phi(v_7) = s_2$, $\phi(v_6) = s_1$, $\phi(v_8) = \phi(v_{10}) = s_1 s_2 s_1$, $\phi(v_9) = s_2 s_1 s_2$, $\phi(v_{11}) = w_0$.

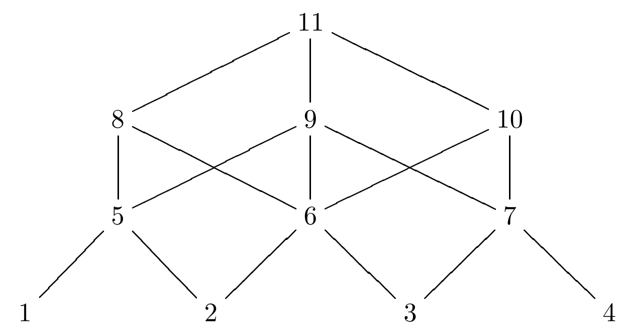
In the case of PSp_4 the orbits are then $\pi v_1 = \pi v_4$, $\pi v_2 = \pi v_3$, $\pi v_5 = \pi v_7$, πv_6, $\pi v_8 = \pi v_{10}$, πv_9, πv_{11}.

In the case of Sp_4 the P_1-orbits are v_1, v_4, $v_2 \cup v_3 \cup v_6$, $v_5 \cup v_8$ (type II), $v_7 \cup v_{10}$ (type II), $v_9 \cup v_{11}$ (type IV) and the P_2-orbits are $v_1 \cup v_2 \cup v_5$, $v_3 \cup v_4 \cup v_7$, $v_6 \cup v_9$ (type II), $v_8 \cup v_{10} \cup v_{11}$.

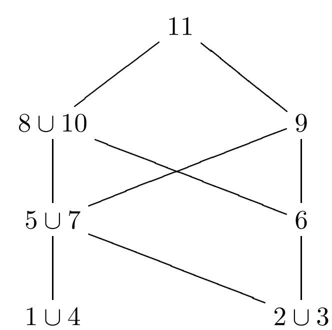
To obtain the correct picture we have to know that s_2 is of type IIIa or IVa for any minimal B-orbit in the PSp_4 case. This follows from the proof in 7.5.3 (with $l = 2$ and $I = \Delta$).

In the pictures below a line joining e.g. 6 and 8 means that $v_6 \leq v_8$.

Sp_4



PSp_4



Next, we determine the groups T_v. Let v be an orbit with $\phi(v) = e$. Choose $x \in \mathcal{V}$ such that $v = BxK$, then $xn_0x^{-1} \in T$ (see 7.5.1). It follows that $x^{-1}Tx \subset K$, since K is the centralizer of n_0. Hence $T_v = T$. The other T_v are now easily determined using the results in 4.2. We write α for α_1, β for α_2. For Sp_4 we have $T_{v_5} = T_{v_7} = \mathrm{Ker}\,(\beta)$, $T_{v_6} = \mathrm{Ker}\,(\alpha)$, $T_{v_8} = T_{v_{10}} = \mathrm{Ker}\,(2\alpha + \beta)$, $T_{v_9} = \mathrm{Ker}\,(\alpha + \beta)$, $T_{v_{11}} = \mathrm{Ker}\,(\beta) \cap \mathrm{Ker}\,(2\alpha + \beta) = \{t \in T \mid t^2 = 1\}$.

And for PSp_4 we have $T_{\pi v_5} = \operatorname{Ker}(\beta)$, $T_{\pi v_6} = \operatorname{Ker}(2\alpha)$, $T_{\pi v_8} = \operatorname{Ker}(2\alpha+\beta)$, $T_{\pi v_9} = \operatorname{Ker}(2(\alpha+\beta))$, $T_{\pi v_{11}} = \operatorname{Ker}(2\alpha) \cap \operatorname{Ker}(2\beta) = \{t \in T \mid t^2 = 1\}$.

Now $D\varepsilon_{\xi,v}$ and $[\mathcal{H}^i A_{\xi,v}]$ can be computed for all ξ, v, i using 3.4.2, 5.2 and 5.3. We give the results for $G = PSp_4$ and the open orbit. Notation is as follows. We have $\hat{X}(T/T_{\pi v_i}) = 0$ for $i = 1, 2$, $\hat{X}(T/T_{\pi v_5}) = \mathbb{Z}_{(p)}\beta/\mathbb{Z}\beta$, $\hat{X}(T/T_{\pi v_6}) = \mathbb{Z}_{(p)}\alpha/2\mathbb{Z}\alpha$, $\hat{X}(T/T_{\pi v_8}) = \mathbb{Z}_{(p)}(2\alpha+\beta)/\mathbb{Z}(2\alpha+\beta)$, $\hat{X}(T/T_{\pi v_9}) = \mathbb{Z}_{(p)}(\alpha+\beta)/2\mathbb{Z}(\alpha+\beta)$, $\hat{X}(T/T_{\pi v_{11}}) = \mathbb{Z}_{(p)}\alpha + \mathbb{Z}_{(p)}\beta/2\mathbb{Z}\alpha + 2\mathbb{Z}\beta$.

