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TOTAL POSITIVITY IN PARTIAL FLAG MANIFOLDS

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ABSTRACT. The projective space of $\mathbf{R}^n$ has a natural open subset: the set of lines spanned by vectors with all coordinates > 0. Such a subset can be defined more generally for any partial flag manifold of a split semisimple real algebraic group. The main result of the paper is that this subset can be defined by algebraic equalities and inequalities.

Let G be a simply connected semisimple algebraic group over $\mathbf{C}$ with a fixed split $\mathbf{R}$-structure. We will often identify a real algebraic variety with its set of $\mathbf{R}$-rational points. This applies, in particular, to G and to the flag manifold $\mathcal{B}$ of G.

In [L2] we have defined (in terms of an "épinglage" of G) the open subsemigroup $G_{>0}$ of totally positive elements of G and a polyhedral open subset $\mathcal{B}_{>0}$ of $\mathcal{B}$ which in some sense plays the same role for $G_{>0}$ as $\mathcal{B}$ for G. More generally, for any partial flag manifold $\mathcal{P}^J$ of G one can define the totally positive part $\mathcal{P}^J_{>0}$. (See [L4] or 1.5.) For $J = \emptyset$ we have $\mathcal{P}^J = \mathcal{B}, \mathcal{P}^J_{>0} = \mathcal{B}_{>0}$.

In this paper we show that $\mathcal{P}^J_{>0}$ is a connected component of an explicitly defined open real algebraic submanifold of $\mathcal{P}^J$. We also show that, in the simply laced case, $\mathcal{P}^J_{>0}$ can be defined by algebraic inequalities involving canonical bases (see [L1]). These results confirm conjectures made in [L4]. In the special case where $J = \emptyset$, they reduce to known results from [L2].

1. PRELIMINARIES

1.1. Let $\mathfrak{g}$ be the Lie algebra of G over $\mathbf{R}$. The given épinglage of G can be specified by giving a set $(e_i, f_i)_{i\in I}$ of Chevalley generators of $\mathfrak{g}$. Then $h_i = [e_i, f_i]$ span the Lie algebra $\mathfrak{t}$ of an R-split maximal torus T of G.

For any $i \in I, a \in \mathbf{R}$, we set

$$x_i(a) = \exp(ae_i) \in G, \quad y_i(a) = \exp(af_i) \in G.$$

Let Y (resp. X) be the free abelian group of all homomorphisms of algebraic groups $\mathbf{R}^* \to T$ (resp. $T \to \mathbf{R}^*$). We write the operations in these groups as addition. Let $\langle,\rangle : Y \times X \to \mathbf{Z}$ be the standard pairing. For $i \in I$, there is a unique element $\check{\alpha}_i \in Y$ whose tangent map takes $1 \in \mathbf{R}$ to h_i. Let $\alpha_i \in X$ be defined by $tx_i(a)t^{-1} = x_i(\alpha_i(t)a)$ for all $a \in \mathbf{R}, t \in T$.

Let X^+ be the set of all $\lambda \in X$ such that $\langle \check{\alpha}_i, \lambda\rangle \in \mathbf{N}$ for all $i \in I$. For $i \in I$ let $\varpi_i \in X$ be defined by $\langle \check{\alpha}_i, \varpi_i\rangle = 1$ and $\langle \check{\alpha}_j, \varpi_i\rangle = 0$ for $j \neq i$. Then $\{\varpi_i | i \in I\}$ is

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a $\mathbf{Z}$-basis of X. For $\lambda \in X$ we set

$$\mathrm{supp}(\lambda) = \{i \in I | \langle \check{\alpha}_i, \lambda \rangle \neq 0\}.$$

If H is a subgroup of G and $g \in G$, we write ${}^gH$ instead of gHg^{-1}.

1.2. Let B^+ be the Borel subgroup of G that contains T and $x_i(a)$ for all $i \in I, a \in \mathbf{R}$. Let B^- be the Borel subgroup of G that contains T and $y_i(a)$ for all $i \in I, a \in \mathbf{R}$. Let U^+, U^- be the unipotent radicals of B^+, B^-. Let $\mathfrak{n}^-$ be the Lie algebra of U^-.

For any subset J of I, let P_J^+ be the subgroup of G generated by B^+ and by $\{y_j(a) | j \in J, a \in \mathbf{R}\}$. Note that $P_\emptyset^+ = B^+$. Let $\mathcal{P}^J$ be the set of subgroups of G of the form ${}^gP_J^+$ for some $g \in G$. We regard $\mathcal{P}^J$ naturally as a real algebraic manifold (a partial flag manifold). Note that $\mathcal{P}^\emptyset = \mathcal{B}$ (the full flag manifold). Let $\pi^J : \mathcal{B} \to \mathcal{P}^J$ be the canonical map, that is, $B \mapsto P$ where $P \in \mathcal{P}^J$ contains B.

1.3. Let $\mathcal{N}$ be the normalizer of T (in G). For $i \in I$, we set

$$\dot{s}_i = y_i(1)x_i(-1)y_i(1) \in \mathcal{N}.$$

Let $W = \mathcal{N}/T$ and let s_i be the image of $\dot{s}_i$ in W. Then W together with $(s_i)_{i \in I}$ is a Coxeter group. Let $l : W \to \mathbf{N}$ be the standard length function. For $w \in W$, let I_w be the set of all sequences $(i_1, i_2, \dots, i_p)$ in I such that $p = l(w)$ and $s_{i_1}s_{i_2}\dots s_{i_p} = w$. Let $\dot{w} = \dot{s}_{i_1}\dot{s}_{i_2}\dots\dot{s}_{i_p} \in \mathcal{N}$ where $(i_1, i_2, \dots, i_p) \in I_w$. This is independent of the choice of $(i_1, i_2, \dots, i_p)$.

