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BASES IN EQUIVARIANT K-THEORY

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Abstract. In this paper we construct a canonical basis for the equivariant K-theory of the flag manifold of a semisimple simply connected $\mathbf{C}$-algebraic group with respect to the action of a maximal torus times $\mathbf{C}^*$. We relate this basis to the canonical basis of the "periodic module" for the affine Hecke algebra. The construction admits a (conjectural) generalization to the case where the flag manifold is replaced by the zero set of a nilpotent vector field.

0. Introduction

0.1. A number of objects in representation theory possess canonical bases with many remarkable properties. This is the case for example for the Hecke algebra attached to a Coxeter group [KL1] and for the highest weight modules of quantized enveloping algebras [L5], [K]. In these cases, the bases in question can be constructed either purely algebraically, or geometrically, using perverse sheaves on an appropriate algebraic variety.

On the other hand, there are other objects in representation theory which seem to possess canonical bases, although it is not clear how to understand this in terms of perverse sheaves and using pure algebra seems to be not strong enough either.

0.2. In this paper we experiment with an alternative method to construct canonical bases in which perverse sheaves are replaced by coherent sheaves, equivariant with respect to a suitable group action. Thus, we try to

(a) interpret the representation theoretic objects as equivariant K-theory $K_H(Y)$ of a suitable variety Y with an action of a reductive group H;
(b) construct an "antilinear involution" $^- : K_H(Y) \to K_H(Y)$;
(c) define an inner product on $K_H(Y)$.

(Note that ingredients of the same type as (b) or (c) were used in the earlier works [KL1], [L5], [K], but unlike these references, here we use them in a K-theory context.)

Having done these steps, we can consider the elements in $K_H(Y)$ that are fixed by the involution (b) and have self-inner product approximately equal to 1 (in a suitable sense); these elements form a candidate for a "signed basis" of $K_H(Y)$. (A signed basis consists of $\pm$ the elements of a basis.)

The fact that equivariant K-theory can be used to realize geometrically certain representation theoretic objects was found in [L1] where the principal series representations of affine Hecke algebras were treated from this point of view. One of

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the main ideas of [L1] was to interpret the parameter v of the Hecke algebra as the standard element of the representation ring of $\mathbf{C}^*$ acting in a natural way on the equivariant K-theory of a space with $\mathbf{C}^*$-action. This turned out to be a common feature of the subsequent use of equivariant K-theory in the representation theory of affine Hecke algebras, see [KL2], [G].

Most of this paper is concerned with carrying out steps (b),(c) for the equivariant K-theory of the variety $\mathcal{B}_y$ of Borel subalgebras of a semisimple Lie algebra containing a fixed nilpotent element y, with respect to a torus (as large as possible) that acts on $\mathcal{B}_y$.

We define the involution (b) as a product of three factors: one is the Serre-Grothendieck duality for coherent sheaves; the second one comes from an involution ("opposition") of $\mathcal{B}_y$ itself; the third one is essentially the action of the longest element in the Hecke algebra action.

(The first factor is a substitute in our case for the Verdier duality in the theory of perverse sheaves; the second and third factors are correcting factors.)

Similarly, we define the inner product (c) in terms of three ingredients: one is the Tor-product of sheaves on the smooth variety (Slodowy) in which $\mathcal{B}_y$ naturally lies; the other two are the same correcting factors that are used in the definition of the involution (b).

0.3. As explained above, the involution and the inner product give rise to a candidate for a signed basis of our equivariant K-theory space. We can show that this is indeed a signed basis:

(a) in the case where the nilpotent is 0;
(b) in the case where the nilpotent is subregular (in type D or E);
(c) in the case where the nilpotent is regular.

(The case (c) is trivial.) In the case (a) we show that the signed basis obtained by the K-theoretic method coincides with the one obtained combinatorially in [L2] for the "periodic Hecke algebra module".

0.4. One of the main motivations of this paper came from the desire to understand geometrically the periodic W-graph constructed in [L6]. In the case where y is a nilpotent element that is regular inside some Levi subalgebra of a parabolic subalgebra, our K-theoretic candidate for a signed basis should conjecturally provide a geometric interpretation of the periodic W-graph in [L6].

0.5. The paper is organized as follows. In Section 1 we discuss affine Hecke algebras by combining the points of view of [L2] and [L4]. One new result here is Lemma 1.22 which describes the effect of the involution $^-$ of the affine Hecke algebra $\mathcal{H}$ on the basis of a large commutative subalgebra of $\mathcal{H}$. In Sections 2 and 3 we discuss the "periodic module" of the affine Hecke algebra (introduced in [L2]), its canonical basis and its "dual basis". (There is a slight difference from [L2] in that, here we consider affine Hecke algebras of simply connected type, while in [L2] we considered affine Hecke algebras of adjoint type.) In Sections 4 and 5 the periodic module is described as a tensor product and the natural inner product on it is described from this point of view. Section 6 is a review of the results about equivariant coherent sheaves that are needed later on.

In Sections 7 and 8 we establish an isomorphism between the affine Hecke algebra and an equivariant K-group of the Steinberg variety of triples. The main results here are 7.25 and 8.6. A result similar to 7.25 and 8.6 appeared in [G] and in [KL2].

(An exposition can also be found in [CG].) While [G] emphasized the algebra structure of the K-group given by convolution, [KL2] emphasized the structure of the K-group as a bimodule over the affine Hecke algebra. Note that [KL2] used topological K-homology instead of algebraic K-theory (which in the present case gives the same result) and the definition of the isomorphism was based on a construction which is not immediately equivalent to the one given here, which is closer to the one in [G]. But, in contrast to the definition in [G], our definition of the isomorphism is symmetric in the two factors of $\mathcal{B} \times \mathcal{B}$ and is better suited for the purposes of this paper. For these reasons, we found it necessary to give a self-contained proof of 7.25 and 8.6 (which differs significantly from the earlier proofs).

One of the byproducts of the analysis in Section 8 is Corollary 8.13, which gives a K-theoretic interpretation of the results in [L3] concerning certain elements of the basis [KL1] of the affine Hecke algebra (corresponding to dominant weights); it describes them as explicit coherent sheaves on the Steinberg variety.

In Section 9, we give a K-theoretic interpretation of the involution $^-$ in [KL1] of the affine Hecke algebra (Proposition 9.12).

In Section 10, the periodic module and its canonical basis [L2] are interpreted in K-theoretic terms. This is done along the steps (a),(b),(c) in 0.2. The interpretation is such that it admits a generalization which corresponds to replacing the nilpotent 0 by an arbitrary nilpotent element. This generalization is discussed in Sections 11 and 12. The main conjecture of the paper is stated in 12.19 (see also 12.22). In Section 13 we verify that conjecture for a subregular nilpotent in type D_4.

Section 14 contains some speculations on possible connections of the matters discussed above with the (unrestricted) representations of Lie algebras in characteristic p.

Contents

1. The affine Hecke algebra

1.1. Let $\mathcal{V}$ be an $\mathbf{R}$-vector space with a given basis $(\alpha_i)_{i\in I}$ which is the set of simple roots of a root system $\mathcal{R}$ in $\mathcal{V}$. Let $(\check{\alpha}_i)_{i\in I}$ be the corresponding set of simple coroots in $\mathrm{Hom}(\mathcal{V}, \mathbf{R})$. Let $\mathcal{R}^+ \subset \mathcal{V}$ be the set of positive roots and let $\check{\mathcal{R}}^+ \subset \mathrm{Hom}(\mathcal{V}, \mathbf{R})$

be the set of positive coroots. We denote by $\check{\alpha}$ the coroot corresponding to $\alpha \in \check{\mathcal{R}}^+$. Let

$$\begin{aligned}
&\nu = |\mathcal{R}^+|,\\
&\mathcal{C}^+ = \{x \in \mathcal{V} | \check{\alpha}_i(x) > 0 \quad \forall i \in I\},\\
&\mathcal{X} = \{x \in \mathcal{V} | \check{\alpha}_i(x) \in \mathbf{Z} \quad \forall i \in I\},\\
&\mathcal{X}_{\text{ad}} = \textstyle\bigoplus_{i \in I} \mathbf{Z}\alpha_i \subset \mathcal{X},\\
&\mathcal{X}_+ = \{x \in \mathcal{X} | \check{\alpha}_i(x) \in \mathbf{N} \quad \forall i \in I\},\\
&\underline{\mathcal{X}} = \mathcal{X}/\mathcal{X}_{\text{ad}} \text{ (a finite group)}.
\end{aligned}$$

Let E be a principal homogeneous space for $\mathcal{V}$; we write the action of $\mathcal{V}$ on E as $(x, e) \mapsto x + e$ for $x \in \mathcal{V}, e \in E$. We regard E naturally as an affine space over $\mathbf{R}$. We assume given a subset $\mathfrak{E}$ of E such that $\mathfrak{E}$ is a principal homogeneous space for $\mathcal{X}$, for the restriction of the $\mathcal{V}$-action to $\mathcal{X}$.

For each $\epsilon \in \mathfrak{E}$ and each $\alpha \in \mathcal{R}^+$, let $\sigma_{\epsilon,\alpha} : E \to E$ be the reflection $x + \epsilon \mapsto (x - \check{\alpha}(x)\alpha) + \epsilon$ for all $x \in \mathcal{V}$. Let $\mathcal{I}$ be the subgroup of the group of affine transformations of E generated by the reflections $\sigma_{\epsilon,\alpha}$ for various ϵ, α as above. The action of $\mathcal{I}$ on E is written as $(\omega, e) \mapsto {}^\omega e$.

Let $H_{\epsilon,\alpha}$ be the fixed point set of $\sigma_{\epsilon,\alpha}$. Let $\mathcal{F}$ be the set of hyperplanes in E of the form $H_{\epsilon,\alpha}$ for some ϵ, α as above. The set of points of E that are not contained in any hyperplane in $\mathcal{F}$ is a union of connected components called *alcoves*. Let X be the set of alcoves.

Let $\mathcal{F}^*$ be the set of affine hyperplanes of E of the form H_{ϵ,α_i} for some $i \in I, \epsilon \in \mathfrak{E}$. The set of points of E that are not contained in any hyperplane in $\mathcal{F}^*$ is a union of connected components called *boxes*.

The set of points of E that belong to exactly one hyperplane in $\mathcal{F}$ is a union of connected components called *faces*. Let S be the set of $\mathcal{I}$-orbits in the set of faces. Then S is finite. If $s \in S$ and F is a face in the $\mathcal{I}$-orbit s, we say that F is of type s. For any alcove A and any $s \in S$, there is a unique face $\delta_s(A)$ of type s such that $\delta_s(A)$ is in the closure of A.

1.2. For $s \in S$ and $A \in X$ we denote by sA the unique alcove $\neq A$ such that $\delta_s(A) = \delta_s(sA)$. Then $A \mapsto sA$ is an involution of X.

The maps $A \to sA$ generate a group of permutations of X which is a Coxeter group (W^a, S) (an affine Weyl group). The action of W^a on X is denoted by $(w, A) \mapsto w(A)$. This action is simply transitive and it commutes with the action of $\mathcal{I}$ on X (which is also simply transitive). Let $l : W^a \to \mathbf{N}$ be the standard length function. Let $\leq$ be the standard partial order on W^a.

1.3. If $x \in \mathcal{X}$ and A is an alcove, then $x + A$ is an alcove; if F is a face of type s, then $x + F$ is face of type, say, $x + s$, where $x + s$ depends only on x, s, not on F. Then $(x, s) \mapsto x + s$ is an $\mathcal{X}$-action on S; it factors through an $\underline{\mathcal{X}}$-action on S.

For $\epsilon \in \mathfrak{E}$, let A_ϵ^+ (resp. A_ϵ^-) be the unique alcove contained in $\mathcal{C}^+ + \epsilon$ (resp. in $-\mathcal{C}^+ + \epsilon$) and having ϵ in its closure. Let S_ϵ be the set of all $s \in S$ such that there exists a face of type s which has ϵ in its closure. Let W_ϵ be the subgroup of W^a generated by S_ϵ (a finite Coxeter group). Let w_ϵ be the unique element of W_ϵ such that $w_\epsilon(A_\epsilon^-) = A_\epsilon^+$.

For $H = H_{\epsilon,\alpha} \in \mathcal{F}$, let $E_H^+ = \check{\alpha}^{-1}(0, \infty) + \epsilon, E_H^- = \check{\alpha}^{-1}(-\infty, 0) + \epsilon$ be the two half spaces determined by H. For $A \in X, s \in S$, let H be the hyperplane in $\mathcal{F}$ that contains $\delta_s(A)$; we say that $A < sA$ if $A \subset E_H^-, sA \subset E_H^+$; we say that $A > sA$ if $A \subset E_H^+, sA \subset E_H^-$.

There is a unique function $d : X \times X \to \mathbf{Z}$ such that $d(A, sA) = 1$ if $A < sA$ and

(a) $d(A,B) + d(B,C) + d(C,A) = 0$

for all $A, B, C \in X$. (See [L2, 1.4].)

Let $\le$ be the partial order on X generated by the relation $A \le B$ if $B = \sigma_{\epsilon,\alpha} A$ for some ϵ, α and $d(A,B) = 1$.

1.4. Let $\mathcal{A} = \mathbf{Z}[v, v^{-1}]$ where v is an indeterminate. The Hecke algebra $\mathcal{H}_{\mathrm{ad}}$ associated to the affine Weyl group W^a is the associative $\mathcal{A}$-algebra which, as an $\mathcal{A}$-module, has basis $\{\tilde{T}_w | w \in W^a\}$ and multiplication defined by the rules

$$(\tilde{T}_s + v^{-1})(\tilde{T}_s - v) = 0, \quad (s \in S),$$
$$\tilde{T}_w \tilde{T}_{w'} = \tilde{T}_{ww'} \text{ if } l(w) + l(w') = l(ww').$$

1.5. Let $^- : \mathcal{H}_{\mathrm{ad}} \to \mathcal{H}_{\mathrm{ad}}$ be the involution of the ring $\mathcal{H}_{\mathrm{ad}}$ such that $\bar{\tilde{T}}_w = \tilde{T}_{w^{-1}}^{-1}$ for all $w \in W_a$ and $\overline{v^n} = v^{-n}$ for all $n \in \mathbf{Z}$.

By [KL1], for any $w \in W^a$, there is a unique element $c'_w \in \mathcal{H}_{\mathrm{ad}}$ such that $\bar{c}'_w = c'_w$ and $c'_w = \sum_{y \in W^a; y \le w} \pi_{y,w} \tilde{T}_y$ where $\pi_{w,w} = 1$ and $\pi_{y,w} \in v^{-1}\mathbf{Z}[v^{-1}]$ for $y < w$.

1.6. Let $p \mapsto p^\circ$ be the ring homomorphism $\mathcal{A} \to \mathcal{A}$ which takes v^n to $(-v)^n$ for any n. Let $\chi \mapsto \chi^\circ$ be the involution of the ring $\mathcal{H}_{\mathrm{ad}}$ which takes $\tilde{T}_s$ to $\tilde{T}_s^{-1}$ for any $s \in S$ and v^n to $(-v)^n$ for any n. This commutes with $^- : \mathcal{H}_{\mathrm{ad}} \to \mathcal{H}_{\mathrm{ad}}$.

1.7. The action of $\underline{\mathcal{X}}$ on S in 1.3 induces an action of $\underline{\mathcal{X}}$ on W^a by Coxeter group automorphisms $(\iota, w) \mapsto {}^\iota w$. Let $\underline{x}$ be the image of $x \in \mathcal{X}$ in $\underline{\mathcal{X}}$. From the definitions we have $({}^{\underline{x}}w)(x + A) = (x + wA)$ for $w \in W^a, x \in \mathcal{X}, A \in X$.

Let $\hat{W}^a$ be the semidirect product of $\underline{\mathcal{X}}$ with W^a. Thus, an element of $\hat{W}^a$ is a product ιw where $w \in W^a, \iota \in \underline{\mathcal{X}}$ and $(\iota w)(\iota' w') = (\iota + \iota')({}^{-\iota'} w) w'$. We extend the length function $l : W^a \to \mathbf{N}$ to a function $l : \hat{W}^a \to \mathbf{N}$ by $l(\iota w) = l(w)$ for $\iota \in \underline{\mathcal{X}}, w \in W^a$. Then $l(w\iota) = l(w)$ automatically holds. We have $l(\iota) = 0$ for $\iota \in \underline{\mathcal{X}}$.

1.8. The Hecke algebra $\mathcal{H}$ associated to $\hat{W}^a$ is the associative $\mathcal{A}$-algebra defined by the generators $\tilde{T}_w$ $(w \in \hat{W}^a)$ and the relations

$$(\tilde{T}_s + v^{-1})(\tilde{T}_s - v) = 0 \quad \text{for } s \in S;$$
$$\tilde{T}_w \tilde{T}_{w'} = \tilde{T}_{ww'} \quad \text{for } w, w' \in \hat{W}^a \text{ with } l(w) + l(w') = l(ww').$$

Then $\{\tilde{T}_w | w \in \hat{W}^a\}$ is an $\mathcal{A}$-basis of $\mathcal{H}$. We identify $\mathcal{H}_{\mathrm{ad}}$ with the subalgebra of $\mathcal{H}$ generated as an $\mathcal{A}$-module by $\{\tilde{T}_w | w \in W^a\}$. We have $\tilde{T}_\iota \tilde{T}_w = \tilde{T}_{\iota w}$ and $\tilde{T}_w \tilde{T}_\iota = \tilde{T}_{w\iota}$ for $\iota \in \underline{\mathcal{X}}, w \in \hat{W}^a$.

Let $^- : \mathcal{H} \to \mathcal{H}$ be the involution of the ring $\mathcal{H}$ such that $\bar{\tilde{T}}_w = \tilde{T}_{w^{-1}}^{-1}$ for all $w \in \hat{W}^a$ and $\overline{v^n} = v^{-n}$ for all $n \in \mathbf{Z}$. For $\iota \in \underline{\mathcal{X}}$ we have $\overline{\tilde{T}_\iota} = \tilde{T}_\iota$.

For any $w \in \hat{W}^a$ we define $c'_w = \tilde{T}_\iota c'_{w_1} = c'_{w_2} \tilde{T}_\iota \in \mathcal{H}$ where $\iota \in \underline{\mathcal{X}}, w_1 \in W^a, w_2 \in W^a$ are such that $w = \iota w_1 = w_2 \iota$ and $c'_{w_1}, c'_{w_2} \in \mathcal{H}_{\mathrm{ad}}$ are as in 1.5. Then $\{c'_w | w \in \hat{W}^a\}$ is an $\mathcal{A}$-basis of $\mathcal{H}$.

1.9. For $\epsilon_1, \epsilon_2 \in \mathfrak{E}$ we denote by $\tau_{\epsilon_2,\epsilon_1}$ the unique element of W^a such that $\tau_{\epsilon_2,\epsilon_1} A^+_{\epsilon_1} = A^+_{\epsilon_2}$.

From the definitions, for $x \in \mathcal{X}$ we have $\underline{x}\tau_{\epsilon_2,\epsilon_1}(-\underline{x}) = \tau_{x+\epsilon_2,x+\epsilon_1}$ in $\hat{W}^a$.

For $\epsilon_1, \epsilon_2, \epsilon_3$ in $\mathfrak{E}$ we have $\tau_{\epsilon_3,\epsilon_2}\tau_{\epsilon_2,\epsilon_1} = \tau_{\epsilon_3,\epsilon_1}$.

Lemma 1.10. *Let $\epsilon \in \mathfrak{E}$. For $x \in \mathcal{X}$ we set $x^\epsilon = (-\underline{x})\tau_{x+\epsilon,\epsilon} \in \hat{W}^a$.*

(a) *The map $x \mapsto x^\epsilon$ is an isomorphism of $\mathcal{X}$ onto a subgroup of $\hat{W}^a$.*
(b) *For any $x, \tilde{x} \in \mathcal{X}$ we have $\underline{\tilde{x}} x^\epsilon (-\underline{\tilde{x}}) = x^{\tilde{x}+\epsilon}$.*

We prove (a). The injectivity of our map is clear. Let $x_1, x_2 \in \mathcal{X}$. We have

$$\begin{aligned} x_1^\epsilon x_2^\epsilon &= (-\underline{x}_1)\tau_{x_1+\epsilon,\epsilon}(-\underline{x}_2)\tau_{x_2+\epsilon,\epsilon} = (-\underline{x}_1)(-\underline{x}_2)\tau_{x_2+x_1+\epsilon,x_2+\epsilon}\tau_{x_2+\epsilon,\epsilon} \\ &= (-\underline{x}_1 - \underline{x}_2)\tau_{x_2+x_1+\epsilon,\epsilon} = (x_1+x_2)^\epsilon. \end{aligned}$$

This proves (a). We prove (b). We have

$$\underline{\tilde{x}} x^\epsilon (-\underline{\tilde{x}}) = (-\underline{x})\underline{\tilde{x}}\tau_{x+\epsilon,\epsilon}(-\underline{\tilde{x}}) = (-\underline{x})\tau_{\tilde{x}+x+\epsilon,\tilde{x}+\epsilon} = x^{\tilde{x}+\epsilon}.$$

The lemma is proved.

1.11. Let W be the subgroup of $\mathrm{Aut}(\mathcal{V})$ generated by the reflections $\sigma_i : \mathcal{V} \to \mathcal{V}$, $\sigma_i(x) = x - \check{\alpha}_i(x)\alpha_i$ with $i \in I$. Note that W together with $\{\sigma_i | i \in I\}$ is a finite Coxeter group. We denote the action of W on $\mathcal{V}$ by $(w, x) \mapsto {}^w x$.

For $w \in W$ let $\mathrm{sgn}_w = (-1)^{l(w)}$ where $l : W \to \mathbf{N}$ is the length function. Let w_0 be the longest element of W.

Given $\epsilon \in \mathfrak{E}$, there is a unique group isomorphism $W \xrightarrow{\sim} W_\epsilon$ which, for any $i \in I$, carries σ_i to $s_i^\epsilon \in S_\epsilon$ where s_i^ϵ is the type of the unique face of A^+_ϵ contained in the hyperplane H_{ϵ,α_i}. It is clear that s_i^ϵ depends only on the $\mathcal{X}_{\mathrm{ad}}$-orbit of ϵ in $\mathfrak{E}$, not on ϵ itself. The same holds for x^ϵ (by 1.10(b)).

Lemma 1.12. *Let $\epsilon \in \mathfrak{E}, i \in I, x \in \mathcal{X}$, and let $x' = {}^{\sigma_i}x \in \mathcal{X}$. Then $s_i^\epsilon x^\epsilon s_i^\epsilon = x'^\epsilon$ in $\hat{W}^a$.*

Let $\epsilon' = x + \epsilon$. Let $\sigma = \sigma_{\epsilon,\alpha_i}, \sigma' = \sigma_{\epsilon',\alpha_i}$. We have $\sigma(\epsilon') = x' + \epsilon$. We must prove that $s_i^\epsilon(-\underline{x})\tau_{\epsilon',\epsilon}s_i^\epsilon = (-\underline{x}')\tau_{\sigma(\epsilon'),\epsilon}$. We have $-\underline{x}' = -\underline{x}$. If F is the face of type s_i^ϵ of A^+_ϵ, then the type of the face $x + F$ of $A^+_{\epsilon'}$ is $s = \underline{x}s_i^\epsilon(-\underline{x})$. It is enough to prove that $s\tau_{\epsilon',\epsilon}s_i^\epsilon = \tau_{\sigma(\epsilon'),\epsilon}$. This follows from

$$s\tau_{\epsilon',\epsilon}s_i^\epsilon A^+_\epsilon = s\tau_{\epsilon',\epsilon}{}^\sigma A^+_\epsilon = {}^\sigma(s\tau_{\epsilon',\epsilon}(A^+_\epsilon)) = {}^\sigma(sA^+_{\epsilon'}) = {}^{\sigma\sigma'}A^+_{\epsilon'} = A^+_{\sigma(\epsilon')} = \tau_{\sigma(\epsilon'),\epsilon}A^+_\epsilon.$$

The lemma is proved.

Lemma 1.13. *Let $\epsilon \in \mathfrak{E}, x \in \mathcal{X}$. For any $i \in I$ we have $l(s_i^\epsilon x^\epsilon s_i^\epsilon) = l(x^\epsilon)$.*

From the definitions, $l(x^\epsilon)$ is the number of hyperplanes in $\mathcal{F}$ that separate A^+_ϵ from $x + A^+_\epsilon$. This is easily seen to be the number $\sum_{\check{\alpha}\in\check{\mathcal{R}}^+} |\check{\alpha}(x)|$. Similarly, if $x' = {}^{\sigma_i}x$, so that $s_i^\epsilon x^\epsilon s_i^\epsilon = x'^\epsilon$, then $l(x'^\epsilon)$ is the number $\sum_{\check{\alpha}\in\check{\mathcal{R}}^+} |\check{\alpha}(x')|$. But this is clearly equal to $\sum_{\check{\alpha}\in\check{\mathcal{R}}^+} |\check{\alpha}(x)|$. The lemma is proved.

Lemma 1.14. *Let $\epsilon \in \mathfrak{E}, x \in \mathcal{X}_+$.*

(a) *For any $s \in S_\epsilon$ we have $l(x^\epsilon s) = l(x^\epsilon) + 1$.*
(b) *For any $s \in S_\epsilon$ such that $sx^\epsilon \neq x^\epsilon s$, we have $l(sx^\epsilon) = l(x^\epsilon) - 1$.*

We set $\tau = \tau_{x+\epsilon,\epsilon}$. To prove (a), it suffices to show that $l(\tau s) = l(\tau) + 1$. This follows from [L2, 3.6].

We prove (b). Let $s' = x + s$. It suffices to show that $l(s'\tau) = l(\tau) - 1$. By assumption we have $s'\tau \neq \tau s$. From 1.13, we have that $l(s'\tau s) = l(\tau)$. It remains to use the following result. Let $w \in W^a, s, s' \in S$ be such that $l(s'ws) = l(w), l(ws) > l(w), s'w \neq ws$; then $l(s'w) < l(w)$. (The proof in [DL, 1.6.4] is applicable to any Coxeter group.) The lemma is proved.

1.15. *In the remainder of this paper we fix an $\mathcal{X}_{\mathrm{ad}}$-orbit $\mathbf{s}$ in $\mathfrak{E}$.*

Following an idea of J. Bernstein, for any $x \in \mathcal{X}$ we set

$$\theta_x = \tilde{T}_{x_2^\epsilon}^{-1}\tilde{T}_{x_1^\epsilon}$$

where $\epsilon \in \mathbf{s}$ and $x_1, x_2 \in \mathcal{X}_+, x = x_1 - x_2$. (Clearly, such x_1, x_2 exist.) We show that θ_x is independent of the choices. The independence of ϵ follows from 1.10(b). To show independence of x_1, x_2 it is enough to verify the following statement.

