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FINITE QUATERNIONIC MATRIX GROUPS

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Abstract. Let $\mathcal{D}$ be a definite quaternion algebra such that its center has degree d over $\mathbb{Q}$. A subgroup G of $GL_n(\mathcal{D})$ is absolutely irreducible if the $\mathbb{Q}$-algebra spanned by the matrices in G is $\mathcal{D}^{n\times n}$. The finite absolutely irreducible subgroups of $GL_n(\mathcal{D})$ are classified for $nd \leq 10$ by constructing representatives of the conjugacy classes of the maximal finite ones. Methods to construct the groups and to deal with the quaternion algebras are developed. The investigation of the invariant rational lattices yields quaternionic structures for many interesting lattices.

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1. Introduction

The rational group algebra of any finite group is a semisimple algebra, hence a direct sum of matrix rings over division algebras. Whereas for any $n \in \mathbb{N}$ there

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exists a finite group G such that $\mathbb{Q}^{n\times n}$ is a direct summand of $\mathbb{Q}G$ (in other words $GL_n(\mathbb{Q})$ has a finite absolutely irreducible subgroup) this is not true for an arbitrary division algebra $\neq \mathbb{Q}$. Clearly the centers of the occurring division algebras are generated by the character values of the corresponding character of G and hence finite abelian extensions of $\mathbb{Q}$. If the center is real, then the involution on $\mathbb{Q}G$ defined by $g \mapsto g^{-1}$ for all $g \in G$ preserves the corresponding direct summand $\mathcal{D}^{n\times n}$ and therefore induces an involution on it. From this one deduces the theorem of Brauer and Speiser, which says that if the center $K := Z(\mathcal{D})$ is a totally real number field, then $\mathcal{D}$ is either K or a quaternion algebra over K.

For fixed n and fixed degree $d := [K : \mathbb{Q}]$ of the real field K over $\mathbb{Q}$ the unit group of the direct summand of $\mathbb{Q}G$ embeds into $GL_{dn}(\mathbb{Q})$ if $\mathcal{D} = K$ is abelian and into $GL_{4dn}(\mathbb{Q})$ if $\mathcal{D}$ is a quaternion algebra over K. The image of G under this homomorphism is a finite subgroup of $GL_{dn}(\mathbb{Q})$ resp. $GL_{4dn}(\mathbb{Q})$ with enveloping algebra $\mathcal{D}^{n\times n}$. Since for given $m \in \mathbb{N}$ the group $GL_m(\mathbb{Q})$ has finitely many conjugacy classes of finite subgroups. This shows that for fixed n and d, there are only finitely many possibilities for $\mathcal{D}$.

This paper deals with the case where $\mathcal{D}$ is a definite quaternion algebra over K. All $\mathcal{D}$ are determined for which $\mathcal{D}^{n\times n}$ is a direct summand of $\mathbb{Q}G$ for a finite group G; in other words, G has an absolutely irreducible representation into $GL_n(\mathcal{D})$, if $n \cdot [K : \mathbb{Q}] \leq 10$. We derive much finer information on the unit group $GL_n(\mathcal{D})$ of $\mathcal{D}^{n\times n}$ by determining all of its absolutely irreducible (cf. Definition 2.1) maximal finite (*a.i.m.f.*), subgroups. The classification results are given in the form of tables containing representatives for the primitive a.i.m.f. subgroups of $GL_n(\mathcal{D})$ and some information on the invariant lattices (cf. Table 6.3, Theorem 12.1, Table 12.7, and the Theorems 12.15, 12.17, 12.19, 13.1, 13.3, 13.5, 14.1, 14.14, 15.1, 15.3, 16.1, 17.1, 18.1, 19.1, and 20.1). The conjugacy classes of the a.i.m.f. subgroups are interrelated via common absolutely irreducible subgroups (cf. Definition 2.12) and we determine the resulting simplicial complexes for $n \leq 7$, $(n, d) \neq (4, 2)$.

Quaternionic matrix groups have already been studied by various authors. For instance in [Ami 55] the finite subgroups of $GL_1(\mathcal{D})$ are classified, [HaS 85] treats the quasisimple finite subgroups of $GL_2(\mathcal{D})$, and A.M. Cohen determines the finite quaternionic reflection groups in [Coh 80]. Quite a few of these reflection groups are a.i.m.f. subgroups (cf. Remark 5.2). The last article is somehow closer to the present paper, since Cohen describes the corresponding root systems. But none of the authors treat the subject from the arithmetic point of view so they do not look at the G-invariant lattices for the various maximal orders in $\mathcal{D}$.

Any subgroup G of $GL_n(\mathcal{D})$ may be considered as a subgroup of $GL_{4dn}(\mathbb{Q})$ via the regular representation of $\mathcal{D}$. The rational irreducible maximal finite subgroups of $GL_m(\mathbb{Q})$ are classified for $m \leq 31$ (cf. [PlN 95], [NeP 95], [Neb 95], [Neb 96], [Neb 96a]). As a consequence of this paper one obtains certain maximal finite subgroups of $GL_m(\mathbb{Q})$ which contain an a.i.m.f. group. So the results give a partial classification of the rational irreducible maximal finite subgroups of $GL_m(\mathbb{Q})$ for the new degrees m=32, 36, and 40 (see Appendix).

Finite subgroups of $GL_m(\mathbb{Q})$ act on Euclidean lattices. In particular the maximal finite groups are automorphism groups of distinguished lattices. The action of an a.i.m.f. subgroup G of $GL_n(\mathcal{D}) \leq GL_{4dn}(\mathbb{Q})$ on such a lattice L defines a Hermitian structure on L as a lattice over its endomorphism ring $End_G(L)$, which is an order in the commuting algebra $C_{\mathbb{Q}^{m\times m}}(G) \cong \mathcal{D}$. Only those lattices L where $End_G(L)$ is a maximal order in $\mathcal{D}$ are investigated. This yields Hermitian structures for many nice lattices. For example for the Leech lattice, the unique even unimodular lattice of dimension 24 without roots, we find, in addition to the two well known structures as Hermitian lattice described in [Tit 80], nine other structures over a maximal order of a definite quaternion algebra $\mathcal{D}$ preserved by an a.i.m.f. subgroup of $GL_n(\mathcal{D})$.

There is a mysterious connection between large class numbers of number fields and the existence of nice lattices. For example the Leech lattice occurs as an invariant lattice of the group $SL_2(23)$ due to the fact that the class number of $\mathbb{Z}[\frac{1+\sqrt{-23}}{2}]$ is 3. The occurence of large prime divisors of the determinants of invariant lattices of maximal finite groups also has an explanation using class groups. In $GL_{16}(\mathbb{Q})$ there are two irreducible maximal finite subgroups fixing no lattice of which the determinant only involves primes dividing the group order. The same phenomenon happens in $GL_{32}(\mathbb{Q})$ where there are at least four such primitive groups. These four groups contain a.i.m.f. subgroups of $GL_2(\mathcal{D})$ where $\mathcal{D}$ is the quaternion algebra with center $\mathbb{Q}[\sqrt{3}, \sqrt{5}]$ ramified only at the four infinite places. The narrow class group of the center of $\mathcal{D}$, which is isomorphic to the group of stable classes of left ideals of a maximal order in $\mathcal{D}$ (cf. [Rei 75]), is of order 2 and the norm of any integral generator has a prime divisor ≥ 11. Related to this the determinants of the integral lattices of the six maximal finite groups have prime divisors ≥ 11.

The article is organized as follows: Chapter 2 contains the fundamental definitions and generalizations of some important properties of rational matrix groups to matrix groups over quaternion algebras. The most important notion is the one of imprimitivity reducing the determination of a.i.m.f. groups to the one of primitive ones. In the next chapter known restrictions on the quaternion algebras that can be Morita equivalent to a direct summand of a group algebra of a finite group are used to introduce a notation for these quaternion algebras. Chapter 4 derives methods to compute representatives of the conjugacy classes of maximal orders in a definite quaternion algebra $\mathcal{D}$ and describes the results for the occurring $\mathcal{D}$ by expanding the mass formulas. Chapter 5 introduces some notation used for finite matrix groups. As an application of the classification of finite subgroups of $GL_2(\mathbb{C})$ and a theorem of Brauer the a.i.m.f. subgroups of $GL_1(\mathcal{D})$ for arbitrary quaternion division algebras $\mathcal{D}$ are determined in Chapter 6. The invariant lattices are only determined if the degree of the center of $\mathcal{D}$ over $\mathbb{Q}$ is ≤ 5 since the class number of $\mathcal{D}$ rapidly increases afterwards.

In this paper one does not know the quaternion algebra $\mathcal{D}$ in advance. So, before one can use arithmetic structures and calculate the a.i.m.f. group G as an automorphism group of a lattice, one has to build up a fairly large subgroup of G to get enough restrictions on $\mathcal{D}$. For this purpose, methods to conclude from the existence of a small normal subgroup N in G, the existence of a (much) bigger one, the generalized Bravais group of N (cf. Definition 7.1), are developed further. For some groups N this generalized Bravais group splits as a tensor factor and reduces the determination of G to the one of $C_G(N)$ which is a maximal finite subgroup in the unit group of the commuting algebra of N. If the enveloping algebra of N is a central simple K-algebra, the general case is not much harder.

The possible normal p-subgroups of primitive a.i.m.f. groups may be derived from a theorem of P. Hall and are essentially extraspecial groups. The investigation of the automorphism groups of the relevant p-groups leads to a determination of the generalized Bravais groups of these groups in Chapter 8. The next chapter contains a table of the occurring quasisimple groups to fix the notation for the irreducible characters and to give the information that is used from the classification of finite simple groups and their character tables in [CCNPW 85].

When building up the primitive maximal finite subgroups of $GL_n(\mathcal{D})$ by normal subgroups one needs not only the maximal finite matrix groups in smaller dimension but the maximal pairs of finite groups together with their normalizers. Some of these "building blocks" are classified in Chapter 10. In Chapter 11 four infinite series of a.i.m.f. groups which come from representations of the groups $SL_2(p)$ for primes p are presented. The last nine chapters deal with the determination of the a.i.m.f. groups of $GL_n(\mathcal{D})$ for definite quaternion algebras $\mathcal{D}$ with $n[Z(\mathcal{D}) : \mathbb{Q}] \leq 10$. There is one chapter for each dimension $n = 2, \ldots, 10$.

The general strategy is as follows: Let G be a primitive a.i.m.f. subgroup of $GL_n(\mathcal{D})$ for some $n \in \mathbb{N}$ and a d-dimensional $\mathbb{Q}$-division algebra $\mathcal{D}$. Then the order of G is bounded in terms of nd (cf. Proposition 2.16). One has only finitely many possibilities for the maximal nilpotent normal subgroup $P := \prod_{p||G|} O_p(G)$ (see Table 8.7). The centralizer $C_G(P)$ is an extension of a direct product of quasisimple groups $Q := C_G(P)^{(\infty)}$ by a subgroup of the outer automorphism group of Q. The possibilities for Q are deduced from the classification of finite simple groups and their character tables in [CCNPW 85] (cf. Table 9.1). So one has a finite list of possible normal subgroups $QP \trianglelefteq G$ with $G/QP \leq Out(QP)$. The methods developed in Chapter 7 now allow one to conclude the existence of a usually much larger normal subgroup $B := \mathcal{B}^{\circ}(QP)$ in G. The possible extensions G of B by outer automorphisms of QP not induced by elements of B can now be handled case by case.

In an appendix the invariants of the lattices of the new maximal finite subgroups of $GL_{32}(\mathbb{Q})$, $GL_{36}(\mathbb{Q})$, and $GL_{40}(\mathbb{Q})$ are displayed in the form of tables.

The computer calculations were mainly done by stand alone C-programs (for solving systems of linear equations, calculating sublattices invariant under an order in $\mathbb{Q}^{n\times n}$ with the algorithm described in [PlH 84], calculating automorphism groups of positive definite lattices as described in [PlS 97], ...) developed at the Lehrstuhl B für Mathematik of the RWTH Aachen (Germany). These algorithms are or will be also available in MAGMA (cf. [MAGMA]). The investigation of the isomorphism type of the matrix groups was done with the help of the two group theory packages GAP (cf. [GAP 94]) and MAGMA. Invariant Hermitian forms for the primitive a.i.m.f. subgroups of $GL_n(\mathcal{D})$ can be obtained from the author's home page, via http://www-math.math.rwth-aachen.de/~LBFM/.

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2. Definitions and first properties

In this paper maximal finite subgroups of the unit group $GL_n(\mathcal{D})$ of $\mathcal{D}^{n\times n}$ for totally definite quaternion algebras $\mathcal{D}$ over totally real number fields will be determined.

Since the representation theoretical methods generalize to arbitrary division algebras $\mathcal{D}$, let $\mathcal{D}$ be a division algebra whose center K is a finite extension of $\mathbb{Q}$.

The module $V := \mathcal{D}^{1\times n}$ is a right module for $\mathcal{D}^{n\times n}$. Its endomorphisms are given by left multiplication with elements of $\mathcal{D}$. For computations it is convenient also to let the endomorphisms act from the right. Then $End_{\mathcal{D}^{n\times n}}(V) \cong \mathcal{D}^{op}$ which we identify with $\mathcal{D}$ in the case of quaternion algebras.

The following definition can also be found in [ShW 86].

Definition 2.1. Let G be a finite group and $\Delta : G \to GL_n(\mathcal{D})$ be a representation of G.

(i) Let L be a subring of K. The *enveloping L-algebra* $\overline{L\Delta(G)}$ is defined as
$$\overline{L\Delta(G)} := \{ \sum_{g\in\Delta(G)} l_g g \mid l_g \in L\} \subseteq \mathcal{D}^{n\times n}.$$

(ii) Δ is called *absolutely irreducible* if the enveloping $\mathbb{Q}$-algebra $\overline{\Delta(G)} := \overline{\mathbb{Q}\Delta(G)}$ of $\Delta(G)$ is $\mathcal{D}^{n\times n}$.

(iii) Δ is called *centrally irreducible* if the enveloping K-algebra $\overline{K\Delta(G)}$ is $\mathcal{D}^{n\times n}$.

(iv) Δ is called *irreducible* if the commuting algebra $C_{\mathcal{D}^{n\times n}}(\Delta(G))$ is a division algebra.

(v) A subgroup $G \leq GL_n(\mathcal{D})$ is called *irreducible* (resp. *centrally irreducible*, *absolutely irreducible*), if its natural representation $id : G \to GL_n(\mathcal{D})$ is irreducible (resp. centrally irreducible, absolutely irreducible).

Being only interested in those groups G, to which the quaternion algebra $\mathcal{D}$ is really attached, only the absolutely irreducible maximal finite (*a.i.m.f.*) subgroups of $GL_n(\mathcal{D})$ will be determined. The irreducible maximal finite subgroups G of $GL_n(\mathcal{D})$ are absolutely irreducible in their enveloping $\mathbb{Q}$-algebra $\overline{G}$, which is a matrix ring over some division algebra $\overline{G} \cong \mathcal{D}'^{m\times m}$ with $m^2 dim_{\mathbb{Q}}(\mathcal{D}')$ dividing $n^2 dim_{\mathbb{Q}}(\mathcal{D})$. The reducible maximal finite subgroups of $GL_n(\mathcal{D})$ can be built up from the irreducible maximal finite subgroups of $GL_l(\mathcal{D})$ for $l < m$.

As in the case $\mathcal{D} = \mathbb{Q}$, the notion of primitivity gives an important reduction in the determination of the maximal finite subgroups.

Definition 2.2. (cf. [Lor 71] (1.3)) Let $G \leq GL_n(\mathcal{D})$ be an irreducible finite group. Consider $V := \mathcal{D}^{1\times n}$ as $\mathcal{D}$-G-bimodule. G is called *imprimitive*, if there exists a decomposition $V = V_1 \oplus \ldots \oplus V_s$ of V as a nontrivial direct sum of $\mathcal{D}$-left modules such that G permutes the V_i (i.e. for all $g \in G$, for all $1 \leq i \leq s, \exists 1 \leq j \leq s$ such that $V_i g \subseteq V_j$). If G is not imprimitive, G is called *primitive*.

If G is an imprimitive group and $V = V_1 \oplus \ldots \oplus V_s$ a nontrivial G-stable decomposition of V as in the definition, the natural representation of G is induced up from the natural representation Δ_1 of the subgroup $U := Stab_G(V_1)$ on V_1. If G is absolutely irreducible, then $\Delta_1 : U \to End_{\mathcal{D}}(V_1)$ is also absolutely irreducible. Especially the imprimitive a.i.m.f. subgroups of $GL_n(\mathcal{D})$, being maximal finite, are wreath products of primitive a.i.m.f. subgroups of $GL_d(\mathcal{D})$ with the full symmetric group $S_{\frac{n}{d}}$ of degree $\frac{n}{d}$ for divisors d of n.

As for $\mathcal{D} = \mathbb{Q}$, the primitive groups have the following frequently used property:

Remark 2.3. Let $G \leq GL_n(\mathcal{D})$ be a primitive finite group and $N \trianglelefteq G$ be a normal subgroup. Then the enveloping K-algebra $\overline{KN} \subseteq \mathcal{D}^{n\times n}$ is simple.

Proof. Assume that $\overline{KN}$ is not simple. Since $\overline{KN}$ is semisimple there exists a decomposition $1 = e_1 + \ldots + e_s$ of $1 \in \overline{KN}$ into centrally primitive idempotents $e_i \in \overline{KN}$ $(1 \leq i \leq s)$. The group G acts by conjugation on N, hence on $\overline{KN}$ and therefore on the set of centrally primitive idempotents in $\overline{KN}$. So the decomposition $V = Ve_1 \oplus \ldots \oplus Ve_s$ is stable under the action of G. By primitivity of G this implies $s = 1$. $\square$

Corollary 2.4. *Let $G \leq GL_n(\mathcal{D})$ be a primitive finite subgroup and $N \trianglelefteq G$ be a normal subgroup.*

(i) *If N is a p-group, then $(p-1) \cdot p^{\alpha}$ divides $dim_{\mathbb{Q}}(K) \cdot n$ for some $\alpha \geq 0$.*
(ii) *If N is abelian, then N is cyclic.*

Notation 2.5. Let G be a primitive subgroup of $GL_n(\mathcal{D})$ and $N \trianglelefteq G$ a normal subgroup of G. By Remark 2.3, the restriction of the natural character of G to N is a multiple of a K-irreducible character χ, where $K := Z(\mathcal{D})$. By the theorem of Skolem-Noether the knowledge of χ is sufficient to identify the conjugacy class of N in $GL_n(\mathcal{D})$. This will be expressed by the phrase G *contains* N *with character* χ.

Invariant Hermitian lattices. For the rest of this chapter assume that $\mathcal{D}$ is a totally definite quaternion algebra, and let $^* : \mathcal{D} \to \mathcal{D}$ be the canonical involution of $\mathcal{D}$ such that $xx^* \in K = Z(\mathcal{D})$ for all $x \in \mathcal{D}$ and such that * induces the identity on the center K. (cf. [Scha 85, (8.11.2)]) We extend * to $\mathcal{D}^{n\times n}$ by applying the involution to the entries of the matrices. Then $g \mapsto (g^*)^t$ where g^t denotes the transposed matrix of g is an involution on the algebra $\mathcal{D}^{n\times n}$.

Then the maximal finite subgroups of $GL_n(\mathcal{D})$ can be described as full automorphism groups of totally positive definite Hermitian lattices as follows:

Definition and Lemma 2.6. *Let G be a finite subgroup of $GL_n(\mathcal{D})$ and $V := \mathcal{D}^{1\times n}$ the natural G-right-module and let $\mathfrak{M}$ be an order in $\mathcal{D} = End_{\mathcal{D}^{n\times n}}(V)$.*

(i) *An $\mathfrak{M}$*-lattice *$L \leq V$ is a finitely generated projective $\mathfrak{M}$-left module with $\mathbb{Q}L = V$.*
(ii) *The set*
$$\mathcal{Z}_{\mathfrak{M}}(G) := \{L \leq V \mid L \text{ is a } \mathfrak{M}\text{-lattice and } Lg \subseteq L\}$$
of G-invariant $\mathfrak{M}$-lattices in V is nonempty.
(iii) *The K vector space*
$$\mathcal{F}(G) := \{F \in \mathcal{D}^{n\times n} \mid F^t = F^* \quad \text{and} \quad gF(g^*)^t = F \quad \text{for all } g \in G\}$$
of G-invariant Hermitian forms contains a totally positive definite form, i.e.
$$\mathcal{F}_{>0}(G) := \{F \in \mathcal{F}(G) \mid \epsilon(F) \text{is positive definite}$$
$$\text{for all embeddings } \epsilon : K \hookrightarrow \mathbb{R}\} \neq \emptyset.$$
(iv) *Let L be an $\mathfrak{M}$-lattice in V and $F \in \mathcal{D}^{n\times n}$ a totally positive definite Hermitian form. The* automorphism group
$$Aut(L, F) := \{g \in GL_n(\mathcal{D}) \mid Lg \subseteq L \text{ and } gF(g^*)^t = F\}$$
of L with respect to F is a finite group.
(v) *The a.i.m.f supergroups of G are of the form $Aut(L, F)$ for some $L \in \mathcal{Z}_{\mathfrak{M}}(G)$ and $F \in \mathcal{F}_{>0}(G)$.*

Proof. (ii) Let $(b_1, \ldots , b_n)$ be a $\mathcal{D}$-basis of V. Then
$$L := \{\sum_{i=1}^{n} m_i b_i g_i \mid m_i \in \mathfrak{M}, g_i \in G\} \in \mathcal{Z}_{\mathfrak{M}}(G).$$

(iii) Choose any totally positive definite Hermitian form $F \in \mathcal{D}^{n\times n}$. Define $F_0 := \sum_{g\in G} gF(g^*)^t$. Then $F_0 \in \mathcal{F}_{>0}(G)$ is totally positive definite.

(iv) Fix an embedding $\epsilon : K \to \mathbb{R}$ and let $m := max\{\epsilon(vF(v^*)^t) \mid v \in S\}$, where S is a finite subset of L generating L. Then the set $\{x \in L \mid \epsilon(xF(x^*)^t) \leq m\}$ is a finite set containing the images of the elements of S under the automorphisms $g \in Aut(L, F)$. Since g is uniquely determined by these images, one has only finitely many possibilities for g.

(v) Follows from (ii)-(iv).

□

In view of 2.6 (v), one may calculate all a.i.m.f. supergroups of a finite subgroup $G \leq GL_n(\mathcal{D})$ as automorphism groups of G-invariant lattices.

The centralizer $C_{GL_n(\mathcal{D})}(G)$ of G in $GL_n(\mathcal{D})$ acts on $\mathcal{Z}_{\mathfrak{M}}(G)$. Two lattices are called isomorphic if they lie in the same orbit under this action. Clearly a system of representatives of isomorphism classes of lattices in $\mathcal{Z}_{\mathfrak{M}}(G)$ suffices to get all a.i.m.f. supergroups. So the theorem of Jordan and Zassenhaus says that one may always find a finite critical set of invariant lattices in the sense of the following definition.

Definition 2.7. A set of lattices $S \subseteq \mathcal{Z}_{\mathfrak{M}}(G)$ is called *critical* (resp. *normal critical*) if for all finite groups H with $G \leq H \leq GL_n(\mathcal{D})$ (resp. $G \trianglelefteq H \leq GL_n(\mathcal{D})$) there is an $L \in S$ and some $F \in \mathcal{F}_{>0}(G)$ such that $H \leq Aut(L, F)$. If $S = \{L\}$ consists of one lattice, L itself is called (*normal*) *critical*.

Definition 2.8. Let $L \in \mathcal{Z}_{\mathfrak{M}}(G)$ and $F \in \mathcal{F}_{>0}(G)$.

(i) The *Hermitian dual lattice* is defined as

$$L^* := \{v \in \mathcal{D}^{1\times n} \mid vFl^t \in \mathfrak{M} \text{ for all } l \in L\}.$$

(ii) If L is integral (i.e. $L^* \supseteq L$), then its Hermitian determinant is $det(L) := |L^*/L|$.

Remark 2.9. For all $L \in \mathcal{Z}_{\mathfrak{M}}(G)$ and $F \in \mathcal{F}_{>0}(G)$ the Hermitian dual lattice L^* is also in $\mathcal{Z}_{\mathfrak{M}}(G)$ (cf. [Neb 98, Lemma 1.1]).

The rational maximal finite supergroups. Recall that $\mathcal{D}$ is a totally definite quaternion algebra with center K and $d := [K : \mathbb{Q}]$. Via the regular representation of $\mathcal{D}$, one may embed $GL_n(\mathcal{D})$ into $GL_{4dn}(\mathbb{Q})$. Therefore it makes sense to ask for the rational maximal finite supergroups $G \leq H \leq GL_{4dn}(\mathbb{Q})$ of an a.i.m.f. subgroup G of $GL_n(\mathcal{D})$. The relation of the lattices is given in the following

Definition 2.10 (cf. [Scha 85, p. 348])**.** Let $\mathfrak{M}$ be an order in $\mathcal{D}$ and L an $\mathfrak{M}$-lattice. Let $F \in \mathcal{D}^{n\times n}$ be a totally positive definite Hermitian form. The corresponding Euclidean $\mathbb{Z}$-lattice L is the set L (considered as $\mathbb{Z}$-module) together with the *trace form* $tr(F) : (v, w) \mapsto tr(vF(w^*)^t)$ where tr is the reduced trace of $\mathcal{D}$ over $\mathbb{Q}$.

Remark 2.11. (i) Since $tr(x) = tr(x^*)$, the trace form $tr(F)$ of a Hermitian form F is a symmetric $\mathbb{Q}$-bilinear form. If F is totally positive definite, then $tr(F)$ is positive definite.

(ii) Let $G \leq GL_n(\mathcal{D})$ be absolutely irreducible. Since $\mathcal{D}$ is totally definite the $\mathbb{Q}$-vector space of G-invariant quadratic forms on $\mathbb{Q}^{4dn}$ is $\{tr(F) \mid F \in \mathcal{F}(G)\} = \{tr(aF_0) \mid a \in K\}$ for any $F_0 \in \mathcal{F}_{>0}(G)$. As in Remark 2.6 one gets that the rational maximal finite supergroups of G are of the form $Aut(L, tr(F)) := \{g \in GL_{4dn}(\mathbb{Q}) \mid Lg = L,\ gtr(F)g^t = tr(F)\}$ for some $F \in \mathcal{F}_{>0}(G)$ and $L \in \mathcal{Z}_{\mathfrak{M}}(G)$, where $\mathfrak{M} := End_G(L)$ is an order in $\mathcal{D}$.

(iii) As the G-invariant Hermitian forms give rise to embeddings of G into an orthogonal group over K, one may also consider the invariant skew-Hermitian forms to get embeddings of G into the symplectic group over K. For an a.i.m.f. subgroup G of $GL_n(\mathcal{D})$ the K vector space $\mathcal{F}(G)$ is of dimension one, whereas the K vector space of invariant skew-Hermitian forms is $\mathcal{D}_0\mathcal{F}(G)$, where $\mathcal{D}_0$ denotes the quaternions of trace 0 in $\mathcal{D}$, is of dimension 3 over K. Therefore the embedding of G into the symplectic group is not unique.

Common absolutely irreducible subgroups. Having found the a.i.m.f. subgroups of $GL_n(\mathcal{D})$, one may interrelate them via common absolutely irreducible subgroups in the sense of the following definition.

Definition 2.12. The simplicial complex $M_n^{irr}(\mathcal{D})$ of a.i.m.f. subgroups of $GL_n(\mathcal{D})$ has the $GL_n(\mathcal{D})$-conjugacy classes of a.i.m.f. groups of degree n as vertices. The $s+1$ vertices $P_0, \dots, P_s$ form an s-simplex, if there exist representatives $G_i \in P_i$ and an absolutely irreducible subgroup $H \leq GL_n(\mathcal{D})$ with $H \leq G_i$ for $i = 0, \dots, s$.

This definition is a straightforward generalization of the definition of $M_n^{irr}(\mathbb{Q})$ for rational irreducible matrix groups (cf. [Ple 91]). One might think of generalizations of this definition to common uniform subgroups U (i.e. $dim_K(\mathcal{F}(U)) = 1$) of a.i.m.f. groups in $GL_n(\mathcal{D})$ and $GL_n(\mathcal{D}')$ for (different) quaternion algebras $\mathcal{D}$ and $\mathcal{D}'$ with the same center K. A second possibility to interrelate a.i.m.f. subgroups of $GL_n(\mathcal{D})$ for different quaternion algebras $\mathcal{D}$ (or even simplices in $M_n^{irr}(\mathcal{D})$) is described in Remark 12.11.

In this paper we determine the simplicial complexes $M_n^{irr}(\mathcal{D})$ for $n \leq 7$ with $(n, [Z(\mathcal{D}) : \mathbb{Q}]) \neq (4, 2)$.

As for the simplicial complexes of rational matrix groups the $\mathcal{D}$-isometry class of the invariant Hermitian forms distinguishes the different components of $M_n^{irr}(\mathcal{D})$. By [Scha 85, Theorem 10.1.7] two Hermitian forms are isometric if and only if their trace forms (cf. Definition 2.10) are isometric quadratic forms over the center $K = Z(\mathcal{D})$. Hence in our situation all totally positive definite Hermitian forms of a given dimension are isometric.

To build up the a.i.m.f. groups, but also to find their common absolutely irreducible subgroups, it is helpful to know divisors of the group order.

Lemma 2.13. *Let $U \leq GL_n(\mathcal{D})$ be an absolutely irreducible subgroup, $\mathfrak{M}$ a maximal order in $\mathcal{D}$ and $L \in \mathcal{Z}_{\mathfrak{M}}(U)$. Let $p \in \mathbb{Z}$ be a prime.*

(i) *If p ramifies in $\mathcal{D}$, then p divides the order of U.*
(ii) *Let F be a U-invariant Hermitian form on L. If p divides $|\mathcal{A}L^*/L|$ for all fractional ideals $\mathcal{A}$ of K such that $\mathcal{A}L^* \supseteq L$, then p divides the order of U.*

Proof. (i) If p ramifies in $\mathcal{D}$, then p divides the discriminant of the maximal orders in $\mathcal{D}^{n\times n}$. The order $\overline{\mathbb{Z}U}$ is contained in some maximal order and hence its discriminant is also divisible by p. Since $\overline{\mathbb{Z}U}$ is an epimorphic image of the group ring $\mathbb{Z}U$, p divides the order of U.

(ii) Since $\mathfrak{M}$ is a maximal order, by [Neb 98, Lemma 1.1] the Hermitian dual lattice L^* is also a $\mathfrak{M} - U$-lattice. Let R be the ring of integers in K and $\wp$ be a prime ideal containing p. Assume that $p \not| |U|$. Then by (i) $\wp$ does not ramify in $\mathcal{D}$ and therefore the two-sided ideals of $R_\wp \otimes_R \mathfrak{M}$ are of the form $R_\wp \otimes_R \mathcal{A}\mathfrak{M}$ for fractional ideals $\mathcal{A}$ of R. If for all fractional ideals $\mathcal{A}$ of R the completion $R_\wp \otimes_R \mathcal{A}L^* \neq R_\wp \otimes_R L$, then the $R_\wp$-order $R_\wp \otimes_R \mathfrak{M}U$ is not a maximal order and therefore p divides $|U|$ which is a contradiction. $\square$

The next lemma may be proved similarly to Lemma (II.7) of [NeP 95]:

Lemma 2.14. *Let $N \trianglelefteq G \leq GL_n(\mathcal{D})$ be a normal subgroup of G with $|G/N| =: s$. Then $s \cdot dim_{\mathbb{Q}}(\overline{\mathbb{Q}N}) \geq dim_{\mathbb{Q}}\overline{\mathbb{Q}G}$.*

Proof. If $G = \bigcup_{i=1}^s Ng_i$ and $(b_1, \ldots, b_m)$ is a $\mathbb{Q}$-basis of $\overline{\mathbb{Q}N}$, then the elements $b_j g_i$ $(1 \leq i \leq s,\ 1 \leq j \leq m)$ generate $\overline{\mathbb{Q}G}$. $\square$

For normal subgroups of index 2 in an absolutely irreducible subgroup G of $GL_n(\mathcal{D})$ one may now strengthen Lemma 2.13:

Lemma 2.15. *Let $N \trianglelefteq G$ be a normal subgroup of index two in an absolutely irreducible subgroup G of $GL_n(\mathcal{D})$. If p is a prime ramifying in $\mathcal{D}$, then p divides the discriminant of the enveloping $\mathbb{Z}$-order $\overline{\mathbb{Z}N}$ of N.*

Proof. If N is already absolutely irreducible, the lemma follows from Lemma 2.13. So assume that N is not absolutely irreducible. Let $g \in G - N$. Then $\overline{\mathbb{Z}G}$ contains the order $\mathcal{O} := \overline{\mathbb{Z}N} \oplus \overline{\mathbb{Z}N}g$ of finite index. The discriminant of $\mathcal{O}$ is $disc(\overline{\mathbb{Z}N})^2$ the square of the discriminant of $\overline{\mathbb{Z}N}$. $\square$

By the formula in [Schu 05] the prime divisors of the order of a finite group G may be bounded in terms of the character degree and the (degree of) the character field of an irreducible faithful character of G.

Proposition 2.16. *Let χ be a faithful irreducible rational character of a finite group G with $\chi(1) = n$. Then the order of G divides*

$$M_n := \prod_{p \leq n+1} p^{\lfloor n/(p-1) \rfloor + \lfloor n/(p(p-1)) \rfloor + \lfloor n/(p^2(p-1)) \rfloor + \ldots}$$

where the product runs over all primes $p \leq n+1$.

In view of Lemma 2.13 (i), one now has only finitely many candidates for quaternion algebras $\mathcal{D}$ such that $GL_n(\mathcal{D})$ has a finite absolutely irreducible subgroup, if one bounds n and the degree of the center of $\mathcal{D}$ over $\mathbb{Q}$.

But there remain too many candidates for $\mathcal{D}$ to be dealt with separately. So the main strategy to find the primitive maximal finite absolutely irreducible subgroups $G \leq GL_n(\mathcal{D})$ will be to build them up using normal subgroups.

The following lemma can be shown to hold as in [Neb 96, Lemma 1.13]:

Lemma 2.17. *Let $N \trianglelefteq G \leq GL_n(\mathcal{D})$ be a normal subgroup of G with $|G/N| = 2$. Assume that $\overline{\mathbb{Q}N}$ and $\overline{\mathbb{Q}G} = \mathcal{D}^{n \times n}$ are simple algebras with centers K resp. K^+, where K is complex and K^+ is the maximal totally real subfield of K. Then the isoclinic group is not a subgroup of $GL_n(\mathcal{D})$.*

Immediately from the theorem of Brauer and Witt one gets the following lemma:

Lemma 2.18. *Let $U \leq G$, χ an irreducible character of G and χ_1 an irreducible constituent of $\chi_{|U}$. Assume that the character fields of χ and χ_1 are equal. If $(\chi_{|U}, \chi_1)$ is odd, then the Schur index of χ is 2 at exactly those primes where χ_1 has Schur index 2.*

3. The Schur subgroup of the Brauer group

Standard references for this section are [Yam 74] and [Rei 75].

Let K be a number field and $Br(K)$ denote the Brauer group of K. If G is a finite group, then by Maschke's theorem, the algebra KG is a finite dimensional semisimple K-algebra, hence $KG = \bigoplus A_i$ decomposes into a direct sum of simple K-algebras $A_i \cong \mathcal{D}_i^{n_i \times n_i}$, which are full matrix rings over K-division algebras $\mathcal{D}_i$. The *Schur subgroup* $S(K)$ of $Br(K)$ consists of the classes $[\mathcal{D}]$ where $\mathcal{D}$ is a central simple K-division algebra for which there is an $n \in \mathbb{N}$ and a finite group G such that $\mathcal{D}^{n\times n}$ is a ring direct summand of KG.

If $\mathcal{D}^{n\times n}$ is a ring direct summand of the rational group algebra $\mathbb{Q}G$ and K contains the center $Z(\mathcal{D})$, then $[K \otimes_{Z(\mathcal{D})} \mathcal{D}] \in S(K)$, so the algebra $\mathbb{Q}G$ contains all information about the Schur subgroups of the Brauer group of algebraic number fields.

Definition 3.1. Let $\mathcal{D}$ be a $\mathbb{Q}$-division algebra. Define

$$\mu(\mathcal{D}) := min\{n \in \mathbb{N} \mid \mathcal{D}^{n\times n} \text{ is a ring direct summand of } \mathbb{Q}G \\ \text{for some finite group } G\} \in \mathbb{N} \cup \{\infty\}.$$

If $\mu(\mathcal{D}) < \infty$, then the center $K := Z(\mathcal{D})$ is the character field of some character of a finite group, hence an abelian extension of $\mathbb{Q}$.

Moreover, by a theorem of Benard and Schacher (cf. [Yam 74], Theorem 6.1), $\mathcal{D}$ has *uniformly distributed invariants*, which means in particular, that the Schur index of the completions $\mathcal{D} \otimes K_\wp$, does not depend on the prime $\wp$ of K but only on the rational prime $\wp \cap \mathbb{Q}$ contained in it.

In this paper, we only treat the case, where K is a (totally) real number field. In this case, the theorem of Brauer and Speiser says, that $\mathcal{D}$ is a quaternion algebra, i.e. all local Schur indices are 1 or 2. So $\mathcal{D}$ is uniquely determined by the set of the rational primes that are contained in primes of K that ramify in $\mathcal{D}$.

Notation 3.2. Let $\mu(\mathcal{D}) < \infty$ and assume that $K := Z(\mathcal{D}) = \mathbb{Q}[\alpha]$ is a (totally) real number field. Let $r(\mathcal{D}) := \{p_1, \dots, p_k\} \subseteq \mathbb{N} \cup \{\infty\}$ be the set of those rational primes that are contained in a prime of K that ramifies in $\mathcal{D}$. Then $\mathcal{D}$ is denoted by $\mathcal{Q}_{\alpha,p_1,\dots,p_k}$.

If $K = \mathbb{Q}$ or $K = \mathbb{Q}[\sqrt{d}]$ is a real quadratic field, then Theorems 7.2, 7.8, and 7.14 of [Yam 74] characterize the set of all central simple K-division algebras $\mathcal{D}$ with $\mu(\mathcal{D}) < \infty$.

The results of this paper in particular give information on $\mu(\mathcal{D})$ for quaternion algebras $\mathcal{D}$ with totally real center and $[Z(\mathcal{D}) : \mathbb{Q}] \cdot \mu(\mathcal{D}) \leq 10$ (cf. Table 4.1). It turns out that in these small dimensions, the p_i are either inert or ramified in $Z(\mathcal{D})$.

It is not true, that for all $n > \mu(\mathcal{D})$ there is a finite group G such that $\mathcal{D}^{n\times n}$ is a ring direct summand of $\mathbb{Q}G$. However, taking wreath products (or tensor products with absolutely irreducible subgroups of $GL_d(\mathbb{Q})$) one shows that this holds for all multiples $n = d \cdot \mu(\mathcal{D})$ of $\mu(\mathcal{D})$. It would be interesting to know, if the ideal generated by the n for which there is a finite group G such that $\mathcal{D}^{n\times n}$ is a ring direct summand of $\mathbb{Q}G$ in general is $\mathbb{Z}$.

4. Algorithms for quaternion algebras

Let $\mathcal{D}$ be a totally definite quaternion algebra over $K = Z(\mathcal{D})$. Let G be an a.i.m.f. subgroup of $GL_n(\mathcal{D})$. To find some distinguished integral lattices on which G acts, we embed $GL_n(\mathcal{D})$ (and hence G) into $GL_{4dn}(\mathbb{Q})$. In the tables we will give rational irreducible maximal finite (*r.i.m.f.*) subgroups of $GL_{4dn}(\mathbb{Q})$ containing G and fixing a G-lattice L for which $End_G(L) \subseteq \mathcal{D}$ is a maximal order. The set of isomorphism classes of such G-lattices is the union of the sets of isomorphism classes of $\mathfrak{M}G$-lattices where $\mathfrak{M}$ runs through a system of representatives of conjugacy classes of maximal orders in $\mathcal{D}$.

To check completeness we use the well known mass formulas developed by M. Eichler [Eic 38] (cf. [Vig 80]).

Let h be the class number of K, D the discriminant of $\mathcal{D}$ over K, and $\mathfrak{M}$ any maximal order in $\mathcal{D}$. Let $(I_i)_{1\leq i\leq s}$ be a system of representatives of left ideal classes of $\mathfrak{M}$, $\mathfrak{M}_i := \{x \in \mathcal{D} \mid I_i x \subseteq I_i\}$ the right order of I_i and $\omega_i := [\mathfrak{M}_i^* : R^*]$ the index of the unit group of R, the ring of integers in K, in the unit group of $\mathfrak{M}_i$. Then one has:

$$\sum_{i=1}^{s} \omega_i^{-1} = 2^{1-d} \cdot |\zeta_K(-1)| \cdot h \cdot \prod_{\wp | D} (n(\wp) - 1)$$

where the product is taken over all primes $\wp$ of R dividing the discriminant D of $\mathcal{D}$ and n denotes the norm of K over $\mathbb{Q}$.

If $\mathfrak{M}_i$ and $\mathfrak{M}_j$ are conjugate in $\mathcal{D}$, one may choose a new representative for the class of I_j to achieve that $\mathfrak{M}_i = \mathfrak{M}_j$. Then $I_i^{-1}I_j$ is a 2-sided $\mathfrak{M}_i$-ideal. Moreover the $\mathfrak{M}$-left ideals I_i and I_j are equivalent, if and only if $I_i^{-1}I_j$ is principal.

So if one reorders the $\mathfrak{M}_i$ such that the first t orders $\mathfrak{M}_1, \dots, \mathfrak{M}_t$ form a system of representatives of conjugacy classes of maximal orders in $\mathcal{D}$ and H_i is the number of isomorphism classes of 2-sided ideals of $\mathfrak{M}_i$ $(1 \leq i \leq t)$, then

$$\sum_{i=1}^{s} \omega_i^{-1} = \sum_{i=1}^{t} \omega_i^{-1} H_i.$$

The occurring quaternion algebras have the additional property of having uniformly distributed invariants (cf. Chapter 3). Therefore the Galois group $Gal(K/\mathbb{Q})$ acts on $\mathcal{D}$:

Choose a K-basis $(1 =: b_1, b_2, b_3, b_4)$ of $\mathcal{D}$. An element $\sigma \in Gal(K/\mathbb{Q})$ defines an automorphism σ of the $\mathbb{Q}$-algebra $\mathcal{D}$ by $\sigma(\sum a_i b_i) := \sum \sigma(a_i) b_i$. By the theorem of Skolem and Noether the class $\sigma Inn(\mathcal{D})$ of the automorphism σ does not depend on the chosen basis. Therefore one gets a well defined action of $Gal(K/\mathbb{Q})$ on the set of conjugacy classes of maximal orders in $\mathcal{D}$. This action preserves ω_i and H_i.

Let $\omega_i^1 := \frac{1}{2}|\{x \in \mathfrak{M}_i \mid xx^* = 1\}|$ be the index of ± 1 in the group of units in $\mathfrak{M}_i$ of norm 1, and $\omega_i^{ns} := N(\mathfrak{M}_i^*)/(R^*)^2$. Then $\omega_i = \omega_i^1 \cdot \omega_i^{ns}$.

If n_i denotes the length of the orbits of the class of $\mathfrak{M}_i$ under $Gal(K/\mathbb{Q})$ one gets Table 4.1 below.

In the first column the degree $d := [K : \mathbb{Q}]$ is given, in the second one the name of the quaternion algebra $\mathcal{D}$ as explained in Notation 3.2. The third column contains the relevant dimensions n and in the last column, the mass formula of $\mathcal{D}$ is expanded. Here the sum is taken over a system of representatives of the orbits of $Gal(K/\mathbb{Q})$ on the conjugacy classes of maximal orders in $\mathcal{D}$.

TABLE 4.1. The totally definite quaternion algebras $\mathcal{Q}$ with $d := [Z(\mathcal{Q}) : \mathbb{Q}] \leq 5$ for which there is an $n \leq \frac{10}{d}$ such that $GL_n(\mathcal{Q})$ has a finite absolutely irreducible subgroup:

d	$\mathcal{D}$	n	$\sum n_i(\omega_i^1 \cdot \omega_i^{ns})^{-1} \cdot H_i$
1	$\mathcal{Q}_{\infty,2}$	$1 \dots 10$	12^{-1}
	$\mathcal{Q}_{\infty,3}$	$1 \dots 10$	6^{-1}
	$\mathcal{Q}_{\infty,5}$	$2, 4, 6, 8, 10$	3^{-1}
	$\mathcal{Q}_{\infty,2,3,5}$	8	$3^{-1} + 3^{-1}$
	$\mathcal{Q}_{\infty,7}$	$3, 4, 6, 8, 9, 10$	2^{-1}
	$\mathcal{Q}_{\infty,11}$	$5, 6, 10$	$2^{-1} + 3^{-1}$
	$\mathcal{Q}_{\infty,13}$	6	1
	$\mathcal{Q}_{\infty,17}$	8	$1 + 3^{-1}$
	$\mathcal{Q}_{\infty,19}$	$9, 10$	$1 + 2^{-1}$
2	$\mathcal{Q}_{\sqrt{2},\infty}$	$1 \dots 5$	24^{-1}
	$\mathcal{Q}_{\sqrt{2},\infty,2,3}$	$2, 4$	1
	$\mathcal{Q}_{\sqrt{2},\infty,2,5}$	4	3^{-1}
	$\mathcal{Q}_{\sqrt{3},\infty}$	$1 \dots 5$	$(12 \cdot 2)^{-1} + (12 \cdot 2)^{-1}$
	$\mathcal{Q}_{\sqrt{5},\infty}$	$1 \dots 5$	60^{-1}
	$\mathcal{Q}_{\sqrt{5},\infty,2,3}$	4	$5^{-1} \cdot 2$
	$\mathcal{Q}_{\sqrt{5},\infty,2,5}$	$2, 4$	5^{-1}
	$\mathcal{Q}_{\sqrt{5},\infty,5,3}$	$2, 4$	$5^{-1} + 3^{-1}$
	$\mathcal{Q}_{\sqrt{6},\infty}$	$2, 4$	$(12 \cdot 2)^{-1} + (6 \cdot 2)^{-1} + (4 \cdot 2)^{-1}$
	$\mathcal{Q}_{\sqrt{7},\infty}$	$3, 4$	$(4 \cdot 2)^{-1} + (3 \cdot 2)^{-1} + (12 \cdot 2)^{-1}$
	$\mathcal{Q}_{\sqrt{10},\infty}$	4	$3^{-1} + 2^{-1} + 12^{-1} + 4^{-1}$
	$\mathcal{Q}_{\sqrt{11},\infty}$	5	$12^{-1} + 2^{-1}$
	$\mathcal{Q}_{\sqrt{13},\infty}$	3	12^{-1}
	$\mathcal{Q}_{\sqrt{15},\infty}$	4	$3^{-1} + (1 \cdot 2)^{-1} + (2 \cdot 2)^{-1} + 6^{-1} +$ $(3 \cdot 2)^{-1} + 2^{-1} + 12^{-1} + (2 \cdot 2)^{-1}$
	$\mathcal{Q}_{\sqrt{17},\infty}$	4	6^{-1}
	$\mathcal{Q}_{\sqrt{21},\infty}$	$3, 4$	$12^{-1} + 6^{-1}$
	$\mathcal{Q}_{\sqrt{33},\infty}$	5	$6^{-1} + 3^{-1}$
3	$\mathcal{Q}_{\theta_7,\infty,7}$	$1 \dots 3$	14^{-1}
	$\mathcal{Q}_{\theta_7,\infty,2}$	2	12^{-1}
	$\mathcal{Q}_{\theta_7,\infty,3}$	2	$6^{-1} + 7^{-1}$
	$\mathcal{Q}_{\theta_9,\infty,3}$	$1 \dots 3$	18^{-1}
	$\mathcal{Q}_{\theta_9,\infty,2}$	2	$12^{-1} + 9^{-1}$
	$\mathcal{Q}_{\omega_{13},\infty,13}$	2	1
	$\mathcal{Q}_{\omega_{19},\infty,19}$	3	$2^{-1} + 1 + 3 \cdot 1$
4	$\mathcal{Q}_{\theta_{15},\infty}$	$1, 2$	$(30 \cdot 2)^{-1} + 60^{-1}$
	$\mathcal{Q}_{\theta_{16},\infty}$	$1, 2$	$16^{-1} + 24^{-1}$
	$\mathcal{Q}_{\theta_{20},\infty}$	$1, 2$	$(20 \cdot 2)^{-1} + (12 \cdot 2)^{-1} + 60^{-1}$
	$\mathcal{Q}_{\theta_{24},\infty}$	$1, 2$	$(24 \cdot 2)^{-1} + (8 \cdot 2)^{-1} + 24^{-1}$
	$\mathcal{Q}_{\eta_{17},\infty}$	2	$6^{-1} + 2 \cdot 12^{-1}$
	$\mathcal{Q}_{\sqrt{2}+\sqrt{5},\infty}$	2	$24^{-1} + 60^{-1}$
	$\mathcal{Q}_{\sqrt{2}+\sqrt{5},\infty,2,5}$	2	$5^{-1} + 2 \cdot 1 \cdot 2$
	$\mathcal{Q}_{\eta_{40},\infty}$	2	$(10 \cdot 2)^{-1} + 60^{-1} + 5^{-1}$ $+(2 \cdot 2)^{-1} + (12 \cdot 2)^{-1} + (4 \cdot 2)^{-1}$
	$\mathcal{Q}_{\sqrt{3}+\sqrt{5},\infty}$	2	$60^{-1} + (12 \cdot 2)^{-1} + (12 \cdot 2)^{-1} + (5 \cdot 2)^{-1}$
	$\mathcal{Q}_{\eta_{48},\infty}$	2	$(6 \cdot 2)^{-1} + (2 \cdot 2)^{-1} + 2 \cdot 3^{-1} + 24^{-1}$ $+(8 \cdot 2)^{-1} + 2 \cdot (1 \cdot 2)^{-1} + (4 \cdot 2)^{-1}$ $+(1 \cdot 2)^{-1} + (8 \cdot 2)^{-1} + (2 \cdot 2)^{-1}$
5	$\mathcal{Q}_{\theta_{11},\infty,11}$	$1, 2$	$22^{-1} + 3^{-1}$
	$\mathcal{Q}_{\theta_{11},\infty,2}$	2	$12^{-1} + 1^{-1} + 11^{-1}$
	$\mathcal{Q}_{\theta_{11},\infty,3}$	2	$6^{-1} + 1^{-1} \cdot 2 + 5 \cdot 1^{-1} + 1^{-1} \cdot 2$
	$\mathcal{Q}_{\sigma_{25},\infty,5}$	2	$3^{-1} + 5 \cdot 3^{-1} \cdot 2 + 5 \cdot 1^{-1} \cdot 2 + 5 \cdot 1^{-1} + 5 \cdot 1^{-1}$

For the algebraic numbers the following notation is used:

Notation 4.2. As usual ζ_m denotes a primitive m-th root of unity in $\mathbb{C}$ and $\sqrt{m}$ a square root of m. Moreover $\theta_m := \zeta_m + \zeta_m^{-1}$ denotes a generator of the maximal totally real subfield of the m-th cyclotomic field. ω_m (resp. η_m, σ_m) denote generators of a subfield K of $\mathbb{Q}[\zeta_m]$ with $Gal(K/\mathbb{Q}) \cong C_3$ (resp. C_4, C_5).

The algorithmic problems in evaluating these formulas are:

a) determine the ideals I_j.
b) decide whether two maximal orders are conjugate in $\mathcal{D}$.
c) determine the length of the orbit of $\mathfrak{M}$ under the Galois group $Gal(K/\mathbb{Q})$.
d) determine $\omega_i^{-1} H_i$.

Problem a) is the major difficulty here. There is of course the well known geometric approach to this question using the Minkowski bound on the norm of a representative of the ideal classes. From the arithmetic point of view one may apply two different strategies to find the ideals I_j:

There is a coarser equivalence relation than conjugacy namely the stable isomorphism cf. [Rei 75, (35.5)]. The theorem of Eichler [Rei 75, (34.9)] says that the reduced norm is an isomorphism of the group of stable isomorphism classes of $\mathfrak{M}$-left ideals onto the narrow class group of the center K. This gives estimates for the norms of the ideals I_j.

A second arithmetic strategy is to look for (commutative, nonfull) suborders $\mathcal{O}$ of $\mathcal{D}$. The number of the maximal orders $\mathfrak{M}_i$ containing $\mathcal{O}$ as a pure submodule can be calculated using the formula [Vig 80, (5.12)].

Example. Let $\mathcal{D} := \mathcal{Q}_{\sqrt{3}+\sqrt{5},\infty}$. Then the narrow class group of $K = \mathbb{Q}[\sqrt{3}+\sqrt{5}]$ has order 2 and is generated by a prime ideal dividing 11. So there are two stable isomorphism classes of $\mathfrak{M}$-ideals one containing the ideal classes of I_1, I_2, and I_3, the other one contains that of of I_4 (in the notation of Table 4.1). The second strategy applied to $\mathcal{O} = \mathbb{Z}[\zeta_5, \sqrt{3}]$ gives that there are two orders $\mathfrak{M}_i$ containing a fifth root of unity, because the class number of $\mathcal{O}$ is 2 (and again a prime ideal dividing 11 generates the class group).

The problems b), c), and d) can be dealt with using the normform of $\mathcal{D}$:

Let $\mathcal{D}$ be a definite quaternion algebra over K and N be its reduced norm which is a quadratic form with associated bilinear form $\langle x, y\rangle = tr(xy^*)$ where tr is the reduced trace and * the canonical involution of $\mathcal{D}$. The special orthogonal group $SO(\mathcal{D}, N) := \{\varphi : \mathcal{D} \to \mathcal{D} \mid N(\varphi(x)) = N(x) \text{ for all } x \in \mathcal{D}, det(\varphi) = 1\}$ is the group of all proper isometries of $\mathcal{D}$ with respect to the quadratic form N. The following proposition is surely well known (cf. [Vig 80, Théorème 3.3]) (cf. also [DuV 64] for a geometric interpretation of the quaternionic conjugation).

Proposition 4.3. *With the notation above one has*

$$SO(\mathcal{D}, N) = \{x \mapsto a_1 x a_2^{-1} \mid a_i \in \mathcal{D}^*, N(a_1) = N(a_2)\}$$

is induced by left multiplication with elements of $\mathcal{D}$ of norm 1 and conjugation with elements of $\mathcal{D}^*$.

Proof. Clearly the mapping $x \mapsto a_1 x a_2^{-1}$ with $a_i \in \mathcal{D}^*$ and $N(a_1) = N(a_2)$ is a proper isometry of the K-vector space $(\mathcal{D}, N)$.

To see the converse inclusion let $\mathcal{D} = \langle 1, i, j, ij = k = -ji\rangle_K$ with $i^2 = a$ and $j^2 = b$ and $\varphi : \mathcal{D} \to \mathcal{D}$ be an isometry of determinant 1 with respect to N. Then

$N(\varphi(1)) = 1$ and after left multiplication by $\varphi(1)^{-1}$ we may assume that $\varphi(1) = 1$. Let $b_2 := \varphi(i)$, $b_3 := \varphi(j)$, and $b_4 := \varphi(k)$. Then $tr(b_i 1) = 0$ and hence $b_i^* = -b_i$ for all $i = 2, 3, 4$, and $b_2^2 = a$, $b_3^2 = b$, $b_4^2 = -ab$. Moreover $tr(b_i b_j^*) = 0 = -tr(b_i b_j)$ and hence $b_i b_j = -b_j b_i$ for all $2 \leq i \neq j \leq 4$. Thus $(b_2 b_3) b_4 = b_4 (b_2 b_3)$ and therefore $b_4 \in K b_2 b_3$ is an element of trace 0 in the field generated by $b_2 b_3$. Since $b_4^2 = (b_2 b_3)^2$, this implies that $b_4 = \pm b_2 b_3$. If $b_4 = b_2 b_3$, then φ is an K-algebra automorphism of $\mathcal{D}$ and hence induced by conjugation with an element of $\mathcal{D}^*$ and we are done. In this case φ is of determinant 1. Hence if $b_4 = -b_2 b_3$, the mapping φ has determinant -1, which is a contradiction. $\square$

Corollary 4.4. *Let $\mathfrak{M}_i$ $(i = 1, 2)$ be two orders in $\mathcal{D}$. Then $\mathfrak{M}_1$ is conjugate to $\mathfrak{M}_2$ if and only if the lattices $(\mathfrak{M}_1, N)$ and $(\mathfrak{M}_2, N)$ are properly isometric.*

Proof. Clearly if the two orders are conjugate the lattices are properly isometric, so we show the converse: let $\varphi : \mathfrak{M}_1 \to \mathfrak{M}_2$ be a proper isometry with respect to N. By the proposition there are elements $a_1, a_2 \in \mathcal{D}^*$ with $N(a_1) = N(a_2)$ such that $a_1 \mathfrak{M}_1 a_2^{-1} = \mathfrak{M}_2$. Since $1 \in \mathfrak{M}_1$ this implies that $a_1 a_2^{-1}$ is an element of norm 1 in $\mathfrak{M}_2$ and hence $\mathfrak{M}_2 = a_1 a_2^{-1} a_2 \mathfrak{M}_1 a_2^{-1} = a_2 \mathfrak{M}_1 a_2^{-1}$ is conjugate to $\mathfrak{M}_1$. $\square$

Since $\ ^*$ is the identity on the subspace K and the negative identity on the 3-dimensional subspace $1^\perp$ consisting of the elements of $\mathcal{D}$ with trace 0, one easily sees that $\ ^*$ is an improper isometry (of determinant -1) of $(\mathcal{D}, N)$. Thus, if one of the orders $\mathfrak{M}_1$ or $\mathfrak{M}_2$ is stable under $\ ^*$, one may omit the word "properly" in the corollary above. Note that this holds particularly for maximal orders.

Corollary 4.5. *Let $\mathfrak{M}$ be an order in $\mathcal{D}$. The group of proper isometries of the lattice $(\mathfrak{M}, N)$ is induced by the transformations of the form $b \mapsto axbx^{-1}$, where $a \in \mathfrak{M}$ is an element of norm 1 and $x \in N_{\mathcal{D}^*}(\mathfrak{M})$ normalizes $\mathfrak{M}$.*

By Corollary 4.5 the order of the isometry group $|Aut(\mathfrak{M}_i, N)| = \omega_i^1 \cdot \omega_i \cdot 2^s \cdot 2 \cdot 2 \cdot H_i^{-1}$, where s is the number of finite primes of K that ramify in $\mathcal{D}$. Now $2\omega_i^1$ is simply the number of shortest vectors of the lattice $(\mathfrak{M}_i, N)$ and can easily be calculated. Hence $\omega_i^{-1} H_i = |Aut(\mathfrak{M}_i, N)|^{-1} \cdot 2^{s+2} \cdot \omega_i^1$.

Corollary 4.6. *Let $\mathfrak{M}$ be an order in $\mathcal{D}$ and $\sigma \in Gal(K/\mathbb{Q})$. Then $\mathfrak{M}$ is conjugate to $\sigma(\mathfrak{M})$, if and only if the R-lattices $(\mathfrak{M}, N)$ and $(\mathfrak{M}, \sigma \circ N)$ are isometric.*

Proof. The corollary follows from Corollary 4.4 and the fact that $tr(\sigma(x)\sigma(y^*)) = \sigma(tr(xy^*))$. $\square$

5. Notation for the finite matrix groups

The notation for the absolutely irreducible maximal finite (*a.i.m.f.*) subgroups of $GL_n(\mathcal{D})$ is similar to the one for the rational irreducible maximal finite (*r.i.m.f.*) subgroups of $GL_n(\mathbb{Q})$.

If $\mathcal{D} = \mathcal{Q}_{\alpha, p_1, \ldots, p_s}$, then the (conjugacy class of an) a.i.m.f. group $G \leq GL_n(\mathcal{D})$ is denoted by ${}_{\alpha, p_1, \ldots, p_s}[G]_n$.

The automorphism groups of root lattices are usually denoted by the name of the corresponding root system $A_n, \ldots, E_8, F_4$.

For the quasisimple groups the notation in [CCNPW 85] is used with the exception that the alternating group of degree m is denoted by Alt_m.

A maximal finite matrix group always contains the negative unit matrix. If G is a matrix group, then $\pm G$ denotes the group generated by G and the negative unit matrix.

The symbols $M_{p+1,i}$ and $A_{p-1}^{(j)}$ ($i, \frac{p+1}{2j} \in \{2,3,4,6\}$ with $2i \mid p-1$) denote (automorphism groups of) lattices of $PSL_2(p)$ as described in Chapter V of [PlN 95].

Let $G \leq GL_n(\mathcal{D})$ and $H \leq GL_m(\mathcal{D}')$ be two irreducible finite matrix groups. Let A be a subalgebra of $C_{\mathcal{D}^{n\times n}}(G)$ such that A^{op} is isomorphic to a subalgebra of $C_{\mathcal{D}'^{m\times m}}(H)$, such that $\mathcal{D}^{n\times n} \otimes_A \mathcal{D}'^{m\times m} \cong \mathcal{D}''^{l\times l}$ is again simple. Then tensoring the natural representations of G and H yields a representation of the direct product $G \times H$. The corresponding matrix group $G \otimes_A H$ is a subgroup of $GL_l(\mathcal{D}')$ and isomorphic to a central product of G and H. If $A \cong \mathbb{Q}[\alpha]$ is a field or $A \cong \mathcal{Q}_\alpha$ is a quaternion algebra, the matrix group $G \otimes_A H$ is abbreviated as $G \underset{\alpha}{\otimes} H$ and if $A = \mathbb{Q}$ as $G \otimes H$.

Already the groups $C_5 \cong G \leq GL_1(Q[\zeta_5])$ and $\tilde{S}_3 \cong H \leq GL_1(\mathcal{Q}_{\infty,3})$ show that this tensor notation needs to be extended. Though $\mathbb{Q}[\zeta_5] \otimes \mathcal{Q}_{\infty,3} \cong \mathbb{Q}[\zeta_5]^{2\times 2}$ the maximal common subalgebra of the two algebras $\mathbb{Q}[\zeta_5]$ and $\mathcal{Q}_{\infty,3}$ is $\mathbb{Q}$. We use the symbol $G \otimes_{\sqrt{5}'} H$ to denote the corresponding subgroup of $GL_2(\mathbb{Q}[\zeta_5])$.

To describe quite frequently occurring extensions of tensor products of matrix groups of index 2 as in Theorem 7.11, we use the notation introduced in [PlN 95, Proposition (II.4)]: The symbols $C \overset{2(p)}{\underset{\alpha}{\otimes}} N$, $C \overset{2(p)}{\underset{\alpha}{⌧}} N$, and $C \overset{2(p)}{\underset{\alpha}{\boxtimes}} N$ denote primitive matrix groups G that are extensions of the tensor product of the two matrix groups N and C by an automorphism x with $x^2 \in C \underset{\alpha}{\otimes} N$. Since N (as well as $C = C_G(N)$) is a normal subgroup of G, one may write the elements of G as tensor products as in [CuR 81, Theorem (11.17)]. Let $x = y \otimes z$. In the first case $G = C \overset{2(p)}{\underset{\alpha}{\otimes}} N$, $y \in \bar{C}$, $z \in \bar{N}$, $\bar{G} = \bar{C} \underset{\alpha}{\otimes} \bar{N}$ and $p \in Z(\bar{G})$ is the norm of y which is also the norm of z (cf. Definition 7.10). If $G = C \overset{2(p)}{\underset{\alpha}{⌧}} N$, then $y \notin \bar{C}$ but still $z \in \bar{N}$ and p is the norm of z. In the last case x induces nontrivial automorphisms on both centers $Z(\bar{N})$ and $Z(\bar{C})$.

If $p = 1$, it is omitted. Also the symbol $\times$ and α is omitted if either N or C is contained in A.

Remark 5.1. As the referee pointed out, one should like to compare the classification of maximal finite subgroups of $GL_n(\mathcal{D})$ with Aschbacher's classification of subgroups of the finite classical groups in [Asc 84].

In Aschbacher's classification the groups in C_1 (reducible groups), C_2 (imprimitive groups), C_3, and C_5 reduce to smaller situations, the same is true here. But cases 3 and 5 are harder to deal with, since Galois groups of abelian extension fields are not necessarily cyclic. One may not always extend the cocylces to overgroups, as one sees from the maximal finite subgroups $[D_{120}.2]_{16,i}$ $i = 1, 2$ of $GL_{16}(\mathbb{Q})$.

The types C_6 and C_8 (extraspecial resp. simple groups) are dealt with in Chapter 8 and 9 of this paper.

The main difficulties are the tensor products (types C_4 and C_7 of Aschbacher's classification). These difficulties lead to Chapter 10, where a first approach is made to classify the possible tensor factors. These factors are not necessarily maximal finite subgroups of the unit group of a smaller algebra as the following example shows.

Consider the symmetry group D_8 of a square. This is an imprimitive maximal finite subgroup of $GL_2(\mathbb{Q})$. It admits an outer automorphism $\alpha \in N_{GL_2(\mathbb{Q})}(D_8)$ satisfying $\alpha^2 = 2I_2$. Similarly the matrix group $L_2(7) \leq Aut(A_6)$ admits an additional outer automorphism $\beta \in N_{GL_6(\mathbb{Q})}(L_2(7))$ with $\beta^2 = 2I_6$. Hence $\alpha \otimes \beta$ also normalizes the tensor product $D_8 \otimes L_2(7) \leq GL_{12}(\mathbb{Q})$ The group $D_8 \overset{2(2)}{\otimes} L_2(7) = \langle D_8 \otimes L_2(7), \frac{1}{2}\alpha \otimes \beta\rangle$ is a maximal finite subgroup of $GL_{12}(\mathbb{Q})$ though the group $\pm L_2(7)$ is not maximal finite in $GL_6(\mathbb{Q})$.

The general phenomenon may be described by the groups $Glide(N)$ of gliding automorphisms as defined in Definition 7.3.

Example 5.2. The quaternionic reflection groups of Table III in [Coh 80] that are a.i.m.f. groups are

$$O_2 = {}_{\infty,3}[SL_2(9)]_2,\ O_3 = {}_{\sqrt{3},\infty,\infty}[2.S_6]_2,$$
$$P_2 = {}_{\infty,2}[(D_8 \otimes Q_8).Alt_5]_2,\ P_3 = {}_{\sqrt{2},\infty,\infty}[(D_8 \otimes Q_8).S_5]_2,$$
$$Q = {}_{\infty,3}[\pm U_3(3)]_3,\ R = {}_{\sqrt{5},\infty,\infty}[2.J_2]_3,$$
$$S_3 = {}_{\infty,2}[2_-^{1+6}.O_6^-(2)]_4,$$
$$T = {}_{\sqrt{5},\infty}[(SL_2(5) \circ SL_2(5) \otimes_{\sqrt{5}} SL_2(5)) : S_3]_4,\ \text{and}\ U = {}_{\infty,2}[\pm U_5(2)]_5.$$

6. The a.i.m.f. subgroups of $GL_1(\mathcal{Q})$

Let $\mathcal{Q}$ be a totally definite quaternion algebra over its center K and assume that K is a (totally real) number field. Let $G \leq GL_1(\mathcal{Q})$ be a finite subgroup such that $\overline{\mathbb{Q}G} = \mathcal{Q}$. The classification of finite subgroups of $PGL_2(\mathbb{C})$ in [Bli 17] shows that $G/Z(G)$ is either abelian, a dihedral group or one of the 3 exceptional groups Alt_3, S_4 or Alt_5. Using this classification one gets the following:

Theorem 6.1. *Let $G \leq GL_1(\mathcal{Q})$ be an a.i.m.f. subgroup of $GL_1(\mathcal{Q})$, where $\mathcal{Q}$ is a totally definite quaternion algebra over a totally real number field $K = Z(\mathcal{Q})$.*

Then K is the maximal totally real subfield of a cyclotomic field.

If $[K : \mathbb{Q}] \leq 2$, then G is one of ${}_{\infty,2}[SL_2(3)]_1$, ${}_{\infty,3}[\tilde{S}_3]_1$, ${}_{\sqrt{2},\infty}[\tilde{S}_4]_1$, ${}_{\sqrt{3},\infty}[Q_{24}]_1$, or ${}_{\sqrt{5},\infty}[SL_2(5)]_1$.

If $[K : \mathbb{Q}] \geq 3$, let m be even such that $K = \mathbb{Q}[\theta_m] \leq \mathbb{Q}[\zeta_m]$. Then $G = Q_{2m} = C_m.C_2 \leq GL_1(\mathcal{Q})$ is a generalized quaternion group, a nonsplit extension of a cyclic group of order m by a C_2. If $\frac{m}{2} = p^\alpha$ is a power of the prime $p \equiv 3 \pmod 4$, then $\mathcal{Q} = \mathcal{Q}_{\theta_m,\infty,p}$ is ramified at the place over p. In all other cases, the quaternion algebra $\mathcal{Q}$ is only ramified at the infinite places of K.

Proof. The possible groups G may immediately be obtained from [Bli 17]. So we only have to compute the local Schur indices of the groups Q_{2m}. To this purpose let p be a prime and $2 < r$ be a divisor of m. Then the restriction χ' of the natural character χ of Q_{2m} to the subgroup Q_{4r} remains irreducible. By the theorem of Brauer [Bra 51] (cf. also [Lor 71]) the Schur index of χ and the one of χ' over $\mathbb{Q}_p[\theta_m]$ are equal. If r is a prime such that p does not divide $2r$, then by Lemma 2.13 the p-adic Schur index of χ' is 1. This is also true if $p = 2$ and r are odd, since then the Sylow 2-subgroup of the cyclic subgroup of index 2 in Q_{4r} is C_2 ([Lor 71, p. 98]). So if $\frac{m}{2}$ is not a prime power, the quaternion algebra $\mathcal{Q}$ is not ramified at any finite prime.

If $\frac{m}{2}$ is a power of some prime $l \equiv 1 \pmod 4$, then $\mathbb{Q}_l$ contains a fourth root of unity. Hence the l-adic Schur index of χ' is 1. (This follows also from the parity of the number of ramified primes, since $[\mathbb{Q}[\theta_l] : \mathbb{Q}]$ is even.) If m is a power of 2, then $m \geq 16$. Since $[\mathbb{Q}_2[\theta_m] : \mathbb{Q}_2]$ is even, $\mathbb{Q}_2[\theta_m]$ splits $\mathbb{Q}_2 \otimes \mathcal{Q}_{\infty,2}$. Choosing $r = 4$ in the consideration above, yields that the 2-adic Schur index of $\chi' = \chi_{|Q_8}$ over $\mathbb{Q}_2[\theta_m]$ is 1. If $\frac{m}{2} = p^\alpha$ is a power of a prime $p \equiv 3 \pmod 4$. Choosing $r = p$, the Schur index of χ' over $\mathbb{Q}_p$, hence also the one over the character field $\mathbb{Q}_p[\chi'] = \mathbb{Q}_p[\theta_p]$ is 2. But now $[\mathbb{Q}_p[\theta_m] : \mathbb{Q}_p[\theta_p]] = p^{\alpha-1}$ is odd, hence the character field $\mathbb{Q}_p[\chi]$ does not split the quaternion algebra $\mathbb{Q}_p \otimes \mathcal{Q}_{\theta_p,\infty,p}$. Therefore $\mathcal{Q}$ is ramified at the place over p, which again also follows from the fact that $[\mathbb{Q}[\theta_m] : \mathbb{Q}]$ is odd in this case. $\square$

Remark 6.2. Let $\mathcal{Q}$ be an indefinite quaternion algebra with totally real center K. If $GL_1(\mathcal{Q})$ has an absolutely irreducible finite subgroup G, then $\mathcal{Q} = K^{2\times 2}$.