If J is a subset of I, we denote by W^J the subgroup of W generated by $\{s_j | j \in J\}$. Let w_0^J be the unique element of maximal length of W^J. We shall write w_0 instead of $w_0^\emptyset$. For $w \in W$, let w^* be defined by $w^* = w_0 w w_0^{-1}$. For $i \in I$, let $i^* \in I$ be defined by $s_{i^*} = (s_i)^*$. Let $J^* = \{j^* | j \in J\}$. We have $(w_0^J)^* = w_0^{J^*}$.

We have a W-action on X in which $s_i \in W$ acts by $\lambda \mapsto \lambda - \langle \check{\alpha}_i, \lambda \rangle \alpha_i$.

For $w \in W$ we write ${}^wB^+, {}^wB^-$ instead of ${}^{\dot{w}}B^+, {}^{\dot{w}}B^-$.

For B, B' in $\mathcal{B}$ there is a unique $w \in W$ such that (B, B') is in the G-orbit on $\mathcal{B} \times \mathcal{B}$ (diagonal action) that contains $(B^+, {}^wB^+)$ (or equivalently $(B^-, {}^{w^*}B^-)$). We then write $\mathrm{pos}(B, B') = w$ and we regard pos as a function $\mathcal{B} \times \mathcal{B} \to W$.

1.4. Let $\mathfrak{U}^-$ be the enveloping algebra of $\mathfrak{n}^-$. We have $\mathfrak{U}^- = \bigoplus_\nu \mathfrak{U}_\nu^-$ where ν runs over $\mathbf{N}[I]$; here the subspaces $\mathfrak{U}_\nu^-$ are defined by $1 \in \mathfrak{U}_0^-$, $e_i \in \mathfrak{U}_i^-, \mathfrak{U}_\nu^- \mathfrak{U}_{\nu'}^- \subset \mathfrak{U}_{\nu+\nu'}^-$. Let $\hat{\mathfrak{U}}^- = \prod_\nu \mathfrak{U}_\nu^-$. We regard $\hat{\mathfrak{U}}^-$ naturally as a completion of $\mathfrak{U}^-$. The algebra structure on $\mathfrak{U}^-$ extends naturally (by continuity) to an algebra structure on $\hat{\mathfrak{U}}^-$.

There is a unique imbedding $U^- \subset \hat{\mathfrak{U}}^-$ compatible with multiplication such that $y_i(a) \in U^-$ corresponds to $\sum_{n \geq 0} (a^n/n!) f_i^n \in \hat{\mathfrak{U}}^-$ for any $i \in I, a \in \mathbf{R}$.

Let $\mathbf{B}$ be the canonical basis of $\mathfrak{U}^-$ (see [L1]). Since $\mathbf{B}$ is compatible with the decomposition $\mathfrak{U}^- = \bigoplus_\nu \mathfrak{U}_\nu^-$, any element of $\hat{\mathfrak{U}}^-$ can be written uniquely as an infinite sum

(a) $\sum_{b \in \mathbf{B}} c_b b$ where $c_b \in \mathbf{R}$.

In particular, any element $u \in U^-$ can be written uniquely as an infinite sum (a).

For any $i \in I$, we define $r_i : \mathbf{B} \to \mathbf{N}$ by

$$b \in \mathfrak{U}^- f_i^{r_i(b)}, b \notin \mathfrak{U}^- f_i^{r_i(b)+1}.$$

1.5. Let $(i_1, i_2, \dots, i_n) \in I_{w_0}$. Let

$$U^+_{>0} = \{x_{i_1}(a_1)x_{i_2}(a_2)\dots x_{i_n}(a_n)|a_1 \in \mathbf{R}_{>0}, \dots, a_n \in \mathbf{R}_{>0}\},$$
$$U^-_{>0} = \{y_{i_1}(a_1)y_{i_2}(a_2)\dots y_{i_n}(a_n)|a_1 \in \mathbf{R}_{>0}, \dots, a_n \in \mathbf{R}_{>0}\}.$$

Then $U^{\pm}_{>0}$ is an open subsemigroup (without 1) of $U^{\pm}$, independent of the choice of $(i_1, i_2, \dots, i_n) \in I_{w_0}$. (See [L2].) Let $U^{\pm}_{\geq 0}$ be the closure of $U^{\pm}_{>0}$ in $U^{\pm}$. Let

$$\mathcal{B}_{>0} = \{{}^u B^+ | u \in U^-_{>0}\} = \{{}^{u'} B^- | u' \in U^+_{>0}\}.$$

(The second equality is proved in [L2, 8.7].) This is an open subset of $\mathcal{B}$. Let $\mathcal{B}_{\geq 0}$ be the closure of $\mathcal{B}_{>0}$ in $\mathcal{B}$. For $J \subset I$ we set

$$\mathcal{P}^J_{>0} = \pi^J(\mathcal{B}_{>0}), \quad \mathcal{P}^J_{\geq 0} = \pi^J(\mathcal{B}_{\geq 0}).$$

Then $\mathcal{P}^J_{>0}$ is open in $\mathcal{P}^J$ and $\mathcal{P}^J_{\geq 0}$ is the closure of $\mathcal{P}^J_{>0}$ in $\mathcal{P}^J$.