If $x, x' \in \mathcal{X}_+$, then $\tilde{T}_{x^\epsilon}\tilde{T}_{x'^\epsilon} = \tilde{T}_{(x+x')^\epsilon}$.

To show this, it is enough to show that $l(x^\epsilon) + l(x'^\epsilon) = l((x+x')^\epsilon)$. By [L2, 3.6], we have

$$l(x^\epsilon) = d(A_\epsilon^+, A_{x+\epsilon}^+), \quad l(x'^\epsilon) = d(A_\epsilon^+, A_{x'+\epsilon}^+) = d(A_{x+\epsilon}^+, A_{x+x'+\epsilon}^+),$$
$$l((x+x')^\epsilon) = d(A_\epsilon^+, A_{x+x'+\epsilon}^+),$$

and it remains to use 1.3(a). The previous argument shows not only that θ_x is well defined, but also that $\theta_x\theta_{x'} = \theta_{x+x'}$ for $x, x' \in \mathcal{X}$. Note that θ_x depends on the choice of $\mathbf{s} \subset \mathfrak{E}$.

1.16. Let $\epsilon \in \mathfrak{E}, x \in \mathcal{X}_+, i \in I$ be such that $\check{\alpha}_i(x) = 1$. Let $s = s_i^\epsilon \in S_\epsilon$. Let

$$y = x^\epsilon s x^\epsilon s = x^\epsilon({}^{\sigma_i}x)^\epsilon = (x + {}^{\sigma_i}x)^\epsilon = (2x - \alpha_i)^\epsilon.$$

Lemma 1.17. *We have $\tilde{T}_y = \tilde{T}_{x^\epsilon}\tilde{T}_s^{-1}\tilde{T}_{x^\epsilon}\tilde{T}_s^{-1}$.*

(Compare [L4].) From our assumption we have $2x - \alpha_i \in \mathcal{X}_+$. Let $\tilde{l} : \mathcal{X} \to \mathbf{Z}$ be the homomorphism given by $\tilde{l}(x') = \sum_{\check{\alpha} \in \check{\mathcal{R}}^+} \check{\alpha}(x')$. As we have seen in the proof of 1.13, for $x' \in \mathcal{X}_+$ we have $\tilde{l}(x') = l(x'^\epsilon)$. Thus we have

$$l(x^\epsilon s x^\epsilon s) = l((2x - \alpha_i)^\epsilon) = \tilde{l}(2x - \alpha_i) = 2\tilde{l}(x) - \tilde{l}(\alpha_i) = 2l(x^\epsilon) - 2.$$

From 1.14(a) we have

(a) $l(x^\epsilon s x^\epsilon) = l(x^\epsilon s x^\epsilon s) + 1 = 2l(x^\epsilon) - 1$.

Since $x \in \mathcal{X}_+$ and $sx^\epsilon \neq x^\epsilon s$ (recall that $\check{\alpha}_i(x) = 1$), from 1.14(b) we have

(b) $l(sx^\epsilon) = l(x^\epsilon) - 1$.

Hence

(c) $l(x^\epsilon) + l(sx^\epsilon) = l(x^\epsilon s x^\epsilon)$

(both sides are $2l(x^\epsilon) - 1$). From (a),(b),(c), we deduce

$$\tilde{T}_{x^\epsilon s x^\epsilon s} = \tilde{T}_{x^\epsilon s x^\epsilon}\tilde{T}_s^{-1} = \tilde{T}_{x^\epsilon}\tilde{T}_{sx^\epsilon}\tilde{T}_s^{-1} = \tilde{T}_{x^\epsilon}\tilde{T}_s^{-1}\tilde{T}_{x^\epsilon}\tilde{T}_s^{-1}.$$

The lemma is proved.

Lemma 1.18. *Let $\epsilon \in \mathbf{s}, i \in I, x \in \mathcal{X}$. Let $s = s_i^\epsilon$.*

(a) *If $\check{\alpha}_i(x) = 0$, then $\tilde{T}_s\theta_x = \theta_x\tilde{T}_s$.*

(b) *If $\check{\alpha}_i(x) = 1$, then $\theta_{\sigma_i x} = \tilde{T}_s^{-1}\theta_x\tilde{T}_s^{-1}$.*

We prove (a). We can write $x = x_1 - x_2$ where $x_1, x_2 \in \mathcal{X}_+$ and $\check{\alpha}_i(x_1) = \check{\alpha}_i(x_2) = 0$. We are then reduced to the case where $x \in \mathcal{X}_+$. In this case, $l(x^\epsilon s) = l(x^\epsilon) + 1$; hence $l(sx^\epsilon) = l(x^\epsilon) + 1$ (since $sx^\epsilon = x^\epsilon s$). We then have

$$\tilde{T}_s \tilde{T}_{x^\epsilon} = \tilde{T}_{sx^\epsilon} = \tilde{T}_{x^\epsilon s} = \tilde{T}_{x^\epsilon} \tilde{T}_s.$$

and (a) follows.

We prove (b). We can write $x = x_1 - x_2$ where $x_1, x_2 \in \mathcal{X}_+$ and $\check{\alpha}_i(x_1) = 1, \check{\alpha}_i(x_2) = 0$. Using (a), we are then reduced to the case where $x = x_1$. Thus we may assume that $x \in \mathcal{X}_+$. Since ${}^{\sigma_i}x = (2x - \alpha_i) - x$ with $2x - \alpha_i$ and x in $\mathcal{X}_+$, we have, using the definition and 1.17,

$$\theta_{{}^{\sigma_i}x} = \tilde{T}_{x^\epsilon}^{-1} \tilde{T}_{(2x-\alpha_i)^\epsilon} = \tilde{T}_{x^\epsilon}^{-1} \tilde{T}_{x^\epsilon} \tilde{T}_s^{-1} \tilde{T}_{x^\epsilon} \tilde{T}_s^{-1} = \tilde{T}_s^{-1} \theta_x \tilde{T}_s^{-1}.$$

The lemma is proved.

1.19. Let $\mathcal{AX}$ be the group algebra of $\mathcal{X}$ with coefficients in $\mathcal{A}$. The basis element of $\mathcal{AX}$ corresponding to $x \in \mathcal{X}$ is denoted by $[x]$. Let $\mathcal{AX}_q$ be the quotient field of $\mathcal{AX}$.

We use the following convention: for $p = \sum_{x \in \mathcal{X}} c_x [x] \in \mathcal{AX}$ (finite sum with $c_x \in \mathcal{A}$) we set $\theta_p = \sum_{x \in \mathcal{X}} c_x \theta_x \in \mathcal{H}$.

We identify W with a standard parabolic subgroup of W^a via the homomorphism $\sigma_i \mapsto s_i^\epsilon$ (see 1.11) where $\epsilon \in \mathbf{s}$. (This identification depends on $\mathbf{s}$ but not on the choice of ϵ in $\mathbf{s}$.) For $w \in W$ we shall denote the corresponding element of W^a again by w. The restriction of the length function of W^a to W is just the standard length function of W. Thus, the elements $\tilde{T}_w$ $(w \in W)$ of $\mathcal{H}_{\mathrm{ad}}$ (or $\mathcal{H}$) are well defined. These elements, and the elements θ_x $(x \in \mathcal{X})$ satisfy the following relations:

(a) $(\tilde{T}_{\sigma_i} + v^{-1})(\tilde{T}_{\sigma_i} - v) = 0, \quad (i \in I)$;
(b) $\tilde{T}_w \tilde{T}_{w'} = \tilde{T}_{ww'}$ if $l(ww') = l(w) + l(w')$;
(c) $\theta_x \tilde{T}_{\sigma_i} - \tilde{T}_{\sigma_i} \theta_{{}^{\sigma_i}x} = (v - v^{-1}) \theta_{\frac{[x] - [{}^{\sigma_i}x]}{1 - [-\alpha_i]}}$;
(d) $\theta_x \theta_{x'} = \theta_{x+x'}$;
(e) $\theta_0 = 1$.

The only relation that needs comment is (c). In that relation the fraction is taken in $\mathcal{AX}_q$, but it actually belongs to $\mathcal{AX}$. To prove (c) we note that for fixed $i \in I$, the set of $x \in \mathcal{X}$ for which (c) holds is a subgroup of $\mathcal{X}$, as one easily verifies. From 1.18, we see that (c) holds for elements $x \in \mathcal{X}$ such that $\check{\alpha}_i(x) \in \{0, 1\}$. Since such elements generate $\mathcal{X}$ as a group, it follows that (c) holds for all x.

1.20. The following result has been stated by J. Bernstein (unpublished; but see [L4] for a proof).

The elements $\tilde{T}_w$ $(w \in W)$ and θ_x $(x \in \mathcal{X})$ with the relations 1.19(a)-(e) *form a presentation of the $\mathcal{A}$-algebra $\mathcal{H}$.*

1.21. In [L4] it is shown that

$$\{\tilde{T}_w \theta_x | w \in W, x \in \mathcal{X}\} \quad \textit{and} \quad \{\theta_x \tilde{T}_w | w \in W, x \in \mathcal{X}\} \textit{are } \mathcal{A}\textit{-bases of } \mathcal{H}.$$

Lemma 1.22. *For any $x \in \mathcal{X}$ we have $\overline{\theta_x} = \tilde{T}_{w_0}^{-1} \theta_{w_0 x} \tilde{T}_{w_0}$.*

First we note that the set of $x \in \mathcal{X}$ for which the lemma holds is a subgroup of $\mathcal{X}$, as an easy verification shows. Hence, it is enough to prove the lemma under the additional assumption that $x \in \mathcal{X}_+$ (such x generate the group $\mathcal{X}$).

Let $x \in \mathcal{X}_+$; then we have also $-^{w_0}x \in \mathcal{X}_+$. Let $\epsilon \in \mathbf{s}$. Under our imbedding $W \subset W^a$, the element $w_0 \in W$ corresponds to $w_\epsilon \in W^a$. We have

$$\overline{\theta_x} = \overline{\tilde{T}_{x^\epsilon}} = \tilde{T}^{-1}_{(-x)^\epsilon}, \quad \theta_{^{w_0}x} = \tilde{T}^{-1}_{(-^{w_0}x)^\epsilon}.$$

We must prove that $\tilde{T}^{-1}_{(-x)^\epsilon} = \tilde{T}^{-1}_{w_0}\tilde{T}^{-1}_{(-^{w_0}x)^\epsilon}\tilde{T}_{w_0}$ or equivalently (after applying $^-$ to both sides), that $\tilde{T}_{x^\epsilon} = \tilde{T}_{w_0}\tilde{T}_{(^{w_0}x)^\epsilon}\tilde{T}^{-1}_{w_0}$, or that $\tilde{T}_{x^\epsilon}\tilde{T}_{w_0} = \tilde{T}_{w_0}\tilde{T}_{(^{w_0}x)^\epsilon}$. In this equation, both sides are $\tilde{T}_{x^\epsilon w_\epsilon} = \tilde{T}_{w_\epsilon(^{w_0}x)^\epsilon}$, since

$$l(w_\epsilon) + l((^{w_0}x)^\epsilon) = l(x^\epsilon) + l(w_\epsilon) = l(x^\epsilon w_\epsilon) = l(w_\epsilon(^{w_0}x)^\epsilon).$$

(The first equality follows from 1.13; the second equality follows from 1.14(a); the third equality is obvious.) The lemma is proved.

1.23. Let $p \mapsto p^\dagger$ be the involution of the $\mathcal{A}$-algebra $\mathcal{A}\mathcal{X}$ defined by $[x]^\dagger = [-x]$ for all $x \in \mathcal{X}$. We extend this to a field involution $^\dagger : \mathcal{A}\mathcal{X}_q \to \mathcal{A}\mathcal{X}_q$.

Let $p \mapsto \bar{p}$ be the involution of the ring $\mathcal{A}\mathcal{X}$ defined by $\overline{v^n[x]} = v^{-n}[x]$ for all $n \in \mathbf{Z}, x \in \mathcal{X}$. Let

$$\rho = \frac{1}{2}\sum_{\alpha \in \mathcal{R}^+} \alpha \in \mathcal{X}, \quad \delta = [-\rho]\prod_{\alpha \in \mathcal{R}^+}([\alpha] - 1) \in \mathcal{A}\mathcal{X},$$
$$\Delta = \prod_{\alpha \in \mathcal{R}^+}(1 - v^{-2}[\alpha]) \in \mathcal{A}\mathcal{X}.$$

Then

$$\Delta^\dagger = \prod_{\alpha \in \mathcal{R}^+}(1 - v^{-2}[-\alpha]) \in \mathcal{A}\mathcal{X}, \quad \bar{\Delta} = \prod_{\alpha \in \mathcal{R}^+}(1 - v^2[\alpha]) \in \mathcal{A}\mathcal{X},$$
$$\bar{\Delta}^\dagger = \prod_{\alpha \in \mathcal{R}^+}(1 - v^2[-\alpha]) \in \mathcal{A}\mathcal{X}.$$

1.24. Let $\chi \mapsto \chi^\blacktriangle$ be the involutive antiautomorphism of the $\mathcal{A}$-algebra $\mathcal{H}$ defined by $\tilde{T}_w \mapsto \tilde{T}_{w^{-1}}$ for all $w \in \hat{W}^a$.

Let $\chi \mapsto \hat{\chi}$ be the involutive antiautomorphism of the $\mathcal{A}$-algebra $\mathcal{H}$ which takes $\tilde{T}_{\sigma_i}$ to itself for any $i \in I$ and takes θ_x to itself for any $x \in \mathcal{X}$.

For $i \in I$, let $i^* \in I$ be defined by $w_0\sigma_i w_0 = \sigma_{i^*}$. Let $\chi \mapsto \chi^\divideontimes$ be the algebra involution of $\mathcal{H}$ which carries $\tilde{T}_{\sigma_i}$ to $\tilde{T}_{\sigma_{i^*}}$ for all $i \in I$ and θ_x to $\theta_{-^{w_0}x}$ for all $x \in \mathcal{X}$.

Let $\chi \mapsto \chi^\triangle$ be the algebra automorphism of $\mathcal{H}$ which carries $\tilde{T}_{\sigma_i}$ to itself for any $i \in I$ and $\overline{\theta_x}$ to θ_{-x} for any $x \in \mathcal{X}$.

1.25. We show that

(a) $$(\chi^\triangle)\hat{} = \chi^\blacktriangle$$

for any $\chi \in \mathcal{H}$. To do this we may assume that χ runs through the set of algebra generators $\tilde{T}_{\sigma_i}, \overline{\theta_x}$ of $\mathcal{H}$. If $\chi = \tilde{T}_{\sigma_i}$, we have $(\chi^\triangle)\hat{} = (\tilde{T}_{\sigma_i})\hat{} = \tilde{T}_{\sigma_i} = \tilde{T}^\blacktriangle_{\sigma_i}$. If $\chi = \overline{\theta_x}$, we have $(\chi^\triangle)\hat{} = (\theta_{-x})\hat{}$. It remains to show that $(\theta_{-x})\hat{} = (\overline{\theta_x})^\blacktriangle$, or equivalently

(b) $$\theta_{-x} = \overline{\theta_x^\blacktriangle}$$

(note that the involutions $^-$ and $^\blacktriangle$ commute). It is easy to check that the set of $x \in \mathcal{X}$ for which (b) holds is a subgroup of $\mathcal{X}$. Hence, it is enough to prove (b) in the case where $-x \in \mathcal{X}_+$. In this case, the left hand side of (b) is $\tilde{T}_w$, where $w = (-x)^\epsilon$. On the other hand, $\theta_x = \theta^{-1}_{-x} = \tilde{T}^{-1}_w$; hence the right hand side of (b) is $\overline{(\tilde{T}^{-1}_w)^\blacktriangle} = \overline{\tilde{T}^{-1}_{w^{-1}}} = \tilde{T}_w$. This proves (b) and hence (a).

2. The $\mathcal{H}$-modules $\mathcal{M}_d, \mathcal{M}_{d'}$

2.1. If M_1, M_2 are $\mathcal{H}_{\rm ad}$-modules (resp. $\mathcal{H}$-modules), a group homomorphism $f : M_1 \to M_2$ is said to be $\mathcal{H}_{\rm ad}$-*antilinear* (resp. $\mathcal{H}$-*antilinear*) if $f(\chi m) = \bar{\chi} f(m)$ for all $m \in M_1$ and all $\chi \in \mathcal{H}_{\rm ad}$ (resp. $\chi \in \mathcal{H}$).

2.2. Let M be the set of all formal sums $m = \sum_{A \in X} m_A A$ where $m_A \in \mathcal{A}$ for all $A \in X$. We regard M as an $\mathcal{A}$-module in the obvious way. For $m \in M$, we set $\text{supp}(m) = \{A \in X | m_A \neq 0\}$.

For any $s \in S$ we define $\tilde{T}_s : M \to M$ by $\tilde{T}_s(\sum_A m_A A) = \sum_A m_A \tilde{T}_s A$ where

$$\tilde{T}_s A = \begin{cases} sA, & \text{if } A < sA, \\ sA + (v - v^{-1})A, & \text{if } sA < A. \end{cases}$$

This defines an $\mathcal{H}_{\rm ad}$-module structure on M. We have

$$\tilde{T}_s^{-1} A = \begin{cases} sA, & \text{if } sA < A, \\ sA - (v - v^{-1})A, & \text{if } A < sA. \end{cases}$$

Let M_c be the $\mathcal{A}$-submodule of M consisting of all elements $m \in M$ such that $\text{supp}(m)$ is a finite set. Then M_c is an $\mathcal{H}_{\rm ad}$-submodule of M.

2.3. For any $\epsilon \in \mathfrak{E}$, we set

$$e_\epsilon = \Sigma' v^{-d(A, A_\epsilon^+)} A \in M_c,$$

where Σ' is sum over (the finite set of) all $A \in X$ such that the closure of A contains ϵ. Let $M_{d'}$ be the $\mathcal{H}_{\rm ad}$-submodule of M_c generated by the elements e_ϵ for various $\epsilon \in \mathfrak{E}$.

By [L2, 2.12], [L6, 4.14], there exists a unique $\mathcal{H}_{\rm ad}$-antilinear map $b : M_{d'} \to M_{d'}$ such that $b(e_\epsilon) = e_\epsilon$ for any $\epsilon \in \mathfrak{E}$. We have $b^2 = 1$.

Let $B \in X$. By [L2, 5.2, 7.3], [L6, 11.2], there exists a unique element $B^\flat = \sum_{A \in X; A \leq B} \Pi_{A,B} A \in M_{d'}$ such that

(a) $$b(B^\flat) = B^\flat,$$

(b) $$\Pi_{B,B} = 1, \quad \Pi_{A,B} \in v^{-1}\mathbf{Z}[v^{-1}] \text{ for } A < B.$$

Let $B \in X$. We can find a unique $\epsilon \in \mathfrak{E}$ such that A_ϵ^+ and B are contained in the same box and a unique element $u \in W^a$ such that $u(A_\epsilon^-) = B$. By [L2, 5.2], we have

$$B^\flat = \sum_{y \in W^a; y \leq u, l(yw_\epsilon) = l(y) + l(w_\epsilon)} \pi_{yw_\epsilon, u} \tilde{T}_y e_\epsilon.$$

By [L2, 8.3],

(c) $$\{B^\flat | B \in X\} \text{ is an } \mathcal{A}\text{-basis of } M_{d'}.$$

2.4. For any $\epsilon \in \mathfrak{E}$, we set

$$e'_\epsilon = \Sigma'(-v)^{-d(A^-_\epsilon, A)} A \in M_c$$

where Σ' is as in 2.3. Let $M'_{d'}$ be the $\mathcal{H}_{\mathrm{ad}}$-submodule of M_c generated by the elements e'_ϵ for various $\epsilon \in \mathfrak{E}$.

Given $\tilde{\epsilon} \in \mathfrak{E}$, we define a bijection $\psi : M \to M$ by $\psi(\sum_A m_A A) = \sum_A m^\circ_A ({}^\omega A)$ where $\omega \in \mathcal{I}$ is the unique element such that ${}^\omega A^-_{\tilde{\epsilon}} = A^+_{\tilde{\epsilon}}$.

For any $\chi \in \mathcal{H}_{\mathrm{ad}}, m \in M$ we have $\psi(\chi m) = \chi^\circ \psi(m)$. (Notation of 1.6.) The proof is entirely similar to that of [L2, 2.10]. From the definition we have $\psi(e_\epsilon) = e'_{{}^\omega\epsilon}$ for any $\epsilon \in \mathfrak{E}$. It follows that ψ restricts to an isomorphism of groups $M_{d'} \xrightarrow{\sim} M'_{d'}$. We can now transfer the results in 2.3 to $M'_{d'}$ via ψ.

Let $B \in X$. We can find a unique $\epsilon \in \mathfrak{E}$ such that A^-_ϵ and B are contained in the same box and a unique element $u \in W^a$ such that $u(A^+_\epsilon) = B$. Then

$$B_\flat := \sum_{y \in W^a; y \le u, l(yw_\epsilon) = l(y) + l(w_\epsilon)} \pi^\circ_{yw_\epsilon, u} \tilde{T}_y^{-1} e'_\epsilon$$

is a finite linear combination of alcoves $A \ge B$ so that the coefficient of A is in $v^{-1}\mathbf{Z}[v^{-1}]$ if $A > B$ and is 1 if $A = B$.

2.5. Let $M_\ge$ be the set of all $m \in M$ such that $\mathrm{supp}(m)$ is contained in a set of the form $\{B \in X | A_0 \le B\}$ for some $A_0 \in X$. This is an $\mathcal{H}_{\mathrm{ad}}$-submodule of M.

For $B \le A$ in X we define $\Pi'_{B,A} \in \mathbf{Z}[v^{-1}]$ inductively by $\Pi'_{B,B} = 1$ and

(a) $$\sum_{D \in X; B \le D \le C} \Pi'_{B,D} \Pi_{D,C} = \delta_{B,C}$$

for $B \le C$ in X. Note that $\Pi'_{B,A} \in v^{-1}\mathbf{Z}[v^{-1}]$ for $B < A$.

For $B \in X$ let $B^\sharp = \sum_{A \in X; B \le A} \Pi'_{B,A} A \in M_\ge$. Let $\tilde{b} : M_\ge \to M_\ge$ be the $\mathcal{H}_{\mathrm{ad}}$-antilinear map defined in [L6, 9.6]. By [L6, 12.2], we have

(b) $\tilde{b}(B^\sharp) = B^\sharp$.

2.6. For any $\kappa \in \mathcal{X}_{\mathrm{ad}}$ we define (cf. [L3, 9.3])

$$\hat{P}(\kappa) = \sum_{\substack{n_1, \dots, n_\nu \in \mathbf{N} \\ n_1\alpha_1 + \dots + n_\nu \alpha_\nu = \kappa}} v^{-2(n_1 + \dots + n_\nu)} \in \mathcal{A}$$

where $\alpha_1, \dots, \alpha_\nu$ is a list of the positive roots. For any $\epsilon \in \mathfrak{E}$, we set

$$\tilde{e}_\epsilon = \sum_{\kappa \in \mathcal{X}_{\mathrm{ad}}} \hat{P}(\kappa) e'_{\kappa + \epsilon} \in M_\ge.$$

Theorem 2.7. *For any $\epsilon \in \mathfrak{E}$ we have $(A^-_\epsilon)^\sharp = \tilde{e}_\epsilon$.*

This follows immediately from [L2, 11.9] and [L3, 6.12, 9.2].

Corollary 2.8. *Let $B \in X$. We associate $\epsilon \in \mathfrak{E}$ and $u \in W^a$ to B as in 2.4. We have*

(a) $$B^\sharp = \sum_{y \in W^a; y \le u, l(yw_\epsilon) = l(y) + l(w_\epsilon)} \pi^\circ_{yw_\epsilon, u} \tilde{T}_y^{-1} \tilde{e}_\epsilon.$$

The proof is by reduction to the special case considered in 2.7. Let $\tilde{B}$ be the right hand side of (a). Using the definitions we have for any $\kappa \in \mathcal{X}_{\mathrm{ad}}$

$$(c'_u)^\circ e'_{\kappa+\epsilon} = \mathfrak{P} \sum_{y \in W^a; y \le u, l(yw_\epsilon) = l(y) + l(w_\epsilon)} \pi^\circ_{yw_\epsilon, u} \tilde{T}_y^{-1} e'_{\kappa+\epsilon}$$

where

$$\mathfrak{P} = \sum_{w \in W_\epsilon} (-v)^{2l(w)-\nu};$$

hence $(c'_u)^\circ \tilde{e}_\epsilon = \mathfrak{P}\tilde{B}$. Since $(c'_u)^\circ$ is fixed by $^- : \mathcal{H}_{\mathrm{ad}} \to \mathcal{H}_{\mathrm{ad}}$ and $\tilde{e}_\epsilon = (A^-_\epsilon)^\sharp$ is fixed by $\tilde{b} : M_\ge \to M_\ge$, it follows that $(c'_u)^\circ \tilde{e}_\epsilon$ is fixed by $\tilde{b}$ (recall that $\tilde{b}$ is $\mathcal{H}_{\mathrm{ad}}$-antilinear). Hence, $\mathfrak{P}\bar{\tilde{B}}$ is fixed by $\tilde{b}$. Since $\bar{\mathfrak{P}} = \mathfrak{P}$, it follows that $\tilde{B}$ is fixed by $\tilde{b}$.

From the definitions we have

$$\tilde{B} = \sum_{\kappa \in \mathcal{X}_{\mathrm{ad}}} \hat{P}(\kappa)(\kappa + B)_\flat.$$

Using the properties of $(\kappa + B)_\flat$ (see 2.4) and the definition of $\hat{P}(\kappa)$, we see that $\tilde{B}$ is a formal linear combination of alcoves $A \ge B$ so that the coefficient of A is in $v^{-1}\mathbf{Z}[v^{-1}]$ if $A > B$ and is 1 if $A = B$. By the characterization of $B^\sharp$ given in [L6, 12.2], we see that $B^\sharp = \tilde{B}$. The corollary is proved.

2.9. Let M_d be the $\mathcal{A}$-submodule of M spanned by $\{B^\sharp | B \in X\}$ (which is then an $\mathcal{A}$-basis of M_d). From [L6, 13.10] (which is applicable in the present case) we see that M_d is an $\mathcal{H}_{\mathrm{ad}}$-submodule of M. From 2.8, we see that M_d is in fact the $\mathcal{H}_{\mathrm{ad}}$-submodule of M generated by the elements $\tilde{e}_\epsilon$ for various $\epsilon \in \mathfrak{E}$.