An element of $\hat{X}(T/T_{\pi v_6})$ will be given by a representative $x\alpha$ in $\mathbb{Z}_{(p)}\alpha$, etc.

The two maps $\hat{X}(T/T_{\pi v_{11}}) \to \mathbb{Z}_{(p)}/\mathbb{Z}$ defined by α and β (4.2.7) are given by $a_\alpha(x\alpha + y\beta) = x - y + \mathbb{Z}$, $a_\beta(x\alpha + y\beta) = y - \frac{1}{2}x + \mathbb{Z}$.

In the formulas below we abbreviate $\varepsilon_{\xi,\pi v_i}$ to $\varepsilon_{\xi,i}$, $\mathcal{J}_{\pi v_{11}}$ to $\mathcal{J}_{11}$, $A_{\xi,\pi v_{11}}$ to $A_{\xi,11}$.

$$
\begin{aligned}
D\varepsilon_{0,11} &= q^{-6}\varepsilon_{0,11} + q^{-6}(1-q)(\varepsilon_{0,8} + \varepsilon_{0,9}) + q^{-6}(1-q)^2(\varepsilon_{0,5} + \varepsilon_{0,6} + \varepsilon_{0,1}) \\
&\quad - q^{-5}(1-q)\varepsilon_{\alpha,6} + q^{-6}(1-q)^3\varepsilon_{0,2} \\
D\varepsilon_{\beta,11} &= q^{-6}\varepsilon_{\beta,11} + q^{-6}(1-q)(\varepsilon_{0,8} + \varepsilon_{\alpha+\beta,9}) + q^{-6}(1-q)^2(\varepsilon_{0,5} + \varepsilon_{\alpha,6} + \varepsilon_{0,1}) \\
&\quad - q^{-5}(1-q)\varepsilon_{0,6} + q^{-6}(1-q)^3\varepsilon_{0,2} \\
D\varepsilon_{\alpha,11} &= q^{-6}\varepsilon_{\alpha,11} + q^{-6}(1-q)\varepsilon_{0,9} - q^{-5}(1-q)(\varepsilon_{0,5} + \varepsilon_{0,6}) - q^{-5}(1-q)^2\varepsilon_{0,2} \\
D\varepsilon_{\alpha+\beta,11} &= q^{-6}\varepsilon_{\alpha+\beta,11} + q^{-6}(1-q)\varepsilon_{\alpha+\beta,9} - q^{-5}(1-q)(\varepsilon_{0,5} + \varepsilon_{\alpha,6}) - q^{-5}(1-q)^2\varepsilon_{0,2} \\
D\varepsilon_{\alpha\pm\frac{1}{2}\beta,11} &= q^{-6}\varepsilon_{\alpha\mp\frac{1}{2}\beta,11} + q^{-6}(1-q)\varepsilon_{\alpha+\frac{1}{2}\beta,8} - q^{-5}(1-q)\varepsilon_{\frac{1}{2}\beta,5} \\
D\varepsilon_{\alpha+y\beta,11} &= q^{-6}\varepsilon_{\alpha-y\beta,11} - q^{-6}(1-q)\mathcal{J}_{11}(-\alpha-y\beta)\varepsilon_{-y\beta,5} \quad \text{if } 2y \notin \mathbb{Z} \\
D\varepsilon_{x\alpha,11} &= q^{-6}\varepsilon_{-x\alpha,11} - q^{-6}(1-q)\mathcal{J}_{11}(-x\alpha)\varepsilon_{(-x-1)\alpha,6} \quad \text{if } x \notin \mathbb{Z} \\
D\varepsilon_{x\alpha+\beta,11} &= q^{-6}\varepsilon_{-x\alpha+\beta,11} - q^{-6}(1-q)\mathcal{J}_{11}(-x\alpha-\beta)\varepsilon_{-x\alpha,6} \quad \text{if } x \notin \mathbb{Z}
\end{aligned}
$$