1.6. For $\lambda \in X^+$, let Λ_λ be a simple algebraic G-module of finite dimension with a non-zero vector η_λ such that $x_i(a)\eta_\lambda = \eta_\lambda$ for all $i \in I, a \in \mathbf{R}$ and $t\eta_\lambda = \lambda(t)\eta_\lambda$ for all $t \in T$. For $B \in \mathcal{B}$ let L^λ_B be the unique B-stable line in Λ_λ. If $J \subset I$ and $\mathrm{supp}(\lambda) \subset I - J$, then for any $P \in \mathcal{P}^J$ there is a unique P-stable line L^λ_P in Λ_λ; we have $L^\lambda_P = L^\lambda_B$ for any $B \in \mathcal{B}$ such that $B \subset P$.

It is known that, if $\mathrm{supp}(\lambda) = I - J$, then $P \mapsto L^\lambda_P$ is an imbedding of $\mathcal{P}^J$ into the projective space of Λ_λ.

Let $f_i : \Lambda_\lambda \to \Lambda_\lambda$ be the linear map such that, for any $a \in \mathbf{R}$, $\exp(af_i) : \Lambda_\lambda \to \Lambda_\lambda$ is given by the action of $y_i(a)$ on the G-module Λ_λ. The maps $f_i : \Lambda_\lambda \to \Lambda_\lambda$ define a $\mathfrak{U}^-$-module structure on Λ_λ. It is clear that this extends naturally (by continuity) to a $\hat{\mathfrak{U}}^-$-module structure on Λ_λ.

If $u \in U^-$, the action of u on the G-module Λ_λ coincides with the action of u in the $\hat{\mathfrak{U}}^-$-module Λ_λ.

1.7. For $\lambda \in X^+$, we have $\Lambda_\lambda = \bigoplus_{\mu \in X} \Lambda^\mu_\lambda$ where

$$\Lambda^\mu_\lambda = \{x \in \Lambda_\lambda | tx = \mu(t)x \quad \forall t \in X\}$$

are the weight spaces.

Let $\mathbf{B}(\lambda)$ be the set of all $b \in \mathbf{B}$ such that $r_i(b) \leq \langle \check{\alpha}_i, \lambda \rangle$ for all $i \in I$. According to [L1, 14.4.11], the map $b \mapsto b\eta_\lambda$ is a bijection of $\mathbf{B}(\lambda)$ onto a basis ${}_\lambda\mathbf{B}$ of Λ_λ (called the canonical basis of Λ_λ). For $b \in \mathbf{B} - \mathbf{B}(\lambda)$ we have $b\eta_\lambda = 0$. The basis ${}_\lambda\mathbf{B}$ is compatible with the weight spaces of Λ_λ. Note that $\eta_\lambda \in {}_\lambda\mathbf{B}$.

The following statement is obtained by combining [L1, 28.1.4] and [L1, 39.1.2]. (Note that the action of $\dot{w}$ on Λ_λ coincides with the action of the operator $T'_{i,1}$ of [L1, 5.2.1], with $v = 1$ on Λ_λ.)

(a) *For any $w \in W$, the vector $\dot{w}(\eta_\lambda)$ is the unique element of ${}_\lambda\mathbf{B}$ which lies in $\Lambda^{w(\lambda)}_\lambda$.*

We set $\xi_\lambda = \dot{w}_0(\eta_\lambda) \in {}_\lambda\mathbf{B}$.

2. Parabolic subgroups of general type

2.1. We fix $J \subset I$. Let $P \in \mathcal{P}^J$. We can find a unique Borel subgroup $B' \subset P$ such that $\mathrm{pos}(B^-, B') = z$ for some $z \in W$ with $l(zw^J_0) = l(z) + l(w^J_0)$. Similarly, we can find a unique Borel subgroup $B'' \subset P$ such that $\mathrm{pos}(B^+, B'') = v$ for some $v \in W$ with $l(vw^J_0) = l(v) + l(w^J_0)$. We say that P is of *general type* if $v = z = w_0w^J_0$

and $\mathrm{pos}(B', B'') = w_0^J$. Let $\mathcal{P}_*^J$ be the set of all $P \in \mathcal{P}^J$ that are of general type. This is an open real algebraic submanifold of $\mathcal{P}^J$.

The following result has been conjectured in [L4, Sec.8]; in the case where $J = \emptyset$ it reduces to [L2, 8.14], while in the case where J has a single element, it reduces to [L4, 8.7].

Proposition 2.2. *$\mathcal{P}_{>0}^J$ is a connected component of $\mathcal{P}_*^J$.*

The proof will be given in 2.6.

Lemma 2.3. *Let $B \in \mathcal{B}_{>0}$. Then B is opposed to B^- and to B^+. (See* [L2, 8.7, 8.8]*.) Hence we can define $B', B'' \in \mathcal{B}$ by*

$$\mathrm{pos}(B^-, B') = w_0 w_0^J, \mathrm{pos}(B', B) = w_0^J, \mathrm{pos}(B^+, B'') = w_0 w_0^J, \mathrm{pos}(B'', B) = w_0^J.$$

The following hold:

(a) $\mathrm{pos}(B', B^+) = w_0$.
(b) $\mathrm{pos}(B'', B^-) = w_0$.
(c) $\mathrm{pos}(B', B'') = w_0^J$.
(d) $B' \in \mathcal{B}_{\geq 0}$.
(e) $B'' \in \mathcal{B}_{\geq 0}$.

We prove (a). Let $(i_1, i_2, \dots, i_n) \in I_{w_0}$ be such that $(i_1, i_2, \dots, i_k) \in I_{w_0 w_0^{J^*}}$ and $(i_{k+1}, i_{k+2}, \dots, i_n) \in I_{w_0^{J^*}}$.