The restriction of $\tilde{b} : M_\ge \to M_\ge$ to M_d will be denoted again by $\tilde{b}$. It is clear that $\tilde{b} : M_d \to M_d$ can be characterized as the unique $\mathcal{H}_{\mathrm{ad}}$-antilinear map $M_d \to M_d$ which maps $\tilde{e}_\epsilon$ to itself for any $\epsilon \in \mathfrak{E}$. We have

(a) $$M_{d'} \subset M_c \subset M_d.$$

(The first inclusion is obvious; the second inclusion follows from [L6, 13.10(b)] which is applicable in the present case.)

2.10. Let $\mathcal{M} = \mathcal{A}[\underline{\mathcal{X}}] \otimes_{\mathcal{A}} M$ where $\mathcal{A}[\underline{\mathcal{X}}]$ is the group algebra of $\underline{\mathcal{X}}$ over $\mathcal{A}$.

For $\iota \in \underline{\mathcal{X}}, m \in M$ we shall write $_\iota m$ instead of $\iota \otimes m \in \mathcal{M}$. In the case where $\iota = \underline{x}$ with $x \in \mathcal{X}$, we shall sometimes write $_x m$ instead of $_{\underline{x}} m$.

Using the definitions, one checks that the operators $\tilde{T}_w : \mathcal{M} \to \mathcal{M}$, $(w \in W^a)$ and $\tilde{T}_{\iota'} : \mathcal{M} \to \mathcal{M}$, $(\iota' \in \underline{\mathcal{X}})$ given by

$$\tilde{T}_w({}_\iota m) = {}_\iota(\tilde{T}_{{}^{-\iota}w}(m)), \quad \tilde{T}_{\iota'}({}_\iota m) = {}_{\iota+\iota'} m,$$

(where $m \in M, \iota \in \underline{\mathcal{X}}$) define an $\mathcal{H}$-module structure on $\mathcal{M}$. Let

$$\mathcal{M}_c = \mathcal{A}[\underline{\mathcal{X}}] \otimes_{\mathcal{A}} M_c, \quad \mathcal{M}_{d'} = \mathcal{A}[\underline{\mathcal{X}}] \otimes_{\mathcal{A}} M_{d'}, \quad \mathcal{M}_d = \mathcal{A}[\underline{\mathcal{X}}] \otimes_{\mathcal{A}} M_d.$$

Then $\mathcal{M}_{d'} \subset \mathcal{M}_c \subset \mathcal{M}_d$ are $\mathcal{H}$-submodules of $\mathcal{M}$. (See 2.9(a).)

Note that $\mathcal{M}_{d'}$ (resp. $\mathcal{M}_d$) is generated as an $\mathcal{H}$-module by the elements $_0 e_\epsilon$ (resp. $_0\tilde{e}_\epsilon$) for various $\epsilon \in \mathfrak{E}$.

There is a unique $\mathcal{H}$-antilinear map $\hat{b} : \mathcal{M}_{d'} \to \mathcal{M}_{d'}$ (resp. $\hat{\tilde{b}} : \mathcal{M}_d \to \mathcal{M}_d$) which maps $_0 e_\epsilon$ to itself (resp. $_0\tilde{e}_\epsilon$ to itself) for any $\epsilon \in \mathfrak{E}$. Indeed, the map $_\iota m \mapsto {}_\iota(b(m))$

(resp. ${}_\iota m \mapsto {}_\iota(\tilde{b}(m))$) has the required property. The uniqueness follows from the previous paragraph.

Let $B \in X$ and $\iota \in \underline{\mathcal{X}}$. There exists a unique element in $\mathcal{M}_{d'}$ which is fixed by $\hat{b}$ and is a finite linear combination of elements ${}_\iota A$ with $A \in X, A \leq B$ such that the coefficient of ${}_\iota A$ is in $v^{-1}\mathbf{Z}[v^{-1}]$ if $A < B$ and is 1 if $A = B$. This element is in fact ${}_\iota B^\flat$.

Moreover, there exists a unique element in $\mathcal{M}_d$ which is fixed by $\hat{\hat{b}}$ and is a formal linear combination of elements ${}_\iota A$ with $A \in X, A \geq B$ such that the coefficient of ${}_\iota A$ is in $v^{-1}\mathbf{Z}[v^{-1}]$ if $A > B$ and is 1 if $A = B$. This element is in fact ${}_\iota B^\sharp$.

Clearly, $\{{}_\iota B^\flat | B \in X, \iota \in \underline{\mathcal{X}}\}$ is an $\mathcal{A}$-basis of $\mathcal{M}_{d'}$ and $\{{}_\iota B^\sharp | B \in X, \iota \in \underline{\mathcal{X}}\}$ is an $\mathcal{A}$-basis of $\mathcal{M}_d$.

Lemma 2.11. *If $\epsilon' = x + \epsilon$ for some $x \in \mathcal{X}_+$, then, with the notation of 1.9, we have $\tilde{T}_{\tau_{\epsilon',\epsilon}} A^+_\epsilon = A^+_{\epsilon'}$ (equality in M).*

Let $w = \tau_{\epsilon',\epsilon}, n = l(w)$. By [L2, 3.6] we have $d(A^+_\epsilon, w(A^+_\epsilon)) = n$. Hence, there exists a sequence $s_1, s_2, \dots, s_n$ in S such that $w = s_n s_{n-1} \dots s_1$ and

$$A^+_\epsilon < s_1 A^+_\epsilon < s_2 s_1 A^+_\epsilon < \cdots < s_n s_{n-1} \dots s_1 A^+_\epsilon = w(A^+_\epsilon).$$

Then $\tilde{T}_{s_1} A^+_\epsilon = s_1 A^+_\epsilon, \tilde{T}_{s_2}(s_1 A^+_\epsilon) = s_2 s_1 A^+_\epsilon, \tilde{T}_{s_3}(s_2 s_1 A^+_\epsilon) = s_3 s_2 s_1 A^+_\epsilon, \dots$; hence $\tilde{T}_w(A^+_\epsilon) = \tilde{T}_{s_n} \dots \tilde{T}_{s_1}(A^+_\epsilon) = w(A^+_\epsilon)$. The lemma follows.

Lemma 2.12. *Let $\epsilon \in \mathfrak{E}$. Let $x \in \mathcal{X}_+$. In the $\mathcal{H}$-module $\mathcal{M}$, we have*

$$\tilde{T}_{x^\epsilon}({}_0A^+_\epsilon) = {}_{-x}A^+_{x+\epsilon}.$$

We set $\tau = \tau_{x+\epsilon,\epsilon}$. Using the definitions and Lemma 2.11, we have $\tilde{T}_{x^\epsilon}({}_0A^+_\epsilon)$ $= \tilde{T}_{-\underline{x}} \tilde{T}_\tau({}_0A^+_\epsilon) = {}_{-x}A_{x+\epsilon}$. The lemma is proved.

Lemma 2.13. *For any $\epsilon \in \mathbf{s}$ and any $x \in \mathcal{X}$ we have in the $\mathcal{H}$-module $\mathcal{M}$:*

$$\theta_x({}_0A^+_\epsilon) = {}_{-x}A^+_{x+\epsilon}.$$

We choose $x_1, x_2 \in \mathcal{X}_+$ such that $x = x_1 - x_2$. It is enough to show that

$$\tilde{T}_{x_1^\epsilon}({}_0A^+_\epsilon) = \tilde{T}_{x_2^\epsilon}({}_{x_2-x_1}A^+_{x_1-x_2+\epsilon}).$$

The left hand side equals ${}_{-x_1}A^+_{x_1+\epsilon}$ (see 2.12). The right hand side equals

$$\begin{aligned}\tilde{T}_{x_2^\epsilon} \tilde{T}_{\underline{x}_2-\underline{x}_1}({}_0A^+_{x_1-x_2+\epsilon}) &= \tilde{T}_{\underline{x}_2-\underline{x}_1} \tilde{T}_{x_2^{x_1-x_2+\epsilon}}({}_0A^+_{x_1-x_2+\epsilon}) \\ &= \tilde{T}_{\underline{x}_2-\underline{x}_1}({}_{-x_2}A^+_{x_1+\epsilon}) = {}_{-x_1}A^+_{x_1+\epsilon}.\end{aligned}$$

(The second equality again follows from 2.12.) The lemma is proved.

2.14. We define an $\mathcal{A}\mathcal{X}$-module structure $\centerdot$ on $\mathcal{M}$ (extending the $\mathcal{A}$-module structure) by

$$[x'] \centerdot {}_\iota \sum_A m_A A = {}_{\iota-\underline{x}'} \sum_A m_A (x' + A).$$

Lemma 2.15. *This $\mathcal{A}\mathcal{X}$-module structure commutes with the $\mathcal{H}$-module structure on $\mathcal{M}$.*

Let $s \in S, x, x' \in \mathcal{X}, A \in X$. We show that $[x'] \centerdot (\tilde{T}_{s x}A) = \tilde{T}_s([x'] \centerdot {}_x A)$. Let $s', s'' \in S$ be defined by $x + s' = s$ and $x - x' + s'' = s$ (hence $s'' = x' + s'$).

Assume first that $s'A > A$ (hence $s''(x + A) > (x + A)$). Then

$$[x'] \centerdot (\tilde{T}_s({}_x A)) = [x'] \centerdot {}_x(s'A) = {}_{x-x'}(x' + s'A)$$
$$= {}_{x-x'}(x' + s')(x' + A) = \tilde{T}_s({}_{x-x'}(x' + A)) = \tilde{T}_s([x'] \centerdot {}_x A).$$

Next assume that $s'A < A$ (hence $s''(x + A) < (x + A)$). Then

$$[x'] \centerdot (\tilde{T}_s({}_x A)) = [x'] \centerdot ({}_x(s'A + (v - v^{-1})A))$$
$$= {}_{x-x'}((x' + s'A) + (v - v^{-1})(x' + A))$$
$$= {}_{x-x'}((x' + s')(x' + A) + (v - v^{-1})(x' + A))$$
$$= \tilde{T}_s({}_{x-x'}(x' + A)) = \tilde{T}_s([x'] \centerdot {}_x A).$$

Next assume that $x, x', x'' \in \mathcal{X}$. We have

$$[x'] \centerdot \tilde{T}_{\underline{x}''}({}_x A) = [x'] \centerdot {}_{x+x''}A = {}_{x+x''-x'}(x' + A)$$
$$= \tilde{T}_{\underline{x}''}({}_{x-x'}(x' + A)) = \tilde{T}_{\underline{x}''}([x'] \centerdot ({}_x A)).$$

The lemma is proved.

Lemma 2.16. *Let $\iota \in \underline{\mathcal{X}}, \epsilon' \in \mathfrak{E}$. In $\mathcal{M}$ we have $\Delta \centerdot {}_\iota \tilde{e}_{\epsilon'} = {}_\iota e'_{\epsilon'}$.*

This is just a reformulation of the definition of $\tilde{e}_{\epsilon'}$.

2.17. Let $i \in I, x \in \mathcal{X}$. In the following computation (in $\mathcal{M}_{d'}$) we use the commutation formula 1.19(c), the equality ${}^{\sigma_i}\rho = \rho - \alpha_i$ and the identity $\tilde{T}_{\sigma_i}({}_0e_\epsilon) = v({}_0e_\epsilon)$:

$$\tilde{T}_{\sigma_i}\theta_{x+\rho}({}_0e_\epsilon) = \theta_{\sigma_i(x+\rho)}\tilde{T}_{\sigma_i}({}_0e_\epsilon) + (v - v^{-1})\theta_{\frac{[x+\rho]-[{}^{\sigma_i}(x+\rho)]}{1-[-\alpha_i]}}({}_0e_\epsilon)$$
$$= (v\theta_{{}^{\sigma_i}x+\rho-\alpha_i} + (v - v^{-1})\theta_{\frac{[x+\rho]-[{}^{\sigma_i}x+\rho-\alpha_i]}{1-[-\alpha_i]}})({}_0e_\epsilon)$$
$$= (v^{-1}\theta_{\frac{[{}^{\sigma_i}x+\rho]-[x+\alpha_i+\rho]}{[\alpha_i]-1}} + v\theta_{\frac{[x+\alpha_i+\rho]-[{}^{\sigma_i}x-\alpha_i+\rho]}{[\alpha_i]-1}})({}_0e_\epsilon).$$

2.18. Let $i \in I, x \in \mathcal{X}$. In the following computation (in $\mathcal{M}_d$) we use the commutation formula 1.19(c), the equality ${}^{\sigma_i}\rho = \rho - \alpha_i$ and the identity $\tilde{T}_{\sigma_i}({}_0\tilde{e}_\epsilon) = -v^{-1}({}_0\tilde{e}_\epsilon)$:

$$\tilde{T}_{\sigma_i}\theta_x({}_0\tilde{e}_\epsilon) = \theta_{{}^{\sigma_i}x}\tilde{T}_{\sigma_i}({}_0\tilde{e}_\epsilon) + (v - v^{-1})\theta_{\frac{[x]-[{}^{\sigma_i}x]}{1-[-\alpha_i]}}({}_0\tilde{e}_\epsilon)$$
$$= (-v^{-1}\theta_{{}^{\sigma_i}x}({}_0\tilde{e}_\epsilon) + (v - v^{-1})\theta_{\frac{[x]-[{}^{\sigma_i}x]}{1-[-\alpha_i]}}({}_0\tilde{e}_\epsilon)$$
$$= (v^{-1}\theta_{\frac{[{}^{\sigma_i}x]-[x+\alpha_i])}{[\alpha_i]-1}} + v\theta_{\frac{[x+\alpha_i]-[{}^{\sigma_i}x+\alpha_i]}{[\alpha_i]-1}})({}_0\tilde{e}_\epsilon).$$

3. Inner product on $\mathcal{M}_d, \mathcal{M}_{d'}$

3.1. Note that $\mathcal{M}_{d'}, \mathcal{M}_c, \mathcal{M}_d$ are $\mathcal{A}\mathcal{X}$-submodules of $\mathcal{M}$ (as in 2.14). Indeed, for $B \in X$ and $x, x' \in \mathcal{X}$, we have

(a) $$[x'] \centerdot {}_x B^\flat = {}_{x-x'}(x' + B)^\flat.$$

(b) $$[x'] \centerdot {}_x B^\sharp = {}_{x-x'}(x' + B)^\sharp.$$

The elements ${}_\iota B^\flat$ (resp. ${}_\iota B^\sharp$) where ι runs over $\underline{\mathcal{X}}$ and B runs over the set of alcoves contained in a fixed box form an $\mathcal{A}\mathcal{X}$-basis of $\mathcal{M}_{d'}$ (resp. of $\mathcal{M}_d$); the number of such elements is well known to be $|W|$. For $\mathcal{M}_{d'}$ this follows from (a) and 2.3(c); for $\mathcal{M}_d$ this is seen in a similar way.

On the other hand, the elements ${}_\iota A$ where ι, A run over the same set as above, clearly form an $\mathcal{AX}$-basis of $\mathcal{M}_c$. Therefore, we see that

(c) *$\mathcal{M}_{d'}, \mathcal{M}_c, \mathcal{M}_d$ are free $\mathcal{AX}$-modules of rank $|W|$.*

Tensoring the inclusions of $\mathcal{AX}$-modules $\mathcal{M}_{d'} \subset \mathcal{M}_c \subset \mathcal{M}_d$ with $\mathcal{AX}_q$ (over $\mathcal{AX}$) we obtain inclusions of $\mathcal{AX}_q$-vector spaces of dimension $|W|$

$$\mathcal{AX}_q \otimes \mathcal{M}_{d'} \subset \mathcal{AX}_q \otimes \mathcal{M}_c \subset \mathcal{AX}_q \otimes \mathcal{M}_d$$

which are necessarily equalities. We denote the resulting (single) $\mathcal{AX}_q$-vector space by $\mathcal{M}_q$. Note that we have naturally $\mathcal{M}_{d'} \subset \mathcal{M}_c \subset \mathcal{M}_d \subset \mathcal{M}_q$.

From 2.15, we see that the $\mathcal{H}$-module structure on $\mathcal{M}_d$ extends by $\mathcal{AX}_q$-linearity to an $\mathcal{H}$-module structure on the $\mathcal{AX}_q$-vector space $\mathcal{M}_q$. This can be further extended to an $\mathcal{H}$-module structure on $\mathcal{M}_q \otimes \mathcal{AX}'_q$ where $\mathcal{AX}'_q$ is an algebraic closure of $\mathcal{AX}_q$. This is a "generic principal series representation" of $\mathcal{H}$; hence

(d) *any $\mathcal{AX}'_q$-subspace of $\mathcal{M}_q \otimes \mathcal{AX}'_q$ which is an $\mathcal{H}$-submodule is either* 0 *or* $\mathcal{M}_q \otimes \mathcal{AX}'_q$.

3.2. Let $(,) : \mathcal{M}_c \times \mathcal{M}_c \to \mathcal{A}$ be the $\mathcal{A}$-bilinear form defined by $({}_\iota A, {}_{\iota'} A') = \delta_{\iota,\iota'}\delta_{A,A'}$ for $\iota, \iota' \in \underline{\mathcal{X}}$ and $A, A' \in X$.

Lemma 3.3. *Let $m, m' \in \mathcal{M}_c, w \in \hat{W}^a, x \in \mathcal{X}$. We have*

(a) $(m, m') = (m', m)$,
(b) $([x] \centerdot m, [x] \centerdot m') = (m, m')$,
(c) $(\tilde{T}_w m, m') = (m, \tilde{T}_{w^{-1}} m')$.

(a) and (b) are obvious. It is enough to prove (c) when either $w \in W^a$ or $l(w) = 0$. If $w \in W^a$, then (c) follows from [L6, 9.2]. It remains to prove (c) in the case where $w = \iota''$ for some $\iota'' \in \underline{\mathcal{X}}$. We may assume that $m = {}_\iota A, m' = {}_{\iota'} A'$ where $\iota, \iota' \in \underline{\mathcal{X}}$ and $A, A' \in X$. We have

$$\begin{aligned}(\tilde{T}_{\iota''} m, m') &= ({}_{\iota''+\iota} A, {}_{\iota'} A') = \delta_{\iota''+\iota,\iota'}\delta_{A,A'} = \delta_{\iota,-\iota''+\iota'}\delta_{A,A'} \\ &= ({}_\iota A, {}_{-\iota''+\iota'} A') = (m, \tilde{T}_{-\iota''} m').\end{aligned}$$

The lemma is proved.

3.4. Let $(|) : \mathcal{M}_c \times \mathcal{M}_c \to \mathcal{AX}$ be the pairing defined by

$$(m|m') = \sum_{x \in \mathcal{X}} (m, [x] \centerdot m')[x].$$

Lemma 3.5. *Let $m, m' \in \mathcal{M}_c, w \in \hat{W}, p \in \mathcal{AX}$. We have*

(a) $(m|m') = (m'|m)^\dagger$,
(b) $(p \centerdot m|m') = (m|p^\dagger \centerdot m') = p(m|m')$,
(c) $(\tilde{T}_w m|m') = (m|\tilde{T}_{w^{-1}} m')$.

(a),(b) follow immediately from 3.3(a),(b), while (c) follows immediately from 3.3(c) together with 2.15.

Lemma 3.6. (a) *Let $\iota, \iota' \in \underline{\mathcal{X}}$ and let $A, A' \in X$. Then $({}_\iota A|{}_{\iota'} A')$ equals $[x'']$ if there exists $x'' \in \mathcal{X}$ such that $\iota = \iota' - \underline{x}'', A = x'' + A'$ and is equal to* 0, *otherwise.*

(b) *Let $A, A' \in X$ be alcoves whose closure contains ϵ. Then $({}_0 A|{}_0 A') = \delta_{A,A'}$.*

(c) *We have* $({}_0e_\epsilon|{}_0e_\epsilon) = \sum_{w\in W} v^{-2l(w)}$.

We prove (a). We have

$$({}_\iota A|{}_{\iota'}A') = \sum_{x''\in\mathcal{X}} ({}_\iota A, {}_{\iota-\underline{x}''}(x''+A'))[x''] = \sum_{\substack{x''\in\mathcal{X}\\ \iota=\iota'-\underline{x}''\\ A=x''+A'}} [x'']$$

and (a) follows. Now (b) follows from (a) since for $x\in\mathcal{X}_{\mathrm{ad}}-\{0\}$ we have $A\neq x+A'$. This proves (b). Now (c) follows immediately from (b) using the definition of e_ϵ. The lemma is proved.

Lemma 3.7. *There exists a unique pairing* $(|) : \mathcal{M}_q\times\mathcal{M}_q\to\mathcal{AX}_q$ *such that*

(i) $(p\,\centerdot\, m|m') = (m|p^\dagger\,\centerdot\, m') = p(m|m')$ *for all* $p\in\mathcal{AX}_q, m, m'\in\mathcal{M}_q$,
(ii) $(\chi m|m') = (m|\chi^\blacktriangle m')$ *for all* $\chi\in\mathcal{H}, m, m'\in\mathcal{M}_q$,
(iii) $({}_0e_\epsilon|{}_0e_\epsilon) = \sum_{w\in W} v^{-2l(w)}$.

($\chi^\blacktriangle$ as in 1.24.) This pairing satisfies automatically $(m|m') = (m'|m)^\dagger$ for all $m, m'\in\mathcal{M}_q$. Moreover, the restriction of this pairing to $\mathcal{M}_c\times\mathcal{M}_c$ coincides with the pairing in 3.4.

Let $(|)_1$ be the pairing $(|) : \mathcal{M}_q\times\mathcal{M}_q\to\mathcal{AX}_q$ obtained by extension of scalars from $\mathcal{AX}$ to $\mathcal{AX}_q$ of the pairing in 3.4. This pairing satisfies (i)-(iii), by 3.5.

Now let $(|)_2$ be another pairing $(|) : \mathcal{M}_q\times\mathcal{M}_q\to\mathcal{AX}_q$ that satisfies (i)-(iii). Since $(|)_1$ is non-singular (see 3.6(b)), there exists an $\mathcal{AX}_q$-linear map $t : \mathcal{M}_q\to\mathcal{M}_q$ such that $(m|m')_2 = (t(m)|m')_1$ for all $m, m'\in\mathcal{M}_q$. For any $\chi\in\mathcal{H}$ we have

$$(t(\chi m)|m')_1 = (\chi m|m')_2 = (m|\chi^\blacktriangle m')_2 = (t(m)|\chi^\blacktriangle m')_1 = (\chi t(m)|m')_1$$

for all m, m'. Since $(|)_1$ is non-singular, it follows that $t(\chi m) = \chi t(m)$ for all $\chi\in\mathcal{H}, m\in\mathcal{M}_q$. Using 3.1(d), we deduce that there exists $\zeta\in\mathcal{AX}_q$ such that $t(m) = \zeta m$ for all $m\in\mathcal{M}_q$. Thus, $(m|m')_2 = \zeta(m|m')_1$ for all $m, m'\in\mathcal{M}_q$. Since (iii) holds for $(|)_1$ and for $(|)_2$, it follows that $\zeta = 1$. Thus the existence and uniqueness of $(|)$ is established. The remaining statements are clear from the construction. The lemma is proved.

3.8. Let $\mathfrak{U}$ be the ring of power series in v^{-1} with coefficients in the group ring $\mathbf{Z}[\mathcal{X}]$. Note that $\mathcal{AX}$ is naturally a subring of $\mathfrak{U}$.

Let p_k be a family of elements of $\mathcal{AX}$ indexed by a (possibly infinite) set K. Write $p_k = \sum_{n\in\mathbf{Z}} p_{k,n}v^n$ where $p_{k,n}\in\mathbf{Z}[\mathcal{X}]$. We say that the sum $\sum_{k\in K} p_k$ is *convergent* in $\mathfrak{U}$ if

(a) for any $n\in\mathbf{Z}$ there are only finitely many $k\in K$ such that $p_{n,k}\neq 0$;
(b) there exists $n_0\in\mathbf{Z}$ such that $p_{n,k} = 0$ for all $n < n_0$ and all $k\in K$.

Then we write $\sum_{k\in K} p_k = p$ where $p = \sum_{n\in\mathbf{Z}}(\sum_{k\in K} p_{n,k})v^n\in\mathfrak{U}$.

Lemma 3.9. *Let $\iota, \iota'\in\underline{\mathcal{X}}$ and let $m = \sum_{A\in X} m_A A, m' = \sum_{A\in X} m'_A A$ be two elements of $M_\geq$. Here $m_A, m'_A\in\mathcal{A}$. Assume that $\Delta\,\centerdot\,{}_\iota m\in\mathcal{M}_c, \Delta\,\centerdot\,{}_{\iota'}m'\in\mathcal{M}_c$. (For example, this assumption is verified if $m, m'\in M_d$, by 2.8, 2.16.)*

(a) *The infinite sum*

$$({}_\iota m|{}_{\iota'}m')' = \sum_{x''\in\mathcal{X};\iota=\iota'-\underline{x}''}\ \sum_{A\in X} m_A m'_{-x''+A}[x'']$$

is convergent in $\mathfrak{U}$; thus, $({}_\iota m|{}_{\iota'}m')'\in\mathfrak{U}$.

(b) *We have* $\Delta\Delta^\dagger({}_\iota m|{}_{\iota'}m')'\in\mathcal{AX}$.

(c) *We have* $\Delta\Delta^\dagger({}_\iota m|_{\iota'}m')' = \Delta\Delta^\dagger({}_\iota m|_{\iota'}m')$ *in* $\mathcal{A}\mathcal{X}_q$.

For any $B \in X$ let $r(B) = \sum_{\kappa\in\mathcal{X}_{\rm ad}} \hat P(\kappa)(\kappa+B) \in M_\ge$. This extends to an $\mathcal{A}$-linear map $r : M_c \to M_\ge$.

We have $\Delta\,\centerdot\,[\underline{x}]m = [\underline{x}]\tilde m, \Delta\,\centerdot\,[\underline{x}']m' = [\underline{x}']\tilde m'$ where $\tilde m, \tilde m' \in M_c$. From the definitions we have $m = r(\tilde m), m' = r(\tilde m')$.

Conversely, if we are given $\tilde m, \tilde m' \in M_c$, then $m = r(\tilde m), m' = r(\tilde m')$ satisfy the assumptions of the lemma. Hence, it is enough to prove the lemma for such m, m'. We may assume that $\tilde m = B, \tilde m' = B'$ for some $B, B' \in X$. We have

$$m_A = \sum_{\kappa;A=\kappa+B} \hat P(\kappa), \quad m'_A = \sum_{\kappa';A=\kappa'+B'} \hat P(\kappa').$$

The infinite sum in (a) is then

$$\text{(d)} \qquad \sum_{\kappa,\kappa'} \sum_{\substack{x''\in\mathcal{X}\\ \iota=\iota'-\underline{x}''\\ \kappa+B=x''+\kappa'+B'}} \hat P(\kappa)\hat P(\kappa')[x''].$$

This is clearly convergent in $\mathfrak{U}$, by the definition of $\hat P(\kappa)$.