TABLE 6.3. The a.i.m.f. subgroups of $GL_1(\mathcal{Q})$, where $\mathcal{Q}$ is a definite quaternion algebra such that $[Z(\mathcal{Q}) : \mathbb{Q}] \leq 5$:

lattice L	$\lvert Aut(L)\rvert$	r.i.m.f. supergroups
${}_{\infty,2}[SL_2(3)]_1$	$2^3 \cdot 3$	F_4
${}_{\infty,3}[\tilde{S}_3]_1$	$2^2 \cdot 3$	A_2^2
${}_{\sqrt{2},\infty}[\tilde{S}_4]_1$	$2^4 \cdot 3$	$E_8,\ F_4^2$
${}_{\sqrt{3},\infty}[Q_{24}]_1$	$2^3 \cdot 3$	$A_2^4,\ F_4^2$
		$A_2 \otimes F_4,\ E_8$
${}_{\sqrt{5},\infty}[SL_2(5)]_1$	$2^3 \cdot 3 \cdot 5$	$E_8,\ [(SL_2(5) \overset{2}{\square} SL_2(5)) : 2]_8$
${}_{\theta_7,\infty,7}[Q_{28}]_1$	$2^2 \cdot 7$	$(A_6)^2,\ (A_6^{(2)})^2$
${}_{\theta_9,\infty,3}[Q_{36}]_1$	$2^2 \cdot 3^2$	$A_2^6,\ E_6^2$
${}_{\theta_{15},\infty}[Q_{60}]_1$	$2^2 \cdot 3 \cdot 5$	$(A_2 \otimes A_4)^2,\ [(SL_2(5) \overset{2}{\square} SL_2(5)) : 2]_8^2,\ E_8^2$
		$A_2 \otimes [(SL_2(5) \overset{2}{\square} SL_2(5)) : 2]_8,\ A_2 \otimes E_8,$ $[SL_2(5) \circ SL_2(5) : 2 \overset{2}{\underset{\sqrt{5}}{\boxtimes}} D_{10}]_{16}$
${}_{\theta_{16},\infty}[Q_{32}]_1$	2^5	(B_{16})
		$F_4^4,\ E_8^2$
${}_{\theta_{20},\infty}[Q_{40}]_1$	$2^3 \cdot 5$	$A_4^4,\ [(SL_2(5) \overset{2}{\square} SL_2(5)) : 2]_8^2,\ E_8^2$
		$A_4 \otimes F_4,\ F_4 \tilde{\otimes} F_4,$ $[SL_2(5) \overset{2(2)}{\underset{\infty,2}{\boxtimes}} 2_-^{1+4}.Alt_5]_{16}$
		$[(SL_2(5) \overset{2}{\square} SL_2(5)) : 2]_8^2,\ E_8^2,$ $[SL_2(5) \circ SL_2(5) : 2 \overset{2}{\underset{\sqrt{5}}{\boxtimes}} D_{10}]_{16}$
${}_{\theta_{24},\infty}[Q_{48}]_1$	$2^4 \cdot 3$	$A_2^8,\ E_8^2,\ [(SL_2(5) \overset{2}{\square} SL_2(5)) : 2]_8^2,$ $(A_2 \otimes F_4)^2,\ F_4^4$
		(F_4^4)
		$(A_2 \otimes F_4)^2,\ F_4 \tilde{\otimes} F_4,\ A_2 \otimes E_8,$ $E_8^2,\ [SL_2(5) \overset{2(2)}{\underset{\infty,2}{\boxtimes}} 2_-^{1+4}.Alt_5]_{16}$
${}_{\theta_{11},\infty,11}[Q_{44}]_1$	$2^2 \cdot 11$	$(A_{10})^2,\ (A_{10}^{(2)})^2,\ (A_{10}^{(3)})^2$
		$[L_2(11) \overset{2(3)}{\otimes} D_{12}]_{20}\ [L_2(11) \overset{2(3)}{\boxtimes} D_{12}]_{20}$

Proof of Remark 6.2. Let $G \leq GL_1(\mathcal{Q})$ be a finite absolutely irreducible subgroup and p be a finite prime ramified in $\mathcal{Q}$. As in the proof of Theorem 6.1 one concludes that $G = {}^{\pm}C_{p^\alpha}.C_2$, $K = \mathbb{Q}[\theta_{p^\alpha}]$ and the finite primes of K ramified in $\mathcal{Q}$ divide p. Since all infinite primes of K are not ramified in $\mathcal{Q}$ and p is totally ramified in K, this contradicts the fact that the number of primes in K that ramify in $\mathcal{Q}$ is even. □

If $\mathcal{Q}$ is a definite quaternion algebra and the degree of the center of $\mathcal{Q}$ over $\mathbb{Q}$ is ≤ 5, the a.i.m.f. subgroups of $GL_1(\mathcal{Q})$ and their r.i.m.f. supergroups are given in Table 6.3. The first column gives a name for the a.i.m.f. group G also indicating the quaternion algebra $\mathcal{Q}$. This entry is followed by the order of G. In the last column the r.i.m.f. supergroups fixing a G-lattice with maximal order as endomorphism ring are given. If there is no such group, at least one r.i.m.f. supergroup of G is specified in brackets. If there is more than one conjugacy class of maximal orders $\mathfrak{M}$ in $\mathcal{Q}$, they are listed in the next lines separated by lines in the third column in the same order as they are displayed in Table 4.1.

7. Normal subgroups of primitive groups

Throughout this chapter let K denote an abelian number field with complex conjugation and R the maximal order in K. Then R is a Dedekind ring.

Let $N \trianglelefteq G$ be a normal subgroup of the primitive group G in $GL_n(K)$. As proved in Chapter 2 (cf. 2.3) the enveloping K-algebra of N is a simple algebra.

Generalizing the notion of a generalized Bravais group (as defined in Definition (II.9) of [NeP 95] for $K = \mathbb{Q}$) to arbitrary number fields K, one may often conclude the existence of a larger normal subgroup $\mathcal{B}_K^\circ(N)$ in G if G is maximal finite.

For this purpose recall the radical idealizer process (cf. [BeZ 85]): Let Λ be an R-order in a simple K-algebra A. The *arithmetic* (*right*) *radical* $AR_r(\Lambda)$ of Λ is defined as the intersection of all those maximal right ideals of Λ which contain the discriminant ideal of Λ. The arithmetic radical is a full R-module in A. Its (*right*) *idealizer* $Id_r(AR_r(\Lambda))$, which is defined as the set of all elements $a \in A$, such that $AR_r(\Lambda)a \subseteq AR_r(\Lambda)$, is again an R-order in A containing Λ. The repeated application of $(Id_r \circ AR_r)$ is called the *radical idealizer process*. It constructs a finite ascending chain of R-orders in A. The maximal element of this chain $(Id_r \circ AR_r)^\infty(\Lambda)$ is necessarily a hereditary order in A (cf. [Rei 75] pp. 356-358).

Definition 7.1. Let $N \leq GL_n(K)$ be a finite group and F an N invariant Hermitian form on K^n. Assume that the algebra $A := \overline{KN}$ is simple. Then the natural A-module K^n decomposes into a direct sum of l copies of an irreducible A-module V. Let $\Lambda := \overline{RN}$, be the R-order generated by the matrices in N and $\Lambda_0 := (Id_r \circ AR_r)^\infty(\Lambda)$ be the hereditary order in A obtained by applying the radical idealizer process to the R-order Λ. Let $L_1, \dots, L_s \subseteq V$ be representatives of the isomorphism classes of the irreducible Λ_0-lattices in V. Then the *generalized Bravais group* of N is defined as

$$\mathcal{B}_K^\circ(N) := \{g \in \overline{KN} \mid L_i g = L_i, 1 \leq i \leq s, gF\overline{g}^t = F\}.$$

If $K = \mathbb{Q}$, the group $\mathcal{B}_{\mathbb{Q}}^\circ(N)$ is also denoted by $\mathcal{B}^\circ(N)$.

Note that the definition of $\mathcal{B}_K^\circ(N)$ does not depend on the choice of $F \in \mathcal{F}_{>0}(N)$, since the elements in $\overline{KN}$ commute with all $F'F^{-1}$ for $F' \in \mathcal{F}(N)$.

As for $K = \mathbb{Q}$ in [NeP 95, Proposition (II.10)] one proves:

Proposition 7.2. *Let k be a subfield of K.*

(i) *If X is a finite subgroup of the unit group $(\overline{kN})^*$ of $\overline{kN}$ with $N \trianglelefteq X$, then $X \leq \mathcal{B}_k^\circ(N)$.*

(ii) *If $\mathcal{D}$ is a central K-division algebra of index s and G is a primitive a.i.m.f. subgroup of $GL_n(\mathcal{D}) \leq GL_{s^2n}(K)$ with $N \trianglelefteq G$, then $N \trianglelefteq \mathcal{B}_k^\circ(N) =: B \trianglelefteq G$. In particular $B = G \cap \overline{kN}$ is the unique maximal finite subgroup of the normalizer of N in $(\overline{kN})^*$. Moreover the centralizer of N in G is $C_G(B) = C_G(N)$.*

Example. The generalization of the definition to arbitrary number fields K provides stronger restrictions on the possible normal subgroups of G. For example, for $K := \mathbb{Q}[\sqrt{2}]$ and $N = Q_8$ one has $\mathcal{B}_K^\circ(N) = \tilde{S}_4$ whereas $\mathcal{B}_\mathbb{Q}^\circ(N) = SL_2(3)$.

In the situation of Proposition 7.2 (i), the a.i.m.f. group G contains the normal subgroup $NC_G(N)$. The quotient group $G/NC_G(N)$ embeds into the outer automorphism group $Out(N)$ of N. The image of $G/C_G(N)$ in $Aut(N)$ contains the group of automorphisms that are induced by $\mathcal{B}_K^\circ(N)$ and is contained in the subgroup $Aut_{stab}(N)$ of $Aut(N)$ consisting of those automorphisms of N, that stabilize the irreducible constituent χ of the natural character of N.

Definition 7.3. Let $N \leq GL_n(K)$ be a finite subgroup of $GL_n(K)$, such that the enveloping algebra $\overline{KN}$ is a simple K-algebra. Let χ be an absolutely irreducible character occurring in the natural character of N.

(i) The automorphism group $Aut(N)$ acts on the set of irreducible characters of N. $Aut_{stab}(N) := Stab_{Aut(N)}(\chi)$ is called the group of *stable automorphisms* of the matrix group N.

(ii) N is called *primitively saturated over* K, if $N \trianglelefteq \mathcal{B}_K^\circ(N)$ and all stable automorphisms of N are induced by conjugation with elements of $\mathcal{B}_K^\circ(N)$.

(iii) The factor group $Glide(N, \chi, K) := Glide_K(N) := Aut_{stab}(N)/\kappa(\mathcal{B}_K^\circ(N))$ of $Aut_{stab}(N)$ modulo the group of automorphisms induced by conjugation with elements of $\mathcal{B}_K^\circ(N)$ is called the group of *gliding automorphisms* of the matrix group N. We set $Glide(N) := Glide_\mathbb{Q}(N)$.

Remark 7.4. Let N, A, Λ_0 be as in Definition 7.1. Since the elements of $Aut_{stab}(N)$ define $Z(A)$-algebra automorphisms of A, the theorem of Skolem and Noether says that there are elements in A^* inducing these automorphisms on A. Since these elements normalize N, they also normalize Λ_0 and therefore act as inclusion preserving bijections on the set of Λ_0-lattices. The subgroup induced by conjugation with elements in $\mathcal{B}_K^\circ(N)$ is the kernel of this action and therefore a normal subgroup of $Aut_{stab}(N)$. Hence $Glide_K(N)$ is well defined.

For a maximal ideal $\wp$ of R let $A_\wp$ and $\Lambda_\wp$ denote the completions of A and Λ_0 at $\wp$. Since $\Lambda_\wp$ is hereditary the $\Lambda_\wp$-lattices in a simple $A_\wp$-module are linearly ordered by inclusion. Since $\mathcal{B}_K^\circ(N)$ stabilizes all $\Lambda_\wp$-lattices one gets an action of $Glide_K(N)$ on this chain of $\Lambda_\wp$-lattices (shifting up or down).

Proposition 7.5. *In the situation of Proposition 7.2 (ii), assume that N is primitively saturated over K and that the center of $\overline{KN}$ is K. If $B := \mathcal{B}_K^\circ(N)$ and $C := C_G(N)$, then $G = BC$.*

Proof. Since G is primitive, the automorphisms of N that are induced by $g \in G$ stabilize χ. Hence there is a $b \in B$ such that $bg \in C$. □

The automorphism groups of the indecomposable root lattices A_n $(n \geq 4)$, E_6, E_7, and E_8 provide examples for primitively saturated groups.

Corollary 7.6. *Let $\mathcal{D}$ be a $\mathbb{Q}$-division algebra and G a primitive irreducible maximal finite subgroup of $GL_n(\mathcal{D})$. Assume that G contains a normal subgroup N isomorphic to either Alt_n $(n \geq 5)$, $U_4(2) = Aut(E_6)'$, $S_6(2) = Aut(E_7)'$, or $2.O_8^+(2) = Aut(E_8)'$ (where the corresponding irreducible constituent χ of the natural character of N is of degree $n-1$, 6, 7, respectively 8). Then $G = \mathcal{B}^\circ(N) \otimes C_G(N)$.*

Proof. In all cases $Aut_{stab}(N)$ is already induced by conjugation with elements of $\mathcal{B}^\circ(N)$. □

Corollary 7.7. *Let $\mathcal{Q}$ be a definite quaternion algebra with center K and G a primitive a.i.m.f. subgroup of $GL_n(\mathcal{Q})$. Then G has no normal subgroup N isomorphic to M_{11}, $2.M_{12}$, or $2.M_{22}$ where the restriction of the natural character of G to N is a multiple of the sum of the two Galois conjugate complex characters of degree 10.*

Proof. Since the whole outer automorphism group of N is already induced by conjugation with elements in $\mathcal{B}^\circ(N)$, the group G is of the form $G = \mathcal{B}^\circ(N)C_G(N)$. In particular the character field of the natural character of G is complex. Therefore G is not an absolutely irreducible subgroup of $GL_n(\mathcal{Q})$. □

The following theorem is a version of a well known theorem of Clifford (cf. [CuR 81, Theorem (11.20)]), which is usually only formulated for algebraically closed fields.

Theorem 7.8. *Let $G \leq GL_n(K)$ be a finite group, $N \trianglelefteq G$ a normal subgroup such that the enveloping K-algebra $A := \overline{KN}$ is central simple. Let $C := C_{\overline{KG}}(A)$ be the commuting algebra of A in $\overline{KG}$. Then the natural representation $\Delta : G \to GL_n(K)$ is a tensor product $\Delta = \Delta_1 \otimes \Delta_2$ of projective representations $\Delta_1 : G \to A^*$ and $\Delta_2 : G \to C^*$.*

Proof. Let $g \in G$. Since $N \trianglelefteq G$, conjugation with g induces a K-algebra automorphism of A. By the theorem of Skolem and Noether, there is an $a \in A^*$, such that $ag =: b \in C$. Hence $g = a \otimes b \in A \otimes C = \overline{KG}$. □

In the situation of Theorem 7.8, $\Delta_1(G)$ is a (not necessarily finite) subgroup of the normalizer of N in the unit group of its enveloping algebra $N_{A^*}(N)$. If one additionally assumes that G is (primitive and) maximal finite and chooses Δ_1 and Δ_2 appropriately, then $B := \mathcal{B}_K^\circ(N) = Ker(\Delta_2)$ is the unique maximal finite subgroup of $N_{A^*}(N)$.

Lemma 7.9. *Let $N \leq GL_n(K)$ be a finite matrix group such that the algebra $\overline{KN}$ is simple with center Z. Let $^-$ also denote the complex conjugation on the abelian number field Z and let Z^+ be the maximal totally real subfield of Z. Let $\alpha \in Glide_K(N)$. Then there is $a \in \overline{KN}^*$ such that a representative of α is induced by conjugation with a. Moreover there is $q \in Z^+$ such that $aF\bar{a}^{tr} = qF$ for all $F \in \mathcal{F}(N)$. The element q is a totally positive element of Z^+ unique up to multiplication with elements of the group $\{z\bar{z} \mid z \in Z\}$.*

Proof. Let $F \in \mathcal{F}_{>0}(N)$ be an N-invariant K-Hermitian form. Then the matrix $aF\bar{a}^t$ is again N-invariant, because a normalizes N. Hence there is a $q \in C := C_{K^{n\times n}}(N)$, such that $aF\bar{a}^t = qF$. Every element $x \in C$ may be written as a sum of a symmetric and a skew symmetric element (with respect to F), i.e. $x = x^+ + x^-$ with $x^+, x^- \in C$ and $x^+F = F\overline{(x^+)}^t$ and $x^-F = -F\overline{(x^-)}^t$. Then clearly $q = aF\bar{a}^tF^{-1}$ is symmetric. Since a commutes with x^+ and x^- one has $x^+q = ax^+F\bar{a}^tF^{-1} = qx^+$ and $x^-q = qx^-$. Hence $q \in Z$ lies in the center of C. If $F' \in \mathcal{F}(N)$ is another N-invariant Hermitian form, then there is $c \in C$ such that $F' = cF$. Then $aF'\bar{a}^t = acF\bar{a}^t = caF\bar{a}^t = cqF = qF'$. Since a is unique up to multiplication with elements of Z and $Fz^tF^{-1} = \bar{z}$ for all $z \in Z$, q is unique up to norms (resp. up to squares if $Z = Z^+$) of elements in Z. Moreover if F is totally positive definite, then $aF\bar{a}^t = qF$ is also totally positive definite, whence q is totally positive. □

Definition 7.10. The element q in the lemma above is called the *norm* of α.

If A is a central simple algebra over a totally real field K, then $\Delta_1(G)/(K^*\mathcal{B}_K^\circ(N))$ is of exponent ≤ 2, as shown in the next theorem. This is somehow an explanation for the fact that the constructions given in Proposition (II.4) of [PlN 95] suffice to describe all r.i.m.f. groups in dimension ≤ 31.

Theorem 7.11. *Let K be a real abelian number field and $N \leq GL_n(K)$ a finite matrix group such that the enveloping algebra $\overline{KN}$ is simple with center K. Assume that $N \trianglelefteq \mathcal{B}_K^\circ(N) =: B$. Then $Glide_K(N)$ is of exponent 1 or 2.*

Proof. Let $\alpha \in Aut_{stab}(N)$. Since $\overline{KN}$ is central simple, there is an $a \in (\overline{KN})^*$, such that α is induced by conjugation with a. Let F be an N-invariant K-quadratic form. By Lemma 7.9 there is $q \in K = Z(\overline{KN})$ such that $aFa^t = qF$. Therefore $a^2q^{-1}F(a^2q^{-1})^t = F$. Since the automorphism α has finite order, there is an $m \in \mathbb{N}$ such that $(a^2q^{-1})^m \in Z(\overline{KN}) = K$. One calculates $F = (a^2q^{-1})^mF((a^2q^{-1})^m)^t = (a^2q^{-1})^{2m}F$. Hence $(a^2q^{-1})^{2m} = 1$ and (a^2q^{-1}) is an element of finite order in $(\overline{KN})^*$ normalizing N. By Proposition 7.2 $a^2q^{-1} \in B$. □

If $Glide_K(N)$ is of order 2 and $\overline{KN}$ is a central simple K-algebra, the primitive a.i.m.f. groups G with normal subgroup N contain a subgroup of index 1 or 2 which is a tensor product $B \otimes C_G(N)$. If $\alpha \in Aut_{stab}(N) - \kappa(B)$ and q are as in the proof above, we call N *nearly tensor decomposing with parameter q*.

Note that Theorem 7.11 is false if one omits the assumption that K is real. One counterexample is provided by the faithful character of degree 144 of the group $3.U_3(5)$ (cf. [CCNPW 85]). The smallest counterexample I know is $N \cong C_3 \times (C_7 : C_3)$. Let $N := \langle z, x, y \mid z^3, x^7, y^3, x^y = x^2\rangle$. Then N has an automorphism s of order 3, with $z^s = z$, $x^s = x$, $y^s = yz$. N has a faithful representation into $GL_3(K)$ where $K := \mathbb{Q}[\sqrt{-3}, \sqrt{-7}]$. The corresponding character χ extends to $\pm N : \langle s\rangle$ but the character value of xs involves further irrationalities. So the order of $Glide_K(N)$ is divisible by 3.

Corollary 7.12. *With the notation of the proof of Theorem 7.11, the element $\alpha \in Glide_K(N)$ is uniquely determined by the class of q in $K^*/(K^*)^2$.*

Proof. Let $\alpha, \beta \in Aut_{stab}(N)$ be induced by conjugation with a resp. $b \in (\overline{KN})^*$ such that $aFa^t = qF$ and $bFb^t = r^2qF$, with $q, r \in K^*$. Replacing b by br^{-1} we assume that $r = 1$. Then $ab^{-1}F(ab^{-1})^t = F$. As in the proof above, the matrix $ab^{-1} \in \overline{KN}$ is an element of finite order normalizing N and hence $ab^{-1} \in \mathcal{B}_K^\circ(N)$. □

8. The normal p-subgroups of primitive groups and their automorphism groups

In this chapter we calculate the generalized Bravais groups and outer automorphism groups of the relevant p-groups N which are candidates for normal p-subgroups of a primitive a.i.m.f. group G. Since all abelian characteristic subgroups of N are cyclic (Corollary 2.4), these groups are classified by P. Hall:

Theorem 8.1 (cf. [Hup 67], p. 357)**.** *Let N be a p-group, such that all abelian characteristic subgroups of N are cyclic.*

If $p > 2$, then N is a central product of a cyclic group and an extraspecial group of exponent p.

If $p = 2$, then N is a central product of an extraspecial 2-group with a cyclic dihedral, generalized quaternion, or quasidihedral 2-group.

If K is an abelian number field, then all of these groups have a (up to automorphism) unique K-irreducible faithful representation. The corresponding matrix group is called an *admissible p-group over K*.

The automorphism groups of the extraspecial groups are well known (cf. [Win 72]). For these groups one finds:

Proposition 8.2. *Let $n \in \mathbb{N}$, p be a prime, and $N = p_+^{1+2n}$ or $N = 2_-^{1+2(n-1)}$ if $p = 2$. If $N \trianglelefteq \mathcal{B}^\circ(N) =: B$, then B is as follows:*

If $p > 2$, then $B = \pm p_+^{1+2n}.Sp_{2n}(p)$.

If $N = 2_+^{1+2n}$, then $B = 2_+^{1+2n}.O_{2n}^+(2)$.

If $N = 2_-^{1+2(n-1)}$, then $B = 2_-^{1+2(n-1)}.O_{2(n-1)}^-(2)$.

Proof. Case $p > 2$: By [Win 72] the subgroup of the outer automorphism group of the extraspecial p-group $p_+^{1+2n} = N$ of exponent p which centralizes the center C_p of N is the symplectic group $Sp_{2n}(p)$. Hence if $N \trianglelefteq B$, then $B \leq \pm N.Sp_{2n}(p)$. In [Wal 62], Wall constructs a lattice of dimension $(p-1) \cdot p^n$ on which $\pm N.Sp_{2n}(p)$ acts. Therefore $\pm N.Sp_{2n}(p) \leq B$.

Case $p = 2$: If $p = 2$, the proposition can be checked directly for $n \leq 3$. So assume $n \geq 4$. Let ϵ be $+$ or $-$. Then by [Win 72] the outer automorphism group of 2_ϵ^{1+2n} is the orthogonal group $SO_{2n}^\epsilon(2) = GO_{2n}^\epsilon(2)$. It contains a subgroup $O_{2n}^\epsilon(2)$ of index 2 (cf. [CCNPW 85, p. xii]).

By [Wal 62] the group $B_1 := 2_+^{1+2n}.O_{2n}^+(2)$ is the full automorphism group of a lattice of dimension 2^n. Now $N = D_8 \otimes \ldots \otimes D_8$ is conjugate in $GL_{2^n}(\mathbb{Q})$ to the tensor product of n copies of D_8. If $D_8 \in GL_2(\mathbb{Q})$ is given in the monomial representation, then $\alpha := \left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$ normalizes D_8 and satisfies $\alpha^2 = 2I_2$. Hence the element $a := \alpha \otimes I_2 \otimes \ldots \otimes I_2$ normalizes N and hence B. Since $a^2 = 2$ is not a square in $\mathbb{Q}^*$, there is no element of finite order in $GL_{2^n}(\mathbb{Q})$ inducing the same automorphism on N as a. Therefore $\langle B_1, a\rangle \cong N.SO_{2n}^+(2)$ is the full holomorph of N and $B_1 = B$ the unique maximal finite subgroup of the normalizer $N_{GL_{2^n}(\mathbb{Q})}(N)$.

If $\epsilon = -$, then $N = 2_-^{1+2(n-1)} = D_8 \otimes \ldots \otimes D_8 \otimes Q_8$ is the centralizer of a subgroup $Q_8 \leq 2_+^{1+2n}$. One finds a subgroup of index 2 in the holomorph of N as a centralizer of the subgroup Q_8 in $2_+^{1+2n}.O_{2n}^+(2)$. Since a normalizes N and lies in the enveloping $\mathbb{Q}$-algebra of N the proposition follows. □

Lemma 8.3. *Let $N := 2_+^{1+2n} \curlyvee C_4 \cong 2_-^{1+2n} \curlyvee C_4$. Then the outer automorphism group $Out(N)$ is isomorphic to $O_{2n+1}(2) \times C_2$.*

Proof. The mapping $q : N/N' \to N'$, $xN' \mapsto x^2$ is a well defined nondegenerate quadratic form on $N/N' \cong \mathbb{F}_2^{2n+1}$. The inner automorphisms induce the identity on N/N' and q is $Out(N)$-invariant. Since every isometry of $(N/N', q)$ can be extended to an automorphism of N, one gets an epimorphism $Aut(N) \to Out(N) \to O_{2n+1}(2)$. The kernel H consists of all automorphisms of N inducing the identity on N/N'. Now $N = Q_8 \curlyvee \ldots \curlyvee Q_8 \curlyvee C_4 = \langle A_1, B_1\rangle \curlyvee \ldots \curlyvee \langle A_n, B_n\rangle \curlyvee \langle A\rangle$ where $\langle A_i, B_i\rangle \cong Q_8$ and A_i and B_i commute with A_j and B_j for $i \neq j$. Let $\alpha \in H$. If $\alpha(A_i) = A^2A_i$ for some i, we multiply α with the inner automorphism κ_{B_i} induced by conjugation with B_i to achieve $\alpha(A_i) = A_i$ for all i. Analogously for B_i. After this α is either the identity or $\alpha = \alpha_0$ where $\alpha_0(A_i) = A_i$, $\alpha_0(B_i) = B_i$ for all i, and $\alpha_0(A) = A^3$. Hence $H/Inn(N) = (\langle\alpha_0\rangle Inn(N))/Inn(N) \trianglelefteq Out(N)$ is a normal subgroup of order 2 of $Out(N)$. Since $O_{2n+1}(2) \cong Aut(N)/H \cong C_{Out(N)}(Z(N)) \trianglelefteq Out(N)$ one has $Out(N) \cong O_{2n+1}(2) \times \langle\alpha_0\rangle$. □

Corollary 8.4. *Let $m, n \in \mathbb{N}$, $m > 1$, p be a prime, and $N = p_+^{1+2n} \curlyvee C_{p^m}$. If $N \trianglelefteq \mathcal{B}^\circ(N) =: B$, then $B = \pm N.Sp_{2n}(p)$.*

Moreover $Out(N) = Sp_{2n}(p) \times Aut(C_{p^m})$.

Proof. If $p > 2$, then $C_{p^m} = Z(N)$ and $p_+^{1+2n} = \Omega_1(N)$ are characteristic subgroups of N. Since the elements in B centralize the center of N, the first statement follows from Proposition 8.2. For $p = 2$, the groups $C_{2^m} = Z(N)$ and $V := 2^{1+2n} \curlyvee C_4 = \Omega_2(N)$ are characteristic subgroups of N. The holomorph of V can be constructed as the centralizer of an element of order 4 in $2_+^{1+2(n+1)}$ in $2_+^{1+2(n+1)}.O^+_{2(n+1)}(2)$. Hence by Lemma 8.3 the subgroup of the automorphism group of N centralizing the center of N is induced by B. Since $O_{2n+1}(2) \cong Sp_{2n}(2)$ (cf. e.g. [Tay 92, Theorem 11.9]), the first statement follows.

The outer automorphism group $Out(N)$ contains a normal subgroup $\kappa(B) = C_{Out(N)}(Z(N)) \cong B/\pm N$, the image of B in $Out(N)$. The automorphisms of $Z(N)$ may be extended to outer automorphisms of N, hence $Out(N)/\kappa(B) \cong Aut(Z(N))$ is isomorphic to the automorphism group of $C_{p^m} = Z(N)$. The kernel of the epimorphism $Out(N) \to Sp_{2n}(p)$ constructed above is a normal complement of $\kappa(B)$ in $Out(N)$ which shows that $Out(N) = \kappa(B) \times Aut(C_p^m)$. □

Lemma 8.5. *Let $m > 3$. The outer automorphism groups of the dihedral, quasidihedral, or generalized quaternion groups U are:*

$$Out(D_{2^m}) \cong Out(Q_{2^m}) \cong C_2 \times C_{2^{m-2}} \text{ and } Out(QD_{2^m}) \cong C_{2^{m-2}}.$$

Proof. In all three cases U has a unique subgroup V isomorphic to $C_{2^{m-1}}$ of index 2 which is therefore characteristic in U. U/V induces a subgroup of order 2 of the automorphism group of V. Since $Aut(V)$ is abelian $Out(U)$ has an epimorphic image $C_{2^{m-2}}$ with kernel H consisting of those outer automorphisms that induce the identity on V modulo inner automorphisms of U.

Let $D_{2^m} = \langle x, y \mid x^{2^{m-1}}, y^2, (xy)^2\rangle$. The elements of order 2 in $D_{2^m} - V$ are x^iy with $1 \leq i \leq 2^{m-1}$. Since $y^x = x^{-2}y$ these form 2 orbits under the group of inner automorphisms of D_{2^m}. Via the absolutely irreducible faithful representation of degree 2 the group D_{2^m} can be viewed as a subset G of the algebra $\mathbb{Q}[\theta_{2^{m-1}}]^{2\times 2}$. If x, y denote the corresponding elements of G, the element $(1 - x)$ normalizes G since $(1 - x)^{-1}y(1 - x) = (1 - x^{-1})(1 - x)^{-1}y = -x^{2^{m-1}-1}y$. Hence the outer automorphism induced by $(1 - x)$ generates H.

Analogous considerations hold for the generalized quaternion group Q_{2^m}.

The elements of the group $QD_{2^m} = \langle x, y \mid x^{2^{m-1}}, y^2, x^y = x^{2^{m-2}-1}\rangle$ that are not in V again form two conjugacy classes. But now one of them consists of elements of order 4, the other one of elements of order 2. Hence here $H = 1$. $\square$

Corollary 8.6. *Let $m, n \in \mathbb{N}$, $m > 3$, and U be one of D_{2^m}, Q_{2^m}, or QD_{2^m}. If $N := 2_+^{1+2n} \curlyvee U$ is a normal subgroup of an a.i.m.f. matrix group G, then $B := \mathcal{B}^\circ(N) = N.O_{2n+1}(2)$ and $Out(N)$ is isomorphic to $O_{2n+1}(2) \times C_2 \times C_{2^{m-2}}$ if $U = D_{2^m}$ or Q_{2^m}, and $Out(N) \cong O_{2n+1}(2) \times C_{2^{m-2}}$ if $U = QD_{2^m}$.*

Proof. Since $m > 3$, U has a unique subgroup V isomorphic to $C_{2^{m-1}}$ of index 2. V is the center of the subgroup $2_+^{1+2n} \curlyvee V =: W$ of N generated by the elements of order 2^{m-1}. Therefore V and W are characteristic subgroups of N and hence normal in G. Thus with Corollary 8.4 $W.O_{2n+1}(2) \cong \mathcal{B}^\circ(W) = G \cap \overline{\mathbb{Q}W}$ is a normal subgroup of G (and therefore also of B). In particular, these automorphisms do extend to automorphisms of N.

If $g \in N - W$, then g induces the Galois automorphism of the center of $\overline{\mathbb{Q}W}$ over the center of $\overline{\mathbb{Q}N}$. Hence $N_{(\overline{\mathbb{Q}N})^*}(W) = \langle N_{(\overline{\mathbb{Q}W})^*}(W), g\rangle$ with the theorem of Skolem and Noether. Hence $B = N.O_{2n+1}(2)$ by Corollary 8.4.

By the same corollary the full automorphism group of W is $C_2 \times C_{2^{m-2}} \times O_{2n+1}(2)$. These automorphisms extend to automorphisms of N. Since a subgroup C_2 of $Out(W)$ is induced by conjugation with elements of U, $Out(N)$ has an epimorphic image $C_{2^{m-2}} \times O_{2n+1}(2)$. Let H be the kernel and $1 \neq \bar{x} \in H$. Then one may choose the representative x of $\bar{x}$ modulo the group of inner automorphisms of N such that x centralizes W (and $N/W \cong C_2$). Then x maps $C := C_N(U) =\cong 2_+^{1+2n}$ and hence $U = C_N(C)$ into themselves. Hence H is a subgroup of the outer automorphism group of U. Since all automorphisms of U can be extended to N, the group H is isomorphic to $Out(U)$ and the corollary follows from Corollary 8.5. $\square$

The results of this section are summarized in the following table.

TABLE 8.7. Let $\mathcal{D}$ be a finite dimensional $\mathbb{Q}$-division algebra and G be a primitive a.i.m.f. group in $GL_d(\mathcal{D})$. Then $O_p(G)$ is one of the following groups:

N	$\mathcal{B}^\circ(N)$	$\bar{N}$	$Glide(N)$
p_+^{1+2n}, $p > 2$	$\pm N.Sp_{2n}(p)$	$\mathbb{Q}[\zeta_p]^{p^n \times p^n}$	1
2_+^{1+2n}	$N.O_{2n}^+(2)$	$\mathbb{Q}^{2^n \times 2^n}$	C_2 (2)
2_-^{1+2n}	$N.O_{2n}^-(2)$	$\mathcal{Q}_{\infty,2}^{2^{n-1} \times 2^{n-1}}$	C_2 (2)
C_{p^m}	$\pm N$	$\mathbb{Q}[\zeta_{p^m}]$	1
$p_+^{1+2n} \curlyvee C_{p^m}$, $m > 1$	$\pm N.Sp_{2n}(p)$	$\mathbb{Q}[\zeta_{p^m}]^{p^n \times p^n}$	1
$2_+^{1+2n} \curlyvee D_{2^m}$, $m > 3$	$\pm N.Sp_{2n}(2)$	$\mathbb{Q}[\theta_{2^{m-1}}]^{2^{n+1} \times 2^{n+1}}$	C_2 $(2 - \theta_{2^{m-1}})$
$2_+^{1+2n} \curlyvee Q_{2^m}$, $m > 3$	$\pm N.Sp_{2n}(2)$	$\mathcal{Q}_{\theta_{2^{m-1}},\infty}^{2^n \times 2^n}$	C_2 $(2 - \theta_{2^{m-1}})$
$2_+^{1+2n} \curlyvee QD_{2^m}$, $m > 3$	$\pm N.Sp_{2n}(2)$	$\mathbb{Q}[\zeta_{2^{m-1}} - \zeta_{2^{m-1}}^{-1}]^{2^{n+1} \times 2^{n+1}}$	1

The first column contains the isomorphism type of the admissible p-group N. The information in the second column is only proved under the assumption that N is a normal subgroup of its generalized Bravais group over $\mathbb{Q}$ (cf. Definition 7.1), which is necessarily the case if N is a normal subgroup of a primitive a.i.m.f. subgroup. Under this assumption, the second column contains the generalized Bravais group of N. The third column gives the enveloping $\mathbb{Q}$-algebra $\overline{N}$ of N and the last column contains the factor group $Glide(N)$ of the subgroup of the automorphism group of N that acts trivially on the center Z of $\overline{N}$. For all a.i.m.f. subgroups G containing N as a normal subgroup the quotient $G/(\mathcal{B}^\circ(N)C_G(N))$ is a subgroup of $Glide(N).Gal(Z/\mathbb{Q})$. If $|Glide(N)| = 2$, a norm (cf. Definition 7.10) of a nontrivial element in $Glide(N)$ is given in brackets.

Definition 8.8. Let $N \trianglelefteq G$. Then N is called *self-centralizing*, if $C_G(N) \le N$.

Proposition 8.9. *Let $\mathcal{D}$ be a definite quaternion algebra with center K. Let G be a primitive a.i.m.f. subgroup of $GL_n(\mathcal{D})$ and $O_2(G)$ a self-centralizing normal subgroup. Then $G = \mathcal{B}^\circ_K(O_2(G))$, $n = 2^{m-1}$ is a power of 2, and $O_2(G)$ is centrally irreducible. Moreover one of the following three possibilities occurs:*

(i) *$K = \mathbb{Q}$, $O_2(G) = 2_-^{1+2m}$, and $G = 2_-^{1+2m}.O_{2m}^-(2)$.*

(ii) *$K = \mathbb{Q}[\sqrt{2}]$ and G is one of $2_-^{1+2m}.GO_{2m}^-(2)$ or $(2_+^{1+2(m-1)} \otimes Q_{16}).O_{2m-1}(2)$.*

(iii) *$K = \mathbb{Q}[\theta_{2^s}]$ with $s > 3$ and $G = (2_+^{1+2(m-1)} \otimes Q_{2^{s+1}}).O_{2m-1}(2)$.*

Proof. By Theorem 8.1 the group $O_2(G)$ is a central product of an extraspecial 2-group with a cyclic, dihedral, quasidihedral, or generalized quaternion group. Since $O_2(G)$ is self-centralizing, $G/O_2(G)$ is a subgroup of $Out(O_2(G))$ with $O_2(G/O_2(G)) = 1$. Hence by Corollary 8.6 and Lemma 8.3 either $G = \mathcal{B}^\circ(O_2(G))$ or $O_2(G) = 2_-^{1+2n}$, $K = \mathbb{Q}[\sqrt{2}]$ and $G = \mathcal{B}^\circ_K(O_2(G)) = 2_-^{1+2m}.GO_{2m}^-(2)$. □

Corollary 8.10. *Let N be an admissible p-group over K and $p^a := |Z(N)|$ the order of the center of N. If N is not an extraspecial 2-group or K contains $\mathbb{Q}[\sqrt{2}]$, then $Glide_K(N) = 1$ and $Aut(N)/Aut_{stab}(N)$ is isomorphic to the Galois group $Gal(K[\zeta_{p^a}]/K)$.*

Lemma 8.11. *Let $G \le GL_n(\mathcal{D})$ be a primitive a.i.m.f. group such that $Fit(G) := \prod_{p||G|} O_p(G)$ is a self-centralizing normal subgroup. Then $Fit(G)$ is irreducible.*

Proof. If $O_2(G)$ is not an extraspecial 2-group, the Lemma follows from Corollary 8.10 and Lemma 2.14.

So assume that $O_2(G)$ is an extraspecial 2-group and let $B := \mathcal{B}^\circ_K(O_2(G))$. Then $N := C_G(B)B$ is a normal subgroup of index 1 or 2 in G.

Let $Z := Z(\overline{KFit(G)})$ be the center of the enveloping K-algebra of $Fit(G)$, $z := [Z : K]$, $f := dim_K(\overline{KFit(G)}) = m^2z$, and $g := dim_K(\overline{KG}) = 4n^2$. By Corollary 8.10, the center of $\overline{KN}$ is a subfield $K \subseteq Z(\overline{KN}) \subseteq Z$, say of degree x over K.

Assume that $Fit(G)$ is reducible. Then $mz < 2n$ and by Lemma 2.14 and Corollary 8.10 $dim_K(\overline{KN}) = (mz)^2x^{-1} \le (mz)^2 \le n^2$. With Lemma 2.14, this contradicts the absolute irreducibility of G. □

The next lemma is useful to exclude cyclic normal subgroups (cf. also Lemma 11.2).

Lemma 8.12. *Let G be a primitive a.i.m.f. group and $3 < p \equiv 3 \pmod 4$ be a prime. If $O_p(G) = C_p$, then $N := C_p : C_{\frac{p-1}{2}}$ is not a normal subgroup of G.*

Proof. Assume that $N \trianglelefteq G$. Since G is primitive, the enveloping algebra of N is $\overline{\mathbb{Q}N} = \mathbb{Q}[\sqrt{\text{-}p}]^{\frac{p-1}{2}\times\frac{p-1}{2}}$. If $C := C_G(N)$, then G/CN embeds into C_2, the outer automorphism group of N. Now N is a subgroup of $M := L_2(p) \subseteq \overline{\mathbb{Q}N}$. Since $N_{\bar{G}^*}(M) > N_{\bar{G}^*}(N)$, the group G also normalizes M and hence $\langle G, M\rangle$ is a proper supergroup of G. $\square$

Remark 8.13. If $p \equiv 1 \pmod 4$, then the outer automorphism group of $C_p : C_{\frac{p-1}{2}} =: N \leq GL_{\frac{p-1}{2}}(\mathbb{Q}[\sqrt{p}])$ is $C_2 \times C_2$, where the additional automorphism is induced by conjugation with $(1-\zeta) \in \overline{\mathbb{Q}N} = \mathbb{Q}[\sqrt{p}]^{\frac{p-1}{2}\times\frac{p-1}{2}}$, where ζ generates the normal p-subgroup of N.

9. The candidates for quasi-semi-simple normal subgroups

In this chapter we list the information used from the classification of finite simple groups and their character tables as given in [CCNPW 85]. Let G be a primitive a.i.m.f. subgroup of $GL_n(\mathcal{D})$. Then the minimal normal subgroups of the centralizer in G of the Fitting group of G are central products of isomorphic quasisimple groups. Such groups are called *quasi-semi-simple*. The candidates for the quasi-semi-simple normal subgroups of G may be derived from the following table:

Table 9.1. Table of the quasisimple matrix groups admitting a homogenous representation into $\mathcal{Q}^{n\times n}$ for a totally definite quaternion algebra $\mathcal{Q}$ with center of degree d over $\mathbb{Q}$ and $d \cdot n \leq 10$:

group N	$\mathcal{B}^\circ(N)$	character of N	$\overline{\mathbb{Q}N}$	$Glide(N)$
Alt_5	$\pm Alt_5$	$\chi_{3a}+\chi_{3b}$	$\mathbb{Q}[\sqrt{5}]^{3\times 3}$	1
Alt_5	$\pm S_5$	χ_4	$\mathbb{Q}^{4\times 4}$	1
Alt_5	$\pm S_6$	χ_5	$\mathbb{Q}^{5\times 5}$	-
$SL_2(5)$	$SL_2(5)$	$2(\chi_{2a}+\chi_{2b})$	$\mathcal{Q}_{\sqrt{5},\infty,\infty}$	1
$SL_2(5)$	$SL_2(9)$	$2\chi_4$	$\mathcal{Q}_{\infty,3}^{2\times 2}$	-
$SL_2(5)$	$SL_2(5)$	$2\chi_6$	$\mathcal{Q}_{\infty,2}^{3\times 3}$	C_2 (2)
$L_2(7)$	$\pm L_2(7)$	$\chi_{3a}+\chi_{3b}$	$\mathbb{Q}[\sqrt{\text{-}7}]^{3\times 3}$	1
$L_2(7)$	$\pm L_2(7)$	χ_6	$\mathbb{Q}^{6\times 6}$	C_2 (2)
$L_2(7)$	$\pm S_6(2)$	χ_7	$\mathbb{Q}^{7\times 7}$	-
$L_2(7)$	$\pm L_2(7):2$	χ_8	$\mathbb{Q}^{8\times 8}$	1

group N	$\mathcal{B}^\circ(N)$	character of N	$\overline{\mathbb{Q}N}$	$Glide(N)$
$SL_2(7)$	$SL_2(7)$	$\chi_{4a}+\chi_{4b}$	$\mathbb{Q}[\sqrt{-7}]^{4\times 4}$	1
$SL_2(7)$	$SL_2(7)$	$2(\chi_{6a}+\chi_{6b})$	$\mathcal{Q}^{3\times 3}_{\sqrt{2},\infty,\infty}$	C_2 $(2+\sqrt{2})$
$SL_2(7)$	$SL_2(7)$	$2\chi_8$	$\mathcal{Q}^{4\times 4}_{\infty,3}$	C_2 (3)
Alt_6	$\pm S_6$	χ_{5a} resp. χ_{5b}	$\mathbb{Q}^{5\times 5}$	1
Alt_6	$\pm S_{10}$	χ_9	$\mathbb{Q}^{9\times 9}$	-
Alt_6	$\pm S_6$	χ_{10}	$\mathbb{Q}^{10\times 10}$	C_2 (2)
$SL_2(9)$	$SL_2(9)$	$2\chi_{4a}$ resp. $2\chi_{4b}$	$\mathcal{Q}^{2\times 2}_{\infty,3}$	C_2 (3)
$SL_2(9)$	$SL_2(9)$	$2(\chi_{8a}+\chi_{8b})$	$\mathcal{Q}^{4\times 4}_{\sqrt{5},\infty,\infty}$	C_2 $(5+2\sqrt{5})$
$SL_2(9)$	$SL_2(9)$	$2(\chi_{10a}+\chi_{10b})$	$\mathcal{Q}^{5\times 5}_{\sqrt{2},\infty,\infty}$	C_2 $(2+\sqrt{2})$
$3.Alt_6$	$\pm 3.Alt_6$	$\chi_{3a}+\chi'_{3a}$ $+\chi_{3b}+\chi'_{3b}$	$\mathbb{Q}[\sqrt{5},\sqrt{-3}]^{3\times 3}$	1
$3.Alt_6$	$\pm 3.Alt_6$	$\chi_6+\chi'_6$	$\mathbb{Q}[\sqrt{-3}]^{6\times 6}$	C_2 (2)
$3.Alt_6$	$\pm 3.M_{10}$	$\chi_9+\chi'_9$	$\mathbb{Q}[\sqrt{-3}]^{9\times 9}$	1
$L_2(8)$	$\pm S_6(2)$	χ_7	$\mathbb{Q}^{7\times 7}$	-
$L_2(8)$	$2.O_8^+(2).2$	χ_8	$\mathbb{Q}^{8\times 8}$	-
$L_2(11)$	$\pm L_2(11)$	$\chi_{5a}+\chi_{5b}$	$\mathbb{Q}[\sqrt{-11}]^{5\times 5}$	1
$L_2(11)$	$\pm L_2(11):2$	χ_{10a}	$\mathbb{Q}^{10\times 10}$	1
$L_2(11)$	$\pm L_2(11)$	χ_{10b}	$\mathbb{Q}^{10\times 10}$	C_2 (3)
$SL_2(11)$	$SL_2(11)$	$\chi_{6a}+\chi_{6b}$	$\mathbb{Q}[\sqrt{-11}]^{6\times 6}$	1
$SL_2(11)$	$SL_2(11)$	$2\chi_{10}$	$\mathcal{Q}^{5\times 5}_{\infty,2}$	C_2 (2)
$SL_2(11)$	$SL_2(11)$	$2(\chi_{10a}+\chi_{10b})$	$\mathcal{Q}^{5\times 5}_{\sqrt{3},\infty,\infty}$	C_2 (2)
$L_2(13)$	$\pm L_2(13)$	$\chi_{7a}+\chi_{7b}$	$\mathbb{Q}[\sqrt{13}]^{7\times 7}$	1
$SL_2(13)$	$SL_2(13)$	$2(\chi_{6a}+\chi_{6b})$	$\mathcal{Q}^{3\times 3}_{\sqrt{13},\infty,\infty}$	1
$SL_2(13)$	$SL_2(13)$	$2\chi_{14}$	$\mathcal{Q}^{7\times 7}_{\infty,2}$	C_2 (2)
$SL_2(17)$	$SL_2(17)$	$2(\chi_{8a}+\chi_{8b})$	$\mathcal{Q}^{4\times 4}_{\sqrt{17},\infty,\infty}$	1
$SL_2(17)$	$SL_2(17)$	$2\chi_{16}$	$\mathcal{Q}^{8\times 8}_{\infty,3}$	C_2 (3)
Alt_7	$\pm S_7$	χ_6	$\mathbb{Q}^{6\times 6}$	1
Alt_7	$\pm Alt_7$	$\chi_{10a}+\chi_{10b}$	$\mathbb{Q}[\sqrt{-7}]^{10\times 10}$	1
$2.Alt_7$	$2.Alt_7$	$\chi_{4a}+\chi_{4b}$	$\mathbb{Q}[\sqrt{-7}]^{4\times 4}$	1
$2.Alt_7$	$2.Alt_7$	$2\chi_{20a}$	$\mathcal{Q}^{10\times 10}_{\infty,3}$	C_2 (3)
$2.Alt_7$	$2.Alt_7$	$2\chi_{20b}$	$\mathcal{Q}^{10\times 10}_{\infty,3}$	C_2 (6)
$3.Alt_7$	$6.U_4(3).2$	$\chi_6+\chi'_6$	$\mathbb{Q}[\sqrt{-3}]^{6\times 6}$	-
$L_2(19)$	$\pm L_2(19)$	$\chi_{9a}+\chi_{9b}$	$\mathbb{Q}[\sqrt{-19}]^{9\times 9}$	1
$SL_2(19)$	$SL_2(19)$	$\chi_{10a}+\chi_{10b}$	$\mathbb{Q}[\sqrt{-19}]^{10\times 10}$	1
$SL_2(19)$	$SL_2(19)$	$2\chi_{18}$	$\mathcal{Q}^{9\times 9}_{\infty,2}$	C_2 (2)
$SL_2(19)$	$SL_2(19)$	$2\chi_{20}$	$\mathcal{Q}^{10\times 10}_{\infty,3}$	C_2 (3)

group N	$\mathcal{B}^\circ(N)$	character of N	$\overline{\mathbb{Q}N}$	$Glide(N)$
$U_3(3)$	$\pm U_3(3)$	$2\chi_6$	$\mathcal{Q}_{\infty,3}^{3\times 3}$	C_2 (3)
$U_3(3)$	$\pm S_6(2)$	χ_7	$\mathbb{Q}^{7\times 7}$	-
$U_3(3)$	$U_3(3)\circ C_4$	$\chi_{7a}+\chi_{7b}$	$\mathbb{Q}[\sqrt{-1}]^{7\times 7}$	1
$SL_2(25)$	$SL_2(25)$	$2\chi_{12}$	$\mathcal{Q}_{\infty,5}^{6\times 6}$	C_2 (5)
M_{11}	$\pm S_{11}$	χ_{10a}	$\mathbb{Q}^{10\times 10}$	-
M_{11}	$\pm M_{11}$	$\chi_{10b}+\chi_{10c}$	$\mathbb{Q}[\sqrt{-2}]^{10\times 10}$	1
Alt_8	$\pm S_6(2)$	χ_7	$\mathbb{Q}^{7\times 7}$	-
$2.Alt_8$	$2.O_8^+(2).2$	χ_8	$\mathbb{Q}^{8\times 8}$	-
$2.L_3(4)$	$2.L_3(4):2_2$	$\chi_{10a}+\chi_{10b}$	$\mathbb{Q}[\sqrt{-7}]^{10\times 10}$	1
$6.L_3(4)$	$6.L_3(4)$	$\chi_6+\chi_6'$	$\mathbb{Q}[\sqrt{-3}]^{6\times 6}$	C_2 (2)
$U_4(2)$	$\pm U_4(2)\circ C_3$	$\chi_{5a}+\chi_{5b}$	$\mathbb{Q}[\sqrt{-3}]^{5\times 5}$	1
$U_4(2)$	$\pm U_4(2):2$	χ_6	$\mathbb{Q}^{6\times 6}$	1
$U_4(2)$	$\pm U_4(2)\circ C_3$	$\chi_{10a}+\chi_{10b}$	$\mathbb{Q}[\sqrt{-3}]^{10\times 10}$	1
$2.U_4(2)$	$2.U_4(2)\circ C_3$	$\chi_{4a}+\chi_{4b}$	$\mathbb{Q}[\sqrt{-3}]^{4\times 4}$	1
$2.U_4(2)$	$2.U_4(2)$	$2\chi_{20}$	$\mathcal{Q}_{\infty,2}^{10\times 10}$	C_2 (2)
$U_3(4)$	$2.G_2(4)$	$2\chi_{12}$	$\mathcal{Q}_{\infty,2}^{6\times 6}$	-
$2.M_{12}$	$2.M_{12}:2$	$\chi_{10a}+\chi_{10b}$	$\mathbb{Q}[\sqrt{-2}]^{10\times 10}$	1
$U_3(5)$	$\pm U_3(5):3$	$2\chi_{20}$	$\mathcal{Q}_{\infty,5}^{10\times 10}$	C_2 (5)
Alt_9	$2.O_8^+(2).2$	χ_8	$\mathbb{Q}^{8\times 8}$	-
$2.Alt_9$	$2.O_8^+(2).2$	χ_{8a} resp. χ_{8b}	$\mathbb{Q}^{8\times 8}$	-
$2.M_{22}$	$2.M_{22}:2$	$\chi_{10a}+\chi_{10b}$	$\mathbb{Q}[\sqrt{-7}]^{10\times 10}$	1
$2.J_2$	$2.J_2$	$2(\chi_{6a}+\chi_{6b})$	$\mathcal{Q}_{\sqrt{5},\infty,\infty}^{3\times 3}$	1
$2.J_2$	$2.J_2$	$2\chi_{14}$	$\mathcal{Q}_{\infty,2}^{7\times 7}$	C_2 (2)
$S_6(2)$	$\pm S_6(2)$	χ_7	$\mathbb{Q}^{7\times 7}$	1
$2.S_6(2)$	$2.O_8^+(2).2$	χ_8	$\mathbb{Q}^{8\times 8}$	-
Alt_{10}	$\pm S_{10}$	χ_9	$\mathbb{Q}^{9\times 9}$	1
$2.U_4(3)$	$2.U_4(3).4$	$2\chi_{20}$	$\mathcal{Q}_{\infty,3}^{10\times 10}$	C_2 (3)
$6.U_4(3)$	$6.U_4(3).2$	$\chi_6+\chi_6'$	$\mathbb{Q}[\sqrt{-3}]^{6\times 6}$	1
$U_5(2)$	$\pm U_5(2)$	$2\chi_{10}$	$\mathcal{Q}_{\infty,2}^{5\times 5}$	C_2 (2)
Alt_{11}	$\pm S_{11}$	χ_{10}	$\mathbb{Q}^{10\times 10}$	1
$2.O_8^+(2)$	$2.O_8^+(2).2$	χ_8	$\mathbb{Q}^{8\times 8}$	1
$2.G_2(4)$	$2.G_2(4)$	$2\chi_{12}$	$\mathcal{Q}_{\infty,2}^{6\times 6}$	C_2 (2)

The first column contains the quasisimple matrix group N, the second column its generalized Bravais group over $\mathbb{Q}$ (cf. Definition 7.1) followed by the character χ of a $\mathbb{Q}$-irreducible constituent of the natural $\overline{\mathbb{Q}N}$-module and the enveloping algebra $\overline{\mathbb{Q}N}$. If G is a primitive maximal finite group in $GL_n(\mathcal{D})$ with normal subgroup N, then G has a normal subgroup $\mathcal{B}^\circ(N)C_G(N)$ such that the factor group $G/(\mathcal{B}^\circ(N)C_G(N))$ embeds into $Glide(N).Gal(\mathbb{Q}[\chi]/\mathbb{Q})$ (cf. Definition 7.3). Especially if $|Glide(N)| = 2$, a norm of a nontrivial element of this group (cf. Definition 7.10) is given in brackets.

10. Some building blocks

By Chapter 7 we may build up the primitive maximal finite matrix groups using normal subgroups that satisfy a certain maximality condition.

Let $\mathcal{D}$ be a definite quaternion algebra with center K and $N = \mathcal{B}^\circ(N)$ be a normal subgroup of a primitive a.i.m.f. subgroup G of $GL_n(\mathcal{D})$. Assume that $\overline{KN}$ is a central simple K-algebra and let $G = (\Delta_1 \otimes \Delta_2)(G)$ be as in Theorem 7.8. Since K is totally real, Theorem 7.11 says that G contains the normal subgroup $U := C_G(N) = ker(\Delta_1)$ with $(\Delta_2(G)K^*)/(\Delta_2(U)K^*) \cong (\Delta_1(G)K^*)/(\Delta_1(N)K^*)$ of exponent 1 or 2. Choose $g_i \in \Delta_2(G)$ such that $(g_1, \dots, g_s)$ maps onto a basis of $(\Delta_2(G)K^*)/(\Delta_2(U)K^*)$. Then there are $q_i \in K^*$ such that $q_i^{-1}g_i^2 \in U$.

Lemma 10.1. *In the situation above, the pair $(U, S = \{g_1, \dots, g_s\})$ satisfies the following maximality condition: For all finite supergroups $V \geq U$ that are contained in $\overline{K\Delta_2(G)} =: A$, such that $g_i \in N_{A^*}(V)$ for all $1 \leq i \leq s$, one has $V = U$.*

Call such a pair (U, S) a *maximal pair* and U a *nearly maximal finite subgroup* of A^*. Note that if (U, S) is a maximal pair then, since $N_{A^*}(U) \subseteq N_{A^*}(\mathcal{B}^\circ(U))$, one has $U = \mathcal{B}^\circ(U)$.

Table 10.2. Assume that A is a quaternion algebra with center $\mathbb{Q}$ and $s = 1$. Then the maximal pairs $(U, \{g\})$ may be derived from the following table:

U	norm(g)	A
$\pm C_3$	2	$\mathcal{Q}_{2,3}$
$\pm C_3$	1	$\mathbb{Q}^{2\times 2}$
C_4	3	$\mathcal{Q}_{2,3}$
C_4	1	$\mathbb{Q}^{2\times 2}$
$\tilde{S}_3$	3	$\mathcal{Q}_{\infty,3}$
$SL_2(3)$	2	$\mathcal{Q}_{\infty,2}$
D_8	2	$\mathbb{Q}^{2\times 2}$
$\pm S_3$	3	$\mathbb{Q}^{2\times 2}$

Here the first column displays the matrix group U, the second column gives a norm of the element $g = g_1$ in the normalizer of U in A^* as defined in Definition 7.10, and the last column the central simple $\mathbb{Q}$-algebra $A = \overline{\Delta_2(G)}$ generated by U and g.

For central simple algebras $\overline{KN}$ the most important situation is that the g_i lie in the enveloping algebra (cf. Theorem 7.11). For the determination of the maximal possibilities for N, the following is helpful:

Remark 10.3. In the situation of Theorem 7.11, let $a \in N_{\overline{KN}^*}(N)$ be normalized such that $aFa^t = qF$ where $q \in R$ is a norm of α (cf. Definition 7.10). Let L be an a^2q^{-1}-invariant RN-lattice and assume that $F \in \mathcal{F}_{>0}(N)$ is integral on L. Then $qL \subseteq La \subseteq L$. If the ideal generated by q and $det(F, L)$ is the ring of integers of K, then F defines a bilinear form $\bar{F} : L/qL \times L/qL \to R/qR$. Since the dual lattice of La with respect to F is $(La)^{\#} = q^{-1}L^{\#}a$, the lattice La corresponds to a maximal isotropic subspace of L/qL.

Let $N(\leq GL_8(\mathbb{Q}))$ be a finite matrix group such that the enveloping $\mathbb{Q}$-algebra $\overline{N}$ is a central simple $\mathbb{Q}$-algebra of dimension 16. Assume that all abelian characteristic subgroups of N are cyclic. If the pair $(N, \{g\})$ with $g \in N_{\overline{\mathbb{Q}N}^*}(N)$ is a maximal pair, then N is one of the groups in the following table:

Table 10.4

N	norm(g)	$\overline{\mathbb{Q}N}$
A_4	1	$\mathbb{Q}^{4\times 4}$
$\pm C_5 : C_4$	5	$\mathbb{Q}^{4\times 4}$
F_4	$1, 2$	$\mathbb{Q}^{4\times 4}$
$C_3 \overset{2}{\square}\ SL_2(3)$	$3, 6$	$\mathbb{Q}^{4\times 4}$
$S_3 \otimes D_8$	$2, 6$	$\mathbb{Q}^{4\times 4}$
$SL_2(5) : 2$	$1, 5$	$\mathcal{Q}_{\infty,5}^{2\times 2}$
$SL_2(5).24$	$1, 5$	$\mathcal{Q}_{\infty,5}^{2\times 2}$
$2_-^{1+4}.Alt_5$	$1, 2$	$\mathcal{Q}_{\infty,2}^{2\times 2}$
$C_3 \overset{2(2)}{\boxtimes} D_8$	$3, 6$	$\mathcal{Q}_{\infty,2}^{2\times 2}$
$S_3 \otimes SL_2(3)$	$1, 2, 3, 6$	$\mathcal{Q}_{\infty,2}^{2\times 2}$
$C_3 \overset{2}{\square}\ SL_2(3)$	$1, 2, 3, 6$	$\mathcal{Q}_{\infty,3}^{2\times 2}$
$SL_2(9)$	$1, 3$	$\mathcal{Q}_{\infty,3}^{2\times 2}$
$\tilde{S}_3 \otimes D_8$	$2, 6$	$\mathcal{Q}_{\infty,3}^{2\times 2}$
$C_3 \overset{2(2)}{\boxtimes} D_8$	$1, 2, 3, 6$	$\mathcal{Q}_{2,3}^{2\times 2}$
$\tilde{S}_3 \otimes_{\sqrt{-3}} SL_2(3)$	$1, 2, 3, 6$	$\mathcal{Q}_{2,3}^{2\times 2}$

In the first column of this table the finite matrix group N is given using the notation of Chapter 5. The last column displays the enveloping $\mathbb{Q}$-algebra $A := \bar{N}$ of N. The second column allows one to read off the elements $g \in N_{A^*}(N)$ such that $(N, \{g\})$ is a maximal pair, since these are modulo N uniquely determined by their norms (cf. Corollary 7.12) . In particular N is a maximal finite subgroup of A^*, if and only if a "1" appears in this column.

Proof. The proof is divided into 5 cases according to the possible enveloping algebras $\overline{\mathbb{Q}N} = \mathcal{Q}^{2\times 2}$ where $\mathcal{Q}$ is either a definite or an indefinite quaternion algebra with center $\mathbb{Q}$. If $\mathcal{Q}$ is definite, Theorem 12.1 below implies that $\mathcal{Q}$ is one of $\mathcal{Q}_{\infty,2}$, $\mathcal{Q}_{\infty,3}$, or $\mathcal{Q}_{\infty,5}$. In the indefinite case it follows from [Sou 94] that $\mathcal{Q}$ is either $\mathbb{Q}^{2\times 2}$ or $\mathcal{Q}_{2,3}$. In the last case, N is already maximal finite, since the 2 maximal finite subgroups of $GL_2(\mathcal{Q}_{2,3})$ are the generalized Bravais groups of their minimal absolutely irreducible subgroups.

If $\overline{\mathbb{Q}N} = \mathbb{Q}^{4\times 4}$, then N is contained in one of the three r.i.m.f. subgroups $G = A_4$, F_4, or A_2^2 of $GL_4(\mathbb{Q})$. Let L be the natural G-lattice.

If $N \leq A_4$, then the order of N is divisible by 5. Hence $N = A_4 \cong {\pm}S_5$ or $N = {\pm}C_5 : C_4$ is the generalized Bravais group of one of the two minimal absolutely irreducible subgroups Alt_5 or $C_5 : C_4$ of A_4. In the first case, $N_{GL_4(\mathbb{Q})}(N) = \mathbb{Q}^*N$, by Corollary 7.6. In the other case, N fixes additionally the lattices $A_4(1-x)$ and $A_4(1-x)^2$ where x generates $O_5(N)$. Hence $N_{GL_4(\mathbb{Q})}(N)$ additionally contains an element $(x + x^{-1})$ of norm 5 inducing a similarity $A_4 \sim A_4(1-x)^2$.

Since the lattice F_4 is 2-modular, the normalizer $N_{GL_4(\mathbb{Q})}(F_4)$ contains an element of norm 2 ([Neb 97, Proposition 3]). Apart from $2L^{\#}$, there is no other sublattice M with $2L \subset M \subset L$ which is similar to L. Moreover 2 and 3 are the only primes dividing the group order. Hence by Remark 10.3, the absolutely irreducible nearly maximal finite subgroups N contained in F_4 are the absolutely irreducible stabilizers of the maximal isotropic subspaces of $L/3L$. There are eight such subspaces lying in one orbit under the action of the group F_4. The stabilizer of such a subspace is $SL_2(3) \overset{2}{\Box} C_3$.

Similarly, the absolutely irreducible nearly maximal finite proper subgroups of A_2^2 stabilize one of the 6 maximal isotropic subspaces of $(A_2^2)/2(A_2^2)$. All these stabilizers are conjugate to $S_3 \otimes D_8$.

Now assume that $\mathcal{Q}$ is a definite quaternion algebra. Then N embeds into one of the six primitive a.i.m.f. groups of Theorem 12.1 or into $SL_2(3) \wr C_2$ or $\tilde{S}_3 \wr C_2$. Let $\mathfrak{M}$ be a maximal order of $\mathcal{Q}$.

If $\mathcal{Q} = \mathcal{Q}_{\infty,5}$, then N embeds into one of $SL_2(5).2$ or $SL_2(5) : 2$. As in the case $N \leq A_4$ the only other possibility for a nearly maximal finite group is ${\pm}C_5.C_4$. But now the outer automorphism of ${\pm}C_5.C_4$ mapping an element x of order 8 in ${\pm}C_5.C_4$ onto $-x$ extends to an automorphism of $SL_2(5).2$ and $SL_2(5) : 2$ stabilizing the character. Hence the nonsplit extension ${\pm}C_5.C_4$ is not a nearly maximal finite group.

If $\mathcal{Q} = \mathcal{Q}_{\infty,2}$, then N is a subgroup of one of the three a.i.m.f. groups $2_-^{1+4}.Alt_5$, $SL_2(3) \otimes S_3$, or $SL_2(3) \wr C_2$. Since $2_-^{1+4}.Alt_5$ has an element of norm 2 in its normalizer, and the stabilizers of the other ten $\mathfrak{M}/2\mathfrak{M}$-subspaces of $L/2L$ corresponding to lattices which are similar to L are not absolutely irreducible, one finds with Remark 10.3 that N is a stabilizer of one of the forty $\mathfrak{M}$-sublattices corresponding to the maximal isotropic subspaces of the $\mathfrak{M}/3\mathfrak{M}$-module $L/3L$, where L is a $\mathfrak{M}2_-^{1+4}Alt_5$-lattice. These lattices lie in one orbit under $2_-^{1+4}.Alt_5$. One calculates $N = C_3 \overset{2(2)}{\boxtimes} D_8$ in this case.

All prime divisors of the order of $G := S_3 \otimes SL_2(3)$ arise as norms of elements of the normalizer of N in $\overline{N}^*$. Since the absolutely irreducible subgroups of G are characteristic in G Corollary 7.12 implies that the group N has no proper nearly maximal finite subgroups.

If N is a subgroup of $SL_2(3) \wr C_2$, the forty maximal isotropic $\mathfrak{M}/3\mathfrak{M}$-subspaces of $L/3L$ fall into two orbits of length 16 and 24. Their stabilizers are not absolutely irreducible. The stabilizers of the thirteen $\mathfrak{M}/2\mathfrak{M}$-subspaces of $L/2L$ corresponding to lattices which are similar to L are either $SL_2(3)\wr C_2$ which has a noncyclic abelian normal subgroup or the subgroup $D_8 \otimes SL_2(3)$ of index 12. Since $\mathcal{B}^{\circ}(D_8 \otimes Q_8) = 2_-^{1+4}.Alt_5$, one finds no groups N here.

In the last case, $\mathcal{Q} = \mathcal{Q}_{\infty,3}$. Now N is a subgroup of one of the a.i.m.f. groups G conjugate to $SL_2(3) \overset{2}{\square} C_3$, $SL_2(9)$, or $\tilde{S}_3 \wr C_2$. All three groups admit an element of norm 3 in their normalizer.

In the first case, G itself admits an element of norm 2 in its normalizer. The minimal absolutely irreducible subgroups of G are $\tilde{S}_4$ and $Q_8 \overset{2}{\square} C_3$. For $p = 2$ and 3, there is only one proper $\mathfrak{M}\tilde{S}_4$-sublattice of L containing pL which is similar to L. This lattice is also fixed by G, hence $N \neq \tilde{S}_4$. Clearly $N \neq Q_8 \overset{2}{\square} C_3$, because $N \neq \mathcal{B}^{\circ}(N) = G$.

In the other two cases, the proper subspaces of the $\mathfrak{M}/3\mathfrak{M}$ module $L/3L$ give rise to thirty-one resp. thirty-seven lattices similar to L. In the first case, their stabilizers are either G or subgroups of index 20 resp. 10 in G normalizing a Sylow 3-subgroup ($\cong C_3 \times C_3$) of G. In the last case, the thirty-seven lattices fall into two orbits. The lattices which are not fixed by G have a reducible stabilizer $\cong C_8$.

Hence by Remark 10.3 N is an absolutely irreducible stabilizer of one of the fifteen $\mathfrak{M}$ sublattices corresponding to the maximal isotropic subspaces of the $\mathfrak{M}/2\mathfrak{M}$-module $L/2L$, where L is a $\mathfrak{M}G$-lattice. In the first case, these subspaces form one orbit. The stabilizer of such a subspace is $\tilde{S}_4$ and not absolutely irreducible. In the last case, the fifteen maximal isotropic subspaces fall into two orbits of length 9 respectively 6 under the action of G. Only a representative of the second orbit has an absolutely irreducible stabilizer $N = D_8 \otimes \tilde{S}_3$. □

11. Special dimensions

There are some cases where it is easy to describe an infinite family of simplicial complexes $M_n^{irr}(\mathcal{D})$. Two of them are dealt with in the next theorem.

Theorem 11.1. (i) *Let* $p \equiv 1 \pmod 4$ *be a prime.* $M_{\frac{p-1}{4}}(\mathcal{Q}_{\sqrt{p},\infty})$ *consists of a single vertex:* ${}_{\sqrt{p},\infty}[SL_2(p)]_{\frac{p-1}{4}}$. *The group* $SL_2(p)$ *fixes an even unimodular* $\mathbb{Z}$*-lattice (of rank* $2(p-1)$*).*

(ii) *Let* p *be a prime. If* $p \equiv 1 \pmod 4$*, then* $M_{\frac{p-1}{2}}(\mathcal{Q}_{\infty,p})$ *consists of one* 1*-dimensional simplex:*

$${}_{\infty,p}[SL_2(p).2]_{\frac{p-1}{2}} \qquad\qquad {}_{\infty,p}[SL_2(p):2]_{\frac{p-1}{2}}$$

where the common absolutely irreducible subgroup of the two a.i.m.f. groups is ${}^{\pm}C_p.C_{p-1}$. *The corresponding* $\mathbb{Z}$*-lattices are unimodular (for the nonsplit extension* ${}_{\infty,p}[SL_2(p).2]_{\frac{p-1}{2}}$*) resp.* p*-modular (for the split extension* ${}_{\infty,p}[SL_2(p):2]_{\frac{p-1}{2}}$*).*

If $p \equiv -1 \pmod 4$*, then* $M_{\frac{p-1}{2}}(\mathcal{Q}_{\infty,p})$ *consists of one single vertex:* ${}_{\infty,p}[{}^{\pm}L_2(p).2]_{\frac{p-1}{2}}$. *The group* ${}_{\infty,p}[{}^{\pm}L_2(p).2]_{\frac{p-1}{2}}$ *fixes an even* p*-modular* $\mathbb{Z}$*-lattice (of rank* $2(p-1)$*).*

To prove the theorem, we need a lemma which is also of independent interest in later chapters.

Lemma 11.2. *Let $\mathcal{D}$ be a definite quaternion algebra with center K and $d := [K : \mathbb{Q}] = 1$ or 2. Let p be an odd prime such that $n := \frac{p-1}{2d} \in \mathbb{N}$. If $G \leq GL_n(\mathcal{D})$ is an a.i.m.f. subgroup, then $O_p(G) = 1$.*

Proof. Assume the $O_p(G) > 1$. Then by the formula in [Schu 05] $P := O_p(G) \cong C_p$ and in the case $d = 2$, $K = \mathbb{Q}[\sqrt{p}]$ and $p \equiv 1 \pmod 4$. Since the commuting algebra $C_{\mathcal{D}^{n\times n}}(P)$ is isomorphic to $\mathbb{Q}[\zeta_p]$, the centralizer $C_G(P) = \pm P$. Now G is absolutely irreducible, so $G/C_G(P) \cong C_{\frac{p-1}{d}}$ is isomorphic to the subgroup of index d in the automorphism group of P. The split extension $\pm P : C_{\frac{p-1}{d}}$ has real Schur index 1, and the nonsplit extension $G = \pm P.C_{\frac{p-1}{d}}$ is a subgroup of ${}_{\sqrt{p},\infty}[SL_2(p)]_{\frac{p-1}{4}}$ (if $d = 2$), ${}_{\infty,p}[SL_2(p).2]_{\frac{p-1}{2}}$ (if $d = 1$ and $p \equiv 1 \pmod 4$), resp. ${}_{\infty,p}[\pm L_2(p).2]_{\frac{p-1}{2}}$ (if $d = 1$ and $p \equiv 3 \pmod 4$) which is a contradiction. □

Proof of Theorem 11.1. From the classification of a.i.m.f. subgroups of $GL_1(\mathcal{Q}_{\sqrt{5},\infty})$ and $GL_{\frac{p-1}{2}}(\mathcal{Q}_{\infty,p})$ for $p \leq 11$ in this paper, the theorem is true for $p \leq 11$. So we may assume $p \geq 13$.

(i) Let $\mathcal{Q} := \mathcal{Q}_{\sqrt{p},\infty}$, $n := \frac{p-1}{4}$, and G be an a.i.m.f. subgroup of $GL_n(\mathcal{Q})$.

Then by Lemma 2.13 p divides the order of G. By Proposition 2.16 the Sylow p-subgroup P of G is $\cong C_p$. Since the degree of the natural character of G is $\frac{p-1}{2}$ [Fei 82, Theorem VIII.7.2] implies that either $G/Z(G) \cong PSL_2(p)$ or $P \trianglelefteq G$ is normal in G. The second case is excluded by Lemma 11.2. Since ± 1 are the only roots of unity contained in the center $\mathbb{Q}[\sqrt{p}]$, the results on the Schur indices of the characters of $SL_2(p)$ in [Fei 83] yield that $G = SL_2(p)$ in the first case. The $SL_2(p)$-invariant $\mathbb{Z}$-lattices are described in [Neb 98].

(ii) Let $-1 \in G \leq GL_{\frac{p-1}{2}}(\mathcal{Q}_{\infty,p})$ be absolutely irreducible.