We have $B = {}^u B^-$ where $u = x_{i_1}(a_1) x_{i_2}(a_2) \dots x_{i_n}(a_n)$ with $a_1, a_2, \dots, a_n$ in $\mathbf{R}_{>0}$. Let

$$u' = x_{i_1}(a_1) x_{i_2}(a_2) \dots x_{i_k}(a_k), \quad u'' = x_{i_{k+1}}(a_{k+1}) x_{i_{k+2}}(a_{k+2}) \dots x_{i_n}(a_n)$$

so that $u = u'u''$. Using [L2, 2.7(d)], we have

$$\mathrm{pos}({}^{u'} B^-, B) = \mathrm{pos}(B^-, {}^{u''} B^-) = \mathrm{pos}(B^-, {}^{w_0^{J^*}} B^-) = w_0^J,$$

$$\mathrm{pos}(B^-, {}^{u'} B^-) = \mathrm{pos}(B^-, {}^{w_0 w_0^{J^*}} B^-) = w_0 w_0^J.$$

Hence $B' = {}^{u'} B^-$. Since $u' \in B^+$, we have

$$\mathrm{pos}(B', B^+) = \mathrm{pos}({}^{u'} B^-, B^+) = \mathrm{pos}(B^-, B^+) = w_0.$$

This proves (a). The proof of (b) is entirely similar.

We prove (c). Since $\mathrm{pos}(B', B) = w_0^J, \mathrm{pos}(B'', B) = w_0^J$, we have $\mathrm{pos}(B', B'') = y$ for some $y \in W^J$. Since $\mathrm{pos}(B^-, B') = w_0 w_0^J$ and $l(w_0 w_0^J) + l(y) = l(w_0 w_0^J y)$, we have $\mathrm{pos}(B^-, B'') = w_0 w_0^J y$. Using (b), we have $w_0 w_0^J y = w_0$ hence $y = w_0^J$. This proves (c).

We prove (e). Let $B_1 \in \mathcal{B}$ be defined by $\mathrm{pos}(B^+, B_1) = w_0 w_0^J, \mathrm{pos}(B_1, B^-) = w_0^J$. Then $B_1 \in \mathcal{B}_{\geq 0}$ by [L2, 8.13].

For u as above we have

$$\mathrm{pos}(B^+, {}^u B_1) = \mathrm{pos}({}^u B^+, {}^u B_1) = w_0 w_0^J, \mathrm{pos}({}^u B_1, B) = \mathrm{pos}({}^u B_1, {}^u B^-) = w_0^J.$$

Hence $B'' = {}^u B_1$. Since $u \in U_{>0}^+$ and $B_1 \in \mathcal{B}_{\geq 0}$, we have ${}^u B_1 \in \mathcal{B}_{\geq 0}$. (See [L2, 8.12].) This proves (e). The proof of (d) is entirely similar.

Lemma 2.4. *Let $B', B'' \in \mathcal{B}$ be such that $\mathrm{pos}(B^-, B') = w_0 w_0^J, \mathrm{pos}(B', B'') = w_0^J$, $\mathrm{pos}(B^+, B'') = w_0 w_0^J$. Let $P = \pi^J(B') = \pi^J(B'')$. Assume that there exists $B \in \mathcal{B}_{\geq 0}$ such that $B \subset P$. Then*

(a) $B' \in \mathcal{B}_{\geq 0}$.
(b) $B'' \in \mathcal{B}_{\geq 0}$.

Assume first that $B \in \mathcal{B}_{>0}$. We have $\mathrm{pos}(B', B) = y$ for some $y \in W^J$. Since $\mathrm{pos}(B^-, B') = w_0 w_0^J$ and $l(w_0 w_0^J) + l(y) = l(w_0 w_0^J y)$, we have $\mathrm{pos}(B^-, B) = w_0 w_0^J y$. Since $B \in \mathcal{B}_{>0}$, we have $\mathrm{pos}(B^-, B) = w_0$. Hence $w_0 w_0^J y = w_0$ and $y = w_0^J$. Thus, B' is as in Lemma 2.3. Similarly, B'' is as in Lemma 2.3. Thus the desired result follows from Lemma 2.3(d),(e).

We now consider the general case. Let $\mathcal{X}$ be the open set of $\mathcal{B}$ consisting of all B_1 such that $\mathrm{pos}(B^-, B_1) \in w_0 w_0^J W^J$. For each $B_1 \in \mathcal{X}$ there is a unique $B_1' \in \mathcal{B}$ such that $\mathrm{pos}(B^-, B_1') = w_0 w_0^J, \mathrm{pos}(B_1', B_1) \in W^J$; moreover, the map $\pi : \mathcal{X} \to \mathcal{B}$ given by $\pi(B_1) = B_1'$ is continuous.

Now let $B \in \mathcal{B}_{\geq 0}$ be such that $B \subset P$. We clearly have $B \in \mathcal{X}$ and $\pi(B) = B'$. There exists a sequence $(B^n)_{n \geq 1}$ in $\mathcal{B}_{>0}$ such that $\lim_{n \to \infty} B^n = B$. Then for each n we have $B^n \in \mathcal{X}$ and by the first part of the argument, we have $\pi(B^n) \in \mathcal{B}_{\geq 0}$. Using the continuity of π and the fact that $\mathcal{B}_{\geq 0}$ is closed in $\mathcal{B}$, it follows that $\pi(B) \in \mathcal{B}_{\geq 0}$. Hence $B' \in \mathcal{B}_{\geq 0}$. Similarly, $B'' \in \mathcal{B}_{\geq 0}$. The lemma is proved.

Lemma 2.5. *Let $B' \in \mathcal{B}_{\geq 0}$ be such that* $\mathrm{pos}(B^-, B') = w_0 w_0^J, \mathrm{pos}(B', B^+) = w_0$. *Then there exists $B \in \mathcal{B}_{>0}$ such that* $\mathrm{pos}(B', B) = w_0^J$.