We have $\Delta = \sum_{y\in\mathcal{X}_{\rm ad}} \tilde P(y)[y]$ (finite sum) where $\tilde P(y) \in \mathcal{A}$ are such that

$$\sum_y \tilde P(y)\hat P(x_0 - y) = \delta_{x_0,0}$$

for any $x_0 \in \mathcal{X}_{\rm ad}$. Hence, if we multiply the power series (d) (in $\mathfrak{U}$) with $\Delta\Delta^\dagger$, we obtain

$$\sum_{\substack{\kappa,\kappa'\\ y,y'}} \sum_{\substack{x''\in\mathcal{X}\\ \iota=\iota'-\underline{x}''\\ \kappa+B=x''+\kappa'+B'}} \hat P(\kappa)\hat P(\kappa')\tilde P(y)\tilde P(y')[x''+y-y'].$$

We make a change of variables $(\kappa, \kappa', y, y', x'') \mapsto (x_0, x'_0, y, y', z)$ where $x_0 = \kappa + y, x'_0 = \kappa' + y', z = x'' + y - y'$. We obtain the sum

$$\begin{aligned}
&\sum_{\substack{x_0,x'_0\\ y,y'}} \sum_{\substack{z\in\mathcal{X}\\ \iota=\iota'-\underline{z}\\ x_0+B=z+x'_0+B'}} \hat P(x_0-y)\hat P(x'_0-y')\tilde P(y)\tilde P(y')[z]\\
&= \sum_{x_0,x'_0} \sum_{\substack{z\in\mathcal{X}\\ \iota=\iota'-\underline{z}\\ x_0+B=z+x'_0+B'}} \delta_{x_0,0}\delta_{x'_0,0}[z] = \sum_{\substack{z\in\mathcal{X}\\ \iota=\iota'-\underline{z}\\ B=z+B'}} [z] = ({}_\iota B|_{\iota'}B').
\end{aligned}$$

The lemma is proved.

Lemma 3.10. *Let* $\iota, \iota' \in \underline{\mathcal{X}}$ *and let* B, B' *be two alcoves in the same box. Then*

$$({}_\iota B^\flat|_{\iota'}B'^\sharp) = \delta_{\iota,\iota'}\delta_{B,B'}.$$

Since both ${}_\iota B^\flat, {}_{\iota'}B'^\sharp$ are contained in $\mathcal{M}_d$, the inner product may be computed by the method of 3.9. With the notation in 2.3, 2.4 we have

$$\begin{aligned}
&({}_\iota B^\flat|{}_{\iota'}B'^\sharp)\\
&= ({}_\iota(\sum_{A\in X}\Pi_{A,B}A)|{}_{\iota'}(\sum_{A\in X}\Pi'_{B,A}A)) = \sum_{\substack{x''\in\mathcal{X}\\ \iota=\iota'-\underline{x}''}}\sum_{A\in X}\Pi_{A,B}\Pi'_{B',-x''+A}[x'']\\
&= \sum_{\substack{x''\in\mathcal{X}\\ \iota=\iota'-\underline{x}''}}\sum_{A\in X}\Pi_{A,B}\Pi'_{x''+B',A}[x''] = \sum_{\substack{x''\in\mathcal{X}\\ \iota=\iota'-\underline{x}''\\ B=x''+B'}}[x''].
\end{aligned}$$

(We have used 2.5(a). We have used the convention that $\Pi_{A,B}=0$ for $A\not\leq B$ and $\Pi'_{B,A}=0$ for $B\not\leq A$.) Since B,B' are alcoves in the same box, we have $B=x''+B' \implies x''=0$. The lemma is proved.

Lemma 3.11. (a) $\mathcal{M}_{d'}=\{m\in\mathcal{M}_q|(m|m')\in\mathcal{AX}\quad\forall m'\in\mathcal{M}_d\}$;
(b) $\mathcal{M}_d=\{m\in\mathcal{M}_q|(m|m')\in\mathcal{AX}\quad\forall m'\in\mathcal{M}_{d'}\}$.

Let $\mathcal{M}'$ be the right hand side of (a). By 3.1 and 3.10, there exists an $\mathcal{AX}$-basis (m_j) of $\mathcal{M}_{d'}$ and an $\mathcal{AX}$-basis (m'_j) of $\mathcal{M}_d$ such that $(m_j|m_{j'})=\delta_{jj'}$ for all j,j'. Hence, $\mathcal{M}_{d'}\subset\mathcal{M}'$. Conversely, let $m\in\mathcal{M}'$. Since (m_j) is an $\mathcal{AX}_q$-basis of $\mathcal{M}_q$, we have $m=\sum_j c_jm_j$ with $c_j\in\mathcal{AX}_q$. We have $c_j=(m|m'_j)$ for all j. By assumption, for any j we have $(m'_j,m)\in\mathcal{AX}$; hence $c_j\in\mathcal{AX}$. Hence, $m\in\mathcal{M}_{d'}$. This proves (a). The proof of (b) is entirely similar. The lemma is proved.

Lemma 3.12. *For any $m\in\mathcal{M}_{d'}, m'\in\mathcal{M}_d$ we have*

$$(b(m)|m')=\overline{(m|\tilde{b}(m'))}.$$

It is enough to prove this for m (resp. m') running through a fixed $\mathcal{AX}$-basis of $\mathcal{M}_{d'}$ (resp. of $\mathcal{M}_d$). The result therefore follows from 3.10.

3.13. Let $\mathbf{Z}((v^{-1}))$ be the ring of power series in v^{-1} with coefficients in $\mathbf{Z}$. Let $p\mapsto p^{(0)}$ be the group homomorphism $\mathbf{Z}[\mathcal{X}]\mapsto\mathbf{Z}$ given by $[x]\mapsto\delta_{x,0}$.

Let $\partial:\mathfrak{U}\mapsto\mathbf{Z}((v^{-1}))$ be the group homomorphism given by $\sum_{n\in\mathbf{Z}}p_nv^n\mapsto\sum_{n\in\mathbf{Z}}p_n^{(0)}v^n$; here $p_n\in\mathbf{Z}[\mathcal{X}]$.

Lemma 3.14. *We have*

(a) $$\{m\in\mathcal{M}_c|\partial(m|m)=1\}=\{\pm{}_\iota B|\iota\in\underline{\mathcal{X}},B\in X\},$$

(b) $$\{m\in\mathcal{M}_d|\tilde{b}(m)=m,\partial(m|m)\in 1+v^{-1}\mathbf{Z}[[v^{-1}]]\}=\{\pm{}_\iota B^\sharp|\iota\in\underline{\mathcal{X}},B\in X\},$$

(c) $$\{m\in\mathcal{M}_{d'}|b(m)=m,\partial(m|m)\in 1+v^{-1}\mathbf{Z}[[v^{-1}]]\}=\{\pm{}_\iota B^\flat|\iota\in\underline{\mathcal{X}},B\in X\}.$$

Let $\iota,\iota'\in\underline{\mathcal{X}}$, and let $B,B'\in X$. By 3.6(a), we have $\partial({}_\iota B|{}_{\iota'}B')=\delta_{\iota,\iota'}\delta_{B,B'}$. We have

$$\begin{aligned}
\partial({}_\iota B^\flat|{}_{\iota'}B'^\flat) &= \sum_{A,A'}\Pi_{A,B}\Pi_{A',B'}\partial({}_\iota A|{}_{\iota'}A') = \sum_{A,A'}\Pi_{A,B}\Pi'_{A',B'}\delta_{\iota,\iota'}\delta_{A,A'}\\
&= \sum_A\Pi_{A,B}\Pi'_{A,B'}\delta_{\iota,\iota'}=\delta_{\iota,\iota'}\delta_{B,B'}+v^{-1}\mathbf{Z}[v^{-1}].
\end{aligned}$$

Using 3.9 we compute

$$\partial({}_\iota B^\sharp|_{\iota'} B'^\sharp) = \sum_{A,A'} \Pi'_{B,A}\Pi'_{B',A'}\partial({}_\iota A|_{\iota'} A') = \sum_{A,A'} \Pi'_{B,A}\Pi'_{B',A'}\delta_{\iota,\iota'}\delta_{A,A'}$$
$$= \sum_{A} \Pi'_{B,A}\Pi'_{B',A}\delta_{\iota,\iota'} = \delta_{\iota,\iota'}\delta_{B,B'} + v^{-1}\mathbf{Z}[[v^{-1}]].$$

These computations show that the right hand sides of (a),(b),(c) are contained in the corresponding left hand sides.

We now show the converse. Let m be in the left hand side of (a). We write $m = \sum_{\iota\in\underline{\mathcal{X}},A\in X} m_{\iota,A}({}_\iota A)$ (finite sum with $m_{\iota,A} \in \mathcal{A}$. By the earlier part of the argument we have $\partial(m|m) = \sum_{\iota,A} m^2_{\iota,A}$. Thus, $\sum_{\iota,A} m^2_{\iota,A} = 1$. This implies that there exists $\iota' \in \underline{\mathcal{X}}, B \in X$ such that $m_{\iota,A} = \pm\delta_{\iota,\iota'}\delta_{A,B}$. Thus, (a) holds.

Next, let m be in the left hand side of (b). We write $m = \sum_{\iota\in\underline{\mathcal{X}},A\in X} m_{\iota,A}({}_\iota A^\sharp)$.

By the earlier part of the argument we have

$$\partial(m|m) = \sum_{\iota,A} m^2_{\iota,A} \mod v^{-1}\mathbf{Z}[[v^{-1}]].$$

Thus,

$$\sum_{\iota,A} m^2_{\iota,A} = 1 \mod v^{-1}\mathbf{Z}[[v^{-1}]].$$

This implies that there exists $\iota' \in \underline{\mathcal{X}}, B \in X$ such that

$$m_{\iota,A} = \pm\delta_{\iota,\iota'}\delta_{A,B} \mod v^{-1}\mathbf{Z}[[v^{-1}]].$$

But $m_{\iota,A} \in \mathcal{A}$ is fixed by ${}^- : \mathcal{A} \to \mathcal{A}$. Hence, $m_{\iota,A} = \pm\delta_{\iota,\iota'}\delta_{A,B}$. Thus, (b) holds.

An entirely similar argument shows that (c) holds. The lemma is proved.

3.15. We have

(a) $$({}_0\tilde{e}_\epsilon|{}_0\tilde{e}_\epsilon) = (\Delta\Delta^\dagger)^{-1}\sum_{w\in W} v^{-2l(w)}.$$

Indeed, using 2.16, we see that this is equivalent to the identity $({}_0e'_\epsilon|{}_0e'_\epsilon) = \sum_{w\in W} v^{-2l(w)}$ which is proved in the same way as 3.6(c).

4. The $\mathcal{H}$-modules $\mathcal{A}\mathcal{X}^{\otimes 2}_{d'}, \mathcal{A}\mathcal{X}^{\otimes 2}_{d}$

4.1. W acts on $\mathcal{A}\mathcal{X}$ by

$$p = \sum_{x\in\mathcal{X}} c_x[x] \mapsto {}^w p = \sum_{x\in\mathcal{X}} c_x[{}^w x] \in \mathcal{A}\mathcal{X}.$$

Let $\mathcal{A}\mathcal{X}^W$ denote the subring of W-invariant elements in $\mathcal{A}\mathcal{X}$. The following result is due to J. Bernstein (unpublished; see [L4] for a proof).

If $p \in \mathcal{A}\mathcal{X}^W$, then θ_p is contained in the centre of $\mathcal{H}$.

4.2. Let $\epsilon \in \mathbf{s}$. For any $x' \in \mathcal{X}$ we have $[x'] \centerdot {}_0A_\epsilon^+ = \theta_{x'}({}_0A_\epsilon^+)$. (Both sides are equal to ${}_{-x'}A_{x'+\epsilon}^+$, see 2.13.) It follows that

$$p \centerdot {}_0A_\epsilon^+ = \theta_p({}_0A_\epsilon^+)$$

for any $p \in \mathcal{AX}$. Assume now that $p \in \mathcal{AX}^W$ in the previous equality. Apply $\tilde{T}_w^{-1}$ (with $w \in W$) to both sides of that equality and use 2.15, 4.1. We obtain

$$p \centerdot (\tilde{T}_w^{-1}({}_0A_\epsilon^+)) = \theta_p(\tilde{T}_w^{-1}({}_0A_\epsilon^+)).$$

Now $T_w^{-1}A_\epsilon^+$ is equal to A', the most general alcove whose closure contains ϵ. Since ϵ is arbitrary in $\mathbf{s}$, we obtain that $p \centerdot {}_0A' = \theta_p({}_0A')$ for any $A' \in X$. It follows that

(a) $$p \centerdot ({}_0m) = \theta_p({}_0m)$$

for any $m \in M$ and $p \in \mathcal{AX}^W$.

4.3. We set

$$\mathcal{AX}^{\otimes 2} = \mathcal{AX} \otimes_{\mathcal{AX}^W} \mathcal{AX}.$$

This is an $\mathcal{AX}^W$-algebra in a natural way.

In the remainder of this paper we fix $\epsilon \in \mathbf{s}$.

Lemma 4.4. *There is a unique $\mathcal{A}$-linear map $f : \mathcal{AX}^{\otimes 2} \to \mathcal{M}_{d'}$ such that*

(a) $$f([x] \otimes [x']) = v^\nu \theta_{x+\rho}([x' + \rho] \centerdot {}_0e_\epsilon)$$

for any $x, x' \in \mathcal{X}$. Moreover, f is an isomorphism of $\mathcal{A}$-modules.

The fact that (a) defines a linear map f as claimed follows from 3.1, 4.2. Let $\mathrm{Im}(f)$ be the image of f. Using 1.19(c), we see that for $i \in I$ and $x, x' \in \mathcal{X}$, the element $\tilde{T}_{\sigma_i}\theta_x([x'] \centerdot {}_0e_\epsilon)$ is an $\mathcal{A}$-linear combination of

$$\theta_{\sigma_i x}\tilde{T}_{\sigma_i}([x'] \centerdot {}_0e_\epsilon) = v\theta_{\sigma_i x}([x'] \centerdot {}_0e_\epsilon)$$

and of elements $\theta_{\tilde{x}}([x'] \centerdot {}_0e_\epsilon)$. (We use that $\tilde{T}_{\sigma_i}({}_0e_\epsilon) = v_0e_\epsilon$.) Thus,

(b) $$\tilde{T}_{\sigma_i}(\mathrm{Im}(f)) \subset \mathrm{Im}(f).$$

Since $\mathrm{Im}(f)$ is stable under the operators θ_{x_1} for $x_1 \in \mathcal{X}$, from (b) it follows that

(c) $$\mathrm{Im}(f) \text{ is an } \mathcal{H}\text{-submodule of } \mathcal{M}_{d'}.$$

(Recall that the algebra $\mathcal{H}$ is generated by the elements $\theta_{x_1}, \tilde{T}_{\sigma_i}$.) For any $x' \in \mathcal{X}$ we have

(d) $${}_{-x'}e_{x'+\epsilon} = [x'] \centerdot {}_0e_\epsilon \in \mathrm{Im}(f).$$

Applying to the element (d) the operator $\tilde{T}_{\underline{x}'} \in \mathcal{H}$, we get the element ${}_0e_{x'+\epsilon}$ which by (c) must belong to $\mathrm{Im}(f)$. Since any element of $\mathfrak{E}$ is of the form $x' + \epsilon$ for some $x' \in \mathcal{X}$, we see that

(e) $${}_0e_{\epsilon'} \in \mathrm{Im}(f) \text{ for any } \epsilon' \in \mathfrak{E}.$$

Since the elements (e) generate the $\mathcal{H}$-module $\mathcal{M}_{d'}$, we see using (d) that $\mathrm{Im}(f) = \mathcal{M}_{d'}$. Thus, f is surjective. We shall regard $\mathcal{M}_{d'}$ as a $\mathcal{AX}$-module under the $\centerdot$-action; we shall regard $\mathcal{AX}^{\otimes 2}$ as an $\mathcal{AX}$-module by $[x_1]([x] \otimes [x']) = [x] \otimes [x' + x_1]$. Then f is a homomorphism of $\mathcal{AX}$-modules. Since f is surjective, to show that it is an isomorphism, it is enough to show that $\mathcal{M}_{d'}$ and $\mathcal{A}\mathcal{X}^{\otimes 2}$ are free $\mathcal{A}\mathcal{X}$-modules of the same (finite) rank. For $\mathcal{M}_{d'}$, this follows from 3.1(c).

By [P], $\mathcal{A}\mathcal{X}$ is a free $\mathcal{A}\mathcal{X}^W$-module of rank $|W|$. It follows that $\mathcal{A}\mathcal{X}^{\otimes 2}$ is a free $\mathcal{A}\mathcal{X}$-module of rank $|W|$. Thus, f must be an isomorphism. The lemma is proved.

Lemma 4.5. *There is a unique $\mathcal{A}$-linear map $\tilde{f} : \mathcal{A}\mathcal{X}^{\otimes 2} \to \mathcal{M}_d$ such that*

$$\tilde{f}([x] \otimes [x']) = (-1)^\nu v^{-2\nu} \theta_x([x' + 2\rho] \bullet_0 \tilde{e}_\epsilon)$$

for any $x, x' \in \mathcal{X}$. Moreover, $\tilde{f}$ is an isomorphism of $\mathcal{A}$-modules.

The proof is entirely similar to that of the previous lemma; we replace $e_\epsilon, v, \dots$ by $\tilde{e}_\epsilon, -v^{-1}, \dots$ respectively.

Lemma 4.6. *In the $\mathcal{H}$-module structure on $\mathcal{A}\mathcal{X}^{\otimes 2}$ obtained from that of $\mathcal{M}_{d'}$ via the isomorphism in 4.4, we have, for $i \in I, x, x', x'' \in \mathcal{X}$:*

(a) $$\tilde{T}_{\sigma_i}([x] \otimes [x']) = \frac{v^{-1}([{}^{\sigma_i}x] - [x + \alpha_i]) + v([x + \alpha_i] - [{}^{\sigma_i}x - \alpha_i])}{[\alpha_i] - 1} \otimes [x'],$$

(b) $$\theta_{x''}([x] \otimes [x']) = [x + x''] \otimes [x'].$$

This $\mathcal{H}$-module structure on $\mathcal{A}\mathcal{X}^{\otimes 2}$ is denoted by $\mathcal{A}\mathcal{X}^{\otimes 2}_{d'}$.

To prove (a) we may assume, by 2.15, that $x' = -\rho$. In this case, (a) follows from the computation in 2.17. The proof of (b) is immediate. The lemma is proved.

Lemma 4.7. *In the $\mathcal{H}$-module structure on $\mathcal{A}\mathcal{X}^{\otimes 2}$ obtained from that of $\mathcal{M}_d$ via the isomorphism in 4.5, we have, for $i \in I, x, x', x'' \in \mathcal{X}$:*

(a) $$\tilde{T}_{\sigma_i}([x] \otimes [x']) = \frac{v^{-1}([{}^{\sigma_i}x] - [x + \alpha_i]) + v([x + \alpha_i] - [{}^{\sigma_i}x + \alpha_i])}{[\alpha_i] - 1} \otimes [x'],$$

(b) $$\theta_{x''}([x] \otimes [x']) = [x + x''] \otimes [x'].$$

This $\mathcal{H}$-module structure on $\mathcal{A}\mathcal{X}^{\otimes 2}$ is denoted by $\mathcal{A}\mathcal{X}^{\otimes 2}_{d}$.

To prove (a) we may assume, by 2.15, that $x' = -2\rho$. In this case, (a) follows from the computation in 2.18. The proof of (b) is immediate. The lemma is proved.

4.8. By transferring the $\mathcal{H}$-antilinear map $\hat{b} : \mathcal{M}_{d'} \to \mathcal{M}_{d'}$ to $\mathcal{A}\mathcal{X}^{\otimes 2}_{d'}$ via the isomorphism in 4.4, we obtain an $\mathcal{H}$-antilinear map $\mathcal{A}\mathcal{X}^{\otimes 2}_{d'} \to \mathcal{A}\mathcal{X}^{\otimes 2}_{d'}$ which is denoted again by $\hat{b}$. From the definition, this map keeps fixed each of the elements $v^{-\nu}[-\rho] \otimes [x']$ (with $x' \in \mathcal{X}$), which correspond to ${}_{-x'-\rho}e_{x'+\rho+\epsilon} \in \mathcal{M}_{d'}$.

Similarly, by transferring the $\mathcal{H}$-antilinear map $\hat{\hat{b}} : \mathcal{M}_d \to \mathcal{M}_d$ to $\mathcal{A}\mathcal{X}^{\otimes 2}_d$ via the isomorphism in 4.5, we obtain an $\mathcal{H}$-antilinear map $\mathcal{A}\mathcal{X}^{\otimes 2}_d \to \mathcal{A}\mathcal{X}^{\otimes 2}_d$ which is denoted again by $\hat{\hat{b}}$. From the definition, this map keeps fixed each of the elements $(-1)^\nu v^{2\nu}[0] \otimes [x']$ (with $x' \in \mathcal{X}$) which correspond to ${}_{-x'}\tilde{e}_{x'+2\rho+\epsilon} \in \mathcal{M}_d$.

Lemma 4.9. *Let $x, x' \in \mathcal{X}$. We have*

(a) $$\hat{b}([x] \otimes [x']) = v^{-\nu} \tilde{T}_{w_0}^{-1}([{}^{w_0}x - 2\rho] \otimes [x']) \quad \text{in } \mathcal{A}\mathcal{X}^{\otimes 2}_{d'},$$

(b) $$\hat{\hat{b}}([x] \otimes [x']) = (-1)^\nu v^{3\nu} \tilde{T}_{w_0}^{-1}([{}^{w_0}x] \otimes [x']) \quad \text{in } \mathcal{A}\mathcal{X}^{\otimes 2}_{d}.$$

We prove (a). In the $\mathcal{H}$-module $\mathcal{AX}_{d'}^{\otimes 2}$ we have $\tilde{T}_{\sigma_i}([-\rho]\otimes[x']) = v[-\rho]\otimes[x']$ for any $i \in I$. Applying this repeatedly, we see that $\tilde{T}_{w_0}([-\rho]\otimes[x']) = v^\nu[-\rho]\otimes[x']$.

Since $\hat{b}$ is $\mathcal{H}$-antilinear and keeps $[-\rho]\otimes[x']$ fixed, we have

$$\begin{aligned}
&\hat{b}([x]\otimes[x']) = \hat{b}(\theta_{x+\rho}([-\rho]\otimes[x'])) = \overline{\theta_{x+\rho}}\hat{b}([-\rho]\otimes[x'])\\
&= v^{-2\nu}\tilde{T}_{w_0}^{-1}\theta_{^{w_0}(x+\rho)}\tilde{T}_{w_0}([-\rho]\otimes[x']) = v^{-2\nu}v^\nu\tilde{T}_{w_0}^{-1}\theta_{^{w_0}x-\rho}([-\rho]\otimes[x'])\\
&= v^{-\nu}\tilde{T}_{w_0}^{-1}([^{w_0}x-2\rho]\otimes[x'])
\end{aligned}$$

where we have used 1.22. This proves (a). The proof of (b) is entirely similar.

Lemma 4.10. (a) *The $\mathcal{A}$-linear map $h : \mathcal{AX}_{d'}^{\otimes 2} \to \mathcal{AX}_{d}^{\otimes 2}$ given by $[x]\otimes[x'] \mapsto \bar{\Delta}[x]\otimes[x']$ is $\mathcal{H}$-linear.*

(b) *The $\mathcal{A}$-linear map $h' : \mathcal{AX}_{d}^{\otimes 2} \to \mathcal{AX}_{d'}^{\otimes 2}$ given by $[x]\otimes[x'] \mapsto \bar{\Delta}^\dagger[x]\otimes[x']$ is $\mathcal{H}$-linear.*

To prove (a) it is enough to show that, for any $i \in I$ and for any $x, x' \in \mathcal{X}$, we have

$$\begin{aligned}
&\bar{\Delta}\frac{v^{-1}([^{\sigma_i}x]-[x+\alpha_i]) + v([x+\alpha_i]-[^{\sigma_i}x-\alpha_i])}{[\alpha_i]-1}\otimes[x']\\
&= \frac{v^{-1}(^{\sigma_i}\bar{\Delta}[^{\sigma_i}x]-\bar{\Delta}[x+\alpha_i]) + v(\bar{\Delta}[x+\alpha_i]-{}^{\sigma_i}\bar{\Delta}[^{\sigma_i}x+\alpha_i])}{[\alpha_i]-1}\otimes[x'].
\end{aligned}$$

Since ${}^{\sigma_i}\bar{\Delta} = \bar{\Delta}\frac{1-v^2[-\alpha_i]}{1-v^2[\alpha_i]}$, this is equivalent to the identity

$$\begin{aligned}
&v^{-1}([^{\sigma_i}x]-[x+\alpha_i]) + v([x+\alpha_i]-[^{\sigma_i}x-\alpha_i])\\
&= v^{-1}(\frac{1-v^2[-\alpha_i]}{1-v^2[\alpha_i]}[^{\sigma_i}x]-[x+\alpha_i]) + v([x+\alpha_i]-\frac{1-v^2[-\alpha_i]}{1-v^2[\alpha_i]}[^{\sigma_i}x+\alpha_i])
\end{aligned}$$

in $\mathcal{AX}_q$, which is easily verified. This proves (a). One can prove (b) in the same way as (a). Alternatively, we can argue as follows. Since $\bar{\Delta}\bar{\Delta}^\dagger \in \mathcal{AX}^W$, the composition $hh' : \mathcal{AX}_{d}^{\otimes 2} \to \mathcal{AX}_{d}^{\otimes 2}$ is the $\mathcal{H}$-linear map given by

$$[x]\otimes[x'] \mapsto (\bar{\Delta}\bar{\Delta}^\dagger[x])\otimes[x'] = [x]\otimes(\bar{\Delta}\bar{\Delta}^\dagger)[x'].$$

Hence, for any $\chi \in \mathcal{H}, m \in \mathcal{AX}_{d}^{\otimes 2}$ we have $hh'(\chi m) = \chi hh'(m) = h(\chi h'(m))$. (The last equality follows from (a).) Since h is injective, it follows that $h'(\chi m) = \chi h'(m)$. The lemma is proved.

Lemma 4.11. *We have a commutative diagram of $\mathcal{H}$-modules*

$$\begin{array}{ccc}
\mathcal{AX}_{d'}^{\otimes 2} & \longrightarrow & \mathcal{AX}_{d}^{\otimes 2}\\
{\scriptstyle f}\downarrow & & {\scriptstyle \tilde{f}}\downarrow\\
\mathcal{M}_{d'} & \longrightarrow & \mathcal{M}_d
\end{array}$$

where the lower horizontal map is the obvious inclusion, $f, \tilde{f}$ are as in 4.4, 4.5 and the upper horizontal map is given by $[x]\otimes[x'] \mapsto \bar{\Delta}[x]\otimes[x']$.