Then by Lemma 2.13 p divides the order of G. By Proposition 2.16 the Sylow p-subgroup of G has order p. Let $\mathfrak{M}$ be a maximal order in $\mathcal{Q}_{\infty,p}$, $\mathcal{P}$ the maximal two-sided ideal of $\mathfrak{M}$ containing p, and $L \in \mathcal{Z}_{\mathfrak{M}}(G)$ a $\mathfrak{M}G$-lattice in the natural G-module $\mathcal{Q}_{\infty,p}^{1\times\frac{p-1}{2}}$. Then $\bar{L} := L/\mathcal{P}L$ is a $\mathbb{F}_{p^2}G$-module of dimension $\frac{p-1}{2}$. Since the kernel of the action of G on $\bar{L}$ coincides with the one on $\mathcal{P}L/pL$, this kernel is contained in $O_p(G)$. By Lemma 11.2 $O_p(G) = 1$, so $\bar{L}$ is a faithful $\mathbb{F}_{p^2}G$-module and [Fei 82, Theorem (VIII.3.3)] implies that G is of type $L_2(p)$, i.e. the unique composition factor $O^{p'}(G)/(O^{p'}(G) \cap O_{p'}(G))$ of G of order divisible by p is either isomorphic to $L_2(p)$ or to C_p. Here $O^{p'}(G)$ is the smallest normal subgroup of G of index prime to p and $O_{p'}(G)$ the largest normal subgroup of G of order prime to p. Let g be an element of order p in G. Then $C_G(g)$ embeds into $GL_1(\mathbb{Q}[\zeta_p])$, hence is $\langle \pm g\rangle$. Therefore g acts fixed point freely on $O_{p'}(G)/\langle \pm 1\rangle$. By Thompson's theorem (cf. [Hup 67], p. 505) $O_{p'}(G)/\langle \pm 1\rangle$ is nilpotent. Let $r \neq p$ be a prime and A an abelian normal r-subgroup of G. Then A is cyclic by Corollary 2.4 and the enveloping algebra of A is contained in $\mathbb{Q}^{(p-1)\times(p-1)}$. Hence by Corollary 2.4 $r < p$ and g centralizes A. Therefore $r = 2$ and $A \leq \langle \pm 1\rangle$. Hence $O_{p'}(G) = O_2(G)$ and the maximal abelian normal subgroup of G is $\langle \pm 1\rangle$.

If $O_2(G) > \pm 1$, then Proposition 8.9 gives a contradiction to the fact that the p-adic Schur index of the natural representation of G is 2.

Hence $O_2(G) = {\pm}1$ and $O^{p'}(G)$ is one of ${\pm}L_2(p)$, $SL_2(p)$, or ${\pm}C_p$. By Lemma 11.2 the latter possibility does not occur. [Fei 83, Theorem 6.1] yields that the p-adic Schur indices of the representations of $SL_2(p)$ are 1. Hence the $\mathbb{C}$-constituents of the restriction of the natural representation of G to $O^{p'}(G)$ are of degree $\frac{p-1}{2}$. Using the character tables in [Schu 07] one concludes that if $p \equiv -1 \pmod 4$, then G is the unique extension of ${\pm}L_2(p) \le GL_{\frac{p-1}{2}}(\mathbb{Q}[\sqrt{-p}])$ by $C_2 \cong Out(L_2(p))$ with real Schur index 2, and if $p \equiv 1 \pmod 4$, then G is one of the two extensions of the matrix group $SL_2(p)$ of (i).

For $p \equiv 1 \pmod 4$, the $SL_2(p).2$-invariant and $SL_2(p):2$-invariant $\mathbb{Z}$-lattices are described in [Neb 98, Remark 2.5]. For their determinants see [Tie 97, Section 5]. If $p \equiv 3 \pmod 4$, the natural representation of the group $L_2(p) \le {}_{\infty,p}[{\pm}L_2(p).2]_{\frac{p-1}{2}}$ is a globally irreducible representation ([Gro 90, Chapter 11]). The $\mathbb{Z}[\frac{1+\sqrt{-p}}{2}]L_2(p)$-lattices are unimodular Hermitian lattices over $\mathbb{Z}[\frac{1+\sqrt{-p}}{2}]$. The lattices of which the endomorphism ring is a maximal order $\mathfrak{M}$ of $\mathcal{Q}_{\infty,p}$ containing $\mathbb{Z}[\frac{1+\sqrt{-p}}{2}]$ and which are preserved by the a.i.m.f. group ${}_{\infty,p}[{\pm}L_2(p).2]_{\frac{p-1}{2}}$ are scalar extensions of these unimodular lattices and therefore also unimodular Hermitian. Since the discriminant of $\mathfrak{M}$ is generated by $\sqrt{-p}$, they become p-modular $\mathbb{Z}$-lattices. □

12. The a.i.m.f. subgroups of $GL_2(\mathcal{Q})$

$$Z(\mathcal{Q}) = \mathbb{Q}$$

Theorem 12.1. *Let $\mathcal{Q}$ be a definite quaternion algebra with center $\mathbb{Q}$ and G be a maximal finite primitive absolutely irreducible subgroup of $GL_2(\mathcal{Q})$. Then $\mathcal{Q}$ is one of $\mathcal{Q}_{\infty,2}$, $\mathcal{Q}_{\infty,3}$, or $\mathcal{Q}_{\infty,5}$ and G is conjugate to one of the groups in the following table.*

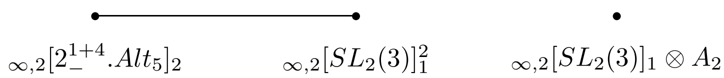
List of the primitive a.i.m.f. subgroups of $GL_2(\mathcal{Q})$.

lattice L	$\lvert Aut(L)\rvert$	r.i.m.f. supergroups
${}_{\infty,2}[2_-^{1+4}.Alt_5]_2$	$2^7\cdot 3\cdot 5$	E_8
${}_{\infty,2}[SL_2(3)]_1 \otimes A_2$	$2^4\cdot 3^2$	$A_2 \otimes F_4$
${}_{\infty,3}[SL_2(9)]_2$	$2^4\cdot 3^2\cdot 5$	E_8
${}_{\infty,3}[SL_2(3) \overset{2}{\square} C_3]_2$	$2^4\cdot 3^2$	F_4^2
${}_{\infty,5}[SL_2(5).2]_2$	$2^4\cdot 3\cdot 5$	E_8
${}_{\infty,5}[SL_2(5):2]_2$	$2^4\cdot 3\cdot 5$	$[(SL_2(5) \overset{2}{\square} SL_2(5)):2]_8$

Proof. If G contains a quasi-semi-simple normal subgroup, [HaS 85] and [CCNPW 85] show that either $G^{(\infty)} \cong SL_2(5)$ and $G = {}_{\infty,5}[SL_2(5).2]_2$ or $G = {}_{\infty,5}[SL_2(5):2]_2$ or $G = G^{(\infty)} = {}_{\infty,3}[SL_2(9)]_2$. Now assume that G contains no quasi-semi-simple normal subgroup. By Lemma 11.2 $O_5(G) = 1$. If $O_3(G) = 1$, then $O_2(G)$ is a self-centralizing normal subgroup of G. With Proposition 8.9 one finds that $G = \mathcal{B}^\circ(O_2(G)) = {}_{\infty,2}[2_-^{1+4}.Alt_5]_2$.

If $O_3(G) > 1$, then $O_3(G) \cong C_3$, G contains $C := C_G(O_3(G))$ of index two, and C is an absolutely irreducible subgroup of $(\mathbb{Q}[\zeta_3] \otimes \mathcal{Q})^*$. Using [Bli 17] one finds that C is one of $C_3 \circ SL_2(3)$ or $C_3 \otimes D_8$. In both cases, one has two possible automorphisms of C yielding each a unique extension $G = C.2$ in $GL_2(\mathcal{Q})$ (cf. Lemma 2.17). In the first case, G is one of ${}_{\infty,2}[SL_2(3)]_1 \otimes A_2$ or ${}_{\infty,3}[SL_2(3) \stackrel{2}{\Box} C_3]_2$. In the second case, one finds no groups G, since $\tilde{S}_3 \otimes D_8$ is imprimitive and $C_3 \stackrel{2(2)}{\boxtimes} D_8$ is a proper subgroup of ${}_{\infty,2}[2_-^{1+4}.Alt_5]_2$. $\Box$

Theorem 12.2. *$M_2(\mathcal{Q}_{\infty,2})^{irr}$ is as follows.*

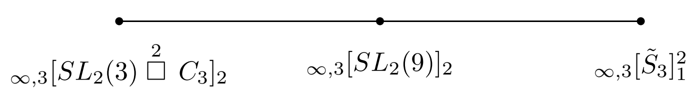


List of the maximal simplices in $M_2^{irr}(\mathcal{Q}_{\infty,2})$

simplex	a common subgroup
$({}_{\infty,2}[2_-^{1+4}.Alt_5]_2, {}_{\infty,2}[SL_2(3)]_1^2)$	$D_8 \otimes Q_8$

Proof. The a.i.m.f. subgroups of $GL_2(\mathcal{Q}_{\infty,2})$ can be deduced from Theorem 12.1 and Theorem 6.1. The completeness of the list of maximal simplices in $M_2^{irr}(\mathcal{Q}_{\infty,2})$ follows from the fact, that the unique minimal absolutely irreducible subgroup of ${}_{\infty,2}[SL_2(3)]_1 \otimes A_2$ is $S_3 \otimes Q_8$ and does not fix any 3-unimodular lattice with maximal order as an endomorphism ring. $\Box$

Theorem 12.3. *$M_2(\mathcal{Q}_{\infty,3})^{irr}$ is as follows.*



List of the maximal simplices in $M_2^{irr}(\mathcal{Q}_{\infty,3})$

simplex	a common subgroup
$({}_{\infty,3}[SL_2(9)]_2, {}_{\infty,3}[\tilde{S}_3]_1^2)$	$(\pm C_3 \times C_3).C_4$
$({}_{\infty,3}[SL_2(9)]_2, {}_{\infty,3}[SL_2(3) \stackrel{2}{\Box} C_3]_2)$	$\tilde{S}_4$

Proof. The a.i.m.f. subgroups of $GL_2(\mathcal{Q}_{\infty,3})$ can be deduced from Theorem 12.1 and Theorem 6.1. The completeness of the list of maximal simplices in $M_2^{irr}(\mathcal{Q}_{\infty,3})$ follows from the fact, that the minimal absolutely irreducible subgroups of ${}_{\infty,3}[SL_2(3) \stackrel{2}{\Box} C_3]_2$ are $\tilde{S}_4$ and $Q_8 \stackrel{2}{\Box} C_3$. Whereas the first group also embeds into ${}_{\infty,3}[SL_2(9)]_2$, the second one has a unique a.i.m.f. supergroup. $\Box$

Theorem 12.4. *$M_2(\mathcal{Q}_{\infty,5})^{irr}$ is as follows.*

List of the maximal simplices in $M_2^{irr}(\mathcal{Q}_{\infty,5})$

simplex	a common subgroup
$({}_{\infty,5}[SL_2(5).2]_2, {}_{\infty,5}[SL_2(5):2]_2)$	$(\pm C_5).C_4$

Proof. The theorem follows immediately from Theorem 12.1 and Theorem 6.1. □

$Z(\mathcal{Q})$ real quadratic.

Theorem 12.5. *Let G be an absolutely irreducible maximal finite subgroup of $GL_2(\mathcal{Q})$, where $\mathcal{Q}$ is a totally definite quaternion algebra with center K and $[K:\mathbb{Q}]=2$. Assume that G has a quasi-semi-simple normal subgroup. Then $\mathcal{Q}$ is isomorphic to $\mathcal{Q}_{\sqrt{5},\infty}$ or $\mathcal{Q}_{\sqrt{3},\infty}$ and G is conjugate to one of*

$${}_{\sqrt{5},\infty}[SL_2(5) \underset{\sqrt{5}}{\otimes} D_{10}]_2,\ {}_{\sqrt{5},\infty}[SL_2(5)]_1 \otimes A_2,\ {}_{\sqrt{3},\infty}[2.S_6]_2,$$

or the imprimitive group ${}_{\sqrt{5},\infty}[SL_2(5)]_1^2$.

Proof. Let G be such a maximal finite group and $N \trianglelefteq G$ a quasi-semi-simple normal subgroup. If G is imprimitive Theorem 6.1 implies that $G = {}_{\sqrt{5},\infty}[SL_2(5)]_1^2$. Assume now that G is primitive. By Table 9.1 N is either $SL_2(5)$ or $SL_2(9)$ (cf. also [HaS 85]). Assume first that N is $SL_2(5)$. Then the enveloping algebra of N is $\overline{\mathbb{Q}N} = \mathcal{Q}_{\sqrt{5},\infty}$. If $K \neq \mathbb{Q}[\sqrt{5}]$, then N is an irreducible subgroup of $GL_2(\mathcal{Q})$. The centralizer $C_G(N)$ embeds into $C_{GL_2(\mathcal{Q})}(N) \cong K[\sqrt{5}]^*$. Since $K[\sqrt{5}]$ is a totally real field, one gets that $C_G(N) = \pm 1$. Therefore G contains N of index 2 and by Lemma 2.14 the enveloping $\mathbb{Q}$-algebra of G is of dimension 8 or 16, contradicting the assumption that G is an absolutely irreducible subgroup of $GL_2(\mathcal{Q})$.

Hence $K = \mathbb{Q}[\sqrt{5}]$ is the center of the enveloping algebra $\overline{\mathbb{Q}N}$. Then N is primitively saturated over K and hence $G = N \underset{K}{\otimes} C$, for some centrally irreducible maximal finite subgroup of $(C_{\mathcal{Q}^{2\times 2}}(N))^*$. The commuting algebra of N is an indefinite quaternion algebra with center $\mathbb{Q}[\sqrt{5}]$ ramified at those primes on which $\mathcal{Q}$ ramifies. The classification of finite subgroups of $GL_2(\mathbb{C})$ ([Bli 17]) now shows that $C_{\mathcal{Q}^{2\times 2}}(\mathbb{Q}N)$ is isomorphic to $\mathbb{Q}[\sqrt{5}]^{2\times 2}$. Moreover $C_G(N)$ is one of $\pm D_{10}$, $\pm S_3$, or D_8. One computes that the two groups ${}_{\sqrt{5},\infty}[SL_2(5) \underset{\sqrt{5}}{\otimes} D_{10}]_2$ and ${}_{\sqrt{5},\infty}[SL_2(5)]_1 \otimes A_2$ are maximal finite whereas the third group is a proper subgroup of the imprimitive maximal finite group ${}_{\sqrt{5},\infty}[SL_2(5)]_1^2$.

If $N = SL_2(9)$, the enveloping algebra $\overline{\mathbb{Q}N}$ of N is $\mathcal{Q}_{\infty,3}^{2\times 2}$. Therefore the centralizer $C := C_G(N)$ has to be contained in the center of $GL_2(\mathcal{Q}) \cong K^*$. Since K is totally real, one has $C = \pm 1$. The factor group G/N is a subgroup of $Glide(N) = C_2$ and [CCNPW 85] implies that $\mathcal{Q} = \mathcal{Q}_{\sqrt{3},\infty}$ and $G = N.2 = {}_{\sqrt{3},\infty}[2.S_6]_2$. □

Theorem 12.6. *Let G be an absolutely irreducible maximal finite subgroup of $GL_2(\mathcal{Q})$, where $\mathcal{Q}$ is a totally definite quaternion algebra with center K and $[K:\mathbb{Q}]=2$. Assume that G has no quasi-semi-simple normal subgroup. Then $\mathcal{Q}$ is isomorphic to $\mathcal{Q}_{\sqrt{2},\infty}$, $\mathcal{Q}_{\sqrt{2},\infty,2,3}$, $\mathcal{Q}_{\sqrt{3},\infty}$, $\mathcal{Q}_{\sqrt{5},\infty,2,5}$, $\mathcal{Q}_{\sqrt{5},\infty,5,3}$, or $\mathcal{Q}_{\sqrt{6},\infty}$*

and G is conjugate to one of the primitive groups

$$_{\sqrt{2},\infty}[2_-^{1+4}.S_5]_2,\ _{\sqrt{2},\infty}[\tilde{S}_4]_1 \otimes A_2,\ _{\sqrt{2},\infty,2,3}[C_3 \overset{2(2+\sqrt{2})}{\boxtimes} D_{16}]_2,$$

$$_{\sqrt{2},\infty,2,3}[C_3 \overset{2(2+\sqrt{2})}{\Box} Q_{16}]_2,\ _{\sqrt{3},\infty}[D_{24} \overset{2(2)}{\otimes} SL_2(3)]_2,\ _{\sqrt{5},\infty,2,5}[C_5 \overset{2(2)}{\boxtimes} D_8],$$

$$_{\sqrt{5},\infty,2,5}[C_5 \overset{2(2)}{\boxtimes} SL_2(3)],\ _{\sqrt{5},\infty,5,3}[\pm C_5 \overset{2(3)}{\boxtimes} S_3],$$

$$_{\sqrt{5},\infty,5,3}[C_5 \overset{2(3)}{\boxtimes} \tilde{S}_3],\ \text{or}\ _{\sqrt{6},\infty}[GL_2(3) \overset{2}{\boxtimes} C_3]_2,$$

or to one of the imprimitive groups $_{\sqrt{2},\infty}[\tilde{S}_4]_1^2$ *or* $_{\sqrt{3},\infty}[C_{12}.C_2]_1^2$.

Proof. The imprimitive a.i.m.f. groups may be determined with Theorem 6.1 so assume that G is primitive. If p is a prime with $O_p(G) \neq 1$, then by Corollary 2.4 one has that $p \in \{2,3,5\}$.

Assume first that $O_5(G) \neq 1$. Then $N := O_5(G) \cong C_5$. The centralizer $C := C_G(O_5(G))$ embeds into the commuting algebra $C_{\mathcal{Q}^{2\times 2}}(O_5(G))$ which is either isomorphic to an indefinite quaternion algebra $\mathcal{Q}'$ with center $\mathbb{Q}[\zeta_5]$ if $K = \mathbb{Q}[\sqrt{5}]$ or to $K[\zeta_5]$ if the center K of $\mathcal{Q}$ is not the maximal real subfield of $\mathbb{Q}[\zeta_5]$. In the latter case one finds $C = \pm C_5$ which contradicts the assumption that G is absolutely irreducible. Therefore $K = \mathbb{Q}[\sqrt{5}]$. Since the prime divisors of $|G|$ lie in $\{2,3,5\}$ the only finite places, on which $\mathcal{Q}$ is ramified contain one of these 3 primes. Therefore, $\mathbb{Q}[\zeta_5]$ splits $\mathcal{Q}$ and one has $\mathcal{Q}' = \mathbb{Q}[\zeta_5]^{2\times 2}$. Moreover G contains C of index 2. Hence by Lemma 2.14, C is an absolutely irreducible subgroup of $GL_2(\mathbb{Q}[\zeta_5])$. Using the classification of finite subgroups of $PGL_2(\mathbb{C})$ in [Bli 17] together with the assumption that G contains no quasi-semi-simple normal subgroup, $O_5(G) = C_5$ and $C = \mathcal{B}^\circ(C)$, one finds that C is one of $C_5 \otimes D_8$, $C_5 \otimes_{\sqrt{5}'} SL_2(3)$, $\pm C_5 \otimes S_3$, or $C_5 \otimes_{\sqrt{5}'} \tilde{S}_3$.

If G centralizes $C/O_5(G)$, then G is a proper subgroup of one of the three groups involving $SL_2(5)$ of Theorem 12.5.

If G induces the nontrivial outer automorphism of $C/O_5(G)$, one has $2 = |H^2(C_2, C_2)|$ possible extensions $G = C.2$, only one of which has real Schur index 1, by Lemma 2.17. One computes that G is

$$_{\sqrt{5},\infty,2,5}[C_5 \overset{2(2)}{\boxtimes} D_8],\ _{\sqrt{5},\infty,2,5}[C_5 \overset{2(2)}{\boxtimes} SL_2(3)],\ _{\sqrt{5},\infty,5,3}[\pm C_5 \overset{2(3)}{\boxtimes} S_3],\ _{\sqrt{5},\infty,5,3}[C_5 \overset{2(3)}{\boxtimes} \tilde{S}_3],$$

respectively.

Assume now that $O_5(G) = 1$ and $O_3(G) \neq 1$. Then $O_3(G) \cong C_3$. The centralizer $C := C_G(O_3(G))$ embeds into the commuting algebra $C_{\mathcal{Q}^{2\times 2}}(O_3(G))$ which is isomorphic to an indefinite quaternion algebra $\mathcal{Q}'$ over $K[\zeta_3]$. Since G is absolutely irreducible and contains C of index 2, Lemma 2.14 implies that C is an absolutely irreducible subgroup of $GL_1(\mathcal{Q}')$. Hence $\mathcal{Q}' = K[\zeta_3]^{2\times 2}$. The classification of finite subgroups of $PGL_2(\mathbb{C})$ ([Bli 17]) together with the assumption that G is primitive, has no quasi-semi-simple normal subgroup, and satisfies $O_5(G) = 1$, one finds that C is one of $C_3 \otimes D_{16}$, $C_3 \otimes QD_{16}$, $C_3 \otimes GL_2(3)$, $C_3 \circ Q_{16}$, $C_3 \circ \tilde{S}_4$, or $O_2(C) = C_4 \circ Q_8$ and $C = C_3\mathcal{B}^\circ(O_2(G)) = C_3 \otimes (C_4 \circ SL_2(3)).2$.

If $C = C_3 \otimes D_{16}$ or $C_3 \circ Q_{16}$, the normalizer $N_{(\overline{\mathbb{Q}C})^*}(C)$ of C in the unit group of its enveloping algebra contains $CK[\zeta_3]^*$ of index 2. In the other cases one has a unique outer automorphism of C inducing the Galois automorphism of $K[\zeta_3]$ over the maximal totally real subfield K of $K[\zeta_3]$. Therefore one finds in these cases only two possible extensions $G = C.2$, only one of which has real Schur index 1. One concludes that G is one of $\tilde{S}_3 \otimes D_{16}$, $C_3 \overset{2(2+\sqrt{2})}{\boxtimes} D_{16}$, $C_3 \overset{2}{\boxtimes} QD_{16}$, $C_3 \overset{2}{\boxtimes} GL_2(3)$, $C_3 \overset{2(2+\sqrt{2})}{\square} Q_{16}$, $S_3 \otimes Q_{16}$, $S_3 \otimes \tilde{S}_4$, respectively $C_3 \overset{2}{\boxtimes} (C_4 \circ SL_2(3)).2$. The first group is a proper subgroup of ${}_{\sqrt{2},\infty}[2_-^{1+4}.S_5]_2$, the third is contained in the fourth group ${}_{\sqrt{6},\infty}[GL_2(3) \overset{2}{\boxtimes} C_3]_2$, the sixth one in the seventh group ${}_{\sqrt{2},\infty}[\tilde{S}_4]_1 \otimes A_2$, and the last group is ${}_{\sqrt{3},\infty}[C_3 \overset{2}{\boxtimes} (C_4 \circ SL_2(3)).2]_2 = {}_{\sqrt{3},\infty}[D_{24} \overset{2(2)}{\otimes} SL_2(3)]_2$.

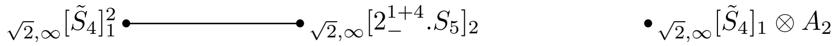
If $O_p(G) = 1$ for all odd primes p, then Proposition 8.9 gives that $O_2(G)$ is one of $D_8 \otimes Q_8$ or $D_8 \otimes Q_{16}$. In the first case $G = {}_{\sqrt{2},\infty}[2_-^{1+4}.S_5]_2$ is maximal finite. In the last case, $O_2(G)$ is an absolutely irreducible subgroup of $GL_2(\mathcal{Q}_{\sqrt{2},\infty})$. Let $\mathfrak{M}$ be the up to conjugacy unique maximal order in $\mathcal{Q}_{\sqrt{2},\infty}$. One computes the Bravais group on a normal critical $\mathfrak{M}O_2(G)$-lattice (cf. Definition 2.7) to be ${}_{\sqrt{2},\infty}[2_-^{1+4}.S_5]_2$, contradicting the assumptions on $O_2(G)$. □

TABLE 12.7. List of the primitive a.i.m.f. subgroups of $GL_2(\mathcal{Q})$ where $\mathcal{Q}$ is a totally definite quaternion algebra over a real quadratic number field $Z(\mathcal{Q})$.

lattice L	$\lvert Aut(L) \rvert$	r.i.m.f. supergroups
${}_{\sqrt{2},\infty}[2_-^{1+4}.S_5]_2$	$2^8 \cdot 3 \cdot 5$	$F_4 \tilde{\otimes} F_4$, E_8^2
${}_{\sqrt{2},\infty}[\tilde{S}_4]_1 \otimes A_2$	$2^5 \cdot 3^2$	$(A_2 \otimes F_4)^2$, $A_2 \otimes E_8$
${}_{\sqrt{2},\infty,2,3}[C_3 \overset{2(2+\sqrt{2})}{\boxtimes} D_{16}]_2$	$2^5 \cdot 3$	$(A_2 \otimes F_4)^2$, A_2^8
${}_{\sqrt{2},\infty,2,3}[C_3 \overset{2(2+\sqrt{2})}{\square} Q_{16}]_2$	$2^5 \cdot 3$	F_4^4, E_8^2
${}_{\sqrt{3},\infty}[D_{24} \overset{2(2)}{\otimes} SL_2(3)]_2$	$2^6 \cdot 3^2$	$(A_2 \otimes F_4)^2$, E_8^2
		$A_2 \otimes E_8$, $F_4 \tilde{\otimes} F_4$
${}_{\sqrt{3},\infty}[2.S_6]_2$	$2^5 \cdot 3^2 \cdot 5$	E_8^2, $[(Sp_4(3) \circ C_3) \underset{\sqrt{-3}}{\overset{2}{\boxtimes}} SL_2(3)]_{16}$
		$[SL_2(9) \underset{\infty,3}{\overset{2(3)}{\otimes}} SL_2(9) : 2]_{16}$, $F_4 \tilde{\otimes} F_4$
${}_{\sqrt{5},\infty}[SL_2(5) \otimes_{\sqrt{5}} D_{10}]_2$	$2^4 \cdot 3 \cdot 5^2$	$[(SL_2(5) \circ SL_2(5)) : 2 \underset{\sqrt{5}}{\overset{2}{\boxtimes}} D_{10}]_{16}$
${}_{\sqrt{5},\infty}[SL_2(5)]_1 \otimes A_2$	$2^4 \cdot 3^2 \cdot 5$	$A_2 \otimes [(SL_2(5) \overset{2}{\square} SL_2(5)) : 2]_8$, $A_2 \otimes E_8$
${}_{\sqrt{5},\infty,2,5}[C_5 \overset{2(2)}{\boxtimes} D_8]$	$2^4 \cdot 5$	A_4^4
${}_{\sqrt{5},\infty,2,5}[C_5 \overset{2(2)}{\boxtimes} SL_2(3)]$	$2^4 \cdot 3 \cdot 5$	E_8^2, $[(SL_2(5) \overset{2}{\square} SL_2(5)) : 2]_8^2$
${}_{\sqrt{5},\infty,5,3}[\pm C_5 \overset{2(3)}{\boxtimes} S_3]$	$2^3 \cdot 3 \cdot 5$	$(A_2 \otimes A_4)^2$
		$(A_2 \otimes A_4)^2$
${}_{\sqrt{5},\infty,5,3}[C_5 \overset{2(3)}{\boxtimes} \tilde{S}_3]$	$2^3 \cdot 3 \cdot 5$	E_8^2, $[(SL_2(5) \overset{2}{\square} SL_2(5)) : 2]_8^2$
		E_8^2, $[(SL_2(5) \overset{2}{\square} SL_2(5)) : 2]_8^2$
${}_{\sqrt{6},\infty}[(S_3 \otimes SL_2(3)).2]_2$	$2^5 \cdot 3^2$	$(A_2 \otimes F_4)^2$, $F_4 \tilde{\otimes} F_4$
		$(A_2 \otimes F_4)^2$, F_4^4, $[(SL_2(5) \overset{2}{\square} SL_2(5)) : 2]_8^2$
		$A_2 \otimes E_8$, E_8^2, $[SL_2(5) \underset{\infty,2}{\overset{2(2)}{\boxtimes}} 2_-^{1+4}.Alt_5]_{16}$

The first column contains representatives G of the conjugacy classes of a.i.m.f. subgroups of $GL_2(\mathcal{Q})$, the second the order of the corresponding groups. In the third column the r.i.m.f. supergroups of G that act on a lattice $L \in \mathcal{Z}_{\mathfrak{M}}(G)$ for some maximal order of $\mathcal{Q}$ are given. There is one line for each conjugacy class of maximal orders in $\mathcal{Q}$ which come in the same order as in Table 4.1.

Theorem 12.8. *$M_2^{irr}(\mathcal{Q}_{\sqrt{2},\infty})$ is as follows.*



List of the maximal simplices in $M_2^{irr}(\mathcal{Q}_{\sqrt{2},\infty})$

simplex	*a common subgroup*
$({}_{\sqrt{2},\infty}[2_-^{1+4}.S_5]_2, {}_{\sqrt{2},\infty}[\tilde{S}_4]_1^2)$	$Q_{16} \otimes D_8$

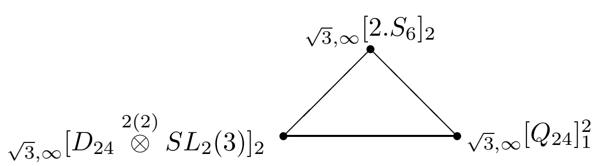
Proof. The completeness of the list of a.i.m.f. subgroups in $GL_2(\mathcal{Q}_{\sqrt{2},\infty})$ follows from Theorems 6.1, 12.5 and 12.6. To see that the list of maximal simplices in $M_2^{irr}(\mathcal{Q}_{\sqrt{2},\infty})$ is complete one has to note that the unique minimal absolutely irreducible subgroup of ${}_{\sqrt{2},\infty}[\tilde{S}_4]_1 \otimes A_2$ is $Q_{16} \otimes S_3$ and not contained in one of the other a.i.m.f. groups. $\square$

Theorem 12.9. *$M_2^{irr}(\mathcal{Q}_{\sqrt{2},\infty,2,3})$ consists of two 0-simplices*

$\bullet \; {}_{\sqrt{2},\infty,2,3}[C_3 \overset{2(2+\sqrt{2})}{\boxtimes} D_{16}]_2$ $\qquad\qquad$ $\bullet \; {}_{\sqrt{2},\infty,2,3}[C_3 \overset{2(2+\sqrt{2})}{\square} Q_{16}]_2$

Proof. The completeness of the list of a.i.m.f. subgroups in $GL_2(\mathcal{Q}_{\sqrt{2},\infty,2,3})$ follows from Theorems 6.1, 12.5, and 12.6. Both a.i.m.f. groups are minimal absolutely irreducible whence the theorem follows. $\square$

Theorem 12.10. *$M_2^{irr}(\mathcal{Q}_{\sqrt{3},\infty})$ is as follows.*



List of the maximal simplices in $M_2^{irr}(\mathcal{Q}_{\sqrt{3},\infty})$

simplex	*a common subgroup*
$({}_{\sqrt{3},\infty}[D_{24} \overset{2(2)}{\otimes} SL_2(3)]_2, {}_{\sqrt{3},\infty}[Q_{24}]_1^2)$	$D_{24} \otimes Q_8$
$({}_{\sqrt{3},\infty}[D_{24} \overset{2(2)}{\otimes} SL_2(3)]_2, {}_{\sqrt{3},\infty}[2.S_6]_2)$	$\tilde{S}_4.2$
$({}_{\sqrt{3},\infty}[Q_{24}]_1^2, {}_{\sqrt{3},\infty}[2.S_6]_2)$	$(\pm C_3 \times C_3).D_8$

Proof. The completeness of the list of a.i.m.f. subgroups in $GL_2(\mathcal{Q}_{\sqrt{3},\infty})$ follows from Theorems 6.1, 12.5, and 12.6. The list of maximal simplices in $M_2^{irr}(\mathcal{Q}_{\sqrt{3},\infty})$ is complete as one can see by computing the lattices of $(\pm C_3 \times C_3).D_8$ and $\tilde{S}_4.2$, the minimal absolutely irreducible subgroups of ${}_{\sqrt{3},\infty}[2.S_6]_2$ of order not divisible by 5. $\square$

Remark 12.11. The simplicial complex $M_2(\mathcal{Q}_{\sqrt{2},\infty})$ contains $M_2(\mathcal{Q}_{\infty,2})$, in the sense, that for every vertex $v \in M_2(\mathcal{Q}_{\infty,2})$ there is a vertex of $v' \in M_2(\mathcal{Q}_{\sqrt{2},\infty})$ with representatives G_v respectively $G_{v'}$, such that $G_v \leq G_{v'}$ and for every simplex $(v_1, \ldots, v_s) \in M_2(\mathcal{Q}_{\infty,2})$, the corresponding simplex $(v_1', \ldots, v_s')$ is a simplex in $M_2(\mathcal{Q}_{\sqrt{2},\infty})$.

In this sense the simplicial complex $M_2(\mathcal{Q}_{\sqrt{3},\infty})$ contains $M_2(\mathcal{Q}_{\infty,3})$.

Theorem 12.12. $M_2^{irr}(\mathcal{Q}_{\sqrt{5},\infty})$ *is as follows.*

$${}_{\sqrt{5},\infty}[SL_2(5) \otimes_{\sqrt{5}} D_{10}]_2 \bullet\!\!-\!\!-\!\!-\!\!-\!\!\bullet\ {}_{\sqrt{5},\infty}[SL_2(5)]_1^2 \qquad\qquad \bullet\ {}_{\sqrt{5},\infty}[SL_2(5)]_1 \otimes A_2$$

List of the maximal simplices in $M_2^{irr}(\mathcal{Q}_{\sqrt{5},\infty})$

simplex	a common subgroup
$({}_{\sqrt{5},\infty}[SL_2(5) \otimes_{\sqrt{5}} D_{10}]_2, {}_{\sqrt{5},\infty}[SL_2(5)]_1^2)$	$Q_{20} \otimes_{\sqrt{5}} D_{10}$

Proof. The completeness of the list of a.i.m.f. subgroups in $GL_2(\mathcal{Q}_{\sqrt{5},\infty})$ follows from Theorems 6.1, 12.5, and 12.6. To see that the list of maximal simplices in $M_2^{irr}(\mathcal{Q}_{\sqrt{5},\infty})$ is complete one has to note that the unique minimal absolutely irreducible subgroup of ${}_{\sqrt{5},\infty}[SL_2(5)]_1 \otimes A_2$ is $Q_{20} \otimes S_3$ and not contained in one of the other a.i.m.f. groups. $\square$

Theorem 12.13. $M_2^{irr}(\mathcal{Q}_{\sqrt{5},\infty,2,5})$ *consists of two* 0*-simplices*

$$\bullet\ {}_{\sqrt{5},\infty,2,5}[C_5 \overset{2(2)}{\boxtimes} D_8]_2 \qquad\qquad \bullet\ {}_{\sqrt{5},\infty,2,5}[C_5 \overset{2(2)}{\boxtimes} SL_2(3)]_2$$

Proof. The completeness of the list of a.i.m.f. subgroups in $GL_2(\mathcal{Q}_{\sqrt{5},\infty,2,5})$ follows from Theorems 6.1, 12.5, and 12.6. The completeness of the list of maximal simplices in $M_2^{irr}(\mathcal{Q}_{\sqrt{5},\infty,2,5})$ follows from the fact that the group ${}_{\sqrt{5},\infty,2,5}[C_5 \overset{2(2)}{\boxtimes} D_8]_2$ is minimal absolutely irreducible. $\square$

Similarly one gets:

Theorem 12.14. *$M_2^{irr}(\mathcal{Q}_{\sqrt{5},\infty,5,3})$ consists of two 0-simplices*

- ${}_{\sqrt{5},\infty,5,3}[\pm C_5 \overset{2(2)}{\boxtimes} S_3]_2$
- ${}_{\sqrt{5},\infty,5,3}[C_5 \overset{2(2)}{\boxtimes} \tilde{S}_3]_2$

$Z(\mathcal{Q})$ real cubic.

Theorem 12.15. *Let $\mathcal{Q}$ be a definite quaternion algebra with center K of degree 3 over $\mathbb{Q}$ and G a primitive a.i.m.f. subgroup of $GL_2(\mathcal{Q})$. Then G is one of the groups in the following table, which is built up as in Table 12.7:*

List of the primitive a.i.m.f. subgroups of $GL_2(\mathcal{Q})$.

lattice L	$\lvert Aut(L)\rvert$	r.i.m.f. supergroups
${}_{\theta_9,\infty,3}[C_9 \underset{\sqrt{-3}}{\overset{2(2)}{\boxtimes}} SL_2(3)]_2$	$2^4 \cdot 3^3$	F_4^6, $[3_+^{1+2} : SL_2(3) \underset{\sqrt{-3}}{\overset{2}{\boxtimes}} SL_2(3)]_{12}^2$
${}_{\theta_9,\infty,2}[D_{18} \otimes SL_2(3)]_2$	$2^4 \cdot 3^3$	$(A_2 \otimes F_4)^3$, $F_4 \otimes E_6$
		$F_4 \otimes E_6$
${}_{\theta_9,\infty,2}[C_9 \overset{2(2)}{\boxtimes} D_8]_2$	$2^4 \cdot 3^2$	E_8^2, $[Sp_4(3) \underset{\sqrt{-3}}{\overset{2}{\boxtimes}} 3_+^{1+2} : SL_2(3)]_{24}$
		(A_2^{12}), (E_6^4)
${}_{\theta_7,\infty,7}[Q_{28}]_1 \otimes A_2$	$2^3 \cdot 3 \cdot 7$	$(A_6 \otimes A_2)^2$, $(A_6^{(2)} \otimes A_2)^2$
${}_{\theta_7,\infty,7}[C_7 \underset{\sqrt{-7}}{\overset{2(3)}{\boxtimes}} \tilde{S}_3]_2$	$2^3 \cdot 3 \cdot 7$	$[6.U_4(3).2^2]_{12}^2$
${}_{\theta_7,\infty,7}[C_7 \overset{2(2)}{\boxtimes} D_8]_2$	$2^4 \cdot 7$	$[L_2(7) \overset{2(2)}{\otimes} D_8]_{12}^2$, $[L_2(7) \overset{2(2)}{\boxtimes} D_8]_{12}^2$
${}_{\theta_7,\infty,2}[D_{14} \otimes SL_2(3)]_2$	$2^4 \cdot 3 \cdot 7$	$A_6 \otimes F_4$, $A_6^{(2)} \otimes F_4$
${}_{\theta_7,\infty,2}[C_7 \overset{2(2)}{\boxtimes} SL_2(3)]_2$	$2^4 \cdot 3 \cdot 7$	$[L_2(7) \overset{2(2)}{\otimes} F_4]_{24}$, $[L_2(7) \overset{2(2)}{\boxtimes} F_4]_{24}$
${}_{\theta_7,\infty,3}[\pm C_7 \overset{2(3)}{\boxtimes} S_3]_2$	$2^3 \cdot 3 \cdot 7$	$[6.U_4(3).2^2]_{12}^2$
		$((A_2 \otimes A_6)^2)$, $((A_2 \otimes A_6^{(2)})^2)$
${}_{\theta_7,\infty,3}[D_{14} \otimes \tilde{S}_3]_2$	$2^3 \cdot 3 \cdot 7$	$(A_2 \otimes A_6)^2$, $(A_2 \otimes A_6^{(2)})^2$
		$([6.U_4(3).2^2]_{12}^2)$
${}_{\omega_{13},\infty,13}[\pm C_{13}.C_4]_2$	$2^3 \cdot 13$	(A_{12}^2)
		$[2.Co_1]_{24}$, $[SL_2(13) \overset{2(2)}{\square} SL_2(3)]_{24}$

Proof. Let $\mathcal{Q}$ be a definite quaternion algebra with center K of degree 3 over $\mathbb{Q}$ and G a primitive a.i.m.f. subgroup of $GL_2(\mathcal{Q})$. Then K is contained in a cyclotomic field of degree ≤ 12, hence $K \cong \mathbb{Q}[\theta_7]$, $\mathbb{Q}[\theta_9]$, or $\mathbb{Q}[\omega_{13}]$, where ω_{13} is a generator of the subfield of degree 3 over $\mathbb{Q}$ of the cyclotomic field $\mathbb{Q}[\zeta_{13}]$ and the $\theta_i = \zeta_i + \zeta_i^{-1}$ generate the maximal real subfield of $\mathbb{Q}[\zeta_i]$ (cf. Notation 4.2). By [CCNPW 85], G has no quasi-semi-simple normal subgroup. If $K = \mathbb{Q}[\omega_{13}]$, then 13 divides $|G|$. One concludes that $O_{13}(G) \cong C_{13}$ and $G = {}_{\omega_{13},\infty,13}[{\pm}C_{13}.C_4]_2$. Now assume that $K = \mathbb{Q}[\theta_7]$. Then 7 divides the order of G. Since the possible normal 2- and 3-subgroups have no automorphism of order 7, one has $O_7(G) \cong C_7$. The centralizer $C := C_G(C_7)$ is a centrally irreducible subgroup of $GL_1(\mathcal{D})$ for a quaternion algebra $\mathcal{D}$ with center $\mathbb{Q}[\zeta_7]$. One only has the possibilities $\mathcal{D} = \mathbb{Q}[\zeta_7]^{2\times 2}$ and $C = {\pm}C_7 \times U$, where U is one of D_8, S_3, or $\tilde{S}_3$, or $\mathcal{D} = \mathcal{Q}_{\zeta_7,2}$, where $C = C_7 \otimes SL_2(3)$. Since $|Out({\pm}U)| = 2$, one has in each case 2 possibilities for $G = C.2$, where there is always a unique extension yielding a representation with real Schur index 2 (cf. Lemma 2.17). Since $Q_{28} \otimes D_8$ is imprimitive, one finds the groups of the proposition. The case $K = \mathbb{Q}[\theta_9]$ is dealt with analogously. $\square$

Corollary 12.16. *Let $\mathcal{Q}$ be a definite quaternion algebra with center K of degree 3 over $\mathbb{Q}$ and G an a.i.m.f. subgroup of $GL_2(\mathcal{Q})$. Then $\mathcal{Q}$ is one of $\mathcal{Q}_{\theta_9,\infty,3}$, $\mathcal{Q}_{\theta_9,\infty,2}$, $\mathcal{Q}_{\theta_7,\infty,7}$, $\mathcal{Q}_{\theta_7,\infty,2}$, $\mathcal{Q}_{\theta_7,\infty,3}$, or $\mathcal{Q}_{\omega_{13},\infty,13}$. The simplicial complexes $M_2^{irr}(\mathcal{Q})$ consist of* 0*-simplices each.*

Proof. Theorems 6.1 and 12.15 prove the completeness of the list of quaternion algebras $\mathcal{Q}$ and of a.i.m.f. subgroups of $GL_2(\mathcal{Q})$. That there are no common absolutely irreducible subgroups, may be easily seen, since the groups ${}_{\theta_9,\infty,2}[C_9 \overset{2(2)}{\boxtimes} D_8]_2$, ${}_{\theta_7,\infty,7}[Q_{28}]_1 \otimes A_2$, ${}_{\theta_7,\infty,7}[C_7 \underset{\sqrt{-7}}{\overset{2(3)}{\boxtimes}} \tilde{S}_3]_2$, ${}_{\theta_7,\infty,7}[C_7 \overset{2(2)}{\boxtimes} D_8]_2$, ${}_{\theta_7,\infty,3}[{\pm}C_7 \overset{2(3)}{\boxtimes} S_3]_2$, and ${}_{\theta_7,\infty,3}[D_{14} \otimes \tilde{S}_3]_2$ are minimal absolutely irreducible and the minimal absolutely subgroups of ${}_{\theta_7,\infty,2}[D_{14} \otimes SL_2(3)]_2$ resp. ${}_{\theta_7,\infty,2}[C_7 \overset{2(2)}{\boxtimes} SL_2(3)]_2$ are $D_{14} \otimes Q_8$ resp. $C_7 \overset{2(2)}{\boxtimes} Q_8$ and not isomorphic.

Let $\mathfrak{M}$ be a maximal order of $\mathcal{Q}_{\theta_9,\infty,3}$ and U a minimal absolutely irreducible subgroup of ${}_{\theta_9,\infty,3}[C_9 \underset{\sqrt{-3}}{\overset{2(2)}{\boxtimes}} SL_2(3)]_2$. Then $U/O_2(U) \cong D_{18}$. Hence the 2-modular constituents of the natural representation of $U \otimes_{\theta_9} \mathfrak{M} \leq GL_{24}(\mathbb{Q})$ are of degree 12, so U cannot fix a 2-modular and a 2-unimodular lattice. So there is no common absolutely irreducible subgroup of ${}_{\theta_9,\infty,3}[Q_{36}]_1^2$ and ${}_{\theta_9,\infty,3}[C_9 \underset{\sqrt{-3}}{\overset{2(2)}{\boxtimes}} SL_2(3)]_2$. $\square$

$Z(\mathcal{Q})$ real quartic.

Theorem 12.17. *Let $\mathcal{Q}$ be a definite quaternion algebra with center K of degree 4 over $\mathbb{Q}$ and G a primitive a.i.m.f. subgroup of $GL_2(\mathcal{Q})$. Then G is one of the groups in the following table:*

List of the primitive a.i.m.f. subgroups of $GL_2(\mathcal{Q})$, where $\mathcal{Q}$ is a definite quaternion algebra with center K and $[K:\mathbb{Q}]=4$.

	${}_{\theta_{16,\infty}}[Q_{32}]_1 \otimes A_2$ ($2^6 \cdot 3$)
O_1	(A_2^{16})
O_2	$(F_4 \otimes A_2)^4$, $(E_8 \otimes A_2)^2$
	${}_{\theta_{16,\infty}}[\tilde{S}_4 \otimes_{\sqrt{2}} D_{32}]_2$ $(2^8 \cdot 3)$
O_1	(F_4^8)
O_2	$(F_4 \tilde{\otimes} F_4)^2$, E_8^4
	${}_{\theta_{24,\infty}}[\tilde{S}_4 \otimes_{\sqrt{2}} D_{48}]_2$ $(2^7 \cdot 3^2)$
O_1	$(A_2 \otimes F_4)^4$, $(A_2 \otimes E_8)^2$ $(F_4 \tilde{\otimes} F_4)^2$, E_8^4
O_2	(E_8^4), $((A_2 \otimes F_4)^4)$
O_3	$(F_4 \tilde{\otimes} F_4)^2$, $(A_2 \otimes E_8)^2$, $A_2 \otimes F_4 \tilde{\otimes} F_4$, $[2_+^{1+10}.O_{10}^+(2)]_{32}$
	${}_{\theta_{20,\infty}}[Q_{40}]_1 \otimes A_2$ $(2^4 \cdot 3 \cdot 5)$
O_1	$(A_2 \otimes A_4)^4$, $(A_2 \otimes [(SL_2(5) \overset{2}{\square} SL_2(5)) : 2]_8)^2$, $(A_2 \otimes E_8)^2$
O_2	$A_2 \otimes A_4 \otimes F_4$, $A_2 \otimes F_4 \tilde{\otimes} F_4$,
O_3	$(A_2 \otimes [(SL_2(5) \overset{2}{\square} SL_2(5)) : 2]_8)^2$, $(A_2 \otimes E_8)^2$,
	$A_2 \otimes [SL_2(5) \circ SL_2(5) : 2 \underset{\sqrt{5}}{\overset{2}{\boxtimes}} D_{10}]_{16}$
	${}_{\theta_{20,\infty}}[SL_2(5) \otimes_{\sqrt{5}} D_{40}]_2$ $(2^5 \cdot 3 \cdot 5^2)$
O_1	E_8^4, $[(SL_2(5) \overset{2}{\square} SL_2(5)) : 2]_8^4$, $[SL_2(5) \circ SL_2(5) : 2 \underset{\sqrt{5}}{\overset{2}{\boxtimes}} D_{10}]_{16}^2$
O_2	$E_8 \otimes F_4$, $[(SL_2(5) \overset{2}{\square} SL_2(5)) : 2]_8 \otimes F_4$, $[2_-^{1+4}.Alt_5 \otimes_{\infty,2} SL_2(5) \underset{\sqrt{5}}{\overset{2(2)}{\boxtimes}} D_{10}]_{32}$
O_3	$[((SL_2(5) \circ SL_2(5)) \underset{\sqrt{5}}{\overset{2}{\boxtimes}} (SL_2(5) \circ SL_2(5))) : S_4]_{32,i}$ $(i=1,2)$
	$[SL_2(5) \circ SL_2(5) : 2 \underset{\sqrt{5}}{\overset{2}{\boxtimes}} D_{10}]_{16}^2$
	${}_{\theta_{20,\infty}}[C_5 \overset{2}{\boxtimes} (C_4 \circ SL_2(3).2)]_2$ $(2^6 \cdot 3 \cdot 5)$
O_1	$(F_4 \tilde{\otimes} F_4)^2$, $(A_4 \otimes F_4)^2$, $[SL_2(5) \underset{\infty,2}{\overset{2(2)}{\boxtimes}} 2_-^{1+4}.Alt_5]_{16}^2$
O_2	$A_4 \otimes E_8$, $[2_+^{1+10}.O_{10}^+(2)]_{32}$, $[SL_2(5) \underset{\infty,2}{\overset{2(2)}{\boxtimes}} 2_-^{1+6}.O_6^-(2)]_{32}$
O_3	$E_8 \otimes F_4$, $[(SL_2(5) \overset{2}{\square} SL_2(5)) : 2]_8 \otimes F_4$, $[2_-^{1+4}.Alt_5 \otimes_{\infty,2} SL_2(5) \underset{\sqrt{5}}{\overset{2(2)}{\boxtimes}} D_{10}]_{32}$
	${}_{\theta_{15,\infty}}[SL_2(5) \otimes_{\sqrt{5}} D_{30}]_2$ $(2^4 \cdot 3^2 \cdot 5^2)$
O_1	$[SL_2(5) \circ SL_2(5) : 2 \underset{\sqrt{5}}{\overset{2}{\boxtimes}} D_{10}]_{16}^2$, $(A_2 \otimes E_8)^2$,
	$(A_2 \otimes [(SL_2(5) \overset{2}{\square} SL_2(5)) : 2]_8)^2$
O_2	$A_2 \otimes [SL_2(5) \circ SL_2(5) : 2 \underset{\sqrt{5}}{\overset{2}{\boxtimes}} D_{10}]_{16}$,
	$[((SL_2(5) \circ SL_2(5)) \underset{\sqrt{5}}{\overset{2}{\boxtimes}} (SL_2(5) \circ SL_2(5))) : S_4]_{32,i}$ $(i=1,2)$

	${}_{\theta_{15},\infty}[C_{15} \overset{2(2)}{\underset{\sqrt{-3}}{\boxtimes}} SL_2(3)]_2 \ (2^4 \cdot 3^2 \cdot 5)$
O_1	$(A_4 \otimes F_4)^2$, $[SL_2(5) \overset{2(3)}{\underset{\infty,3}{\boxtimes}} (SL_2(3) \overset{2}{\square} C_3)]_{16}^2$, $[Sp_4(3) \circ C_3 \overset{2}{\underset{\sqrt{-3}}{\boxtimes}} SL_2(3)]_{16}^2$
O_2	$E_8 \otimes F_4$, $F_4 \otimes [(SL_2(5) \overset{2}{\square} SL_2(5)) : 2]_8$,
	$[(SL_2(5) \otimes_{\sqrt{5}} D_{10}) \overset{2(3)}{\underset{\infty,3}{\boxtimes}} (SL_2(3) \overset{2}{\square} C_3)]_{32}$
	${}_{\theta_{15},\infty}[C_{15} \overset{2(2)}{\boxtimes} D_8]_2 \ (2^4 \cdot 3 \cdot 5)$
O_1	$(F_4 \tilde{\otimes} F_4)^2$, $[D_{120}.(C_4 \times C_2)]_{16}^2$, $[SL_2(5) \overset{2(2)}{\underset{\infty,2}{\boxtimes}} 2_-^{1+4}.Alt_5]_{16}^2$
O_2	$[(2_-^{1+4}.Alt_5 \otimes_{\infty,2} SL_2(5)) \overset{2(2)}{\underset{\sqrt{5}}{\boxtimes}} D_{10}]_{32}$
	$[((SL_2(5) \circ SL_2(5)) : 2 \overset{2(6)}{\underset{\sqrt{5}}{\boxtimes}} (C_3 \overset{2(2)}{\boxtimes} D_8)]_{32,i} \ (i = 1, 2)$
	${}_{\eta_{17},\infty}[\pm C_{17}.C_4]_2 \ (2^3 \cdot 17)$
O_1	$[SL_2(17) \overset{2(3)}{\square} \tilde{S}_3]_{32,i} \ (i = 1, 2)$, $[SL_2(17) \overset{2(3)}{\circ} \tilde{S}_3]_{32}$
O_2	$[2_+^{1+10}.O_{10}^+(2)]_{32}$
	${}_{\sqrt{2},\infty}[\tilde{S}_4]_1 \otimes {}_{\sqrt{5}}[\pm D_{10}]_2 \ (2^5 \cdot 3 \cdot 5)$
O_1	$(A_4 \otimes F_4)^2$, $A_4 \otimes E_8$, $[C_{15} : C_4 \overset{2(2)}{\boxtimes} F_4]_{32}$
O_2	$[SL_2(5) \circ SL_2(5) : 2 \overset{2}{\underset{\sqrt{5}}{\boxtimes}} D_{10}]_{16}^2$, $[(2_-^{1+4}.Alt_5 \otimes_{\infty,2} SL_2(5)) \overset{2(2)}{\underset{\sqrt{5}}{\boxtimes}} D_{10}]_{32}$,
	$[SL_2(9) \otimes D_{10} \overset{2}{\square} SL_2(5)]_{32}$, $[(SL_2(5) \otimes_{\sqrt{5}} D_{10}) \overset{2(3)}{\underset{\infty,3}{\boxtimes}} (SL_2(3) \overset{2}{\square} C_3)]_{32}$
	${}_{\sqrt{5},\infty}[SL_2(5)]_1 \otimes {}_{\sqrt{2}}[D_{16}]_2 \ (2^6 \cdot 3 \cdot 5)$
O_1	$[2_+^{1+10}.O_{10}^+(2)]_{32}$, $[(Sp_4(3) \otimes_{\sqrt{-3}} Sp_4(3)) : 2 \overset{2}{\square} C_3]_{32}$,
	$[SL_2(9).2 \overset{2(2)}{\underset{\infty,2}{\boxtimes}} 2_-^{1+4}.Alt_5]_{32}$, $[SL_2(5) \overset{2(2)}{\underset{\infty,2}{\boxtimes}} 2_-^{1+6}.O_6^-(2)]_{32}$,
	$(F_4 \tilde{\otimes} F_4)^2$, $[SL_2(5) \overset{2(2)}{\underset{\infty,2}{\boxtimes}} 2_-^{1+4}.Alt_5]_{16}^2$
O_2	$[((SL_2(5) \circ SL_2(5)) : 2 \overset{2(6)}{\underset{\sqrt{5}}{\boxtimes}} (C_3 \overset{2(2)}{\boxtimes} D_8)]_{32,i} \ (i = 1, 2)$, $E_8 \otimes F_4$,
	$[(SL_2(5) \overset{2}{\square} SL_2(5)) : 2]_8^4$, $[(SL_2(5) \overset{2}{\square} SL_2(5)) : 2]_8 \otimes F_4$, E_8^4
	${}_{\sqrt{2}+\sqrt{5},\infty,2,5}[C_5 \overset{2(2)}{\boxtimes} D_{16}]_2 \ (2^5 \cdot 5)$
O_1	A_4^8, $(A_4 \otimes F_4)^2$, $[D_{120}.(C_4 \times C_2)]_{16}^2$
O_2	$([D_{120}.(C_4 \times C_2)]_{16}^2)$
	${}_{\sqrt{2}+\sqrt{5},\infty,2,5}[C_5 \overset{2(2)}{\boxtimes} Q_{16}]_2 \ (2^5 \cdot 5)$
O_1	$[SL_2(5) \overset{2(3)}{\underset{\infty,3}{\boxtimes}} SL_2(9)]_{16}^2$, $[SL_2(9) \overset{2(3)}{\underset{\infty,3}{\otimes}} SL_2(9)]_{16}^2$, $[SL_2(5) \overset{2(2)}{\underset{\infty,2}{\boxtimes}} 2_-^{1+4}.Alt_5]_{16}^2$,
	$[Sp_4(3) \circ C_3 \overset{2}{\underset{\sqrt{-3}}{\boxtimes}} SL_2(3)]_{16}^2$, $[SL_2(5) \overset{2(3)}{\underset{\infty,3}{\boxtimes}} (SL_2(3) \overset{2}{\square} C_3)]_{16}^2$,
	E_8^4, $(F_4 \tilde{\otimes} F_4)^2$, $[(SL_2(5) \overset{2}{\square} SL_2(5)) : 2]_8^4$
O_2	$([Sp_4(3) \circ C_3 \overset{2}{\underset{\sqrt{-3}}{\boxtimes}} SL_2(3)]_{16}^2)$

	${}_{\eta_{40},\infty}[C_5 \overset{2}{\boxtimes} QD_{16}]_2$ $(2^5 \cdot 5)$
O_1	A_4^8, $(A_4 \otimes F_4)^2$, $(F_4 \tilde{\otimes} F_4)^2$, E_8^4, $[(SL_2(5) \overset{2}{\square} SL_2(5)) : 2]_8^4$,
	$[SL_2(5) \underset{\infty,2}{\overset{2(2)}{\boxtimes}} 2_-^{1+4}.Alt_5]_{16}^2$
O_2	$[(SL_2(5) \overset{2}{\square} SL_2(5)) : 2]_8^4$, E_8^4, $F_4 \otimes [(SL_2(5) \overset{2}{\square} SL_2(5)) : 2]_8$,
	$F_4 \otimes E_8$, $[SL_2(5) \circ SL_2(5) : 2 \underset{\sqrt{5}}{\overset{2}{\boxtimes}} D_{10}]_{16}^2$,
	$[(2_-^{1+4}.Alt_5 \otimes_{\infty,2} SL_2(5)) \underset{\sqrt{5}}{\overset{2(2)}{\boxtimes}} D_{10}]_{32}$
O_3	(A_4^8), $([(SL_2(5) \overset{2}{\square} SL_2(5)) : 2]_8^4)$, (E_8^4)
O_4	$F_4 \otimes [(SL_2(5) \overset{2}{\square} SL_2(5)) : 2]_8$, $F_4 \otimes E_8$, $[2_+^{1+10}.O_{10}^+(2)]_{32}$
	$[(2_-^{1+4}.Alt_5 \otimes_{\infty,2} SL_2(5)) \underset{\sqrt{5}}{\overset{2(2)}{\boxtimes}} D_{10}]_{32}$, $[SL_2(5) \underset{\infty,2}{\overset{2(2)}{\boxtimes}} 2_-^{1+6}.O_6^-(2)]_{32}$
O_5	$[2_+^{1+10}.O_{10}^+(2)]_{32}$, $(A_4 \otimes F_4)^2$, $[SL_2(5) \underset{\infty,2}{\overset{2(2)}{\boxtimes}} 2_-^{1+6}.O_6^-(2)]_{32}$
	$A_4 \otimes E_8$, $(F_4 \tilde{\otimes} F_4)^2$, $[SL_2(5) \underset{\infty,2}{\overset{2(2)}{\boxtimes}} 2_-^{1+4}.Alt_5]_{16}^2$
O_6	$A_4 \otimes E_8$, $(F_4 \tilde{\otimes} F_4)^2$, $[SL_2(5) \underset{\infty,2}{\overset{2(2)}{\boxtimes}} 2_-^{1+4}.Alt_5]_{16}^2$
	$[2_+^{1+10}.O_{10}^+(2)]_{32}$, $(A_4 \otimes F_4)^2$, $[SL_2(5) \underset{\infty,2}{\overset{2(2)}{\boxtimes}} 2_-^{1+6}.O_6^-(2)]_{32}$

	${}_{\sqrt{5},\infty}[SL_2(5)]_1 \otimes {}_{\sqrt{3}}[D_{24}]_2$ $(2^5 \cdot 3^2 \cdot 5)$
O_1	$(A_2 \otimes E_8)^2$, $(A_2 \otimes [(SL_2(5) \overset{2}{\square} SL_2(5)) : 2]_8)^2$,
	$F_4 \otimes [(SL_2(5) \overset{2}{\square} SL_2(5)) : 2]_8$,
	$[((SL_2(5) \circ SL_2(5)) \underset{\sqrt{5}}{\overset{2}{\boxtimes}} (SL_2(5) \circ SL_2(5))) : S_4]_{32,i}$ $(i = 1, 2)$,
	$F_4 \otimes E_8$, $[((SL_2(5) \circ SL_2(5)) : 2 \underset{\sqrt{5}}{\overset{2(6)}{\boxtimes}} (C_3 \overset{2(2)}{\boxtimes} D_8)]_{32,i}$ $(i = 1, 2)$
O_2	$F_4 \otimes E_8$, $F_4 \otimes [(SL_2(5) \overset{2}{\square} SL_2(5)) : 2]_8$, $A_2 \otimes F_4 \tilde{\otimes} F_4$, $[2_+^{1+10}.O_{10}^+(2)]_{32}$,
	$A_2 \otimes [SL_2(5) \underset{\infty,2}{\overset{2(2)}{\boxtimes}} 2_-^{1+4}.Alt_5]_{16}$, $[SL_2(5) \underset{\infty,2}{\overset{2(2)}{\boxtimes}} 2_-^{1+6}.O_6^-(2)]_{32}$,
	$[(Sp_4(3) \otimes_{\sqrt{-3}} Sp_4(3)) : 2 \overset{2}{\square} C_3]_{32}$, $[SL_2(5) \underset{\infty,3}{\overset{2(3)}{\boxtimes}} (Sp_4(3) \overset{2}{\square} C_3)]_{32}$
O_3	$(A_2 \otimes E_8)^2$, $(A_2 \otimes [(SL_2(5) \overset{2}{\square} SL_2(5)) : 2]_8)^2$, $F_4 \tilde{\otimes} F_4^2$,
	$[SL_2(5) \underset{\infty,2}{\overset{2(2)}{\boxtimes}} 2_-^{1+4}.Alt_5]_{16}^2$, E_8^4,
	$[(SL_2(5) \overset{2}{\square} SL_2(5)) : 2]_8^4$,
	$[SL_2(5) \underset{\infty,3}{\overset{2(3)}{\boxtimes}} (SL_2(3) \overset{2}{\square} C_3)]_{16}^2$, $[Sp_4(3) \circ C_3 \underset{\sqrt{-3}}{\overset{2}{\boxtimes}} SL_2(3)]_{16}^2$
O_4	$[SL_2(5) \circ C_5 \overset{2(3)}{\boxtimes} D_{24}]_{32}$

	${}_{\sqrt{3},\infty}[Q_{24}]_1 \otimes {}_{\sqrt{5}}[{}^{\pm}D_{10}]_2$ $(2^4 \cdot 3 \cdot 5)$
O_1	$[(2_-^{1+4}.Alt_5 \underset{\infty,2}{\otimes} SL_2(5)) \overset{2(2)}{\underset{\sqrt{5}}{\boxtimes}} D_{10}]_{32}$, $[(SL_2(5) \underset{\sqrt{5}}{\otimes} D_{10}) \overset{2(3)}{\underset{\infty,3}{\boxtimes}} (SL_2(3) \overset{2}{\square} C_3)]_{32}$,
	$[SL_2(5) \circ SL_2(5) : 2 \overset{2}{\underset{\sqrt{5}}{\boxtimes}} D_{10}]_{16}^2$, $A_2 \otimes [SL_2(5) \circ SL_2(5) : 2 \overset{2}{\underset{\sqrt{5}}{\boxtimes}} D_{10}]_{16}$
O_2	$A_4 \otimes E_8$, $A_2 \otimes A_4 \otimes F_4$, $[C_{15} : C_4 \overset{2(2)}{\boxtimes} F_4]_{32}$,
	$[(2_-^{1+4}.Alt_5 \underset{\infty,2}{\otimes} SL_2(5)) \overset{2(2)}{\underset{\sqrt{5}}{\boxtimes}} D_{10}]_{32}$
O_3	$(A_2 \otimes A_4)^4$, $(F_4 \otimes A_4)^2$, $[D_{120}.(C_4 \times C_2)]_{16}^2$,
	$[SL_2(5) \circ SL_2(5) : 2 \overset{2}{\underset{\sqrt{5}}{\boxtimes}} D_{10}]_{16}^2$
O_4	$([D_{120}.2]_{16,1}^2)$

	${}_{\sqrt{3}+\sqrt{5},\infty}[C_5 \overset{2}{\boxtimes} D_{24}]_2$ $(2^4 \cdot 3 \cdot 5)$
O_1	$[SL_2(5) \circ C_5 \overset{2(3)}{\boxtimes} D_{24}]_{32}$
O_2	$[SL_2(3) \overset{2(2)}{\circ} C_5 \overset{2(3)}{\boxtimes} D_{24}]_{32}$
O_3	$([D_{120}.2]_{16,1}^2)$
O_4	$[((SL_2(5) \circ SL_2(5)) \overset{2}{\underset{\sqrt{5}}{\boxtimes}} (SL_2(5) \circ SL_2(5))) : S_4]_{32,1}$,
	$[((SL_2(5) \circ SL_2(5)) : 2 \overset{2(6)}{\underset{\sqrt{5}}{\boxtimes}} (C_3 \overset{2(2)}{\boxtimes} D_8)]_{32,1}$

	${}_{\sqrt{3}+\sqrt{5},\infty}[C_5 \overset{2}{\underset{\sqrt{5}'}{\boxtimes}} Q_{24}]_2$ $(2^4 \cdot 3 \cdot 5)$
O_1	$[SL_2(5) \circ C_5 \overset{2(3)}{\underset{\sqrt{5}'}{\boxtimes}} Q_{24}]_{32}$
O_2	$[SL_2(3) \overset{2(2)}{\circ} (C_5 \overset{2(3)}{\underset{\sqrt{5}'}{\boxtimes}} Q_{24})]_{32}$
O_3	$([D_{120}.2]_{16,2}^2)$
O_4	$([(SL_2(5) \overset{2}{\square} SL_2(5)) : 2]_8^4)$, $((F_4 \tilde{\otimes} F_4)^2)$

	${}_{\eta_{48},\infty}[C_3 \overset{2}{\boxtimes} QD_{32}]_2$ $(2^6 \cdot 3)$
O_1	A_2^{16}, $(A_2 \otimes F_4)^4$, F_4^8, E_8^4
O_2	(F_4^8), (A_2^{16})
O_3	(A_2^{16}), (F_4^8), (E_8^4)
O_4	E_8^4, $(F_4 \tilde{\otimes} F_4)^2$, $(A_2 \otimes F_4)^4$, $(A_2 \otimes E_8)^2$
O_5	(A_2^{16}), (F_4^8)
O_6	(A_2^{16}), $[2_+^{1+10}.O_{10}^+(2)]_{32}$
O_7	$(A_2 \otimes E_8)^2$, $(A_2 \otimes F_4)^4$, $(F_4 \tilde{\otimes} F_4)^2$, E_8^4
O_8	$[2_+^{1+10}.O_{10}^+(2)]_{32}$
O_9	(A_2^{16}), (F_4^8)
O_{10}	(A_2^{16}), (F_4^8)

In the first line of each box the a.i.m.f. group G and its order is given. In the next lines some r.i.m.f. supergroups fixing a G-lattice with maximal order as endomorphism ring are given. If I did not find such groups, at least one r.i.m.f. supergroup of G is specified in brackets. If there is more than one isomorphism class of maximal orders in $\mathcal{Q}$ they are listed in the following lines, headed by the symbols O_1, O_2, ... to distinguish the different $\mathbb{Z}$-isomorphism classes of maximal orders, in the same order as they are displayed in Table 4.1.

Proof. Let $\mathcal{Q}$ be a definite quaternion algebra with center K of degree 4 over $\mathbb{Q}$ and G a primitive a.i.m.f. subgroup of $GL_2(\mathcal{Q})$. Then K is contained in a cyclotomic field of degree 8 or 16 over $\mathbb{Q}$. If K is the maximal real subfield of a cyclotomic field of degree 8 over $\mathbb{Q}$, then K is one of $\mathbb{Q}[\theta_{15}]$, $\mathbb{Q}[\theta_{20}]$, $\mathbb{Q}[\theta_{24}]$, or $\mathbb{Q}[\theta_{16}]$.

Assume first that $K = \mathbb{Q}[\theta_{15}]$. If G contains a quasi-semi-simple normal subgroup, then $SL_2(5) \trianglelefteq G$. The centralizer $C := C_G(SL_2(5))$ embeds into the commuting algebra $\mathcal{D} := C_{\mathcal{Q}^{2\times 2}}(SL_2(5))$, which is an indefinite quaternion algebra with center K. Since $SL_2(5)$ is primitively saturated over K the group G is of the form $G = SL_2(5)C$. Hence C is a maximal finite subgroup of $\mathcal{D}^*$ and the enveloping $\mathbb{Q}[\sqrt{5}]$-algebra $\overline{\mathbb{Q}[\sqrt{5}]C}$ of C is $\mathcal{D}$. By the classification of finite subgroups of $PGL_2(\mathbb{C})$ in [Bli 17], this implies that $\mathcal{D} = K^{2\times 2}$ and $C = {\pm}D_{30}$. Hence $G = {}_{\theta_{15},\infty}[SL_2(5) \otimes_{\sqrt{5}} D_{30}]_2$ in this case. Now assume that G does not contain a quasi-semi-simple normal subgroup. Since K is the character field of the natural character of G, an inspection of the relevant groups in Table 8.7 yields that $O_5(G) > 1$. Hence $O_5(G) \cong C_5$ and G contains the normal subgroup $N := \mathcal{B}^{\circ}_K(O_5(G)) = {\pm}C_{15}$. The centralizer $C_G(N) = \mathcal{B}^{\circ}(C_G(N))$ is a centrally irreducible subgroup of $(\mathcal{Q} \otimes_K \mathbb{Q}[\zeta_{15}])^*$ and $G/C_G(N) \cong C_2$. By Theorem 8.1 $C_G(N)$ is either $C_{15} \otimes_{\sqrt{-3}} SL_2(3)$ or $C_{15} \otimes D_8$. In each case there are two possible automorphisms. Since the group $Q_{60} \otimes D_8$ is imprimitive and $D_{30} \otimes SL_2(3)$ embeds into ${}_{\theta_{15},\infty}[SL_2(5) \otimes_{\sqrt{5}} D_{30}]_2$ one finds that G is one of ${}_{\theta_{15},\infty}[C_{15} \overset{2(2)}{\boxtimes} SL_2(3)]_2$ or ${}_{\theta_{15},\infty}[C_{15} \overset{2(2)}{\boxtimes} D_8]_2$.

The case where $K = \mathbb{Q}[\theta_{20}]$ is similar. If G contains a quasi-semi-simple normal subgroup, one easily concludes that $G = {}_{\theta_{20},\infty}[SL_2(5) \otimes_{\sqrt{5}} D_{40}]_2$. If G has no quasi-semi-simple normal subgroup, then as above, $O_5(G) \cong C_5$ and $N := \mathcal{B}^{\circ}_K(O_5(G)) = C_{20} \trianglelefteq G$. Let $C := C_G(N)$.

If $O_2(G) > C_4$, then $O_2(C) = C_4 \otimes D_8 \cong C_4 \circ Q_8$ and $C = \mathcal{B}^{\circ}_K(C) = C_5 \otimes (C_4 \circ SL_2(3).2)$. There is only one possible automorphism and therefore $G = {}_{\theta_{20},\infty}[C_5 \overset{2}{\boxtimes} (C_4 \circ SL_2(3).2)]_2$ in this case.

If $O_3(C) > 1$, one has to remark that $Q_{40} \otimes S_3 \cong C_{20} \underset{\sqrt{-1}}{\overset{2(3)}{\boxtimes}} \tilde{S}_3$ and $D_{40} \otimes \tilde{S}_3 \cong C_{20} \overset{2(3)}{\boxtimes} S_3$. Since the last group is a subgroup of ${}_{\theta_{20},\infty}[SL_2(5) \otimes_{\sqrt{5}} D_{40}]_2$, one finds that G is ${}_{\theta_{20},\infty}[Q_{40}]_1 \otimes A_2$ in this case.

In the last two cases, K does not contain a subfield $\mathbb{Q}[\sqrt{5}]$. Since K is the character field of the natural character of G, one finds that $O_5(G) = 1$ and that G does not contain a quasi-semi-simple normal subgroup.