There is a unique $u \in U^+$ such that $B' = {}^u B^-$. Since $B' \in \mathcal{B}_{\geq 0}$, we must have $u \in U^+_{\geq 0}$, by an argument in the proof of [L2, 8.4]. By [L2, 2.8], there exists $w \in W$, $(i_1, i_2, \dots, i_k) \in I_w$ and $a_1, a_2, \dots, a_k$ in $\mathbf{R}_{>0}$ such that

$$u = x_{i_1}(a_1) x_{i_2}(a_2) \dots x_{i_k}(a_k).$$

By [L2, 2.7(d)], we have $u \in B^- \dot{w} B^-$, hence $\mathrm{pos}(B^-, {}^u B^-) = \mathrm{pos}(B^-, {}^w B^-) = w^*$. Now using $\mathrm{pos}(B^-, {}^u B^-) = w_0 w_0^J$, we see that $w^* = w_0 w_0^J$.

Let $(i_{k+1}, i_{k+2}, \dots, i_n) \in I_{w_0^{J^*}}$ and let $\tilde{u} = x_{i_{k+1}}(1) x_{i_{k+2}}(1) \dots x_{i_n}(1)$. Let $\tilde{B} = {}^{\tilde{u}} B^-$. Clearly, $\mathrm{pos}(B^-, \tilde{B}) = \mathrm{pos}(B^-, {}^{w_0^{J^*}} B^-) = w_0^J$. Since $u\tilde{u} \in U^+_{>0}$, we have ${}^{u\tilde{u}} B^- \in \mathcal{B}_{>0}$. Then $\mathrm{pos}(B', {}^{u\tilde{u}} B^-) = \mathrm{pos}({}^u B^-, {}^{u\tilde{u}} B^-) = \mathrm{pos}(B^-, {}^{\tilde{u}} B^-) = w_0^J$. The lemma is proved.

2.6. **Proof of Proposition 2.2.** The inclusion $\mathcal{P}^J_{>0} \subset \mathcal{P}^J_*$ follows from Lemma 2.3.

Since $\mathcal{P}^J_{>0}$ is an open connected subset of $\mathcal{P}^J_*$, it remains to show that $\mathcal{P}^J_{>0}$ is a closed subset of $\mathcal{P}^J_*$. Now $\mathcal{P}^J_{\geq 0} \cap \mathcal{P}^J_*$ is certainly a closed subset of $\mathcal{P}^J_*$. Hence it is enough to show that $\mathcal{P}^J_{\geq 0} \cap \mathcal{P}^J_* \subset \mathcal{P}^J_{>0}$. Let $P \in \mathcal{P}^J_{\geq 0} \cap \mathcal{P}^J_*$. We associate B', B'' to P as in 2.1. We can find $B \in \mathcal{B}_{\geq 0}$ such that $B \subset P$. The assumptions of Lemma 2.4 are satisfied. We deduce that $B' \in \mathcal{B}_{\geq 0}$. Thus, the assumptions of Lemma 2.5 are satisfied. We deduce that there exists $B_1 \in \mathcal{B}_{>0}$ such that $\mathrm{pos}(B', B_1) = w_0^J$. We have $B_1 \subset P$; hence, $P \in \mathcal{P}^J_{>0}$. Proposition 2.2 is proved.

Note added 2/25/1998. K. Rietsch has informed me that the lemmas in this section could also be proved by arguments in Lemmas 3 and 4 of her paper *An algebraic cell decomposition of the non-negative part of a flag variety*, preprint, http://xxx.lanl.gov/abs/alg-geom/9709035 (1997).

3. Canonical bases and $\mathcal{P}^J_{>0}, \mathcal{P}^J_{\geq 0}$

3.1. In this section we assume that G is of simply laced type. This assumption allows us to use the positivity properties of the canonical bases. We fix $J \subset I$.

If $\lambda \in X^+$ and $x \in \Lambda_\lambda$, we say that $x > 0$ (resp. $x \geq 0$) if all coordinates of x with respect to ${}_\lambda\mathbf{B}$ are > 0 (resp. ≥ 0). If L is a line in Λ_λ, we say that $L > 0$ (resp. $L \geq 0$) if for some $x \in L - \{0\}$ we have $x > 0$ (resp. $x \geq 0$).

Proposition 3.2. *Assume that $\lambda \in X^+$ is such that $\operatorname{supp}(\lambda) \subset I - J$.*

(a) *If $P \in \mathcal{P}^J_{>0}$, then $L^\lambda_P > 0$.*
(b) *If $P \in \mathcal{P}^J_{\geq 0}$, then $L^\lambda_P \geq 0$.*

We prove (a). We argue as in the proof of [L2, 8.17]. We choose $B \subset P$ such that $B \in \mathcal{B}_{>0}$. We have $L^\lambda_P = L^\lambda_B$. Now $L^\lambda_P = L^\lambda_B > 0$ by the first two lines in the proof of [L2, 8.17]. This proves (a).

We prove (b). If P is as in (b), then P is in the closure of $\mathcal{P}^J_{>0}$ in $\mathcal{P}^J$, hence the line L^λ_P is a limit of a sequence of lines $L^\lambda_{P'}$ with $P' \in \mathcal{P}^J_{>0}$ to which (a) is applicable so that $L^\lambda_{P'} \geq 0$. Hence $L^\lambda_P \geq 0$. The proposition is proved.