An equivalent statement is that

$$(-1)^\nu v^{-2\nu}\theta_{\bar{\Delta}}\theta_x([x'+2\rho]\,\bullet_0\tilde{e}_\epsilon) = v^\nu\theta_{x+\rho}([x'+\rho]\,\bullet_0 e_\epsilon)$$

for all $x, x' \in \mathcal{X}$. To prove this we may assume that $x = x' = -\rho$. Thus, we must prove that the images

(a) $(-1)^\nu v^{-2\nu}\theta_{\bar\Delta[-\rho]}([\rho]\bullet{}_0\tilde e_\epsilon), v^\nu{}_0e_\epsilon$

of $[-\rho]\otimes[-\rho]$ under the two possible compositions in the diagram are equal. Since these compositions are $\mathcal{H}$-linear (see 4.10) and the vector $[-\rho]\otimes[\rho]\in\mathcal{A}\mathcal{X}_{d'}^{\otimes 2}$ is in the kernel of $\tilde T_{\sigma_i}-v$ for any $i\in I$, the first vector in (a) is also in the kernel of $\tilde T_{\sigma_i}-v$ for any $i\in I$. Hence,

(b) $\pi^{-1}\Sigma\theta_{\bar\Delta[-\rho]}([\rho]\bullet{}_0\tilde e_\epsilon)=\theta_{\bar\Delta[-\rho]}([\rho]\bullet{}_0\tilde e_\epsilon)$

where $\Sigma=\sum_{w\in W}v^{-l(w)}\tilde T_w^{-1}$ and $\pi=\sum_{w\in W}v^{-2l(w)}$. To show that the two vectors in (a) are equal, it is enough to show that they become equal after applying $\Delta\bullet$ to them. Hence, using 2.16, it is enough to prove that

$$(-1)^\nu v^{-2\nu}\theta_{\bar\Delta[-\rho]}({}_0e'_\epsilon)=v^\nu\Delta[-\rho]\bullet{}_0e_\epsilon,$$

or equivalently, that

$$\theta_{\bar\Delta[-\rho]}\Sigma^*({}_0A_\epsilon^+)=v^{3\nu}\Sigma\theta_{\Delta[-\rho]}({}_0A_\epsilon^+)$$

(the last expression is equal to $v^{3\nu}\Sigma(\Delta[-\rho]\bullet{}_0A_\epsilon^+)$ by 4.2). Here,

$$\Sigma^*=\sum_{w\in W}\mathrm{sgn}_w v^{l(w)-\nu}\tilde T_w^{-1}.$$

Hence, it is enough to prove the equality

(c) $$\theta_{\bar\Delta[-\rho]}\Sigma^*=v^{3\nu}\Sigma\theta_{\Delta[-\rho]}$$

in $\mathcal{H}$. Now applying $\Delta[\rho]\bullet$ to the two sides of (b), and using again 2.16, we obtain $\pi^{-1}\Sigma\theta_{\bar\Delta[-\rho]}({}_0e'_\epsilon)=\theta_{\bar\Delta[-\rho]}({}_0e'_\epsilon)$; hence

$$\pi^{-1}\Sigma\theta_{\bar\Delta[-\rho]}\Sigma^*({}_0A_\epsilon^+)=\theta_{\bar\Delta[-\rho]}\Sigma^*({}_0A_\epsilon^+).$$

From this we deduce that

(d) $\pi^{-1}\Sigma\theta_{\bar\Delta[-\rho]}\Sigma^*=\theta_{\bar\Delta[-\rho]}\Sigma^*$

as elements of $\mathcal{H}$. To this equality we apply the antiautomorphism of the ring $\mathcal{H}$ such that $\theta_x\mapsto\theta_x$ for all $x\in\mathcal{X}$, $\tilde T_{\sigma_i}\mapsto\tilde T_{\sigma_i}$ for all $i\in I$ and $v^n\mapsto(-v)^{-n}$ for all n. This carries π to $v^{2\nu}\pi$, Σ to $v^\nu\Sigma^*$, Σ^* to $(-v)^\nu\Sigma$, and $\theta_{\bar\Delta[-\rho]}$ to $\theta_{\Delta[-\rho]}$. Hence, from (d) we get the equality

(e) $v^{-\nu}\pi^{-1}\Sigma\theta_{\Delta[-\rho]}\Sigma^*=\Sigma\theta_{\Delta[-\rho]}$.

In view of (d),(e), the desired equality (c) is equivalent to:

$$\Sigma\theta_{\bar\Delta[-\rho]}\Sigma^*=v^{2\nu}\Sigma\theta_{\Delta[-\rho]}\Sigma^*.$$

To prove this, we set $j_p=\Sigma\theta_p\Sigma^*$ for any $p\in\mathcal{A}\mathcal{X}$. The argument in [L3, 7.3] shows that $j_{{}^wp}=\mathrm{sgn}_w j_p$ for any $w\in W, p\in\mathcal{A}\mathcal{X}$. In particular, $j_{\bar\Delta[-\rho]}=(-1)^\nu j_{{}^{w_0}(\bar\Delta[-\rho])}$. Hence, to show the desired equality $j_{\bar\Delta[-\rho]}=v^{2\nu}j_{\Delta[-\rho]}$, it suffices to show that

$${}^{w_0}(\bar\Delta[-\rho])=(-1)^\nu v^{2\nu}\Delta[-\rho].$$

This is immediate from the definitions. The lemma is proved.

5. Inner product on $\mathcal{AX}^{\otimes 2}_{d'}, \mathcal{AX}^{\otimes 2}_{d}$

5.1. Let $(,) : \mathcal{AX} \times \mathcal{AX} \to \mathcal{AX}^W$ be the symmetric pairing defined by

(a) $(p, p') = \delta^{-1} \sum_{w \in W} \mathrm{sgn}_w {}^w(pp'[\rho])$.

(δ is as in 1.23. The fraction in the right hand side is taken in $\mathcal{AX}_q$, but in fact it belongs to $\mathcal{AX}$ and even to $\mathcal{AX}^W$.) This pairing is clearly $\mathcal{AX}^W$-bilinear. Hence, by extending the scalars from $\mathcal{AX}^W$ to $\mathcal{AX}$ we obtain an $\mathcal{AX}$-bilinear pairing $(:) : \mathcal{AX}^{\otimes 2} \times \mathcal{AX}^{\otimes 2} \to \mathcal{AX}$ such that $(p \otimes q : p' \otimes q') = (p, p')qq'$ for $p, p', q, q' \in \mathcal{AX}$. ($\mathcal{AX}^{\otimes 2}$ is regarded as an $\mathcal{AX}$-module by $p'(p \otimes q) = p \otimes (p'q)$.)

Given $\chi \in \mathcal{H}$, we will sometimes write ${}^{d'}\chi$ (resp. ${}^{d}\chi$) for the action of χ in $\mathcal{AX}^{\otimes 2}_{d'}$ (resp. $\mathcal{AX}^{\otimes 2}_{d}$).

Lemma 5.2. *For any $\chi \in \mathcal{H}$ and $\xi, \xi' \in \mathcal{AX}^{\otimes 2}$, we have*

(a) $$({}^{d}\hat{\chi}\xi : \xi') = (\xi, {}^{d'}\chi\xi').$$

Let $\mathcal{H}_1$ be the set of all $\chi \in \mathcal{H}$ such that (a) holds for any ξ, ξ'. Clearly, $\mathcal{H}_1$ is a subalgebra of $\mathcal{H}$. Therefore, it is enough to show that χ_1 contains $\tilde{T}_{\sigma_i}$ for any $i \in I$ and θ_x for any $x \in \mathcal{X}$. The fact that $\theta_x \in \mathcal{H}_1$ is immediate. It remains to show that for $i \in I$, $\chi = \tilde{T}_{\sigma_i}$ satisfies (a). We may assume that $\xi = [x] \otimes [y], \xi' = [x'] \otimes [y']$ where $x, x', y, y' \in \mathcal{X}$. The equality to be proved is then

$$\begin{aligned}&\Big(\frac{v^{-1}([{}^{\sigma_i}x] - [x + \alpha_i]) + v([x + \alpha_i] - [{}^{\sigma_i}x + \alpha_i])}{[\alpha_i] - 1}, [x']\Big)\\ &= \Big([x], \frac{v^{-1}([{}^{\sigma_i}x'] - [x' + \alpha_i]) + v([x' + \alpha_i] - [{}^{\sigma_i}x' - \alpha_i])}{[\alpha_i] - 1}\Big).\end{aligned}$$

From the definitions, we see that, if p_1, p_2, p_3, p_4 are elements of $\mathcal{AX}$, then, in order to have $(p_1, p_2) = (p_3, p_4)$, it is sufficient that

$$p_1p_2[\rho] - p_3p_4[\rho] \text{ is fixed by } \sigma_i.$$

Hence, it is enough to show that

$$\begin{aligned}&\frac{v^{-1}([{}^{\sigma_i}x + x' + \rho] - [x + x' + \rho + \alpha_i]) + v([x + x' + \rho + \alpha_i] - [{}^{\sigma_i}x + x' + \rho + \alpha_i])}{[\alpha_i] - 1}\\ &- \frac{v^{-1}([x + {}^{\sigma_i}x' + \rho] - [x + x' + \rho + \alpha_i]) + v([x + x' + \rho + \alpha_i] - [x + {}^{\sigma_i}x' + \rho - \alpha_i])}{[\alpha_i] - 1}\end{aligned}$$

is fixed by σ_i, or that

$$\frac{v^{-1}[{}^{\sigma_i}x + x' + \rho] - v[{}^{\sigma_i}x + x' + \rho + \alpha_i] - v^{-1}[x + {}^{\sigma_i}x' + \rho] + v[x + {}^{\sigma_i}x' + \rho - \alpha_i]}{[\alpha_i] - 1}$$

is fixed by σ_i. This is easily checked. The lemma is proved.

Lemma 5.3. *Let $\chi \in \mathcal{H}$ and let $\xi, \xi' \in \mathcal{AX}^{\otimes 2}$. Then*

(a) $$((\bar{\Delta} \otimes 1)({}^{d'}\hat{\chi})\xi : \xi') = ((\bar{\Delta} \otimes 1)\xi : {}^{d'}\chi\xi'),$$

(b) $$((\bar{\Delta}^{\dagger} \otimes 1)({}^{d}\hat{\chi})\xi : \xi') = ((\bar{\Delta}^{\dagger} \otimes 1)\xi : {}^{d}\chi\xi').$$

Using 4.10, the identities to be proved can be rewritten as

$$({}^d\hat{\chi}(\bar{\Delta}\otimes 1)\xi : \xi') = ((\bar{\Delta}\otimes 1)\xi : {}^{d'}\chi\xi'),$$

$$({}^{d'}\hat{\chi}(\bar{\Delta}^\dagger\otimes 1)\xi : \xi') = ((\bar{\Delta}^\dagger\otimes 1)\xi : {}^{d}\chi\xi').$$

These are special cases of 5.2(a). The lemma is proved.

5.4. Let $\diamond : \mathcal{A}\mathcal{X}^{\otimes 2} \to \mathcal{A}\mathcal{X}^{\otimes 2}$ be the $\mathcal{A}$-linear map defined by

$$\diamond(p_1 \otimes p_2) = {}^{w_0}p_1^\dagger \otimes p_2^\dagger$$

for $p_1, p_2 \in \mathcal{A}\mathcal{X}$. Clearly, $\diamond$ is $\mathcal{A}\mathcal{X}$-semilinear with respect to the involution $p \mapsto p^\dagger$ of $\mathcal{A}\mathcal{X}$. From the definitions (see 1.24) we have (both in $\mathcal{A}\mathcal{X}^{\otimes 2}_{d'}$ and in $\mathcal{A}\mathcal{X}^{\otimes 2}_{d}$):

(a) $\chi^{⁕} \diamond (\xi) = \diamond(\chi\xi)$

for all $\chi \in \mathcal{H}, \xi \in \mathcal{A}\mathcal{X}^{\otimes 2}$. From 1.22, we see that

(b) $(\bar{\theta}_x)^{⁕} = \overline{\theta_{-{}^{w_0}x}}$

for all $x \in X$. Hence, (a) implies

(c) $\overline{\theta_{-{}^{w_0}x}} \diamond (\xi) = \diamond(\bar{\theta}_x\xi)$

for all $x \in \mathcal{X}$ (both in $\mathcal{A}\mathcal{X}^{\otimes 2}_{d'}$ and in $\mathcal{A}\mathcal{X}^{\otimes 2}_{d}$).

Lemma 5.5. *Let $\xi, \xi' \in \mathcal{A}\mathcal{X}^{\otimes 2}$. We have $(\xi : \diamond(\xi')) = (\diamond(\xi) : \xi')^\dagger$.*

We may assume that $\xi = [x]\otimes[y], \xi' = [x']\otimes[y']$ where $x, x', y, y' \in \mathcal{X}$. We have

$$(\xi : \diamond(\xi')) = ([x], [-{}^{w_0}x'])[y - y'], \quad (\diamond(\xi) : \xi') = ([-{}^{w_0}x], [x'])[-y + y'].$$

Thus, it is enough to show that $([x], [-{}^{w_0}x']) = ([-{}^{w_0}x], [x'])^\dagger$. We have

$$([x], [-{}^{w_0}x']) = \delta^{-1}\sum_{w\in W}\mathrm{sgn}_w {}^w[x - {}^{w_0}x' + \rho] = \delta^{-1}\sum_{w'\in W}\mathrm{sgn}_{w'w_0} {}^{w'w_0}[x - {}^{w_0}x' + \rho]$$
$$= \delta^{\dagger -1}\sum_{w'\in W}\mathrm{sgn}_{w'} {}^{w'}[-{}^{w_0}x + x' + \rho]^\dagger = ([-{}^{w_0}x], [x'])^\dagger.$$

The lemma is proved.

5.6. Let $\Gamma_{d'} : \mathcal{A}\mathcal{X}^{\otimes 2}_{d'} \to \mathcal{A}\mathcal{X}^{\otimes 2}_{d'}, \Gamma_d : \mathcal{A}\mathcal{X}^{\otimes 2}_{d} \to \mathcal{A}\mathcal{X}^{\otimes 2}_{d}$ be the $\mathcal{A}$-linear maps defined by

$$\Gamma_{d'}(\xi) = {}^{d'}\tilde{T}_{w_0} \diamond (\xi), \quad \Gamma_d(\xi) = {}^{d}\tilde{T}_{w_0} \diamond (\xi).$$

Lemma 5.7. *We have*

(a) $$\Gamma_{d'}({}^{d'}\chi\xi) = {}^{d'}\chi^{\triangle}\Gamma_{d'}\xi, \quad \Gamma_d({}^{d}\chi\xi) = {}^{d}\chi^{\triangle}\Gamma_d\xi$$

for all $\chi \in \mathcal{H}, \xi \in \mathcal{A}\mathcal{X}^{\otimes 2}$. ($\chi^{\triangle}$ as in 1.24.)

To prove (a) for $\Gamma_{d'}$ it is enough to consider the case where $\chi = \tilde{T}_{\sigma_i}$ or $\chi = \overline{\theta_x}$. In the first case, we have by 5.4(a) (the $\mathcal{H}$ actions are on $\mathcal{A}\mathcal{X}^{\otimes 2}_{d'}$):

$$\Gamma_{d'}(\tilde{T}_{\sigma_i}\xi) = \tilde{T}_{w_0} \diamond (\tilde{T}_{\sigma_i}\xi) = \tilde{T}_{w_0}\tilde{T}_{\sigma_{i^*}} \diamond (\xi) = \tilde{T}_{\sigma_i}\tilde{T}_{w_0} \diamond (\xi) = \tilde{T}_{\sigma_i}\Gamma_{d'}\xi.$$

In the second case we have by 5.4(c) and 1.22:

$$\Gamma_{d'}(\overline{\theta_x}\xi) = \tilde{T}_{w_0} \diamond (\overline{\theta_x}\xi) = \tilde{T}_{w_0}\overline{\theta_{-{}^{w_0}x}} \diamond (\xi) = \theta_{-x}\tilde{T}_{w_0} \diamond (\xi) = \theta_{-x}\Gamma_{d'}\xi.$$

This proves (a) for $\Gamma_{d'}$. The proof for Γ_d is entirely similar.

5.8. For $\xi, \xi' \in \mathcal{A}\mathcal{X}^{\otimes 2}$ we set

(a) $$(\xi \mid \xi')_{d'} = (-v)^{-\nu}(\xi : (\bar{\Delta} \otimes 1)\Gamma_{d'}(\xi')) \in \mathcal{A}\mathcal{X},$$

(b) $$(\xi \mid \xi')_d = (-v)^{-5\nu}(\Delta\Delta^\dagger)^{-1}(\xi : (\bar{\Delta}^\dagger \otimes 1)\Gamma_d(\xi')) \in \mathcal{A}\mathcal{X}_q.$$

Lemma 5.9. *For $\xi, \xi' \in \mathcal{A}\mathcal{X}^{\otimes 2}$, we have*

$$((\bar{\Delta} \otimes 1)\xi \mid (\bar{\Delta} \otimes 1)\xi')_d = (\xi \mid \xi')_{d'}.$$

An equivalent statement is:

$$(-v)^{-5\nu}(\Delta\Delta^\dagger)^{-1}((\bar{\Delta} \otimes 1)\xi : (\bar{\Delta}^\dagger \otimes 1)\Gamma_d((\bar{\Delta} \otimes 1)\xi')) = (-v)^{-\nu}(\xi : (\bar{\Delta} \otimes 1)\Gamma_{d'}(\xi')).$$

The left hand side equals

$$\begin{aligned}
&(-v)^{-5\nu}(\Delta\Delta^\dagger)^{-1}((\bar{\Delta}^\dagger\bar{\Delta} \otimes 1)\xi : \Gamma_d((\bar{\Delta} \otimes 1)\xi')) \\
&= (-v)^{-5\nu}(\Delta\Delta^\dagger)^{-1}((1 \otimes \bar{\Delta}^\dagger\bar{\Delta})\xi : {}^d\tilde{T}_{w_0} \diamond ((\bar{\Delta} \otimes 1)\xi')) \\
&= (-v)^{-5\nu}(\Delta\Delta^\dagger)^{-1}\bar{\Delta}^\dagger\bar{\Delta}(\xi : {}^d\tilde{T}_{w_0}(\bar{\Delta} \otimes 1) \diamond (\xi')) \\
&= (-v)^{-5\nu}v^{4\nu}(\xi : (\bar{\Delta} \otimes 1)^{d'}\tilde{T}_{w_0} \diamond (\xi')) = (\xi \mid \xi')_{d'}.
\end{aligned}$$

(The first equality holds since $\bar{\Delta}^\dagger\bar{\Delta} \in \mathcal{A}\mathcal{X}^W$; the third equality holds due to 4.10 and the identity $\bar{\Delta}^\dagger\bar{\Delta} = v^{4\nu}\Delta\Delta^\dagger$.) The lemma is proved.

Lemma 5.10. *For $\xi, \xi' \in \mathcal{A}\mathcal{X}^{\otimes 2}$ we have $(\xi \mid \xi')_{d'} = (\xi' \mid \xi)^\dagger_{d'}$, $(\xi \mid \xi')_d = (\xi' \mid \xi)^\dagger_d$,*

We have

$$\begin{aligned}
&(\xi : (\bar{\Delta} \otimes 1)\tilde{T}_{w_0} \diamond (\xi')) = ((\bar{\Delta} \otimes 1)\xi : \tilde{T}_{w_0} \diamond (\xi')) = ((\bar{\Delta} \otimes 1)\xi : \diamond\tilde{T}_{w_0}(\xi')) \\
&= (\diamond(\bar{\Delta} \otimes 1)\xi : \tilde{T}_{w_0}(\xi'))^\dagger = ((\bar{\Delta} \otimes 1) \diamond (\xi) : \tilde{T}_{w_0}(\xi'))^\dagger = ((\bar{\Delta} \otimes 1)\tilde{T}_{w_0} \diamond (\xi) : \xi')^\dagger \\
&= (\xi' : (\bar{\Delta} \otimes 1)\tilde{T}_{w_0} \diamond (\xi))^\dagger
\end{aligned}$$

where the action of $\tilde{T}_{w_0}$ is in $\mathcal{A}\mathcal{X}^{\otimes 2}_{d'}$. (The first and last equalities are obvious. The second equality follows from 5.4(a). The third equality follows from 5.5. The fourth equality follows from 5.4(a), since $x \mapsto -^{w_0}x$ is a permutation of $\mathcal{R}^+$. The fifth equality follows from 5.3.) Thus the lemma is proved in the case of $\mathcal{A}\mathcal{X}^{\otimes 2}_{d'}$. The proof in the case of $\mathcal{A}\mathcal{X}^{\otimes 2}_d$ is entirely similar. The lemma is proved.

Lemma 5.11. *Let $p \in \mathcal{A}\mathcal{X}$. For $\xi, \xi' \in \mathcal{A}\mathcal{X}^{\otimes 2}$ we have*

(a) $((1 \otimes p)\xi \mid \xi')_{d'} = p(\xi \mid \xi')_{d'} = (\xi \mid (1 \otimes p^\dagger)\xi')_{d'}$,
(b) $((1 \otimes p)\xi \mid \xi')_d = p(\xi \mid \xi')_d = (\xi \mid (1 \otimes p^\dagger)\xi')_d$.

The first equality in (a) is obvious from the definition. The second equality in (a) follows from the first using 5.10. The same applies to (b).

Lemma 5.12. *Let $\chi \in \mathcal{H}$. For any $\xi, \xi' \in \mathcal{A}\mathcal{X}^{\otimes 2}$ we have*

(a) $$(\xi \mid {}^{d'}\chi\xi')_{d'} = ({}^{d'}\chi^\blacktriangle\xi \mid \xi')_{d'},$$

(b) $$(\xi \mid {}^{d}\chi\xi')_d = ({}^{d}\chi^\blacktriangle\xi \mid \xi')_d.$$

($\chi^\blacktriangle$ as in 1.24.)

We prove (a). Using 5.3 and 5.7 we have

$$\begin{aligned}(\xi \mid {}^{d'}\chi\xi')_{d'} &= (-v)^{-\nu}(\xi : (\bar{\Delta}\otimes 1)\Gamma_{d'}(\chi\xi')) = (-v)^{-\nu}(\xi : (\bar{\Delta}\otimes 1)\chi^{\Delta}\Gamma_{d'}(\xi'))\\ &= (-v)^{-\nu}((\chi^{\Delta})\hat{}\xi : (\bar{\Delta}\otimes 1)\Gamma_{d'}(\xi')) = (-v)^{-\nu}((\chi^{\Delta})\hat{}\xi \mid \xi').\end{aligned}$$

It remains to use 1.25(a). The proof of (b) is entirely similar. The lemma is proved.

Lemma 5.13. *We have in $\mathcal{A}\mathcal{X}$:*

(a) $$\sum_{w\in W}\mathrm{sgn}_w{}^w(\bar{\Delta}[-\rho]) = (-1)^{\nu}\sum_{w\in W}v^{2l(w)}\delta.$$

For any subset J of $\mathcal{R}^+$ we set $\alpha_J = \sum_{\alpha\in J}\alpha$. From the definition we have

$$\bar{\Delta}[-\rho] = \sum_{x\in\mathcal{X}}c_x[x]$$

where

$$c_x = \sum_{J\in\mathcal{R}^+;\alpha_J = x+\rho}(-1)^{|J|}v^{2|J|}.$$

For any $x\in\mathcal{X}$ we denote by x_0 the unique element of $\mathcal{X}_+$ in the W-orbit of x. The properties (b),(c) below are easily verified:

(b) For $u\in W$ we have $c_{{}^u\rho} = \mathrm{sgn}_u(-1)^{\nu}v^{2\nu-2l(u)}$.

(c) If $c_x\neq 0$ and $\check{\alpha}_i(x_0)>0$ for all i, then $x = {}^u\rho$ for a unique $u\in W$.

The left hand side of (a) is $\sum_{x\in\mathcal{X};w\in W}\mathrm{sgn}_w c_x[{}^w x]$. Clearly, this is equal to the sum restricted to $x\in\mathcal{X}\cap\mathcal{C}^+$. Hence, it is equal to

$$\begin{aligned}&\sum_{u,w\in W}\mathrm{sgn}_w c_{{}^u\rho}[{}^{wu}\rho] = \sum_{u,w\in W}\mathrm{sgn}_{wu}(-1)^{\nu}v^{2\nu-2l(u)}[{}^{wu}\rho]\\ &= \sum_{u,w'\in W}\mathrm{sgn}_{w'}(-1)^{\nu}v^{2\nu-2l(u)}[{}^{w'}\rho] = (-1)^{\nu}\sum_{u\in W}v^{2\nu-2l(u)}\sum_{w'\in W}\mathrm{sgn}_{w'}[{}^{w'}\rho]\\ &= (-1)^{\nu}\sum_{w\in W}v^{2l(w)}\delta.\end{aligned}$$

The lemma is proved.

Lemma 5.14. *We have*

$$(v^{-\nu}[-\rho]\otimes[-\rho] \mid v^{-\nu}[-\rho]\otimes[-\rho])_{d'} = ({}_0e_{\epsilon}|{}_0e_{\epsilon}).$$

We have in $\mathcal{A}\mathcal{X}^{\otimes 2}_{d'}$:

$$\begin{aligned}&(v^{-\nu}[-\rho]\otimes[-\rho] \mid v^{-\nu}[-\rho]\otimes[-\rho])_{d'}\\ &= v^{-2\nu}(-v)^{-\nu}([-\rho]\otimes[-\rho] : (\bar{\Delta}\otimes 1)\tilde{T}_{w_0}\diamond([-\rho]\otimes[-\rho]))\\ &= (-v)^{-3\nu}([-\rho]\otimes[-\rho] : (\bar{\Delta}\otimes 1)\tilde{T}_{w_0}([-\rho]\otimes[\rho]))\\ &= (-v)^{-3\nu}v^{\nu}([-\rho]\otimes[-\rho] : (\bar{\Delta}\otimes 1)[-\rho]\otimes[\rho])\\ &= (-1)^{\nu}v^{-2\nu}([-\rho],\bar{\Delta}[-\rho]) = (-1)^{\nu}v^{-2\nu}\delta^{-1}\sum_{w\in W}\mathrm{sgn}_w{}^w(\bar{\Delta}[-\rho])\\ &= v^{-2\nu}\sum_{w\in W}v^{2l(w)}.\end{aligned}$$

(We have used 5.13.) We now use 3.6(c). The lemma is proved.