If $K = \mathbb{Q}[\theta_{24}]$ clearly $O_3(G) \neq 1$. Hence $O_3(G) = C_3$ and G contains a normal subgroup $N := \mathcal{B}_K^\circ(O_3(G)) = C_{24}$. One concludes that $O_2(G) = C_8 \circ Q_8$ and $C_G(N) = \mathcal{B}_K^\circ(C_{24} \otimes_{\sqrt{-1}} Q_8) = C_{48} \otimes_{\sqrt{2}} \tilde{S}_4$. Now $\tilde{S}_4$ is primitively saturated over K and therefore $G = {}_{\theta_{24},\infty}[\tilde{S}_4 \otimes_{\sqrt{2}} D_{48}]_2$

In the last case $K = \mathbb{Q}[\theta_{16}]$. If $O_3(G) > 1$, then $O_3(G) \cong C_3$ and $C_G(O_3(G))$ is an absolutely irreducible subgroup of $\mathbb{Q}[\sqrt{-3}] \otimes \mathcal{Q}$. Since $\tilde{S}_3 \otimes D_{32} = C_{16} \overset{2(3)}{\boxtimes} S_3$ embeds into ${}_{\theta_{16},\infty}[\tilde{S}_4 \otimes_{\sqrt{2}} D_{32}]_2$, G is ${}_{\theta_{16},\infty}[Q_{32}]_1 \otimes A_2 = {}_{\theta_{16},\infty}[C_{16} \overset{2(3)}{\boxtimes} \tilde{S}_3]_2$.

If $O_3(G) = 1$, then $O_2(G)$ is a self-centralizing normal subgroup of G and $G = \mathcal{B}^\circ(O_2(G)) = {}_{\theta_{16},\infty}[\tilde{S}_4 \otimes_{\sqrt{2}} D_{32}]_2$ with Proposition 8.9.

Now we consider the case, where K does not embed into a cyclotomic field of degree 8 over $\mathbb{Q}$. Since K is real and of degree 4, this implies that K is contained in one of the cyclotomic fields $\mathbb{Q}[\zeta_i]$ for $i = 17, 40, 60, 48$ of degree 16 over $\mathbb{Q}$ and that K is one of $\mathbb{Q}[\eta_{17}]$, $\mathbb{Q}[\sqrt{2}, \sqrt{5}]$, $\mathbb{Q}[\eta_{40}]$, $\mathbb{Q}[\sqrt{3}, \sqrt{5}]$, or $\mathbb{Q}[\eta_{48}]$. The fields $\mathbb{Q}[\eta_i]$ denote subfields of $\mathbb{Q}[\zeta_i]$ with $Gal(\mathbb{Q}[\eta_i]/\mathbb{Q}) \cong C_4$.

In all cases i divides the exponent of G. If $K = \mathbb{Q}[\eta_i]$ is a cyclic extension of $\mathbb{Q}$, then K is generated by a single character value. So in these cases G contains an element x of order i. Since K is the character field of the natural character of G, the whole Galois group $\Gamma := Gal(\mathbb{Q}[\zeta_i]/\mathbb{Q}[\eta_i])$ is induced by conjugation with elements in the normalizer $N_G(\langle x \rangle)$. Hence G contains the irreducible subgroup ${}^{\pm}C_i.\Gamma$. Computing the automorphism group of the invariant lattices one gets $G = {}^{\pm}C_i.\Gamma$ is one of ${}_{\eta_{17},\infty}[{}^{\pm}C_{17}.C_4]_2$, ${}_{\eta_{40},\infty}[C_5 \overset{2}{\boxtimes} QD_{16}]_2$, respectively ${}_{\eta_{48},\infty}[C_3 \overset{2}{\boxtimes} QD_{32}]_2$.

Now let $K = \mathbb{Q}[\sqrt{2}, \sqrt{5}]$. If G contains a quasi-semi-simple normal subgroup one easily concludes that $G = {}_{\sqrt{5},\infty}[SL_2(5)]_1 \otimes {}_{\sqrt{2}}[D_{16}]_2$. Otherwise G contains a normal subgroup $N \cong C_5$. The centralizer $C_G(N)$ is an absolutely irreducible subgroup of $GL_1(\mathcal{Q} \otimes \mathbb{Q}[\zeta_5])$ and G contains $C_G(N)$ of index 2.

Hence, clearly $O_3(G) = 1$ and by Table 8.7 $O_2(G)$ is one of Q_8, D_{16}, or Q_{16}. In the first case, G contains the normal subgroup $\mathcal{B}_K^\circ(O_2(G)) = \tilde{S}_4$. This group is primitively saturated over K and therefore $G = {}_{\sqrt{2},\infty}[\tilde{S}_4]_1 \otimes {}_{\sqrt{5}}[{}^{\pm}D_{10}]_2$.

In the other two cases the elements in $G - C_G(N)$ may induce two different automorphisms. Since the groups $Q_{20} \otimes D_{16}$ resp. $D_{10} \otimes Q_{16}$ are contained in ${}_{\sqrt{5},\infty}[SL_2(5)]_1 \otimes {}_{\sqrt{2}}[D_{16}]_2$ resp. ${}_{\sqrt{2},\infty}[\tilde{S}_4]_1 \otimes {}_{\sqrt{5}}[{}^{\pm}D_{10}]_2$, one finds that G is one of ${}_{\sqrt{2}+\sqrt{5},\infty,2,5}[C_5 \overset{2(2)}{\boxtimes} D_{16}]_2$ resp. ${}_{\sqrt{2}+\sqrt{5},\infty,2,5}[C_5 \overset{2(2)}{\boxtimes} Q_{16}]_2$.

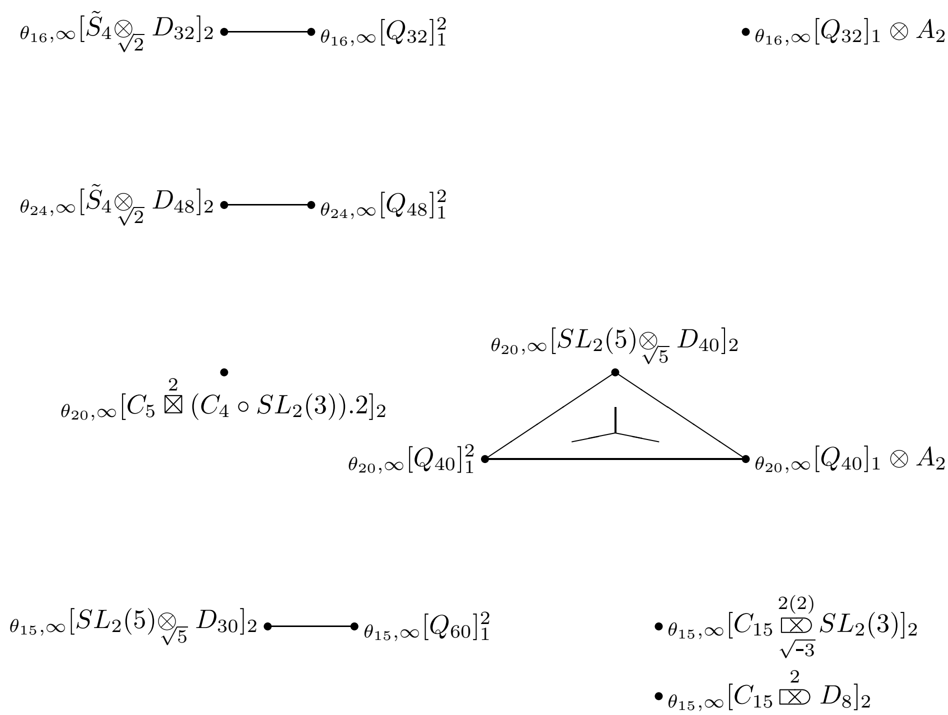
In the case $K = \mathbb{Q}[\sqrt{5}, \sqrt{3}]$, one analogously gets that G is one of

$$ {}_{\sqrt{5},\infty}[SL_2(5)]_1 \otimes {}_{\sqrt{3}}[D_{24}]_2,\ {}_{\sqrt{3},\infty}[Q_{24}]_1 \otimes {}_{\sqrt{5}}[{}^{\pm}D_{10}]_2, $$

$$ {}_{\sqrt{3}+\sqrt{5},\infty}[C_5 \overset{2}{\boxtimes} D_{24}]_2,\ \text{or}\ {}_{\sqrt{3}+\sqrt{5},\infty}[C_5 \underset{\sqrt{5}'}{\overset{2}{\boxtimes}} Q_{24}]_2. $$

□

Theorem 12.18. *Let $\mathcal{Q}$ be a definite quaternion algebra with center K and $[K : \mathbb{Q}] = 4$. If G is an a.i.m.f. subgroup of $GL_2(\mathcal{Q})$, then $\mathcal{Q}$ is one of $\mathcal{Q}_{\theta_{16},\infty}$, $\mathcal{Q}_{\theta_{24},\infty}$, $\mathcal{Q}_{\theta_{20},\infty}$, $\mathcal{Q}_{\theta_{15},\infty}$, $\mathcal{Q}_{\eta_{17},\infty}$, $\mathcal{Q}_{\sqrt{2}+\sqrt{5},\infty}$, $\mathcal{Q}_{\sqrt{2}+\sqrt{5},\infty,2,5}$, $\mathcal{Q}_{\eta_{40},\infty}$, $\mathcal{Q}_{\sqrt{3}+\sqrt{5},\infty}$, or $\mathcal{Q}_{\eta_{40},\infty}$. If K is not the maximal real subfield of a cyclotomic field, the simplicial complexes $M_2^{irr}(\mathcal{Q})$ consist of zero simplices. For $K = \mathbb{Q}[\theta_i]$ $(i = 16, 24, 20, 15)$ the simplicial complexes $M_2^{irr}(\mathcal{Q})$ are as follows.*



List of the maximal simplices in $M_2^{irr}(\mathcal{Q})$

simplex	a common subgroup
$({}_{\theta_{16,\infty}}[\tilde{S}_4 \otimes_{\sqrt{2}} D_{32}]_2, {}_{\theta_{16,\infty}}[Q_{32}]_1^2)$	$Q_{32} \otimes D_8$
$({}_{\theta_{24,\infty}}[\tilde{S}_4 \otimes_{\sqrt{2}} D_{48}]_2, {}_{\theta_{24,\infty}}[Q_{48}]_1^2)$	$Q_{48} \otimes D_8$
$({}_{\theta_{20,\infty}}[SL_2(5) \otimes_{\sqrt{5}} D_{40}]_2, {}_{\theta_{20,\infty}}[Q_{40}]_1^2, {}_{\theta_{20,\infty}}[C_5 \overset{2}{\boxtimes} (C_4 \circ SL_2(3)).2]_2)$	$Q_{40} \otimes D_8$
$({}_{\theta_{15,\infty}}[SL_2(5) \otimes_{\sqrt{5}} D_{30}]_2, {}_{\theta_{15,\infty}}[Q_{60}]_1^2)$	$Q_{20} \otimes_{\sqrt{5}} D_{30}$

$Z(\mathcal{Q})$ real quintic.

Analogously one finds:

Theorem 12.19. *Let $\mathcal{Q}$ be a definite quaternion algebra with center K of degree 5 over $\mathbb{Q}$ and G a primitive a.i.m.f. subgroup of $GL_2(\mathcal{Q})$. Then G is one of the groups in the following table, which is built up as Table 12.7. The simplicial complexes consist of zero simplices each.*

List of the primitive a.i.m.f. subgroups of $GL_2(\mathcal{Q})$, where $\mathcal{Q}$ is a definite quaternion algebra with center K and $[K:\mathbb{Q}]=5$.

lattice L	$\lvert Aut(L)\rvert$	some r.i.m.f. supergroups
${}_{\theta_{11,\infty,11}}[Q_{44}]_1 \otimes A_2$	$2^3\cdot 3\cdot 11$	$(A_2\otimes A_{10})^2$, $(A_2\otimes A_{10}^{(2)})^2$, $(A_2\otimes A_{10}^{(3)})^2$
		$([L_2(11)\overset{2(3)}{\otimes} D_{12}]_{20}^2)$, $([L_2(11)\overset{2(3)}{\boxtimes} D_{12}]_{20}^2)$
${}_{\theta_{11,\infty,11}}[C_{11}\overset{2(2)}{\boxtimes} SL_2(3)]_2$	$2^4\cdot 3\cdot 11$	$[SL_2(11)\overset{2(2)}{\circ} SL_2(3)]_{20}^2$, $[U_5(2)\overset{2(2)}{\circ} SL_2(3)]_{20}^2$
		$[L_2(11)\otimes_{\sqrt{-11}} SL_2(3)\otimes S_3).2]_{40}$
${}_{\theta_{11,\infty,11}}[\pm C_{11}\overset{2(3)}{\boxtimes} S_3]_2$	$2^3\cdot 3\cdot 11$	$[L_2(11)\overset{2(3)}{\boxtimes} D_{12}]_{20}^2$, $[L_2(11)\overset{2(3)}{\otimes} D_{12}]_{20}^2$
		$((A_2\otimes A_{10})^2)$
${}_{\theta_{11,\infty,2}}[\pm C_{11}\overset{2(2)}{\boxtimes} D_8]_2$	$2^4\cdot 11$	$[SL_2(11)\underset{\infty,2}{\overset{2(2)}{\otimes}} 2_-^{1+4}.Alt_5]_{40}$, $[U_5(2)\underset{\infty,2}{\overset{2(2)}{\otimes}} 2_-^{1+4}.Alt_5]_{40}$
		$([2.M_{12}.2\otimes_{\sqrt{-2}} GL_2(3)]_{40})$
		$((A_{10}^4))$, $(((A_{10}^{(2)})^4))$, $(((A_{10}^{(3)})^4))$
${}_{\theta_{11}}[\pm D_{22}]_1\otimes{}_{\infty,2}[SL_2(3)]_1$	$2^4\cdot 3\cdot 11$	$A_{10}\otimes F_4$, $A_{10}^{(2)}\otimes F_4$, $A_{10}^{(3)}\otimes F_4$
		$(A_{10}\otimes F_4)$
		$[SL_2(11)\overset{2(S)}{\circ} L_2(3)]_{20}^2$
${}_{\theta_{11}}[\pm D_{22}]_1\otimes{}_{\infty,3}[\tilde{S}_3]_1$	$2^3\cdot 3\cdot 11$	$(A_{10}\otimes A_2)^2$, $(A_{10}^{(2)}\otimes A_2)^2$, $(A_{10}^{(3)}\otimes A_2)^2$
		$((A_{10}\otimes A_2)^2)$
		$((A_{10}\otimes A_2)^2)$
		$((A_{10}\otimes A_2)^2)$
${}_{\theta_{11,\infty,3}}[\pm C_{11}\overset{2(3)}{\boxtimes}\tilde{S}_3]_2$	$2^3\cdot 3\cdot 11$	$[L_2(11)\overset{2(3)}{\boxtimes} D_{12}]_{20}^2$, $[L_2(11)\overset{2(3)}{\otimes} D_{12}]_{20}^2$
		$([L_2(11)\overset{2(3)}{\boxtimes} D_{12}]_{20}^2)$, $([L_2(11)\overset{2(3)}{\otimes} D_{12}]_{20}^2)$
		$([L_2(11)\overset{2(3)}{\boxtimes} D_{12}]_{20}^2)$, $([L_2(11)\overset{2(3)}{\otimes} D_{12}]_{20}^2)$
		$([L_2(11)\overset{2(3)}{\boxtimes} D_{12}]_{20}^2)$, $([L_2(11)\overset{2(3)}{\otimes} D_{12}]_{20}^2)$
${}_{\sigma_{25,\infty,5}}[\pm C_{25}.C_4]_2$	$2^3\cdot 5^2$	E_8^5, $[(SL_2(5)\overset{2}{\square} SL_2(5)):2]_8^5$
		(E_8^5), $([(SL_2(5)\overset{2}{\square} SL_2(5)):2]_8^5)$
		(E_8^5), $([(SL_2(5)\overset{2}{\square} SL_2(5)):2]_8^5)$
		(E_8^5), $([(SL_2(5)\overset{2}{\square} SL_2(5)):2]_8^5)$
		(E_8^5), $([(SL_2(5)\overset{2}{\square} SL_2(5)):2]_8^5)$

13. The a.i.m.f. subgroups of $GL_3(\mathcal{Q})$

$$Z(\mathcal{Q}) = \mathbb{Q}$$

Theorem 13.1. *Let $\mathcal{Q}$ be a definite quaternion algebra with center $\mathbb{Q}$ and G be a primitive a.i.m.f. subgroup of $GL_3(\mathcal{Q})$. Then G is one of the groups in the following table.*

List of the primitive a.i.m.f. subgroups of $GL_3(\mathcal{Q})$.

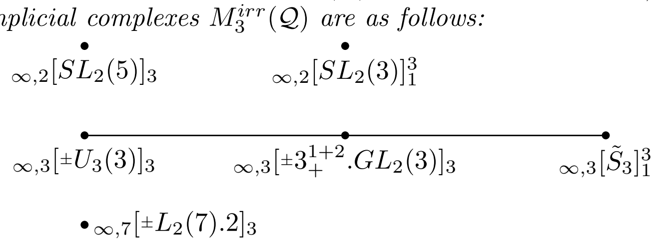
lattice L	$\lvert Aut(L)\rvert$	r.i.m.f. supergroups
${}_{\infty,2}[SL_2(5)]_3$	$2^3 \cdot 3 \cdot 5$	$[SL_2(5) \overset{2(2)}{\circ} SL_2(3)]_{12}$
${}_{\infty,3}[{\pm}U_3(3)]_3$	$2^6 \cdot 3^3 \cdot 7$	$[6.U_4(3).2^2]_{12}$
${}_{\infty,3}[{\pm}3^{1+2}_{+}.GL_2(3)]_3$	$2^5 \cdot 3^4$	E_6^2
${}_{\infty,7}[{\pm}L_2(7).2]_3$	$2^5 \cdot 3 \cdot 7$	$(A_6^{(2)})^2$

Proof. Let $\mathcal{Q}$ be a definite quaternion algebra with center $\mathbb{Q}$ and G be a primitive a.i.m.f. subgroup of $GL_3(\mathcal{Q})$. Assume that $1 \neq N \trianglelefteq G$ is a quasi-semi-simple normal subgroup of G. With [CCNPW 85] one finds that N is one of Alt_5, $SL_2(5)$, $L_2(7)$, or $U_3(3)$. The centralizer $C := C_G(N)$ in G of N embeds into the commuting algebra $C_{\mathcal{Q}^{3\times 3}}(N)$, which is isomorphic to $\mathbb{Q}[\sqrt{5}]$, $\mathbb{Q}$, $\mathbb{Q}[\sqrt{-7}]$, resp. $\mathbb{Q}$ in the respective cases. Therefore $C = {\pm}1$ in all cases and $G/{\pm}N$ embeds into C_2, the outer automorphism group of N. This gives a contradiction in the first case, since both groups $({\pm}Alt_5).2$ are subgroups of $GL_6(\mathbb{Q})$. In the second and fourth cases, one finds that $N = G = {}_{\infty,2}[SL_2(5)]_3$ resp. $N = G = {}_{\infty,3}[{\pm}U_3(3)]_3$, because the extensions of the natural character of N to $N.2$ are not rational. In the third case $G = {}_{\infty,7}[{\pm}L_2(7).2]_3$ has to be isomorphic to a nonsplit extension of ${\pm}N$ by C_2.

Now assume that G does not contain a quasi-semi-simple normal subgroup and let p be a prime with $O_p(G) \neq 1$. Then by Corollary 2.4 one has $p \in \{2, 3, 7\}$. By Lemma 11.2 $O_7(G) = 1$.

Therefore Proposition 8.11 gives that $O_3(G) \neq 1$. From Table 8.7 one gets that $O_3(G)$ is one of C_3, C_9, or 3^{1+2}_{+}. In the first two cases, $C_G(O_3(G)) = {\pm}O_3(G)$ and G contains $C_G(O_3(G))$ of index 2, contradicting the irreducibility of G. In the last case, G contains the generalized Bravais group $\mathcal{B}^{\circ}(O_3(G)) = {\pm}3^{1+2}_{+} : SL_2(3)$ of index 2. The split extension is a subgroup of $GL_6(\mathbb{Q})$, so G has to be isomorphic to the nonsplit extension $G = {}_{\infty,3}[{\pm}3^{1+2}_{+}.GL_2(3)]_3$. $\square$

Theorem 13.2. *Let $\mathcal{Q}$ be a definite quaternion algebra with center $\mathbb{Q}$ and let G be an a.i.m.f. subgroup of $GL_3(\mathcal{Q})$. Then $\mathcal{Q}$ is one of $\mathcal{Q}_{\infty,2}$, $\mathcal{Q}_{\infty,3}$, or $\mathcal{Q}_{\infty,7}$. The simplicial complexes $M_3^{irr}(\mathcal{Q})$ are as follows:*



List of maximal simplices in $M_3^{irr}(\mathcal{Q}_{\infty,3})$:

simplex	a common subgroup
$({}_{\infty,3}[\pm U_3(3)], {}_{\infty,3}[\pm 3_+^{1+2}.GL_2(3)]_3)$	$3_+^{1+2} : C_8$
$({}_{\infty,3}[\tilde{S}_3]_1^3, {}_{\infty,3}[\pm 3_+^{1+2}.GL_2(3)]_3)$	$(\pm 3_+^{1+2}).C_2$

Proof. Theorems 13.1 and 6.1 prove the completeness of the list of quaternion algebras $\mathcal{Q}$ and of a.i.m.f. subgroups of $GL_3(\mathcal{Q})$. The completeness of the list of maximal simplices in $M_3^{irr}(\mathcal{Q})$ for the respective quaternion algebras $\mathcal{Q}$ can be seen as follows: $M_3^{irr}(\mathcal{Q}_{\infty,2})$ consists of two 0-simplices, because the group ${}_{\infty,2}[SL_2(5)]_3$ is minimal absolutely irreducible. The unique minimal absolutely irreducible subgroup of ${}_{\infty,3}[\pm U_3(3)]_3$ is $3_+^{1+2} : C_8$ as one sees from the list of maximal subgroups of $U_3(3)$ in [CCNPW 85]. This group does not embed into ${}_{\infty,3}[\tilde{S}_3]_1^3$ so the list of maximal simplices in $M_3^{irr}(\mathcal{Q}_{\infty,3})$ is complete. □

$Z(\mathcal{Q})$ real quadratic.

Theorem 13.3. *Let $\mathcal{Q}$ be a definite quaternion algebra with center K, such that $[K : \mathbb{Q}] = 2$ and let G be a primitive a.i.m.f. subgroup of $GL_3(\mathcal{Q})$. Then G is conjugate to one of the groups in the following table, which is built up as Table 12.7.*

List of the primitive a.i.m.f. subgroups of $GL_3(\mathcal{Q})$.

lattice L	$\lvert Aut(L)\rvert$	r.i.m.f. supergroups
${}_{\sqrt{2},\infty}[SL_2(7)]_3$	$2^4 \cdot 3 \cdot 7$	$[SL_2(7) \circ \tilde{S}_4]_{24}$
${}_{\sqrt{2},\infty}[SL_2(5).2]_3$	$2^4 \cdot 3 \cdot 5$	$[SL_2(5) \overset{2(2)}{\circ} SL_2(3)]_{12}^2$, $[SL_2(5) \underset{\infty,2}{\overset{2(2)}{\boxtimes}} 2_-^{1+4'}.Alt_5]_{24}$
${}_{\sqrt{3},\infty}[(\pm U_3(3)).2]_3$	$2^7 \cdot 3^3 \cdot 7$	$[6.U_4(3).2^2]_{12}^2$, $[6.U_4(3).2 \underset{\sqrt{-3}}{\overset{2}{\boxtimes}} SL_2(3)]_{24}$
		$[2.Co_1]_{24}$, $[(C_4 \circ SL_2(3)).2 \underset{\sqrt{-1}}{\overset{2(3)}{\boxtimes}} U_3(3)]_{24}$
${}_{\sqrt{3},\infty}[C_4 \overset{2}{\boxtimes} 3_+^{1+2} : SL_2(3)]_3$	$2^6 \cdot 3^4$	E_6^4, $[3_+^{1+2} : SL_2(3) \underset{\sqrt{-3}}{\overset{2(2)}{\boxtimes}} SL_2(3)]_{12}^2$
		$F_4 \otimes E_6$, $[Sp_4(3) \underset{\sqrt{-3}}{\overset{2}{\boxtimes}} 3_+^{1+2} : SL_2(3)]_{24}$
${}_{\sqrt{5},\infty}[2.J_2]_3$	$2^8 \cdot 3^3 \cdot 5^2 \cdot 7$	$[2.J_2 \overset{2}{\Box} SL_2(5)]_{24}$, $[2.Co_1]_{24}$
${}_{\sqrt{5},\infty}[Alt_5 \otimes_{\sqrt{5}} SL_2(5)]_3$	$2^5 \cdot 3^2 \cdot 5^2$	$[(SL_2(5) \circ SL_2(5)) : 2 \underset{\sqrt{5}}{\overset{2}{\boxtimes}} Alt_5]_{24,i}$ $(i = 1, 2)$
${}_{\sqrt{7},\infty}[C_4 \overset{2}{\boxtimes} L_2(7)]_3$	$2^6 \cdot 3 \cdot 7$	$(A_6^{(2)})^4$, $[L_2(7) \overset{2(2)}{\boxtimes} D_8]_{12}^2$, $[6.U_4(3).2^2]_{12}^2$
		$[2.Co_1]_{24}$, $[6.U_4(3).2 \underset{\sqrt{-3}}{\overset{2}{\boxtimes}} SL_2(3)]_{24}$
		$[L_2(7) \overset{2(2)}{\boxtimes} F_4]_{24}$, $A_6^{(2)} \otimes F_4$, $[(C_4 \circ SL_2(3)).2 \underset{\sqrt{-1}}{\overset{2(3)}{\boxtimes}} U_3(3)]_{24}$
${}_{\sqrt{13},\infty}[SL_2(13)]_3$	$2^3 \cdot 3 \cdot 7 \cdot 13$	$[SL_2(13) \overset{2(2)}{\Box} SL_2(3)]_{24}$, $[2.Co_1]_{24}$
${}_{\sqrt{21},\infty}[\pm C_3 \overset{2}{\boxtimes} L_2(7)]_3$	$2^5 \cdot 3^2 \cdot 7$	$(A_2 \otimes A_6^{(2)})^2$, $[6.U_4(3).2^2]_{12}^2$
		$[2.Co_1]_{24}$, $[L_2(7) \overset{2(2)}{\boxtimes} F_4]_{24}$

Proof. Let G be a primitive a.i.m.f. subgroup of $GL_3(\mathcal{Q})$. Assume that $1 \neq N \trianglelefteq G$ is a quasi-semi-simple normal subgroup of G. With [CCNPW 85] one finds that N is one of Alt_5, $SL_2(5)$ (2 groups), $L_2(7)$, $SL_2(7)$, $3.Alt_6$, $SL_2(13)$, $U_3(3)$, or $2.J_2$. If N is isomorphic to $SL_2(7)$, $SL_2(13)$, or $2.J_2$, one computes that $G = N$ is an a.i.m.f. group.

If N is $U_3(3)$ or $SL_2(5)$ (where the restriction of the natural character of G to N is $2\chi_6$), the centralizer $C_G(N)$ embeds into $C_{\mathcal{Q}^{3\times 3}}(N) = Z(\mathcal{Q}) = K$. Since K is a (totally) real field, one concludes that $C_G(N) = \pm 1$ and $G/\pm N \cong C_2$ is isomorphic to the outer automorphism group of N. Using [CCNPW 85] one finds that $G = {}_{\sqrt{2},\infty}[SL_2(5).2]_3$ resp. $G = {}_{\sqrt{3},\infty}[(\pm U_3(3)).2]_3$.

If $N = SL_2(5)$, where the restriction of the natural character χ of G to N contains χ_{2a}, one has $\mathcal{Q} = \mathcal{Q}_{\sqrt{5},\infty}$ and $\chi_{|N} = 3\chi_{2a}$. The centralizer $C_G(N)$, embedding into $C_{\mathcal{Q}^{3\times 3}}(N) = \mathbb{Q}[\sqrt{5}]^{3\times 3}$, is either ± 1 or Alt_5. Since 3 does not divide the order of the outer automorphism group of N, the first possibility contradicts the irreducibility of G. In the second case one computes $G = NC_G(N) = {}_{\sqrt{5},\infty}[Alt_5 \otimes_{\sqrt{5}} SL_2(5)]_3$.

Now assume that G contains a simple normal subgroup N isomorphic to Alt_5. Since the maximal real subfield of $\mathcal{Q}$ is the center $K = Z(\mathcal{Q})$, one finds that $K \cong \mathbb{Q}[\sqrt{5}]$ and the restriction of the natural character of G to N is (w.l.g.) $2\chi_{3a}$. The centralizer $C := C_G(N)$ embeds into $GL_1(\mathcal{Q})$. Therefore it is isomorphic to a subgroup of $SL_2(5)$. Since the outer automorphism of N induces the Galois automorphism on the center of the enveloping algebra $\mathbb{Q}N$ one concludes that $G = CN = {}_{\sqrt{5},\infty}[Alt_5 \otimes_{\sqrt{5}} SL_2(5)]_3$.

Now let $N = L_2(7)$ be a normal subgroup of G. The centralizer $C_G(N)$ embeds into $K[\sqrt{-7}]$ and therefore is one of ± 1, C_4, or $\pm C_3$. Since G contains $NC_G(N)$ of index 2, one concludes, that in the first case G cannot be absolutely irreducible, because that character field is only $\mathbb{Q}$. In the remaining two cases one constructs G to be ${}_{\sqrt{7},\infty}[C_4 \overset{2}{\boxtimes} L_2(7)]_3$ resp. ${}_{\sqrt{21},\infty}[\pm C_3 \overset{2}{\boxtimes} L_2(7)]_3$.

In the last case $N = 3.Alt_6$ and $C_G(N) = \pm C_3$. Using [CCNPW 85] one finds that $G = \pm 3.PGL_2(9)$ is not maximal finite but contained in ${}_{\sqrt{5},\infty}[2.J_2]_3$.

Assume for the rest of the proof, that G does not contain a quasi-semi-simple normal subgroup. By Corollary 2.4 $O_p(G) = 1$ for $p \notin \{2,3,5,7,13\}$ and by Lemma 11.2 $O_{13}(G) = 1$.

If $O_7(G) \neq 1$, then $O_7(G) = C_7$. Because $K = Z(\mathcal{Q})$ is a real quadratic number field, one has that $C := C_G(O_7(G))$ embeds into $K[\zeta_7]$, hence is one of $\pm C_7$, C_{28} or $\pm C_{21}$ and G contains C of index 6. In the first case, the character field of the natural character of G is $\mathbb{Q}$ contradicting the absolute irreducibility of G. In the other two cases one has a unique possibility for $G \leq GL_3(\mathcal{Q})$. Both groups are not maximal finite but contained in ${}_{\sqrt{7},\infty}[C_4 \overset{2}{\boxtimes} L_2(7)]_3$ resp. ${}_{\sqrt{21},\infty}[\pm C_3 \overset{2}{\boxtimes} L_2(7)]_3$.

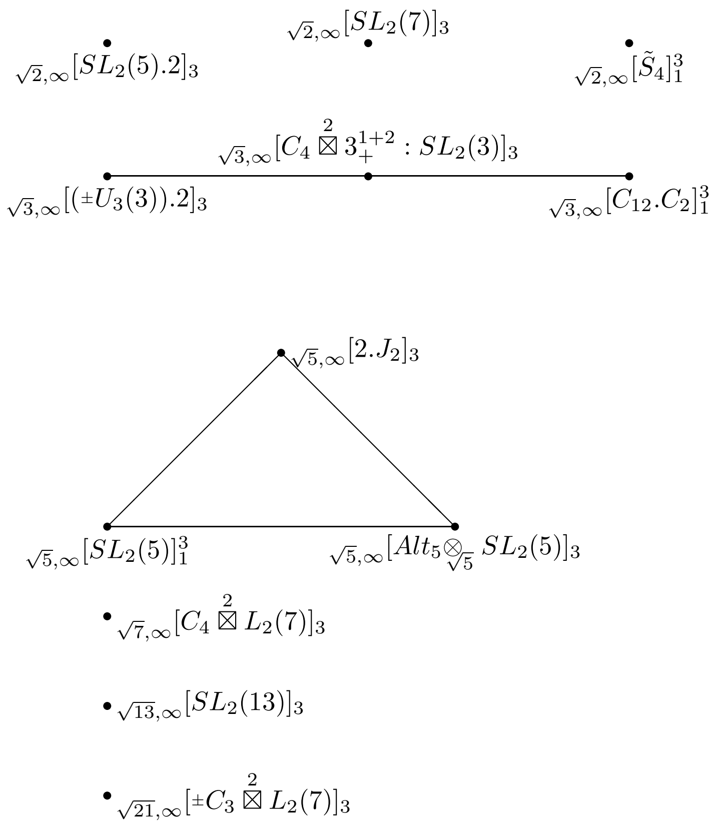
Next assume that $O_5(G) \neq 1$. Then $O_5(G) = C_5$ and $K = Z(\mathcal{Q})$ is isomorphic to $\mathbb{Q}[\sqrt{5}]$. The centralizer $C_G(O_5(G))$ embeds into $C_{\mathcal{Q}^{3\times 3}}(O_5(G)) = \mathbb{Q}[\zeta_5]^{3\times 3}$. Since G does not contain a quasi-semi-simple normal subgroup and 3 does not divide the order of the automorphism group of $O_5(G)$, this contradicts the irreducibility of G.

Assume now, that $O_3(G) > 1$. Then $O_3(G)$ is one of C_3, C_9, or 3_+^{1+2}. In the first case $C_G(O_3(G))$ embeds into $K[\zeta_3]^{3\times 3}$. Since G does not contain a quasi-semi-simple normal subgroup and 3 does not divide the order of the automorphism group of $O_5(G)$, this contradicts the irreducibility of G. In the second case, $C := C_G(O_3(G))$ embeds into $K[\zeta_9]$, hence is one of $\pm C_9$ or C_{36}. The assumption that

$O_3(G) = C_9$ implies in both cases that the index of C in G is not divisible by 3, which contradicts the irreducibility of G. In the last case, the group G contains the normal subgroup $B := \mathcal{B}^\circ(O_3(G)) = \pm 3_+^{1+2} : SL_2(3)$. The centralizer $C_G(B)$ embeds into $K[\zeta_3]$, hence is one of $\pm C_3$ or C_{12}. The first possibility contradicts the absolutely irreducibility of G, and in the second case, $G = {}_{\sqrt{3},\infty}[C_4 \overset{2}{\boxtimes} 3_+^{1+2} : SL_2(3)]_3$.

If $O_p(G) = 1$ for all odd primes p, $O_2(G)$ is self-centralizing in G contradicting Proposition 8.9. $\square$

Theorem 13.4. *Let $\mathcal{Q}$ be a definite quaternion algebra with real quadratic center K and let G be an a.i.m.f. subgroup of $GL_3(\mathcal{Q})$. Then $\mathcal{Q}$ is one of $\mathcal{Q}_{\sqrt{2},\infty}$, $\mathcal{Q}_{\sqrt{3},\infty}$, $\mathcal{Q}_{\sqrt{5},\infty}$, $\mathcal{Q}_{\sqrt{7},\infty}$, $\mathcal{Q}_{\sqrt{13},\infty}$, or $\mathcal{Q}_{\sqrt{21},\infty}$. The simplicial complexes $M_3^{irr}(\mathcal{Q})$ are as follows:*



List of maximal simplices in $M_3^{irr}(\mathcal{Q}_{\sqrt{3},\infty})$:

simplex	a common subgroup
$({}_{\sqrt{3},\infty}[(\pm U_3(3)).2]_3, {}_{\sqrt{3},\infty}[C_4 \overset{2}{\boxtimes} \pm 3_+^{1+2} : SL_2(3)]_3)$	$(\pm 3_+^{1+2} : C_8).C_2$
$({}_{\sqrt{3},\infty}[C_{12}.C_2]_1^3, {}_{\sqrt{3},\infty}[C_4 \overset{2}{\boxtimes} \pm 3_+^{1+2} : SL_2(3)]_3)$	$C_4 \overset{2}{\boxtimes} 3_+^{1+2}$

List of maximal simplices in $M_3^{irr}(\mathcal{Q}_{\sqrt{5},\infty})$:

simplex	a common subgroup
$({}_{\sqrt{5},\infty}[2.J_2]_3, {}_{\sqrt{5},\infty}[Alt_5 \otimes_{\sqrt{5}} SL_2(5)]_3)$	$Alt_5 \otimes Q_8$
$({}_{\sqrt{5},\infty}[SL_2(5)]_1^3, {}_{\sqrt{5},\infty}[Alt_5 \otimes_{\sqrt{5}} SL_2(5)]_3)$	$Q_{20} \otimes Alt_4$
$({}_{\sqrt{5},\infty}[2.J_2]_3, {}_{\sqrt{5},\infty}[SL_2(5)]_1^3)$	$(\pm C_5 \times C_5).D_{12}$

Proof. Theorems 13.3 and 6.1 prove the completeness of the list of quaternion algebras $\mathcal{Q}$ and of a.i.m.f. subgroups of $GL_3(\mathcal{Q})$. The completeness of the list of maximal simplices in $M_3^{irr}(\mathcal{Q})$ for the respective quaternion algebras $\mathcal{Q}$ can be seen as follows:

$M_3^{irr}(\mathcal{Q}_{\sqrt{2},\infty})$ consists of three 0-simplices, because the groups ${}_{\sqrt{2},\infty}[SL_2(7)]_3$ and ${}_{\sqrt{2},\infty}[SL_2(5).2]_3$ are minimal absolutely irreducible groups.

The unique minimal absolutely irreducible subgroup of ${}_{\sqrt{3},\infty}[(\pm U_3(3)).2]_3$ is $(\pm 3_+^{1+2} : C_8).2$ as one sees from the list of maximal subgroups of $U_3(3)$ in [CCNPW 85]. Therefore, there is no common absolutely irreducible subgroup of

$$ {}_{\sqrt{3},\infty}[(\pm U_3(3)).2]_3 \quad \text{and} \quad {}_{\sqrt{3},\infty}[C_{12}.C_2]_1^3 $$

and one sees that the list of maximal simplices in $M_3^{irr}(\mathcal{Q}_{\sqrt{3},\infty})$ is complete.

From the list of maximal subgroups in [CCNPW 85] one finds that the absolutely irreducible maximal subgroups of ${}_{\sqrt{5},\infty}[2.J_2]_3$ are $\pm 3.PGL_2(9)$, $SL_2(3) \otimes Alt_5$, and $(\pm C_5 \times C_5).D_{12}$. The first group has no absolutely irreducible subgroup of which the only nonabelian composition factors are isomorphic to Alt_5. The unique minimal absolutely irreducible subgroup of the second group is $Q_8 \otimes Alt_5$, and the two minimal absolutely irreducible subgroups ($(\pm C_5 \times C_5).C_6$ and $(\pm C_5 \times C_5).S_3$) of the third group do not embed into ${}_{\sqrt{5},\infty}[Alt_5 \otimes_{\sqrt{5}} SL_2(5)]_3$. □

$Z(\mathcal{Q})$ real cubic.

Theorem 13.5. *Let $\mathcal{Q}$ be a definite quaternion algebra with center K of degree 3 over $\mathbb{Q}$ and G a primitive a.i.m.f. subgroup of $GL_3(\mathcal{Q})$. Then G is conjugate to one of* ${}_{\theta_9,\infty,3}[\pm C_9 \underset{\sqrt{-3}}{\overset{2}{\boxtimes}} 3_+^{1+2} : SL_2(3)]_3$, ${}_{\theta_7,\infty,7}[\pm C_7 \underset{\sqrt{-7}}{\overset{2}{\boxtimes}} L_2(7)]_3$, *or* ${}_{\omega_{19},\infty,19}[\pm C_{19}.C_6]_3$.

List of the primitive a.i.m.f. subgroups of $GL_3(\mathcal{Q})$.

lattice L	$\lvert Aut(L)\rvert$	some r.i.m.f. supergroups
${}_{\theta_9,\infty,3}[\pm C_9 \underset{\sqrt{-3}}{\overset{2}{\boxtimes}} 3_+^{1+2} : SL_2(3)]_3$	$2^5 \cdot 3^5$	$[\pm 3^{1+4} : Sp_4(3).2]_{18}^2$, E_6^6
${}_{\theta_7,\infty,7}[\pm C_7 \underset{\sqrt{-7}}{\overset{2}{\boxtimes}} L_2(7)]_3$	$2^5 \cdot 3 \cdot 7^2$	$[\pm L_2(7) \underset{\sqrt{-7}}{\overset{2}{\boxtimes}} L_2(7)]_{18}^2$
${}_{\omega_{19},\infty,19}[\pm C_{19}.C_6]_3$	$2^2 \cdot 3 \cdot 19$	A_{18}^2, $(A_{18}^{(5)})^2$

Proof. Let $\mathcal{Q}$ be a definite quaternion algebra with center K of degree 3 over $\mathbb{Q}$ and let G be a primitive a.i.m.f. subgroup of $GL_3(\mathcal{Q})$. Then K is contained in a cyclotomic field of degree ≤ 18, hence $K \cong \mathbb{Q}[\theta_7]$, $\mathbb{Q}[\theta_9]$, $\mathbb{Q}[\omega_{13}]$, or $\mathbb{Q}[\omega_{19}]$, where θ_i are generators of the maximal totally real subfield of the corresponding cyclotomic field $\mathbb{Q}[\zeta_i]$ and the ω_i generators of the subfield of degree 3 over $\mathbb{Q}$ of the corresponding cyclotomic field $\mathbb{Q}[\zeta_i]$ (cf. notation 4.2). By Table 9.1 the only possibility for a quasi-semi-simple normal subgroup N of G is $N = L_2(7)$. If $L_2(7) \trianglelefteq G$, then clearly

$$K = \mathbb{Q}[\theta_7],\ C_G(L_2(7)) = {}^{\pm}C_7,\ \text{and } G = {}_{\theta_7,\infty,7}[{}^{\pm}C_7 \overset{2}{\underset{\sqrt{-7}}{\boxtimes}} L_2(7)]_3.$$

Assume that $K \neq \mathbb{Q}[\omega_{13}]$. Then 13 divides $|G|$ and one concludes that $O_{13}(G) \cong C_{13}$. To get the character field K, 4 has to divide the degree of the natural character of G, which is a contradiction. If $K = \mathbb{Q}[\omega_{19}]$, one similarly gets that $G = {}_{\omega_{19},\infty,19}[{}^{\pm}C_{19}.C_6]_3$. Now assume that $K = \mathbb{Q}[\theta_7]$. Then 7 divides the order of G. Since the possible normal 2- and 3-subgroups have no automorphism of order 7, one has either $L_2(7) \trianglelefteq G$ (which is dealt with above) or $O_7(G) \cong C_7$. In the last case $\mathbb{Q}[\zeta_7]$ is a maximal subfield of $\mathcal{Q}$ and the centralizer $C_G(C_7)$ is a centrally irreducible subgroup of $GL_3(\mathbb{Q}[\zeta_7])$. Since $O_7(G) \neq C_7 \times C_7$, this implies $L_2(7) \trianglelefteq G$. One finds $G = {}_{\theta_9,\infty,3}[{}^{\pm}C_9 \overset{2}{\underset{\sqrt{-3}}{\boxtimes}} 3_+^{1+2} : SL_2(3)]_3$ completely analogous, if $K = \mathbb{Q}[\theta_9]$. □

Corollary 13.6. *Let $\mathcal{Q}$ be a definite quaternion algebra with center K of degree 3 over $\mathbb{Q}$ and G an a.i.m.f. subgroup of $GL_3(\mathcal{Q})$. Then $\mathcal{Q}$ is one of $\mathcal{Q}_{\theta_9,\infty,3}$, $\mathcal{Q}_{\theta_7,\infty,7}$, or $\mathcal{Q}_{\omega_{19},\infty,19}$. The simplicial complexes $M_3^{irr}(\mathcal{Q})$ are as follows:*

$${}_{\theta_9,\infty,3}[{}^{\pm}C_9 \overset{2}{\underset{\sqrt{-3}}{\boxtimes}} 3_+^{1+2} : SL_2(3)]_3 \bullet\!\!\longrightarrow\!\!\!\!\!\!-\!\!\bullet\ {}_{\theta_9,\infty,3}[{}^{\pm}C_9.C_2]_1^3$$

$${}_{\theta_7,\infty,7}[{}^{\pm}C_7 \overset{2}{\underset{\sqrt{-7}}{\boxtimes}} L_2(7)]_3 \bullet\!\!\longrightarrow\!\!\!\!\!\!-\!\!\bullet\ {}_{\theta_7,\infty,7}[{}^{\pm}C_7.C_2]_1^3$$

$${}_{\omega_{19},\infty,19}[{}^{\pm}C_{19}.C_6]_3 \bullet$$

simplex	a common subgroup
$({}_{\theta_9,\infty,3}[{}^{\pm}C_9 \overset{2}{\underset{\sqrt{-3}}{\boxtimes}} 3_+^{1+2} : SL_2(3)]_3,\ {}_{\theta_9,\infty,3}[{}^{\pm}C_9.C_2]_1^3)$	${}^{\pm}C_9 \overset{2}{\underset{\sqrt{-3}}{\boxtimes}} 3_+^{1+2}$
$({}_{\theta_7,\infty,7}[{}^{\pm}C_7 \overset{2}{\underset{\sqrt{-7}}{\boxtimes}} L_2(7)]_3,\ {}_{\theta_7,\infty,7}[{}^{\pm}C_7.C_2]_1^3)$	${}^{\pm}C_7 \overset{2}{\underset{\sqrt{-7}}{\boxtimes}} C_7 : C_3$

Proof. Theorems 6.1 and 13.5 give the list of quaternion algebras $\mathcal{Q}$ and the a.i.m.f. subgroups of $GL_3(\mathcal{Q})$. Since all simplicial complexes $M_3^{irr}(\mathcal{Q})$ for the respective quaternion algebras $\mathcal{Q}$ consist of one simplex, it is clear that the list of maximal simplices in $M_3^{irr}(\mathcal{Q})$ is complete. □

14. The a.i.m.f. subgroups of $GL_4(\mathcal{Q})$

$$Z(\mathcal{Q}) = \mathbb{Q}$$

Theorem 14.1. *Let $\mathcal{Q}$ be a totally definite quaternion algebra with center $\mathbb{Q}$ and G be a maximal finite primitive absolutely irreducible subgroup of $GL_4(\mathcal{Q})$. Then $\mathcal{Q}$ is one of $\mathcal{Q}_{\infty,2}$, $\mathcal{Q}_{\infty,3}$, $\mathcal{Q}_{\infty,5}$, or $\mathcal{Q}_{\infty,7}$. The primitive a.i.m.f. subgroups G of $GL_4(\mathcal{Q})$ are given in the following table:*

List of the primitive a.i.m.f. subgroups of $GL_4(\mathcal{Q})$.

lattice L	$\lvert Aut(L)\rvert$	r.i.m.f. supergroups
${}_{\infty,2}[2_-^{1+4}Alt_5]_2 \otimes A_2$	$2^8 \cdot 3^2 \cdot 5$	$A_2 \otimes E_8$
${}_{\infty,2}[SL_2(3)]_1 \otimes A_4$	$2^6 \cdot 3^2 \cdot 5$	$A_4 \otimes F_4$
${}_{\infty,2}[2_-^{1+6}.O_6^-(2)]_4$	$2^{13} \cdot 3^4 \cdot 5$	$F_4 \tilde{\otimes} F_4$
${}_{\infty,2}[SL_2(5) \overset{2(2)}{\boxtimes} D_8]_4$	$2^6 \cdot 3 \cdot 5$	$[SL_2(5) \underset{\infty,2}{\overset{2(2)}{\boxtimes}} 2_-^{1+4'}.Alt_5]_{16}$
${}_{\infty,3}[\tilde{S}_3]_1 \otimes A_4$	$2^5 \cdot 3^2 \cdot 5$	$(A_2 \otimes A_4)^2$
${}_{\infty,3}[\tilde{S}_3]_1 \otimes F_4$	$2^8 \cdot 3^3$	$(A_2 \otimes F_4)^2$
${}_{\infty,3}[Sp_4(3) \overset{2}{\square} C_3]_4$	$2^8 \cdot 3^5 \cdot 5$	E_8^2
${}_{\infty,3}[SL_2(5) \overset{2(3)}{\boxtimes} S_3]_4$	$2^5 \cdot 3^2 \cdot 5$	$[(SL_2(5) \overset{2}{\square} SL_2(5)) : 2]_8^2$
${}_{\infty,3}[SL_2(7)]_4$	$2^4 \cdot 3 \cdot 7$	$[SL_2(7) \overset{2(3)}{\circ} \tilde{S}_3]_{16}$
${}_{\infty,5}[SL_2(5).2]_2 \otimes A_2$	$2^5 \cdot 3^2 \cdot 5$	$A_2 \otimes E_8$
${}_{\infty,5}[SL_2(5) : 2]_2 \otimes A_2$	$2^5 \cdot 3^2 \cdot 5$	$A_2 \otimes [(SL_2(5) \overset{2}{\square} SL_2(5)) : 2]_8$
${}_{\infty,5}[SL_2(5) \underset{\sqrt{5}}{\overset{2}{\boxtimes}} D_{10}]_4$	$2^5 \cdot 3 \cdot 5^2$	$[(SL_2(5) \circ SL_2(5)) : 2 \underset{\sqrt{5}}{\overset{2}{\boxtimes}} D_{10}]_{16}$
${}_{\infty,7}[SL_2(7).2]_4$	$2^5 \cdot 3 \cdot 7$	(B_{16})
${}_{\infty,7}[2.S_7]_4$	$2^5 \cdot 3^2 \cdot 5 \cdot 7$	E_8^2

The proof of this theorem is split up into eight lemmata. For the rest of this paragraph let $\mathcal{Q}$ be a definite quaternion algebra with center $\mathbb{Q}$ and let G be a primitive a.i.m.f. subgroup of $GL_4(\mathcal{Q})$. Then G has a complex representation of degree 8 of which the character values lie in $\mathbb{Q}$. By [Schu 05] this implies that the prime divisors of the order of G and hence the finite primes, at which $\mathcal{Q}$ ramifies, lie in $\{2, 3, 5, 7\}$.

Lemma 14.2. *If the order of G is divisible by 7, then G is one of ${}_{\infty,3}[SL_2(7)]_4$, ${}_{\infty,7}[SL_2(7).2]_4$, or ${}_{\infty,7}[2.S_7]_4$.*

Proof. Assume that 7 divides $|G|$. Since $O_7(G) = 1$ and the possible normal p-subgroups of G have no automorphism of order 7 (cf. Chapter 8), G contains a quasi-semi-simple normal subgroup N of order divisible by 7. According to [CCNPW 85] N is one of $SL_2(7)$ (2 representations) or $2.Alt_7$ (cf. Table 9.1).

If N is conjugate to $SL_2(7)$, where the enveloping $\mathbb{Q}$-algebra of N is $\mathcal{Q}_{\infty,3}^{4\times 4}$, the group N is already an absolutely irreducible subgroup of $GL_4(\mathcal{Q}_{\infty,3})$. One computes that $G = N = {}_{\infty,3}[SL_2(7)]_4$.

Next assume that N is conjugate to $SL_2(7)$, where the enveloping $\mathbb{Q}$-algebra of N is $\mathbb{Q}[\sqrt{-7}]^{4\times 4}$. Then the centralizer $C_G(N)$ embeds into $C_{\mathcal{Q}^{4\times 4}}(N) \cong \mathbb{Q}[\sqrt{-7}]$, hence is ± 1. Therefore G is isomorphic to $SL_2(7).2$. Since there is an element x of order 7 in G, such that $\chi(x) \in \mathbb{Q}$ for all irreducible characters χ of G and $\chi_o(x) = -1$ for the natural character χ_o (of degree 8) of G, Theorem A of [Fei 83] implies that $\mathcal{Q}$ can only be ramified at ∞ and 7. Let $\mathfrak{M}$ denote the maximal order in $\mathcal{Q}_{\infty,7}$, which is unique up to conjugacy. Then N fixes up to isomorphism five $\mathfrak{M}$-lattices, three of which form a set of normal critical lattices (in the sense of Definition 2.7), and a one-dimensional space of Hermitian forms. The automorphism group on the three normal critical lattices is ${}_{\infty,7}[SL_2(7).2]_4$ whereas the automorphism groups of the two $\mathfrak{M}N$-lattices, which are not invariant under the outer automorphism of N are conjugate to ${}_{\infty,7}[2.S_7]_4$. One concludes that G is conjugate to ${}_{\infty,7}[SL_2(7).2]_4$ in this case.

If N is isomorphic to $2.Alt_7$, one concludes as above that $G = N.2$ and $\mathcal{Q}$ is isomorphic to $\mathcal{Q}_{\infty,7}$. Since N contains the subgroup $SL_2(7)$ of the last case, one gets that G is conjugate to ${}_{\infty,7}[2.S_7]_4$ in this case. □

We now may assume that 7 does not divide the order of G. Hence the only finite primes on which $\mathcal{Q}$ ramifies lie in $\{2, 3, 5\}$.

Lemma 14.3. *If G contains a quasisimple normal subgroup N isomorphic to $Sp_4(3)$, then G is ${}_{\infty,3}[Sp_4(3) \overset{2}{\square} C_3]_4$.*

Proof. If $Sp_4(3) \cong N \trianglelefteq G$, then G contains the normal subgroup $\mathcal{B}^\circ(N) = Sp_4(3) \circ C_3 = NC_G(N)$ of index 2. Since there is an element x of order 3 in G, such that $\chi(x) \in \mathbb{Q}$ for all irreducible characters χ of G and $\chi_o(x) = -1$ for the natural character χ_o (of degree 8) of G, Theorem A of [Fei 83] implies that $\mathcal{Q} = \mathcal{Q}_{\infty,3}$. Let $\mathfrak{M}$ denote the maximal order in $\mathcal{Q}_{\infty,3}$, which is unique up to conjugacy. N fixes only one isomorphism class of $\mathfrak{M}$-lattices and a one-dimensional space of Hermitian forms and therefore has at most one a.i.m.f. supergroup. One computes that G is ${}_{\infty,3}[Sp_4(3) \overset{2}{\square} C_3]_4$. □

Imediately from Corollary 7.6 one gets:

Lemma 14.4. *If G contains a normal subgroup N isomorphic to Alt_5, then G is ${}_{\infty,2}[SL_2(3)]_1 \otimes A_4$ or ${}_{\infty,3}[\tilde{S}_3]_1 \otimes A_4$.*

Lemma 14.5. *G does not contain a normal subgroup $SL_2(9)$.*

Proof. Assume that $SL_2(9) \cong N \trianglelefteq G$. Then the restriction of the natural character of G to N is $4\chi_{4a}$ (or $4\chi_{4b}$) and the centralizer $C := C_G(N)$ embeds into $C_{\mathcal{Q}^{4\times 4}}(N)$ which is an indefinite quaternion algebra $\mathcal{C}$ with center $\mathbb{Q}$. Since the two outer automorphisms of N, not contained in $S_6 \leq Aut(N)$ interchange the two characters χ_{4a} and χ_{4b}, the group G contains CN of index ≤ 2. Since $\mathcal{B}^\circ(C_3 \circ SL_2(9)) = C_3 \circ Sp_4(3)$ one has $O_3(C) = 1$. If $G = CN$, then C is an a.i.m.f. subgroup of $GL_1(\mathcal{C})$. Hence by Corollary 6.2 $\mathcal{C} \cong \mathbb{Q}^{2\times 2}$, $\mathcal{Q} \cong \mathcal{Q}_{\infty,3}$, $C = D_8$ and $G = {}_{\infty,3}[SL_2(9)]_2 \otimes D_8$ is imprimitive and contained in ${}_{\infty,3}[SL_2(9)]_2^2$. Hence G contains CN of index 2. Since $O_3(C) = 1$, one finds $O_2(C) \cong C_4$ and $G = C_4 \overset{2(3)}{\sqsubset\!\sqsupset} N$ is contained in ${}_{\infty,2}[2_-^{1+6}.O_6^-(2)]_4$. □

Lemma 14.6. *If G contains a normal subgroup N isomorphic to $SL_2(5)$, then G is one of*

$$_{\infty,2}[SL_2(5) \overset{2(2)}{\boxtimes} D_8]_4,\ _{\infty,3}[SL_2(5) \overset{2(3)}{\boxtimes} S_3]_4,\ _{\infty,5}[SL_2(5).2]_2 \otimes A_2,$$

$$_{\infty,5}[SL_2(5):2]_2 \otimes A_2,\ \textit{or}\ _{\infty,5}[SL_2(5) \underset{\sqrt{5}}{\overset{2}{\boxtimes}} D_{10}]_4.$$

Proof. By Table 9.1 the restriction of the natural character of G to N is $4(\chi_{2a}+\chi_{2b})$. The centralizer $C := C_G(N)$ embeds into $\mathcal{C} := C_{\mathcal{Q}^{4\times 4}}(N)$ which is an indefinite quaternion algebra with center $\mathbb{Q}[\sqrt{5}]$. Moreover, the center of the enveloping algebra $\overline{\mathbb{Q}CN}$ is $\mathbb{Q}[\sqrt{5}]$ and G contains CN of index 2. With Lemma 2.14 this implies that $dim_{\mathbb{Q}}(\overline{\mathbb{Q}CN}) = 32$. Therefore $dim_{\mathbb{Q}[\sqrt{5}]}(\overline{\mathbb{Q}[\sqrt{5}]C}) = 4$. One concludes that $\mathcal{C}$ is isomorphic to $\mathbb{Q}[\sqrt{5}]^{2\times 2}$ and C is one of D_{10}, S_3, or D_8. Let α be an element of $G - CN$. In the first case, α does not centralize C. Computing the two possible extensions $CN.2 = G$ one finds that they are isomorphic and G is conjugate to $_{\infty,5}[SL_2(5) \underset{\sqrt{5}}{\overset{2}{\boxtimes}} D_{10}]_4$ in this case. In the other two cases one has two possibilities: Either α centralizes C or it induces the unique nontrivial outer automorphism of C. If α centralizes C one concludes that G is one of

$$_{\infty,5}[SL_2(5).2]_2 \otimes A_2 \text{ or } _{\infty,5}[SL_2(5):2]_2 \otimes A_2,$$

since the two groups $_{\infty,5}[SL_2(5).2]_2 \otimes D_8$ resp. $_{\infty,5}[SL_2(5):2]_2 \otimes D_8$ are imprimitive and contained in $_{\infty,5}[SL_2(5).2]_2^2$ resp. $_{\infty,5}[SL_2(5):2]_2^2$.

If α does not centralize C, one finds in each case two nonisomorphic extensions: $_{\infty,3}[SL_2(5) \overset{2(3)}{\boxtimes} S_3]_4$ and a proper subgroup of $_{\infty,3}[Sp_4(3) \overset{2}{\square} S_3]_4$ resp. $_{\infty,2}[SL_2(5) \overset{2(2)}{\boxtimes} D_8]_4$ and a proper subgroup of $_{\infty,2}[2_-^{1+6}.O_6^-(2)]_4$. Hence G is conjugate to $_{\infty,3}[SL_2(5) \overset{2(3)}{\boxtimes} S_3]_4$ resp. $_{\infty,2}[SL_2(5) \overset{2(2)}{\boxtimes} D_8]_4$ in these cases. □

Lemma 14.7. *If G does not contain a quasi-semi-simple normal subgroup, then $O_5(G) = 1$.*

Proof. Assume that $O_5(G) > 1$. Then $O_5(G) \cong C_5$. Since $\mathbb{Q}[\zeta_5]$ splits all possible quaternion algebras $\mathcal{Q}$ (which are $\mathcal{Q}_{\infty,2}$, $\mathcal{Q}_{\infty,3}$, $\mathcal{Q}_{\infty,5}$, and $\mathcal{Q}_{\infty,2,3,5}$ since by Lemma 14.2 ramification at 7 is excluded) the centralizer $C := C_G(O_5(G))$ embeds into $C_{\mathcal{Q}^{4\times 4}}(O_5(G)) \cong \mathbb{Q}[\zeta_5]^{2\times 2}$. Moreover G contains C of index 4. Applying Lemma 2.14 two times, one sees that the enveloping algebra $\overline{\mathbb{Q}C}$ is isomorphic to $\mathbb{Q}[\zeta_5]^{2\times 2}$.

If $O_3(C) \neq 1$, then C is one of $\pm C_5 \otimes S_3$ or $C_5 \otimes \tilde{S}_3$. Since the outer automorphism groups of $\tilde{S}_3$ resp. $\pm S_3$ are $\cong C_2$ and $\mathcal{Q}$ is totally definite one concludes that G contains one of the groups $Q_{20} \otimes S_3$ or $D_{10} \otimes \tilde{S}_3$ of index 2. Computing the possible extensions one finds that G is not maximal finite but contained in $_{\infty,5}[SL_2(5).2]_2 \otimes A_2$ and $_{\infty,5}[SL_2(5):2]_2 \otimes A_2$ resp. $_{\infty,3}[Sp_4(3) \overset{2}{\square} C_3]_4$ and $_{\infty,3}[SL_2(5) \overset{2(3)}{\boxtimes} S_3]_4$ or $_{\infty,3}[\tilde{S}_3]_1 \otimes A_4$ resp. $_{\infty,5}[SL_2(5) \underset{\sqrt{5}}{\overset{2}{\boxtimes}} D_{10}]_4$.

If $O_3(G) = 1$, then C is either D_8 or $SL_2(3)$ and as above one finds that G is a proper subgroup of $_{\infty,5}[SL_2(5).2]_2^2$ and $_{\infty,5}[SL_2(5):2]_2^2$ resp. $_{\infty,2}[2_-^{1+6}.O_6^-(2)]_4$ and $_{\infty,2}[SL_2(5) \overset{2(2)}{\boxtimes} D_8]_4$ or $_{\infty,2}[SL_2(3)]_1 \otimes A_4$ resp. $_{\infty,5}[SL_2(5) \underset{\sqrt{5}}{\overset{2}{\boxtimes}} D_{10}]_4$. □

Lemma 14.8. *If G does not contain a quasi-semi-simple normal subgroup, $O_5(G) = 1$, and $O_3(G) \neq 1$, then G is conjugate to ${}_{\infty,2}[2_-^{1+4}.Alt_5]_2 \otimes A_2$ or ${}_{\infty,3}[\tilde{S}_3]_1 \otimes F_4$.*

Proof. Assume that $O_3(G) > 1$. Then $O_3(G) \cong C_3$. Since $\mathbb{Q}[\zeta_3]$ splits all possible quaternion algebras $\mathcal{Q}$ (which are $\mathcal{Q}_{\infty,2}$, $\mathcal{Q}_{\infty,3}$, $\mathcal{Q}_{\infty,5}$, and $\mathcal{Q}_{\infty,2,3,5}$ since by Lemma 14.2 ramification at 7 is excluded) the centralizer $C := C_G(O_3(G))$ embeds into $C_{\mathcal{Q}^{4\times 4}}(O_3(G)) \cong \mathbb{Q}[\zeta_3]^{2\times 2}$. Moreover G contains C of index 2. Lemma 2.14 implies that the enveloping algebra $\overline{\mathbb{Q}C}$ is isomorphic to $\mathbb{Q}[\zeta_3]^{4\times 4}$.

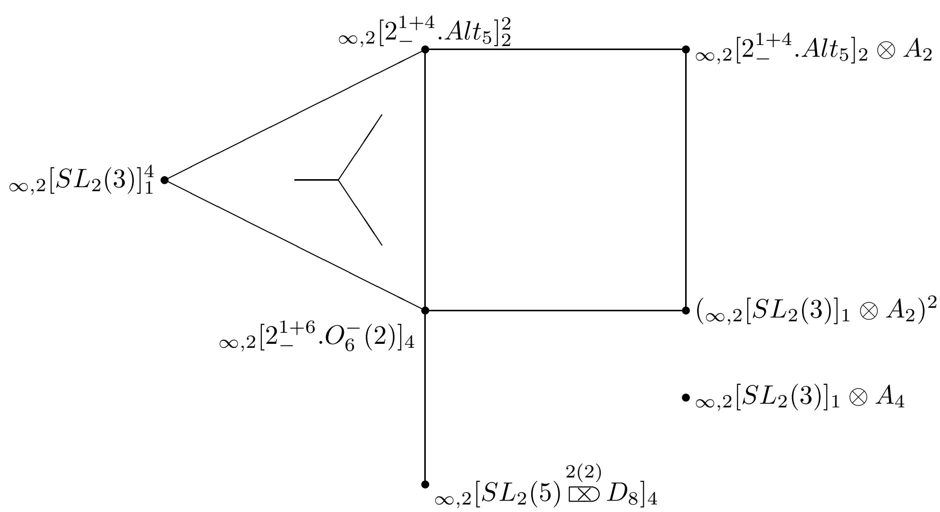
With Theorem 8.1 one finds that $O_2(G)$ is one of Q_8, $Q_8 \circ Q_8$, or $Q_8 \otimes D_8$. and C is one of $C_3 \otimes GL_2(3)$, $C_3 \circ \tilde{S}_4$, $C_3 \otimes F_4$ or $C_3 \circ 2_-^{1+4}.Alt_5$. Constructing the possible extensions one gets that G is either one of the two groups in the lemma or a proper subgroup of ${}_{\infty,2}[2_-^{1+6}.O_6^-(2)]_4$ or ${}_{\infty,3}[Sp_4(3) \overset{2}{\square} C_3]_4$. $\square$

Proposition 8.9 yields the following:

Lemma 14.9. *If G does not contain a quasi-semi-simple normal subgroup and $O_p(G) = 1$ for all odd primes p, then G is conjugate to ${}_{\infty,2}[2_-^{1+6}.O_6^-(2)]_4$.*

Proof of Theorem 14.1. Assume first that G contains a quasi-semi-simple normal subgroup N. According to Table 9.1 N is one of Alt_5, $SL_2(5)$, $SL_2(9)$, $SL_2(7)$ (2 representations), $2.Alt_7$, or $Sp_4(3)$. These cases are dealt with in Lemma 14.4, 14.6, 14.5, 14.2, respectively 14.3. The remaining three lemmata treat the case, that G does not contain a quasi-semi-simple normal subgroup. $\square$

Theorem 14.10. *$M_4^{irr}(\mathcal{Q}_{\infty,2})$ is as follows:*



List of the maximal simplices in $M_4^{irr}(\mathcal{Q}_{\infty,2})$

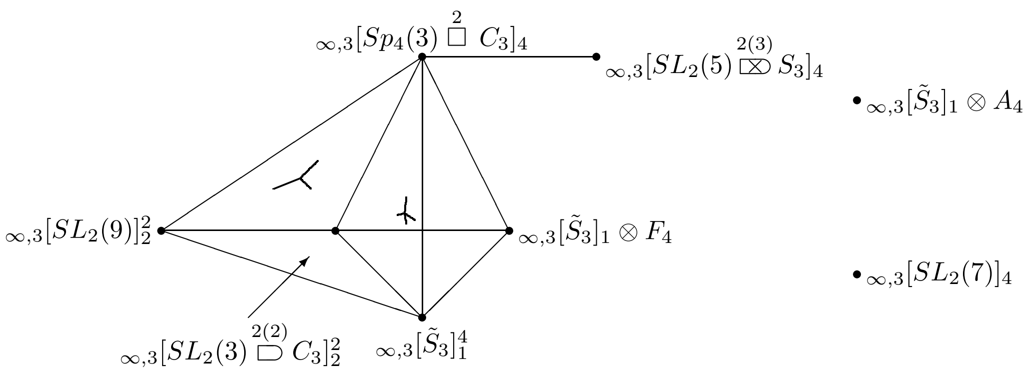
simplex	a common subgroup
$({}_{\infty,2}[2_-^{1+4}.Alt_5]_2^2,\ {}_{\infty,2}[SL_2(3)]_1^4, {}_{\infty,2}[2_-^{1+6}.O_6^-(2)]_4)$	$D_8 \otimes D_8 \otimes Q_8$
$({}_{\infty,2}[2_-^{1+6}.O_6^-(2)]_4, {}_{\infty,2}[SL_2(5) \overset{2(2)}{\boxtimes} D_8]_4)$	$Q_{20} \overset{2(2)}{\boxtimes} D_8$
$(({}_{\infty,2}[SL_2(3)]_1 \otimes A_2)^2, {}_{\infty,2}[2_-^{1+6}.O_6^-(2)]_4)$	$((S_3 \times S_3) \otimes SL_2(3)).2$
$(({}_{\infty,2}[SL_2(3)]_1 \otimes A_2)^2, {}_{\infty,2}[2_-^{1+4}.Alt_5]_2 \otimes A_2)$	$A_2 \otimes D_8 \otimes Q_8$
$({}_{\infty,2}[2_-^{1+4}.Alt_5]_2^2, {}_{\infty,2}[2_-^{1+4}.Alt_5]_2 \otimes A_2)$	$(C_3 \wr C_2) \overset{2(2)}{\boxtimes} D_8$

Proof. The list of a.i.m.f. subgroups of $GL_4(\mathcal{Q}_{\infty,2})$ is obtained from Theorems 14.1, 12.1, and 6.1. The vertex ${}_{\infty,2}[SL_2(3)]_1 \otimes A_4$ forms a component on its own, as it can be seen from the proof of Theorem (VI.13) in [NeP 95]. There it is shown that for every absolutely irreducible subgroup $U \leq GL_{16}(\mathbb{Q})$ of $Aut(F_4 \otimes A_4)$ the degrees of the 5-modular constituents of the natural representation of U are divisible by 4. Assume that there is a common absolutely irreducible subgroup $V \leq GL_4(\mathcal{Q}_{\infty,2})$ of ${}_{\infty,2}[SL_2(3)]_1 \otimes A_4$ and one of the other a.i.m.f. subgroups of $GL_4(\mathcal{Q}_{\infty,2})$. Let $H \cong SL_2(3)$ be the unit group of the endomorphism ring $\mathfrak{M}$ of the V-lattice ${}_{\infty,2}[SL_2(3)]_1 \otimes A_4$. Then the group $H \circ V \leq GL_{16}(\mathbb{Q})$ is an absolutely irreducible subgroup of $Aut(F_4 \otimes A_4)$ acting on the $\mathbb{Z}$-lattices of the r.i.m.f. supergroups that one obtains from the $\mathfrak{M}$-lattices of the a.i.m.f. supergroups of V. Hence $H \circ V$ fixes a 5-unimodular $\mathbb{Z}$-lattice or a $\mathbb{Z}$-lattice with elementary divisors $2^8 \cdot 5^8$ contradicting Theorem (VI.13) of [NeP 95].

Now consider the vertex $G := {}_{\infty,2}[SL_2(5) \overset{2(2)}{\boxtimes} D_8]_4$. The minimal absolutely irreducible subgroup of this group is easily seen to be $Q_{20} \overset{2(2)}{\boxtimes} D_8$. Since the only other a.i.m.f. supergroup of this group is ${}_{\infty,2}[2_-^{1+6}.O_6^-(2)]_4$, the list of maximal simplices in $M_4^{irr}(\mathcal{Q}_{\infty,2})$ with vertex G is complete.