$$
D\varepsilon_{x\alpha+y\beta,11}
\left.\begin{aligned}
&= q^{-6}\varepsilon_{-x\alpha-y\beta,11} + q^{-6}(1-q)\varepsilon_{-y(2\alpha+\beta),8} && \text{if } y - \tfrac{1}{2}x \in \mathbb{Z} \\
&= q^{-6}\varepsilon_{-x\alpha-y\beta,11} + q^{-6}(1-q)\varepsilon_{-y(\alpha+\beta),9} && \text{if } x - y \in \mathbb{Z} \\
&= q^{-6}\varepsilon_{-x\alpha-y\beta,11} && \text{if } y - \tfrac{1}{2}x \notin \mathbb{Z},\ x - y \notin \mathbb{Z}
\end{aligned}\right\} \begin{array}{l} y \notin \mathbb{Z}, \\ x \notin 1 + 2\mathbb{Z} \end{array}
$$

$$
\begin{aligned}
&[\mathcal{H}^{-6}A_{0,11}] = \sum_{v\in V} \varepsilon_{0,v} \\
&[\mathcal{H}^{-6}A_{\beta,11}] = \varepsilon_{\beta,11} + \varepsilon_{0,8} + \varepsilon_{\alpha+\beta,9} + \varepsilon_{0,5} + \varepsilon_{\alpha,6} + \varepsilon_{0,1} + \varepsilon_{0,2} \\
&[\mathcal{H}^{-6}A_{\alpha,11}] = \varepsilon_{\alpha,11} + \varepsilon_{0,9},\ [\mathcal{H}^{-4}A_{\alpha,11}] = q\varepsilon_{0,1} \\
&[\mathcal{H}^{-6}A_{\alpha+\beta,11}] = \varepsilon_{\alpha+\beta,11} + \varepsilon_{\alpha+\beta,9},\ [\mathcal{H}^{-4}A_{\alpha+\beta,11}] = q\varepsilon_{0,1} \\
&[\mathcal{H}^{-6}A_{\alpha\pm\frac{1}{2}\beta,11}] = \varepsilon_{\alpha\pm\frac{1}{2}\beta,11} + \varepsilon_{\alpha+\frac{1}{2}\beta,8} \\
&[\mathcal{H}^{-6}A_{\alpha+y\beta,11}] = \varepsilon_{\alpha+y\beta,11}, [\mathcal{H}^{-5}A_{\alpha+y\beta,11}] = \mathcal{J}_{11}(\alpha+y\beta)\varepsilon_{y\beta,5} \quad \text{if } 2y \notin \mathbb{Z} \\
&[\mathcal{H}^{-6}A_{x\alpha,11}] = \varepsilon_{x\alpha,11},\ [\mathcal{H}^{-5}A_{x\alpha,11}] = \mathcal{J}_{11}(x\alpha)\varepsilon_{(x+1)\alpha,6} \quad \text{if } x \notin \mathbb{Z} \\
&[\mathcal{H}^{-6}A_{x\alpha+\beta,11}] = \varepsilon_{x\alpha+\beta,11},\ [\mathcal{H}^{-5}A_{x\alpha+\beta,11}] = \mathcal{J}_{11}(x\alpha+\beta)\varepsilon_{x\alpha,6} \text{ if } x \notin \mathbb{Z}
\end{aligned}
$$

$$
[\mathcal{H}^{-6}A_{x\alpha+y\beta,11}]
\left.\begin{aligned}
&= \varepsilon_{x\alpha+y\beta,11} + \varepsilon_{y(2\alpha+\beta),8} && \text{if } y - \tfrac{1}{2}x \in \mathbb{Z} \\
&= \varepsilon_{x\alpha+y\beta,11} + \varepsilon_{y(\alpha+\beta),9} && \text{if } x - y \in \mathbb{Z} \\
&= \varepsilon_{x\alpha+y\beta,11} && \text{if } y - \tfrac{1}{2}x \notin \mathbb{Z},\ x - y \notin \mathbb{Z}
\end{aligned}\right\} \begin{array}{l} y \notin \mathbb{Z}, \\ x \notin 1 + 2\mathbb{Z}. \end{array}
$$

The $[\mathcal{H}^i A_{\xi,11}]$ which have not been written down are zero.

The condition $\langle \hat{\phi}_{\pi v_{11}}\xi, \check{\alpha}\rangle = \langle \hat{\phi}_{\pi v_{11}}\xi, \check{\beta}\rangle = 0$ of 7.1.2 is satisfied by $\xi = 0$, α, β, $\alpha+\beta$, $\pm\frac{1}{2}\beta$, $\alpha \pm \frac{1}{2}\beta$.

For the ξ for which a Jacobi sum appears in the coefficients we have $a_\alpha(\xi) \neq 0$, $a_\beta(\xi) \neq 0$, but the conclusion of Lemma 7.4.1 does not hold.

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