Proposition 5.15. *All maps in the commutative diagram 4.11 respect the inner products* $(|)_{d'}$ *on* $\mathcal{AX}_{d'}^{\otimes 2}$, $(|)_d$ *on* $\mathcal{AX}_d^{\otimes 2}$, $(|)$ *on* $\mathcal{M}_{d'}$ *in 3.4 and the inner product on* $\mathcal{M}_d$ *given by the restriction of the one in 3.6.*

For the inclusion $\mathcal{M}_{d'} \to \mathcal{M}_d$, this is obvious from 3.6. For the map $\mathcal{AX}_{d'}^{\otimes 2} \to \mathcal{AX}_d^{\otimes 2}$, this follows from 5.9. For f this follows from the uniqueness statement in 3.7, using 5.11, 5.12, 5.14. From these facts, it follows that $\tilde{f}$ is automatically compatible with the inner products. The proposition is proved.

5.16. From 5.15 and 3.15 we deduce that

$$((-1)^\nu v^{2\nu}[0] \otimes [-2\rho] \mid (-1)^\nu v^{2\nu}[0] \otimes [-2\rho])_d = (\Delta\Delta^\dagger)^{-1} \sum_{w \in W} v^{-2l(w)}.$$

This can also be proved directly by arguments similar to those in 5.14.

Corollary 5.17. *For any* $x \in \mathcal{X}$ *we have*

$$(\theta_x({}_0e_\epsilon)|{}_0e_\epsilon) = (-1)^\nu v^{-2\nu}([x-\rho], \bar{\Delta}[-\rho]).$$

We apply the identity 5.15(a) with $\xi = v^{-\nu}[x-\rho] \otimes [-\rho], \xi' = v^{-\nu}[-\rho] \otimes [-\rho]$. Note that, as in the proof of 5.15, we have $(\xi \mid \xi')_d = (-1)^\nu v^{-2\nu}([x-\rho], \bar{\Delta}[-\rho])$. On the other hand, $(f(\xi)|f(\xi')) = (\theta_x({}_0e_\epsilon)|{}_0e_\epsilon)$.

6. Generalities on coherent sheaves

6.1. In this paper, unless otherwise specified, all algebraic varieties are assumed to be quasiprojective over $\mathbf{C}$ and all algebraic groups are assumed to be affine over $\mathbf{C}$. Let H be an algebraic group. By an *H-variety* we mean an algebraic variety V with a given algebraic action $H \times V \to V$ such that there exists a smooth algebraic variety V' with an action of H and an H-equivariant closed imbedding $V' \to V$.

Let V be an H-variety. Let $\mathrm{Coh}_H(V)$ be the abelian category of H-equivariant coherent sheaves on V; see [T1]. Let $\mathrm{Vec}_H(V)$ be the category of H-equivariant vector bundles on X. Any object in $\mathrm{Vec}_H(V)$ gives rise to an object of $\mathrm{Coh}_H(V)$ (by taking the sheaf of sections) and will often be identified with this object of $\mathrm{Coh}_H(V)$.

The trivial line bundle on V as an object of $\mathrm{Vec}_H(V)$ or $\mathrm{Coh}_H(V)$ is generally denoted by $\mathbf{C}$.

Let $K_H(V)$ be the Grothendieck group of the abelian category $\mathrm{Coh}_H(V)$. This is naturally a module over R_H, the Grothendieck group of finite dimensional H-modules. (We have $R_H = K_H(\text{point})$ in a natural way.)

If $f : V \to V'$ is an H-equivariant morphism of H-varieties, then for any $F \in \mathrm{Vec}_H(V')$ (resp. $F \in \mathrm{Coh}_H(V')$) the inverse image $f^*(F) \in \mathrm{Vec}_H(V)$ (resp. $f^*(F) \in \mathrm{Coh}_H(V)$) is well defined. If in addition f is smooth, then $F \mapsto f^*(F)$ is exact; hence it induces an R_H-linear map $f^* : K_H(V') \to K_H(V)$.

If $f : V \to V'$ is a proper H-equivariant morphism, then for any $F \in \mathrm{Coh}_H(V)$ the higher direct image sheaves $R^n f_*(F)$ are naturally objects of $\mathrm{Coh}_H(V')$ and are zero for large $|n|$; moreover, the assignment $F \mapsto \sum_{n \in \mathbf{Z}} (-1)^n R^n f_*(F)$ defines an R_H-linear map $f_* : K_H(V) \to K_H(V')$ (direct image).

If $E \in \mathrm{Vec}_H(V)$, then $F \mapsto F \otimes E$ defines an R_H-linear map $K_H(V) \to K_H(V)$.

6.2. Let V be an H-variety and let V' be a closed H-stable subvariety. Let $f : V' \to V$ be the imbedding. Let $\mathrm{Coh}_H(V; V')$ be the subcategory of $\mathrm{Coh}_H(V)$ whose objects are those $F \in \mathrm{Coh}_H(V)$ such that the support of F is contained in V'. Let $K_H(V; V')$ be the Grothendieck group of $\mathrm{Coh}_H(V; V')$. If $F \in \mathrm{Coh}_H(V')$, then $R^n f_*(F) = 0$ for $n \neq 0$ and $R^0 f_*(F) \in \mathrm{Coh}_H(V; V')$. Moreover, $F \mapsto R^0 f_*(F)$ defines an isomorphism $K_H(V') \xrightarrow{\sim} K_H(V; V')$. Hence, in this case $f_* : K_H(V') \to K_H(V)$ may be identified with the map $K_H(V; V') \to K_H(V)$ induced by the obvious inclusion $\mathrm{Coh}_H(V; V') \subset \mathrm{Coh}_H(V)$.

6.3. Let V be a smooth H-variety and let V_1, V_2 be closed H-stable subvarieties of V. Let $F_1 \in \mathrm{Coh}_H(V; V_1), F_2 \in \mathrm{Coh}_H(V; V_2)$. By an H-equivariant version of the Hilbert syzygy theorem, we can find complexes of locally free sheaves

$$\cdots \to F_1^p \to F_1^{p+1} \to \dots, \quad \cdots \to F_2^p \to F_2{}^{p+1} \to \dots,$$

in $\mathrm{Coh}_H(V)$ such that F_1^p, F_2^p are zero for $p > 0$ and for $|p|$ large, and the cohomology sheaves are zero except in degree 0 where they are F_1, F_2 respectively. We consider the complex of locally free sheaves obtained by taking the tensor product of the two complexes above:

$$\cdots \to F_1^0 \otimes F_2^{-1} \oplus F_1^{-1} \otimes F_2^0 \to F_1^0 \otimes F_2{}^0 \to 0 \to \dots.$$

The p-th cohomology sheaf $E^p \in \mathrm{Coh}_H(V)$ of this complex satisfies $E^p = 0$ for $p > 0$ and for $|p|$ large and the support of E^p is contained in $V_1 \cap V_2$; thus, $E^p \in \mathrm{Coh}_H(V; V_1 \cap V_2)$. Then $F_1 \circledcirc_V F_2 = \sum_{p \in \mathbf{Z}} (-1)^p E^p \in K_H(V; V_1 \cap V_2)$ is independent of the choices made and it extends to an R_H-bilinear pairing

$$\circledcirc_V : K_H(V; V_1) \times K_H(V; V_2) \to K_H(V; V_1 \cap V_2).$$

6.4. **Tor-product.** Let V, V_1, V_2 be as in 6.3. Let $F_1' \in \mathrm{Coh}_H(V_1), F_2' \in \mathrm{Coh}_H(V_2)$. Let f_1, f_2, f_{12} be the inclusions of $V_1, V_2, V_1 \cap V_2$ in V. Let $F_1 = R^0 f_{1*}(F_1'), F_2 = R^0 f_{2*}(F_2')$. Then $F_1 \circledcirc_V F_2 \in K_H(V; V_1 \cap V_2)$ is well defined. Applying to it the inverse of the isomorphism $K_H(V_1 \cap V_2) \xrightarrow{\sim} K_H(V; V_1 \cap V_2)$ in 6.2, we obtain an element

$$F_1' \otimes_V^L F_2' \in K_H(V_1 \cap V_2).$$

This construction defines an R_H-bilinear pairing

$$K_H(V_1) \times K_H(V_2) \to K_H(V_1 \cap V_2)$$

denoted by $\xi, \xi' \mapsto \xi \otimes_V^L \xi'$ and called *Tor-product*. This definition goes back to Serre [S2].

6.5. Let V, V' be smooth H-varieties and let $f : V \to V'$ be a smooth H-equivariant morphism. Let V_1, V_2 be closed subvarieties of V and let V_1', V_2' be closed subvarieties of V'. Assume that $V_2 = f^{-1}(V_2'), f(V_1) \subset V_1'$. Let

$$f_1 : V_1 \to V_1', f_2 : V_2 \to V_2', f_{12} : V_1 \cap V_2 \to V_1' \cap V_2'$$

be the restrictions of f. Assume that f_1 (hence f_{12}) is proper. Let $F \in \mathrm{Coh}_H(V_1)$, $F' \in \mathrm{Coh}_H(V_2')$. We have $f_{12*}(F \otimes_V^L f_2^*(F')) = f_{1*}(F) \otimes_{V'}^L F'$. Here $F \otimes_V^L f_2^*(F') \in K_H(V_1 \cap V_2)$ (resp. $f_{1*}(F) \otimes_{V'}^L F' \in K_H(V_1' \cap V_2')$) is relative to V, V_1, V_2 (resp. to V', V_1', V_2').

6.6. Let V, V' be smooth H-varieties and let $f : V \to V'$ be a smooth H-equivariant morphism. Let V'_1, V'_2 be closed subvarieties of V' and let $V_1 = f^{-1}(V'_1)$, $V_2 = f^{-1}(V'_2)$. Let

$$f_1 : V_1 \to V'_1, f_2 : V_2 \to V'_2, f_{12} : V_1 \cap V_2 \to V'_1 \cap V'_2$$

be the restrictions of f. Let $F \in \mathrm{Coh}_H(V'_1), F' \in \mathrm{Coh}_H(V'_2)$. We have

$$f_{12}^*(F \otimes_{V'}^L F') = f_1^* F \otimes_V^L f_2^* F'.$$

Here $F \otimes_{V'}^L F' \in K_H(V'_1 \cap V'_2)$ is relative to V', V'_1, V'_2 and $f_1^* F \otimes_V^L f_2^* F' \in K_H(V_1 \cap V_2)$ is relative to V, V_1, V_2.

6.7. In the setup of 6.4, let us assume that $V_2 = V$ and that $F'_2 \in \mathrm{Vec}_H(V)$. Let $\tilde{F}'_2 \in \mathrm{Vec}_H(V_1)$ be the restriction of F'_2 to V_1. Then $F'_1 \otimes_V^L F'_2 = F'_1 \otimes \tilde{F}'_2$. Here $F'_1 \otimes \tilde{F}'_2$ is the usual tensor product of a coherent sheaf with a locally free coherent sheaf on V_1.

6.8. For any smooth H-variety V we denote by $\Omega_V \in \mathrm{Vec}_H(V)$ the line bundle of top exterior differential forms on V. (Here the "top degree" of a differential form may vary from one connected component to another.)

6.9. Let V be a smooth H-variety and let V' be a closed H-stable subvariety of V. We define a group homomorphism

$$D_{V;V'} : K_H(V;V') \to K_H(V;V')$$

as follows. Let $F \in \mathrm{Coh}_H(V;V')$. We can find a complex of locally free sheaves

(a) $\dots \to F^p \to F^{p+1} \to \dots$

in $\mathrm{Vec}_H(V)$ such that F^p are zero for $p > 0$ and for $|p|$ large, and the p-th cohomology sheaf is zero except in degree 0 where it is F. We consider the complex of locally free sheaves $\dots \to \tilde{F}^p \to \tilde{F}^{p+1} \to \dots$ in $\mathrm{Coh}_H(V)$ where, for any connected component V_j of V of dimension n, we have $\tilde{F}^p|_{V_j} = \mathrm{Hom}(F^{-n-p}|_{V_j}, \Omega_{V_j})$; the maps in the complex are the transposes of those in (a).

The pth cohomology sheaf $E^p \in \mathrm{Coh}_H(V)$ of this complex satisfies $E^p = 0$ for $|p|$ large and the support of E^p is contained in V'; thus, $E^p \in \mathrm{Coh}_H(V;V')$. We set $D_{V;V'}(F) = \sum_{p \in \mathbf{Z}} (-1)^p E^p \in K_H(V;V')$. This is independent of the choices and defines the required homomorphism.

6.10. **Serre-Grothendieck duality.** Let V' be an H-variety. We define a group homomorphism

$$D_{V'} : K_H(V') \to K_H(V')$$

(Serre-Grothendieck duality) as follows. We choose an H-equivariant closed imbedding of V' into a smooth H-variety V. Let $f : V' \to V$ be the inclusion. Let $F' \in \mathrm{Coh}_H(V')$. Let $F = R^0 f_*(F') \in K_H(V;V')$. Then $D_{V;V'}(F) \in K_H(V;V')$ is well defined. Applying to it the inverse of the isomorphism $K_H(V') \xrightarrow{\sim} K_H(V;V')$ in 6.3, we obtain an element of $K_H(V;V')$ denoted by $D_{V'}(F')$. One shows that $F' \mapsto D_{V'}(F')$ is a well defined homomorphism $K_H(V') \to K_H(V')$, independent of the choice of imbedding $V' \subset V$. This definition goes back to [S1], [Gr]; see also [Ha].

6.11. Let $f : V \to V'$ be a proper H-equivariant morphism of H-varieties. The following diagram is commutative.

$$\begin{array}{ccc} K_H(V) & \xrightarrow{f_*} & K_H(V') \\ {\scriptstyle D_V}\downarrow & & {\scriptstyle D_{V'}}\downarrow \\ K_H(V) & \xrightarrow{f_*} & K_H(V') \end{array}$$

6.12. Let $f : V \to V'$ be a smooth H-equivariant morphism of H-varieties. Assume that the fibres of f are connected of dimension n. Let Ω_f be the nth exterior power of the cotangent bundle along the fibres of f (a line bundle in $\mathrm{Vec}_H(V)$). For any $\xi' \in K_H(V')$ we have

$$D_V(f^*\xi) = (-)^n f^*(D_{V'}\xi) \otimes \Omega_f.$$

6.13. If V is a smooth connected H-variety of dimension n and if $E \in \mathrm{Vec}_H(V)$, then $D_V(E) = (-1)^n \mathrm{Hom}(E, \Omega_V)$.

6.14. Let V, V_1, V_2 be as in 6.3. Assume that V has pure dimension n. For $\xi_1 \in K_H(V; V_1), \xi_2 \in K_H(V; V_2)$ we have

$$(-1)^n D_{V;V_1\cap V_2}(\xi_1 \circledcirc_V \xi_2) \otimes \Omega_V = (D_{V;V_1}\xi_1) \circledcirc_V (D_{V;V_2}\xi_2) \in K_H(V; V_1 \cap V_2).$$

6.15. Let V, V_1, V_2 be as in 6.3. Assume that V has pure dimension n. For $\xi_1 \in K_H(V_1), \xi_2 \in K_H(V_2)$ we have

$$(-1)^n D_{V_1\cap V_2}(\xi_1 \otimes_V^L \xi_2) \otimes \Omega_V = (D_{V_1}\xi_1) \otimes_V^L (D_{V_2}\xi_2) \in K_H(V_1 \cap V_2).$$

6.16. For any H-variety V, the map $D_V : K_H(V) \to K_H(V)$ is semilinear with respect to the involution of R_H which takes any H-module to the dual module. Moreover, $D_V D_V = 1$.

7. The homomorphism $\mathcal{H} \to K_{\mathcal{G}}(Z)$

7.1. Let G be a connected, semisimple, simply connected algebraic group. Let $\mathfrak{g}$ be the Lie algebra of G and let $\mathfrak{g}_n$ be the variety of nilpotent elements in $\mathfrak{g}$. Let $\mathcal{B}$ be the variety of all Borel subalgebras of $\mathfrak{g}$.

For any parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ we denote by $\mathfrak{n}_\mathfrak{p}$ the nil-radical of $\mathfrak{p}$.

A parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is said to be *almost minimal* if the variety of Borel subalgebras contained in $\mathfrak{p}$ is 1-dimensional. Let I be a finite set indexing the G-orbits on the set of almost minimal parabolic subalgebras (for the adjoint action). A parabolic subalgebra in the G-orbit indexed by i is said to have type i. Let $\mathcal{P}_i$ be the variety of all parabolic subalgebras of type i. Let $\pi_i : \mathcal{B} \to \mathcal{P}_i$ be the morphism defined by $\pi_i(\mathfrak{b}) = \mathfrak{p}$ where $\mathfrak{b} \in \mathcal{B}, \mathfrak{p} \in \mathcal{P}_i, \mathfrak{b} \subset \mathfrak{p}$.

7.2. Let $\mathbf{X}$ be the set of isomorphism classes of algebraic G-equivariant line bundles on $\mathcal{B}$ where G acts on $\mathcal{B}$ by the adjoint action. Then $\mathbf{X}$ is a finitely generated free abelian group under the operation given by tensor product of line bundles. For $L, L' \in \mathbf{X}$ we shall often write LL' instead of $L \otimes L'$; L^n instead of $L^{\otimes n}$ (if $n \in \mathbf{N}$); L^{-n} instead of the dual of $L^{\otimes n}$ (if $n \in \mathbf{N}$). For each $i \in I$, let $L_i \in \mathbf{X}$ be the tangent bundle along the fibres of $\pi_i : \mathcal{B} \to \mathcal{P}_i$. Then L_i^{-1} is the cotangent bundle along the fibres of $\pi_i : \mathcal{B} \to \mathcal{P}_i$.

Given $i \in I$ and $L \in \mathbf{X}$, we define an integer m by the requirement that the Euler characteristic of any fibre of π_i (a projective line) with coefficients in the restriction of L to that fibre (regarded as a coherent sheaf) is $m+1$. We set $m=\check{\alpha}_i(L)\in\mathbf{Z}$. Then $\check{\alpha}_i:\mathbf{X}\to\mathbf{Z}$ is a group homomorphism such that

(a) for $L\in\mathbf{X}$, we have $\check{\alpha}_i(L)=0$ if and only if $L=\pi_i^*(\tilde{L})$ for some G-equivariant line bundle $\tilde{L}$ on $\mathcal{P}_i$;

(b) $\check{\alpha}_i(L_i)=2$.

Let $\mathcal{X}$ be a free abelian group (in additive notation) with a given isomorphism $\mathcal{X}\xrightarrow{\sim}\mathbf{X}$ denoted by $x\mapsto L_x$. (Thus, $L_xL_{x'}=L_{x+x'}$ for $x,x'\in\mathcal{X}$.) Let $\alpha_i\in\mathcal{X}$ be defined by $L_{\alpha_i}=L_i$. Then $L_{-\alpha_i}=L_i^{-1}$. The composition $\mathcal{X}\to\mathbf{X}\xrightarrow{\check{\alpha}_i}\mathbf{Z}$ is denoted again by $\check{\alpha}_i$. Let $\mathcal{V}=\mathbf{R}\otimes\mathcal{X}$. Then $(\mathcal{V}=E,\mathcal{X},I,\alpha_i\in\mathcal{X},\check{\alpha}_i:\mathcal{X}\to\mathbf{Z})$ is as in 1.1. Hence, all definitions and results of Sections 1-5 are applicable. (We take $\epsilon=0$.) The finite Coxeter group W with its generators $\{\sigma_i|i\in I\}$ becomes a group acting on $\mathbf{X}$ by $w:L\mapsto {}^wL$ where ${}^{\sigma_i}L=LL_i^{-\check{\alpha}_i(L)}$.

7.3. Let $w\mapsto\mathcal{O}_w$ be the bijection between W and the set of G-orbits on $\mathcal{B}\times\mathcal{B}$ (diagonal action) characterized by properties (a)-(c) below:

(a) $\mathcal{O}_1$ is the diagonal in $\mathcal{B}\times\mathcal{B}$;

(b) if $\mathfrak{b},\mathfrak{b}'\in\mathcal{B}$, we have $(\mathfrak{b},\mathfrak{b}')\in\mathcal{O}_{\sigma_i}$ if and only if $\mathfrak{b}\neq\mathfrak{b}'$ and $\pi_i(\mathfrak{b})=\pi_i(\mathfrak{b}')$;

(c) if $(\mathfrak{b},\mathfrak{b}')\in\mathcal{O}_w,(\mathfrak{b}',\mathfrak{b}'')\in\mathcal{O}_{w'},l(ww')=l(w)+l(w')$, then $(\mathfrak{b},\mathfrak{b}'')\in\mathcal{O}_{ww'}$.

7.4. Let $w\in W$. Let $p_1,p_2:\mathcal{O}_w\to\mathcal{B}$ be the first and second projection. Let $L\in\mathbf{X}$. Then the G-equivariant line bundles $p_1^*L,p_2^*({}^{w^{-1}}L)$ on $\mathcal{O}_w$ are isomorphic.

7.5. Let

$$\Lambda=\{(y,\mathfrak{b})\in\mathfrak{g}_n\times\mathcal{B}|y\in\mathfrak{b}\},$$
$$Z=\{(y,\mathfrak{b},\mathfrak{b}')\in\mathfrak{g}_n\times\mathcal{B}\times\mathcal{B}|y\in\mathfrak{b}\cap\mathfrak{b}'\},$$
$$\Lambda^{aab}=\{(y,\mathfrak{b};y',\mathfrak{b}';y'',\mathfrak{b}'')\in\Lambda^3|y=y'\},$$
$$\Lambda^{abb}=\{(y,\mathfrak{b};y',\mathfrak{b}';y'',\mathfrak{b}'')\in\Lambda^3|y'=y''\},$$
$$\Theta=\{(y,\mathfrak{b};y',\mathfrak{b}';y'',\mathfrak{b}'')\in\Lambda^3|y=y'=y''\},$$
$$\mathcal{G}=G\times\mathbf{C}^*.$$

We regard $\mathcal{B}$ and $\mathfrak{g}_n$ as $\mathcal{G}$-varieties with $\mathcal{G}$-action

$$(g,\lambda):\mathfrak{b}\mapsto Ad(g)\mathfrak{b}\quad\text{and}\quad(g,\lambda):y\mapsto\lambda^{-2}Ad(g)y$$

respectively. We regard $\mathcal{B}\times\mathcal{B}$ and $\mathfrak{g}_n\times\mathcal{B}$ as $\mathcal{G}$-varieties where $\mathcal{G}$ acts simultaneously on both factors. We regard Λ as a $\mathcal{G}$-variety with the $\mathcal{G}$-action given by restriction of the $\mathcal{G}$-action on $\mathfrak{g}_n\times\mathcal{B}$. We regard $\Lambda\times\mathcal{B},\Lambda^2,\Lambda^3$ as $\mathcal{G}$-varieties where $\mathcal{G}$ acts simultaneously on each factor Λ. We regard Z as a $\mathcal{G}$-variety with the $\mathcal{G}$-action given by restriction of the $\mathcal{G}$-action on $\Lambda\times\mathcal{B}$. (We regard Z as a closed subvariety of $\Lambda\times\mathcal{B}$ by $(y,\mathfrak{b},\mathfrak{b}')\mapsto(y,\mathfrak{b};\mathfrak{b}')$.) We regard $\Lambda^{aab},\Lambda^{abb},\Theta$ as $\mathcal{G}$-varieties with the $\mathcal{G}$-action given by restriction of the $\mathcal{G}$-action on Λ^3.

7.6. For $L\in\mathbf{X}$, we regard L as a $\mathcal{G}$-equivariant line bundle on $\mathcal{B}$ with trivial $\mathbf{C}^*$-action. The inverse image of L under the second projection $pr_2:\Lambda\to\mathcal{B}$ is a $\mathcal{G}$-equivariant line bundle on $\mathcal{B}$ (since that projection is $\mathcal{G}$-equivariant). We denote this inverse image again by L. For $L,L'\in\mathbf{X}$ we denote by $L\boxtimes L'$ the external tensor product of L,L'. This is naturally a $\mathcal{G}$-equivariant line bundle on $\mathcal{B}\times\mathcal{B}$, or on $\Lambda\times\mathcal{B}$, or on Λ^2.

The restriction of $L\boxtimes L'$ from Λ^2 (resp. $\mathcal{B}\times\mathcal{B},\Lambda\times\mathcal{B}$) to various $\mathcal{G}$-stable subvarieties of Λ^2 (resp. $\mathcal{B}\times\mathcal{B},\Lambda\times\mathcal{B}$) will be denoted again by $L\boxtimes L'$.

Similarly, for $L,L',L''\in\mathbf{X}$ we denote by $L\boxtimes L'\boxtimes L''$ the external tensor product of L,L',L''. This is naturally a $\mathcal{G}$-equivariant line bundle on $\mathcal{B}\times\mathcal{B}\times\mathcal{B}$ or on Λ^3.

The restriction of $L \boxtimes L' \boxtimes L''$ from Λ^3 to various $\mathcal{G}$-stable subvarieties of Λ^3 will be denoted again by $L \boxtimes L' \boxtimes L''$.

We always identify $K_{\mathcal{G}}(\Lambda)$ with $K_{\mathcal{G}}(\mathcal{B})$ via $pr_2^* : K_{\mathcal{G}}(\mathcal{B}) \xrightarrow{\sim} K_{\mathcal{G}}(\Lambda)$. Similarly, we identify $K_{\mathcal{G}}(\Lambda \times \mathcal{B})$ with $K_{\mathcal{G}}(\mathcal{B} \times \mathcal{B})$ via the isomorphism $(pr_2 \times 1)^* : K_{\mathcal{G}}(\mathcal{B} \times \mathcal{B}) \xrightarrow{\sim} K_{\mathcal{G}}(\Lambda \times \mathcal{B})$ induced by $pr_2 \times 1 : \Lambda \times \mathcal{B} \to \mathcal{B} \times \mathcal{B}$ (a $\mathcal{G}$-equivariant vector bundle).

7.7. For any $n \in \mathbf{Z}$, let $v^n : \mathcal{G} \to \mathbf{C}^*$ be the homomorphism $(g, \lambda) \mapsto \lambda^n$. If H is a closed subgroup of $\mathcal{G}$, we denote the restriction of $v^n : \mathcal{G} \to \mathbf{C}^*$ to H again by v^n.

We may regard v^n as an H-equivariant line bundle on a point, or as an H-equivariant line bundle on any H-variety V' (the inverse image under the map from V' to the point of the previous line bundle). Hence, for any $F \in \mathrm{Vec}_H(V')$ (resp. $F \in \mathrm{Coh}_H(V')$), the tensor product $v^n F = v^n \otimes F$ is well defined in $\mathrm{Vec}_H(V')$ (resp. $\mathrm{Coh}_H(V')$). In this way $K_H(V')$ becomes an $\mathcal{A}$-module. (Note that, what in 1.4 was an indeterminate v, now has a concrete K-theoretic meaning.)