To finish the proof it remains to show that there are no other simplices with one vertex $G := ({}_{\infty,2}[SL_2(3)]_1 \otimes A_2)^2$ or $H := {}_{\infty,2}[2_-^{1+4}.Alt_5]_2 \otimes A_2$ and one vertex in $\{{}_{\infty,2}[2_-^{1+4}.Alt_5]_2^2, {}_{\infty,2}[SL_2(3)]_1^4, {}_{\infty,2}[2_-^{1+6}.O_6^-(2)]_4\}$. Assume that there is such an additional simplex and let U be an absolutely irreducible common subgroup of the groups belonging to the vertices of the simplex. First assume that one of the vertices of the simplex is H. Let $\mathfrak{M}$ be the maximal order in $\mathcal{Q}_{\infty,2}$ and $L \in \mathcal{Z}_{\mathfrak{M}}(H)$ be a natural $\mathfrak{M}$-lattice of U. For $p = 2$ and 3 let L_p be the full preimage of the Sylow p-subgroup of the finite abelian group $L^{\#}/L$. Then for both primes $p = 2$ and 3, the $\mathfrak{M}/p\mathfrak{M}U$-module L_p/L is not simple, hence U fixes a $\mathfrak{M}$-lattice M_p with $L \subset M_p \subset L_p$. Computing the stabilizers in H of all the possible lattices one finds no such absolutely irreducible group U. In an analogous way, one checks the completeness of the list of maximal simplices in $M_4^{irr}(\mathcal{Q}_{\infty,2})$ with vertex G. Since the unique G-orbit of lattices M_2 having an absolutely irreducible stabilizer $Stab_G(M_2)$ satisfies $M_2 \sim {}_{\infty,2}[2_-^{1+4}.Alt_5]_2 \otimes A_2$ one concludes that every simplex with vertex G not listed in the theorem also contains a vertex H. Therefore the list of maximal simplices in $M_4^{irr}(\mathcal{Q}_{\infty,2})$ with vertex G is complete. $\square$

Theorem 14.11. *$M_4^{irr}(\mathcal{Q}_{\infty,3})$ is as follows.*



List of the maximal simplices in $M_4^{irr}(\mathcal{Q}_{\infty,3})$

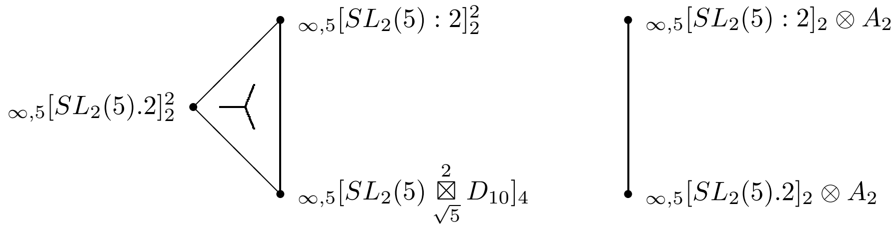
simplex	a common subgroup
$({}_{\infty,3}[SL_2(3) \overset{2(2)}{\sqsubset} C_3]_2^2{}_{\infty,3}[\tilde{S}_3]_1^4,$ ${}_{\infty,3}[\tilde{S}_3]_1 \otimes F_4, {}_{\infty,3}[Sp_4(3) \overset{2}{\square} C_3]_4)$	$GL_2(3) \otimes \tilde{S}_3$
$({}_{\infty,3}[SL_2(9)]_2^2, {}_{\infty,3}[SL_2(3) \overset{2(2)}{\sqsubset} C_3]_2^2{}_{\infty,3}[Sp_4(3) \overset{2}{\square} C_3]_4)$	$D_8 \otimes \tilde{S}_4$
$({}_{\infty,3}[\tilde{S}_3]_1^4, {}_{\infty,3}[SL_2(9)]_2^2)$	$((\pm C_3 \times C_3).C_4) \wr C_2$
$({}_{\infty,3}[Sp_4(3) \overset{2}{\square} C_3]_4, {}_{\infty,3}[SL_2(5) \overset{2(3)}{\boxtimes} S_3]_4)$	$Q_{20} \overset{2(3)}{\boxtimes} S_3$

Proof. The list of a.i.m.f. subgroups of $GL_4(\mathcal{Q}_{\infty,3})$ is obtained from Theorems 14.1, 12.1, and 6.1. The group ${}_{\infty,3}[SL_2(7)]_4$ forms a simplex on its own, because it is minimal absolutely irreducible. The minimal absolutely irreducible subgroups U of ${}_{\infty,3}[\tilde{S}_3]_1 \otimes A_4$ either satisfy $U^{(\infty)} = Alt_5$ or $U = \tilde{S}_3 \otimes C_5 : C_4$. In both cases U is not a subgroup of one of the other a.i.m.f. groups. The minimal absolutely irreducible subgroup of ${}_{\infty,3}[SL_2(5) \overset{2(3)}{\boxtimes} S_3]_4$ is $Q_{20} \overset{2(3)}{\boxtimes} S_3$ and its only other a.i.m.f. supergroup is ${}_{\infty,3}[Sp_4(3) \overset{2}{\square} C_3]_4$. To prove the theorem it remains to show that the list of maximal simplices in $M_4^{irr}(\mathcal{Q}_{\infty,3})$ with vertex $G := {}_{\infty,3}[SL_2(9)]_2^2$ is complete. Assume that there is an absolutely irreducible subgroup $U \leq G$ such that the a.i.m.f. supergroups of U lie not in one of the two maximal simplices in $M_4^{irr}(\mathcal{Q}_{\infty,3})$ with vertex G listed in the theorem. Then U embeds into one of ${}_{\infty,3}[\tilde{S}_3]_1^4$ or ${}_{\infty,3}[\tilde{S}_3]_1 \otimes F_4$ and hence the order of U is not divisible by 5. Moreover U contains a normal subgroup $N \trianglelefteq U$ of index 2, such that the restriction of the natural representation Δ of U to N is $\Delta_{|N} = \Delta_1 + \Delta_2$ with $\Delta_i(N) \leq {}_{\infty,3}[SL_2(9)]_2$ absolutely irreducible ($i = 1, 2$). Therefore $\Delta_1(N)$ is one of the two absolutely irreducible subgroups of ${}_{\infty,3}[SL_2(9)]_2$ of order not divisible by 5, which are $\tilde{S}_4$ and $(\pm C_3 \times C_3).C_4$. By Lemma 2.14 one also finds that the enveloping algebra $\overline{\mathbb{Q}N}$ of N is $\mathcal{Q}_{\infty,3}^{2\times 2} \oplus \mathcal{Q}_{\infty,3}^{2\times 2}$. Hence Δ_1 and Δ_2 are inequivalent. Let $\mathfrak{M}$ be the maximal order in $\mathcal{Q}_{\infty,3}$ and $L \in \mathcal{Z}_{\mathfrak{M}}(\Delta_1(N))$.

If $\Delta_1(N) = (\pm C_3 \times C_3).C_4$, then $2L$ is a maximal $\mathfrak{M}\Delta_1(N)$ sublattice of L. Hence U cannot embed into one of ${}_{\infty,3}[SL_2(3) \overset{2(2)}{\sqsubset} C_3]_2^2$ or ${}_{\infty,3}[\tilde{S}_3]_1 \otimes F_4$. Therefore U is a subgroup of ${}_{\infty,3}[Sp_4(3) \overset{2}{\square} C_3]_4$ in this case. Since this primitive a.i.m.f. group has a normal subgroup $\cong C_3$, there is a normal subgroup N_1 of U of index 2 such that $\overline{\mathbb{Q}N_1} \cong \mathbb{Q}[\sqrt{-3}]^{4\times 4}$ Therefore $N_1 \cap N =: N_2$ is a normal subgroup of index 2 in N such that the enveloping algebra $\overline{\mathbb{Q}\Delta_1(N_2)}$ is $\mathbb{Q}[\sqrt{-3}]^{2\times 2}$. But $\Delta_1(N)$ has only one subgroup of index 2 and the enveloping algebra of this subgroup is isomorphic to $\mathcal{Q}_{\infty,3} \oplus \mathcal{Q}_{\infty,3}$ which is a contradiction.

Hence $\Delta_1(N) = \tilde{S}_4$. If $\wp_3$ denotes the maximal ideal of $\mathfrak{M}$ containing 3, then $L/\wp L$ is a simple $\mathbb{F}_9\Delta_1(N)$-module. Since Δ_1 and Δ_2 are inequivalent, one concludes that U cannot fix one of the $\mathfrak{M}$-lattices of ${}_{\infty,3}[\tilde{S}_3]_1 \otimes F_4$ or ${}_{\infty,3}[\tilde{S}_3]_1^4$. $\square$

Theorem 14.12. *$M_4^{irr}(\mathcal{Q}_{\infty,5})$ is as follows.*



List of the maximal simplices in $M_4^{irr}(\mathcal{Q}_{\infty,5})$

simplex	a common subgroup
$({}_{\infty,5}[SL_2(5).2]_2^2, {}_{\infty,5}[SL_2(5):2]_2^2, {}_{\infty,5}[SL_2(5) \underset{\sqrt{5}}{\overset{2}{\boxtimes}} D_{10}]_4)$	$Q_{20} \underset{\sqrt{5}}{\overset{2}{\boxtimes}} D_{10}$
$({}_{\infty,5}[SL_2(5):2]_2 \otimes A_2, {}_{\infty,5}[SL_2(5).2]_2 \otimes A_2)$	$(\pm C_5.C_4) \otimes A_2$

Proof. The list of a.i.m.f. subgroups of $GL_4(\mathcal{Q}_{\infty,5})$ may be obtained from Theorems 14.1, 12.1, and 6.1. To see that the list of maximal simplices in $M_4^{irr}(\mathcal{Q}_{\infty,5})$ is complete, one has to note, that the minimal uniform subgroup of both groups ${}_{\infty,5}[SL_2(5):2]_2 \otimes A_2$ and ${}_{\infty,5}[SL_2(5).2]_2 \otimes A_2$ is $(\pm C_5.C_4) \otimes A_2$. Since this group does not embed into one of the other 3 a.i.m.f. groups one easily deduces the theorem. $\square$

Theorem 14.13. *$M_4^{irr}(\mathcal{Q}_{\infty,7})$ is as follows.*

Proof. By Theorems 14.1, 12.1, and 6.1. $M_4^{irr}(\mathcal{Q}_{\infty,7})$ has two vertices. These two a.i.m.f. groups have no common absolutely irreducible subgroup since both groups are minimal absolutely irreducible as one sees from the list of maximal subgroups of the two groups given in [CCNPW 85]. $\square$

$$Z(\mathcal{Q}) \quad \text{real quadratic.}$$

Theorem 14.14. *Let $\mathcal{Q}$ be a definite quaternion algebra with center K, such that $[K:\mathbb{Q}] = 2$ and let G be a primitive a.i.m.f. subgroup of $GL_4(\mathcal{Q})$. Then G is conjugate to one of the a.i.m.f. groups given in the following table.*

The table is built up as the one in Theorem 12.17.

List of the primitive a.i.m.f. subgroups of $GL_4(\mathcal{Q})$, where $\mathcal{Q}$ is a definite quaternion algebra over a real quadratic field.

${}_{\sqrt{2},\infty}[\tilde{S}_4]_1 \otimes A_4 \ (2^7 \cdot 3^2 \cdot 5)$ $(F_4 \otimes A_4)^2,\ E_8 \otimes A_4$
${}_{\sqrt{2},\infty}[2_-^{1+4}.S_5]_2 \otimes A_2 \ (2^9 \cdot 3^2 \cdot 5)$ $(E_8 \otimes A_2)^2,\ (F_4 \otimes A_2)^4$
${}_{\infty,3}[SL_2(9)]_2 \otimes {}_{\sqrt{2}}[D_{16}]_2 \ (2^7 \cdot 3^2 \cdot 5)$ $[(Sp_4(3) \otimes_{\sqrt{-3}} Sp_4(3)) : 2 \overset{2}{\square} C_3]_{32},\ [SL_2(9).2 \overset{2(2)}{\underset{\infty,2}{\boxtimes}} 2_-^{1+4}.Alt_5]_{32}$
${}_{\infty,5}[SL_2(5).2]_2 \otimes {}_{\sqrt{2}}[D_{16}]_2 \ (2^7 \cdot 3 \cdot 5)$ $[SL_2(5) \overset{2(2)}{\underset{\infty,2}{\boxtimes}} 2_-^{1+6}.O_6^-(2)]_{32},\ [SL_2(5) \overset{2(2)}{\underset{\infty,2}{\boxtimes}} 2_-^{1+4}.Alt_5]_{16}^2$
${}_{\sqrt{2},\infty}[2_-^{1+6}.O_6^-(2).2]_4 \ (2^{14} \cdot 3^4 \cdot 5)$ $[2_+^{1+10}.O_{10}^+(2)]_{32},\ (F_4 \tilde{\otimes} F_4)^2$
${}_{\sqrt{2},\infty,2,3}[C_3 \overset{2(2+\sqrt{2})}{\sqsubset\!\supset} \tilde{S}_4 \otimes_{\sqrt{2}} D_{16}]_4 \ (2^8 \cdot 3^2)$ $E_8^4,\ (F_4 \tilde{\otimes} F_4)^2$
${}_{\sqrt{2},\infty,2,3}[C_3 \overset{2(2+\sqrt{2})}{\boxtimes} \tilde{S}_4 \circ Q_{16}]_4 \ (2^8 \cdot 3^2)$ $(A_2 \otimes E_8)^2,\ (A_2 \otimes F_4)^4$
${}_{\sqrt{2},\infty,2,5}[SL_2(5) \overset{2(2+\sqrt{2})}{\boxtimes} D_{16}]_{4,1} \ (2^7 \cdot 3 \cdot 5)$ (E_8^4)
${}_{\sqrt{2},\infty,2,5}[SL_2(5) \overset{2(2+\sqrt{2})}{\boxtimes} D_{16}]_{4,2} \ (2^7 \cdot 3 \cdot 5)$ $([(SL_2(5) \overset{2}{\square} SL_2(5)) : 2]_8^4)$
${}_{\sqrt{2},\infty,2,5}[D_{10} \overset{2(2+\sqrt{2})}{\boxtimes} Q_{16}]_4 \ (2^6 \cdot 5)$ (A_4^8)

	${}_{\sqrt{3},\infty}[Q_{24}]_1 \otimes A_4\ (2^6 \cdot 3^2 \cdot 5)$
O_1	$(A_4 \otimes A_2)^4,\ (A_4 \otimes F_4)^2$
O_2	$F_4 \otimes A_4 \otimes A_2,\ E_8 \otimes A_4$
	${}_{\sqrt{3},\infty}[SL_2(7):2]_4\ (2^5 \cdot 3 \cdot 7)$
O_1	$[SL_2(7) \overset{2(3)}{\circ} \tilde{S}_3]_{16}^2,\ [SL_2(7) \overset{2(3)}{\underset{\infty,3}{\otimes}} (SL_2(3) \overset{2}{\Box}\ C_3)]_{32}$
O_2	$[(SL_2(3) \circ C_4).2 \overset{2(3)}{\underset{\sqrt{-1}}{\boxtimes}} SL_2(7)]_{32},\ [SL_2(7) \overset{2(3)}{\underset{\infty,3}{\otimes}} SL_2(9)]_{32}$
	${}_{\sqrt{3},\infty}[Sp_4(3) \overset{2}{\underset{\sqrt{-3}}{\boxtimes}} C_{12}]_4\ (2^9 \cdot 3^5 \cdot 5)$
O_1	$E_8^4,\ [(Sp_4(3) \circ C_3) \overset{2}{\underset{\sqrt{-3}}{\boxtimes}} SL_2(3)]_{16}^2$
O_2	$F_4 \otimes E_8,\ [(Sp_4(3) \otimes_{\sqrt{-3}} Sp_4(3)):2 \overset{2}{\Box}\ C_3]_{32}$
	${}_{\infty,5}[SL_2(5).2]_2 \otimes {}_{\sqrt{3}}[D_{24}]_2\ (2^6 \cdot 3^2 \cdot 5)$
O_1	$[(SL_2(5) \overset{2}{\Box}\ SL_2(5)):2]_8^4,\ [SL_2(5) \overset{2(3)}{\underset{\infty,3}{\boxtimes}} (SL_2(3) \overset{2}{\Box}\ C_3)]_{16}^2$
O_2	$[(SL_2(5) \overset{2}{\Box}\ SL_2(5)):2]_8 \otimes F_4,\ [SL_2(5) \overset{2(3)}{\underset{\infty,3}{\boxtimes}} (Sp_4(3) \overset{2}{\Box}\ C_3)]_{32}$
	${}_{\sqrt{3},\infty}[D_8 \otimes D_8 \otimes C_4.S_6 \overset{2}{\boxtimes} C_3]_4\ (2^{11} \cdot 3^3 \cdot 5)$
O_1	$(F_4 \tilde{\otimes} F_4)^2,\ (A_2 \otimes E_8)^2$
O_2	$(F_4 \tilde{\otimes} F_4) \otimes A_2,\ [2_+^{1+10}.O_{10}^+(2)]_{32}$
	${}_{\sqrt{3},\infty}[SL_2(5) \overset{2(2+\sqrt{3})}{\boxtimes} D_{24}]_4\ (2^6 \cdot 3^2 \cdot 5)$
O_1	$[SL_2(5) \overset{2(2)}{\underset{\infty,2}{\boxtimes}} 2_-^{1+6}.O_6^-(2)]_{32},\ [SL_2(5) \overset{2(2)}{\underset{\infty,2}{\boxtimes}} 2_-^{1+4}.Alt_5]_{16} \otimes A_2$
O_2	$[SL_2(5) \overset{2(2)}{\underset{\infty,2}{\boxtimes}} 2_-^{1+4}.Alt_5]_{16}^2,\ ([(SL_2(5) \overset{2}{\Box}\ SL_2(5)):2]_8 \otimes A_2)^2$
	${}_{\sqrt{3},\infty}[D_{10} \overset{2}{\boxtimes} Q_{24}]_4\ (2^5 \cdot 3 \cdot 5)$
O_1	$[(SL_2(5) \circ SL_2(5)):2 \overset{2}{\underset{\sqrt{5}}{\boxtimes}} D_{10}]_{16}^2,\ [D_{120}.(C_4 \times C_2)]_{16}^2$
O_2	$[C_{15}:C_4 \overset{2(2)}{\boxtimes} F_4]_{32},\ [2_-^{1+4}.Alt_5 \otimes_{\infty,2} SL_2(5) \overset{2}{\underset{\sqrt{5}}{\boxtimes}} D_{10}]_{32}$

${}_{\sqrt{5},\infty}[SL_2(5)]_1 \otimes F_4 \ (2^9 \cdot 3^3 \cdot 5)$ $[(SL_2(5) \overset{2}{\square} SL_2(5)) : 2]_8 \otimes F_4,\ F_4 \otimes E_8$
${}_{\sqrt{5},\infty}[SL_2(5) \otimes_{\sqrt{5}} D_{10}]_2 \otimes A_2 \ (2^5 \cdot 3^2 \cdot 5^2)$ $[(SL_2(5) \circ SL_2(5)) : 2 \underset{\sqrt{5}}{\overset{2}{\boxtimes}} D_{10}]_{16} \otimes A_2$
${}_{\sqrt{5},\infty}[SL_2(9)]_4 \ (2^4 \cdot 3^2 \cdot 5)$ $[SL_2(9) \overset{2}{\square} SL_2(5)]_{32},\ ([4.L_3(4).2^2]_{32,1},\ [4.L_3(4).2^2]_{32,2})$
${}_{\infty,3}[SL_2(9)]_2 \otimes {}_{\sqrt{5}}[\pm D_{10}]_2 \ (2^5 \cdot 3^2 \cdot 5^2)$ $[SL_2(9) \otimes D_{10} \overset{2}{\square} SL_2(5)]_{32}$
${}_{\sqrt{5},\infty}[(SL_2(5) \circ SL_2(5) \otimes_{\sqrt{5}} SL_2(5)) : S_3]_4 \ (2^8 \cdot 3^4 \cdot 5^3)$ $[((SL_2(5) \circ SL_2(5)) \underset{\sqrt{5}}{\overset{2}{\boxtimes}} (SL_2(5) \circ SL_2(5))) : S_4]_{32,i} \ (i = 1, 2)$
${}_{\infty,3}[SL_2(3) \overset{2}{\square} C_3]_2 \otimes {}_{\sqrt{5}}[\pm D_{10}]_2 \ (2^5 \cdot 3^2 \cdot 5)$ $[(SL_2(5) \otimes_{\sqrt{5}} D_{10}) \underset{\infty,3}{\overset{2(3)}{\boxtimes}} (SL_2(3) \overset{2}{\square} C_3)]_{32}$
${}_{\infty,2}[2_-^{1+4}.Alt_5]_2 \otimes {}_{\sqrt{5}}[\pm D_{10}]_2 \ (2^8 \cdot 3 \cdot 5^2)$ $[(2_-^{1+4}.Alt_5 \otimes_{\infty,2} SL_2(5)) \underset{\sqrt{5}}{\overset{2(2)}{\boxtimes}} D_{10}]_{32}$
${}_{\sqrt{5},\infty}[SL_2(5)]_1 \otimes_{\sqrt{5}} {}_{\sqrt{5},2,5}[C_5 \underset{\sqrt{5}'}{\overset{2(2)}{\boxtimes}} SL_2(3)]_2 \ (2^6 \cdot 3^2 \cdot 5^2)$ $[(SL_2(5) \circ SL_2(5)) : 2 \underset{\sqrt{5}}{\overset{2}{\boxtimes}} D_{10}]_{16}^2$
${}_{\sqrt{5},\infty}[SL_2(5)]_1 \otimes_{\sqrt{5}} {}_{\sqrt{5},2,5}[C_5 \overset{2(2)}{\boxtimes} D_8]_2 \ (2^6 \cdot 3 \cdot 5^2)$ $E_8^4,\ [(SL_2(5) \overset{2}{\square} SL_2(5)) : 2]_8^4$
${}_{\sqrt{5},\infty,2,5}[C_5 \overset{2(2)}{\boxtimes} F_4]_4 \ (2^8 \cdot 3^2 \cdot 5)$ $(A_4 \otimes F_4)^2$
${}_{\sqrt{5},\infty,2,5}[C_5 \underset{\sqrt{5}'}{\overset{2(2)}{\boxtimes}} 2_-^{1+4}.Alt_5]_4 \ (2^8 \cdot 3^2 \cdot 5)$ $(F_4 \tilde{\otimes} F_4)^2,\ [SL_2(5) \underset{\infty,2}{\overset{2(2)}{\boxtimes}} 2_-^{1+4}.Alt_5]_{16}^2$

	${}_{\sqrt{5},\infty,2,5}[C_5 \overset{2(2)}{\boxtimes} D_8]_2 \otimes A_2$ $(2^5 \cdot 3 \cdot 5)$
	$(A_2 \otimes A_4)^4$
	${}_{\sqrt{5},\infty,2,5}[C_5 \underset{\sqrt{5}'}{\overset{2(2)}{\boxtimes}} SL_2(3)]_2 \otimes A_2$ $(2^5 \cdot 3^2 \cdot 5)$
	$(A_2 \otimes E_8)^2$, $(A_2 \otimes [(SL_2(5) \overset{2}{\square} SL_2(5)) : 2]_8)^2$
	${}_{\sqrt{5},\infty,2,5}[C_5 \underset{\sqrt{5}}{\overset{2(2)}{\boxtimes}} (C_3 \overset{2(2)}{\boxtimes} D_8)]_{32}$ $(2^5 \cdot 3 \cdot 5)$
	$[D_{120}.(C_4 \times C_2)]^2_{16}$
	${}_{\sqrt{5},\infty,2,5}[C_5 \underset{\sqrt{5}'}{\overset{2(2)}{\boxtimes}} (SL_2(3) \overset{2}{\square} C_3)]_4$ $(2^5 \cdot 3^2 \cdot 5)$
	$[(Sp_4(3) \circ C_3) \underset{\sqrt{-3}}{\overset{2}{\boxtimes}} SL_2(3)]^2_{16}$, $[SL_2(5) \underset{\infty,3}{\overset{2(3)}{\boxtimes}} (SL_2(3) \overset{2}{\square} C_3)]^2_{16}$

	${}_{\sqrt{5},\infty,5,3}[C_5 \overset{2(3)}{\boxtimes} SL_2(9)]_4$ $(2^5 \cdot 3^2 \cdot 5^2)$
O_1	$[SL_2(9) \overset{2(3)}{\otimes} SL_2(9) : 2]^2_{16}$, $[SL_2(5) \overset{2(3)}{\otimes} SL_2(9)]^2_{16}$
O_2	$[(Sp_4(3) \otimes_{\sqrt{-3}} Sp_4(3)) : 2 \overset{2}{\square} C_3]_{32}$, $[SL_2(5) \overset{2(3)}{\boxtimes} (Sp_4(3) \overset{2}{\square} C_3)]_{32}$
	${}_{\sqrt{5},\infty}[SL_2(5)]_1 \otimes_{\sqrt{5}} {}_{\sqrt{5},5,3}[C_5 \underset{\sqrt{5}'}{\overset{2(3)}{\boxtimes}} \tilde{S}_3]_2$ $(2^5 \cdot 3^2 \cdot 5^2)$
O_1	$[(SL_2(5) \circ SL_2(5)) : 2 \underset{\sqrt{5}}{\overset{2}{\boxtimes}} D_{10}]^2_{16}$
O_2	$[(SL_2(5) \circ SL_2(5)) : 2 \underset{\sqrt{5}}{\overset{2}{\boxtimes}} D_{10}]^2_{16}$
	${}_{\sqrt{5},\infty}[SL_2(5)]_1 \otimes_{\sqrt{5}} {}_{\sqrt{5},5,3}[C_5 \overset{2(3)}{\boxtimes} S_3]_2$ $(2^5 \cdot 3^2 \cdot 5^2)$
O_1	$(A_2 \otimes E_8)^2$, $(A_2 \otimes [(SL_2(5) \overset{2}{\square} SL_2(5)) : 2]_8)^2$
O_2	$(A_2 \otimes E_8)^2$, $(A_2 \otimes [(SL_2(5) \overset{2}{\square} SL_2(5)) : 2]_8)^2$
	${}_{\sqrt{5},\infty,5,3}[C_5 \underset{\sqrt{5}}{\overset{2(3)}{\boxtimes}} (C_3 \overset{2(2)}{\boxtimes} D_8)]_{32}$ $(2^5 \cdot 3 \cdot 5)$
O_1	$[D_{120}.(C_4 \times C_2)]^2_{16}$, $([C_{15} : C_4 \overset{2(2)}{\boxtimes} F_4]_{32})$
O_2	$[D_{120}.(C_4 \times C_2)]^2_{16}$
	${}_{\sqrt{5},\infty,5,3}[C_5 \underset{\sqrt{5}'}{\overset{2(3)}{\boxtimes}} (C_3 \overset{2(2)}{\boxtimes} D_8)]_{32}$ $(2^5 \cdot 3 \cdot 5)$
O_1	$[SL_2(5) \underset{\infty,2}{\overset{2(2)}{\boxtimes}} 2^{1+4}_{-}.Alt_5]^2_{16}$, $(F_4 \tilde{\otimes} F_4)^2$
O_2	$[SL_2(5) \underset{\infty,2}{\overset{2(2)}{\boxtimes}} 2^{1+4}_{-}.Alt_5]^2_{16}$, $(F_4 \tilde{\otimes} F_4)^2$
	${}_{\sqrt{5},\infty,5,3}[C_5 \underset{\sqrt{5}'}{\overset{2(3)}{\boxtimes}} (SL_2(3) \overset{2}{\square} C_3)]_4$ $(2^5 \cdot 3^2 \cdot 5)$
O_1	$[(Sp_4(3) \circ C_3) \underset{\sqrt{-3}}{\overset{2}{\boxtimes}} SL_2(3)]^2_{16}$, $[SL_2(5) \underset{\infty,3}{\overset{2(3)}{\boxtimes}} (SL_2(3) \overset{2}{\square} C_3)]^2_{16}$
O_2	$[(Sp_4(3) \circ C_3) \underset{\sqrt{-3}}{\overset{2}{\boxtimes}} SL_2(3)]^2_{16}$, $[SL_2(5) \underset{\infty,3}{\overset{2(3)}{\boxtimes}} (SL_2(3) \overset{2}{\square} C_3)]^2_{16}$
	${}_{\sqrt{5},\infty,5,3}[C_5 \overset{2(3)}{\boxtimes} (SL_2(3) \overset{2}{\square} C_3)]_4$ $(2^5 \cdot 3^2 \cdot 5)$
O_1	$(A_4 \otimes F_4)^2$
O_2	$(A_4 \otimes F_4)^2$

	${}_{\sqrt{5},\infty,2,3}[C_5 \overset{2(6)}{\underset{\sqrt{5}'}{\boxtimes}} (\tilde{S}_3 \otimes D_8)]_{32}\ (2^5 \cdot 3 \cdot 5)$
	(E_8^4), $([(SL_2(5) \overset{2}{\square} SL_2(5)) : 2]_8^4)$
	${}_{\sqrt{5},\infty,2,3}[C_5 \overset{2(6)}{\boxtimes} (S_3 \otimes D_8)]_{32}\ (2^5 \cdot 3 \cdot 5)$
	$((A_2 \otimes A_4)^4)$
	${}_{\sqrt{5},\infty,2,3}[C_5 \overset{2(6)}{\underset{\sqrt{5}}{\boxtimes}} (C_3 \overset{2(2)}{\boxtimes} D_8)]_{32}\ (2^5 \cdot 3 \cdot 5)$
	$[D_{120}.(C_4 \times C_2)]_{16}^2$, $[((SL_2(5) \circ SL_2(5)) : 2 \overset{2(6)}{\underset{\sqrt{5}}{\boxtimes}} (C_3 \overset{2(2)}{\boxtimes} D_8)]_{32,1}$
	${}_{\sqrt{5},\infty,2,3}[C_5 \overset{2(6)}{\underset{\sqrt{5}'}{\boxtimes}} (C_3 \overset{2(2)}{\boxtimes} D_8)]_{32}\ (2^5 \cdot 3 \cdot 5)$
	$([SL_2(5) \overset{2(2)}{\underset{\infty,2}{\boxtimes}} 2_-^{1+4}.Alt_5]_{16}^2)$
	${}_{\sqrt{5},\infty,2,3}[C_5 \overset{2(6)}{\underset{\sqrt{5}'}{\boxtimes}} S_3 \otimes SL_2(3))]_4\ (2^5 \cdot 3^2 \cdot 5)$
	$((A_2 \otimes E_8)^2)$, $((A_2 \otimes [(SL_2(5) \overset{2}{\square} SL_2(5)) : 2]_8)^2)$
	${}_{\sqrt{5},\infty,2,3}[C_5 \overset{2(6)}{\underset{\sqrt{5}}{\boxtimes}} (\tilde{S}_3 \otimes_{\sqrt{-3}} SL_2(3))]_4\ (2^5 \cdot 3^2 \cdot 5)$
	$[((SL_2(5) \circ SL_2(5)) \overset{2}{\underset{\sqrt{5}}{\boxtimes}} (SL_2(5) \circ SL_2(5))) : S_4]_{32,1}$
	${}_{\sqrt{5},\infty,2,3}[C_5 \overset{2(6)}{\boxtimes} (SL_2(3) \overset{2}{\square} C_3)]_4\ (2^5 \cdot 3^2 \cdot 5)$
	$((A_4 \otimes F_4)^2)$
	${}_{\sqrt{5},\infty,2,3}[C_5 \overset{2(6)}{\underset{\sqrt{5}'}{\boxtimes}} (SL_2(3) \overset{2}{\square} C_3)]_4\ (2^5 \cdot 3^2 \cdot 5)$
	$([SL_2(5) \overset{2(3)}{\underset{\infty,3}{\boxtimes}} (SL_2(3) \overset{2}{\square} C_3)]_{16}^2)$

	${}_{\sqrt{6},\infty}[(SL_2(9) \otimes D_8).2]_4\ (2^7 \cdot 3^2 \cdot 5)$
O_1	$[2_+^{1+10}.O_{10}^+(2)]_{32}$, $[(Sp_4(3) \otimes_{\sqrt{-3}} Sp_4(3)) : 2 \overset{2}{\square} C_3]_{32}$
O_2	$[(Sp_4(3) \otimes_{\sqrt{-3}} Sp_4(3)) : 2 \overset{2}{\square} C_3]_{32}$, E_8^4
O_3	$(F_4 \tilde{\otimes} F_4)^2$, $[SL_2(9).2 \overset{2(2)}{\underset{\infty,2}{\boxtimes}} 2_-^{1+4}.Alt_5]_{32}$
	${}_{\sqrt{6},\infty}[(S_3 \otimes 2_-^{1+4}.Alt_5).2]_4\ (2^9 \cdot 3^2 \cdot 5)$
O_1	$(A_2 \otimes E_8)^2$, $[2_+^{1+10}.O_{10}^+(2)]_{32}$
O_2	$(A_2 \otimes E_8)^2$, $[SL_2(5) \overset{2(2)}{\underset{\infty,2}{\boxtimes}} 2_-^{1+4}.Alt_5]_{16}^2$, E_8^4
O_3	$(F_4 \tilde{\otimes} F_4)^2$, $A_2 \otimes F_4 \tilde{\otimes} F_4$, $[SL_2(5) \overset{2(2)}{\underset{\infty,2}{\boxtimes}} 2_-^{1+6}.O_6^-(2)]_{32}$
	${}_{\sqrt{6},\infty}[(S_3 \otimes F_4).2]_4\ (2^9 \cdot 3^3)$
O_1	$(F_4 \tilde{\otimes} F_4)^2$, $A_2 \otimes F_4 \tilde{\otimes} F_4$, $[SL_2(5) \overset{2(2)}{\underset{\infty,2}{\boxtimes}} 2_-^{1+6}.O_6^-(2)]_{32}$
O_2	$(A_2 \otimes F_4)^4$, $(F_4 \tilde{\otimes} F_4)^2$
O_3	$(A_2 \otimes E_8)^2$, $[2_+^{1+10}.O_{10}^+(2)]_{32}$

	${}_{\sqrt{7},\infty}[2.Alt_7 \overset{2}{\boxtimes} C_4]_4\ (2^6\cdot 3^2\cdot 5\cdot 7)$
O_1	E_8^4, $(F_4\tilde{\otimes}F_4)^2$, $[2.Alt_7 \underset{\sqrt{-7}}{\overset{2(3)}{\boxtimes}} \tilde{S}_3]_{16}^2$
O_2	$[(2.Alt_7 \underset{\sqrt{-7}}{\overset{2}{\boxtimes}} 2.Alt_7):2]_{32}$, $[2.Alt_7 \underset{\sqrt{-7}}{\overset{2(3)}{\boxtimes}} (SL_2(3) \overset{2}{\Box} C_3)]_{32}$
O_3	$F_4\otimes E_8$, $[2_+^{1+10}.O_{10}^+(2)]_{32}$

	${}_{\sqrt{7},\infty}[SL_2(7) \overset{2}{\boxtimes} C_4]_4\ (2^6\cdot 3\cdot 7)$
O_1	$(F_4\tilde{\otimes}F_4)^2$, $[SL_2(7) \underset{\sqrt{-7}}{\overset{2(3)}{\boxtimes}} \tilde{S}_3]_{16}^2$
O_2	$[SL_2(7) \underset{\sqrt{-7}}{\overset{2}{\boxtimes}} 2.Alt_7]_{32}$, $[SL_2(7) \underset{\sqrt{-7}}{\overset{2(3)}{\boxtimes}} (SL_2(3) \overset{2}{\Box} C_3)]_{32}$
O_3	$[2_+^{1+10}.O_{10}^+(2)]_{32}$

	${}_{\sqrt{10},\infty}[SL_2(5) \overset{2}{\boxtimes} D_{16}]_4\ (2^7\cdot 3\cdot 5)$
O_1	E_8^4, $[(SL_2(5) \overset{2}{\Box}\ SL_2(5)):2]_8^4$, $F_4\otimes[(SL_2(5) \overset{2}{\Box}\ SL_2(5)):2]_8$,
	$F_4\otimes E_8$, $[(SL_2(5)\circ SL_2(5)):2 \underset{\sqrt{5}}{\overset{2(6)}{\boxtimes}} (C_3 \overset{2(2)}{\boxtimes} D_8)]_{32,i} (i=1,2)$
O_2	$[SL_2(5) \underset{\infty,2}{\overset{2(2)}{\boxtimes}} 2_-^{1+6}.O_6^-(2)]_{32}$, $[(Sp_4(3)\otimes_{\sqrt{-3}} Sp_4(3)):2 \overset{2}{\Box}\ C_3]_{32}$,
	$(F_4\tilde{\otimes}F_4)^2$, $[SL_2(5) \underset{\infty,3}{\overset{2(3)}{\boxtimes}} (Sp_4(3) \overset{2}{\Box}\ C_3)]_{32}$, $[2_+^{1+10}.O_{10}^+(2)]_{32}$,
	$[SL_2(9).2 \underset{\infty,2}{\overset{2(2)}{\boxtimes}} 2_-^{1+4}.Alt_5]_{32}$, $[SL_2(5) \underset{\infty,2}{\overset{2(2)}{\boxtimes}} 2_-^{1+4}.Alt_5]_{16}^2$
O_3	$\sim$
O_4	$\sim$

	${}_{\sqrt{10},\infty}[D_{10} \overset{2}{\boxtimes} \tilde{S}_4]_4\ (2^6\cdot 3\cdot 5)$
O_1	$[SL_2(9)\otimes D_{10} \overset{2}{\Box}\ SL_2(5)]_{32}$, $[(SL_2(5)\circ SL_2(5)):2 \underset{\sqrt{5}}{\overset{2}{\boxtimes}} D_{10}]_{16}^2$,
	$[(2_-^{1+4}.Alt_5 \otimes_{\infty,2} SL_2(5)) \underset{\sqrt{5}}{\overset{2}{\boxtimes}} D_{10}]_{32}$, $[(SL_2(5)\otimes_{\sqrt{5}} D_{10}) \underset{\infty,3}{\overset{2(3)}{\boxtimes}} (SL_2(3) \overset{2}{\Box}\ C_3)]_{32}$
O_2	$(A_4\otimes F_4)^2$, $A_4\otimes E_8$, $[C_{15}:C_4 \overset{2(2)}{\boxtimes} F_4]_{32}$
O_3	$\sim$
O_4	$\sim$

	${}_{\sqrt{15},\infty}[SL_2(5) \overset{2}{\boxtimes} D_{24}]_{4,1}\ (2^6\cdot 3^2\cdot 5)$
O_1	$(A_2\otimes E_8)^2$, $[(SL_2(5)\circ SL_2(5)):2 \underset{\sqrt{5}}{\overset{2(6)}{\boxtimes}} (C_3 \overset{2(2)}{\boxtimes} D_8)]_{32,2}$
O_2	$A_2\otimes(F_4\tilde{\otimes}F_4)$, $[SL_2(5) \underset{\infty,3}{\overset{2(3)}{\boxtimes}} (Sp_4(3) \overset{2}{\Box}\ C_3)]_{32}$
O_3	E_8^4, $[SL_2(5) \underset{\infty,2}{\overset{2(2)}{\boxtimes}} 2_-^{1+4}.Alt_5]_{16}^2$
O_4	E_8^4, $[SL_2(5) \underset{\infty,2}{\overset{2(2)}{\boxtimes}} 2_-^{1+4}.Alt_5]_{16}^2$
O_5	$(A_2\otimes E_8)^2$, $[SL_2(5) \underset{\infty,3}{\overset{2(3)}{\boxtimes}} (C_3 \overset{2}{\Box}\ SL_2(3))]_{16}^2$
O_6	$(F_4\tilde{\otimes}F_4)^2$, $[(SL_2(5) \overset{2}{\Box}\ SL_2(5)):2]_8^4$
O_7	$[2_+^{1+10}.O_{10}^+(2)]_{32}$, $F_4\otimes[(SL_2(5) \overset{2}{\Box}\ SL_2(5)):2]_8$
O_8	$[SL_2(5) \underset{\infty,2}{\overset{2(2)}{\boxtimes}} 2_-^{1+6}.O_6^-(2)]_{32}$, $F_4\otimes E_8$

	${}_{\sqrt{15},\infty}[SL_2(5) \overset{2}{\boxtimes} D_{24}]_{4,2}$ $(2^6 \cdot 3^2 \cdot 5)$
O_1	$(A_2 \otimes [(SL_2(5) \overset{2}{\square}\ SL_2(5)) : 2]_8)^2$, $[(SL_2(5) \circ SL_2(5)) : 2 \underset{\sqrt{5}}{\overset{2(6)}{\boxtimes}} (C_3 \overset{2(2)}{\boxtimes} D_8)]_{32,1}$
O_2	$[SL_2(5) \underset{\infty,2}{\overset{2(2)}{\boxtimes}} 2_-^{1+4}.Alt_5]_{16} \otimes A_2$, $[(Sp_4(3) \otimes_{\sqrt{-3}} Sp_4(3)) : 2 \overset{2}{\square}\ C_3]_{32}$
O_3	$(F_4 \tilde{\otimes} F_4)^2$, $[(SL_2(5) \overset{2}{\square}\ SL_2(5)) : 2]_8^4$
O_4	$(F_4 \tilde{\otimes} F_4)^2$, $[(SL_2(5) \overset{2}{\square}\ SL_2(5)) : 2]_8^4$
O_5	$(A_2 \otimes [(SL_2(5) \overset{2}{\square}\ SL_2(5)) : 2]_8)^2$, $[SL_2(5) \underset{\infty,3}{\overset{2(3)}{\boxtimes}} (Sp_4(3) \overset{2}{\square}\ C_3)]_{32}$
O_6	$[SL_2(5) \underset{\infty,2}{\overset{2(2)}{\boxtimes}} 2_-^{1+4}.Alt_5]_{16}^2$, E_8^4
O_7	$[SL_2(5) \underset{\infty,2}{\overset{2(2)}{\boxtimes}} 2_-^{1+6}.O_6^-(2)]_{32}$, $F_4 \otimes E_8$
O_8	$[2_+^{1+10}.O_{10}^+(2)]_{32}$, $F_4 \otimes [(SL_2(5) \overset{2}{\square}\ SL_2(5)) : 2]_8$
	${}_{\sqrt{15},\infty}[SL_2(5) \overset{2}{\boxtimes} D_{24}]_{4,3}$ $(2^6 \cdot 3^2 \cdot 5)$
O_1	$F_4 \otimes E_8$, $[(SL_2(5) \circ SL_2(5)) : 2 \underset{\sqrt{5}}{\overset{2(6)}{\boxtimes}} (C_3 \overset{2(2)}{\boxtimes} D_8)]_{32,1}$
O_2	$[2_+^{1+10}.O_{10}^+(2)]_{32}$, $F_4 \otimes [(SL_2(5) \overset{2}{\square}\ SL_2(5)) : 2]_8$
O_3	$(A_2 \otimes [(SL_2(5) \overset{2}{\square}\ SL_2(5)) : 2]_8)^2$, $[(Sp_4(3) \circ C_3) \underset{\sqrt{-3}}{\overset{2}{\boxtimes}} SL_2(3)]_{16}^2$
O_4	$(A_2 \otimes [(SL_2(5) \overset{2}{\square}\ SL_2(5)) : 2]_8)^2$, $[(Sp_4(3) \circ C_3) \underset{\sqrt{-3}}{\overset{2}{\boxtimes}} SL_2(3)]_{16}^2$
O_5	$(F_4 \tilde{\otimes} F_4)^2$, $[(SL_2(5) \overset{2}{\square}\ SL_2(5)) : 2]_8^4$
O_6	$(A_2 \otimes E_8)^2$, $[SL_2(5) \underset{\infty,3}{\overset{2(3)}{\boxtimes}} (SL_2(3) \overset{2}{\square}\ C_3)]_{16}^2$
O_7	$A_2 \otimes F_4 \tilde{\otimes} F_4$, $[SL_2(5) \underset{\infty,3}{\overset{2(3)}{\boxtimes}} (Sp_4(3) \overset{2}{\square}\ C_3)]_{32}$
O_8	$[SL_2(5) \underset{\infty,2}{\overset{2(2)}{\boxtimes}} 2_-^{1+4}.Alt_5]_{16} \otimes A_2$, $[(Sp_4(3) \otimes_{\sqrt{-3}} Sp_4(3)) : 2 \overset{2}{\square}\ C_3]_{32}$
	${}_{\sqrt{15},\infty}[SL_2(5) \overset{2}{\boxtimes} D_{24}]_{4,4}$ $(2^6 \cdot 3^2 \cdot 5)$
O_1	$F_4 \otimes [(SL_2(5) \overset{2}{\square}\ SL_2(5)) : 2]_8$, $[(SL_2(5) \circ SL_2(5)) : 2 \underset{\sqrt{5}}{\overset{2(6)}{\boxtimes}} (C_3 \overset{2(2)}{\boxtimes} D_8)]_{32,1}$
O_2	$[SL_2(5) \underset{\infty,2}{\overset{2(2)}{\boxtimes}} 2_-^{1+6}.O_6^-(2)]_{32}$, $F_4 \otimes E_8$
O_3	$(A_2 \otimes E_8)^2$, $[SL_2(5) \underset{\infty,3}{\overset{2(3)}{\boxtimes}} (SL_2(3) \overset{2}{\square}\ C_3)]_{16}^2$
O_4	$(A_2 \otimes E_8)^2$, $[SL_2(5) \underset{\infty,3}{\overset{2(3)}{\boxtimes}} (SL_2(3) \overset{2}{\square}\ C_3)]_{16}^2$
O_5	E_8^4, $[SL_2(5) \underset{\infty,2}{\overset{2(2)}{\boxtimes}} 2_-^{1+4}.Alt_5]_{16}^2$
O_6	$(A_2 \otimes [(SL_2(5) \overset{2}{\square}\ SL_2(5)) : 2]_8)^2$, $[(Sp_4(3) \circ C_3) \underset{\sqrt{-3}}{\overset{2}{\boxtimes}} SL_2(3)]_{16}^2$
O_7	$A_2 \otimes [SL_2(5) \underset{\infty,2}{\overset{2(2)}{\boxtimes}} 2_-^{1+4}.Alt_5]_{16}$, $[(Sp_4(3) \otimes_{\sqrt{-3}} Sp_4(3)) : 2 \overset{2}{\square}\ C_3]_{32}$
O_8	$A_2 \otimes F_4 \tilde{\otimes} F_4$, $[SL_2(5) \underset{\infty,3}{\overset{2(3)}{\boxtimes}} (Sp_4(3) \overset{2}{\square}\ C_3)]_{32}$

	${}_{\sqrt{15},\infty}[D_{10} \overset{2}{\boxtimes} Q_{24}]_{4,1}$ $(2^5 \cdot 3 \cdot 5)$
O_1	$[2_-^{1+4}.Alt_5 \underset{\infty,2}{\otimes} SL_2(5) \overset{2}{\underset{\sqrt{5}}{\boxtimes}} D_{10}]_{32}$, $[(SL_2(5) \circ SL_2(5)) : 2 \overset{2}{\underset{\sqrt{5}}{\boxtimes}} D_{10}]_{16}^2$
O_2	$[2_-^{1+4}.Alt_5 \underset{\infty,2}{\otimes} SL_2(5) \overset{2}{\underset{\sqrt{5}}{\boxtimes}} D_{10}]_{32}$, $A_4 \otimes E_8$
O_3	$[D_{120}.(C_4 \times C_2)]_{16}^2$, $(A_2 \otimes A_4)^4$
O_4	$[D_{120}.(C_4 \times C_2)]_{16}^2$, $(A_2 \otimes A_4)^4$
O_5	$[2_-^{1+4}.Alt_5 \underset{\infty,2}{\otimes} SL_2(5) \overset{2}{\underset{\sqrt{5}}{\boxtimes}} D_{10}]_{32}$, $(A_4 \otimes F_4)^2$
O_6	$[D_{120}.(C_4 \times C_2)]_{16}^2$, $(A_2 \otimes A_4)^4$
O_7	$A_2 \otimes A_4 \otimes F_4$, $[C_{15} : C_4 \overset{2(2)}{\boxtimes} F_4]_{32}$
O_8	$A_2 \otimes A_4 \otimes F_4$, $[C_{15} : C_4 \overset{2(2)}{\boxtimes} F_4]_{32}$
	${}_{\sqrt{15},\infty}[D_{10} \overset{2}{\boxtimes} Q_{24}]_{4,2}$ $(2^5 \cdot 3 \cdot 5)$
O_1	$[2_-^{1+4}.Alt_5 \underset{\infty,2}{\otimes} SL_2(5) \overset{2}{\underset{\sqrt{5}}{\boxtimes}} D_{10}]_{32}$, $[(SL_2(5) \circ SL_2(5)) : 2 \overset{2}{\underset{\sqrt{5}}{\boxtimes}} D_{10}]_{16}^2$
O_2	$A_2 \otimes A_4 \otimes F_4$, $[C_{15} : C_4 \overset{2(2)}{\boxtimes} F_4]_{32}$
O_3	$(F_4 \otimes A_4)^2$, $[(SL_2(5) \circ SL_2(5)) : 2 \overset{2}{\underset{\sqrt{5}}{\boxtimes}} D_{10}]_{16}^2$
O_4	$(A_4 \otimes F_4)^2$, $[(SL_2(5) \circ SL_2(5)) : 2 \overset{2}{\underset{\sqrt{5}}{\boxtimes}} D_{10}]_{16}^2$
O_5	$[D_{120}.(C_4 \times C_2)]_{16}^2$, $[(SL_2(5) \circ SL_2(5)) : 2 \overset{2}{\underset{\sqrt{5}}{\boxtimes}} D_{10}]_{16}^2$
O_6	$(F_4 \otimes A_4)^2$, $[(SL_2(5) \circ SL_2(5)) : 2 \overset{2}{\underset{\sqrt{5}}{\boxtimes}} D_{10}]_{16}^2$
O_7	$[2_-^{1+4}.Alt_5 \underset{\infty,2}{\otimes} SL_2(5) \overset{2}{\underset{\sqrt{5}}{\boxtimes}} D_{10}]_{32}$, $A_4 \otimes E_8$
O_8	$[2_-^{1+4}.Alt_5 \underset{\infty,2}{\otimes} SL_2(5) \overset{2}{\underset{\sqrt{5}}{\boxtimes}} D_{10}]_{32}$, $A_4 \otimes E_8$
	${}_{\sqrt{17},\infty}[SL_2(17)]_4$ $(2^5 \cdot 3^2 \cdot 17)$
	$[SL_2(17) \overset{2(3)}{\Box} \tilde{S}_3]_{32,i} (i = 1, 2)$
	${}_{\sqrt{21},\infty}[2.Alt_7 \overset{2}{\boxtimes} C_3]_4$ $(2^5 \cdot 3^3 \cdot 5 \cdot 7)$
O_1	$(A_2 \otimes E_8)^2$, $[2.Alt_{10}]_{16}^2$, $[2.Alt_7 \overset{2(3)}{\underset{\sqrt{-7}}{\boxtimes}} \tilde{S}_3]_{16}^2$
O_2	$[2_+^{1+10}.O_{10}^+(2)]_{32}$, $[(2.Alt_7 \overset{2}{\underset{\sqrt{-7}}{\boxtimes}} 2.Alt_7) : 2]_{32}$
	${}_{\sqrt{21},\infty}[SL_2(7) \overset{2}{\boxtimes} C_3]_4$ $(2^5 \cdot 3^2 \cdot 7)$
O_1	$[SL_2(7) \overset{2(3)}{\underset{\sqrt{-7}}{\boxtimes}} \tilde{S}_3]_{16}^2$
O_2	$[SL_2(7) \overset{2}{\underset{\sqrt{-7}}{\boxtimes}} 2.Alt_7]_{32}$

In this table, the symbol $\sim$ means that the r.i.m.f. supergroups acting on the O_3G- and O_4G-lattices are the same as the ones for O_2.

The proof is split into twelve lemmata which are organized according to the different candidates for quasi-semi-simple normal subgroups and normal p-subgroups. For the rest of this section let $\mathcal{Q}$ be a definite quaternion algebra with center K and let G be a primitive a.i.m.f. subgroup of $GL_4(\mathcal{Q})$. Assume that $1 \neq N \trianglelefteq G$ is a quasi-semi-simple normal subgroup of G. By Table 9.1 and Lemma 7.2 N is one of Alt_5, $SL_2(5)$, $SL_2(5) \circ SL_2(5)$, $SL_2(5) \circ SL_2(5) \otimes_{\sqrt{5}} SL_2(5)$, $SL_2(7)$ (2 groups), $SL_2(9)$ (2 groups), $SL_2(17)$, $2.Alt_7$, or $Sp_4(3) = 2.U_4(2)$.

The first lemma deals with the absolutely irreducible candidates for normal subgroups N.

Lemma 14.15. *If G contains a normal subgroup N isomorphic to $SL_2(9)$ with character χ_{8a} (or χ_{8b}) resp. $SL_2(17)$ with character χ_{8a} (or χ_{8b}), then G is conjugate to ${}_{\sqrt{5},\infty}[SL_2(9)]_4$ resp. ${}_{\sqrt{17},\infty}[SL_2(17)]_4$.*

If G contains a normal subgroup N conjugate to $SL_2(5) \circ SL_2(5) \otimes_{\sqrt{5}} SL_2(5)$, then $G = {}_{\sqrt{5},\infty}[(SL_2(5) \circ SL_2(5) \otimes_{\sqrt{5}} SL_2(5)) : S_3]_4$.

Proof. In both cases N is already absolutely irreducible. One computes that $G = \mathcal{B}^{\circ}(N)$ is maximal finite. □

The next two lemmata deal with primitively saturated groups.

Lemma 14.16. *If G contains a normal subgroup $N \cong Alt_5$ with character χ_4, then G is one of ${}_{\sqrt{2},\infty}[\tilde{S}_4]_1 \otimes A_4$ or ${}_{\sqrt{3},\infty}[Q_{24}]_1 \otimes A_4$.*

Proof. By Proposition 7.5 G is of the form $A_4 \otimes H$, where $H \leq GL_1(\mathcal{Q})$ is a primitive a.i.m.f. group. Hence by Theorem 6.1 H is one of ${}_{\sqrt{2},\infty}[\tilde{S}_4]_1$, ${}_{\sqrt{3},\infty}[Q_{24}]_1$, or ${}_{\sqrt{5},\infty}[SL_2(5)]_1$. The lemma follows because $G = {}_{\sqrt{5},\infty}[SL_2(5)]_1 \otimes A_4$ is contained in ${}_{\sqrt{5},\infty}[(SL_2(5) \circ SL_2(5) \otimes_{\sqrt{5}} SL_2(5)) : S_3]_4$. □

Lemma 14.17. *N is not conjugate to $SL_2(5) \circ SL_2(5)$.*

Proof. Assume that G contains a normal subgroup $N = SL_2(5) \circ SL_2(5)$. The enveloping algebra of N is $\mathbb{Q}[\sqrt{5}]^{4\times 4}$. Hence $K = \mathbb{Q}[\sqrt{5}]$. Since $B := \mathcal{B}^{\circ}(N) = SL_2(5) \circ SL_2(5) : 2$ is primitively saturated over K, the group $G = BC$, where $C := C_G(N)$ is a centrally irreducible maximal finite subgroup of $GL_1(\mathcal{Q})$. Hence $\mathcal{Q} = \mathcal{Q}_{\sqrt{5},\infty} = K \otimes \mathcal{Q}_{\infty,2} = K \otimes \mathcal{Q}_{\infty,3}$ and $C = SL_2(5)$. But this contradicts Lemma 14.15. □

Now we come to the centrally irreducible groups N.

Lemma 14.18. *If G contains a normal subgroup N isomorphic to $SL_2(7)$ with character χ_8, then G is conjugate to ${}_{\sqrt{3},\infty}[SL_2(7).2]_4$.*

Proof. The group N is a centrally irreducible subgroup of $GL_4(\mathcal{Q})$. Therefore $C_G(N) \subseteq K$ is ± 1. Since G is absolutely irreducible, it contains N of index 2. With [CCNPW 85] one gets $G = {}_{\sqrt{3},\infty}[SL_2(7).2]_4$. □

There are three candidates N, for which the centralizer $C_G(N)$ is contained in the character field $K[\chi(N)]$ of a constituent of the natural character of N.

Lemma 14.19. *If G contains a normal subgroup N isomorphic to $SL_2(7)$, $2.Alt_7$, resp. $Sp_4(3)$, with character $\chi_{4a}+\chi_{4b}$, then G is conjugate to one of*

$$_{\sqrt{7},\infty}[SL_2(7) \overset{2}{\boxtimes} C_4]_4 \text{ or } _{\sqrt{21},\infty}[SL_2(7) \overset{2}{\boxtimes} C_3]_4,$$

$$_{\sqrt{7},\infty}[2.Alt_7 \overset{2}{\boxtimes} C_4]_4 \text{ or } _{\sqrt{21},\infty}[2.Alt_7 \overset{2}{\boxtimes} C_3]_4,$$

resp. $_{\sqrt{3},\infty}[Sp_4(3) \underset{\sqrt{-3}}{\overset{2}{\boxtimes}} C_{12}]_4$.

Proof. In all cases $C_G(N) =: C$ is contained in the extension of K by the character values of the natural character of N. Hence in the first two cases $C \leq K[\sqrt{-7}]^*$ and $C \leq K[\sqrt{-3}]^*$ in the last case. In all cases G contains the normal subgroup $\mathcal{B}^\circ(N)C$ of index 2. Since the dimension of the enveloping $\mathbb{Q}$-algebra of N is 32, Lemma 2.14 implies that C is not contained in N. Hence in the first two cases C is one of C_4 or C_3 and $K=\mathbb{Q}[\sqrt{7}]$ or $\mathbb{Q}[\sqrt{21}]$. By Lemma 2.17 there is a unique extension $G=(N\otimes C_4).2$ or $G=(N\otimes C_3).2$ with real Schur index 2. The maximality of these four groups is checked with Remark 2.6. In the last case, $\mathcal{B}^\circ(N) = Sp_4(3)\circ C_3$ has a nontrivial normal 3-subgroup. The primitivity of G implies that $C=C_{12}$ and $K=\mathbb{Q}[\sqrt{3}]$. Again $G = {}_{\sqrt{3},\infty}[Sp_4(3) \underset{\sqrt{-3}}{\overset{2}{\boxtimes}} C_{12}]_4$ is unique. □

In the next case $\overline{KN}$ is a proper central simple K-subalgebra of $\mathcal{Q}^{4\times 4}$.

Lemma 14.20. *If G contains a normal subgroup $N=SL_2(9)$ with character χ_4, then G is conjugate to one of $_{\infty,3}[SL_2(9)]_2 \otimes {}_{\sqrt{2}}[D_{16}]_2$, $_{\infty,3}[SL_2(9)]_2 \otimes {}_{\sqrt{5}}[D_{10}]_2$, $_{\sqrt{5},\infty,3,5}[C_5 \overset{2(3)}{\boxtimes} SL_2(9)]_4$, or $_{\sqrt{6},\infty}[(SL_2(9)\otimes D_8).2]_4$.*

Proof. Since $\overline{KN}$ is central simple, by [Rei 75, 7.11] the algebra $\overline{\mathbb{Q}G} = \mathcal{Q}^{4\times 4}$ is a tensor product $\mathcal{Q}^{4\times 4} = \overline{KN} \otimes_K A$, where A is the commuting algebra of N, an indefinite quaternion algebra over K. Let $B := \mathcal{B}^\circ_K(N)$ and $C := C_G(N)$. Then $O_3(C)=1$, because $\mathcal{B}^\circ(N\circ C_3) = Sp_4(3)\circ C_3$. Distinguish 2 cases:

a) $K=\mathbb{Q}[\sqrt{3}]$. Then $B=2.S_6$ is primitively saturated. By Lemma 7.5, $G = B \otimes_K C$ for some centrally irreducible maximal finite subgroup $C\leq A$. Using the classification of finite subgroups of $GL_2(\mathbb{C})$ in [Bli 17], one finds that $C=D_{24}$ (which contains $\pm S_3$ and D_8) contradicting $O_3(C)=1$.

b) Now let $K\neq \mathbb{Q}[\sqrt{3}]$. Then $B=N=SL_2(9)$ and G contains the normal subgroup BC of index ≤ 2. Assume first, that C is a centrally irreducible subgroup of A. Then $A=K^{2\times 2}$ by Remark 6.2 and $C\leq GL_2(K)$ is a dihedral group with $O_3(C)=1$. Hence $C=\pm D_{10}$, D_{16}, or D_8. In the first two cases C is an absolutely irreducible subgroup of $GL_2(K)$ for $K=\mathbb{Q}[\sqrt{5}]$ resp. $\mathbb{Q}[\sqrt{2}]$. Computing the automorphism groups of the NC-lattices one finds that G is $_{\infty,3}[SL_2(9)]_2 \otimes {}_{\sqrt{5}}[D_{10}]_2$ resp. $_{\infty,3}[SL_2(9)]_2 \otimes {}_{\sqrt{2}}[D_{16}]_2$. In the third case $NC = SL_2(9)\otimes D_8$ is not absolutely irreducible. Since $Out(NC) = C_2\times C_2$, the group $G=NC.2$ is one of $_{\sqrt{3},\infty,\infty}[2.S_6]_2 \otimes [D_8]_2$ (and imprimitive), $_{\infty,3}[SL_2(9)]_2 \otimes {}_{\sqrt{2}}[D_{16}]_2$ (leading to a bigger C), or the a.i.m.f. group $_{\sqrt{6},\infty}[(SL_2(9)\otimes D_8).2]_4$, because in each case there is a unique extension with real character field. If C is not centrally irreducible, then C is cyclic. The conditions $G=NC.2$ and $O_3(C)=1$ imply that C is one of C_8 or C_5 and $G = C_8 \overset{2(3)}{\boxtimes} SL_2(9) = D_{16} \overset{C_2}{\diamond} 2.S_6$ or $G = C_5 \overset{2(3)}{\boxtimes} SL_2(9)$. Since 3 is a norm in $\mathbb{Q}[\zeta_8]/\mathbb{Q}[\sqrt{2}]$ but not in $\mathbb{Q}[\zeta_5]/\mathbb{Q}[\sqrt{5}]$, the algebra $\mathcal{Q}$ is $\mathcal{Q}_{\sqrt{2},\infty}$ in the first case and $\mathcal{Q}_{\sqrt{5},\infty,3,5}$ in the second case. Whereas the second group is maximal finite, the first one is a subgroup of ${}_{\sqrt{2},\infty}[2_-^{1+6}.O_6^-(2).2]_4$. $\square$

The last and most fruitful case is the one where G has a normal subgroup $N \cong SL_2(5)$. This case is split up into two lemmata according to whether N is primitively saturated over K or not.

Lemma 14.21. *If $K = \mathbb{Q}[\sqrt{5}]$ and G contains a normal subgroup $N \cong SL_2(5)$ with character χ_{2a} (or χ_{2b}), then G is ${}_{\sqrt{5},\infty}[SL_2(5)]_1 \otimes_{\sqrt{5}} C$, where C is one of F_4, ${}_{\sqrt{5}}[D_{10}]_2 \otimes A_2$, ${}_{\sqrt{5},3,5}[C_5 \overset{2(3)}{\boxtimes} S_3]_2$, ${}_{\sqrt{5},3,5}[C_5 \overset{2(3)}{\boxtimes} \tilde{S}_3]_2$, ${}_{\sqrt{5},2,5}[C_5 \overset{2(2)}{\boxtimes} D_8]_2$, or ${}_{\sqrt{5},2,5}[C_5 \overset{2(2)}{\boxtimes} SL_2(3)]_2$.*

Proof. N is primitively saturated over $K = \mathbb{Q}[\sqrt{5}]$. Hence by Lemma 7.5 $G = NC$, for some primitive centrally irreducible maximal finite subgroup $C := C_G(N) \leq \mathcal{D}^*$, where $\mathcal{D} := C_{\mathcal{Q}^{4\times 4}}(N)$. By the formula in [Schu 05] (cf. Proposition 2.16), the only primes dividing the order of G are 2, 3, and 5. If C is not absolutely irreducible in $\mathcal{D}^*$, then C is a maximal finite subgroup of $GL_4(\mathbb{Q})$, because $\mathbb{Q}[\sqrt{5}]$ splits the possible p-adic Schur indices at $p = 2$, 3, and 5. Using Lemma 14.16 and the classification of maximal finite subgroups of $GL_4(\mathbb{Q})$ (cf. e.g. [BBNWZ 78]), one gets that $G = {}_{\sqrt{5},\infty}[SL_2(5)]_1 \otimes F_4$. Now assume that C is absolutely irreducible. Then the character field of the natural character of C is $K = \mathbb{Q}[\sqrt{5}]$. By Lemmas 14.17 and 14.15 C has no normal subgroup $SL_2(5)$ or $SL_2(5) \circ SL_2(5)$. With [CCNPW 85] one finds that C is soluble. An inspection of the possible normal p-subgroups yields $O_5(C) = C_5$. The centralizer $D := C_C(O_5(C))$ embeds into $C_{\mathcal{D}}(C)$ which is a quaternion algebra over $\mathbb{Q}[\zeta_5]$ and C contains D of index 2. Since C is absolutely irreducible, [Bli 17] yields the possibilities $D = {\pm}C_5E$, where E is one of $SL_2(3)$, D_8, S_3, or $\tilde{S}_3$.

If $C_C(E) > {\pm}C_5$, then C is one of ${\pm}C_5.C_2 \otimes_{\sqrt{-1}} SL_2(3)$, ${\pm}D_{10} \otimes D_8$, ${\pm}C_5.C_2 \otimes_{\sqrt{-1}} \tilde{S}_3$, or ${\pm}D_{10} \otimes S_3$. Now the first and third groups are not maximal finite, but contained in $SL_2(5) \otimes_{\sqrt{-1}} SL_2(3)$ resp. $SL_2(5) \otimes_{\sqrt{-1}} \tilde{S}_3$ and the second group is imprimitive. So $G = {}_{\sqrt{5},\infty}[SL_2(5)]_1 \otimes_{\sqrt{5}} {}_{\sqrt{5}}[{\pm}D_{10} \otimes S_3]_2$ in this case.

If $C_C(E) = {\pm}C_5$, then $G = {\pm}C_5 \overset{2(p)}{\boxtimes} E$, for some square free $p \in \mathbb{N}_{>1}$. Since $|Glide(E)| = 2$ and the enveloping algebra of E is central simple, the outer automorphism and p are unique. By Lemma 2.17 there is a unique extension G in $GL_4(\mathbb{R})$, in each of the four cases. $\square$

Lemma 14.22. *If $K \neq \mathbb{Q}[\sqrt{5}]$ and G contains a normal subgroup $N \cong SL_2(5)$ with character $\chi_{2a} + \chi_{2b}$, then G is conjugate to one of*

$$ {}_{\infty,5}[SL_2(5).2]_2 \otimes {}_{\sqrt{2}}[D_{16}]_2,\ {}_{\sqrt{2},\infty,2,5}[SL_2(5) \overset{2(2+\sqrt{2})}{\boxtimes} D_{16}]_{4,1}, $$

$$ {}_{\sqrt{2},\infty,2,5}[SL_2(5) \overset{2(2+\sqrt{2})}{\boxtimes} D_{16}]_{4,2},\ {}_{\infty,5}[SL_2(5).2]_2 \otimes {}_{\sqrt{3}}[D_{24}]_2, $$

$$ {}_{\sqrt{3},\infty}[SL_2(5) \overset{2(2+\sqrt{3})}{\boxtimes} D_{24}]_4,\ {}_{\sqrt{10},\infty}[SL_2(5) \overset{2}{\boxtimes} D_{16}]_4, $$

or ${}_{\sqrt{15},\infty}[SL_2(5) \overset{2}{\boxtimes} D_{24}]_{4,i}$ $(1 \leq i \leq 4)$.

Proof. Let $A := C_{\mathcal{Q}^{4\times 4}}(N)$ be the commuting algebra of N. Then A is an indefinite quaternion algebra with center $K[\sqrt{5}]$. The centralizer $C := C_G(N)$ embeds into A with $\overline{\mathbb{Q}[\sqrt{5}]C} = A$. By [Bli 17] this implies that C is one of D_{16} or D_{24}. In both cases, the outer automorphism group of C is isomorphic to $C_2 \times C_2$.

Assume first that $C \cong D_{16}$. Let $D_{16} = \langle x, y \mid x^8, y^2, (xy)^2\rangle$. Then $Out(D_{16}) = \langle \alpha, \beta\rangle$, with $\alpha(x) = x^{-1}, \alpha(y) = x^3 y$ and $\beta(x) = x^3, \beta(y) = y$ (cf. Lemma 8.5). Then $\alpha^2 = \beta^2 = id$, but $(\alpha\beta)^2$ is the conjugation by x. Since $\alpha\beta$ does not fix x, this implies that there is no extension $D_{16}.2$, where $\alpha\beta$ is an inner automorphism. Note that the action of α on the epimorphic image C is induced by conjugation with $y(1-x)$ and hence by an inner automorphism of $\overline{\mathbb{Q}C}$. If γ denotes the outer automorphism of $SL_2(5)$, then G/NC induces one of γ, $\gamma\alpha$, or $\gamma\beta$ on the central product NC. In all cases there are $2 = |H^2(C_2, Z(NC) \cong C_2)|$ extensions $NC.2$. Only in the last case do they lead to isomorphic groups, because there is an element of norm -1 in $\mathbb{Q}[\sqrt{2}]$. Since the group ${}_{\infty,5}[SL_2(5):2]_2 \otimes {}_{\sqrt{2}}[D_{16}]_2$ is contained in ${}_{\sqrt{2},\infty}[2_-^{1+6}.O_6^-(2).2]_4$ the group G is one of the a.i.m.f. groups

$$ {}_{\infty,5}[SL_2(5).2]_2 \otimes {}_{\sqrt{2}}[D_{16}]_2,\ {}_{\sqrt{2},\infty,2,5}[SL_2(5) \overset{2(2+\sqrt{2})}{\boxtimes} D_{16}]_{4,1}, $$

$$ {}_{\sqrt{2},\infty,2,5}[SL_2(5) \overset{2(2+\sqrt{2})}{\boxtimes} D_{16}]_{4,2}, \text{ or } {}_{\sqrt{10},\infty}[SL_2(5) \overset{2}{\boxtimes} D_{16}]_4. $$

The case $C \cong D_{24}$ is similar. Here all the groups $C.2$ exist and one has eight different groups $NC.2$, Since ${}_{\infty,5}[SL_2(5):2]_2 \otimes {}_{\sqrt{3}}[D_{24}]_2$ is contained in ${}_{\sqrt{3},\infty}[Sp_4(3) \underset{\sqrt{-3}}{\overset{2}{\boxtimes}} C_{12}]_4$ and one extension $SL_2(5) \overset{2(2+\sqrt{3})}{\boxtimes} D_{24}$ a proper subgroup of ${}_{\sqrt{3},\infty}[D_8 \otimes D_8 \otimes C_4.S_6 \overset{2}{\boxtimes} C_3]_4$ the group G is one of the six a.i.m.f. groups ${}_{\infty,5}[SL_2(5).2]_2 \otimes {}_{\sqrt{3}}[D_{24}]_2$, ${}_{\sqrt{3},\infty}[SL_2(5) \overset{2(2+\sqrt{3})}{\boxtimes} D_{24}]_4$, or ${}_{\sqrt{15},\infty}[SL_2(5) \overset{2}{\boxtimes} D_{24}]_{4,i}$ $(1 \le i \le 4)$. □

For the rest of this section we assume that G does not contain a quasi-semi-simple normal subgroup. By Lemma 11.2 and Corollary 2.4 one has $O_p(G) = 1$ for $p > 5$ and $O_p(G) \le C_p$ for $p = 3, 5$.

Lemma 14.23. *If G does not contain a quasi-semi-simple normal subgroup and $O_p(G) = 1$ for all odd primes p, then $O_2(G) = 2_-^{1+6} = D_8 \otimes D_8 \otimes Q_8$ and $G = {}_{\sqrt{2},\infty}[2_-^{1+6}.O_6^-(2).2]_4$.*

Proof. By Proposition 8.9 $O_2(G)$ is one of 2_-^{1+6} or $Q_8 \circ Q_8 \otimes Q_{16}$. In the first case $G = {}_{\sqrt{2},\infty}[2_-^{1+6}.O_6^-(2).2]_4$ is maximal finite. In the other case N is already irreducible. The Bravais group on a normal critical N-lattice (cf. Definition 2.7) is ${}_{\sqrt{2},\infty}[2_-^{1+4}.S_5]_2^2$ contradicting the primitivity of G. □

Lemma 14.24. *If G does not contain a quasi-semi-simple normal subgroup, $O_5(G) = 1$, and $O_3(G) = C_3$, then G is one of*

$$ {}_{\sqrt{2},\infty}[2_-^{1+4}.S_5]_1 \otimes A_2,\ {}_{\sqrt{2},\infty,2,3}[C_3 \overset{2(2+\sqrt{2})}{\boxtimes} (\tilde{S}_4 \otimes_{\sqrt{2}} D_{16})]_4, $$

$$ {}_{\sqrt{2},\infty,2,3}[C_3 \overset{2(2+\sqrt{2})}{\boxtimes} (\tilde{S}_4 \circ Q_{16})]_4,\ {}_{\sqrt{3},\infty}[D_8 \otimes D_8 \otimes C_4.S_6 \overset{2}{\boxtimes} C_3]_4, $$

$$ {}_{\sqrt{6},\infty}[(S_3 \otimes 2_-^{1+4}.Alt_5).2]_4, \text{ or } {}_{\sqrt{6},\infty}[(S_3 \otimes F_4).2]_4. $$

Proof. The centralizer $C := C_G(O_3(G))$ is an absolutely irreducible subgroup of $(\mathbb{Q}[\zeta_3] \otimes \mathcal{Q}^{2\times 2})^*$ and $G/C \cong C_2$. Table 8.7 gives that $O_2(G) = O_2(C)$ is one of $Q_8 \circ Q_8 \otimes C_4$, $D_8 \otimes Q_8$, $Q_8 \circ Q_8$, $Q_8 \otimes D_{16}$, $Q_8 \underset{\sqrt{-2}}{\otimes} QD_{16}$, or $Q_8 \circ Q_{16}$.

In the first case $K = \mathbb{Q}[\sqrt{3}]$ and G contains $B := \mathcal{B}^\circ(C) = C_3 \otimes (C_4 \otimes Q_8 \circ Q_8).S_6$ of index 2. Hence G is conjugate to ${}_{\sqrt{3},\infty}[D_8 \otimes D_8 \otimes C_4.S_6 \overset{2}{\boxtimes} C_3]_4$ in this case.

If $O_2(C) = Q_8 \otimes D_8$, then G contains $B = C_3 \circ 2_-^{1+4}.Alt_5$ of index 2^2. Hence $C_G(O_2(C)) = C_G(\mathcal{B}^\circ(O_2(G)) = {\pm}S_3$, and G is one of the two groups

$$ {}_{\sqrt{6},\infty}[(S_3 \otimes 2_-^{1+4}.Alt_5).2]_4 \text{ or } {}_{\sqrt{2},\infty}[2_-^{1+4}.S_5]_1 \otimes A_2. $$

Note that in both cases there is a unique extension with real character field.

In the case $O_2(G) = Q_8 \circ Q_8$ one similarly finds that G contains $\tilde{S}_3 \otimes F_4$ of index 2. Since the group $F_4.2 \otimes \tilde{S}_3$ is contained in ${}_{\sqrt{2},\infty}[2_-^{1+6}.O_6^-(2).2]_4$, G is conjugate to ${}_{\sqrt{6},\infty}[(S_3 \otimes F_4).2]_4$ in this case.