3.3. Let $\zeta = \sum_{j \in J} \varpi_j \in X$. For $i \in I - J$ we set $N_i = \langle \check{\alpha}_i, w_0^J(\zeta)\rangle$. It is easy to see that $N_i \geq 0$. We have the following partial converse to 3.2.

Theorem 3.4. *Assume that $\lambda \in X^+$ is such that $\langle \check{\alpha}_i, \lambda\rangle \geq N_i + 1$ for all $i \in I - J$ and $\langle \check{\alpha}_j, \lambda\rangle = 0$ for all $j \in J$. (In particular, $\operatorname{supp}(\lambda) = I - J$.) Let $P \in \mathcal{P}^J$.*

(a) *We have $P \in \mathcal{P}^J_{>0}$ if and only if $L^\lambda_P > 0$.*
(b) *We have $P \in \mathcal{P}^J_{\geq 0}$ if and only if $L^\lambda_P \geq 0$.*

The proof will be given in 3.12.

Lemma 3.5. *We fix $i \in I$. For any $\nu = \sum_{i' \in I} \nu_{i'} i'$ we set*

$$\phi(\nu) = \sum_{i' \in I; \langle \check{\alpha}_i, \alpha_{i'}\rangle = -1} \nu_{i'} \in \mathbf{N}.$$

If $k > \phi(\nu)$ and $x \in \mathfrak{U}^-_\nu$, we have $f_i^k x \in \mathfrak{U}^- f_i^{k-\phi(\nu)}$.

We argue by induction on $\sum_{i' \in I} \nu_{i'}$. If $\nu = 0$, the result is trivial. Assume now that $\nu \neq 0$. Then we may assume that $x = f_{i'} x'$ for some $i' \in I$ and some $x' \in \mathfrak{U}_{\nu'}$ where $\nu' = \nu - i' \in \mathbf{N}[I]$ and that the result is true for (x', ν') instead of (x, ν). If $\langle \check{\alpha}_i, \alpha_{i'}\rangle \neq -1$, then $\phi(\nu) = \phi(\nu')$ and

$$f_i^k x = f_i^k f_{i'} x' = f_{i'} f_i^k x' \in f_{i'} \mathfrak{U}^- f_i^{k-\phi(\nu')} \subset \mathfrak{U}^- f_i^{k-\phi(\nu)}.$$

If $\langle \check{\alpha}_i, \alpha_{i'}\rangle = -1$, then $\phi(\nu) = \phi(\nu') + 1$. Using Serre's relations we see that $f_i^k f_{i'} \in \mathfrak{U}^- f_i^{k-1}$ provided that $k \geq 2$. If $k > \phi(\nu)$, then $k \geq 2$ and $k - 1 > \phi(\nu')$; hence, by the induction hypothesis we have

$$f_i^k x = f_i^k f_{i'} x' \in \mathfrak{U}^- f_i^{k-1} x' \in \mathfrak{U}^- f_i^{k-1-\phi(\nu')} = \mathfrak{U}^- f_i^{k-\phi(\nu)}.$$

The lemma is proved.

Lemma 3.6. *Let b^J be the unique element of $\mathbf{B}$ such that $b^J \eta_\zeta = (\dot{w}_0^J)\eta_\zeta$. For any $i \in I - J$ and any $k > N_i$ we have $f_i^k b^J \in \mathfrak{U}^- f_i$.*

Let $\nu = \sum_{i' \in I} \nu_{i'} i'$ be such that $b^J \in \mathfrak{U}^-_\nu$. We have $\eta_\zeta \in \Lambda^\zeta_\zeta$ and $(\dot{w}_0^J)\eta_\zeta \in \Lambda_\zeta^{w_0^J(\zeta)}$. Hence $\zeta - \sum_{i' \in I} \nu_{i'} \alpha_{i'} = w_0^J(\zeta)$. In particular,

(a) $\nu_{i'} = 0$ for $i' \notin J$.

Since $\nu_i = 0$, the number $\phi(\nu)$ in 3.5 is given by

$$\phi(\nu) = -\langle \check{\alpha}_i, \sum_{i' \in I} \nu_{i'} \alpha_{i'} \rangle.$$

The last expression is equal to $\langle \check{\alpha}_i, w_0^J(\zeta) - \zeta \rangle = \langle \check{\alpha}_i, w_0^J(\zeta) \rangle = N_i$. Thus, $\phi(\nu) = N_i$ and the desired result follows from Lemma 3.5.

Lemma 3.7. *If $j \in J$, then $f_j b^J$ is a linear combination of elements $b' \in \mathbf{B}$ with $r_{j'}(b') \geq 2$ for some $j' \in J$.*

$f_j (w_0^J)\dot{} \eta_\zeta$ is equal to $\pm (w_0^J)\dot{} e_{\tilde{j}} \eta_\zeta$ for some $\tilde{j} \in J$. But $e_{\tilde{j}} \eta_\zeta = 0$. Thus, $f_j (w_0^J)\dot{} \eta_\zeta = 0$. Hence $f_j b^J \eta_\zeta = 0$. It follows that $f_j b^J$ is a linear combination with non-zero coefficients of elements $b' \in \mathbf{B}$ such that either $r_{j'}(b') \geq 2$ for some $j' \in J$ or $r_i(b') \geq 1$ for some $i \in I - J$. The second alternative does not occur since (by 3.6(a)) b^J and $f_j b^J$ belong to the subalgebra of $\mathfrak{U}^-$ generated by $\{f_{j'} | j' \in J\}$. The lemma follows.