7.8. We have an isomorphism of rings

(a) $\mathcal{A}\mathcal{X} \xrightarrow{\sim} K_{\mathcal{G}}(\mathcal{B}) = K_{\mathcal{G}}(\Lambda)$

given by $v^n[x] \mapsto v^n L_x$ for $n \in \mathbf{Z}, x \in \mathcal{X}$. Here $v^n[x]$ is as in 1.19 and $v^n L_x$ is as in 7.7. Taking the inverse image under the map of $\mathcal{B}$ to the point we obtain an imbedding $R_{\mathcal{G}} \to K_{\mathcal{G}}(\mathcal{B})$ which identifies

(b) $R_{\mathcal{G}} = \mathcal{A}\mathcal{X}^W \subset \mathcal{A}\mathcal{X}$.

We have an isomorphism of rings

(c) $\mathcal{A}\mathcal{X}^{\otimes 2} \xrightarrow{\sim} K_{\mathcal{G}}(\mathcal{B} \times \mathcal{B}) = K_{\mathcal{G}}(\Lambda \times \mathcal{B})$

given by $v^n[x] \otimes [x'] \mapsto v^n L_x \boxtimes L_{x'}$ for $n \in \mathbf{Z}, x \in \mathcal{X}, x' \in \mathcal{X}$. Here, $v^n L_x \boxtimes L_{x'}$ is as in 7.6, 7.7. The proof of (c) is exactly as in [KL2].

7.9. We define $\pi_{12} : \Lambda^{aab} \to Z, \quad \pi_{23} : \Lambda^{abb} \to Z, \quad \pi_{13} : \Theta \to Z$ by

$$\pi_{12}(y, \mathfrak{b}; y', \mathfrak{b}'; y'', \mathfrak{b}'') = (y, \mathfrak{b}, \mathfrak{b}'),$$
$$\pi_{23}(y, \mathfrak{b}; y', \mathfrak{b}'; y'', \mathfrak{b}'') = (y'', \mathfrak{b}', \mathfrak{b}''),$$
$$\pi_{13}(y, \mathfrak{b}; y', \mathfrak{b}'; y'', \mathfrak{b}'') = (y, \mathfrak{b}, \mathfrak{b}'').$$

Note that p_{12}, p_{13} are smooth morphisms and p_{23} is a proper morphism; these morphisms are compatible with the $\mathcal{G}$-actions. For $F \in \mathrm{Coh}_{\mathcal{G}}(Z), F' \in \mathrm{Coh}_{\mathcal{G}}(Z)$, we set

(a) $F \star F' = p_{13*}(p_{12}^* F \otimes_{\Lambda^3}^L p_{23}^* F') \in K_{\mathcal{G}}(Z)$.

(The Tor-product is relative to the smooth $\mathcal{G}$-variety Λ^3 and its closed subvarieties $\Lambda^{aab}, \Lambda^{abb}$ with intersection Θ.)

The convolution product $\star$ has first appeared in the work of Kashiwara and Tanisaki [KT] (in a non-equivariant setting); subsequently, this convolution product and variations of it (as in 10.1) have been used in the equivariant setting by Ginzburg [G].

The assignment $F, F' \mapsto F \star F'$ extends to an $R_{\mathcal{G}}$-bilinear pairing

$$K_{\mathcal{G}}(Z) \times K_{\mathcal{G}}(Z) \to K_{\mathcal{G}}(Z)$$

denoted by $\xi, \xi' \mapsto \xi * \xi'$. This may be regarded as a multiplication law on $K_{\mathcal{G}}(Z)$ which is associative, as a routine argument using Λ^4 shows.

7.10. For any $L \in \mathbf{X}$ we regard L as an object in $\mathrm{Vec}_{\mathcal{G}}(\Lambda)$ as in 7.6. Taking its direct image under the closed imbedding $\Lambda \subset Z$ given by $(y, \mathfrak{b}) \mapsto (y, \mathfrak{b}, \mathfrak{b})$ we obtain an object ${}^!L \in \mathrm{Coh}_{\mathcal{G}}(Z)$. From the definitions (and using 6.5) we see that

$$ {}^!L \star F = (L \boxtimes \mathbf{C}) \otimes F, \quad F \star ({}^!L) = (\mathbf{C} \boxtimes L) \otimes F $$

for any $F \in \mathrm{Coh}_{\mathcal{G}}(Z)$. Here $L \boxtimes \mathbf{C}, \mathbf{C} \boxtimes L$ are regarded as $\mathcal{G}$-equivariant line bundles on Z as in 7.6. In particular, ${}^!\mathbf{C}$ is the unit element of the ring $K_{\mathcal{G}}(Z)$.

7.11. For $F \in \mathrm{Coh}_{\mathcal{G}}(Z), F' \in \mathrm{Coh}_{\mathcal{G}}(\Lambda \times \mathcal{B})$, we define

$$ F \underset{1}{\star} F' = q_{13*}(q_{12}^* F \otimes^L_{\Lambda^2 \times \mathcal{B}} q_{23}^* F') \in K_{\mathcal{G}}(\Lambda \times \mathcal{B}). $$

(The Tor-product is relative to the smooth $\mathcal{G}$-variety $\Lambda^2 \times \mathcal{B}$ and its closed subvarieties $Z \times \mathcal{B}, \Lambda^2 \times \mathcal{B}$ with intersection $Z \times \mathcal{B}$.) Here

$q_{12} : Z \times \mathcal{B} \to Z$ is $(y, \mathfrak{b}, \mathfrak{b}'; \mathfrak{b}'') \mapsto (y, \mathfrak{b}, \mathfrak{b}')$,
$q_{23} : \Lambda^2 \times \mathcal{B} \to \Lambda \times \mathcal{B}$ is $(y, \mathfrak{b}; y', \mathfrak{b}'; \mathfrak{b}'') \mapsto (y', \mathfrak{b}'; \mathfrak{b}'')$,
$q_{13} : Z \times \mathcal{B} \to \Lambda \times \mathcal{B}$ is $(y, \mathfrak{b}; y, \mathfrak{b}'; \mathfrak{b}'') \mapsto (y, \mathfrak{b}; \mathfrak{b}'')$.

This extends to an $R_{\mathcal{G}}$-bilinear pairing

$$ \underset{1}{\star} : K_{\mathcal{G}}(Z) \times K_{\mathcal{G}}(\Lambda \times \mathcal{B}) \to K_{\mathcal{G}}(\Lambda \times \mathcal{B}) $$

which may be regarded as a $K_{\mathcal{G}}(Z)$-module structure on $K_{\mathcal{G}}(\Lambda \times \mathcal{B})$, as a routine argument using $\Lambda^3 \times \mathcal{B}$ shows.

7.12. Let $L \in \mathbf{X}$. From the definitions (and using 6.5) we see that

$$ {}^!L \underset{1}{\star} F' = (L \boxtimes \mathbf{C}) \otimes F' $$

for any $F' \in \mathrm{Coh}_{\mathcal{G}}(\Lambda \times \mathcal{B})$. Here the line bundle $L \boxtimes \mathbf{C}$ is regarded as a $\mathcal{G}$-equivariant line bundle on $\Lambda \times \mathcal{B}$ as in 7.6.

7.13. We regard $\mathcal{B} \times \mathcal{B}$ as a closed subvariety of Z (hence of Λ^2) by $(\mathfrak{b}, \mathfrak{b}') \mapsto (0, \mathfrak{b}, \mathfrak{b}')$. For $F \in \mathrm{Coh}_{\mathcal{G}}(Z), F' \in \mathrm{Coh}_{\mathcal{G}}(\mathcal{B} \times \mathcal{B})$, we define

$$ F \underset{2}{\star} F' = r_{13*}(q_{12}^* F \otimes^L_{\Lambda^3} r_{23}^* F') \in K_{\mathcal{G}}(\Lambda \times \Lambda). $$

(The Tor-product is relative to the smooth $\mathcal{G}$-variety $\Lambda^2 \times \mathcal{B}$ and its closed $\mathcal{G}$-stable subvarieties $Z \times \mathcal{B}, \Lambda \times \mathcal{B} \times \mathcal{B}$ with intersection $\mathcal{B} \times \mathcal{B} \times \mathcal{B}$.) Here q_{12} is as in 7.11,

$$ r_{23} : \Lambda \times \mathcal{B} \times \mathcal{B} \to \mathcal{B} \to \mathcal{B} \text{ is } (y, \mathfrak{b}; \mathfrak{b}', \mathfrak{b}'') \mapsto (\mathfrak{b}', \mathfrak{b}''), $$
$$ r_{13} : \mathcal{B} \times \mathcal{B} \times \mathcal{B} \to \mathcal{B} \times \mathcal{B} \text{ is } (\mathfrak{b}, \mathfrak{b}', \mathfrak{b}'') \mapsto (\mathfrak{b}, \mathfrak{b}''). $$

This extends to an $R_{\mathcal{G}}$-bilinear pairing

$$ \underset{2}{\star} : K_{\mathcal{G}}(Z) \times K_{\mathcal{G}}(\mathcal{B} \times \mathcal{B}) \to K_{\mathcal{G}}(\mathcal{B} \times \mathcal{B}) $$

which may be regarded as a $K_{\mathcal{G}}(Z)$-module structure on $K_{\mathcal{G}}(\mathcal{B} \times \mathcal{B})$, as a routine argument using $\Lambda^3 \times \mathcal{B}$ shows.

From the definitions (and using 6.5) we see that, for $L \in \mathbf{X}$,

(a) ${}^!L \underset{2}{\star} F' = (L \boxtimes \mathbf{C}) \otimes F'$

for any $F' \in \mathrm{Coh}_{\mathcal{G}}(\mathcal{B} \times \mathcal{B})$.

7.14. We regard $\mathcal{B} \times \mathcal{B}$ as a closed subvariety of Z as in 7.13 and Z as a closed subvariety of Λ^2 as in 7.5. Let $j : \mathcal{B} \times \mathcal{B} \to Z$ and $\tilde{j} : Z \to \Lambda \times \mathcal{B}$ be the inclusions. Using the definitions and 6.5 we see that

(a) $j_* : K_{\mathcal{G}}(\mathcal{B} \times \mathcal{B}) \to K_{\mathcal{G}}(Z)$ is $K_{\mathcal{G}}(Z)$-linear.
(b) $\tilde{j}_* : K_{\mathcal{G}}(Z) \to K_{\mathcal{G}}(\Lambda \times \mathcal{B})$ is $K_{\mathcal{G}}(Z)$-linear.

($K_{\mathcal{G}}(Z)$ is regarded as a $K_{\mathcal{G}}(Z)$-module via left multiplication in the ring $K_{\mathcal{G}}(Z)$.)

Lemma 7.15. *The maps $j_* : K_{\mathcal{G}}(\mathcal{B} \times \mathcal{B}) \to K_{\mathcal{G}}(Z)$ and $\tilde{j}_* : K_{\mathcal{G}}(Z) \to K_{\mathcal{G}}(\Lambda \times \mathcal{B})$ in 7.14 are injective.*

We fix a Borel subgroup B_0 of G with Lie algebra $\mathfrak{b}_0$. Let T be a maximal torus in B_0. Let $Z_0 = \{(y, \mathfrak{b}) \in \Lambda | y \in \mathfrak{b}_0\}$. Let $j' : \mathcal{B} \to Z_0$ be defined by $\mathfrak{b} \mapsto (0, \mathfrak{b})$ and let $\tilde{j}' : Z_0 \to \Lambda$ be the obvious inclusion. These are closed imbeddings. Note that Z_0, Λ are $T \times \mathbf{C}^*$-stable subvarieties of $Z, \Lambda \times \mathcal{B}$, via $(y, \mathfrak{b}) \mapsto (y, \mathfrak{b}, \mathfrak{b}_0)$.

By general results in [T1], we have canonically

$$K_{\mathcal{G}}(Z) = K_{T\times\mathbf{C}^*}(Z_0),\ K_{\mathcal{G}}(\Lambda \times \mathcal{B}) = K_{T\times\mathbf{C}^*}(\Lambda)$$

and $j_*, \tilde{j}_*$ correspond to

$$j'_* : K_{T\times\mathbf{C}^*}(\mathcal{B}) \to K_{T\times\mathbf{C}^*}(Z_0), \tilde{j}'_* : K_{T\times\mathbf{C}^*}(Z_0) \to K_{T\times\mathbf{C}^*}(\Lambda).$$

It is then enough to show that $j'_*, \tilde{j}'_*$ are injective. Now the $\mathbf{C}^*$-action on $Z, \Lambda \times \mathcal{B}$ (restriction of the $\mathcal{G}$-action) leaves Z_0, Λ stable and on these subvarieties has the same fixed point set, namely $\{(0, \mathfrak{b}) | \mathfrak{b} \in \mathcal{B}\}$.

From the concentration theorem [T2], it then follows that $j'_*, \tilde{j}'_*$ become isomorphisms after tensoring with the field of fractions of $R_{T\times\mathbf{C}^*}$. To show that $j'_*, \tilde{j}'_*$ are injective, it is therefore enough to show that the $R_{T\times\mathbf{C}^*}$-modules

$$K_{T\times\mathbf{C}^*}(\mathcal{B}), \quad K_{T\times\mathbf{C}^*}(Z_0), \quad K_{T\times\mathbf{C}^*}(\Lambda)$$

are projective of rank $|W|$. This follows by a standard argument using the decomposition of $\mathcal{B}$ into B_0-orbits (Bruhat cells), the decomposition of Z_0 into $T \times \mathbf{C}^*$-stable cells

$$Z_0 = \bigsqcup_{w\in W} Z_{0,w}, \quad Z_{0,w} = \{(y, \mathfrak{b}) \in Z_0 | (\mathfrak{b}, \mathfrak{b}_0) \in \mathcal{O}_w\}$$

and the analogous decomposition of Λ into $T \times \mathbf{C}^*$-stable cells

$$\Lambda = \bigsqcup_{w\in W} \Lambda_w, \quad \Lambda_w = \{(y, \mathfrak{b}) \in \Lambda | (\mathfrak{b}, \mathfrak{b}_0) \in \mathcal{O}_w\}.$$

The lemma is proved.

7.16. Let $i \in I$. The smooth subvariety $\bar{\mathcal{O}}_i = \mathcal{O}_{\sigma_i} \cup \mathcal{O}_1$ of $\mathcal{B} \times \mathcal{B}$ is $\mathcal{G}$-stable. Let $\pi'_i : \bar{\mathcal{O}}_i \to \mathcal{P}_i$ be the morphism defined by $\pi'_i(\mathfrak{b}, \mathfrak{b}') = \pi_i(\mathfrak{b}) = \pi_i(\mathfrak{b}')$.

Let $\bar{Z}_i$ be the set of all triples $(y, \mathfrak{b}, \mathfrak{b}') \in Z$ such that $(\mathfrak{b}, \mathfrak{b}') \in \bar{\mathcal{O}}_i$ and y belongs to $\mathfrak{n}_{\mathfrak{p}}$, where $\mathfrak{p} = \pi_i(\mathfrak{b}) = \pi_i(\mathfrak{b}')$. Thus $\bar{Z}_i$ is naturally a line bundle over $\bar{\mathcal{O}}_i$. In particular, it is a smooth variety of dimension 2ν. Note that $\bar{Z}_i$ is a $\mathcal{G}$-stable closed subvariety of Z.

7.17. Let $L', L'' \in \mathbf{X}$. We regard $L' \boxtimes L''$ as a $\mathcal{G}$-equivariant line bundle on $\bar{Z}_i$ (a closed subvariety of $Z \subset \Lambda \times \mathcal{B}$) as in 7.6. Taking its direct image under the closed imbedding $\bar{Z}_i \subset Z$, we obtain an object $L' \boxtimes_i L'' \in \mathrm{Coh}_{\mathcal{G}}(Z)$.

Lemma 7.18. *Let $L, \tilde{L} \in \mathbf{X}$ and let $L \boxtimes \tilde{L} \in \mathrm{Vec}_{\mathcal{G}}(\Lambda \times \mathcal{B})$ be as in 7.6. In the $K_{\mathcal{G}}(Z)$-module $K_{\mathcal{G}}(\Lambda \times \mathcal{B})$ we have*

(a) $(L' \boxtimes_i L'') \underset{1}{\star} (L \boxtimes \tilde{L}) = \frac{(1-vL_i)L'({}^{\sigma_i}(L''L)L_i^{-1}-L''L)}{L_i^{-1}-1} \boxtimes \tilde{L}$.

The fraction above (in the quotient field of the group algebra $\mathcal{A}[\mathbf{X}]$) is actually in $\mathcal{A}[\mathbf{X}]$ and the right hand side of (a) is interpreted as the corresponding $\mathbf{Z}$-linear combination of elements $v^k L_1 \boxtimes \tilde{L}$ taken in $K_{\mathcal{G}}(\Lambda \times \mathcal{B})$.

A similar interpretation holds for the following identity in $K_{\mathcal{G}}(\mathcal{B}) = K_{\mathcal{G}}(\Lambda)$, in which $L \in \mathbf{X}$ and $m = \check{\alpha}_i(L) \in \mathbf{Z}$:

(b) $\pi_i^* \pi_{i*}(L) = L\frac{L_i^{-m-1}-1}{L_i^{-1}-1} = L\frac{{}^{\sigma_i}LL^{-1}L_i^{-1}-1}{L_i^{-1}-1} = \frac{{}^{\sigma_i}LL_i^{-1}-L}{L_i^{-1}-1}$.

(This follows directly from the definitions.) By definition, we have

$$
\begin{aligned}
(L' \boxtimes_i L'') \underset{1}{\star} (L \boxtimes \tilde{L}) &= q_{13*}((L' \boxtimes L'' \boxtimes \mathbf{C}) \otimes^{L}_{\Lambda^2 \times \mathbf{C}} (\mathbf{C} \boxtimes L \boxtimes \tilde{L})) \\
\text{(c)} \qquad &= q_{13*}((L' \boxtimes L'' \boxtimes \mathbf{C}) \otimes (\mathbf{C} \boxtimes L \boxtimes \tilde{L})) = q'_{13*}(L' \boxtimes L''L \boxtimes \tilde{L}).
\end{aligned}
$$

Here $q_{13} : Z \times \mathcal{B} \to \Lambda \times \mathcal{B}$ is as in 7.11 and $q'_{13} : \bar{Z}_i \times \mathcal{B} \to \Lambda \times \mathcal{B}$ is the restriction of q_{13} under $\bar{Z}_i \times \mathcal{B} \subset Z \times \mathcal{B}$. The first $\mathbf{C} \boxtimes L \boxtimes \tilde{L}$ is in $\mathrm{Vec}_{\mathcal{G}}(\Lambda^2 \times \mathcal{B})$, the second $\mathbf{C} \boxtimes L \boxtimes \tilde{L}$ is in $\mathrm{Vec}_{\mathcal{G}}(Z \times \mathcal{B})$; the second equality in (c) follows from 6.7.

Let Λ_i be the closed subvariety of Λ consisting of all $(y, \mathfrak{b})$ such that $y \in \mathfrak{n}_{\mathfrak{p}}$, where $\mathfrak{p} = \pi_i(\mathfrak{b})$. We can factor $q'_{13} = jp$ where $p : \bar{Z}_i \times \mathcal{B} \to \Lambda_i \times \mathcal{B}$ is given by

$$p(y, \mathfrak{b}, \mathfrak{b}'; \mathfrak{b}'') = (y, \mathfrak{b}; \mathfrak{b}'')$$

and $j : \Lambda_i \times \mathcal{B} \to \Lambda \times \mathcal{B}$ is the inclusion. Then

$$(L' \boxtimes_i L'') \underset{1}{\star} (L \boxtimes \tilde{L}) = j_* p_* (L' \boxtimes L''L \boxtimes \tilde{L}) = j_*(L' \otimes p' * (\mathbf{C} \boxtimes L''L) \boxtimes \tilde{L})$$

where $p' : \bar{Z}_i \to \Lambda_i$ is $(y, \mathfrak{b}, \mathfrak{b}') \mapsto (y, \mathfrak{b})$. We have a cartesian diagram

$$
\begin{tikzcd}
\bar{Z}_i \arrow[r, "p'"] \arrow[d] & \Lambda_i \arrow[d] \\
\mathcal{B} \arrow[r, "\pi_i"] & \mathcal{P}_i
\end{tikzcd}
$$

in which the left vertical arrow is $(y, \mathfrak{b}, \mathfrak{b}') \mapsto \mathfrak{b}$ and the right vertical arrow is $(y, \mathfrak{b}) \mapsto \pi_i(\mathfrak{b})$. The horizontal arrows are proper and the vertical arrows are smooth. Hence, $p'_*(\mathbf{C} \boxtimes L''L) = \pi_i^* \pi_{i*}(L''L)$ and, using (b) for $L''L$ instead of L we have

$$p'_*(\mathbf{C} \boxtimes L''L) = \frac{{}^{\sigma_i}(L''L)L_i^{-1} - L''L}{L_i^{-1} - 1}.$$

Hence,

$$
\begin{aligned}
(L' \boxtimes_i L'') \underset{1}{\star} (L \boxtimes \tilde{L}) =& j_*(\frac{L'({}^{\sigma_i}(L''L)L_i^{-1} - L''L)}{L_i^{-1} - 1} \boxtimes \tilde{L}) \\
&= j'_*(\frac{L'({}^{\sigma_i}(L''L)L_i^{-1} - L''L)}{L_i^{-1} - 1}) \boxtimes \tilde{L}
\end{aligned}
$$

where $j' : \Lambda_i \to \Lambda$ is the inclusion. Now j' is the imbedding of the zero section of a line bundle coming from the line bundle on $\mathcal{B}$ whose fibre at $\mathfrak{b}$ is $\mathfrak{n}_\mathfrak{b}/\mathfrak{n}_\mathfrak{p}$ where $\mathfrak{p} = \pi_i(\mathfrak{b})$. Hence, j'_* is multiplication by $(1 - v^2 L_i)$. The lemma follows.

7.19. **The line bundle $\mathcal{L}_i$.** We fix $i \in I$. We choose $L', L'' \in \mathbf{X}$ such that $\check\alpha_i(L') = \check\alpha_i(L'') = -1$ and $L'L'' = L_i^{-1} \in \mathbf{X}$. We show that the restriction of $L' \boxtimes L''$ from $\mathcal{B} \times \mathcal{B}$ to $\bar{\mathcal{O}}_i$ is independent of the choice of L', L''.

Another choice for L', L'' is $L'\mathcal{L}, L''\mathcal{L}^{-1}$ where $\mathcal{L} \in \mathbf{X}$ is such that $\check\alpha_i(\mathcal{L}) = 0$. We must show that $L'\mathcal{L} \boxtimes L''\mathcal{L}^{-1} = L' \boxtimes L''$ have the same restriction to $\bar{\mathcal{O}}_i$, or equivalently, that

(a) $(\mathcal{L} \boxtimes \mathcal{L}^{-1})_{\bar{\mathcal{O}}_i} = \mathbf{C}$.

By 7.2(a), there exists a G-equivariant line bundle $\tilde{\mathcal{L}}$ on $\mathcal{P}_i$ such that $\mathcal{L} = \pi_i^* \tilde{\mathcal{L}}$. The restriction of $\mathcal{L} \boxtimes \mathcal{L}^{-1} = \pi_i^* \tilde{\mathcal{L}} \boxtimes \pi_i^* \tilde{\mathcal{L}}^{-1}$ to $\bar{\mathcal{O}}_i$ is then

$$\pi_i'^* \tilde{\mathcal{L}} \otimes \pi_i'^* \tilde{\mathcal{L}}^{-1} = \pi_i'^* (\tilde{\mathcal{L}} \otimes \tilde{\mathcal{L}}^{-1}) = \pi_i'^* \mathbf{C} = \mathbf{C}$$

where $\pi_i' : \bar{\mathcal{O}}_i \to \mathcal{P}_i$ as in 7.16; our assertion is verified.

The line bundle $(L' \boxtimes L'')_{\bar{\mathcal{O}}_i}$ above will be denoted by $\mathcal{L}_i$. It is a canonically defined $\mathcal{G}$-equivariant line bundle (with trivial $\mathbf{C}^*$-action). The inverse image of $\mathcal{L}_i$ under the canonical map $\bar{Z}_i \to \bar{\mathcal{O}}_i$ is again denoted by $\mathcal{L}_i$. It is the same as the restriction of $L' \boxtimes L'' \in \mathrm{Vec}_{\mathcal{G}}(\Lambda^2)$ to $\mathbf{Z}_i$.

7.20. Taking direct image of the line bundle $v^{-1}\mathcal{L}_i \in \mathrm{Vec}_{\mathcal{G}}(\bar{Z}_i)$ under the closed imbedding $\bar{Z}_i \subset Z$, we obtain an object $\mathbf{a}_i \in \mathrm{Coh}_{\mathcal{G}}(Z)$.

We now reformulate Lemma 7.18 in the case where L', L'' are as in 7.19.

Lemma 7.21. *Let $L, \tilde{L} \in \mathbf{X}$ and let $L \boxtimes \tilde{L} \in \mathrm{Vec}_{\mathcal{G}}(\Lambda \times \mathcal{B})$ be as in 7.6. In the $K_{\mathcal{G}}(Z)$-module $K_{\mathcal{G}}(\Lambda \times \mathcal{B})$ (see 7.11) we have*

$$\mathbf{a}_i \underset{1}{\star} (L \boxtimes \tilde{L}) = \frac{v^{-1}(1 - v^2 L_i)(L - {}^{\sigma_i}L)}{L_i - 1} \boxtimes \tilde{L}.$$

Using ${}^{\sigma_i}L'' = L''L_i = L'^{-1}$, we have

$$\frac{L'({}^{\sigma_i}(L''L)L_i^{-1} - L''L)}{L_i^{-1} - 1} = \frac{L'L'^{-1}\,{}^{\sigma_i}LL_i^{-1} - L'L''L}{L_i^{-1} - 1} = \frac{{}^{\sigma_i}LL_i^{-1} - LL_i^{-1}}{L_i^{-1} - 1} = \frac{L - {}^{\sigma_i}L}{L_i - 1}.$$

Hence, the lemma follows from 7.18.

Lemma 7.22. *Let $L, \tilde{L} \in \mathbf{X}$ and let $L \boxtimes \tilde{L} \in \mathrm{Vec}_{\mathcal{G}}(\mathcal{B} \times \mathcal{B})$ be as in 7.6. In the $K_{\mathcal{G}}(Z)$-module $K_{\mathcal{G}}(\mathcal{B} \times \mathcal{B})$ (see 7.13) we have*

(a) $\mathbf{a}_i \underset{2}{\star} (L \boxtimes \tilde{L}) = \frac{v^{-1}(L - v^2 LL_i - {}^{\sigma_i}L + v^2({}^{\sigma_i}LL_i^{-1}))}{L_i - 1} \boxtimes \tilde{L}$.

Here the right hand side of (a) is interpreted in a way similar to 7.18(a).