If $O_2(C) = Q_8 \otimes D_{16}$, then $K = \mathbb{Q}[\sqrt{2}]$ and G contains $B = \tilde{S}_4 \underset{\sqrt{2}}{\otimes} D_{16} \circ C_3$ of index 2. If the elements in $G - B$ induce an outer automorphism of $\mathcal{B}^\circ(O_2(G))$, then G is conjugate to ${}_{\sqrt{2},\infty,2,3}[C_3 \overset{2(2+\sqrt{2})}{\Box} (\tilde{S}_4 \underset{\sqrt{2}}{\otimes} D_{16})]_4$. If they don't, then $C_G(O_2(G)) \cong {\pm}S_3$ and G is contained in ${}_{\sqrt{2},\infty}[2_-^{1+4}.S_5]_2 \otimes A_2$.

Assume now that $O_2(G) = Q_8 \underset{\sqrt{-2}}{\otimes} QD_{16}$. Then $K = \mathbb{Q}[\sqrt{6}]$ and $G/B \cong C_2$. Hence there is a unique group $G = B.2$ with real Schur index 2. This group is not maximal finite but contained in ${}_{\sqrt{6},\infty}[(S_3 \otimes 2_-^{1+4}.Alt_5).2]_4$.

In the last case $O_2(G) = Q_8 \circ Q_{16}$. Now $K = \mathbb{Q}[\sqrt{2}]$ and $G = B.2$ is conjugate to ${}_{\sqrt{2},\infty,2,3}[C_3 \overset{2(2+\sqrt{2})}{\boxtimes} (\tilde{S}_4 \circ Q_{16})]_4$, because $\tilde{S}_4 \circ Q_{16} \otimes \tilde{S}_3$ is contained in ${}_{\sqrt{2}}[F_4.2]_4 \otimes {}_{\infty,3}[\tilde{S}_3]_1$. $\square$

The last and most complicated case is the case $O_5(G) > 1$. In this case $O_5(G) \cong C_5$. Recall that we assume for the rest of this chapter, that G is a primitive a.i.m.f. group of $GL_4(\mathcal{Q})$, $K = Z(\mathcal{Q})$ a real quadratic field and that G does not contain a quasi-semi-simple normal subgroup. As for the case $SL_2(5) \trianglelefteq G$, there are two essentially different situations: $K = \mathbb{Q}[\sqrt{5}]$ and $K \neq Q[\sqrt{5}]$ which are treated separately.

Lemma 14.25. *If $K = \mathbb{Q}[\sqrt{5}]$ and $O_5(G) = C_5$, then G is one of*

$$ {}_{\sqrt{5},\infty,2,5}[C_5 \overset{2(2)}{\boxtimes} F_4]_4, \quad {}_{\sqrt{5},\infty,2,5}[C_5 \underset{\sqrt{5}'}{\overset{2(2)}{\boxtimes}} 2_-^{1+4}.Alt_5]_4, $$

$$ {}_{\infty,2}[2_-^{1+4}.Alt_5]_2 \otimes {}_{\sqrt{5}}[{\pm}D_{10}]_2, \quad {}_{\infty,3}[SL_2(3) \overset{2}{\Box} C_3]_2 \otimes {}_{\sqrt{5}}[{\pm}D_{10}]_2, $$

$$ A_2 \otimes {}_{\sqrt{5},\infty,2,5}[C_5 \overset{2(2)}{\boxtimes} D_8]_2, \quad {}_{\sqrt{5},\infty,2,3}[C_5 \overset{2(6)}{\boxtimes} S_3 \otimes D_8]_{32}, $$

$$ {}_{\sqrt{5},\infty,2,3}[C_5 \underset{\sqrt{5}'}{\overset{2(6)}{\boxtimes}} \tilde{S}_3 \otimes D_8]_{32}, \quad {}_{\sqrt{5},\infty,2,5}[C_5 \underset{\sqrt{5}}{\overset{2(2)}{\boxtimes}} (C_3 \overset{2(2)}{\boxtimes} D_8)]_{32}, $$

$$ {}_{\sqrt{5},\infty,5,3}[C_5 \underset{\sqrt{5}}{\overset{2(3)}{\boxtimes}} (C_3 \overset{2(2)}{\boxtimes} D_8)]_{32}, \quad {}_{\sqrt{5},\infty,2,3}[C_5 \underset{\sqrt{5}}{\overset{2(6)}{\boxtimes}} (C_3 \overset{2(2)}{\boxtimes} D_8)]_{32}, $$

$$ {}_{\sqrt{5},\infty,5,3}[C_5 \underset{\sqrt{5}'}{\overset{2(3)}{\boxtimes}} (C_3 \overset{2(2)}{\boxtimes} D_8)]_{32}, \quad {}_{\sqrt{5},\infty,2,3}[C_5 \underset{\sqrt{5}'}{\overset{2(6)}{\boxtimes}} (C_3 \overset{2(2)}{\boxtimes} D_8)]_{32}, $$

$$
{}_{\sqrt{5},\infty,5,3}[C_5 \overset{2(3)}{\boxtimes} (SL_2(3) \overset{2}{\square} C_3)]_4,\ \ {}_{\sqrt{5},\infty,2,3}[C_5 \overset{2(6)}{\boxtimes} (SL_2(3) \overset{2}{\square} C_3)]_4,
$$

$$
{}_{\sqrt{5},\infty,2,5}[C_5 \overset{2(2)}{\underset{\sqrt{5}'}{\boxtimes}} (SL_2(3) \overset{2}{\square} C_3)]_4,\ \ {}_{\sqrt{5},\infty,5,3}[C_5 \overset{2(3)}{\underset{\sqrt{5}'}{\boxtimes}} (SL_2(3) \overset{2}{\square} C_3)]_4,
$$

$$
{}_{\sqrt{5},\infty,2,3}[C_5 \overset{2(6)}{\underset{\sqrt{5}'}{\boxtimes}} (SL_2(3) \overset{2}{\square} C_3)]_4,\ \ {}_{\sqrt{5},\infty,2,3}[C_5 \overset{2(6)}{\underset{\sqrt{5}}{\boxtimes}} \tilde{S}_3 \otimes_{\sqrt{-3}} SL_2(3)]_4,
$$

$$
A_2 \otimes {}_{\sqrt{5},\infty,2,5}[C_5 \overset{2(2)}{\underset{\sqrt{5}'}{\boxtimes}} SL_2(3)]_2 \ \ or\ \ {}_{\sqrt{5},\infty,2,3}[C_5 \overset{2(6)}{\underset{\sqrt{5}'}{\boxtimes}} S_3 \otimes SL_2(3)]_4.
$$

Proof. The centralizer $C := C_G(O_5(G))$ is an absolutely irreducible subgroup of $(\mathbb{Q}[\zeta_5] \otimes_K \mathcal{Q}^{2\times 2})^*$ and $G/C \cong C_2$.

Assume first that $O_3(C) = 1$. Using Table 8.7 one finds that $O_2(G) = O_2(C)$ is one of $Q_8 \circ Q_8$ or $Q_8 \otimes D_8$ and G contains C of index 2. Since $Q_{20} \otimes F_4$ is a subgroup of ${}_{\sqrt{5},\infty}[SL_2(5)]_1 \otimes F_4$, G is ${}_{\sqrt{5},\infty,2,5}[C_5 \overset{2(2)}{\boxtimes} F_4]_4$ in the first case. The second case leads to the two a.i.m.f. groups ${}_{\sqrt{5},\infty,2,5}[C_5 \overset{2(2)}{\underset{\sqrt{5}'}{\boxtimes}} 2_-^{1+4}.Alt_5]_4$ and ${}_{\sqrt{5}}[D_{10}]_2 \otimes {}_{\infty,2}[2_-^{1+4}.Alt_5]_2$.

Now assume that $O_3(C) \neq 1$. Then $O_3(C) \cong C_3$ and $C \cong C_5 \times H$. Since $O_5(H) = 1$ and H does not contain a quasi-semi-simple normal subgroup and 5 does not divide the order of the automorphism groups of the possible normal 2-subgroups, $\overline{\mathbb{Q}H}$ is a central simple $\mathbb{Q}$-algebra of dimension 16. Table 10.4 yields that H is one of $S_3 \otimes D_8$, $\tilde{S}_3 \otimes D_8$, $C_3 \overset{2(2)}{\boxtimes} D_8$ (2 groups), $S_3 \otimes SL_2(3)$, $\tilde{S}_3 \otimes_{\sqrt{-3}} SL_2(3)$, or $C_3 \overset{2(2)}{\boxtimes} SL_2(3)$ (2 groups). In all cases $Glide(H)$ is isomorphic to $C_2 \times C_2$. If $G = C_G(H)H$, then $C_G(H)$ is one of ${}^{\pm}D_{10}$ or Q_{20} according to the real Schur index of an absolutely irreducible constituent of the natural character of H. In the second case G is not maximal finite but contained in $SL_2(5)H$. In the first case H is a maximal finite subgroup of its enveloping algebra. Therefore the only possibility for H is $H = {}_{\infty,3}[SL_2(3) \overset{2}{\square} C_3]_2$. One checks that $G = {}_{\infty,3}[SL_2(3) \overset{2}{\square} C_3]_2 \otimes_{\sqrt{5}} [{}^{\pm}D_{10}]_2$ is maximal finite.

If $C_G(H) = {}^{\pm}C_5$, then one has for each group H three possible automorphisms. By Lemma 2.17 there is at most one extension $G = C.2$ in each case. Constructing the twenty-four groups $G = H.2$ one finds that G is one of the groups in the Lemma.

More precisely, the three nontrivial "outer" elements in $N_{\overline{\mathbb{Q}H}^*}(H)$ may be distinguished via their norms, which are 2, 3, respectively 6. If one considers of the isoclinic pairs the group H with real Schur index 1 first, one finds from Table 10.4 that in the first and second case, the automorphism with norm 3 yields imprimitive groups. In the fourth and fifth case the normalizer element of norm 2 yields a proper subgroup of ${}_{\sqrt{5},\infty,2,5}[C_5 \overset{2(2)}{\underset{\sqrt{5}'}{\boxtimes}} 2_-^{1+4}.Alt_5]_4$ resp. ${}_{\sqrt{5},\infty,2,5}[C_5 \overset{2(2)}{\boxtimes} F_4]_4$.

In some cases, G is not maximal finite due to the fact, that an outer element normalizes $SL_2(5) \leq (\mathbb{Q}[\sqrt{5}] \otimes \overline{\mathbb{Q}H})^*$. These cannot be read off directly from Table 10.4 and are the following: In the second case, additionally the automorphism with norm 2 gives rise to a group contained in ${}_{\sqrt{5},\infty}[SL_2(5)]_1 \otimes_{\sqrt{5}} {}_{\sqrt{5},2,5}[C_5 \overset{2(2)}{\boxtimes} D_8]_2$. In the seventh case, the automorphisms of prime norm 2 resp. 3 yield proper subgroups of

$$ {}_{\sqrt{5},\infty}[SL_2(5)]_1 \otimes_{\sqrt{5}} {}_{\sqrt{5},2,5}[C_5 \overset{2(2)}{\underset{\sqrt{5}'}{\boxtimes}} SL_2(3)]_2 \text{ resp. } {}_{\sqrt{5},\infty}[SL_2(5)]_1 \otimes_{\sqrt{5}} {}_{\sqrt{5},5,3}[C_5 \overset{2(3)}{\underset{\sqrt{5}'}{\boxtimes}} \tilde{S}_3]_2. $$

Finally the group $SL_2(3) \otimes C_5 \overset{2(3)}{\underset{\sqrt{5}'}{\boxtimes}} S_3$ is not maximal finite but contained in

${}_{\sqrt{5},\infty}[SL_2(5)]_1 \otimes_{\sqrt{5}} {}_{\sqrt{5},3,5}[C_5 \overset{2(3)}{\underset{\sqrt{5}'}{\boxtimes}} \tilde{S}_3]_2$. □

Lemma 14.26. *If $K \neq \mathbb{Q}[\sqrt{5}]$ and $O_5(G) = C_5$, then G is one of*

$$ {}_{\sqrt{2},\infty,2,5}[D_{10} \overset{2(2+\sqrt{2})}{\boxtimes} Q_{16}]_4, \quad {}_{\sqrt{3},\infty}[D_{10} \overset{2}{\boxtimes} Q_{24}]_4, $$
$$ {}_{\sqrt{10},\infty}[D_{10} \overset{2}{\boxtimes} \tilde{S}_4]_4, \quad {}_{\sqrt{15},\infty}[D_{10} \overset{2}{\boxtimes} Q_{24}]_{4,1}, \quad {}_{\sqrt{15},\infty}[D_{10} \overset{2}{\boxtimes} Q_{24}]_{4,2}. $$

Proof. The centralizer $C := C_G(O_5(G))$ is an absolutely irreducible subgroup of $(\mathbb{Q}[\zeta_5] \otimes \mathcal{Q})^* \cong GL_2(K[\zeta_5])$ and $G/C \cong C_4$. From the classification of finite subgroups of $PGL_2(\mathbb{C})$ one concludes that $C = C_5 \times H$, where H is one of D_{16}, Q_{16}, $\tilde{S}_4$, Q_{24}, or D_{24}. In all cases the exponent of $Out(H)$ is 2. So $C_G(H) > {}^{\pm}C_5$ contains one of ${}^{\pm}D_{10}$ or Q_{20}, according to the real Schur index of an absolutely irreducible constituent of the natural character of H. Since $Glide(H)$ does not contain an element of norm 5, one concludes that G is not maximal in the second case, but an additional quasi-semi-simple normal subgroup $SL_2(5)$ arises. This excludes the first and last case. In the other three cases, G is clearly not of the form $HC_G(H)$, since otherwise $G = C_5 : C_4 \otimes H$ is contained in $A_4 \otimes H$. Hence $HC_G(H) = D_{10}H$ and $G = HC_G(H).2$. Fixing the outer automorphism one has two possible extensions in each case. They lead to isomorphic groups G.

If $H = Q_{16}$, then $Out(H) \cong C_2 \times C_2$. Since one outer automorphism does not give rise to an extension $H.2$ (cf. proof of Lemma 14.22), two groups G need to be considered. The group $D_{10} \overset{2}{\boxtimes} Q_{16}$ is contained in ${}_{\sqrt{10},\infty}[D_{10} \overset{2}{\boxtimes} \tilde{S}_4]_4$ so G is conjugate to ${}_{\sqrt{2},\infty,2,5}[D_{10} \overset{2(2+\sqrt{2})}{\boxtimes} Q_{16}]_4$ in this case.

If $H = \tilde{S}_4$ and $K = \mathbb{Q}[\sqrt{2}]$, then H is primitively saturated over K. Hence $G = C_G(H)H$ is not maximal finite. One finds that G is ${}_{\sqrt{10},\infty}[D_{10} \overset{2}{\boxtimes} \tilde{S}_4]_4$ in this case.

Finally, if $H = Q_{24}$, then $Out(H) \cong C_2 \times C_2$. Here three different automorphisms have to be considered. They yield the three a.i.m.f. groups ${}_{\sqrt{3},\infty}[D_{10} \overset{2}{\boxtimes} Q_{24}]_4$, ${}_{\sqrt{15},\infty}[D_{10} \overset{2}{\boxtimes} Q_{24}]_{4,1}$, and ${}_{\sqrt{15},\infty}[D_{10} \overset{2}{\boxtimes} Q_{24}]_{4,2}$. □

15. The a.i.m.f. subgroups of $GL_5(\mathcal{Q})$

$$Z(\mathcal{Q}) = \mathbb{Q}$$

Theorem 15.1. *Let $\mathcal{Q}$ be a definite quaternion algebra with center $\mathbb{Q}$ and let G be a primitive a.i.m.f. subgroup of $GL_5(\mathcal{Q})$. Then G is conjugate to one of the groups in the following table.*

List of the primitive a.i.m.f. subgroups of $GL_5(\mathcal{Q})$.

lattice L	$\lvert Aut(L)\rvert$	r.i.m.f. supergroups
${}_{\infty,2}[{}^{\pm}U_5(2)]_5$	$2^{11} \cdot 3^5 \cdot 5 \cdot 11$	$[U_5(2) \overset{2(2)}{\circ} SL_2(3)]_{20}$
${}_{\infty,2}[SL_2(11)]_5$	$2^3 \cdot 3 \cdot 5 \cdot 11$	$[SL_2(11) \overset{2(2)}{\circ} SL_2(3)]_{20}$
$A_5 \otimes {}_{\infty,2}[SL_2(3)]_1$	$2^7 \cdot 3^3 \cdot 5$	$A_5 \otimes F_4$
${}_{\infty,3}[{}^{\pm}U_4(2) \overset{2}{\Box} C_3]_5$	$2^8 \cdot 3^5 \cdot 5$	$[{}^{\pm}U_4(2) \overset{2}{\Box} C_3]_{10}^2$
${}_{\infty,11}[{}^{\pm}L_2(11).2]_5$	$2^4 \cdot 3 \cdot 5 \cdot 11$	$(A_{10}^{(2)})^2$
		$[L_2(11) \overset{2(3)}{\boxtimes} D_{12}]_{20}$

Proof. Let $\mathcal{Q}$ be a definite quaternion algebra with center $\mathbb{Q}$ and let G be a primitive a.i.m.f. subgroup of $GL_5(\mathcal{Q})$. Assume that $1 \neq N \trianglelefteq G$ is a quasi-semi-simple normal subgroup of G. By Table 9.1 N is one of Alt_6, $L_2(11)$, $SL_2(11)$, $U_4(2)$, or $U_5(2)$. The centralizer $C := C_G(N)$ in G of N embeds into the commuting algebra $C_{\mathcal{Q}^{5\times 5}}(N)$, which is isomorphic to $\mathcal{Q}$, $\mathcal{Q}$, $\mathbb{Q}[\sqrt{-11}]$, $\mathbb{Q}$, $\mathbb{Q}[\sqrt{-3}]$, resp. $\mathbb{Q}$ in the respective cases. In the first case $G = \mathcal{B}^{\circ}(N)C$ is one of $A_5 \otimes {}_{\infty,2}[SL_2(3)]_1$ or $A_5 \otimes {}_{\infty,3}[\tilde{S}_3]_1$ by Corollary 7.6 and Theorem 6.1. Whereas the first group is a maximal finite subgroup of $GL_5(\mathcal{Q}_{\infty,2})$, the second group is a proper subgroup of ${}_{\infty,3}[{}^{\pm}U_4(2) \overset{2}{\Box} C_3]_5$.

In cases 3 and 5, N is already absolutely irreducible and lattice sparse. Its unique a.i.m.f. supergroup is ${}_{\infty,2}[{}^{\pm}U_5(2)]_5$, resp. ${}_{\infty,2}[SL_2(11)]_5$.

In the other two cases, C is contained in $\mathcal{B}^{\circ}(N)$, which is a normal subgroup of index $2 = \lvert Out(N)\rvert$ in G. Since the commuting algebra $C_{\mathcal{Q}^{5\times 5}}(N)$ is an imaginary quadratic field, there is in both cases only one absolutely irreducible subgroup $G = \mathcal{B}^{\circ}(N).2 \leq GL_5(\mathcal{Q})$. One computes $G = {}_{\infty,11}[{}^{\pm}L_2(11).2]_5$ resp. $G = {}_{\infty,3}[{}^{\pm}U_4(2) \overset{2}{\Box} C_3]_5$.

Now assume that G has no quasi-semi-simple normal subgroup. Since the possible normal p-subgroups of G, which embed into $GL_1(\mathcal{Q})$ do not admit an automorphism of order 5, one has $O_{11}(G) \cong C_{11}$, contradicting Lemma 11.2. $\square$

Theorem 15.2. *Let $\mathcal{Q}$ be a definite quaternion algebra with center $\mathbb{Q}$ and let G be an a.i.m.f. subgroup of $GL_5(\mathcal{Q})$. Then $\mathcal{Q}$ is one of $\mathcal{Q}_{\infty,2}$, $\mathcal{Q}_{\infty,3}$, or $\mathcal{Q}_{\infty,11}$. The simplicial complexes $M_5^{irr}(\mathcal{Q})$ are as follows:*

$\bullet$ ${}_{\infty,2}[SL_2(11)]_5$ $\qquad$ $\bullet$ $A_5 \otimes {}_{\infty,2}[SL_2(3)]_1$ $\qquad$ ${}_{\infty,2}[SL_2(3)]_1^5$ $\bullet$—$\bullet$ ${}_{\infty,2}[{}^{\pm}U_5(2)]_5$

${}_{\infty,3}[{}^{\pm}U_4(2) \overset{2}{\Box} C_3]_5$ $\bullet$—$\bullet$ ${}_{\infty,3}[\tilde{S}_3]_1^5$

$\bullet$ ${}_{\infty,11}[({}^{\pm}L_2(11)).2]_5$

List of maximal simplices in $M_5^{irr}(\mathcal{Q}_{\infty,2})$:

simplex	a common subgroup
$({}_{\infty,2}[{}^{\pm}U_5(2)]_5,\ {}_{\infty,2}[SL_2(3)]_1^5)$	$2^{4+4}:(Alt_5\times C_3)$

List of maximal simplices in $M_5^{irr}(\mathcal{Q}_{\infty,3})$:

simplex	a common subgroup
$({}_{\infty,3}[{}^{\pm}U_4(2)\overset{2}{\square}C_3]_5,\ {}_{\infty,3}[\tilde{S}_3]_1^5)$	$2^4:Alt_5\overset{2}{\square}C_3$

Proof. Theorems 15.1 and 6.1 prove the completeness of the list of quaternion algebras $\mathcal{Q}$ and of a.i.m.f. subgroups of $GL_5(\mathcal{Q})$. One only has to show the completeness of the list of maximal simplices in $M_5^{irr}(\mathcal{Q}_{\infty,2})$, because for the other two quaternion algebras $\mathcal{Q}$, the simplicial complex $M_5^{irr}(\mathcal{Q})$ consists of a single simplex: Let $\mathfrak{M}_2$ denote a maximal order of $\mathcal{Q}_{\infty,2}$. Since the group ${}_{\infty,2}[SL_2(11)]_5$ fixes a lattice of determinant divisible by 11, the minimal absolutely irreducible subgroups of the group ${}_{\infty,2}[SL_2(11)]_5$ are of order divisible by 11 (cf. Lemma 2.13). Going through the list of maximal subgroups of $L_2(11)$ in [CCNPW 85] one sees that ${}_{\infty,2}[SL_2(11)]_5$ is minimal absolutely irreducible. Hence ${}_{\infty,2}[SL_2(11)]_5$ forms a 0-simplex in $M_5^{irr}(\mathcal{Q}_{\infty,2})$. The minimal absolutely irreducible subgroups of $A_5\otimes{}_{\infty,2}[SL_2(3)]_1$ are $Alt_5\otimes Q_8$ and $C_4\overset{2}{\boxtimes}Alt_5$. Both groups do not embed into any other a.i.m.f. subgroup of $GL_5(\mathcal{Q}_{\infty,2})$, since they do not fix any 3-unimodular $\mathfrak{M}_2$-lattice. Therefore the list of maximal simplices in $M_5^{irr}(\mathcal{Q}_{\infty,2})$ is complete. $\square$

$Z(\mathcal{Q})$ real quadratic.

Theorem 15.3. *Let G be a primitive absolutely irreducible maximal finite subgroup of $GL_5(\mathcal{Q})$, where $\mathcal{Q}$ is a totally definite quaternion algebra with center K and $[K:\mathbb{Q}]=2$. Then $\mathcal{Q}$ is isomorphic to $\mathcal{Q}_{\sqrt{5},\infty}$, $\mathcal{Q}_{\sqrt{2},\infty}$, $\mathcal{Q}_{\sqrt{3},\infty}$, $\mathcal{Q}_{\sqrt{11},\infty}$, or $\mathcal{Q}_{\sqrt{33},\infty}$ and G is conjugate to one of the groups given in the table below, which is built up as Table 12.7:*

Proof. Let $\mathcal{Q}$ be a definite quaternion algebra with center $K:=Z(\mathcal{Q})$ a real quadratic field. Let G be a primitive absolutely irreducible subgroup of $GL_5(\mathcal{Q})$, and p be a prime such that $O_p(G)\neq 1$. Then either $O_p(G)$ is a subgroup of $GL_1(\mathcal{Q})$ or $p=5$ and $O_5(G)\cong C_{25}$ or 5_+^{1+2}, or $p=11$ and $O_{11}(G)\cong C_{11}$.

If $O_5(G)\cong C_{25}$, then $C_G(O_5(G))={}^{\pm}O_5(G)$. But 5 divides the index of the abelian normal subgroup $O_5(G)$ in G, because G is absolutely irreducible. This contradicts the assumption $O_5(G)\cong C_{25}$ (and also the primitivity of G).

Now let $O_5(G)\cong 5_+^{1+2}$. Then $K=\mathbb{Q}[\sqrt{5}]$. The inclusion $5_+^{1+2}\trianglelefteq B:={}^{\pm}5_+^{1+2}:SL_2(5)\leq GL_5(\mathbb{Q}[\zeta_5])$ together with $Out(5_+^{1+2})\cong GL_2(5)$ then shows that G contains a normal subgroup B of index 2. There is a unique extension $B.2$ with real Schur index 2. Hence $G={}_{\sqrt{5},\infty}[{}^{\pm}5_+^{1+2}:SL_2(5).2]_5$ in this case.

If $O_{11}(G)\cong C_{11}$, then $C_G(O_{11}(G))$ is isomorphic to one of C_{22}, C_{44}, or C_{66}. In the first case, G is not absolutely irreducible and in the other two cases, G is a proper subgroup of ${}_{\sqrt{11},\infty}[C_4\overset{2}{\square}L_2(11)]_5$ resp. ${}_{\sqrt{33},\infty}[{}^{\pm}C_3\overset{2}{\square}L_2(11)]_5$.

List of the primitive a.i.m.f. subgroups of $GL_5(\mathcal{Q})$ where $\mathcal{Q}$ is a totally definite quaternion algebra over a real quadratic number field $Z(\mathcal{Q})$.

lattice L	$\lvert Aut(L)\rvert$	some r.i.m.f. supergroups
${}_{\sqrt{5},\infty}[SL_2(5)]_1 \otimes A_5$	$2^7 \cdot 3^3 \cdot 5^2$	$A_5 \otimes E_8$, $A_5 \otimes H_4$
${}_{\sqrt{5},\infty}[\pm 5_+^{1+2} : SL_2(5).2]_5$	$2^5 \cdot 3 \cdot 5^4$	$[\pm 5_+^{1+2} : SL_2(5).2 \overset{2}{\square} SL_2(5)]_{40}$
${}_{\sqrt{2},\infty}[SL_2(11).2]_5$	$2^4 \cdot 3 \cdot 5 \cdot 11$	$[SL_2(11) \overset{2(2)}{\circ} SL_2(3)]_{20}^2$, $[SL_2(11) \underset{\infty,2}{\overset{2(2)}{\otimes}} 2_-^{1+4}.Alt_5]_{40}$
${}_{\sqrt{2},\infty}[\pm U_5(2).2]_5$	$2^{12} \cdot 3^5 \cdot 5 \cdot 11$	$[\pm U_5(2) \overset{2(2)}{\circ} SL_2(3)]_{20}^2$, $[U_5(2) \underset{\infty,2}{\overset{2(2)}{\otimes}} 2_-^{1+4}.Alt_5]_{40}$
${}_{\sqrt{2},\infty}[SL_2(9)]_5$	$2^4 \cdot 3^2 \cdot 5$	$[2.U_4(2) \overset{2(2)}{\circ} SL_2(3)]_{40}$
${}_{\sqrt{2},\infty}[\tilde{S}_4]_1 \otimes A_5$	$2^8 \cdot 3^3 \cdot 5$	$A_5 \otimes E_8$, $(A_5 \otimes F_4)^2$
${}_{\sqrt{3},\infty}[SL_2(11)]_5$	$2^3 \cdot 3 \cdot 5 \cdot 11$	$[SL_2(11) \overset{2(3)}{\sqsubset\!\sqsupset} C_{12}.C_2]_{40}$
		$[SL_2(11) \overset{2(2)}{\sqsubset\!\sqsupset} SL_2(3)]_{40}$
${}_{\sqrt{3},\infty}[C_{12} \underset{\sqrt{-3}}{\overset{2}{\boxtimes}} U_4(2)]_5$	$2^9 \cdot 3^5 \cdot 5$	$[\pm C_3 \overset{2}{\square} U_4(2)]_{10}^4$, $[\pm U_5(2) \overset{2(2)}{\circ} SL_2(3)]_{20}^2$
		$[\pm C_3 \overset{2}{\square} U_4(2)]_{10} \otimes F_4$, $[U_5(2) \underset{\infty,2}{\overset{2(2)}{\otimes}} 2_-^{1+4}.Alt_5]_{40}$
${}_{\sqrt{11},\infty}[C_4 \overset{2}{\square} L_2(11)]_5$	$2^5 \cdot 3 \cdot 5 \cdot 11$	$(A_{10}^{(3)})^4$ $[\pm U_5(2) \overset{2(2)}{\circ} SL_2(3)]_{20}^2$
		$A_{10}^{(3)} \otimes F_4$, $[U_5(2) \underset{\infty,2}{\overset{2(2)}{\otimes}} 2_-^{1+4}.Alt_5]_{40}$
${}_{\sqrt{33},\infty}[\pm C_3 \overset{2}{\square} L_2(11)]_5$	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	$(A_{10}^{(3)} \otimes A_2)^2$, $[\pm U_5(2) \overset{2(2)}{\circ} SL_2(3)]_{20}^2$, $[L_2(11) \overset{2(3)}{\boxtimes} D_{12}]_{20}^2$
		$[(L_2(11) \otimes_{\sqrt{-11}} SL_2(3) \otimes S_3).2]_{40}$

Assume now, that for all primes p, $O_p(G)$ is a subgroup of $GL_1(\mathcal{Q})$. Then the automorphism group of $O_p(G)$ is soluble and its order is not divisible by 5. Since G is absolutely irreducible, the last term of the derived series of $G^{(\infty)}$ is a quasi-semi-simple group. Table 9.1 implies that $G^{(\infty)}$ is one of Alt_6, $SL_2(9)$, $L_2(11)$, $SL_2(11)$ (2 matrix groups), $U_4(2)$, or $U_5(2)$.

The two groups ${}_{\sqrt{2},\infty}[SL_2(9)]_5$ and ${}_{\sqrt{3},\infty}[SL_2(11)]_5$ are a.i.m.f. groups.

In the first case, $\mathcal{B}^\circ(Alt_6) \cong \pm S_6 = A_5$ is a primitively saturated absolutely irreducible subgroup of $GL_5(\mathbb{Q})$. Therefore Corollary 7.6 says that G is a tensor product $\pm S_6 \otimes U$, where U is a maximal finite subgroup of $GL_1(\mathcal{Q})$. By Theorem 6.1 U is one of ${}_{\sqrt{5},\infty}[SL_2(5)]_1$, ${}_{\sqrt{2},\infty}[\tilde{S}_4]_1$, or ${}_{\sqrt{3},\infty}[C_{12}.C_2]_1$. Whereas in the first two cases, G is maximal finite, the last group ${}_{\sqrt{3},\infty}[C_{12}.C_2]_1 \otimes A_5$ is contained in the maximal finite group ${}_{\sqrt{3},\infty}[C_{12} \underset{\sqrt{-3}}{\overset{2}{\boxtimes}} U_4(2)]_5$. (Note that $\pm C_3 \circ U_4(2) = \mathcal{B}^\circ(Alt_6 \otimes C_3)$.)

Now assume that $G^{(\infty)} \cong L_2(11)$. Then the centralizer $C := C_G(G^{(\infty)})$ embeds into $GL_1(K[\sqrt{-11}])$, and $G : G^{(\infty)}C = 2$. If $C = \pm 1$, then the center of the enveloping algebra of G is $\mathbb{Q}$ and therefore G is not absolutely irreducible in $GL_5(\mathcal{Q})$. Hence the biquadratic field $K[\sqrt{-11}]$ contains a root of unity. Therefore $K = \mathbb{Q}[\sqrt{11}]$ and $C \cong C_4$ or $K = \mathbb{Q}[\sqrt{33}]$ and $C \cong \pm C_3$. By Lemma 2.17 in both cases, there is a unique extension $G = G^{(\infty)}C.C_2$ with real Schur index. Hence G is one of ${}_{\sqrt{11},\infty}[C_4 \overset{2}{\boxtimes} L_2(11)]_5$ or ${}_{\sqrt{33},\infty}[\pm C_3 \overset{2}{\boxtimes} L_2(11)]_5$.

Next let $G^{(\infty)}$ be $SL_2(11)$, where the restriction of the natural character of G to $G^{(\infty)}$ is χ_{10}. Then the centralizer $C := C_G(G^{(\infty)})$ embeds into $GL_1(K)$, hence is trivial, and $G : G^{(\infty)} = 2$. There is a unique extension $G = SL_2(11).2$ with real character field. Therefore $G = {}_{\sqrt{2},\infty}[SL_2(11).2]_5$.

The case $G^{(\infty)} = U_5(2)$ is completely analogous.

The remaining case is $G^{(\infty)} \cong U_4(2)$. Then G contains the normal subgroup $B := \mathcal{B}^{\circ}(G^{(\infty)}) = \pm C_3 \circ U_4(2)$. Moreover $C := C_G(B) = C_G(G^{(\infty)}) \leq GL_1(K[\sqrt{-3}])$ and G contains BC of index 2. If $C \leq B$, then the character field of the natural character of G is $\mathbb{Q}$, contradicting the fact that G is absolutely irreducible in $GL_5(\mathcal{Q})$. Hence $C \cong C_{12}$ and $K = \mathbb{Q}[\sqrt{3}]$. The unique extension $G = BC.2$ in $GL_5(\mathcal{Q})$ is ${}_{\sqrt{3},\infty}[C_{12} \underset{\sqrt{-3}}{\overset{2}{\boxtimes}} U_4(2)]_5$. $\square$

Theorem 15.4. *$M_5^{irr}(\mathcal{Q}_{\sqrt{2},\infty})$ is as follows.*

• ${}_{\sqrt{2},\infty}[\tilde{S}_4]_1^5$ — • ${}_{\sqrt{2},\infty}[\pm U_5(2).2]_5$

• ${}_{\sqrt{2},\infty}[SL_2(9)]_5$

• ${}_{\sqrt{2},\infty}[SL_2(11).2]_5$

• ${}_{\sqrt{2},\infty}[\tilde{S}_4]_1 \otimes A_5$

List of the maximal simplices in $M_5^{irr}(\mathcal{Q}_{\sqrt{2},\infty})$

simplex	*a common subgroup*
$({}_{\sqrt{2},\infty}[\pm U_5(2).2]_5, {}_{\sqrt{2},\infty}[\tilde{S}_4]_1^5)$	$(\pm 2^{4+4}.(C_3 \times Alt_5)).2$

Proof. The completeness of the list of a.i.m.f. subgroups in $GL_5(\mathcal{Q}_{\sqrt{2},\infty})$ follows from Theorems 6.1 and 15.3. So we only have to show that the list of maximal simplices in $M_5^{irr}(\mathcal{Q}_{\sqrt{2},\infty})$ is complete. The two a.i.m.f. groups ${}_{\sqrt{2},\infty}[SL_2(9)]_5$ and ${}_{\sqrt{2},\infty}[SL_2(11).2]_5$ are minimal absolutely irreducible.

So we only have to deal with $G := {}_{\sqrt{2},\infty}[\tilde{S}_4]_1 \otimes A_5$. The minimal absolutely irreducible subgroups V of G contain a normal subgroup N of index ≤ 2 of the form $N := U \otimes Alt_5$, where U is of index ≤ 2 in one of the two absolutely irreducible subgroups Q_{16} or $\tilde{S}_4$ of $\tilde{S}_4$. The minimality of V implies that U is of order 8 or 16. Hence the 3-modular defect of V is one. Let $\mathfrak{M}$ be a maximal order of $\mathcal{Q}_{\sqrt{2},\infty}$. Then the $\mathfrak{M}V$-lattices are fixed under the group $H := Q_{16} \circ V$, where Q_{16} is the Sylow 2-subgroup of the unit group of $\mathfrak{M}$. The group H is an absolutely irreducible subgroup of $GL_{20}(\mathbb{Q}[\sqrt{2}])$. Since 3 is inert in $\mathbb{Q}[\sqrt{2}]$, one concludes that the 3-modular constituents of H are of degree 8 and 32. Therefore V does not fix a 3-unimodular lattice. $\square$

Theorem 15.5. *$M_5^{irr}(\mathcal{Q}_{\sqrt{3},\infty})$ is as follows.*

$\bullet\ {}_{\sqrt{3},\infty}[C_{12}.C_2]_1^5$

$\bullet\ {}_{\sqrt{3},\infty}[C_{12} \overset{2}{\underset{\sqrt{-3}}{\boxtimes}} U_4(2)]_5$

$\bullet\ {}_{\sqrt{3},\infty}[SL_2(11)]_5$

List of the maximal simplices in $M_5^{irr}(\mathcal{Q}_{\sqrt{3},\infty})$

simplex	*a common subgroup*
$({}_{\sqrt{3},\infty}[C_{12} \overset{2}{\underset{\sqrt{-3}}{\boxtimes}} U_4(2)]_5,\ {}_{\sqrt{3},\infty}[C_{12}.C_2]_1^5)$	$C_{12} \overset{2}{\underset{\sqrt{-3}}{\boxtimes}} 2^4.Alt_5$

Proof. The completeness of the list of a.i.m.f. subgroups in $GL_5(\mathcal{Q}_{\sqrt{3},\infty})$ follows from Theorems 6.1 and 15.3. The list of maximal simplices in $M_5^{irr}(\mathcal{Q}_{\sqrt{3},\infty})$ is complete, because ${}_{\sqrt{3},\infty}[SL_2(11)]_5$ is minimal absolutely irreducible. $\square$

Theorem 15.6. $M_5^{irr}(\mathcal{Q}_{\sqrt{5},\infty})$ *is as follows.*

$\bullet\ {}_{\sqrt{5},\infty}[SL_2(5)]_1^5$

$\bullet\ {}_{\sqrt{5},\infty}[(\pm 5_+^{1+2} : SL_2(5)).2]_5$

$\bullet\ {}_{\sqrt{5},\infty}[SL_2(5)]_1 \otimes A_5$

List of the maximal simplices in $M_5^{irr}(\mathcal{Q}_{\sqrt{5},\infty})$

simplex	*a common subgroup*
$({}_{\sqrt{5},\infty}[(\pm 5_+^{1+2} : SL_2(5)).2]_5,\ {}_{\sqrt{5},\infty}[SL_2(5)]_1^5)$	$(\pm 5_+^{1+2}).2$

Proof. To see the completeness of the list of maximal simplices in $M_5^{irr}(\mathcal{Q}_{\sqrt{5},\infty})$ it suffices to show that there is no common absolutely irreducible subgroup of $G := {}_{\sqrt{5},\infty}[SL_2(5)]_1 \otimes A_5$ and one of the other two maximal finite subgroups of $GL_5(\mathcal{Q}_{\sqrt{5},\infty})$. The minimal absolutely irreducible subgroups U of G are $\pm C_5.C_2 \otimes Alt_5$ and $\pm C_5 \overset{2}{\boxtimes} Alt_5$. If $\mathfrak{M}$ is a maximal order of $\mathcal{Q}_{\sqrt{5},\infty}$, then the 3-modular constituents of the natural representation of $U\mathfrak{M}$ are of degree 8 and 32. So both groups U do not embed into ${}_{\sqrt{5},\infty}[(\pm 5_+^{1+2} : SL_2(5)).2]_5$ or ${}_{\sqrt{5},\infty}[SL_2(5)]_1^5$. $\square$

Theorem 15.7. *The two simplicial complexes* $M_5(\mathcal{Q}_{\sqrt{11},\infty})^{irr}$ *and* $M_5(\mathcal{Q}_{\sqrt{33},\infty})^{irr}$ *consists of one* 0*-simplex each.*

16. The a.i.m.f. subgroups of $GL_6(\mathcal{Q})$

Theorem 16.1. *Let $\mathcal{Q}$ be a definite quaternion algebra with center $\mathbb{Q}$ and let G be a primitive a.i.m.f. subgroup of $GL_6(\mathcal{Q})$. Then G is conjugate to one of the groups in the table below, which is built up as Table 12.7.*

List of the primitive a.i.m.f. subgroups of $GL_6(\mathcal{Q})$.

lattice L	$\|Aut(L)\|$	r.i.m.f. supergroups
${}_{\infty,2}[2.G_2(4)]_6$	$2^{13}\cdot 3^3\cdot 5^2\cdot 7\cdot 13$	$[2.Co_1]_{24}$
${}_{\infty,2}[(\pm 3).PGL_2(9)]_6$	$2^5\cdot 3^3\cdot 5$	$[(\pm 3).PGL_2(9) \overset{2(2)}{\circ} SL_2(3)]_{24}$
${}_{\infty,2}[3_+^{1+2}:SL_2(3) \overset{2(2)}{\boxtimes} D_8]_6$	$2^7\cdot 3^4$	$[Sp_4(3) \underset{\sqrt{-3}}{\overset{2}{\boxtimes}} (3_+^{1+2}:SL_2(3))]_{24}$
${}_{\infty,2}[SL_2(5) \overset{2(2)}{\otimes} D_8]_6$	$2^6\cdot 3\cdot 5$	$[SL_2(5) \underset{\infty,2}{\overset{2(2)}{\boxtimes}} 2_-^{1+4}.Alt_5]_{24}$
${}_{\infty,2}[L_2(7) \overset{2(2)}{\otimes} SL_2(3)]_6$	$2^7\cdot 3^2\cdot 7$	$[L_2(7) \overset{2(2)}{\otimes} F_4]_{24}$
${}_{\infty,2}[L_2(7) \overset{2(2)}{\boxtimes} SL_2(3)]_6$	$2^7\cdot 3^2\cdot 7$	$[L_2(7) \overset{2(2)}{\boxtimes} F_4]_{24}$
${}_{\infty,2}[C_4 \overset{2(3)}{\subset\!\supset} U_3(3)]_6$	$2^8\cdot 3^3\cdot 7$	$[(SL_2(3)\circ C_4) \underset{\sqrt{-1}}{\overset{2(3)}{\boxtimes}} U_3(3)]_{24}$
${}_{\infty,2}[SL_2(5)]_3\otimes A_2$	$2^4\cdot 3^2\cdot 5$	$A_2\otimes [SL_2(5) \overset{2(2)}{\circ} SL_2(3)]_{12}$
${}_{\infty,2}[SL_2(3)]_1\otimes E_6$	$2^{10}\cdot 3^4\cdot 5$	$E_6\otimes F_4$
${}_{\infty,2}[SL_2(3)]_1\otimes A_6$	$2^7\cdot 3^3\cdot 5\cdot 7$	$A_6\otimes F_4$
${}_{\infty,2}[SL_2(3)]_1\otimes A_6^{(2)}$	$2^7\cdot 3^2\cdot 7$	$A_6^{(2)}\otimes F_4$
${}_{\infty,2}[SL_2(3)]_1\otimes M_{6,2}$	$2^6\cdot 3^2\cdot 5$	$M_{6,2}\otimes F_4$
${}_{\infty,3}[6.U_4(3).2^2]_6$	$2^{10}\cdot 3^7\cdot 5\cdot 7$	$[6.U_4(3).2^2]_{12}^2$
${}_{\infty,3}[C_3 \overset{2(2)}{\subset\!\supset} SL_2(5)]_6$	$2^4\cdot 3^2\cdot 5$	$[SL_2(5) \overset{2(2)}{\circ} SL_2(3)]_{12}^2$
${}_{\infty,3}[3_+^{1+2}:SL_2(3) \overset{2(2)}{\boxtimes} SL_2(3)]_6$	$2^7\cdot 3^5$	$[3_+^{1+2}:SL_2(3) \overset{2(2)}{\boxtimes} SL_2(3)]_{12}^2$
${}_{\infty,3}[\tilde{S}_3]_1\otimes A_6$	$2^6\cdot 3^3\cdot 5\cdot 7$	$(A_6\otimes A_2)^2$
${}_{\infty,3}[\tilde{S}_3]_1\otimes A_6^{(2)}$	$2^6\cdot 3^2\cdot 7$	$(A_6^{(2)}\otimes A_2)^2$
${}_{\infty,3}[\tilde{S}_3]_1\otimes M_{6,2}$	$2^5\cdot 3^2\cdot 5$	$(M_{6,2}\otimes A_2)^2$
${}_{\infty,5}[SL_2(25)]_6$	$2^4\cdot 3\cdot 5^2\cdot 13$	$[2.Co_1]_{24}$
${}_{\infty,5}[2.J_2.2]_6$	$2^9\cdot 3^3\cdot 5^2\cdot 7$	$[2.Co_1]_{24}$
${}_{\infty,5}[2.J_2:2]_6$	$2^9\cdot 3^3\cdot 5^2\cdot 7$	$[2.J_2 \overset{2}{\Box} SL_2(5)]_{24}$
${}_{\infty,5}[Alt_5 \underset{\sqrt{5}}{\overset{2}{\boxtimes}} SL_2(5)]_{6,1}$	$2^6\cdot 3^2\cdot 5^2$	$[(SL_2(5)\circ SL_2(5)):2 \underset{\sqrt{5}}{\overset{2}{\boxtimes}} Alt_5]_{24,1}$
${}_{\infty,5}[Alt_5 \underset{\sqrt{5}}{\overset{2}{\boxtimes}} SL_2(5)]_{6,2}$	$2^6\cdot 3^2\cdot 5^2$	$[(SL_2(5)\circ SL_2(5)):2 \underset{\sqrt{5}}{\overset{2}{\boxtimes}} Alt_5]_{24,2}$
${}_{\infty,7}[L_2(7) \overset{2(2)}{\boxtimes} D_8]_6$	$2^7\cdot 3\cdot 7$	$[L_2(7) \overset{2(2)}{\boxtimes} D_8]_{12}^2$
${}_{\infty,7}[L_2(7) \overset{2(3)}{\boxtimes} \tilde{S}_3]_6$	$2^6\cdot 3^2\cdot 7$	$[6.U_4(3).2^2]_{12}^2$
${}_{\infty,7}[\pm L_2(7).2]_3\otimes A_2$	$2^6\cdot 3^2\cdot 7$	$(A_2\otimes A_6^{(2)})^2$
${}_{\infty,11}[SL_2(11).2]_6$	$2^4\cdot 3\cdot 5\cdot 11$	(B_{24})
		$[2.Co_1]_{24}$
${}_{\infty,13}[SL_2(13).2]_6$	$2^4\cdot 3\cdot 7\cdot 13$	$[2.Co_1]_{24}$
${}_{\infty,13}[SL_2(13):2]_6$	$2^4\cdot 3\cdot 7\cdot 13$	$[SL_2(13) \overset{2(2)}{\subset\!\supset} SL_2(3)]_{24}$

The proof is split into seventeen lemmata. For the rest of this section let $\mathcal{Q}$ be a definite quaternion algebra with center $\mathbb{Q}$ and G be a primitive a.i.m.f. subgroup of $GL_6(\mathcal{Q})$.

Lemma 16.2. *The last term $G^{(\infty)}$ of the derived series is either* 1 *or a quasi-semi-simple group. If G is soluble, then G is one of* $_{\infty,2}[3_+^{1+2} : SL_2(3) \overset{2(2)}{\boxtimes} D_8]_6$ *or* $_{\infty,3}[3_+^{1+2} : SL_2(3) \overset{2(2)}{\boxtimes} SL_2(3)]_6$.

Proof. Since the possible normal p-subgroups of G embed into $GL_{12}(\mathbb{Q})$ and $GL_6(\mathcal{Q})$, Theorem 8.1 together with Lemma 2.18 leave the following possibilities for $O_p(G)$: C_{13}, C_7, C_3, C_9, 3_+^{1+2}, C_2, C_4, D_8, Q_8, C_8, or QD_{16}. The automorphism groups of these groups are soluble, so the first part of the lemma follows.

The first case is excluded by Lemma 11.2 and the case $O_7(G) \cong C_7$ is excluded with the help of Lemma 8.12.

If $O_3(G) = C_9$, then $C := C_G(O_3(G))$ embeds into $GL_2(\mathbb{Q}[\zeta_9])$, because $\mathbb{Q}[\zeta_9]$ splits all possible quaternion algebras. With [Bli 17] one finds that C is one of $C_9 \otimes D_8$ or $C_9 \otimes_{\sqrt{-3}} \mathcal{B}^\circ(Q_8) = C_9 \otimes_{\sqrt{-3}} SL_2(3)$, and G contains C of index 6. In both cases 3 does not divide the order of the outer automorphism group of $C/O_3(G)$, so one concludes that $O_3(G) > C_9$ which is a contradiction.

Now assume that $O_3(G) = 3_+^{1+2}$. Then G contains a normal subgroup $B := \mathcal{B}^\circ(O_3(G)) \cong \pm 3_+^{1+2} : SL_2(3)$. Similarly as above one has that $C := C_G(O_3(G))$ is one of $C_3 \otimes D_8$ or $C_3 \circ SL_2(3)$. Moreover G contains BC of index 2. Let $\alpha \in G - BC$. In each case one has two possibilities, either α induces a nontrivial outer automorphism on $\mathcal{B}^\circ(O_2(G))$ or not. For each possibility there is a unique group G with real Schur index 2. For the first possibility one computes that G is $_{\infty,2}[3_+^{1+2} : SL_2(3) \overset{2(2)}{\boxtimes} D_8]_6$ or $_{\infty,3}[3_+^{1+2} : SL_2(3) \overset{2(2)}{\boxtimes} SL_2(3)]_6$. For the second possibility, one computes that G is either a proper subgroup of the imprimitive a.i.m.f. group $_{\infty,3}[\pm 3_+^{1+2}.GL_2(3)]_3^2$ or a proper subgroup of $E_6 \otimes {}_{\infty,2}[SL_2(3)]_1$.

In the other cases, $\mathcal{B}^\circ(O_p(G))$ does not admit an outer automorphism of order 3. One concludes that G being absolutely irreducible, has to contain a quasi-semi-simple normal subgroup (cf. Lemma 8.11). □

Lemma 16.3. *If G contains a subgroup U conjugate to $3.Alt_6$, where the restriction of the natural character of G to U is $\chi_6 + \chi_6'$, then G is one of* $_{\infty,2}[2.G_2(4)]_6$, $_{\infty,2}[(\pm 3).PGL_2(9)]_6$, *or* $_{\infty,3}[6.U_4(3).2^2]_6$.

Proof. The last term of the derived series $G^{(\infty)}$ has to contain $3.Alt_6$, hence is one of $3.Alt_6$, $6.L_3(4)$, $6.U_4(3)$, or $2.G_2(4)$.

First we prove that $\mathcal{Q}$ is either $\mathcal{Q}_{\infty,3}$ or $\mathcal{Q}_{\infty,2}$.

If $G^{(\infty)} = 2.G_2(4)$, then $G^{(\infty)}$ is already an absolutely irreducible subgroup of $GL_6(\mathcal{Q}_{\infty,2})$, hence in this case it is clear that $\mathcal{Q} = \mathcal{Q}_{\infty,2}$. In the other three cases, one has $O_3(G) = C_3$ and the enveloping algebra of U coincides with the one of $C_G(O_3(G)) =: C$. The discriminant of the enveloping $\mathbb{Z}$-order of U is $2^{18} \cdot 3^{11+36}$. Therefore, 2 and 3 are the only primes, which may divide the discriminant of the enveloping $\mathbb{Z}$-order of C. Since C is a normal subgroup of G of index 2, Lemma 2.15 together with the fact that the number of ramified primes is even and $\mathcal{Q}$ is ramified at ∞, implies that $\mathcal{Q}$ is either $\mathcal{Q}_{\infty,2}$ or $\mathcal{Q}_{\infty,3}$.

Let $\mathfrak{M}_2$ (resp. $\mathfrak{M}_3$) denote a maximal order of $\mathcal{Q}_{\infty,2}$ (resp. $\mathcal{Q}_{\infty,3}$).

The group U is a uniform subgroup of $GL_6(\mathcal{Q})$ and fixes up to isomorphism $6 \cdot 6 = 36$ $\mathfrak{M}_2$-lattices. Computing the automorphism groups of the relevant lattices one finds that ${}_{\infty,2}[2.G_2(4)]_6$ and ${}_{\infty,2}[(\pm 3).PGL_2(9)]_6$ are the only primitive a.i.m.f. supergroups of U in $GL_6(\mathcal{Q}_{\infty,2})$.

Similarly U fixes up to isomorphism $2 \cdot 6 = 12$ $\mathfrak{M}_3$-lattices and one finds ${}_{\infty,3}[6.U_4(3).2^2]_6$ as the only primitive a.i.m.f. supergroup of U in $GL_6(\mathcal{Q}_{\infty,3})$. □

The next lemma follows also from Theorem 11.2.

Lemma 16.4. *If $G^{(\infty)}$ is $SL_2(13)$, then G is one of ${}_{\infty,13}[SL_2(13).2]_6$ or ${}_{\infty,13}[SL_2(13):2]_6$.*

Proof. The centralizer $C_G(G^{(\infty)})$ embeds into $\mathbb{Q}[\sqrt{13}]$, and is therefore ± 1. Hence G contains $G^{(\infty)}$ of index $2 = |Out(G^{(\infty)})|$. One computes that both groups G are a.i.m.f. groups in $GL_6(\mathcal{Q}_{\infty,13})$. □

Similarly one finds

Lemma 16.5. *If $G^{(\infty)}$ is $2.J_2$, then G is one of ${}_{\infty,5}[2.J_2.2]_6$ or ${}_{\infty,5}[2.J_2:2]_6$.*

Lemma 16.6. *If $G^{(\infty)}$ is $SL_2(11)$, then G is conjugate to ${}_{\infty,11}[SL_2(11).2]_6$.*

Proof. The centralizer $C_G(G^{(\infty)})$ embeds into $\mathbb{Q}[\sqrt{-11}]$, and is therefore ± 1. Hence G contains $G^{(\infty)}$ of index $2 = |Out(G^{(\infty)})|$. By Lemma 2.17 there is at most one extension $G^{(\infty)}.2$ which is an a.i.m.f. group of $GL_6(\mathcal{Q})$. One computes that G is conjugate to ${}_{\infty,11}[SL_2(11).2]_6$. □

Lemma 16.7. *$G^{(\infty)}$ is not isomorphic to $SL_2(7)$.*

Proof. Assume that $G^{(\infty)}$ is isomorphic to $SL_2(7)$. Then the restriction of the natural character of G to $G^{(\infty)}$ is $2(\chi_{6a} + \chi_{6b})$. The centralizer $C_G(G^{(\infty)})$ embeds into $\mathbb{Q}[\sqrt{2}]$ hence is ± 1 and $G = G^{(\infty)}$ or $G = G^{(\infty)}.2$. The lemma follows, since the second group is no subgroup of $GL_6(\mathcal{Q})$, because the character field is of degree 4 over $\mathbb{Q}$, and $G^{(\infty)}$ is not absolutely irreducible. □

Lemma 16.8. *If $G^{(\infty)}$ is $SL_2(25)$, then $G = G^{(\infty)}$ is conjugate to ${}_{\infty,5}[SL_2(25)]_6$.*

Proof. The group $SL_2(25)$ has two characters of degree 12. The corresponding representations lead to conjugate groups in $GL_6(\mathcal{Q}_{\infty,5})$, since the characters are interchanged by an outer automorphism of $G^{(\infty)}$. The absolutely irreducible subgroup of $GL_6(\mathcal{Q}_{\infty,5})$ fixes up to isomorphism 1 lattice. Since it is the full automorphism group of this lattice, the lemma follows. □

Lemma 16.9. *If $G^{(\infty)}$ is $U_4(2)$, then G is conjugate to $E_6 \otimes {}_{\infty,2}[SL_2(3)]_1$.*

Proof. Let $B := \mathcal{B}^\circ(G^{(\infty)}) = Aut(E_6)$. By Corollary 7.6 $G = B \otimes C_G(B)$ is a tensor product and $C_G(B)$ is an a.i.m.f. subgroup of $GL_1(\mathcal{Q})$. From Proposition 6.1 one gets that G is one of $E_6 \otimes {}_{\infty,2}[SL_2(3)]_1$ or $E_6 \otimes {}_{\infty,3}[\tilde{S}_3]_1$. In the last case, G is not maximal finite, but a proper subgroup of ${}_{\infty,3}[6.U_4(3).2^2]_6$. (Note that $\mathcal{B}^\circ(C_3 \otimes U_4(2)) = 6.U_4(3).2$.) □

Similarly one finds

Lemma 16.10. *If $G^{(\infty)}$ is Alt_7, then G is conjugate to one of $A_6 \otimes {}_{\infty,2}[SL_2(3)]_1$ or $A_6 \otimes {}_{\infty,3}[\tilde{S}_3]_1$.*

Lemma 16.11. *$G^{(\infty)}$ is not isomorphic to $SL_2(9)$.*

Proof. Assume that $G^{(\infty)} = SL_2(9)$. Lemma 2.18 implies that $\mathcal{Q} = \mathcal{Q}_{\infty,3}$. The centralizer $C_G(G^{(\infty)})$ embeds into $GL_3(\mathbb{Q})$. The primitivity of G implies that $C = \pm 1$. But then G is not irreducible, because $3 \nmid |Out(SL_2(9))|$. □

Lemma 16.12. *If G contains a normal subgroup N conjugate to $SL_2(5)$, where the restriction of the natural character of G to N is $3(\chi_{2a} + \chi_{2b})$, then G is one of the two isoclinic groups ${}_{\infty,5}[Alt_5 \overset{2}{\underset{\sqrt{5}}{\boxtimes}} SL_2(5)]_{6,1}$ or ${}_{\infty,5}[Alt_5 \overset{2}{\underset{\sqrt{5}}{\boxtimes}} SL_2(5)]_{6,2}$.*

Proof. The centralizer $C := C_G(N)$ is a centrally irreducible subgroup of $GL_3(\mathbb{Q}[\sqrt{5}])$ hence $C = Alt_5$. Moreover G contains CN of index 2. Computing the two possible extensions one finds that G is one of the two groups in the lemma. □

Lemma 16.13. *If G contains a normal subgroup N isomorphic to Alt_5, then G is one of*

$$M_{6,2} \otimes {}_{\infty,2}[SL_2(3)]_1,\ M_{6,2} \otimes {}_{\infty,3}[\tilde{S}_3]_1,$$

$${}_{\infty,5}[Alt_5 \overset{2}{\underset{\sqrt{5}}{\boxtimes}} SL_2(5)]_{6,1} \text{ or } {}_{\infty,5}[Alt_5 \overset{2}{\underset{\sqrt{5}}{\boxtimes}} SL_2(5)]_{6,2}.$$

Proof. The centralizer $C := C_G(N)$ is a centrally irreducible subgroup of $GL_1(\mathbb{Q}[\sqrt{5}] \otimes \mathcal{Q})$ and G contains CN of index 2. Assume first that $C_G(C) > \pm N$. Then $C_G(C) \cong \pm N.2$ and G is a tensor product $G = C_G(C) \otimes C$, where C is an a.i.m.f. subgroup of $GL_1(\mathcal{Q})$. Since the nonsplit extension $S_5 \curlywedge C_4$ is a monomial subgroup of $GL_6(\mathbb{Q})$ (cf. [PlN 95, (V.3)]), one finds that G is one of $M_{6,2} \otimes {}_{\infty,2}[SL_2(3)]_1$ or $M_{6,2} \otimes {}_{\infty,3}[\tilde{S}_3]_1$. Now let $C_G(C) = \pm N$. Using the classification of a.i.m.f. subgroups of $GL_1(\mathcal{Q}')$ for definite quaternion algebras $\mathcal{Q}'$ with center $\mathbb{Q}$ or $\mathbb{Q}[\sqrt{5}]$, one finds that $C = \mathcal{B}^\circ(C)$ is one of $SL_2(3)$, $\tilde{S}_3$, Q_{20}, or $SL_2(5)$. In the last case, the lemma follows from Lemma 16.12, whereas in the first three cases both extensions $G = NC_G(N).2$ are proper subgroups of one of the two groups of Lemma 16.12. □

Lemma 16.14. *If $N := G^{(\infty)} = U_3(3)$, then G is conjugate to ${}_{\infty,2}[C_4 \overset{2(3)}{\sqsubset} U_3(3)]_6$.*

Proof. The centralizer $C := C_G(N)$ embeds into an indefinite quaternion algebra with center $\mathbb{Q}$. Since $\mathcal{B}^\circ(N \circ C_3) = 6.U_4(3).2$, one concludes that $O_3(C) = 1$. One has the following possibilities for $O_2(G) = O_2(C)$: ± 1, C_4, or D_8. The first possibility immediately yields a contradiction, then G contains N of index 2 and Lemma 2.14 implies that $dim(\overline{\mathbb{Q}G}) \leq 2dim(\overline{\mathbb{Q}N}) = 2 \cdot 36 < 144$. Therefore G is not absolutely irreducible in this case. In the other two cases G contains a normal subgroup $N \circ C_4$. The discriminant of the enveloping $\mathbb{Z}$-order of $N \circ C_4$ is $3^{12} \cdot 2^{36}$. Therefore Lemma 2.15 implies that $\mathcal{Q}$ is one of $\mathcal{Q}_{\infty,2}$ or $\mathcal{Q}_{\infty,3}$, if G contains $N \circ C_4$ of index two. If $[G : (N \circ C_4)] > 2$, then $O_2(C) = D_8$ and $NO_2(C)$ is already an absolutely irreducible subgroup of $GL_6(\mathcal{Q}_{\infty,3})$. So in this case $\mathcal{Q} = \mathcal{Q}_{\infty,3}$.

Let $\mathfrak{M}_2$ (resp. $\mathfrak{M}_3$) denote a maximal order of $\mathcal{Q}_{\infty,2}$ (resp. $\mathcal{Q}_{\infty,3}$).

Then $N \circ C_4$ fixes only one $\mathfrak{M}_2$-lattice. The automorphism group of this lattice is ${}_{\infty,2}[C_4 \overset{2(3)}{\sqsubset} U_3(3)]_6$. Hence G is conjugate to this a.i.m.f. group, if $\mathcal{Q} = \mathcal{Q}_{\infty,2}$. If $\mathcal{Q} = \mathcal{Q}_{\infty,3}$, then $N \circ C_4$ fixes up to isomorphism 6 $\mathfrak{M}_3$-lattices. The automorphism group of the normalized lattices is conjugate to ${}_{\infty,3}[{}^{\pm}U_3(3)]_3^2$ contradicting the primitivity of G. □

Lemma 16.15. *If $G^{(\infty)}$ is conjugate to $SL_2(5)$, where the restriction of the natural character of G to $G^{(\infty)}$ is $2\chi_6$, then G is one of ${}_{\infty,2}[SL_2(5) \overset{2(2)}{\otimes} D_8]_6$, $A_2 \otimes {}_{\infty,2}[SL_2(5)]_3$, or ${}_{\infty,3}[C_3 \overset{2(2)}{\sqsupset} SL_2(5)]_6$.*

Proof. Let $G^{(\infty)}$ be conjugate to $SL_2(5)$ as described in the lemma. Then the centralizer $C := C_G(G^{(\infty)})$ embeds into $GL_1(\mathcal{Q}')$, where $\mathcal{Q}'$ is an indefinite quaternion algebra with center $\mathbb{Q}$. Hence C is soluble. Moreover G contains $CG^{(\infty)}$ of index $\leq 2 = |Out(G^{(\infty)})|$. By Lemma 2.14 this implies that $C \neq {}^{\pm}1$. Therefore one either has $O_3(C) = C_3$ or $O_3(C) = 1$ and $C = C_4$ or D_8. The discriminant of the enveloping $\mathbb{Z}$-order of $G^{(\infty)}$ is $(5^8 \cdot 2^{5+9})^2$. Using Lemma 2.15 one excludes in all cases that $\mathcal{Q}$ is ramified at 5. Hence $\mathcal{Q}$ is one of $\mathcal{Q}_{\infty,2}$ or $\mathcal{Q}_{\infty,3}$, where the latter possibility only occurs if $O_3(G) = C_3$. Let $\mathfrak{M}_2$ (resp. $\mathfrak{M}_3$) denote a maximal order of $\mathcal{Q}_{\infty,2}$ (resp. $\mathcal{Q}_{\infty,3}$). In the first case $N := SL_2(5) \circ C_3$ is a normal subgroup of G. The Bravais group of a normal critical $\mathfrak{M}_2 N$-lattice (cf. Definition 2.7) is conjugate to $A_2 \otimes {}_{\infty,2}[SL_2(5)]_3$. So G is conjugate to this group, if $\mathcal{Q} = \mathcal{Q}_{\infty,2}$. If $\mathcal{Q} = \mathcal{Q}_{\infty,3}$, then every $\mathfrak{M}_3 N$-lattice is normal critical. One concludes that G is conjugate to ${}_{\infty,3}[C_3 \overset{2(2)}{\sqsupset} SL_2(5)]_6$ in this case. If $O_3(C) = 1$, then G contains a normal subgroup $N := SL_2(5) \circ C_4$. Moreover $\mathcal{Q} = \mathcal{Q}_{\infty,2}$. The automorphism group of a normal critical $\mathfrak{M}_2 N$-lattice is conjugate to ${}_{\infty,2}[SL_2(5) \overset{2(2)}{\otimes} D_8]_6$. Therefore $G = {}_{\infty,2}[SL_2(5) \overset{2(2)}{\otimes} D_8]_6$. □

Similarly one finds

Lemma 16.16. *If $G^{(\infty)}$ is conjugate to $L_2(7)$, where the restriction of the natural character of G to $G^{(\infty)}$ contains χ_6, then G is conjugate to ${}_{\infty,2}[L_2(7) \overset{2(2)}{\otimes} SL_2(3)]_6$.*

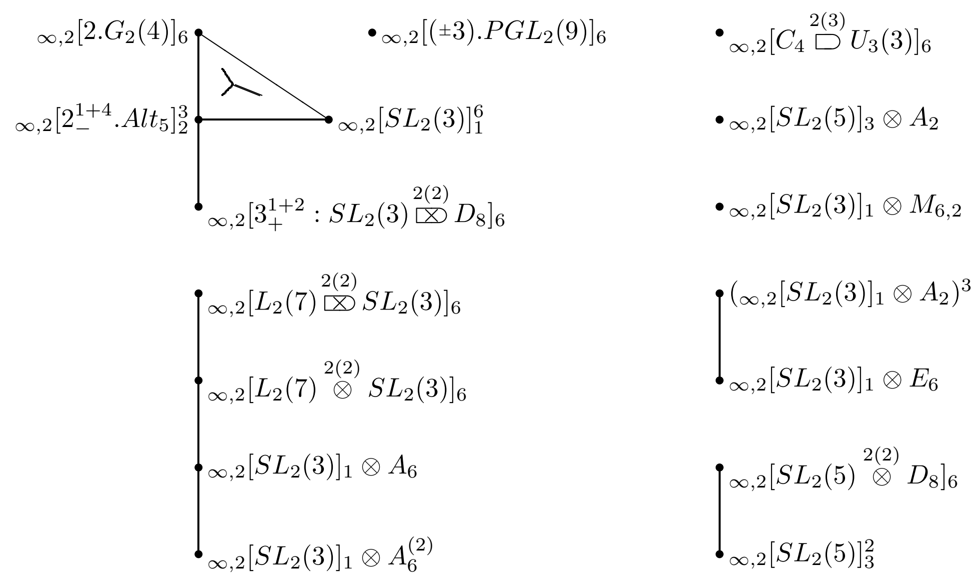
Lemma 16.17. *If $G^{(\infty)}$ is conjugate to $L_2(7)$, with character $\chi_{3a} + \chi_{3b}$, then G is conjugate to one of ${}_{\infty,2}[L_2(7) \overset{2(2)}{\boxtimes} SL_2(3)]_6$, ${}_{\infty,2}[SL_2(3)]_1 \otimes A_6^{(2)}$, ${}_{\infty,3}[L_2(7) \overset{2(3)}{\boxtimes} S_3]_6$, ${}_{\infty,3}[\tilde{S}_3]_1 \otimes A_6^{(2)}$, ${}_{\infty,7}[L_2(7) \overset{2(2)}{\boxtimes} D_8]_6$, ${}_{\infty,7}[L_2(7) \overset{2(3)}{\boxtimes} \tilde{S}_3]_6$, or ${}_{\infty,7}[{}^{\pm}L_2(7).2]_3 \otimes A_2$.*

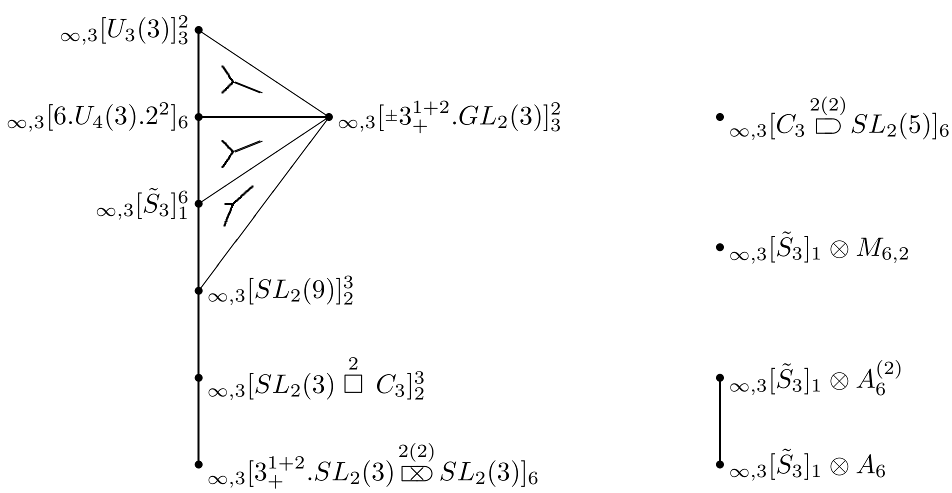
Proof. The centralizer $C := C_G(G^{(\infty)})$ embeds into a quaternion algebra $\mathcal{Q}'$ with center $\mathbb{Q}[\sqrt{-7}]$, more precisely $\mathcal{Q}' = \mathbb{Q}[\sqrt{-7}]^{2\times 2}$ if 2 is not ramified in $\mathcal{Q}$ and $\mathcal{Q}' = \mathcal{Q}_{\sqrt{-7},2,2}$, if 2 ramifies in $\mathcal{Q}$. Because G is absolutely irreducible, G contains $CG^{(\infty)}$ of index 2 and C is a centrally irreducible subgroup of $GL_1(\mathcal{Q}')$. The classification of the finite subgroup of $PGL_2(\mathbb{C})$ in [Bli 17] implies that C is one of $SL_2(3)$, D_8, S_3, or $\tilde{S}_3$. Since the enveloping algebra of C is central simple, one has $2 = |N_{\overline{\mathbb{Q}C}^*}(C)/C_{\overline{\mathbb{Q}C}^*}(C)|$ possible automorphisms. By Lemma 2.17 there is for each automorphism a unique extension with real Schur index 2. Since $D_8 \otimes {}_{\infty,7}[{}^{\pm}L_2(7).2]_3$ is imprimitive, G is one of the seven groups in the Lemma. □

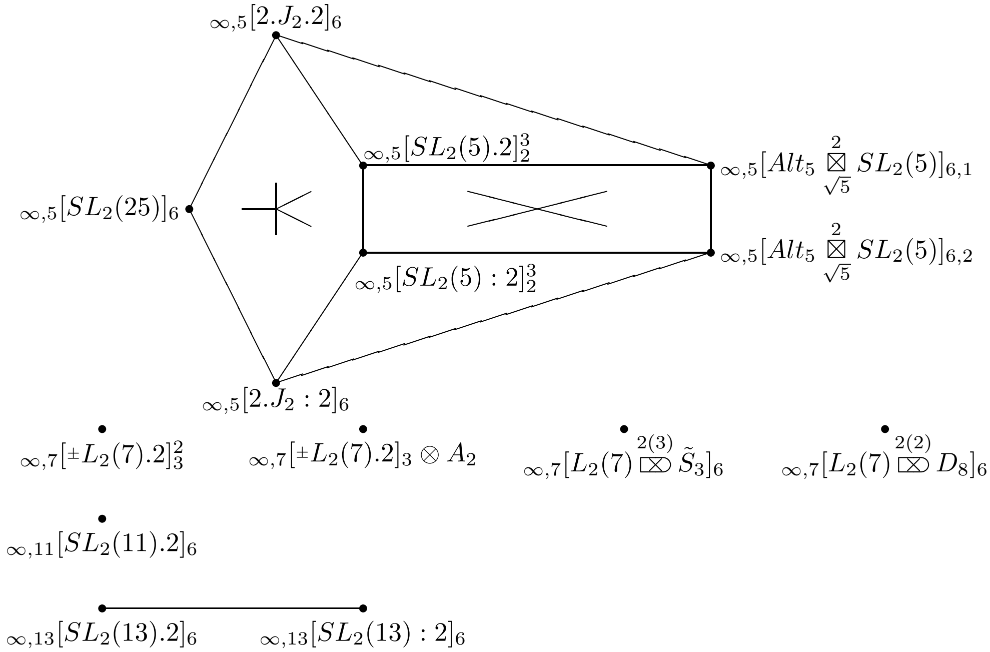
Lemma 16.18. *$G^{(\infty)}$ is not conjugate to $3.Alt_6$, where the restriction of the natural representation of G to $G^{(\infty)}$ is $\chi_{3a} + \chi'_{3a} + \chi_{3b} + \chi'_{3b}$.*

Proof. Assume that $G^{(\infty)}$ is conjugate to $3.Alt_6$. Then $C_G(G^{(\infty)}) = {}^{\pm}C_3$ is contained in ${}^{\pm}G^{(\infty)}$ and G contains ${}^{\pm}G^{(\infty)}$ of index $2^2 = |Out(G^{(\infty)})|$. Since $\mathcal{Q}$ is positive definite, G contains the unique extension $N := {}^{\pm}3.PGL_2(9)$ with real Schur index 2 (cf. Lemma 2.17) of index 2. The Bravais group of a normal critical $\mathbb{Z}N$-lattice is $2.J_2$ contradicting the assumption that $G^{(\infty)} = 3.Alt_6$. □

Theorem 16.19. *Let $\mathcal{Q}$ be a definite quaternion algebra with center $\mathbb{Q}$ and G be an a.i.m.f. subgroup of $GL_6(\mathcal{Q})$. Then $\mathcal{Q}$ is one of $\mathcal{Q}_{\infty,2}$, $\mathcal{Q}_{\infty,3}$, $\mathcal{Q}_{\infty,5}$, $\mathcal{Q}_{\infty,7}$, $\mathcal{Q}_{\infty,11}$, or $\mathcal{Q}_{\infty,13}$. The simplicial complexes $M_6^{irr}(\mathcal{Q})$ are as follows:*







List of maximal simplices in $M_6^{irr}(\mathcal{Q}_{\infty,2})$:

simplex	a common subgroup
$({}_{\infty,2}[2.G_2(4)]_6,\ {}_{\infty,2}[2_-^{1+4}.Alt_5]_2^3,\ {}_{\infty,2}[SL_2(3)]_1^6)$	$Alt_4 \otimes D_8 \otimes Q_8$
$({}_{\infty,2}[3_+^{1+2} : SL_2(3) \overset{2(2)}{\boxtimes} D_8]_6,\ {}_{\infty,2}[2_-^{1+4}.Alt_5]_2^3)$	$3_+^{1+2} \overset{2(2)}{\boxtimes} D_8$
$({}_{\infty,2}[SL_2(5) \overset{2(2)}{\otimes} D_8]_6,\ {}_{\infty,2}[SL_2(5)]_3^2)$	$SL_2(5) \otimes D_8$
$((A_2 \otimes {}_{\infty,2}[SL_2(3)]_1)^3,\ E_6 \otimes {}_{\infty,2}[SL_2(3)]_1)$	$3_+^{1+2} : 2 \otimes Q_8$
$({}_{\infty,2}[L_2(7) \overset{2(2)}{\boxtimes} SL_2(3)]_6,\ {}_{\infty,2}[L_2(7) \overset{2(2)}{\otimes} SL_2(3)]_6)$	$C_7 : C_3 \overset{2(2)}{\boxtimes} SL_2(3)$
$(A_6 \otimes {}_{\infty,2}[SL_2(3)]_1,\ {}_{\infty,2}[L_2(7) \overset{2(2)}{\otimes} SL_2(3)]_6)$	$L_2(7) \otimes Q_8$
$(A_6 \otimes {}_{\infty,2}[SL_2(3)]_1,\ A_6^{(2)} \otimes {}_{\infty,2}[SL_2(3)]_1)$	$C_7 : C_6 \otimes SL_2(3)$

List of maximal simplices in $M_6^{irr}(\mathcal{Q}_{\infty,3})$:

simplex	a common subgroup
$({}_{\infty,3}[U_3(3)]_3^2,\ {}_{\infty,3}[6.U_4(3).2^2]_6,\ {}_{\infty,3}[\pm 3_+^{1+2}.GL_2(3)]_3^2)$	$3_+^{1+2} : C_8 \otimes S_3$
$({}_{\infty,3}[6.U_4(3).2^2]_6,\ {}_{\infty,3}[\pm 3_+^{1+2}.GL_2(3)]_3^2,\ {}_{\infty,3}[\tilde{S}_3]_1^6)$	$\pm 3_+^{1+2}.C_2 \otimes S_3$
$({}_{\infty,3}[\tilde{S}_3]_1^6,\ {}_{\infty,3}[\pm 3_+^{1+2}.GL_2(3)]_3^2,\ {}_{\infty,3}[SL_2(9)]_2^3)$	$(\pm C_3^4).C_{12}$
$({}_{\infty,3}[SL_2(3) \overset{2}{\square}\ C_3]_2^3,\ {}_{\infty,3}[SL_2(9)]_2^3)$	$\tilde{S}_4 \otimes Alt_4$
$({}_{\infty,3}[3_+^{1+2} : SL_2(3) \overset{2(2)}{\boxtimes} SL_2(3)]_6,\ {}_{\infty,3}[SL_2(3) \overset{2}{\square}\ C_3]_2^3)$	$3_+^{1+2} \overset{2(2)}{\boxtimes} Q_8$
$(A_6 \otimes {}_{\infty,3}[\tilde{S}_3]_1,\ A_6^{(2)} \otimes {}_{\infty,3}[\tilde{S}_3]_1)$	$C_7 : C_6 \otimes \tilde{S}_3$

List of maximal simplices in $M_6^{irr}(\mathcal{Q}_{\infty,5})$:

simplex	a common subgroup
$({}_{\infty,5}[SL_2(5).2]_2^3, {}_{\infty,5}[SL_2(5) : 2]_2^3,$ ${}_{\infty,5}[2.J_2.2]_6, {}_{\infty,5}[2.J_2 : 2]_6, {}_{\infty,5}[SL_2(25)]_6)$	$(\pm C_5 \times C_5).C_{12}$
$({}_{\infty,5}[SL_2(5).2]_2^3, {}_{\infty,5}[SL_2(5) : 2]_2^3,$ ${}_{\infty,5}[Alt_5 \underset{\sqrt{5}}{\overset{2}{\boxtimes}} SL_2(5)]_{6,1}, {}_{\infty,5}[Alt_5 \underset{\sqrt{5}}{\overset{2}{\boxtimes}} SL_2(5)]_{6,2})$	$Q_{20} \overset{2}{\boxtimes}\ Alt_4$
$({}_{\infty,5}[2.J_2.2]_6, {}_{\infty,5}[Alt_5 \underset{\sqrt{5}}{\overset{2}{\boxtimes}} SL_2(5)]_{6,1})$	$Alt_5 \overset{2}{\sqsupset} Q_8$
$({}_{\infty,5}[2.J_2 : 2]_6, {}_{\infty,5}[Alt_5 \underset{\sqrt{5}}{\overset{2}{\boxtimes}} SL_2(5)]_{6,2})$	$Alt_5 \overset{2}{\sqsupset} Q_8$

List of maximal simplices in $M_6^{irr}(\mathcal{Q}_{\infty,13})$:

simplex	a common subgroup
$({}_{\infty,13}[SL_2(13) : 2]_6,\ {}_{\infty,13}[SL_2(13).2]_6)$	$\pm C_{13}.C_{12}$

Theorems 16.1, 13.1, 12.1, and 6.1 prove the completeness of the list of quaternion algebras $\mathcal{Q}$ and of a.i.m.f. subgroups of $GL_6(\mathcal{Q})$. So it remains to prove the completeness of the list of maximal simplices in $M_6^{irr}(\mathcal{Q})$. That the simplices listed do exist, can be easily seen by computing the automorphism groups of the invariant lattices of the groups listed in the column "a common subgroup".