Lemma 3.8. *Let $u \in U^-$. Assume that $u \eta_\lambda > 0$ in Λ_λ where λ is as in 3.4. Then $u (w_0^J)\dot{} \eta_\zeta \geq 0$ in Λ_ζ. Moreover, the projection of $u (w_0^J)\dot{} \eta_\zeta$ onto the $w_0(\zeta)$-weight space of Λ_ζ is $\neq 0$.*

Using the imbedding $U^- \subset \hat{\mathfrak{U}}^-$ (see 1.4), we can write $u = \sum_{b \in \mathbf{B}} c_b b$ (infinite sum) where $c_b \in \mathbf{R}$. Our assumption is that $c_b > 0$ for any $b \in \mathbf{B}$ such that

(a) $r_j(b) = 0$ for all $j \in J$ and $r_i(b) \leq N_i$ for all $i \in I - J$.

The product ub^J in $\hat{\mathfrak{U}}^-$ can be written as an infinite sum $ub^J = \sum_{b' \in \mathbf{B}} d_{b'} b'$ with $d_{b'} \in \mathbf{R}$. We must show that $d_{b'} \geq 0$ for any $b' \in \mathbf{B}$ such that

(b) $r_j(b') \leq 1$ for all $j \in J$ and $r_i(b') = 0$ for all $i \in I - J$.

We must also show that

(c) $d_{b'} \neq 0$ where $b' \in \mathbf{B}$ satisfies $b' \eta_\zeta = \xi_\zeta$.

We have $bb^J = \sum_{b' \in \mathbf{B}} g_{b,b'} b'$ (finite sum) where $g_{b,b'} \in \mathbf{Z}$. Hence

(d) $d_{b'} = \sum_{b \in \mathbf{B}^+} c_b g_{b,b'}$.

If for some $i \in I - J$, b satisfies $r_i(b) \geq N_i + 1$, then $bb^J \in \mathfrak{U}^- f_i^{N_i+1} b^J \in \mathfrak{U}^- f_i$ (see Lemma 3.6). Hence (by [L1, 14.3.2(b)]) bb^J is a linear combination of elements $b' \in \mathbf{B}^+$ with $r_i(b') \geq 1$. Thus, $g_{b,b'} = 0$ whenever b' is as in (b).

If for some $j \in J$, b satisfies $r_j(b) \geq 1$, then $bb^J \in \mathfrak{U}^- f_j b^J$; hence, is a linear combination of elements $b' \in \mathbf{B}$ with $r_{j'}(b') \geq 2$ for some $j' \in J$ (see Lemma 3.7). Thus, $g_{b,b'} = 0$ whenever b' is as in (b).

We see that, if b' is as in (b), then (c) can be rewritten as

(e) $d_{b'} = \sum_b c_b g_{b,b'}$ where b runs over the $b \in \mathbf{B}$ that satisfy (a).

For such b we have $c_b > 0$ by our assumption. Since $g_{b,b'} \geq 0$ for all b, b' (see [L1, 14.4.13]), it follows that $d_{b'} \geq 0$ for all b' as in (b).

It remains to verify (c). Since $b^J \eta_\zeta$ is a non-zero vector in a weight space of Λ_ζ, we must have $\xi_\zeta \in \mathfrak{U}^- b^J \eta_\zeta$. Hence we can find $b \in \mathbf{B}$ such that $bb^J \eta_\zeta$ is a non-zero multiple of ξ_ζ. Hence, if b' is as in (c), we have $g_{b,b'} \neq 0$. Since $g_{b,b'} \geq 0$, by the earlier argument, it follows that $g_{b,b'} > 0$. Then, in (e), the contribution of b to $d_{b'}$ is $c_b g_{b,b'} > 0$. Since the contribution of the other b in (e) is ≥ 0, it follows that $d_{b'} > 0$. The lemma is proved.

Lemma 3.9. (a) *Let $B \in \mathcal{B}$ be such that the projection of L_B^λ onto the $w_0(\lambda)$-weight space of Λ_λ is non-zero and the projection of L_B^ζ onto the $w_0(\zeta)$-weight space of Λ_ζ is non-zero. Then* $\mathrm{pos}(B^+, B) = w_0$.

(b) *Let $B \in \mathcal{B}$ be such that the projection of L_B^λ onto the λ-weight space of Λ_λ is non-zero and the projection of L_B^ζ onto the ζ-weight space of Λ_ζ is non-zero. Then* $\mathrm{pos}(B^-, B) = w_0$.

We prove (a). We have $B = {}^g B^+$ where $g = u\dot{w}$ with $u \in U^+, w \in W$.

If $\dot{w}\eta_\lambda \neq \xi_\lambda$, then $u\dot{w}\eta_\lambda \in L_B^\lambda$ is contained in the sum of weight spaces of Λ_λ corresponding to weights strictly higher than $w_0(\lambda)$, contradicting our assumption. Similarly, if $\dot{w}\eta_\zeta \neq \xi_\zeta$, then $u\dot{w}\eta_\zeta \in L_B^\zeta$ is contained in the sum of weight spaces of Λ_ζ corresponding to weights strictly higher than $w_0(\zeta)$, contradicting our assumption. Thus, we must have $\dot{w}\eta_\lambda = \xi_\lambda$ and $\dot{w}\eta_\zeta = \xi_\zeta$. Hence $\dot{w}_0^{-1}\dot{w}$ stabilizes both η_λ and η_ζ. Since $\langle \check{\alpha}_i, \lambda + \zeta \rangle > 0$ for all $i \in I$, we deduce that $w_0^{-1}w = 1$. Hence $\mathrm{pos}(B^+, B) = w_0$. This proves (a). The proof of (b) is entirely similar. The lemma is proved.