To make the formulations of the proofs not so lengthy, we introduce some notation for imprimitive groups.

Notation 16.20. Let $G = H \wr S_d = (H_1 \times \ldots \times H_d) : S_d$ be an imprimitive subgroup of $GL_n(\mathcal{D})$ and Δ its natural representation.

For a subgroup $U \leq G$ the restriction of Δ to the stabilizer

$$S_1(U) := \{u \in U \mid h_1 u \in H_1 \text{ for all } h_1 \in H_1\}$$

of the first component has a summand $\Delta_1 : S_1(U) \to H_1$. Define

$$\pi_1(U) := \Delta_1(S_1(U)) \leq H_1.$$

Then $\Delta_{|U}$ is induced up from Δ_1, hence Frobenius reciprocity implies that if U is absolutely irreducible, then $\pi_1(U)$ is an absolutely irreducible subgroup of $GL_{\frac{n}{d}}(\mathcal{D})$.

The *base group* of U is defined as the intersection of U with $H_1 \times \ldots \times H_d \trianglelefteq G$. Clearly this is a normal subgroup of U.

The proof is split up into several lemmata according to the different quaternion algebras $\mathcal{Q}_{\infty,p}$. Since $M_6^{irr}(\mathcal{Q}_{\infty,13})$ and $M_6^{irr}(\mathcal{Q}_{\infty,11})$ consist of one simplex, it suffices to consider $p \leq 7$.

Lemma 16.21. *$M_6^{irr}(\mathcal{Q}_{\infty,7})$ consists of four 0-simplices.*

Proof. The minimal absolutely irreducible subgroups of the three primitive a.i.m.f. groups ${}_{\infty,7}[L_2(7) \overset{2(2)}{\boxtimes} D_8]_6$, ${}_{\infty,7}[L_2(7) \overset{2(3)}{\boxtimes} \tilde{S}_3]_6$, and $A_2 \otimes {}_{\infty,7}[{}^{\pm}L_2(7).2]_3$ are the normalizers in these groups of the Sylow 7-subgroups. The lemma follows by computing the invariant lattices of these three groups. $\square$

Lemma 16.22. *The list of maximal simplices in $M_6^{irr}(\mathcal{Q}_{\infty,5})$ given in Theorem 16.19 is complete.*

Proof. Since the other five a.i.m.f. groups in $GL_6(\mathcal{Q}_{\infty,5})$ form a 4-simplex, it suffices to consider the minimal absolutely irreducible subgroups U of one of the two groups G ${}_{\infty,5}[Alt_5 \underset{\sqrt{5}}{\overset{2}{\boxtimes}} SL_2(5)]_{6,1}$ and ${}_{\infty,5}[Alt_5 \underset{\sqrt{5}}{\overset{2}{\boxtimes}} SL_2(5)]_{6,2}$. Let $N \trianglelefteq U$ be the intersection of U with the normal subgroup $Alt_5 \trianglelefteq G$ and $M := U \cap SL_2(5)$. Then the restriction of the natural representation Δ of U to N is of degree 1 or 3. If it is of degree 1, the index of NM in U is divisible by 3. Since 5 divides the order of U and subgroups of Alt_5 and $SL_2(5)$, of which the order is a multiple of 5 have no nontrivial factor group of order divisible by 3, one concludes that $U \cong SL_2(5).2$ is a full subdirect product. But this contradicts the fact that U is absolutely irreducible. Hence $\Delta_{|N}$ is of degree 3, and N is one of Alt_4 or Alt_5. In the first case, 5 divides the order of the centralizer $C := C_U(N)$ and the minimality of U implies that $U = Q_{20} \overset{2}{\boxtimes} Alt_4$.

In the second case, $C \leq SL_2(5)$ is a centrally irreducible subgroup of $\mathcal{Q}_{\sqrt{5},\infty}$. This leaves the possibilities $C = Q_{20}$, Q_8, $SL_2(3)$, and $\tilde{S}_3$. Moreover U contains NC of index 2. In the first case U contains $Q_{20} \overset{2}{\boxtimes} Alt_4$ from above and the third possibility $C = SL_2(3)$ gives groups containing $NQ_8.2$. An inspection of the lattices of the remaining groups yields the lemma. $\square$

Proposition 16.23. *The list of maximal simplices in $M_6^{irr}(\mathcal{Q}_{\infty,3})$ given in Theorem 16.19 is complete.*

The proof is divided into seven lemmata, which are organized according to the largest primes dividing the determinant of an invariant primitive lattice of the a.i.m.f. group.

Let S denote a maximal simplex of $M_6^{irr}(\mathcal{Q}_{\infty,3})$ not listed in Theorem 16.19.

Lemma 16.24. *S contains no vertex $A_6 \otimes {}_{\infty,3}[\tilde{S}_3]_1$ or $A_6^{(2)} \otimes {}_{\infty,3}[\tilde{S}_3]_1$.*

Proof. Let U be a minimal absolutely irreducible subgroup of one of the two a.i.m.f. groups of the lemma. Then by Lemma 2.13 7 divides the order of U. If U contains a normal subgroup of order 7, then $U = C_7 : C_6 \otimes {}_{\infty,3}[\tilde{S}_3]_1$ is a common subgroup of the 2 groups. If the Sylow 7-subgroup of U is not normal in U, then the minimality of U implies that $U^{(\infty)} = L_2(7)$, where the restriction of the natural character of U to $U^{(\infty)}$ is $4\chi_6$, where $\chi_6 + 1$ is a permutation character of $L_2(7)$. Since the corresponding permutation representation does not extend to $L_2(7).2$, one concludes that $U = L_2(7) \otimes {}_{\infty,3}[\tilde{S}_3]_1$, which has only ${}_{\infty,3}[\tilde{S}_3]_1 \otimes A_6$ as an a.i.m.f. supergroup. $\square$

Lemma 16.25. *S contains no vertex* $_{\infty,3}[C_3 \overset{2(2)}{\square} SL_2(5)]_6$ *or* $M_{6,2} \otimes {}_{\infty,3}[\tilde{S}_3]_1$.

Proof. Let U be a minimal absolutely irreducible subgroup of one of the two groups G. Then by Lemma 2.13 5 divides $|U|$. Since for both groups, the normalizer of a Sylow 5-subgroup is reducible, one concludes that $U^{(\infty)} = G^{(\infty)}$. In both cases one finds that $U = G$ is minimal absolutely irreducible. $\square$

Lemma 16.26. *The irreducible subgroups* $V \leq \mathcal{B}^\circ(3_+^{1+2}) = 3_+^{1+2} : SL_2(3) \leq GL_3(\mathbb{Q}[\zeta_3])$ *satisfy* 3_+^{1+2} *or* $3_-^{1+2} \leq O_3(V)$.

Proof. Let V be an irreducible subgroup of $G := 3_+^{1+2} : SL_2(3)$. If $N := V \cap O_3(G) = 1$, then V is isomorphic to a subgroup of $SL_2(3)$ and reducible. Therefore N, being nontrivial, contains the center of G and is one of $O_3(G)$, $C_3 \times C_3$ or C_3. In the first case, $3_+^{1+2} \leq O_3(V)$. In the second case V/N stabilizes a flag in $\mathbb{F}_3^2 = O_3(G)/Z(G)$, hence is an abelian subgroup of $SL_2(3)$. Since the degree of the natural character of V is 3 and N is an abelian normal subgroup of V, one has $O_3(V) = 3_-^{1+2}$ in this case. The last case again contradicts the irreducibility of V, being contained in $C_3 \times SL_2(3)$. $\square$

Lemma 16.27. *S contains no vertex* $G := {}_{\infty,3}[3_+^{1+2} : SL_2(3) \overset{2(2)}{\boxtimes} SL_2(3)]_6$.

Proof. Let $\mathfrak{M}$ be a maximal order in $\mathcal{Q}_{\infty,3}$. Let U be a minimal absolutely irreducible subgroup of G. The natural representation of U is of the form $\Delta_1 \otimes \Delta_2$, where $\Delta_1(U)$ has a subgroup V of index 2 such that $V \leq \pm 3_+^{1+2}.SL_2(3)$ is an irreducible subgroup of $GL_3(\mathbb{Q}[\zeta_3])$. By Lemma 16.26 $O_3(V)$ contains an extraspecial 3-group. In particular the 2-modular constituents of the natural representation of $V\mathfrak{M}$ are of degree 12. Comparing the determinants of the invariant integral lattices one sees that the only other a.i.m.f. group, into which U might embed, is $_{\infty,3}[SL_2(3) \overset{2}{\square} C_3]_3^2$. $\square$

Lemma 16.28. *S contains no vertex* $G := {}_{\infty,3}[SL_2(3) \overset{2}{\square} C_3]_2^3$.

Proof. Let $\mathfrak{M}$ be a maximal order in $\mathcal{Q}_{\infty,3}$. Let U be a minimal absolutely irreducible subgroup of G. With the notation introduced in 16.20 the group $\pi_1(U) \leq {}_{\infty,3}[SL_2(3) \overset{2}{\square} C_3]_2$ is an absolutely irreducible subgroup of $GL_2(\mathcal{Q}_{\infty,3})$. Hence $\pi_1(U)$ contains one of the two minimal absolutely irreducible subgroups $Q_8 \overset{2}{\square} C_3$ or $\tilde{S}_4$ of $_{\infty,3}[SL_2(3) \overset{2}{\square} C_3]_2$ (cf. proof of Theorem 12.3). In particular the 3-modular constituents of $\mathfrak{M}\pi_1(U)$ are of degree 4.

The intersection N of U with the base group of G is a normal subgroup $N \trianglelefteq U$ with $U/N \cong C_3$ or S_3. In the first case, $\pi_1(U) = \pi_1(N)$ and the Clifford theory implies that the degrees of the 3-modular constituents of U are divisible by 4. With Lemma 16.27 one sees that U does not define a new simplex. In the second case, $\pi_1(N)$ is a normal subgroup of index 2 in $\pi_1(U)$. Assuming that the 3-modular constituents of $\mathfrak{M}\pi_1(N)$ are not all of degree 4, one only finds the possibility that $\pi_1(N) = C_{12}.C_2 = S_3 \curlywedge^{C_2} C_8$. But then the 2-modular constituents of $\mathfrak{M}\pi_1(N)$ are of degree 4 which implies that U does not define a new simplex. $\square$

Lemma 16.29. *There is no common absolutely irreducible subgroup of* $G := {}_{\infty,3}[U_3(3)]_3^2$ *and one of* $_{\infty,3}[\tilde{S}_3]_1^6$ *or* $_{\infty,3}[SL_2(9)]_2^3$.

Proof. Let U be an absolutely irreducible subgroup of G. Then U contains a normal subgroup $N \trianglelefteq U$ of index 2, such that the restriction of the natural representation Δ of U to N is the sum of two inequivalent absolutely irreducible representations $\Delta_{|N} = \Delta_1 + \Delta_2$. The groups $\Delta_1(N) \cong \Delta_2(N)$ are absolutely irreducible subgroups of ${}_{\infty,3}[U_3(3)]_3$. Hence $\Delta_1(N)$ is either $\cong U_3(3)$ or $3_+^{1+2} : C_8$. Both groups have no subgroup of index 3 or 6. Therefore U is not a subgroup of ${}_{\infty,3}[\tilde{S}_3]_1^6$ or ${}_{\infty,3}[SL_2(9)]_2^3$. $\square$

Proposition 16.23 now follows from the next lemma:

Lemma 16.30. *There is no common absolutely irreducible subgroup of* $G := {}_{\infty,3}[SL_2(9)]_2^3$ *and* $H := {}_{\infty,3}[6.U_4(3).2^2]_6$.

Proof. Let U be an absolutely irreducible subgroup of G. Then $\pi_1(U)$ is one of the four absolutely irreducible subgroups $\pm 3^2.C_4$, $\tilde{S}_4$, $SL_2(5)$, or $SL_2(9)$ of $\pi_1(G) = {}_{\infty,3}[SL_2(9)]_2$.

If $\pi_1(U) = \tilde{S}_4$, then $\pi_1(U)$ is a subgroup of ${}_{\infty,3}[SL_2(3) \overset{2}{\square} C_3]_2$ and the lemma follows from Lemma 16.28.

Since U is a subgroup of H the centralizer $N := C_U(O_3(H)) \trianglelefteq U$ in U of $O_3(H)$ is a subgroup of U of index 2 with commuting algebra $C_{\mathcal{Q}_{\infty,3}^{6\times 6}}(N) \cong \mathbb{Q}[\zeta_3]$. Since N is normal, $\pi_1(N) \trianglelefteq \pi_1(U)$ is a subgroup of index 2 with $C_{\mathcal{Q}_{\infty,3}^{2\times 2}}(\pi_1(N)) \cong \mathbb{Q}[\zeta_3]$.

Since in the first case the unique subgroup V of index 2 in $\pi_1(U)$ has commuting algebra $C_{\mathcal{Q}_{\infty,3}^{2\times 2}}(V) = \mathbb{Q} \oplus \mathbb{Q}$ and the last two groups $\pi_1(U)$ are perfect, this is a contradiction. $\square$

Proposition 16.31. *The list of maximal simplices in* $M_6^{irr}(\mathcal{Q}_{\infty,2})$ *given in Theorem 16.19 is complete.*

This proposition is proved in the rest of this chapter which concludes the proof of Theorem 16.19.

The first lemma is easily checked with help of Lemma 2.13 and [CCNPW 85].

Lemma 16.32. (i) *The minimal absolutely irreducible subgroup of* ${}_{\infty,2}[SL_2(3)]_1 \otimes M_{6,2}$ *is* $S_5 \otimes Q_8$.

(ii) *The minimal absolutely irreducible subgroups of* ${}_{\infty,2}[SL_2(3)]_1 \otimes A_6$ *are* $C_7 : C_6 \otimes Q_8$, $C_7 : C_6 \curlywedge^{C_3} SL_2(3)$, *and* $L_2(7) \otimes Q_8$.

(iii) *The minimal absolutely irreducible subgroups of* ${}_{\infty,2}[SL_2(3)]_1 \otimes A_6^{(2)}$ *embed into* ${}_{\infty,2}[SL_2(3)]_1 \otimes A_6$.

(iv) *The group* ${}_{\infty,2}[SL_2(5)]_3 \otimes A_2$ *is minimal absolutely irreducible.*

(v) *The group* ${}_{\infty,2}[(\pm 3).PGL_2(9)]_6$ *is minimal absolutely irreducible.*

(vi) *The minimal absolutely irreducible subgroup of* ${}_{\infty,2}[C_4 \overset{2(3)}{\sqsupset} U_3(3)]_6$ *is the normalizer of the Sylow 3-subgroup* $C_4 \overset{2(3)}{\sqsupset} 3_+^{1+2} : C_8$.

Corollary 16.33. *The restriction of* $M_6^{irr}(\mathcal{Q}_{\infty,2})$ *to the set* $\{{}_{\infty,2}[L_2(7) \overset{2(2)}{\otimes} SL_2(3)]_6$, ${}_{\infty,2}[L_2(7) \overset{2(2)}{\boxtimes} SL_2(3)]_6$, $A_6^{(2)} \otimes {}_{\infty,2}[SL_2(3)]_1$, $A_6 \otimes {}_{\infty,2}[SL_2(3)]_1$, ${}_{\infty,2}[SL_2(3)]_1 \otimes M_{6,2}$, $A_2 \otimes {}_{\infty,2}[SL_2(5)]_3$, ${}_{\infty,2}[(\pm 3).PGL_2(9)]_6$, ${}_{\infty,2}[C_4 \overset{2(3)}{\sqsupset} U_3(3)]_6\}$ *consists of full simplices and is as given in Theorem 16.19.*

Proof. After computing the a.i.m.f. supergroups of the minimal absolutely irreducible subgroups given in Lemma 16.32, it suffices to show, that there is no common absolutely irreducible group of one of the first two groups and one further a.i.m.f. group not mentioned in the corollary. Let U be such an absolutely irreducible group. Then by Lemma 2.13 the order of U is divisible by 7. Hence the only other a.i.m.f. group into which U may embed is ${}_{\infty,2}[2.G_2(4)]_6$. Therefore $C_U(C_7) \leq \pm C_3$ and U is clearly not an absolutely irreducible subgroup of one of the first two groups. □

Lemma 16.34. *The restriction of $M_6^{irr}(\mathcal{Q}_{\infty,2})$ to the set $\{{}_{\infty,2}[SL_2(5) \overset{2(2)}{\otimes} D_8]_6, {}_{\infty,2}[SL_2(5)]_3^2\}$ consists of full simplices and is a one dimensional simplex.*

Proof. Let U be an absolutely irreducible subgroup of one of the two a.i.m.f. groups G in the lemma and one other a.i.m.f. group. Then $U^{(\infty)} = SL_2(5)$ (or $SL_2(5) \times SL_2(5)$), where the restriction of the natural character of G to U is $2\chi_6$ (or $\chi_6+\chi_6'$). The second possibility is clearly impossible, hence $U^{(\infty)} = SL_2(5)$. The only other a.i.m.f. groups into which U may embed are ${}_{\infty,2}[2.G_2(4)]_6$ and ${}_{\infty,2}[SL_2(3)]_1^6$. Since U is absolutely irreducible, the centralizer of $U^{(\infty)}$ in U is a nontrivial 2-group. One concludes that the 5-modular constituents of the absolutely irreducible group $U \circ SL_2(3) \leq GL_{24}(\mathbb{Q})$ are of degree 8 and 16. So $U \circ SL_2(3)$ does not fix a 5-unimodular $\mathbb{Z}$-lattice. □

Lemma 16.35. *The only maximal simplex in $M_6^{irr}(\mathcal{Q}_{\infty,2})$ with vertex $G := {}_{\infty,2}[3_+^{1+2} : SL_2(3) \overset{2(2)}{\boxtimes} D_8]_6$ is $(G, {}_{\infty,2}[2_-^{1+4}.Alt_5]_2^3)$.*

Proof. Let U be an absolutely irreducible subgroup of G. Then $U = V \overset{2(2)}{\boxtimes} D_8$, where $V \leq 3_+^{1+2} : SL_2(3)$ is an absolutely irreducible subgroup of $GL_3(\mathbb{Q}[\zeta_3])$. By Lemma 16.26 $O_3(V)$ either contains $O_3(G)$ or is 3_-^{1+2}. Let $\mathfrak{M}$ be a maximal order in $\mathcal{Q}_{\infty,2}$, and $SL_2(3)$ its unit group. The natural representation of $O_3(V) \otimes_{\sqrt{-3}} SL_2(3)$ has two different 2-modular constituents of degree 6. These are interchanged by the outer automorphism of $V \otimes D_8$ inducing the Galois automorphism on the center of $O_3(V)$. Therefore the 2-modular constituents of $U \circ SL_2(3)$ are of degree 12. So the only a.i.m.f. groups into which U may embed are $G, {}_{\infty,2}[2_-^{1+4}.Alt_5]_2^3$, and ${}_{\infty,2}[2.G_2(4)]_6$. Since the order of the latter group is not divisible by 3^4 and the normalizer of its Sylow 3-subgroup is not absolutely irreducible, the lemma follows. □

Lemma 16.36. *There is no common absolutely irreducible subgroup of ${}_{\infty,2}[SL_2(3)]_1 \otimes E_6$ or $({}_{\infty,2}[SL_2(3)]_1 \otimes A_2)^3$ and one of ${}_{\infty,2}[2.G_2(4)]_6$, ${}_{\infty,2}[2_-^{1+4}.Alt_5]_2^3$, or ${}_{\infty,2}[SL_2(3)]_1^6$.*

Proof. Let V be an absolutely irreducible subgroup of $E_6 \otimes {}_{\infty,2}[SL_2(3)]_1$ or $(A_2 \otimes {}_{\infty,2}[SL_2(3)]_1)^3$, $\mathfrak{M}$ a maximal order of $\mathcal{Q}_{\infty,2}$ and $U \cong SL_2(3)$ the unit group of $\mathfrak{M}$. Assume that V embeds into one of the last three groups of the lemma. Then the degrees of the 3-modular constituents of the natural representation of $V \circ U$ are not all divisible by 4. We claim that

$(\star)$ $\qquad (U \circ V)/(O_3(U \circ V))$ contains a normal 2-subgroup $\geq C_4 \circ Q_8$.

Assume first that $V \leq E_6 \otimes {}_{\infty,2}[SL_2(3)]_1$. Then the natural representation of V is a tensor product $\Delta_1 \otimes \Delta_2$ with $\Delta_1(V) \leq E_6 \leq GL_6(\mathbb{Q})$ and $\Delta_2(V) \leq SL_2(3) \leq$

$GL_1(\mathcal{Q}_{\infty,2})$ absolutely irreducible. Clearly, $\Delta_1(V)$ does not contain $U_4(2)$, hence is soluble and contained in one of the two absolutely irreducible maximal subgroups $\pm 3^{1+2}.2.S_4$ or $\pm 3^3 : (S_4 \times C_2)$ ([CCNPW 85]) of E_6. Moreover $\Delta_2(V)$, being absolutely irreducible, contains the normal two subgroup $Q_8 \leq SL_2(3)$. One concludes that $O_2(V/O_3(V))$ contains a subgroup C_4, hence $(\star)$.

Assume now that $V \leq ({}_{\infty,2}[SL_2(3)]_1 \otimes A_2)^3$. Then $\pi_1(V)$ is an absolutely irreducible subgroup of ${}_{\infty,2}[SL_2(3)]_1 \otimes A_2$, hence contains $S_3 \otimes Q_8$. The first component of the base group of V is a normal subgroup of index ≤ 2 in $\pi_1(U)$, hence contains C_4. Again one sees that $O_2(V/O_3(V))$ contains a subgroup C_4, hence $(\star)$.

Since $C_4 \circ Q_8$ is an irreducible subgroup of $GL_4(\mathbb{F}_3)$ the 3-modular constituents of $V \circ U$ have degree divisible by 4, which gives a contradiction. $\square$

Since the first two and the last three groups of the lemma above form a full simplex in $M_6^{irr}(\mathcal{Q}_{\infty,2})$, one now gets Proposition 16.31

17. The a.i.m.f. subgroups of $GL_7(\mathcal{Q})$

Theorem 17.1. *Let $\mathcal{Q}$ be a definite quaternion algebra with center $\mathbb{Q}$ and G be a primitive a.i.m.f. subgroup of $GL_7(\mathcal{Q})$. Then G is conjugate to one of the groups in the following table.*

List of the primitive a.i.m.f. subgroups of $GL_7(\mathcal{Q})$.

lattice L	$\|Aut(L)\|$	r.i.m.f. supergroups
${}_{\infty,2}[\pm U_3(3) \overset{2}{\square} C_4]_7$	$2^8 \cdot 3^3 \cdot 7$	$[U_3(3) \underset{\sqrt{-1}}{\overset{2}{\boxtimes}} (Q_8 \circ C_4).S_3]_{28}$
${}_{\infty,2}[SL_2(13)]_7$	$2^3 \cdot 3 \cdot 7 \cdot 13$	$[SL_2(13) \overset{2(2)}{\circ} SL_2(3)]_{28}$
${}_{\infty,2}[2.J_2]_7$	$2^8 \cdot 3^3 \cdot 5^2 \cdot 7$	$[2.J_2 \overset{2(2)}{\circ} SL_2(3)]_{28}$
${}_{\infty,2}[SL_2(3)]_1 \otimes E_7$	$2^{12} \cdot 3^5 \cdot 5 \cdot 7$	$F_4 \otimes E_7$
${}_{\infty,3}[\tilde{S}_3]_1 \otimes E_7$	$2^{11} \cdot 3^5 \cdot 5 \cdot 7$	$(A_2 \otimes E_7)^2$

Proof. Let $\mathcal{Q}$ be a definite quaternion algebra with center $\mathbb{Q}$ and G be a primitive a.i.m.f. subgroup of $GL_7(\mathcal{Q})$. Assume that $1 \neq N \trianglelefteq G$ is a quasi-semi-simple normal subgroup of G. With Table 9.1 one finds that $B := \mathcal{B}^\circ(N)$ is one of $SL_2(13)$, $U_3(3) \circ C_4$, $\pm S_6(2)$, or $2.J_2$. The centralizer $C := C_G(N) = C_G(B)$ in G of N embeds into the commuting algebra $C_{\mathcal{Q}^{7\times 7}}(N)$, which is isomorphic to $\mathbb{Q}$, $\mathbb{Q}[\sqrt{-1}]$, $\mathcal{Q}$, or $\mathbb{Q}$ in the respective cases. If $B = SL_2(13)$ or $B = 2.J_2$, the group B is already absolutely irreducible and one computes and concludes that $G = B$ is ${}_{\infty,2}[SL_2(13)]_7$ or ${}_{\infty,2}[2.J_2]_7$.

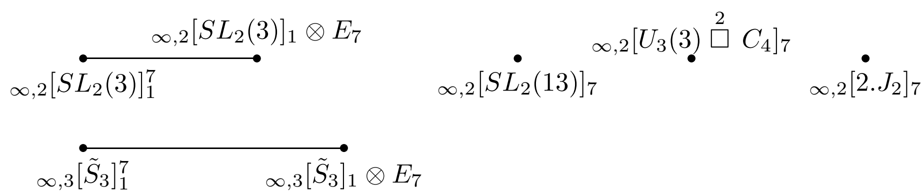
If $B = U_3(3) \circ C_4$, then $C \cong C_4$ is contained in B and G contains B of index $2 = |Out(N)|$. Since the commuting algebra of B is isomorphic to an imaginary quadratic field, one finds only one group $G = B.2$ in $GL_7(\mathcal{Q})$. Hence G is ${}_{\infty,2}[\pm U_3(3) \overset{2}{\square} C_4]_7$ in this case.

If $N = S_6(2)$, then $B = \pm S_6(2) = Aut(E_7)$ is tensor decomposing. With Theorem 6.1 one finds that G is conjugate to one of $E_7 \otimes {}_{\infty,2}[SL_2(3)]_1$ or $E_7 \otimes {}_{\infty,3}[\tilde{S}_3]_1$.

Now assume that G does not contain a quasi-semi-simple normal subgroup. Since there are no nilpotent groups having a character of degree 7 or 14, the largest nilpotent normal subgroup of G embeds into $GL_1(\mathcal{Q})$, hence is one of $\pm C_3$ or Q_8.

This leads to a contradiction, because both groups clearly have no automorphism of order 7. $\square$

Theorem 17.2. *Let $\mathcal{Q}$ be a definite quaternion algebra with center $\mathbb{Q}$ and G be an a.i.m.f. subgroup of $GL_7(\mathcal{Q})$. Then $\mathcal{Q}$ is one of $\mathcal{Q}_{\infty,2}$ or $\mathcal{Q}_{\infty,3}$. The simplicial complexes $M_7^{irr}(\mathcal{Q})$ are as follows:*



List of maximal simplices in $M_7^{irr}(\mathcal{Q}_{\infty,2})$:

simplex	a common subgroup
$({}_{\infty,2}[SL_2(3)]_1 \otimes E_7,\ {}_{\infty,2}[SL_2(3)]_1^7)$	$L_2(7) \otimes SL_2(3)$

List of maximal simplices in $M_7^{irr}(\mathcal{Q}_{\infty,3})$:

simplex	a common subgroup
$({}_{\infty,3}[\tilde{S}_3]_1 \otimes E_7,\ {}_{\infty,3}[\tilde{S}_3]_1^7)$	$L_2(7) \otimes \tilde{S}_3$

Proof. Theorems 17.1 and 6.1 prove the completeness of the list of quaternion algebras $\mathcal{Q}$ and of a.i.m.f. subgroups of $GL_7(\mathcal{Q})$. One only has to show the completeness of the list of maximal simplices in $M_7^{irr}(\mathcal{Q}_{\infty,2})$, because the simplicial complex $M_7^{irr}(\mathcal{Q}_{\infty,3})$ consists of a single simplex: The group $_{\infty,2}[SL_2(13)]_7$ fixes a $\mathbb{Z}$-lattice of determinant divisible by 13. So the minimal absolutely irreducible subgroups of the group $_{\infty,2}[SL_2(13)]_7$ are of order divisible by 13 (Lemma 2.13). Since the orders of the maximal subgroups of $L_2(13)$ are not divisible by $7 \cdot 13$, one concludes that $_{\infty,2}[SL_2(13)]_7$ is minimal absolutely irreducible. Hence $_{\infty,2}[SL_2(13)]_7$ forms a 0-simplex in $M_7^{irr}(\mathcal{Q}_{\infty,2})$. By [CCNPW 85] the maximal subgroups of $2.J_2$ of order divisible by 7 are $C_2 \times U_3(3)$ and $(C_2 \times L_2(7)).2$. Since the last group has no irreducible character of degree 14, and the unique irreducible character of $U_3(3)$ of degree 14 belongs to an orthogonal representation, one sees that $_{\infty,2}[2.J_2]_7$ is minimal absolutely irreducible. Similarly, the restriction of the characters χ_{7a} and χ_{7b} of $U_3(3)$ to a maximal subgroup of $U_3(3)$ become reducible, because these characters are constituents of the permutation character of $U_3(3)$ associated to its unique subgroup of order divisible by 7. One concludes that $(\pm U_3(3)).2$ is the unique minimal absolutely irreducible subgroup of $_{\infty,2}[U_3(3) \overset{2}{\Box} C_4]_7$. Since this subgroup does not embed into one of the other a.i.m.f. groups, $_{\infty,2}[U_3(3) \overset{2}{\Box} C_4]_7$ forms a component on its own in $M_7^{irr}(\mathcal{Q}_{\infty,2})$. The remaining two a.i.m.f. groups form a 1-simplex in $M_7^{irr}(\mathcal{Q}_{\infty,2})$, so the proof is complete. $\square$

18. The a.i.m.f. subgroups of $GL_8(\mathcal{Q})$

Theorem 18.1. *Let $\mathcal{Q}$ be a definite quaternion algebra with center $\mathbb{Q}$ and G a primitive a.i.m.f. subgroup of $GL_8(\mathcal{Q})$. Then G is conjugate to one of the groups listed in the following table:*

List of the primitive a.i.m.f. subgroups of $GL_8(\mathcal{Q})$.

lattice L	$\lvert Aut(L)\rvert$	r.i.m.f. supergroups
${}_{\infty,2}[2_-^{1+8}.O_8^-(2)]_8$	$2^{21}\cdot 3^4\cdot 5\cdot 7\cdot 17$	$[2_+^{1+10}.O_{10}^+(2)]_{32}$
${}_{\infty,2}[2_-^{1+4}.Alt_5]_2\otimes A_4$	$2^{10}\cdot 3^2\cdot 5^2$	$A_4\otimes E_8$
${}_{\infty,2}[SL_2(3)]_1\otimes A_4\otimes A_2$	$2^7\cdot 3^3\cdot 5$	$A_2\otimes A_4\otimes F_4$
${}_{\infty,2}[2_-^{1+6}.O_6^-(2)]_4\otimes A_2$	$2^{14}\cdot 3^5\cdot 5$	$A_2\otimes F_4\tilde{\otimes}F_4$
${}_{\infty,2}[SL_2(5)\overset{2(2)}{\boxtimes}D_8]_4\otimes A_2$	$2^7\cdot 3^2\cdot 5$	$A_2\otimes[SL_2(5)\underset{\infty,2}{\overset{2(2)}{\boxtimes}}2_-^{1+4'}.Alt_5]_{16}$
${}_{\infty,2}[SL_2(3)]_1\otimes E_8$	$2^{16}\cdot 3^6\cdot 5^2\cdot 7$	$F_4\otimes E_8$
${}_{\infty,2}[SL_2(3)]_1\otimes[(SL_2(5)\overset{2}{\square}SL_2(5)):2]_8$	$2^8\cdot 3^3\cdot 5^2$	$F_4\otimes[(SL_2(5)\overset{2}{\square}SL_2(5)):2]_8$
${}_{\infty,2}[SL_2(3)]_1\otimes M_{8,3}$	$2^7\cdot 3^2\cdot 7$	$F_4\otimes M_{8,3}$
${}_{\infty,2}[C_4\overset{2(3)}{\sqsubset\!\sqsupset}SL_2(7)]_8$	$2^6\cdot 3\cdot 7$	$[SL_2(3)\circ C_4.2\underset{\sqrt{-1}}{\overset{2(3)}{\boxtimes}}SL_2(7)]_{32}$
${}_{\infty,2}[Sp_4(3)\circ C_3\overset{2(2)}{\boxtimes}D_8]_8$	$2^{10}\cdot 3^6\cdot 5$	$[(Sp_4(3)\otimes_{\sqrt{-3}}Sp_4(3)):2\overset{2}{\square}C_3]_{32}$
${}_{\infty,2}[SL_2(5)\overset{2(2)}{\boxtimes}F_4]_8$	$2^{10}\cdot 3^3\cdot 5$	$[SL_2(5)\underset{\infty,2}{\overset{2(2)}{\boxtimes}}2_-^{1+6}.O_6^-(2)]_{32}$
${}_{\infty,2}[SL_2(5)\overset{2(3)}{\boxtimes}(C_3\overset{2(2)}{\boxtimes}D_8)]_8$	$2^7\cdot 3^2\cdot 5$	$[SL_2(5)\underset{\infty,3}{\overset{2(3)}{\boxtimes}}(Sp_4(3)\overset{2}{\square}C_3)]_{32}$
${}_{\infty,2}[SL_2(5)\underset{\sqrt{5}}{\overset{2(2)}{\boxtimes}}(D_{10}\otimes D_8)]_8$	$2^7\cdot 3\cdot 5^2$	$[2_-^{1+4}.Alt_5\otimes_{\infty,2}SL_2(5)\underset{\sqrt{5}}{\overset{2}{\boxtimes}}D_{10}]_{32}$
${}_{\infty,2}[D_{10}\overset{2(6)}{\boxtimes}(C_3\overset{2}{\square}SL_2(3))]_8$	$2^6\cdot 3^2\cdot 5$	$[C_{15}:C_4\overset{2(2)}{\boxtimes}F_4]_{32}$
${}_{\infty,3}[\tilde{S}_3]_1\otimes E_8$	$2^{15}\cdot 3^6\cdot 5^2\cdot 7$	$(A_2\otimes E_8)^2$
${}_{\infty,3}[\tilde{S}_3]_1\otimes[(SL_2(5)\overset{2}{\square}SL_2(5)):2]_8$	$2^7\cdot 3^3\cdot 5^2$	$(A_2\otimes[(SL_2(5)\overset{2}{\square}SL_2(5)):2]_8)^2$
${}_{\infty,3}[\tilde{S}_3]_1\otimes M_{8,3}$	$2^6\cdot 3^2\cdot 7$	$(A_2\otimes M_{8,3})^2$
${}_{\infty,3}[SL_2(9)]_2\otimes A_4$	$2^7\cdot 3^3\cdot 5^2$	$A_4\otimes E_8$
${}_{\infty,3}[SL_2(9)]_2\otimes F_4$	$2^{11}\cdot 3^4\cdot 5$	$F_4\otimes E_8$
${}_{\infty,3}[SL_2(3)\overset{2}{\square}C_3]_2\otimes A_4$	$2^7\cdot 3^3\cdot 5$	$(A_4\otimes F_4)^2$
${}_{\infty,3}[Sp_4(3)\circ C_3\underset{\sqrt{-3}}{\overset{2(2)}{\boxtimes}}SL_2(3)]_8$	$2^{10}\cdot 3^6\cdot 5$	$[Sp_4(3)\circ C_3\underset{\sqrt{-3}}{\overset{2(2)}{\boxtimes}}SL_2(3)]_{16}^2$
${}_{\infty,3}[SL_2(7)\overset{2(3)}{\otimes}S_3]_8$	$2^6\cdot 3^2\cdot 7$	$[SL_2(7)\overset{2(3)}{\circ}\tilde{S}_3]_{16}^2$
${}_{\infty,3}[SL_2(17)]_8$	$2^5\cdot 3^2\cdot 17$	$[SL_2(17)\overset{2(3)}{\circ}\tilde{S}_3]_{16}$
${}_{\infty,3}[SL_2(7)\overset{2(3)}{\boxtimes}S_3]_8$	$2^6\cdot 3^2\cdot 7$	$[SL_2(7)\underset{\sqrt{-7}}{\overset{2(3)}{\boxtimes}}\tilde{S}_3]_{16}^2$
${}_{\infty,3}[2.Alt_7\overset{2(3)}{\boxtimes}S_3]_8$	$2^6\cdot 3^3\cdot 5\cdot 7$	$[2.Alt_7\underset{\sqrt{-7}}{\overset{2(3)}{\boxtimes}}\tilde{S}_3]_{16}^2$
${}_{\infty,3}[SL_2(5)\overset{2(3)}{\boxtimes}(SL_2(3)\overset{2}{\square}C_3)]_8$	$2^7\cdot 3^3\cdot 5$	$[SL_2(5)\underset{\infty,3}{\overset{2(3)}{\boxtimes}}(SL_2(3)\overset{2}{\square}C_3)]_{16}^2$
${}_{\infty,3}[SL_2(5)\overset{2(2)}{\boxtimes}(C_3\overset{2(2)}{\boxtimes}D_8)]_8$	$2^7\cdot 3^2\cdot 5$	$[SL_2(5)\underset{\infty,2}{\overset{2(2)}{\boxtimes}}2_-^{1+4}.Alt_5]_{16}^2$
${}_{\infty,3}[C_3\overset{2(2)}{\sqsubset\!\sqsupset}2_-^{1+6}.O_6^-(2)]_8$	$2^{14}\cdot 3^5\cdot 5$	$(F_4\tilde{\otimes}F_4)^2$
${}_{\infty,3}[SL_2(5)\underset{\sqrt{5}}{\overset{2(3)}{\boxtimes}}(D_{10}\otimes S_3)]_8$	$2^6\cdot 3^2\cdot 5^2$	$[(SL_2(5)\circ SL_2(5)):2\underset{\sqrt{5}}{\overset{2}{\boxtimes}}D_{10}]_{16}^2$
${}_{\infty,3}[D_{10}\overset{2(6)}{\boxtimes}(C_3\overset{2(2)}{\boxtimes}D_8)]_8$	$2^6\cdot 3\cdot 5$	$[D_{120}.(C_4\times C_2)]_{16}^2$

lattice L	$\lvert Aut(L)\rvert$	r.i.m.f. supergroups
$_{\infty,5}[2.S_6]_8$	$2^5\cdot 3^2\cdot 5$	$[SL_2(9)\overset{2}{\Box}SL_2(5)]_{32}$
$_{\infty,5}[SL_2(5).2]_2\otimes F_4$	$2^{10}\cdot 3^3\cdot 5$	$F_4\otimes E_8$
$_{\infty,5}[SL_2(5):2]_2\otimes F_4$	$2^{10}\cdot 3^3\cdot 5$	$F_4\otimes[(SL_2(5)\overset{2}{\Box}SL_2(5)):2]_8$
$_{\infty,5}[D_{10}\overset{2(3)}{\boxtimes}SL_2(9)]_8$	$2^6\cdot 3^2\cdot 5^2$	$[SL_2(9)\otimes D_{10}\overset{2}{\Box}SL_2(5)]_{32}$
$_{\infty,5}[SL_2(5)\underset{\sqrt{5}}{\overset{2}{\boxtimes}}D_{10}]_4\otimes A_2$	$2^6\cdot 3^2\cdot 5^2$	$A_2\otimes[(SL_2(5)\circ SL_2(5)):2\underset{\sqrt{5}}{\overset{2}{\boxtimes}}D_{10}]_{16}$
$_{\infty,5}[((SL_2(5)\circ SL_2(5))\underset{\sqrt{5}}{\overset{2}{\boxtimes}}SL_2(5)):S_3]_{8,1}$	$2^9\cdot 3^4\cdot 5^3$	G_1
$_{\infty,5}[((SL_2(5)\circ SL_2(5))\underset{\sqrt{5}}{\overset{2}{\boxtimes}}SL_2(5)):S_3]_{8,2}$	$2^9\cdot 3^4\cdot 5^3$	G_2
$_{\infty,5}[SL_2(5)\overset{2(6)}{\boxtimes}(C_3\overset{2(2)}{\boxtimes}D_8)]_{8,1}$	$2^7\cdot 3^2\cdot 5$	$[(SL_2(5)\circ SL_2(5)):2\underset{\sqrt{5}}{\overset{2(6)}{\boxtimes}}(C_3\overset{2(2)}{\boxtimes}D_8)]_{32,1}$
$_{\infty,5}[SL_2(5)\overset{2(6)}{\boxtimes}(C_3\overset{2(2)}{\boxtimes}D_8)]_{8,2}$	$2^7\cdot 3^2\cdot 5$	$[(SL_2(5)\circ SL_2(5)):2\underset{\sqrt{5}}{\overset{2(6)}{\boxtimes}}(C_3\overset{2(2)}{\boxtimes}D_8)]_{32,2}$
$_{\infty,5}[D_{10}\overset{2(2)}{\boxtimes}2_-^{1+4}.Alt_5]_8$	$2^9\cdot 3\cdot 5^2$	$[(2_-^{1+4}.Alt_5\otimes_{\infty,2}SL_2(5))\underset{\sqrt{5}}{\overset{2(2)}{\boxtimes}}D_{10}]_{32}$
$_{\infty,5}[D_{10}\overset{2(3)}{\boxtimes}(C_3\overset{2}{\Box}SL_2(3))]_8$	$2^6\cdot 3^2\cdot 5$	$[(SL_2(5)\otimes_{\sqrt{5}}D_{10})\underset{\infty,3}{\overset{2(3)}{\boxtimes}}(SL_2(3)\overset{2}{\Box}C_3)]_{32}$
$_{\infty,5}[SL_2(5).2]_2\otimes_{\sqrt{-3}}\ _{\infty,3}[\tilde{S}_3]_1\otimes_{\sqrt{-3}}\ _{\infty,2}[SL_2(3)]_1$	$2^7\cdot 3^3\cdot 5$	$[Sp_4(3)\circ C_3\underset{\sqrt{-3}}{\overset{2(2)}{\boxtimes}}SL_2(3)]_{16}^2$
$_{\infty,5}[SL_2(5):2]_2\otimes_{\sqrt{-3}}\ _{\infty,3}[\tilde{S}_3]_1\otimes_{\sqrt{-3}}\ _{\infty,2}[SL_2(3)]_1$	$2^7\cdot 3^3\cdot 5$	$[SL_2(5)\underset{\infty,3}{\overset{2(3)}{\boxtimes}}(SL_2(3)\overset{2}{\Box}C_3)]_{16}^2$
$_{\infty,5}[SL_2(5).2]_2\otimes_{\sqrt{-3}}\ _{2,3}[C_3\overset{2(2)}{\boxtimes}D_8]_2$	$2^7\cdot 3^2\cdot 5$	E_8^4
$_{\infty,5}[SL_2(5):2]_2\otimes_{\sqrt{-3}}\ _{2,3}[C_3\overset{2(2)}{\boxtimes}D_8]_2$	$2^7\cdot 3^2\cdot 5$	$[(SL_2(5)\overset{2}{\Box}SL_2(5)):2]_8^4$
$_{\infty,2,3,5}[SL_2(5)\overset{2(6)}{\boxtimes}(S_3\otimes D_8)]_{8,1}$	$2^7\cdot 3^2\cdot 5$	$(F_4\tilde{\otimes}F_4)^2$
$_{\infty,2,3,5}[SL_2(5)\overset{2(6)}{\boxtimes}(S_3\otimes D_8)]_{8,2}$	$2^7\cdot 3^2\cdot 5$	$[SL_2(5)\underset{\infty,2}{\overset{2(2)}{\boxtimes}}2_-^{1+4}.Alt_5]_{16}^2$
$_{\infty,2,3,5}[SL_2(5)\overset{2(6)}{\boxtimes}(SL_2(3)\overset{2}{\Box}C_3)]_{8,1}$	$2^7\cdot 3^3\cdot 5$	$(A_2\otimes E_8)^2$
$_{\infty,2,3,5}[SL_2(5)\overset{2(6)}{\boxtimes}(SL_2(3)\overset{2}{\Box}C_3)]_{8,2}$	$2^7\cdot 3^3\cdot 5$	$(A_2\otimes[(SL_2(5)\overset{2}{\Box}SL_2(5)):2]_8)^2$
$_{3,5}[\pm D_{10}\overset{2(3)}{\boxtimes}S_3]_4\otimes_{\sqrt{-3}}\ _{\infty,2}[SL_2(3)]_1$	$2^6\cdot 3^2\cdot 5$	$(A_4\otimes F_4)^2$
$_{\infty,2,3,5}[D_{10}\overset{2(2)}{\boxtimes}(C_3\overset{2}{\Box}SL_2(3))]_8$	$2^6\cdot 3^2\cdot 5$	$[(SL_2(5)\circ SL_2(5)):2\underset{\sqrt{5}}{\overset{2}{\boxtimes}}D_{10}]_{16}^2$
$_{2,5}[D_{10}\overset{2(2)}{\boxtimes}D_8]_4\otimes_{\sqrt{-3}}\ _{\infty,3}[\tilde{S}_3]_1$	$2^6\cdot 3\cdot 5$	$[D_{120}.(C_4\times C_2)]_{16}^2$
$_{\infty,2,3,5}[D_{10}\overset{2(3)}{\boxtimes}(C_3\overset{2(2)}{\boxtimes}D_8)]_8$	$2^6\cdot 3\cdot 5$	$(A_2\otimes A_4)^2$

Here for $i=1,2$ $G_i:=[((SL_2(5)\circ SL_2(5))\underset{\sqrt{5}}{\overset{2}{\boxtimes}}(SL_2(5)\circ SL_2(5))):S_4]_{32,i}$.

lattice L	$\lvert Aut(L)\rvert$	r.i.m.f. supergroups
${}_{\infty,7}[SL_2(7).2]_4 \otimes A_2$	$2^6 \cdot 3^2 \cdot 7$	$(A_2 \otimes E_8)^2$
${}_{\infty,7}[SL_2(7) \overset{2(2)}{\boxtimes} D_8]_8$	$2^7 \cdot 3 \cdot 7$	$(F_4 \tilde{\otimes} F_4)^2$
${}_{\infty,7}[SL_2(7) \underset{\sqrt{-7}}{\overset{2(3)}{\boxtimes}} \tilde{S}_3]_8$	$2^6 \cdot 3^2 \cdot 7$	$[SL_2(7) \underset{\sqrt{-7}}{\overset{2(3)}{\boxtimes}} \tilde{S}_3]_{16}^2$
${}_{\infty,7}[2.S_7]_4 \otimes A_2$	$2^6 \cdot 3^3 \cdot 5 \cdot 7$	$(A_2 \otimes E_8)^2$
${}_{\infty,7}[Alt_7 \underset{\sqrt{-7}}{\overset{2(3)}{\boxtimes}} \tilde{S}_3]_8$	$2^6 \cdot 3^3 \cdot 5 \cdot 7$	$[Alt_7 \underset{\sqrt{-7}}{\overset{2(3)}{\boxtimes}} \tilde{S}_3]_{16}^2$
${}_{\infty,7}[Alt_7 \overset{2(3)}{\boxtimes} D_8]_8$	$2^7 \cdot 3^2 \cdot 5 \cdot 7$	$(F_4 \tilde{\otimes} F_4)^2$
${}_{\infty,17}[SL_2(17).2]_8$	$2^6 \cdot 3^2 \cdot 17$	$[SL_2(17) \overset{2(3)}{\sqsupset} \tilde{S}_3]_{32,1}$
		$[SL_2(17) \overset{2(3)}{\sqsupset} \tilde{S}_3]_{32,1}$
${}_{\infty,17}[SL_2(17) : 2]_8$	$2^6 \cdot 3^2 \cdot 17$	$[SL_2(17) \overset{2(3)}{\sqsupset} \tilde{S}_3]_{32,2}$
		$[SL_2(17) \overset{2(3)}{\sqsupset} \tilde{S}_3]_{32,2}$

The proof is split into several lemmata. For the rest of this chapter let $\mathcal{Q}$ be a definite quaternion algebra with center $\mathbb{Q}$ and G be a primitive a.i.m.f. subgroup of $GL_8(\mathcal{Q})$.

By Table 9.1 and Lemma 7.2 the possibilities for quasi-semi-simple normal subgroups N of G are $SL_2(5)$, $SL_2(5) \circ SL_2(5)$, $SL_2(5) \circ SL_2(5) \otimes_{\sqrt{5}} SL_2(5)$, Alt_5, $L_2(7)$, $SL_2(7)$ (2 matrix groups), $SL_2(9)$ (2 matrix groups), $SL_2(17)$ (2 matrix groups), $2.Alt_7$, $Sp_4(3) = 2.U_4(2)$, and $2.O_8^+(2)$.

First we treat the tensor decomposing normal subgroups N.

Lemma 18.2. *If G contains a normal subgroup $N \cong Alt_5$, then G is one of*

$$ {}_{\infty,2}[2_-^{1+4}.Alt_5]_2 \otimes A_4,\ {}_{\infty,2}[SL_2(3)]_1 \otimes A_2 \otimes A_4, $$

$$ {}_{\infty,3}[SL_2(3) \overset{2}{\square} C_3]_2 \otimes A_4,\ \textit{or}\ {}_{\infty,3}[SL_2(9)]_2 \otimes A_4. $$

Proof. By Corollary 7.6 G is of the form $A_4 \otimes H$, where $H \leq GL_2(\mathcal{Q})$ is a primitive a.i.m.f. group. Hence by Theorem 12.1 H is one of

$$ {}_{\infty,2}[(D_8 \otimes Q_8).Alt_5]_2,\ {}_{\infty,2}[SL_2(3)]_1 \otimes A_2,\ {}_{\infty,3}[SL_2(9)]_2, $$

$$ {}_{\infty,3}[SL_2(3) \overset{2}{\square} C_3]_2,\ {}_{\infty,5}[SL_2(5).2]_2 \text{ or } {}_{\infty,5}[SL_2(5){:}2]_2. $$

In the last two cases G is a proper subgroup of

$$ {}_{\infty,5}[((SL_2(5) \circ SL_2(5)) \underset{\sqrt{5}}{\overset{2}{\boxtimes}} SL_2(5)) : S_3]_{8,1} $$

resp. ${}_{\infty,5}[((SL_2(5) \circ SL_2(5)) \underset{\sqrt{5}}{\overset{2}{\boxtimes}} SL_2(5)) : S_3]_{8,2}$ □

Similarly one gets the next two lemmata:

Lemma 18.3. *If G contains a normal subgroup $N \cong L_2(7)$, then G is one of* ${}_{\infty,2}[SL_2(3)]_1 \otimes M_{8,3}$ *or* ${}_{\infty,3}[\tilde{S}_3]_1 \otimes M_{8,3}$.

Lemma 18.4. *If G contains a normal subgroup $N \cong 2.O_8^+(2)$, then G is one of* ${}_{\infty,2}[SL_2(3)]_1 \otimes E_8$ *or* ${}_{\infty,3}[\tilde{S}_3]_1 \otimes E_8$.

The next lemma deals with the absolutely irreducible quasi-semi-simple normal subgroups N:

Lemma 18.5. *If G contains a normal subgroup N isomorphic to $SL_2(17)$ with character χ_{16}, then G is conjugate to* ${}_{\infty,3}[SL_2(17)]_8$.

Proof. N is already absolutely irreducible. One finds $G = \mathcal{B}^\circ(N)$. □

Next we treat those candidates for normal subgroups N in G, such that $C_G(N)$ has to be contained in $\mathcal{B}^\circ(N)$.

Lemma 18.6. *If G has a normal subgroup N isomorphic to one of $SL_2(9)$, $SL_2(5) \circ SL_2(5) \otimes_{\sqrt{5}} SL_2(5)$, or $SL_2(17)$ with character $\chi_{8a} + \chi_{8b}$, then G is conjugate to one of*

$$ {}_{\infty,5}[2.S_6],\ {}_{\infty,5}[((SL_2(5) \circ SL_2(5)) \overset{2}{\underset{\sqrt{5}}{\boxtimes}} SL_2(5)) : S_3]_{8,1}, $$

$$ {}_{\infty,5}[((SL_2(5) \circ SL_2(5)) \overset{2}{\underset{\sqrt{5}}{\boxtimes}} SL_2(5)) : S_3]_{8,2},\ {}_{\infty,17}[SL_2(17).2]_8 \text{ or } {}_{\infty,17}[SL_2(17) : 2]_8. $$

Proof. In all cases the centralizer $C_G(N)$ embeds into the enveloping algebra of N and hence is contained in $\mathcal{B}^\circ(N)$. Assume first, that N is isomorphic to $SL_2(9)$. Since the character field of the extension of the character χ_{8a} to $2.PGL_2(9)$ is of degree 4 over $\mathbb{Q}$ (cf. [CCNPW 85]), G is isomorphic to $2.S_6$. (Note that the outer automorphism of S_6 interchanges the two isoclinism classes of groups $2.S_6$, so there is only one group to be considered.) Hence $G = {}_{\infty,5}[2.S_6]$.

If $N = SL_2(5) \circ SL_2(5) \otimes_{\sqrt{5}} SL_2(5)$, then $\mathcal{B}^\circ(N) = N : S_3$ and one computes that G is one of the two extensions $(N : S_3).2$. In the last case, $G = N.2$ is one of the two extensions of $SL_2(17)$ by $Out(SL_2(17)) \cong C_2$. □

Lemma 18.7. *If G contains a normal subgroup $N \cong SL_2(7)$ with character χ_8, then G is one of* ${}_{\infty,3}[SL_2(7) \overset{2(3)}{\otimes} S_3]_8$ *or* ${}_{\infty,2}[C_4 \overset{2(3)}{\sqsupset} SL_2(7)]_8$.

Proof. By Table 9.1 the group N is nearly tensor decomposing over $\mathbb{Q}$ with parameter 3. Since $\mathcal{B}^\circ(N) = N$ by 10.1 G is either $G = NC$ where $C := C_G(N)$ is an a.i.m.f. subgroup of $GL_1(\mathcal{D})$, where $\mathcal{D}$ is an indefinite quaternion algebra with center $\mathbb{Q}$ such that $(C, 3, \mathcal{D})$ is not a maximal triple or of the form $B \overset{2(3)}{\otimes} C$ or $B \overset{2(3)}{\boxtimes} C$ where $(C, 3, \mathcal{D})$ is a maximal triple. Since the group $SL_2(7) \otimes D_8$ is imprimitive using Table 10.2 one finds that G is one of ${}_{\infty,3}[SL_2(7) \overset{2(3)}{\otimes} S_3]_8$ or ${}_{\infty,2}[C_4 \overset{2(3)}{\sqsupset} SL_2(7)]_8$. □

Lemma 18.8. *If G contains a normal subgroup $N \cong SL_2(9)$ with character χ_4, then G is one of* ${}_{\infty,3}[SL_2(9)]_2 \otimes A_4$, ${}_{\infty,3}[SL_2(9)]_2 \otimes F_4$, *or* ${}_{\infty,5}[D_{10} \overset{2(3)}{\boxtimes} SL_2(9)]_8$.

Proof. As in the last lemma N is nearly tensor decomposing over $\mathbb{Q}$ with parameter 3. Hence G is either $G = NC$ where $C := C_G(N)$ is an a.i.m.f. subgroup of $GL_2(\mathcal{D})$, where $\mathcal{D}$ is an indefinite quaternion algebra with center $\mathbb{Q}$ such that $(C, 3, \mathcal{D})$ is not a maximal triple or of the form $B \overset{2(3)}{\otimes} C$ or $B \overset{2(3)}{\boxtimes} C$ where $(C, 3, \mathcal{D})$ is a maximal triple. Moreover $\mathcal{B}^\circ(SL_2(9) \circ C_3) = Sp_4(3) \circ C_3$ implies that in both cases $O_3(C) = 1$. Since the group ${}_{\infty,2}[(C_4 \circ SL_2(3)).2 \underset{\sqrt{-1}}{\overset{2(3)}{\boxtimes}} SL_2(9)]_8$ is contained in ${}_{\infty,2}[2_-^{1+8}.O_8^-(2)]_8$, Table 10.4 implies that G is one of the three a.i.m.f. groups in the Lemma. □

In the next three cases, the center of the enveloping algebra of N is an imaginary quadratic field.

Lemma 18.9. *If G has a normal subgroup $N \cong SL_2(7)$ with character $\chi_{4a} + \chi_{4b}$, then G is conjugate to one of ${}_{\infty,3}[SL_2(7) \overset{2(3)}{\boxtimes} S_3]_8$, ${}_{\infty,7}[SL_2(7).2]_4 \otimes A_2$, ${}_{\infty,7}[SL_2(7) \overset{2(2)}{\boxtimes} D_8]_8$, or ${}_{\infty,7}[SL_2(7) \underset{\sqrt{-7}}{\overset{2(3)}{\boxtimes}} \tilde{S}_3]_8$.*

Proof. The centralizer $C := C_G(N)$ is a centrally irreducible subgroup of $GL_1(\mathcal{D})$ where $\mathcal{D}$ is a quaternion algebra over $\mathbb{Q}[\sqrt{-7}]$ and G contains NC of index 2. Using the classification of finite subgroups of $GL_2(\mathbb{C})$ in [Bli 17], one finds that C is one of D_8, $\pm S_3$, $SL_2(3)$, or $\tilde{S}_3$. Distinguish 2 cases:

a) $C_G(C) > \pm N$. Then $G = (N.2)C$ is one of ${}_{\infty,7}[SL_2(7).2]_4 \otimes [D_8]_2$ or ${}_{\infty,7}[SL_2(7).2]_4 \otimes A_2$. In the first case, G is imprimitive contained in ${}_{\infty,7}[SL_2(7).2]_4^2$.

b) $C_G(C) = \pm N$. The groups C are nearly tensor decomposing over $\mathbb{Q}$ with parameter 2, 3, 2, resp. 3. Since $C = C_G(N)$ Lemma 10.1 implies that G is one of ${}_{\infty,7}[SL_2(7) \overset{2(2)}{\boxtimes} D_8]_8$, ${}_{\infty,2}[SL_2(7) \overset{2(2)}{\boxtimes} SL_2(3)]_8$, ${}_{\infty,3}[SL_2(7) \overset{2(3)}{\boxtimes} S_3]_8$, or ${}_{\infty,7}[SL_2(7) \underset{\sqrt{-7}}{\overset{2(3)}{\boxtimes}} \tilde{S}_3]_8$. The second group fixes a 32-dimensional extremal unimodular lattice with maximal order as an endomorphism ring. By [BaN 97] this yields that the second group is contained in ${}_{\infty,2}[2_-^{1+8}.O_8^-(2)]_8$. □

Completely analogous one finds:

Lemma 18.10. *If G has a normal subgroup $N \cong 2.Alt_7$ with character $\chi_{4a} + \chi_{4b}$, then G is conjugate to one of*

$$ {}_{\infty,3}[2.Alt_7 \overset{2(3)}{\boxtimes} S_3]_8,\ {}_{\infty,7}[2.S_7]_4 \otimes A_2, $$

$$ {}_{\infty,7}[2.Alt_7 \overset{2(2)}{\boxtimes} D_8]_8 \ \text{or}\ {}_{\infty,7}[2.Alt_7 \underset{\sqrt{-7}}{\overset{2(3)}{\boxtimes}} \tilde{S}_3]_8. $$

Lemma 18.11. *If G has a normal subgroup $N \cong 2.U_4(2) = Sp_4(3)$ with character $\chi_{4a} + \chi_{4b}$, then G is conjugate to*

$$ {}_{\infty,2}[Sp_4(3) \circ C_3 \overset{2(2)}{\boxtimes} D_8]_8 \ \text{or}\ {}_{\infty,3}[Sp_4(3) \circ C_3 \underset{\sqrt{-3}}{\overset{2(2)}{\boxtimes}} SL_2(3)]_8. $$

Proof. Let $B := \mathcal{B}^\circ(N) \cong \pm C_3 \circ N$. The centralizer $C := C_G(N)$ is a centrally irreducible subgroup of $GL_1(\mathcal{D})$ where $\mathcal{D}$ is a quaternion algebra over $\mathbb{Q}[\sqrt{-3}]$ and G contains BC of index 2. Moreover $O_3(B) = C_3$ implies $O_3(C) = 1$. Hence C is either D_8 or $SL_2(3)$ and the lemma follows as above. □

Lemma 18.12. *If $N = SL_2(5) \circ SL_2(5)$ is a normal subgroup of G, then G is one of ${}_{\infty,2}[SL_2(3)]_1 \otimes [(SL_2(5) \overset{2}{\square} SL_2(5)) : 2]_8$ or ${}_{\infty,3}[\tilde{S}_3]_1 \otimes [(SL_2(5) \overset{2}{\square} SL_2(5)) : 2]_8$.*

Proof. G contains the normal subgroup $B := \mathcal{B}^{\circ}(N) = N : 2$. The centralizer $C := C_G(N)$ embeds into $\mathbb{Q}[\sqrt{5}] \otimes \mathcal{Q}$. Since the primes dividing $|G|$ are ≤ 5, $\mathbb{Q}[\sqrt{5}]$ splits all possible Schur indices of $\mathcal{Q}$ at a finite prime. Moreover G contains CB of index $2 = [\mathbb{Q}[\sqrt{5}] : \mathbb{Q}]$ and C is a centrally irreducible subgroup of $GL_1(\mathcal{Q}_{\sqrt{5},\infty})$. Hence C is one of $SL_2(5)$, $SL_2(3)$, $\tilde{S}_3$, or Q_{20}. The first case contradicts Lemma 18.6. The lemma follows since the groups $(SL_2(5) \circ SL_2(5)) : 2 \overset{2(2)}{\boxtimes} SL_2(3)$ (2 extensions), $(SL_2(5) \circ SL_2(5)) : 2 \overset{2(3)}{\boxtimes} \tilde{S}_3$ (2 extensions), $(SL_2(5) \circ SL_2(5)) : 2 \underset{\sqrt{5}}{\overset{2}{\boxtimes}} Q_{20}$ are contained in one of the groups ${}_{\infty,5}[((SL_2(5) \circ SL_2(5)) \underset{\sqrt{5}}{\overset{2}{\boxtimes}} SL_2(5)) : S_3]_{8,i}$ $(i = 1, 2)$. $\square$

Lemma 18.13. *If $N := SL_2(5)$ is the only quasi-semi-simple normal subgroup of G, then G is one of the following twenty a.i.m.f. groups*

$$
\begin{gathered}
{}_{\infty,5}[SL_2(5).2]_2 \otimes F_4, \quad {}_{\infty,5}[SL_2(5):2]_2 \otimes F_4, \\
{}_{\infty,5}[SL_2(5).2]_2 \otimes_{\sqrt{-3}} {}_{\infty,3}[\tilde{S}_3]_1 \otimes_{\sqrt{-3}} {}_{\infty,2}[SL_2(3)]_1, \\
{}_{\infty,5}[SL_2(5):2]_2 \otimes_{\sqrt{-3}} {}_{\infty,3}[\tilde{S}_3]_1 \otimes_{\sqrt{-3}} {}_{\infty,2}[SL_2(3)]_1, \\
{}_{\infty,5}[SL_2(5).2]_2 \otimes_{\sqrt{-3}} {}_{2,3}[C_3 \overset{2(2)}{\boxtimes} D_8]_2, \quad {}_{\infty,5}[SL_2(5):2]_2 \otimes_{\sqrt{-3}} {}_{2,3}[C_3 \overset{2(2)}{\boxtimes} D_8]_2, \\
{}_{\infty,2}[SL_2(5) \overset{2(2)}{\boxtimes} F_4]_8, \quad {}_{\infty,2,3,5}[SL_2(5) \overset{2(6)}{\boxtimes} (SL_2(3) \overset{2}{\square} C_3)]_{8,1}, \\
{}_{\infty,2,3,5}[SL_2(5) \overset{2(6)}{\boxtimes} (SL_2(3) \overset{2}{\square} C_3)]_{8,2}, \quad {}_{\infty,2}[SL_2(5) \overset{2(2)}{\boxtimes} D_8]_4 \otimes A_2, \\
{}_{\infty,3}[SL_2(5) \overset{2(3)}{\boxtimes} (SL_2(3) \overset{2}{\square} C_3)]_8, \quad {}_{\infty,2,3,5}[SL_2(5) \overset{2(6)}{\boxtimes} (S_3 \otimes D_8)]_{8,1}, \\
{}_{\infty,2,3,5}[SL_2(5) \overset{2(6)}{\boxtimes} (S_3 \otimes D_8)]_{8,2}, \quad {}_{\infty,3}[SL_2(5) \overset{2(2)}{\boxtimes} (C_3 \overset{2(2)}{\boxtimes} D_8)]_8, \\
{}_{\infty,2}[SL_2(5) \overset{2(3)}{\boxtimes} (C_3 \overset{2(2)}{\boxtimes} D_8)]_8, \quad {}_{\infty,5}[SL_2(5) \overset{2(6)}{\boxtimes} (C_3 \overset{2(2)}{\boxtimes} D_8)]_{8,1}, \\
{}_{\infty,5}[SL_2(5) \overset{2(6)}{\boxtimes} (C_3 \overset{2(2)}{\boxtimes} D_8)]_{8,2}, \quad {}_{\infty,5}[SL_2(5) \underset{\sqrt{5}}{\overset{2}{\boxtimes}} D_{10}]_4 \otimes A_2, \\
{}_{\infty,3}[SL_2(5) \underset{\sqrt{5}}{\overset{2(3)}{\boxtimes}} (D_{10} \otimes S_3)]_8 \textit{ or } {}_{\infty,2}[SL_2(5) \underset{\sqrt{5}}{\overset{2(2)}{\boxtimes}} (D_{10} \otimes D_8)]_8.
\end{gathered}
$$

Proof. Let $C := C_G(N)$. Then C embeds into $\mathbb{Q}[\sqrt{5}]^{4\times 4}$ (again since $\mathbb{Q}[\sqrt{5}]$ splits all possible finite Schur indices) and G contains the group NC of index 2. By 2.14 C is a centrally irreducible subgroup of $GL_4(\mathbb{Q}[\sqrt{5}])$. Distinguish two cases:

a) $C_G(C) > N$. Then $C_G(C) = N.2$ is one of the two extensions of N by $Out(N)$ and $G = N.2 \otimes C$, where C is an a.i.m.f. subgroup of $\mathcal{Q}_{\infty,5} \otimes \mathcal{Q}$. Hence C is either a r.i.m.f. subgroup of $GL_4(\mathbb{Q})$, thus $C = F_4$ by Lemma 18.2 or a 3-parametric irreducible Bravais group in $GL_8(\mathbb{Q})$. By [Sou 94] C is one of $SL_2(3) \otimes_{\sqrt{-3}} \tilde{S}_3$ (B_{21}) or $C_3 \overset{2(2)}{\boxtimes} D_8$ $(B_{19} \sim B_{20})$. Hence G is one of the first six groups of the lemma.

b) $C_G(C) = N$. The groups C will be constructed according to their possible normal p-subgroups:

(i) Assume first that $O_3(C) = O_5(C) = 1$. Then Table 8.7 together with the central irreducibility of C implies that $O_2(C) = Q_8 \circ Q_8$ and G contains $N \otimes F_4$

of index 2. Moreover the elements in $G - CN$ induce an outer automorphism of F_4. Since one of the two extensions $SL_2(5) \overset{2(2)}{\boxtimes} F_4$ embeds into ${}_{\infty,2}[2^{1+8}_{-}.O_8^{-}(2)]_8$ the group G is ${}_{\infty,2}[SL_2(5) \overset{2(2)}{\boxtimes} F_4]_8$ in this case.

(ii) Now assume that $O_3(C) > 1$ and $O_5(C) = 1$. Then $O_3(C) \cong C_3$ and $C_C(O_3(C))$ is a centrally irreducible subgroup of $GL_2(\mathbb{Q}[\sqrt{5}, \zeta_3])$. Hence $O_2(C)$ is one of Q_8 or D_8 and C is one of $SL_2(3) \otimes_{\sqrt{-3}} \tilde{S}_3$, $SL_2(3) \overset{2}{\square} C_3$, $D_8 \otimes S_3$, or $C_3 \overset{2(2)}{\boxtimes} D_8$. Note that by Lemma 2.17 in each case there is a unique extension $C = C_C(O_3(C)).2$ with real Schur index 1. For all four candidates for C, the outer automorphism group $Out(C)$ is isomorphic to $C_2 \times C_2$, hence one has to consider three nontrivial outer automorphisms which can be distinguished via the determinants of the elements in $\overline{\mathbb{Q}C}$ inducing the automorphism by conjugation (cf. Corollary 7.12). In each case there are two extensions $G = NC.2$. Hence one has to construct twenty-four candidates for a.i.m.f. groups G. The groups $SL_2(5) \underset{\infty,2}{\overset{2(2)}{\boxtimes}} SL_2(3) \otimes \tilde{S}_3$ (both extensions) and $SL_2(5) \underset{\infty,3}{\overset{2(3)}{\boxtimes}} \tilde{S}_3 \otimes SL_2(3)$ (both extensions) clearly embed into ${}_{\infty,3}[\tilde{S}_3]_1 \otimes E_8$, ${}_{\infty,3}[\tilde{S}_3]_1 \otimes [(SL_2(5) \overset{2}{\square} SL_2(5)) : 2]_8$, ${}_{\infty,2}[SL_2(3)]_1 \otimes E_8$, respectively ${}_{\infty,2}[SL_2(3)]_1 \otimes [(SL_2(5) \overset{2}{\square} SL_2(5)) : 2]_8$. Also the two groups $SL_2(5) \underset{\sqrt{5}}{\overset{2(6)}{\boxtimes}} (SL_2(3) \otimes_{\sqrt{-3}} \tilde{S}_3)$ are not maximal finite but contained in the respective groups ${}_{\infty,5}[((SL_2(5) \circ SL_2(5)) \underset{\sqrt{5}}{\overset{2}{\boxtimes}} SL_2(5)) : S_3]_{8,i}$ $(i = 1, 2)$.

The two groups $SL_2(5) \overset{2(2)}{\boxtimes} (SL_2(3) \overset{2}{\square} C_3)$ may be enlarged to the respective groups $SL_2(5) \overset{2(2)}{\boxtimes} F_4$. One of the extensions $SL_2(5) \overset{2(3)}{\boxtimes} (SL_2(3) \overset{2}{\square} C_3)$ is contained in ${}_{\infty,3}[Sp_4(3) \circ C_3 \underset{\sqrt{-3}}{\overset{2(2)}{\boxtimes}} SL_2(3)]_8$.

The two groups ${}_{\infty,3}[SL_2(5) \overset{2(3)}{\boxtimes} S_3]_4 \otimes D_8$ are imprimitive and one of the groups $SL_2(5) \overset{2(2)}{\boxtimes} D_8 \otimes A_2$ is contained in ${}_{\infty,2}[2^{1+6}_{-}.O_6^{-}(2)]_4 \otimes A_2$. One of the extensions $SL_2(5) \underset{\sqrt{-1}}{\overset{2(2)}{\boxtimes}} (C_3 \overset{2(2)}{\boxtimes} D_8)$ is contained in ${}_{\infty,3}[C_3 \overset{2(2)}{\sqsupset} 2^{1+6}_{-}.O_6^{-}(2)]_8$ and one of the groups $SL_2(5) \underset{\sqrt{-1}}{\overset{2(3)}{\boxtimes}} (C_3 \overset{2(2)}{\boxtimes} D_8)$ is a proper subgroup of ${}_{\infty,2}[Sp_4(3) \circ C_3 \overset{2(2)}{\boxtimes} D_8]_8$. Hence G is one of the ten groups numbered 8–17 of the lemma.

(iii) Now assume that $O_5(C) > 1$. Then $O_5(C) \cong C_5$ and $C_C(O_5(C))$ is a centrally irreducible subgroup of $GL_2(\mathbb{Q}[\zeta_5])$ and hence of the form $H \otimes C_5$, where H is one of S_3, $\tilde{S}_3$, D_8, or $SL_2(3)$. Moreover C contains $C_C(O_5(C))$ of index 2. Since the outer automorphism group of H is C_2 in all cases, C is one of ${}^{\pm}D_{10} \otimes S_3$, ${}^{\pm}C_5 \overset{2(3)}{\boxtimes} S_3$, $Q_{20} \circ \tilde{S}_3$, $C_5 \underset{\sqrt{5}'}{\overset{2(3)}{\boxtimes}} \tilde{S}_3$, $D_{10} \otimes D_8$, $C_5 \overset{2(2)}{\boxtimes} D_8$, $Q_{20} \circ SL_2(3)$, or $C_5 \underset{\sqrt{5}'}{\overset{2(2)}{\boxtimes}} SL_2(3)$. In the four cases where $C_C(H) = {}^{\pm}C_5$ the outer automorphism group of C is cyclic of order 4 yielding no possibilities for primitive groups $G = NC.2 \leq GL_8(\mathcal{Q})$. In the other four cases $Out_{stab}(C) \cong C_2$ and one has two possible extensions $G = NC.2$. But now they lead to isomorphic groups. In all cases where a normal subgroup Q_{20}

is involved, one may enlarge $NC.2$ by replacing Q_{20} by $SL_2(5)$. Since the group ${}_{\infty,5}[SL_2(5) \overset{2}{\underset{\sqrt{5}}{\boxtimes}} D_{10}]_4 \otimes D_8$ is imprimitive, G is one of the last three groups of the Lemma. $\square$

For the rest of this chapter we assume that G does not contain a quasi-semi-simple normal subgroup. By Lemma 11.2 $O_{17}(G) = 1$.

Immediately from Proposition 8.9 one finds

Lemma 18.14. *If $O_3(G) = O_5(G) = 1$, then $G = {}_{\infty,2}[2_-^{1+8}.O_8^-(2)]_8$.*

Lemma 18.15. *If $O_5(G) = 1$ and $O_3(G) > 1$, then G is one of*

$$ {}_{\infty,2}[2_-^{1+6}.O_6^-(2)]_4 \otimes A_2 \ \textit{or}\ {}_{\infty,3}[C_3 \overset{2(2)}{\square} 2_-^{1+6}.O_6^-(2)]_8. $$

Proof. Then $O_3(G) \cong C_3$ and $C := C_G(O_3(G))$ is a normal subgroup of index 2 in G. Moreover C is a centrally irreducible normal subgroup of $GL_4(\mathcal{Q} \otimes \mathbb{Q}[\zeta_3])$, whence $O_2(C) = 2_-^{1+6}$ or 2_+^{1+6}. Let $B := \mathcal{B}^\circ(O_2(G))$. Then G contains the normal subgroup $O_3(G)B$ of index two. The enveloping algebra of B is a central simple $\mathbb{Q}$-algebra and B fixes up to isomorphism 2 lattices. With Corollary 7.12 one finds that $Glide(B)$ is (at most) C_2. The group $2_+^{1+6}.Alt_8 \otimes \tilde{S}_3$ is contained in ${}_{\infty,3}[\tilde{S}_3]_1 \otimes E_8$ and $C_3 \overset{2(2)}{\boxtimes} 2_+^{1+6}.Alt_8$ has the a.i.m.f. supergroup ${}_{\infty,2}[2_-^{1+8}.O_8^-(2)]_8$. So G is one of the two groups in the lemma. $\square$

Lemma 18.16. *If $O_5(G) > 1$ and $O_3(G) = 1$, then G is conjugate to*

$$ {}_{\infty,5}[D_{10} \overset{2(2)}{\boxtimes} 2_-^{1+4}.Alt_5]_8. $$

Proof. If $O_5(G) > 1$, then $O_5(G) \cong C_5$ and $C := C_G(O_5(G))$ is a centrally irreducible subgroup of $GL_4(\mathbb{Q}[\zeta_5])$. Moreover $G/C \cong C_4 \cong Out(C_5)$. Let $B := \mathcal{B}^\circ(O_2(G))$. Table 8.7 gives that $C = C_5 B$, since B is one of F_4 or $2_-^{1+4}.Alt_5$. In both cases $Glide(B) \cong C_2$, hence G contains the normal subgroup $Q_{20} \otimes F_4$ resp. $D_{10} \otimes 2_-^{1+4}.Alt_5$ of index 2. The first possibility leads to groups contained in ${}_{\infty,5}[SL_2(5).2]_2 \otimes F_4$ or ${}_{\infty,2}[SL_2(5) \overset{2(2)}{\boxtimes} F_4]_8$. In the second case G is the a.i.m.f. group of the lemma, since $(C_5 : C_4) \otimes 2_-^{1+4}.Alt_5$ is contained in ${}_{\infty,2}[2_-^{1+4}.Alt_5]_2 \otimes A_4$. $\square$

Lemma 18.17. *If $O_5(G) > 1$ and $O_3(G) > 1$, then G is conjugate to one of*

$$ {}_{2,5}[D_{10} \overset{2(2)}{\boxtimes} D_8]_4 \otimes_{\sqrt{-3}} {}_{\infty,3}[\tilde{S}_3]_1,\ {}_{3,5}[{}^{\pm}D_{10} \overset{2(3)}{\boxtimes} S_3]_4 \otimes_{\sqrt{-3}} {}_{\infty,2}[SL_2(3)]_1, $$

$$ {}_{\infty,2,3,5}[D_{10} \overset{2(2)}{\boxtimes} (C_3 \overset{2}{\square}\ SL_2(3))]_8,\ {}_{\infty,5}[D_{10} \overset{2(3)}{\boxtimes} (C_3 \overset{2}{\square}\ SL_2(3))]_8, $$

$$ {}_{\infty,2}[D_{10} \overset{2(6)}{\boxtimes} (C_3 \overset{2}{\square}\ SL_2(3))]_8,\ {}_{\infty,2,3,5}[D_{10} \overset{2(3)}{\boxtimes} (C_3 \overset{2(2)}{\boxtimes} D_8)]_8, $$

or ${}_{\infty,3}[D_{10} \overset{2(6)}{\boxtimes} (C_3 \overset{2(2)}{\boxtimes} D_8)]_8$.

Proof. As in the previous lemma $O_5(G) = C_5$ and $C := C_G(O_5(G))$ is a centrally irreducible subgroup of $GL_4(\mathbb{Q}[\zeta_5])$. One finds that C is of the form $C_5 \times H$ where H does not admit an outer automorphism of order 4. Hence G contains a normal subgroup $Q_{20}H$ or $D_{10}H$ of index 2. In the first case, one has the same candidates for H as in the proof of the Lemma 18.13 b) (ii). In all four cases the enveloping $\mathbb{Q}$-algebra of H is central simple and $Glide(H)$ does not contain an element of norm

5. One concludes that G is not maximal but contained in one of the groups of Lemma 18.13. In the second case H is one of $SL_2(3) \otimes S_3$, $SL_2(3) \overset{2}{\Box} C_3$, $D_8 \otimes \tilde{S}_3$, or $C_3 \overset{2(2)}{\boxtimes} D_8$. As in part b) (ii) of Lemma 18.13 $Out(H) \cong C_2 \times C_2$ in all cases. Since the groups $C_5 : C_4 \otimes H$ are contained in the corresponding groups $A_4 \otimes H$, one has to consider three automorphisms in each case. But now the two possible extensions $D_{10}H.2$ lead to isomorphic groups. The group $D_{10} \overset{2(2)}{\boxtimes} SL_2(3) \otimes A_2$ is contained in ${}_{\infty,5}[D_{10} \underset{\sqrt{5}}{\overset{2}{\boxtimes}} SL_2(5)]_4 \otimes A_2$ and $D_{10} \overset{2(6)}{\boxtimes} (SL_2(3) \otimes S_3)$ is contained in ${}_{\infty,3}[(D_{10} \otimes S_3) \underset{\sqrt{5}}{\overset{2(3)}{\boxtimes}} SL_2(5)]_8$.

Clearly $D_{10} \overset{2(3)}{\boxtimes} \tilde{S}_3 \otimes D_8$ is imprimitive and $D_{10} \overset{2(6)}{\boxtimes} (S_3 \otimes D_8)$ is a subgroup of ${}_{\infty,2}[(D_{10} \otimes D_8) \underset{\sqrt{5}}{\overset{2(2)}{\boxtimes}} SL_2(5)]_8$.

With Corollary 7.12 one gets that $D_{10} \overset{2(2)}{\boxtimes} (C_3 \overset{2(2)}{\boxtimes} D_8)$ is contained in $D_{10} \overset{2(2)}{\boxtimes} 2_-^{1+4}.Alt_5$.

Since the other groups are a.i.m.f. groups, one gets the Lemma. □

19. The a.i.m.f. subgroups of $GL_9(\mathcal{Q})$

Theorem 19.1. *Let $\mathcal{Q}$ be a definite quaternion algebra with center $\mathbb{Q}$ and G a primitive a.i.m.f. subgroup of $GL_9(\mathcal{Q})$. Then G is conjugate to one of the groups in the following table.*

List of the primitive a.i.m.f. subgroups of $GL_9(\mathcal{Q})$

lattice L	$\lvert Aut(L)\rvert$	r.i.m.f. supergroups
${}_{\infty,2}[SL_2(19)]_9$	$2^3 \cdot 3^3 \cdot 5 \cdot 19$	$[SL_2(19) \overset{2(2)}{\circ} SL_2(3)]_{36}$
${}_{\infty,2}[SL_2(3)]_1 \otimes A_9$	$2^3 \cdot 3 \cdot 10!$	$F_4 \otimes A_9$
${}_{\infty,3}[\pm 3_+^{1+4}.Sp_4(3).2]_9$	$2^9 \cdot 3^9 \cdot 5$	$[\pm 3_+^{1+4}.Sp_4(3).2]_{18}^2$
${}_{\infty,3}[\pm 3.Alt_6.2^2]_9$	$2^6 \cdot 3^3 \cdot 5$	$[\pm 3.Alt_6.2^2]_{18}^2$
${}_{\infty,3}[\tilde{S}_3]_1 \otimes A_9$	$2^2 \cdot 3 \cdot 10!$	$(A_2 \otimes A_9)^2$
${}_{\infty,7}[\pm L_2(7) \underset{\sqrt{-7}}{\overset{2}{\boxtimes}} L_2(7)]_9$	$2^9 \cdot 3^2 \cdot 7^2$	$[\pm L_2(7) \underset{\sqrt{-7}}{\overset{2}{\boxtimes}} L_2(7)]_{18}^2$
${}_{\infty,19}[\pm L_2(19).2]_9$	$2^4 \cdot 3^3 \cdot 5 \cdot 19$	$((A_{18}^{(5)})^2)$
		$(A_{18}^{(5)})^2$

Proof. Let $\mathcal{Q}$ be a definite quaternion algebra with center $\mathbb{Q}$ and G be a primitive a.i.m.f. subgroup of $GL_9(\mathcal{Q})$. Assume that $1 \neq N \trianglelefteq G$ is a quasi-semi-simple normal subgroup of G. With Table 9.1 one finds that $B := \mathcal{B}^\circ(N)$ is one of $SL_2(5)$, $\pm L_2(7)$, $\pm 3.Alt_6$, $\pm L_2(19)$, $SL_2(19)$, $\pm U_3(3)$, or $\pm S_{10}$. In the last case, G is one of ${}_{\infty,2}[SL_2(3)]_1 \otimes A_9$, ${}_{\infty,3}[\tilde{S}_3]_1 \otimes A_9$, by Corollary 7.6. If $B = SL_2(19)$ the group B is already absolutely irreducible and one computes and concludes that $G = B$ is ${}_{\infty,2}[SL_2(19)]_9$.

If $B = {}^{\pm}U_3(3)$ or $B = SL_2(5)$, then $\mathcal{Q} = \mathcal{Q}_{\infty,3}$ resp. $\mathcal{Q}_{\infty,2}$ and the centralizer $C_G(B)$ is an absolutely irreducible subgroup of $GL_3(\mathbb{Q})$. One concludes that G is imprimitive in these two cases.

If $B = {}^{\pm}L_2(19)$, the centralizer $C_G(B)$ is ${}^{\pm}1$ and $G = B.2 = {}_{\infty,19}[{}^{\pm}L_2(19).2]_9$.

If $B = {}^{\pm}3.Alt_6$, then $\mathcal{Q} = \mathcal{Q}_{\infty,3}$. If $\mathfrak{M}$ denotes a maximal order in $\mathcal{Q}$, then $\mathcal{Z}_{\mathfrak{M}}(B)$ contains only one isomorphism class of lattices. For $L \in \mathcal{Z}_{\mathfrak{M}}(B)$ one calculates that the Hermitian automorphism group of L is $G = {}_{\infty,3}[3.Alt_6.2^2]_9$.

In the last case, $B = {}^{\pm}L_2(7)$. The centralizer $C := C_G(B)$ is an absolutely irreducible subgroup of $GL_3(\mathbb{Q}[\sqrt{-7}])$. One concludes that either G is imprimitive or C is one of ${}^{\pm}C_7 : C_3$ or ${}^{\pm}L_2(7)$. One finds that $G = {}_{\infty,7}[{}^{\pm}L_2(7) \underset{\sqrt{-7}}{\overset{2}{\boxtimes}} L_2(7)]_9$ in this case.

Now assume that G does not contain a quasi-semi-simple normal subgroup. Then the Fitting subgroup of G is a self-centralizing normal subgroup. By Table 8.7 one has the following possibilities for $Fit(G)$: ${}^{\pm}C_{19}$, ${}^{\pm}C_7$, ${}^{\pm}3_+^{1+2} \curlyvee C_9$, or ${}^{\pm}3_+^{1+4}$, because 3 does not divide the order of $Out(Fit(G))/\mathcal{B}^\circ(Fit(G))$ and 9 does not divide the degree of the corresponding irreducible character of $Fit(G)$ in the other cases. In the first case, G is a proper subgroup of ${}_{\infty,19}[{}^{\pm}L_2(19).2]_9$ by Lemma 11.2. The second and third case lead to reducible groups and in the last case, G contains $\mathcal{B}^\circ(Fit(G)) = {}^{\pm}3_+^{1+4}.Sp_4(3)$ of index 2. One concludes that $G = {}_{\infty,3}[{}^{\pm}3_+^{1+4}.Sp_4(3).2]_9$. □

20. The a.i.m.f. subgroups of $GL_{10}(\mathcal{Q})$

Theorem 20.1. *Let $\mathcal{Q}$ be a definite quaternion algebra with center $\mathbb{Q}$ and G a primitive a.i.m.f. subgroup of $GL_{10}(\mathcal{Q})$. Then G is one of the groups listed in the following table:*

List of the primitive a.i.m.f. subgroups of $GL_{10}(\mathcal{Q})$.

lattice L	$\lvert Aut(L)\rvert$	r.i.m.f. supergroups
$_{\infty,2}[2.U_4(2)]_{10}$	$2^7\cdot 3^4\cdot 5$	$[2.U_4(2)\overset{2(2)}{\circ}SL_2(3)]_{40}$
$_{\infty,2}[SL_2(11)]_5\otimes A_2$	$2^4\cdot 3^2\cdot 5\cdot 11$	$A_2\otimes[SL_2(11)\overset{2(2)}{\circ}SL_2(3)]_{20}$
$_{\infty,2}[SL_2(11)\overset{2(2)}{\otimes}D_8]_{10}$	$2^6\cdot 3\cdot 5\cdot 11$	$[SL_2(11)\overset{2(2)}{\otimes}2_-^{1+4}.Alt_5]_{40}$
$_{\infty,2}[\pm U_5(2)]_5\otimes A_2$	$2^{12}\cdot 3^6\cdot 5\cdot 11$	$A_2\otimes[\pm U_5(2)\overset{2(2)}{\circ}SL_2(3)]_{20}$
$_{\infty,2}[U_5(2)\overset{2(2)}{\otimes}D_8]_{10}$	$2^{14}\cdot 3^5\cdot 5\cdot 11$	$[\pm U_5(2)\overset{2(2)}{\otimes}2_-^{1+4}.Alt_5]_{40}$
$_{\infty,2}[S_6\overset{2(2)}{\otimes}SL_2(3)]_{10}$	$2^8\cdot 3^3\cdot 5$	$F_4\overset{2(2)}{\otimes}[\pm S_6]_{10}$
$_{\infty,2}[SL_2(3)]_1\otimes[\pm U_4(2)\overset{2}{\square}C_3]_{10}$	$2^{10}\cdot 3^6\cdot 5$	$F_4\otimes[\pm U_4(2)\overset{2}{\square}C_3]_{10}$
$_{\infty,2}[SL_2(3)]_1\otimes A_{10}$	$2^3\cdot 3\cdot 11!$	$A_{10}\otimes F_4$
$_{\infty,2}[SL_2(3)]_1\otimes A_{10}^{(2)}$	$2^6\cdot 3^2\cdot 5\cdot 11$	$A_{10}^{(2)}\otimes F_4$
$_{\infty,2}[SL_2(3)]_1\otimes A_{10}^{(3)}$	$2^6\cdot 3^2\cdot 5\cdot 11$	$A_{10}^{(3)}\otimes F_4$
$_{\infty,2}[2_-^{1+4}.Alt_5]_2\otimes A_5$	$2^7\cdot 3\cdot 5\cdot 6!$	$A_5\otimes E_8$
$_{\infty,3}[C_3\overset{2(2)}{\boxtimes}SL_2(11)]_{10}$	$2^4\cdot 3^2\cdot 5\cdot 11$	$[SL_2(11)\overset{2(2)}{\circ}SL_2(3)]_{20}^2$
$_{\infty,3}[(C_3\circ U_4(2))\underset{\sqrt{-3}}{\overset{2(2)}{\boxtimes}}SL_2(3)]_{10}$	$2^{10}\cdot 3^6\cdot 5$	$[\pm U_5(2)\overset{2(2)}{\circ}SL_2(3)]_{20}^2$
$_{\infty,3}[C_3\overset{2(2)}{\boxtimes}\pm U_5(2)]_{10}$	$2^{12}\cdot 3^6\cdot 5\cdot 11$	$[\pm U_5(2)\overset{2(2)}{\circ}SL_2(3)]_{20}^2$
$_{\infty,3}[\pm U_4(2)\overset{2}{\square}C_3]_{10}$	$2^8\cdot 3^5\cdot 5$	$[\pm U_4(2)\overset{2}{\square}C_3]_{20}^2$
$_{\infty,3}[2.U_4(3).4]_{10}$	$2^{10}\cdot 3^6\cdot 5\cdot 7$	$[2.U_4(3).4\overset{2(3)}{\circ}\tilde{S}_3]_{40}$
$_{\infty,3}[SL_2(19)]_{10}$	$2^3\cdot 3^2\cdot 5\cdot 19$	$[SL_2(19)\overset{2(3)}{\circ}\tilde{S}_3]_{40}$
$_{\infty,3}[2.Alt_7]_{10a}$	$2^4\cdot 3^2\cdot 5\cdot 7$	$[2.Alt_7\overset{2(3)}{\circ}\tilde{S}_3]_{40}$
$_{\infty,3}[2.Alt_7]_{10b}$	$2^4\cdot 3^2\cdot 5\cdot 7$	$[2.Alt_7\circ\tilde{S}_3]_{40}$
$_{\infty,3}[L_2(11)\overset{2(3)}{\otimes}\tilde{S}_3]_{10}$	$2^5\cdot 3^2\cdot 5\cdot 11$	$[L_2(11)\overset{2(3)}{\otimes}D_{12}]_{20}^2$
$_{\infty,3}[L_2(11)\overset{2(3)}{\boxtimes}\tilde{S}_3]_{10}$	$2^5\cdot 3^2\cdot 5\cdot 11$	$[L_2(11)\overset{2(3)}{\boxtimes}D_{12}]_{20}^2$
$_{\infty,3}[\tilde{S}_3]_1\otimes A_{10}$	$2^2\cdot 3\cdot 11!$	$(A_{10}\otimes A_2)^2$
$_{\infty,3}[\tilde{S}_3]_1\otimes A_{10}^{(2)}$	$2^5\cdot 3^2\cdot 5\cdot 11$	$(A_{10}^{(2)}\otimes A_2)^2$
$_{\infty,3}[\tilde{S}_3]_1\otimes A_{10}^{(3)}$	$2^5\cdot 3^2\cdot 5\cdot 11$	$(A_{10}^{(3)}\otimes A_2)^2$
$_{\infty,3}[SL_2(9)]_2\otimes A_5$	$2^4\cdot 3^2\cdot 5\cdot 6!$	$A_5\otimes E_8$
$_{\infty,5}[\pm U_3(5):3]_{10}$	$2^5\cdot 3^3\cdot 5^3\cdot 7$	$[\pm U_3(5):3\overset{2}{\square}C_3]_{40}$
$_{\infty,5}[\pm 5_+^{1+2}:SL_2(5).4]_{10}$	$2^6\cdot 3\cdot 5^4$	$[\pm 5_+^{1+2}:SL_2(5).2\overset{2}{\square}SL_2(5)]_{40}$
$_{\infty,5}[SL_2(5).2]_2\otimes A_5$	$2^4\cdot 3\cdot 5\cdot 6!$	$A_5\otimes E_8$
$_{\infty,5}[SL_2(5):2]_2\otimes A_5$	$2^4\cdot 3\cdot 5\cdot 6!$	$A_5\otimes[(SL_2(5)\overset{2}{\square}SL_2(5)):2]_8$
$_{\infty,7}[2.S_7]_{10}$	$2^5\cdot 3^2\cdot 5\cdot 7$	$((\Lambda^3 A_6)^2)$
$_{\infty,7}[2.L_3(4).2^2]_{10}$	$2^9\cdot 3^2\cdot 5\cdot 7$	$[2.L_3(4).2^2]_{20}^2$
$_{\infty,11}[L_2(11)\underset{\sqrt{-11}}{\overset{2(2)}{\boxtimes}}SL_2(3)]_{10}$	$2^6\cdot 3^2\cdot 5\cdot 11$	$[U_5(2)\overset{2(2)}{\circ}SL_2(3)]_{20}^2$
		$[(L_2(11)\otimes_{\sqrt{-11}}SL_2(3)\otimes S_3).2]_{40}$
$_{\infty,11}[L_2(11)\overset{2(3)}{\boxtimes}S_3]_{10}$	$2^5\cdot 3^2\cdot 5\cdot 11$	$[L_2(11)\overset{2(3)}{\boxtimes}D_{12}]_{20}^2$
		$(A_{10}^{(3)}\otimes A_2)^2$
$_{\infty,11}[\pm L_2(11).2]_5\otimes A_2$	$2^5\cdot 3^2\cdot 5\cdot 11$	$(A_{10}^{(3)}\otimes A_2)^2$
		$[L_2(11)\overset{2(3)}{\boxtimes}D_{12}]_{20}^2$
$_{\infty,19}[SL_2(19).2]_{10}$	$2^4\cdot 3^2\cdot 5\cdot 19$	(B_{40})
		(B_{40})

The proof is split into lemmata. For the rest of this chapter let $\mathcal{Q}$ be a definite quaternion algebra with center $\mathbb{Q}$ and G be a primitive a.i.m.f. subgroup of $GL_{10}(\mathcal{Q})$.

By Table 9.1 and Lemma 7.2 the possibilities for quasi-semi-simple normal subgroups N of G are $SL_2(5)$, Alt_6 (2 matrix groups), $SL_2(9)$ (2 matrix groups), $L_2(11)$ (3 matrix groups), $SL_2(11)$ (2 matrix groups), Alt_7, $2.Alt_7$ (2 matrix groups), $SL_2(19)$ (2 matrix groups), M_{11}, $2.L_3(4)$, $U_4(2)$ (2 matrix groups), $2.U_4(2)$, $2.M_{12}$, $U_3(5)$, $2.M_{22}$, $2.U_4(3)$, $U_5(2)$, and Alt_{11}. By Corollary 7.7, N is not conjugate to one of M_{11}, $2.M_{12}$, or $2.M_{22}$.

Lemma 20.2. *G has no normal subgroup $SL_2(11)$ with character $\chi_{10a} + \chi_{10b}$.*

Proof. Assume that G has a normal subgroup N conjugate to $SL_2(11)$, where the restriction of the natural character of G to N is $\chi_{10a} + \chi_{10b}$. Then G contains the normal subgroup $NC_G(N)$ of index ≤ 2. Since the outer automorphism of N does not interchange the two Galois conjugate characters χ_{10a} and χ_{10b}, the character field of the natural character of G is $\mathbb{Q}[\sqrt{3}]$. Therefore G is not absolutely irreducible. $\square$

Lemma 20.3. *If G contains a normal subgroup $N \cong Alt_6$ with character χ_5, then G is one of ${}_{\infty,2}[2_-^{1+4}.Alt_5]_2 \otimes A_5$, ${}_{\infty,3}[SL_2(9)]_2 \otimes A_5$, ${}_{\infty,5}[SL_2(5).2]_2 \otimes A_5$, or ${}_{\infty,5}[SL_2(5):2]_2 \otimes A_5$.*

Proof. By Corollary 7.6 G is of the form $A_5 \otimes H$, where $H \leq GL_2(\mathcal{Q})$ is a primitive a.i.m.f. group. Hence by Theorem 12.1 H is one of ${}_{\infty,2}[(D_8 \otimes Q_8).Alt_5]_2$, ${}_{\infty,2}[SL_2(3)]_1 \otimes A_2$, ${}_{\infty,3}[SL_2(9)]_2$, ${}_{\infty,3}[SL_2(3) \overset{2}{\square} C_3]_2$, ${}_{\infty,5}[SL_2(5).2]_2$, or ${}_{\infty,5}[SL_2(5):2]_2$. If $H = {}_{\infty,2}[SL_2(3)]_1 \otimes A_2$, then $G = {}_{\infty,2}[SL_2(3)]_1 \otimes (A_2 \otimes A_5)$ is contained in ${}_{\infty,2}[SL_2(3)]_1 \otimes [\pm U_4(2) \overset{2}{\square} C_3]_{10}$.

If $H = {}_{\infty,3}[SL_2(3) \overset{2}{\square} C_3]_2$, one computes that G is a proper subgroup of ${}_{\infty,3}[(C_3 \circ U_4(2)) \overset{2(2)}{\underset{\sqrt{-3}}{\boxtimes}} SL_2(3)]_{10}$. $\square$

Similarly one gets the next two lemmata:

Lemma 20.4. *If G contains a normal subgroup $N \cong Alt_{11}$ with character χ_{10}, then G is one of ${}_{\infty,2}[SL_2(3)]_1 \otimes A_{10}$ or ${}_{\infty,3}[\tilde{S}_3]_1 \otimes A_{10}$.*

Lemma 20.5. *If G contains a normal subgroup $N \cong L_2(11)$ with character χ_{10a}, then G is one of ${}_{\infty,2}[SL_2(3)]_1 \otimes A_{10}^{(2)}$ or ${}_{\infty,3}[\tilde{S}_3]_1 \otimes A_{10}^{(2)}$.*

Lemma 20.6. *If G contains a normal subgroup $N \cong SL_2(9)$ with character χ_4, then G is conjugate to ${}_{\infty,3}[SL_2(9)]_2 \otimes A_5$.*

Proof. By 2.18 one has $\mathcal{Q} = \mathcal{Q}_{\infty,3}$ and $C := C_G(N)$ embeds into $GL_5(\mathbb{Q})$. Since G contains NC of index ≤ 2, C is an absolutely irreducible subgroup of $GL_5(\mathbb{Q})$. Therefore $C^{(\infty)}$ is one of Alt_5 or Alt_6. Since in the first case $C \ntrianglelefteq \mathcal{B}^\circ(C)$, one has $C^{(\infty)} = Alt_6$ and the lemma follows from 20.3. $\square$

Similarly one gets

Lemma 20.7. *If G contains a normal subgroup $N \cong SL_2(5)$ with character $\chi_{2a} + \chi_{2b}$, then G is conjugate to one of ${}_{\infty,5}[SL_2(5):2]_2 \otimes A_5$ or ${}_{\infty,5}[SL_2(5).2]_2 \otimes A_5$.*

Proof. Now $C := C_G(N)$ is a centrally irreducible subgroup of $GL_5(\mathbb{Q}[\sqrt{5}])$. Again $C^{(\infty)}$ is one of Alt_5 or Alt_6 and the Lemma follows from 20.3. $\square$

The next lemma deals with the absolutely irreducible candidates for normal subgroups N.

Lemma 20.8. *If G contains a normal subgroup N isomorphic to $2.Alt_7$ with character χ_{20a}, $2.Alt_7$ with character χ_{20b}, $SL_2(19)$ with character χ_{20}, $2.U_4(2)$ with character χ_{20}, $U_3(5)$ with character χ_{20}, resp. $2.U_4(3)$ with character χ_{20}, then G is conjugate to one of ${}_{\infty,3}[2.Alt_7]_{10a}$, ${}_{\infty,3}[2.Alt_7]_{10b}$, ${}_{\infty,3}[SL_2(19)]_{10}$, ${}_{\infty,2}[2.U_4(2)]_{10}$, ${}_{\infty,5}[{}^{\pm}U_3(5):3]_{10}$, resp. ${}_{\infty,3}[2.U_4(3).4]_{10}$.*

Proof. In all cases N is already absolutely irreducible. One finds $G = \mathcal{B}^{\circ}(N)$. $\square$

Next we treat those candidates for normal subgroups N in G, such that $C_G(N)$ has to be contained in $\mathcal{B}^{\circ}(N)$.

Lemma 20.9. *G has no normal subgroup N isomorphic to $SL_2(9)$ with character $\chi_{10a} + \chi_{10b}$. If G contains a normal subgroup N isomorphic to Alt_7 with character $\chi_{10a} + \chi_{10b}$, $SL_2(19)$ with character $\chi_{10a} + \chi_{10b}$, $2.L_3(4)$ with character $\chi_{10a} + \chi_{10b}$, resp. $U_4(2)$ with character $\chi_{10a} + \chi_{10b}$, then $G = \mathcal{B}^{\circ}(N).2$ is conjugate to ${}_{\infty,7}[2.S_7]_{10}$, ${}_{\infty,19}[SL_2(19).2]_{10}$, ${}_{\infty,7}[2.L_3(4).2^2]_{10}$, resp. ${}_{\infty,3}[{}^{\pm}U_4(2) \overset{2}{\square} C_3]_{10}$.*

Proof. In all cases the centralizer $C_G(N)$ embeds into the enveloping algebra of N and hence is contained in $\mathcal{B}^{\circ}(N)$. Assume first, that N is isomorphic to $SL_2(9)$. Since the character field of the extension of the character χ_{10a} to $2.PGL_2(9)$ is of degree 4 over $\mathbb{Q}$ (cf. [CCNPW 85]), G is isomorphic to $2.S_6$. (Note that the outer automorphism of S_6 interchanges the two isoclinism classes of groups $2.S_6$, so there is only one group to be considered.) But then G is not maximal finite, since it is contained in the a.i.m.f. group ${}_{\infty,2}[2.U_4(2)]_{10}$.

In all the other cases $Glide(N) = 1$ and there is an automorphism of N inducing the Galois automorphism of the character field $\mathbb{Q}[\chi]$. Since $\mathbb{Q}[\chi]$ is an imaginary quadratic number field, Remark (I.13) of [Neb 96] implies that there is a unique extension $G = \mathcal{B}^{\circ}(N).2$ with real Schur index 2. Computing the automorphism groups of the G-invariant lattices, one finds that in all cases G is a maximal finite subgroup of $GL_{10}(\mathcal{Q})$. $\square$

Lemma 20.10. *If G has a normal subgroup $N \cong Alt_6$ with character χ_{10}, then G is conjugate to ${}_{\infty,2}[S_6 \overset{2(2)}{\otimes} SL_2(3)]_{10}$.*

Proof. By Table 9.1 the group N is nearly tensor decomposing over $\mathbb{Q}$ with parameter 2. Let $B := \mathcal{B}^{\circ}(N) = {}^{\pm}S_6$ and $C := C_G(N) = C_G(B)$. By 10.1 G is either $G = BC$ where C is an a.i.m.f. subgroup of $GL_1(\mathcal{Q})$ such that $(C, 2, \mathcal{Q})$ is not a maximal triple or of the form $B \overset{2(2)}{\otimes} C$ or $B \overset{2(2)}{\boxtimes} C$ where $(C, 2, \mathcal{Q})$ is a maximal triple. Table 10.2 G is one of ${}_{\infty,3}[{}^{\pm}S_6 \otimes \tilde{S}_3]_{10}$ or ${}_{\infty,2}[{}^{\pm}S_6 \overset{2(2)}{\otimes} SL_2(3)]_{10}$. The first group is not maximal finite but contained in ${}_{\infty,3}[{}^{\pm}U_4(2) \overset{2}{\square} C_3]_{10}$. $\square$

Similarly one finds the next lemma, because ${}_{\infty,2}[L_2(11) \otimes SL_2(3)]_{10}$ is contained in $A_{10} \otimes {}_{\infty,2}[SL_2(3)]_1$.

Lemma 20.11. *If G has a normal subgroup $N \cong L_2(11)$ with character χ_{10}, then G is conjugate to ${}_{\infty,3}[L_2(11) \overset{2(3)}{\otimes} \tilde{S}_3]_{10}$.*

The next two lemmata deal with similar situations, where now the enveloping algebra of N is a matrix ring over a definite quaternion algebra over $\mathbb{Q}$.

Lemma 20.12. *If G has a normal subgroup $N \cong SL_2(11)$ with character $2\chi_{10}$, then G is conjugate to one of ${}_{\infty,2}[SL_2(11)]_5 \otimes A_2$, ${}_{\infty,2}[SL_2(11) \overset{2(2)}{\otimes} D_8]_{10}$, or ${}_{\infty,3}[C_3 \overset{2(2)}{\boxtimes} SL_2(11)]_{10}$.*

Proof. By Table 9.1 the group N is nearly tensor decomposing over $\mathbb{Q}$ with parameter 2. By 10.1 G is either $G = NC$ where $C := C_G(N)$ is an a.i.m.f. subgroup of $GL_1(\mathcal{D})$ such that $(C, 2, \mathcal{D})$ is not a maximal triple or of the form $N \overset{2(2)}{\otimes} C$ or $N \overset{2(2)}{\boxtimes} C$ where $(C, 2, \mathcal{D})$ is a maximal triple and $\mathcal{D}$ is an indefinite quaternion algebra over $\mathbb{Q}$. By Table 10.2 G is one of the three groups in the lemma. $\square$

Completely analogously one gets

Lemma 20.13. *If G has a normal subgroup $N \cong U_5(2)$ with character $2\chi_{10}$, then G is conjugate to one of ${}_{\infty,2}[{}^{\pm}U_5(2)]_5 \otimes A_2$, ${}_{\infty,2}[{}^{\pm}U_5(2) \overset{2(2)}{\otimes} D_8]_{10}$, or ${}_{\infty,3}[C_3 \overset{2(2)}{\boxtimes} {}^{\pm}U_5(2)]_{10}$.*

In the last two cases, the center of the enveloping algebra of N is an imaginary quadratic field. Hence here the situation is not so tight.

Lemma 20.14. *If G has a normal subgroup $N \cong L_2(11)$ with character $\chi_{5a} + \chi_{5b}$, then G is conjugate to one of ${}_{\infty,2}[SL_2(3)]_1 \otimes A_{10}^{(3)}$, ${}_{\infty,3}[L_2(11) \overset{2(3)}{\boxtimes} \tilde{S}_3]_{10}$, ${}_{\infty,3}[\tilde{S}_3]_1 \otimes A_{10}^{(3)}$, ${}_{\infty,11}[L_2(11) \underset{\sqrt{-11}}{\overset{2(2)}{\boxtimes}} SL_2(3)]_{10}$, ${}_{\infty,11}[L_2(11) \overset{2(3)}{\boxtimes} S_3]_{10}$, or ${}_{\infty,11}[{}^{\pm}L_2(11).2]_5 \otimes A_2$.*

Proof. The centralizer $C := C_G(N)$ is a centrally irreducible subgroup of $GL_1(\mathcal{D})$ where $\mathcal{D}$ is a quaternion algebra over $\mathbb{Q}[\sqrt{-11}]$ and G contains NC of index 2. Using the classification of finite subgroups of $GL_2(\mathbb{C})$ in [Bli 17], one finds that C is one of D_8, ${}^{\pm}S_3$, $SL_2(3)$, or $\tilde{S}_3$. Distinguish 2 cases:

a) $C_G(C) > {}^{\pm}N$. Then $G = ({}^{\pm}N.2)C$ is one of ${}_{\infty,11}[{}^{\pm}L_2(11).2]_5 \otimes [D_8]_2$, ${}_{\infty,11}[{}^{\pm}L_2(11).2]_5 \otimes A_2$, ${}_{\infty,2}[SL_2(3)]_1 \otimes A_{10}^{(3)}$, or ${}_{\infty,3}[\tilde{S}_3]_1 \otimes A_{10}^{(3)}$. In the first case, G is imprimitive and contained in ${}_{\infty,11}[{}^{\pm}L_2(11).2]_5^2$.

b) $C_G(C) = {}^{\pm}N$. The groups C are nearly tensor decomposing over $\mathbb{Q}$ with parameter 2, 3, 2, resp. 3. Since $C = C_G(N)$ Lemma 10.1 implies that G is one of ${}_{\infty,2}[L_2(11) \overset{2(2)}{\boxtimes} D_8]_{10}$, ${}_{\infty,11}[L_2(11) \underset{\sqrt{-11}}{\overset{2(2)}{\boxtimes}} SL_2(3)]_{10}$, ${}_{\infty,11}[L_2(11) \overset{2(3)}{\boxtimes} S_3]_{10}$, or ${}_{\infty,3}[L_2(11) \overset{2(3)}{\boxtimes} \tilde{S}_3]_{10}$. Note that 3 is decomposed and 2 is inert in $\mathbb{Q}[\sqrt{-11}]$. The first group is not maximal finite but contained in ${}_{\infty,2}[{}^{\pm}U_5(2) \overset{2(2)}{\otimes} D_8]_{10}$. $\square$

Lemma 20.15. *If G has a normal subgroup $N \cong U_4(2)$ with character $\chi_{5a} + \chi_{5b}$, then G is conjugate to ${}_{\infty,2}[SL_2(3)]_1 \otimes [{}^{\pm}U_4(2) \overset{2}{\square} C_3]_{10}$ or ${}_{\infty,3}[(C_3 \circ U_4(2)) \underset{\sqrt{-3}}{\overset{2(2)}{\boxtimes}} SL_2(3)]_{10}$.*

Proof. Let $B := \mathcal{B}^\circ(N) \cong {}^{\pm}C_3 \circ N$. The centralizer $C := C_G(N)$ is a centrally irreducible subgroup of $GL_1(\mathcal{D})$ where $\mathcal{D}$ is a quaternion algebra over $\mathbb{Q}[\sqrt{-3}]$ and G contains BC of index 2. As in the last lemma C is one of D_8, ${}^{\pm}S_3$, $SL_2(3)$, or $\tilde{S}_3$. But now $O_3(B) \cong C_3$ and therefore all three groups ${}_{\infty,3}[\tilde{S}_3]_1 \otimes [{}^{\pm}U_4(2) \overset{2}{\Box} C_3]_{10}$, $A_2 \otimes {}_{\infty,3}[{}^{\pm}U_4(2) \overset{2}{\Box} C_3]_5$, and $D_8 \otimes {}_{\infty,3}[{}^{\pm}U_4(2) \overset{2}{\Box} C_3]_5$ are imprimitive and one only finds one a.i.m.f. group in the case $C_G(C) > B$. In the case $C_G(C) = B$ one again uses 10.1 to deduce that G is one of

$$_{\infty,2}[{}^{\pm}U_4(2) \circ C_3 \overset{2(2)}{\boxtimes} D_8]_{10},\ {}_{\infty,3}[{}^{\pm}U_4(2) \circ C_3 \underset{\sqrt{-3}}{\overset{2(2)}{\boxtimes}} SL_2(3)]_{10},$$

$$_{\infty,3}[{}^{\pm}U_4(2) \circ C_3 \overset{2(3)}{\boxtimes} S_3]_{10} \text{ or } {}_{\infty,3}[{}^{\pm}U_4(2) \circ C_3 \underset{\sqrt{-3}}{\overset{2(3)}{\boxtimes}} \tilde{S}_3]_{10}.$$

The first group is not maximal finite but contained in ${}_{\infty,2}[{}^{\pm}U_5(2) \overset{2(2)}{\otimes} D_8]_{10}$ and the last two groups are imprimitive. $\square$

For the rest of the proof of 20.1 we assume that G contains no quasi-semi-simple normal subgroup. Then the Fitting subgroup $Fit(G) := \prod_{p||G|} O_p(G)$ is a self-centralizing normal subgroup of G, and hence an irreducible subgroup of $GL_{10}(\mathcal{Q})$ by Lemma 8.11. From Table 8.7 one gets that $Fit(G)$ is one of ${}^{\pm}C_{25}$, ${}^{\pm}5_+^{1+2}$, of ${}^{\pm}C_{11}$.

The first possibility immediately leads to a contradiction.

Lemma 20.16. *If* $O_5(G) = 5_+^{1+2}$, *then* $G = {}_{\infty,5}[{}^{\pm}5_+^{1+2} : SL_2(5).4]_{10}$.

Proof. Then G contains the group $B := \mathcal{B}^\circ(O_5(G)) = {}^{\pm}5_+^{1+2} : SL_2(5)$ with $G/B \cong C_4(= Gal(\mathbb{Q}[\zeta_5]/\mathbb{Q}))$. Since the split extension $B : C_4$ has real Schur index 1 one concludes $G = {}_{\infty,5}[{}^{\pm}5_+^{1+2} : SL_2(5).4]_{10}$. $\square$

Lemma 20.17. $O_{11}(G) \neq C_{11}$.

Proof. The centralizer $C := C_G(O_{11}(G))$ is a centrally irreducible subgroup of $GL(\mathcal{D})$ where $\mathcal{D}$ is a quaternion algebra over $\mathbb{Q}[\zeta_{11}]$. As in 20.14 C is one of the groups D_8, ${}^{\pm}S_3$, $SL_2(3)$, or $\tilde{S}_3$. Since these groups have no automorphism of order 5, G contains the group ${}^{\pm}C_{11} : C_5C$ of index 2. In all cases, G is a proper subgroup of one of the groups of 20.14. $\square$

21. Appendix

Some invariants of the occurring primitive r.i.m.f. subgroups of $GL_{32}(\mathbb{Q})$, that are not tensor products:

lattice L	$det(L)$	$min(L)$	$\lvert L_{min}\rvert$	$\lvert Aut(L)\rvert$	lattice sparse
$[2_+^{1+10}.O_{10}^+(2)]_{32}$	1	4	146880	$2^{31}\cdot 3^5\cdot 5^2\cdot 7\cdot 17\cdot 31$	$p\neq 2$
$[((SL_2(5)\circ SL_2(5))\overset{2}{\underset{\sqrt{5}}{\boxtimes}}(SL_2(5)\circ SL_2(5))):S_4]_{32,1}$	5^{16}	8	21600	$2^{13}\cdot 3^5\cdot 5^4$	$+$
$[((SL_2(5)\circ SL_2(5))\overset{2}{\underset{\sqrt{5}}{\boxtimes}}(SL_2(5)\circ SL_2(5))):S_4]_{32,2}$	1	4	43200+ 103680	$2^{13}\cdot 3^5\cdot 5^4$	$p\neq 5$
$[4.L_3(4).2^2]_{32,1}$	5^{16}	8	11520+ 10080	$2^{10}\cdot 3^2\cdot 5\cdot 7$	$\mathbb{Q}[\sqrt{-5}]$, $+$ $+$
$[4.L_3(4).2^2]_{32,2}$	1	4	8064+ 20160+ $2\cdot 23040$+ 32256+ 40320	$2^{10}\cdot 3^2\cdot 5\cdot 7$	$\mathbb{Q}[\sqrt{-5}]$ $+$
$[SL_2(17)\overset{2(3)}{\sqsupset}\tilde{S}_3]_{32,1}$	1	4	$3\cdot 4896$+ $4\cdot 14688$	$2^7\cdot 3^3\cdot 17$	$p\neq 3,17$
$[SL_2(17)\overset{2(3)}{\sqsupset}\tilde{S}_3]_{32,2}$	17^{16}	12	1632	$2^7\cdot 3^3\cdot 17$	$p\neq 3$
$[(2.Alt_7\overset{2}{\underset{\sqrt{-7}}{\boxtimes}}2.Alt_7):2]_{32}$	7^{16}	8	5040	$2^9\cdot 3^4\cdot 5^2\cdot 7^2$	$+$
$[2.Alt_7\overset{2(3)}{\underset{\sqrt{-7}}{\boxtimes}}(SL_2(3)\overset{2}{\square}C_3)]_{32}$	$2^{16}\cdot 7^{16}$	12	6720	$2^8\cdot 3^4\cdot 5\cdot 7$	$+$
$[(Sp_4(3)\otimes_{\sqrt{-3}}Sp_4(3)):2\overset{2}{\square}C_3]_{32}$	3^{16}	6	9600	$2^{15}\cdot 3^9\cdot 5^2$	$+$
$[SL_2(5)\overset{2(2)}{\underset{\infty,2}{\boxtimes}}2_-^{1+6}.O_6^-(2)]_{32}$	5^{16}	8	21600	$2^{16}\cdot 3^5\cdot 5^2$	$p\neq 2$
$[SL_2(9).2\overset{2(2)}{\underset{\infty,2}{\boxtimes}}2_-^{1+4}.Alt_5]_{32}$	$2^{16}\cdot 3^{16}$	8	7200	$2^{12}\cdot 3^3\cdot 5^2$	$+$
$[SL_2(5)\overset{2(3)}{\underset{\infty,3}{\boxtimes}}(Sp_4(3)\overset{2}{\square}C_3)]_{32}$	$3^{16}\cdot 5^{16}$	12	4800	$2^{11}\cdot 3^6\cdot 5^2$	$+$
$[SL_2(17)\overset{2(3)}{\circ}\tilde{S}_3]_{32}$	17^4	6	233376*	$2^7\cdot 3^3\cdot 17$	$p\neq 3$
$[SL_2(7)\overset{2}{\underset{\sqrt{-7}}{\boxtimes}}2.Alt_7]_{32}$	$2^8\cdot 7^{16}$	12	47040	$2^8\cdot 3^3\cdot 5\cdot 7^2$	$+$
$[SL_2(9)\otimes D_{10}\overset{2}{\square}SL_2(5)]_{32}$	$3^{16}\cdot 5^8$	8	3600	$2^8\cdot 3^3\cdot 5^3$	$p\neq 5$
$[SL_2(7)\overset{2(3)}{\underset{\sqrt{-7}}{\boxtimes}}(SL_2(3)\overset{2}{\square}C_3)]_{32}$	$2^{12}\cdot 7^{16}$	10	1344	$2^8\cdot 3^3\cdot 7$	$p\neq 2$
$[SL_2(7)\overset{2(3)}{\underset{\infty,3}{\otimes}}(SL_2(3)\overset{2}{\square}C_3)]_{32}$	$2^{16}\cdot 7^8$	6	1344	$2^8\cdot 3^3\cdot 7$	$p\neq 3$

* not decomposed into orbits under $Aut(L)$

lattice L	$det(L)$	$min(L)$	$\|L_{min}\|$	$\|Aut(L)\|$	lattice sparse
$[(SL_2(3) \circ C_4).2 \underset{\sqrt{-1}}{\overset{2(3)}{\boxtimes}} SL_2(7)]_{32}$	$2^{16} \cdot 3^{16} \cdot 7^8$	12	672+ 1344	$2^9 \cdot 3^2 \cdot 7$	+
$[SL_2(7) \underset{\infty,3}{\overset{2(3)}{\otimes}} SL_2(9)]_{32}$	7^8	6	6720+ 20160	$2^8 \cdot 3^3 \cdot 5 \cdot 7$	$p \neq 3$
$[SL_2(9) \overset{2}{\square} SL_2(5)]_{32}$	$2^8 \cdot 5^8$	6	4800	$2^7 \cdot 3^3 \cdot 5^2$	$p \neq 5$
$[((SL_2(5) \circ SL_2(5)) : 2 \underset{\sqrt{5}}{\overset{2(6)}{\boxtimes}} (C_3 \overset{2(2)}{\boxtimes} D_8)]_{32,1}$	$2^{16} \cdot 3^{16}$	8	1440+ 2880	$2^{10} \cdot 3^3 \cdot 5^2$	$p \neq 5$
$[((SL_2(5) \circ SL_2(5)) : 2 \underset{\sqrt{5}}{\overset{2(6)}{\boxtimes}} (C_3 \overset{2(2)}{\boxtimes} D_8)]_{32,2}$	$2^{16} \cdot 3^{16} \cdot 5^{16}$	16	1440	$2^{10} \cdot 3^3 \cdot 5^2$	+
$[C_{15} : C_4 \overset{2(2)}{\boxtimes} F_4]_{32}$	$3^{16} \cdot 5^8$	8	1440+ 2160	$2^{10} \cdot 3^3 \cdot 5$	$p \neq 2, 5$
$[(2_-^{1+4}.Alt_5 \underset{\infty,2}{\otimes} SL_2(5)) \underset{\sqrt{5}}{\overset{2(2)}{\boxtimes}} D_{10}]_{32}$	$2^{16} \cdot 5^8$	8	9600+ 12000+ 14400	$2^{11} \cdot 3^2 \cdot 5^3$	$p \neq 5$
$[(SL_2(5) \underset{\sqrt{5}}{\otimes} D_{10}) \underset{\infty,3}{\overset{2(3)}{\boxtimes}} (SL_2(3) \overset{2}{\square} C_3)]_{32}$	$2^{16} \cdot 3^{16} \cdot 5^8$	12	4800+ 5760	$2^8 \cdot 3^3 \cdot 5^2$	$p \neq 5$
$[SL_2(5) \circ (C_5 \overset{2}{\boxtimes} D_{24})]_{32}$	11^8	6	$3 \cdot 1440$ +2400	$2^6 \cdot 3^2 \cdot 5^2$	$\mathbb{Q}[\sqrt{3}, \sqrt{5}]$ $p \neq 5$
$[SL_2(3) \overset{2(2)}{\circ} (C_5 \overset{2}{\boxtimes} D_{24})]_{32}$	11^8	6	$5 \cdot 720$ +960	$2^7 \cdot 3^2 \cdot 5$	$\mathbb{Q}[\sqrt{3}, \sqrt{5}]$ $p \neq 5$
$[SL_2(5) \circ (C_5 \underset{\sqrt{5}'}{\overset{2}{\boxtimes}} Q_{24})]_{32}$	$5^8 \cdot 11^8$	8	1440	$2^6 \cdot 3^2 \cdot 5^2$	$\mathbb{Q}[\sqrt{3}, \sqrt{5}]$, +
$[SL_2(3) \overset{2(2)}{\circ} (C_5 \underset{\sqrt{5}'}{\overset{2}{\boxtimes}} Q_{24})]_{32}$	$5^8 \cdot 11^8$	8	720	$2^7 \cdot 3^2 \cdot 5$	$\mathbb{Q}[\sqrt{3}, \sqrt{5}]$, +

The tables are organized as follows:

First a name of the rational irreducible maximal finite (r.i.m.f.) subgroup G of $GL_{32}(\mathbb{Q})$ resp. of a invariant lattice L of minimal determinant is given (cf. Section 5). The next columns indicate the abelian invariants of the discriminant group $L^{\#}/L$ of L, the minimum of the square lengths of the nonzero vectors in L and the number of these minimal vectors decomposed into orbit lengths under G. The fourth column gives the order of G and the last column allows one to deduce some information on the lattice of G-invariant lattices. A + in this column indicates that G is lattice sparse, that is that all invariant lattices are obtained from L by multiplying with invertible elements in the commuting algebra of G (which is $\mathbb{Q}$ except for the groups 4 and 5 and the last four groups), taking duals with respect to positive definite invariant quadratic forms (which are unique up to scalar multiples except for the last four groups), and taking intersections and sums. If G is not lattice sparse the primes p are indicated such that all G-sublattices of L of p-power index can be obtained by a combination of the four operations above.

The next two tables are built up similarly.

Theorem 21.1. *The groups in this table are maximal finite subgroups of* $GL_{32}(\mathbb{Q})$.

Proof. For all groups except the last four groups G the theorem follows easily by showing that G is the automorphism group of all its invariant lattices. Only the last four groups are not uniform. Let H be a r.i.m.f. supergroup of one of these groups G. As for the other groups one easily shows that the space of invariant quadratic forms of H is a proper subspace of $\mathcal{F}(G)$. Therefore $C_{\mathbb{Q}^{32\times 32}}(H)$ is isomorphic to one of the proper subfields of $\mathbb{Q}[\sqrt{3}, \sqrt{5}]$ and either H is uniform or $dim(\mathcal{F}(H)) = 2$ and H satisfies the conditions of [NeP 95, Theorem (II.4)]. Thus there is $(F, L) \in \mathcal{F}_{>0}(H) \times \mathcal{Z}(H)$ such that F is integral on L and the prime divisors of the determinant $det(F, L)$ of a Gram matrix of F on L divide the group order $|H|$. By the formula in [Schu 05] the largest prime which may divide the order of H is 31. Since the determinants of the integral positive definite lattices $(F, L) \in \mathcal{F}_{>0}(G) \times \mathcal{Z}(G)$ which involve only prime divisors ≤ 31 are divisible by 11, one concludes that 11 divides $|H|$. Moreover H is primitive, because G is primitive. Since the possible normal p-subgroups of H do not admit an automorphism of order 11 it follows that H has a quasi-semi-simple normal subgroup. Let $(F, L) \in \mathcal{F}_{>0}(H) \times \mathcal{Z}(H)$ be an H-invariant integral lattice of minimal determinant. The 11-modular representation $\delta : G \to GL_8(11)$ obtained from the action of G on $L^{\#}/L$ is faithful because 11 does not divide the order of G and extends to a representation of H. So H has an image $\bar{H}$ with $G \leq \bar{H} \leq GL_8(11)$. Then the determination of the minimal degrees of a projective representation of a finite Chevalley group in nondefining characteristic in [LaS 74] resp. [SeZ 93] show that the simple composition factors of H are contained in [CCNPW 85]. One now gets the result from the classification of the nonabelian finite simple groups and [CCNPW 85]. □

Only one primitive r.i.m.f. group of $GL_{36}(\mathbb{Q})$ whose lattices are not tensor products turns up:

lattice L	$det(L)$	$min(L)$	$\lvert L_{min}\rvert$	$\lvert Aut(L)\rvert$	lattice sparse
$[SL_2(19) \overset{2(2)}{\circ} SL_2(3)]_{36}$	$2^{18} \cdot 19^8$	10	4104	$2^6 \cdot 3^3 \cdot 5 \cdot 19$	+

Some invariants of the occurring primitive r.i.m.f. groups of dimension 40, that are not tensor products:

lattice L	$det(L)$	$min(L)$	$\|L_{min}\|$	$\|Aut(L)\|$	lattice sparse
$[\pm U_3(5) : 3 \overset{2}{\square} C_3]_{40}$	5^{20}	8	10500	$2^6 \cdot 3^4 \cdot 5^3 \cdot 7$	$p \neq 3$
$[\pm 5_+^{1+2} : SL_2(5).2 \overset{2}{\square} SL_2(5)]_{40}$	5^4	4	3600	$2^8 \cdot 3^2 \cdot 5^5$	$p \neq 5$
$[SL_2(11) \overset{2(2)}{\underset{\infty,2}{\otimes}} 2_-^{1+4}.Alt_5]_{40}$	11^8	6	13200	$2^{10} \cdot 3^2 \cdot 5^2 \cdot 11$	$p \neq 2$
$[U_5(2) \overset{2(2)}{\underset{\infty,2}{\otimes}} 2_-^{1+4}.Alt_5]_{40}$	1	4	39600	$2^{18} \cdot 3^6 \cdot 5^2 \cdot 11$	$p \neq 2$
$[2.U_4(2) \overset{2(2)}{\circ} SL_2(3)]_{40}$	$2^8 \cdot 3^8$	6	960+ 11520+ 12960+ 17280+ 25920	$2^{10} \cdot 3^5 \cdot 5$	$p \neq 2$
$[SL_2(11) \overset{2(3)}{\sqsupset} C_{12}.C_2]_{40}$	$11^8 \cdot 2^{20}$	8	1320	$2^6 \cdot 3^2 \cdot 5 \cdot 11$	$p \neq 11$
$[SL_2(11) \overset{2(2)}{\sqsupset} SL_2(3)]_{40}$	11^8	6	$2 \cdot 1320$ +3960 +5280 +7920	$2^6 \cdot 3^2 \cdot 5 \cdot 11$	$p \neq 2, 11$
$[SL_2(19) \overset{2(3)}{\circ} \tilde{S}_3]_{40}$	$3^{20} \cdot 19^8$	10	4104	$2^5 \cdot 3^3 \cdot 5 \cdot 19$	$+$
$[2.Alt_7 \overset{2(3)}{\circ} \tilde{S}_3]_{40}$	$3^8 \cdot 7^8$	6	5040 +3360	$2^6 \cdot 3^3 \cdot 5 \cdot 7$	$p \neq 3$
$[2.Alt_7 \circ \tilde{S}_3]_{40}$	$2^{12} \cdot 3^8$	6	$3 \cdot 1680$ +10080	$2^5 \cdot 3^3 \cdot 5 \cdot 7$	$p \neq 2, 3$
$[2.U_4(3).4 \overset{2(3)}{\circ} \tilde{S}_3]_{40}$	3^{20}	6	3360	$2^{12} \cdot 3^7 \cdot 5 \cdot 7$	$+$
$F_4 \overset{2(2)}{\otimes} [\pm S_6]_{10}$	$2^{16} \cdot 3^{16}$	6	960 +1440	$2^{12} \cdot 3^4 \cdot 5$	$p \neq 2$
$[(L_2(11) \otimes_{\sqrt{-11}} SL_2(3) \otimes S_3).2]_{40}$	2^{20}	6	$2 \cdot 2640$ +15840 $+2 \cdot 31680$	$2^7 \cdot 3^3 \cdot 5 \cdot 11$	$p \neq 3, 11$
$[2.M_{12}.2 \otimes_{\sqrt{-2}} GL_2(3)]_{40}$	2^{20}	6	21120 +63360	$2^{11} \cdot 3^4 \cdot 5 \cdot 11$	$\mathbb{Q}[\sqrt{-2}]$

References

[Ami 55] S. A. Amitsur, *Finite subgroups of division rings.* Trans. Amer. Math. Soc. **80** (1955) 361-386. MR **17:**577c

[Asc 84] M. Aschbacher, *On the maximal subgroups of the finite classical groups.* Invent. Math. **76** (1984) 469-514. MR **86a:**20054

[BaN 97] C. Bachoc, G. Nebe, *Classification of two genera of 32-dimensional lattices over the Hurwitz order.* Exp. Math. **6**, no. 2 (1997) 151-162. CMP 98:02

[BeZ 85] H. Benz and H. Zassenhaus, *Über verschränkte Produktordnungen.* J. Number Theory **20** (1985) 282-298. MR **86k:**11072

[Bli 17] H. F. Blichfeldt, *Finite Collineation groups.* The University of Chicago Science Series (1917).

[Bra 51] R. Brauer, *On the algebraic structure of group rings.* J. Math. Soc. Japan **3** (1951) 237-251. MR **13:**442b

[BBNWZ 78] H. Brown, R. Bülow, J. Neubüser, H. Wondratschek and H. Zassenhaus, *Crystallographic Groups of Four-Dimensional Space.* Wiley, 1977. MR **58:**4109

[Coh 80] A. M. Cohen, *Finite quaternionic reflection groups.* J. Alg. **64** (1980) 293-324. MR **81g:**20093

[CCNPW 85] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *Atlas of finite groups.* Oxford University Press, 1985. MR **88g:**20025

[CoS 88] J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices and Groups.* Springer-Verlag, 1988. MR **89a:**11067

[CuR 81] C. W. Curtis and I. Reiner *Methods of Representation Theory.* Wiley Classics, MR **90k:**20001

[DuV 64] Patrick Du Val, *Homographies Quaternions and Rotations.* Oxford Math. Monographs (1964). MR **29:**6361

[Eic 38] M. Eichler, *Über die Idealklassenzahl hyperkomplexer Systeme.* Math. Zeitschrift **43** (1938) 481-494.

[Fei 74] W. Feit, *On integral representations of finite groups.* Proc. London Math Soc. (3) **29** (1974) 633-683. MR **51:**10448

[Fei 82] W. Feit, *The Representation Theory of Finite Groups.* North-Holland (1982). MR **83g:**20001

[Fei 83] W. Feit, *The Computations of some Schur indices.* Israel J. Math. **46**, no. 4 (1983) 274-300. MR **85j:**20004

[GAP 94] M. Schönert (ed.), *Groups, Algorithms and Programming.* Lehrstuhl D für Mathematik, RWTH Aachen, Germany (1994) available via anonymous ftp on the server ftp://ftp-gap.dcs.st-and.ac.uk/pub/gap

[Gro 90] B. Gross, *Group representations and Lattices.* J. Amer. Math. Soc. **3** (1990) 929-960. MR **92a:**11077

[HaS 85] B. Hartley and M. A. Shahabi Shojaei, *Finite quasisimple groups of* 2×2 *matrices over a division ring.* Math. Proc. Cambridge Phil. Soc. **97** (1985) 415-420. MR **86h:**20073

[HoP 89] D. F. Holt and W. Plesken, *Perfect Groups.* Oxford University Press, 1989. MR **91c:**20029

[Hup 67] B. Huppert, *Endliche Gruppen I.* Springer-Verlag 1967. MR **37:**302

[JLPW 95] C. Jansen, K. Lux, R. A. Parker and R. A. Wilson, *An atlas of Brauer Characters.* Clarendon Press, Oxford, 1995. MR **96k:**20016

[KANT 93] M. Pohst e.a., *KANT-V2.* Computer Algebra in Deutschland (edited by Fachgruppe Computeralgebra der GI, DMV and GAMM), (1993) 212–218.

[Lan 70] S. Lang, *Algebraic Number Theory.* Addison-Wesley 1970. MR **44:**181

[LaS 74] V. Landazuri, G.M. Seitz, *On the minimal degrees of projective representations of the finite Chevalley groups.* J. Alg. **32** (1974) 418-443. MR **50:**13299

[Lor 71] F. Lorenz, *Über die Berechnung der Schurschen Indizes von Charakteren endlicher Gruppen.* J. Number Theory **3** (1971) 60-103. MR **43:**3357

[MAGMA] W. Bosma and J. Cannon, *Magma V2.* Handbook of Magma functions, University of Sydney, 1997.

[Neb 95] G. Nebe, *Endliche rationale Matrixgruppen vom Grad 24.*, Dissertation, RWTH Aachen (1995).

[Neb 96] G. Nebe, *Finite subgroups of* $GL_{24}(\mathbb{Q})$. Exp. Math. **5**, no. 3 (1996) 163-195. MR **98b:**20080

[Neb 96a] G. Nebe, *Finite subgroups of* $GL_n(\mathbb{Q})$ *for* $25 \leq n \leq 31$. Comm. Alg. **24(7)** (1996) 2341-2397. MR **97e:**20066

[Neb 98] G. Nebe, *Some cyclo-quaternionic lattices.* J. Algebra **199** (1998) 472-498. CMP 98:06

[Neb 97] G. Nebe, *The normaliser action and strongly modular lattices.*, Enseign. Math. **43** (1997) 67-76. CMP 97:15

[NeP 95] G. Nebe and W. Plesken, *Finite rational matrix groups of degree 16.* Mem. Amer. Math. Soc. **116**, no. 556 (1995). MR **95k:**20081

[O'Me 62] O. T. O'Meara, *Introduction to Quadratic Forms.*, 3rd corrected printing, Springer 1973.

[Ple 91] W. Plesken, *Some applications of representation theory.* Progress in Mathematics, vol 95: Representation Theory of Finite Groups and Finite-Dimensional Algebras, (ed. by G. O. Michler, C. M. Ringel), (1991) 477-496. MR **92k:**20019

[PlH 84] W. Plesken and W. Hanrath, *The Lattices of Six-Dimensional Euclidean Space.* Math. Comp. **43**, no. 168 (1984) 573-587. MR **85m:**11036

[PlN 95] W. Plesken and G. Nebe, *Finite rational matrix groups.* Mem. Amer. Math. Soc. **116**, no. 556 (1995). MR **95k:**20081

[PlP 85] W. Plesken and M. Pohst, *Constructing Integral Lattices With Prescribed Minimum. I.* Math. Comp. **45**, no. 171 (1985) 209-221. MR **87e:**11077

[PlS 97] W. Plesken and B. Souvignier, *Computing isometries of lattices.* J. Symbolic Comp. **24** no. 3-4 (1997). CMP 98:05

[Rei 75] I. Reiner, *Maximal Orders.* Academic Press, 1975. MR **52:**13910

[Scha 85] W. Scharlau, *Quadratic and Hermitian Forms.* Grundlehren der mathematischen Wissenschaften 270, Springer-Verlag (1985). MR **86k:**11022

[Schu 05] I. Schur, *Über eine Klasse von endlichen Gruppen linearer Substitutionen.* in I. Schur: *Gesammelte Abhandlungen I.* Springer Verlag (1973) 128-142. MR **57:**2858a

[Schu 07] I. Schur, Über die Darstellung der endlichen Gruppen durch gebrochen lineare Substitutionen, pp. 198-250 in I. Schur: *Gesammelte Abhandlungen I.* Springer-Verlag 1973 resp. *J. reine u. angew. Math.* **132** (1907) 85-137. MR **57:**2858a

[SeZ 93] G. M. Seitz and A. E. Zalesskii, *On the minimal degrees of projective representations of the finite Chevalley Groups, II.* J. Alg.**158** (1993) 233-243. MR **94h:**20017

[ShW 86] M. Shirvani and B. A. F. Wehrfritz, *Skew Linear Groups.* London Math. Soc. Lecture Notes **118** (1986). MR **89h:**20001

[Sou 94] B. Souvignier, *Irreducible finite integral matrix groups of degree 8 and 10.* Math. Comp. **63**, no. 207 (1994) 335-350. MR **94i:**20091

[Sri 68] B. Srinivasan, *The characters of the finite symplectic group $Sp(4,q)$.* Trans. Amer. Math. Soc. **131** (1968) 488-525. MR **36:**3897

[Swa 62] R. G. Swan, *Projective modules over group rings and maximal orders.* Ann, Math. **76**, no. 1 (1962) 55-61. MR **25:**3066

[Tay 92] D. E. Taylor, *The geometry of the classical groups.* Sigma Series in Pure Mathematics, Heldermann Verlag, Berlin (1992). MR **94d:**20028

[Tie 97] P. H. Tiep, *Globally Irreducible Representations of Finite Groups and Integral Lattices.* Geometriae Dedicata **64** (1997) 85-123. CMP 97:08

[Tit 80] J. Tits, *Four presentations of Leech's lattice.* Finite simple groups II, Academic Press (1980) 303-307.

[Vig 80] M.-F. Vignéras, *Arithmétique des Algèbres de Quaternions.* Springer Lecture Notes 800 (1980). MR **82i:**12016

[Was 82] L. C. Washington, *Introduction to Cyclotomic Fields.* Springer-Verlag 1982. MR **85g:**11001

[Wat 62] G. L. Watson, *Transformations of a Quadratic Form which do not Increase the Class-Number.* Proc. London Math. Soc. (3), 12 (1962) 577-587. MR **26:**81

[Wal 62] G. E. Wall, *On the Clifford Collineation, Transform and Similarity Groups (IV).* Nagoya Math. J. **21** (1962) 199-222. MR **26:**6242

[Win 72] D. L. Winter, *The automorphism group of an extraspecial p-group.* Rocky mountains J. Math. Vol. 2, No 2 (1972) 159-168 MR **45:**6911

[Wit 52] E. Witt, *Die algebraische Struktur des Gruppenringes einer endlichen Gruppe über einem Zahlkörper.* J. Reine Angew. Math. **190** (1952) 231-245. MR **14:**845a

[Yam 74] T. Yamada, *The Schur subgroup of the Brauer group.* Springer Lecture Notes 397 (1974) MR **50:**456

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