Lemma 3.10. (a) *Let $B \in \mathcal{B}$ be such that $L_B^\lambda > 0$ and $L_B^\zeta > 0$. Then $B \in \mathcal{B}_{>0}$.*

(b) *Let $B \in \mathcal{B}$ be such that $L_B^\lambda \geq 0$ and $L_B^\zeta \geq 0$. Then $B \in \mathcal{B}_{\geq 0}$.*

We prove (a). Let $\mathcal{Y}$ be the set of all $B \in \mathcal{B}$ such that $L_B^\lambda > 0$ and $L_B^\zeta > 0$. If $B \in \mathcal{Y}$, then B satisfies the assumptions of 3.9(a) and 3.9(b) hence B is opposed to both B^+ and B^-. From this point, the proof of (a) proceeds exactly as the proof of 8.17(a) in [L2].

Now (b) follows from (a) using the same argument as the one used in [L2] to deduce 8.17(b) from 8.17(a). The lemma is proved.

Lemma 3.11. *Let $B \in \mathcal{B}$ be such that* $\mathrm{pos}(B^-, B) = w_0 w_0^J$. *Assume that $L_B^\lambda > 0$ in Λ_λ where λ is as in 3.4. Then $B \in \mathcal{B}_{\geq 0}$ and* $\mathrm{pos}(B^+, B) = w_0$.

We have $B = {}^g B^-$ where $g = u\dot{w}_0^{-1}(w_0^{J^*})^{\cdot}$ and $u \in U^-$. The line $L_B^\lambda = gL_{B^-}^\lambda$ contains the vector

$$g\xi_\lambda = u\dot{w}_0^{-1}(w_0^{J^*})^{\cdot}\xi_\lambda = u(w_0^J)^{\cdot}\dot{w}_0^{-1}\xi_\lambda = u(w_0^J)^{\cdot}\eta_\lambda = u\eta_\lambda.$$

Since $L_B^\lambda > 0$, we have $u\eta_\lambda > 0$ or $-u\eta_\lambda > 0$. The second alternative cannot hold since $u\eta_\lambda$ is equal to η_λ plus a linear combination of elements in lower weight spaces. Thus,

$$u\eta_\lambda > 0.$$

The line $L_B^\zeta = gL_{B^-}^\zeta$ contains the vector

$$g\xi_\zeta = u\dot{w}_0^{-1}(w_0^{J^*})^{\cdot}\xi_\zeta = u(w_0^J)^{\cdot}\dot{w}_0^{-1}\xi_\zeta = u(w_0^J)^{\cdot}\eta_\zeta.$$

This vector is ≥ 0, by Lemma 3.8. Thus,

(a) $L_B^\zeta \geq 0$.

From Lemma 3.8 we see also that

(b) the projection of L_B^ζ onto the $w_0(\zeta)$-weight space of Λ_ζ is $\neq 0$.

This statement remains true if ζ is replaced by λ (since $L_B^\lambda > 0$).

Using (a) together with $L_B^\lambda > 0$, we see that the assumptions of Lemma 3.10(b) are satisfied; hence, $B \in \mathcal{B}_{\geq 0}$.

Using (b) and the analogous statement for λ, we see that the assumptions of Lemma 3.9(a) are satisfied; hence, $\mathrm{pos}(B^+, B) = w_0$. The lemma is proved.

3.12. **Proof of Theorem 3.4.** We prove 3.4(a). We attach B', B'' to P as in 2.1. Since the projections of L_P^λ onto the λ-weight space and the $w_0(\lambda)$-weight space are non-zero, it follows (as in the proof of 3.9) that $\mathrm{pos}(B^-, B') = w_0 w_0^J$ and $\mathrm{pos}(B^+, B'') = w_0 w_0^J$. By Lemma 3.9, we have $B' \in \mathcal{B}_{\geq 0}$ and $\mathrm{pos}(B^+, B') = w_0$. Now using Lemma 2.5, we see that there exists $B \in \mathcal{B}_{>0}$ such that $\mathrm{pos}(B', B) = w_0^J$. Then $B \subset P$ and $P \in \mathcal{P}^J_{>0}$. This proves 3.4(a).

Now 3.4(b) follows from 3.4(a) using the same argument as the one used in [L2] to deduce 8.17(b) from 8.17(a). Theorem 3.4 is proved.

Note added 2/25/1998. One can show (K. Rietsch) that the conclusion of Theorem 3.4 holds for any $\lambda \in X^+$ such that $\mathrm{supp}(\lambda) = I - J$.

3.13. The following is a reformulation of a result in [L3]. Let $u \in U^-$. Write $u = \sum_{b \in \mathbf{B}} c_b b$ (infinite sum) with $c_b \in \mathbf{R}$ as in 1.4(a). Then:

(a) We have $u \in U^-_{>0}$ if and only if $c_b > 0$ for all $b \in \mathbf{B}$.

(b) We have $u \in U^-_{\geq 0}$ if and only if $c_b \geq 0$ for all $b \in \mathbf{B}$.

This can be easily deduced from [L2, 5.4], using the definitions.

References

[L1] G. Lusztig, *Introduction to quantum groups*, Progr. in Math. 110, Birkhäuser, Boston, 1993. MR **94m:**17016

[L2] G. Lusztig, *Total positivity in reductive groups*, Lie Theory and Geometry: in honor of B. Kostant, Progr. in Math. 123, Birkhäuser, Boston, 1994, pp. 531-568. MR **96m:**20071

[L3] G. Lusztig, *Total positivity and canonical bases*, Algebraic groups and Lie groups (G. I. Lehrer, ed.), Cambridge Univ. Press, 1997, pp. 281-295.

[L4] G. Lusztig, *Introduction to total positivity*, Positivity in Lie theory: open problems, De Gruyter (to appear).

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