Let $k : \mathcal{B} \times \mathcal{B} \to \Lambda \times \mathcal{B}$ be the closed imbedding given by $(\mathfrak{b}, \mathfrak{b}') \mapsto (0, \mathfrak{b}; \mathfrak{b}')$. Now $k_* : K_{\mathcal{G}}(\mathcal{B} \times \mathcal{B}) \to K_{\mathcal{G}}(\Lambda \times \mathcal{B})$ is compatible with the $K_{\mathcal{G}}(Z)$-module structures in 7.11, 7.13. (We have $k_* = \tilde{j}_* j_*$, where $\tilde{j}_*, j_*$ are $K_{\mathcal{G}}(Z)$-module homomorphisms; see 7.14.) Moreover, k_* is injective (see 7.15). Under the identification $K_{\mathcal{G}}(\Lambda \times \mathcal{B}) = K_{\mathcal{G}}(\mathcal{B} \times \mathcal{B})$ (see 7.6), the map k_* becomes the map $K_{\mathcal{G}}(\mathcal{B} \times \mathcal{B}) \to K_{\mathcal{G}}(\mathcal{B} \times \mathcal{B})$ given by multiplication by $\bar{\Delta} \boxtimes \mathbf{C}$, where $\bar{\Delta}$ is as in 1.23. Note that

$${}^{\sigma_i}\bar{\Delta} = \bar{\Delta} \frac{1 - v^2 L_i^{-1}}{1 - v^2 L_i}.$$

where the W-action on $\mathcal{A}[\mathbf{X}]$ is given by extending linearly the W-action on $\mathbf{X}$.

Let γ be the left hand side of (a). Since k_* is $K_{\mathcal{G}}$-linear, we have

(b) $(\bar{\Delta} \boxtimes \mathbf{C})\gamma = \mathbf{a}_i \underset{1}{\star} ((\bar{\Delta} \boxtimes \mathbf{C}) \otimes (L \boxtimes \tilde{L})) = \mathbf{a}_i \underset{1}{\star} (\bar{\Delta} L \boxtimes \tilde{L}))$.

Now using 7.21 to compute the right hand side of (b), we see that

$$\begin{aligned}(\bar{\Delta} \boxtimes \mathbf{C})\gamma &= \frac{v^{-1}(1 - v^2 L_i)(\bar{\Delta} L - {}^{\sigma_i}\bar{\Delta}^{\sigma_i} L)}{L_i - 1} \boxtimes \tilde{L} \\ &= \frac{v^{-1}\bar{\Delta}(1 - v^2 L_i)(L - {}^{\sigma_i}L\frac{1-v^2L_i^{-1}}{1-v^2L_i})}{L_i - 1} \boxtimes \tilde{L} \\ &= (\bar{\Delta} \boxtimes \mathbf{C})\frac{v^{-1}(L - v^2 L L_i - {}^{\sigma_i}L + v^2({}^{\sigma_i}L L_i^{-1}))}{L_i - 1} \boxtimes \tilde{L}.\end{aligned}$$

Since multiplication by $\bar{\Delta} \boxtimes \mathbf{C}$ on $K_{\mathcal{G}}(\mathcal{B} \times \mathcal{B})$ is injective, (see 7.8(c)) it follows that

$$\gamma = \frac{v^{-1}(L - v^2 L L_i - {}^{\sigma_i}L + v^2({}^{\sigma_i}L L_i^{-1}))}{L_i - 1} \boxtimes \tilde{L}.$$

(This could have been also deduced from the arguments in 4.10.) The lemma is proved.

Lemma 7.23. *There is a unique $\mathcal{H}$-module structure on the $\mathcal{A}$-module $K_{\mathcal{G}}(\mathcal{B} \times \mathcal{B})$ given by*

(i) $(-\tilde{T}_{\sigma_i} - v^{-1}) \cdot (L \boxtimes \tilde{L}) = \mathbf{a}_i \underset{2}{\star} (L \boxtimes \tilde{L})$ *for $i \in I$ and $L, \tilde{L} \in \mathbf{X}$;*

(ii) $\theta_x \cdot (L \boxtimes \tilde{L}) = {}^!L_x \underset{2}{\star} (L \boxtimes \tilde{L})$ *for $x \in \mathcal{X}$ and $L, \tilde{L} \in \mathbf{X}$.*

Using 7.13, 7.22, we see that the following statement is equivalent to the lemma.

There is a unique $\mathcal{H}$-module structure on the $\mathcal{A}$-module $K_{\mathcal{G}}(\mathcal{B} \times \mathcal{B})$ given by

$\tilde{T}_i \cdot (L \boxtimes \tilde{L}) = \frac{v^{-1}({}^{\sigma_i}L - L L_i) + v(L L_i - {}^{\sigma_i}L L_i^{-1})}{L_i - 1} \boxtimes \tilde{L}$ *for $i \in I$ and $L, \tilde{L} \in \mathbf{X}$;*

$\theta_x \cdot (L \boxtimes \tilde{L}) = (L_x L \boxtimes \tilde{L})$ *for $x \in \mathcal{X}$ and $L, \tilde{L} \in \mathbf{X}$.*

This follows immediately from Lemma 4.6, using the identification $K_{\mathcal{G}}(\mathcal{B} \times \mathcal{B}) = \mathcal{A}\mathcal{X}^{\otimes 2}$ in 7.8(c).

Lemma 7.24. *There is a unique $\mathcal{H}$-module structure on the $\mathcal{A}$-module $K_{\mathcal{G}}(\Lambda \times \mathcal{B})$ given by*

(i) $(-\tilde{T}_{\sigma_i} - v^{-1}) \cdot (L \boxtimes \tilde{L}) = \mathbf{a}_i \underset{1}{\star} (L \boxtimes \tilde{L})$ *for $i \in I$ and $L, \tilde{L} \in \mathbf{X}$;*

(ii) $\theta_x \cdot (L \boxtimes \tilde{L}) = {}^!L_x \underset{1}{\star} (L \boxtimes \tilde{L})$ *for $x \in \mathcal{X}$ and $L, \tilde{L} \in \mathbf{X}$.*

Using 7.11, 7.12, 7.21, we see that the following statement is equivalent to the lemma.

There is a unique $\mathcal{H}$-module structure on the $\mathcal{A}$-module $K_{\mathcal{G}}(\Lambda \times \mathcal{B})$ given by

$\tilde{T}_i \cdot (L \boxtimes \tilde{L}) = \frac{v^{-1}({}^{\sigma_i}L - L L_i) + v(L L_i - {}^{\sigma_i}L L_i)}{L_i - 1} \boxtimes \tilde{L}$ *for $i \in I$ and $L, \tilde{L} \in \mathbf{X}$;*

$\theta_x \cdot (L \boxtimes \tilde{L}) = (L_x L \boxtimes \tilde{L})$ *for $x \in \mathcal{X}$ and $L, \tilde{L} \in \mathbf{X}$.*

This follows immediately from Lemma 4.7, using the identification $K_{\mathcal{G}}(\Lambda \times \mathcal{B}) = K_{\mathcal{G}}(\mathcal{B} \times \mathcal{B}) = \mathcal{A}\mathcal{X}^{\otimes 2}$ in 7.6, 7.8(c).

Proposition 7.25. *The assignment $-\tilde{T}_i - v^{-1} \mapsto \mathbf{a}_i \in K_{\mathcal{G}}(Z), \theta_x \mapsto {}^!L_x \in K_{\mathcal{G}}(Z)$ is a homomorphism of $\mathcal{A}$-algebras $\mathcal{H} \to K_{\mathcal{G}}(Z)$.*

This follows from Lemma 7.24, since the map $\tilde{j}_*$ in 7.14 is compatible with the $K_{\mathcal{G}}(Z)$-module structures and is injective (see 7.15).

8. The isomorphism $\mathcal{H} \xrightarrow{\sim} K_{\mathcal{G}}(Z)$

8.1. We fix $i \in I$. Let $\mathfrak{Z}$ be a locally closed $\mathcal{G}$-stable subset of Z. We say that $\mathfrak{Z}$ is *left i-saturated* if the following holds:

$$(y, \mathfrak{b}', \mathfrak{b}'') \in \mathfrak{Z}, (y, \mathfrak{b}) \in \Lambda, (\mathfrak{b}, \mathfrak{b}') \in \mathcal{O}_{\sigma_i} \implies (y, \mathfrak{b}, \mathfrak{b}'') \in \mathfrak{Z}.$$

Assume that $\mathfrak{Z}$ is a $\mathcal{G}$-stable, left i-saturated closed subset of Z. We define a homomorphism $r_i : K_{\mathcal{G}}(\mathfrak{Z}) \to K_{\mathcal{G}}(\mathfrak{Z})$ by $F \mapsto \tilde{p}_{13*}(\tilde{p}_{12}^*(v^{-1}\mathcal{L}_i) \otimes^L_{\Lambda^3} \tilde{p}_{23}^* F)$. The Tor-product is relative to the smooth variety Λ^3 and its closed subvarieties $\bar{Z}_i \times \Lambda, \Lambda \times \mathfrak{Z}$ with intersection

$$\mathfrak{V} = \{(y, \mathfrak{b}, \mathfrak{b}', \mathfrak{b}'') | (y, \mathfrak{b}, \mathfrak{b}') \in \bar{Z}_i, (y, \mathfrak{b}', \mathfrak{b}'') \in \mathfrak{Z}\}.$$

Here $\tilde{p}_{12} : \bar{Z}_i \times \Lambda \to \bar{Z}_i, \tilde{p}_{23} : \Lambda \times \mathfrak{Z} \to \mathfrak{Z}$ are the obvious projections and $\tilde{p}_{13} : \mathfrak{V} \to \mathfrak{Z}$ is given by $\tilde{p}_{13}(y, \mathfrak{b}, \mathfrak{b}', \mathfrak{b}'') = (y, \mathfrak{b}, \mathfrak{b}'')$ (this is well defined since $\mathfrak{Z}$ is left i-saturated). The imbedding $\bar{Z}_i \times \Lambda \subset \Lambda^3$ is the composition $\bar{Z}_i \times \Lambda \subset Z \times \Lambda \subset \Lambda^3$. The imbedding $\Lambda \times \mathfrak{Z} \subset Z$ is the composition $\Lambda \times \mathfrak{Z} \subset \Lambda \times Z \subset \Lambda^3$.

The following two properties of the map r_i follow using the definition and 6.5.

(a) If $\mathfrak{Z}', \mathfrak{Z}$ are two $\mathcal{G}$-stable, left i-saturated closed subsets of Z with $\mathfrak{Z}' \subset \mathfrak{Z}$, then the maps $r_i : K_{\mathcal{G}}(\mathfrak{Z}') \to K_{\mathcal{G}}(\mathfrak{Z}')$ and $r_i : K_{\mathcal{G}}(\mathfrak{Z}) \to K_{\mathcal{G}}(\mathfrak{Z})$ are compatible with the direct image map $K_{\mathcal{G}}(\mathfrak{Z}') \to K_{\mathcal{G}}(\mathfrak{Z})$ induced by the inclusion $\mathfrak{Z}' \subset \mathfrak{Z}$.

(b) For $\mathfrak{Z} = Z$, $r_i : K_{\mathcal{G}}(Z) \to K_{\mathcal{G}}(Z)$ is just left multiplication by $\mathbf{a}_i$ (see 7.20).

8.2. For $w \in W$ we set $\mathcal{O}_{\geq w} = \bigcup_{w'; w' \geq w} \mathcal{O}_w$. We set

$$Z_w = \{(y, \mathfrak{b}, \mathfrak{b}') \in Z | (\mathfrak{b}, \mathfrak{b}') \in \mathcal{O}_w\},$$
$$Z_{\leq w} = \bigcup_{w'; w' \leq w} Z_{w'}, \quad Z_{< w} = \bigcup_{w'; w' < w} Z_{w'}.$$

8.3. If $i \in I$ is such that $s_i w < w$, then $Z_{\leq w}$ is left i-saturated; hence $r_i : K_{\mathcal{G}}(Z_{\leq w}) \to K_{\mathcal{G}}(Z_{\leq w})$ is well defined.

Lemma 8.4. *Let $i \in I$ be such that $s_i w < w$. Let $j : Z_{\leq \sigma_i w} \to Z_{\leq w}$ and $h : Z_{\leq w} \to Z_w$ be the inclusions. Let $f : K_{\mathcal{G}}(Z_{\leq \sigma_i w}) \to K_{\mathcal{G}}(Z_w)$ be the composition*

$$K_{\mathcal{G}}(Z_{\leq \sigma_i w}) \xrightarrow{j_*} K_{\mathcal{G}}(Z_{\leq w}) \xrightarrow{r_i} K_{\mathcal{G}}(Z_{\leq w}) \xrightarrow{h^*} K_{\mathcal{G}}(Z_w).$$

Then f is surjective.

Let $L', L'' \in \mathbf{X}$ be as in 7.19. Since Z_w is a vector bundle over $\mathcal{B}$, the $\mathcal{A}$-module $K_{\mathcal{G}}(Z_w)$ is generated by $\{v^{-1} L' \boxtimes L | L \in \mathbf{X}\}$. Hence, it is enough to show that, if $L \in X$, then $v^{-1} L' \boxtimes L \in \mathrm{Vec}_{\mathcal{G}}(Z_w)$ is in the image of f.

Let $F = L''^{-1} \boxtimes L \in \mathrm{Vec}_{\mathcal{G}}(Z_{\leq s_i w})$.

We shall use the notation of 8.1 with $\mathfrak{Z} = Z_{\leq \sigma_i w}$ except for $\tilde{p}_{13}$ which is not defined since this $\mathfrak{Z}$ is not left i-saturated. Instead, the formula which defined $\tilde{p}_{13}$ in 8.1, gives a map $q : \mathfrak{V} \to Z_{\leq w}$. In this context, we can form

$$v^{-1}(L' \boxtimes L'' \boxtimes \mathbf{C}) \otimes^L_{\Lambda^3} (\mathbf{C} \boxtimes L''^{-1} \boxtimes L) \in K_{\mathcal{G}}(\mathfrak{V}),$$
$$q_*(v^{-1}(L' \boxtimes L'' \boxtimes \mathbf{C}) \otimes^L_{\Lambda^3} (\mathbf{C} \boxtimes L''^{-1} \boxtimes L)) \in K_{\mathcal{G}}(Z_{\leq w}).$$

From the definitions we have

$$q_*(v^{-1}(L' \boxtimes L'' \boxtimes \mathbf{C}) \otimes^L_{\Lambda^3} (\mathbf{C} \boxtimes L''^{-1} \boxtimes L)) = r_i(\tilde{F})$$

where r_i is defined as in 8.1 in terms of $Y = Z_{\leq w}$ (which is left i-saturated) and $\tilde{F} \in \mathrm{Coh}_{\mathcal{G}} Z_{\leq w}$ is the direct image of F under the closed imbedding $Z_{\leq s_i w} \subset Z_{\leq w}$.

Let $k : Z_w \to Z_{\le w}$ be the (open) inclusion. Then

$$\begin{aligned} k^* r_i(\tilde{F}) &= k^* q_*(v^{-1}(L' \boxtimes L'' \boxtimes \mathbf{C}) \otimes^L_{\Lambda^3} (\mathbf{C} \boxtimes L''^{-1} \boxtimes L)) \\ &= q'_* k_1^*(v^{-1}(L' \boxtimes L'' \boxtimes \mathbf{C}) \otimes^L_{\Lambda^3} (\mathbf{C} \boxtimes L''^{-1} \boxtimes L)), \end{aligned}$$

where $q' : V' \to Z_w$ (a proper map) and $k_1 : V' \to \mathfrak{V}$ (an open imbedding) are defined by

$$\begin{aligned} V' &= \{(x, \mathfrak{b}, \mathfrak{b}', \mathfrak{b}'') | (x, \mathfrak{b}, \mathfrak{b}') \in \bar{Z}_i, (x, \mathfrak{b}', \mathfrak{b}'') \in Z_{\le \sigma_i w}, (\mathfrak{b}, \mathfrak{b}'') \in \mathcal{O}_w\} \\ &= \{(x, \mathfrak{b}, \mathfrak{b}', \mathfrak{b}'') | (x, \mathfrak{b}, \mathfrak{b}') \in Z_{\sigma_i}, (x, \mathfrak{b}', \mathfrak{b}'') \in Z_{\sigma_i w}\}, \\ q'(x, \mathfrak{b}, \mathfrak{b}', \mathfrak{b}'') &= (x, \mathfrak{b}, \mathfrak{b}''), \quad k_1(x, \mathfrak{b}, \mathfrak{b}', \mathfrak{b}'') = (x, \mathfrak{b}, \mathfrak{b}', \mathfrak{b}''). \end{aligned}$$

Note that q' is in fact an isomorphism. Let

$$U = \{(x, \mathfrak{b}; x', \mathfrak{b}'; x'', \mathfrak{b}'') \in \Lambda^3 | (\mathfrak{b}, \mathfrak{b}') \in \mathcal{O}_{\ge s_i}, (\mathfrak{b}', \mathfrak{b}'') \in \mathcal{O}_{\ge \sigma_i w}\}.$$

This is an open subset of Λ^3. We have

$$\begin{aligned} U \cap (\bar{Z}_i \times \Lambda) &= U \cap (Z_{\sigma_i} \times \Lambda), \\ U \cap (\Lambda \times Z_{\le \sigma_i w}) &= U \cap (\Lambda \times Z_{\sigma_i w}), \\ U \cap \mathfrak{V} &= V'. \end{aligned}$$

By 6.6, we have

(a) $k_1^*(v^{-1}(L' \boxtimes L'' \boxtimes \mathbf{C}) \otimes^L_{\Lambda^3} (\mathbf{C} \boxtimes L''^{-1} \boxtimes L)) = v^{-1}(L' \boxtimes L'' \boxtimes \mathbf{C}) \otimes^L_U (\mathbf{C} \boxtimes L''^{-1} \boxtimes L)$

where $\otimes^L_U$ is computed in the smooth variety U, relative to its closed subvarieties $U \cap (Z_{\sigma_i} \times \Lambda), U \cap (\Lambda \times Z_{\sigma_i w})$ and we regard

$$L' \boxtimes L'' \boxtimes \mathbf{C} \in \mathrm{Vec}_{\mathcal{G}}(U \cap (Z_{\sigma_i} \times \Lambda)), \quad \mathbf{C} \boxtimes L''^{-1} \boxtimes L \in \mathrm{Vec}_{\mathcal{G}}(U \cap (\Lambda \times Z_{\sigma_i w})).$$

Now the subvarieties $U \cap (Z_{\sigma_i} \times \Lambda)$, $U \cap (\Lambda \times Z_{\sigma_i w})$ of the smooth variety U of dimension 6ν are smooth of dimension 4ν and intersect transversally in the smooth variety Z_w of dimension 2ν. It follows that the right hand side of (a) is just the vector bundle obtained by restricting the two vector bundles to the intersection and then taking the usual tensor product. We obtain the vector bundle $v^{-1} L' \boxtimes L \in \mathrm{Vec}_{\mathcal{G}}(Z_w)$. Thus, the last vector bundle is in the image of f. The lemma is proved.

Lemma 8.5. *Assume that $\sigma_i w < w$. Let j, h be as in 8.4, let $j' : Z_{<w} \to Z_{\le w}$ be the inclusion and let $r_i : K_{\mathcal{G}}(Z_{\le w}) \to K_{\mathcal{G}}(Z_{\le w})$ be as in 8.1. For any $\xi \in K_{\mathcal{G}}(Z_{\le w})$ there exists $\xi' \in K_{\mathcal{G}}(Z_{\le \sigma_i w})$ and $\xi'' \in K_{\mathcal{G}}(Z_{<w})$ such that $\xi = r_i(j_*(\xi')) + j'_*(\xi'')$.*

By Lemma 8.4, there exists $\xi' \in K_{\mathcal{G}}(Z_{\le \sigma_i w})$ such that $\xi - r_i(j_*(\xi'))$ is in the kernel of $K_{\mathcal{G}}(Z_{\le w}) \xrightarrow{h^*} K_{\mathcal{G}}(Z_w)$; hence in the image of $j'_* : K_{\mathcal{G}}(Z_{<w}) \to K_{\mathcal{G}}(Z_{\le w})$. (Note that $Z_{<w}$ is closed in $Z_{\le w}$ with complement Z_w.) The lemma is proved.

Theorem 8.6. *The homomorphism $\mathcal{H} \to K_{\mathcal{G}}(Z)$ in 7.25 is an isomorphism of $\mathcal{A}$-algebras.*

Using 8.5, it follows easily (as in [KL2, 3.21]) that this map is surjective.

We shall regard $\mathcal{H}$ as an $\mathcal{AX}$-module by $[x] : \chi \mapsto \chi\theta_x$; we shall regard $K_{\mathcal{G}}(Z)$ as an $\mathcal{AX}$-module by $[x] : \xi \mapsto \xi \star {}^! L_x = \xi \otimes (\mathbf{C} \boxtimes L_x)$.

Our map $\mathcal{H} \to K_{\mathcal{G}}(Z)$ is compatible with these $\mathcal{AX}$-module structures. Hence, to show that it is an isomorphism, it is enough to show that $\mathcal{H}$ and $K_{\mathcal{G}}(Z)$ are projective $\mathcal{AX}$-modules of the same (finite) rank. Now $\mathcal{H}$ is a free $\mathcal{AX}$-module of rank $|W|$ by 1.21. The fact that $K_{\mathcal{G}}(Z)$ is a projective $\mathcal{A}\mathcal{X}$-module of rank $|W|$ is shown using the partition $Z = \bigcup_{w \in W} Z_w$. The theorem is proved.

Let us identify the algebras $\mathcal{H}, K_{\mathcal{G}}(Z)$ via the isomorphism in the theorem. Then the element $\tilde{T}_w \in \mathcal{H}$ can be regarded as an element of $K_{\mathcal{G}}(Z)$.

8.7. We assume that we are in the setup of Lemma 8.4. Let

$$\tilde{Z} = \bigcup_{w' \in W; w' \leq w; w' \neq w; w' \neq \sigma_i w} Z_w.$$

Note that $\tilde{Z}$ is a left i-saturated, $\mathcal{G}$-stable closed subset of Z.

In the following arguments we write $()_*$ for the direct image map induced by an obvious closed imbedding and $()^*$ for the inverse image map induced by an obvious open imbedding. We show that

(a) the composition $K_{\mathcal{G}}(Z_{<\sigma_i w}) \xrightarrow{()_*} K_{\mathcal{G}}(Z_{\leq \sigma_i w}) \xrightarrow{f} K_{\mathcal{G}}(Z_w)$ is 0.

This equals the composition

$$K_{\mathcal{G}}(Z_{<\sigma_i w}) \xrightarrow{()_*} K_{\mathcal{G}}(\tilde{Z}) \xrightarrow{()_*} K_{\mathcal{G}}(Z_{\leq w}) \xrightarrow{r_i} K_{\mathcal{G}}(Z_{\leq w}) \xrightarrow{h^*} K_{\mathcal{G}}(Z_w)$$

and also the composition

$$K_{\mathcal{G}}(Z_{<\sigma_i w}) \xrightarrow{()_*} K_{\mathcal{G}}(\tilde{Z}) \xrightarrow{r_i} K_{\mathcal{G}}(\tilde{Z}) \xrightarrow{()_*} K_{\mathcal{G}}(Z_{\leq w}) \xrightarrow{()^*} K_{\mathcal{G}}(Z_w).$$

(See 8.1(a).) But the composition of the last two maps is zero since $\tilde{Z} \cap Z_w = \emptyset$.

Thus, (a) is verified. Using (a) and the exact sequence

(b) $0 \to K_{\mathcal{G}}(Z_{<\sigma_i w}) \xrightarrow{()_*} K_{\mathcal{G}}(Z_{\leq \sigma_i w}) \xrightarrow{()^*} K(Z_{\sigma_i w}) \to 0$

we see that there is a unique map $\tilde{f} : K_{\mathcal{G}}(Z_{\sigma_i w}) \to K_{\mathcal{G}}(Z_w)$ such that the composition

(c) $K_{\mathcal{G}}(Z_{\leq \sigma_i w}) \xrightarrow{()^*} K_{\mathcal{G}}(Z_{\sigma_i w}) \xrightarrow{\tilde{f}} K_{\mathcal{G}}(Z_w)$

is equal to f.

Lemma 8.8. *Let $L \in \mathbf{X}$. The map $\tilde{f}$ takes $\mathbf{C} \boxtimes L \in \mathrm{Vec}_{\mathcal{G}}(Z_{\sigma_i w})$ to $v^{-1}\mathbf{C} \boxtimes L \in \mathrm{Vec}_{\mathcal{G}}(Z_w)$. In particular, it is an isomorphism.*

The proof of Lemma 8.4 shows that $\tilde{f}$ takes $L''^{-1} \boxtimes L$ to $v^{-1}L' \boxtimes L$. By 7.4, we have $L''^{-1} \boxtimes L = \mathbf{C} \boxtimes ({}^{w^{-1}\sigma_i}(L''^{-1}) \otimes L)$ as objects of $\mathrm{Vec}_{\mathcal{G}}(Z_{\sigma_i w})$ and $L' \boxtimes L = \mathbf{C} \boxtimes ({}^{w^{-1}}L' \otimes L)$ as objects of $\mathrm{Vec}_{\mathcal{G}}(Z_w)$.

It is then enough to show that ${}^{\sigma_i}(L''^{-1}) = L'$ or that ${}^{\sigma_i}L'' \otimes L' = \mathbf{C}$. But ${}^{\sigma_i}L'' \otimes L' = L'' L_i L' = \mathbf{C}$. The lemma is proved.

Lemma 8.9. *Let $w \in W$. Let $j_w : Z_{\leq w} \to Z$ be the inclusion.*

(a) *$j_{w*} : K_{\mathcal{G}}(Z_{\leq w}) \to K_{\mathcal{G}}(Z)$ is injective.*
(b) *$\tilde{T}_w$ is in the image of j_{w*}.*
(c) *Let $\xi_w \in K_{\mathcal{G}}(Z_{\leq w})$ be the unique element such that $j_{w*}(\xi_w) = \tilde{T}_w$. The restriction of ξ_w to the open subset Z_w of $Z_{\leq w}$ is $(-v)^{-l(w)}\mathbf{C}$.*

(a) is proved by standard arguments (compare [KL2, 3.17]).

We prove (b) and (c). For $w = 1$ this is clear. Assume now that $l(w) \geq 1$ and that the result is known for elements of strictly smaller length. Let $i \in I$ be such that $\sigma_i w < w$. Let $w' = \sigma_i w$. By the induction hypothesis, $\tilde{T}_{w'} = j_{w'*}(\xi_{w'})$ for a unique $\xi_{w'} \in K_{\mathcal{G}}(Z_{\leq w'})$. Let j, h be as in 8.4. Then

$$\begin{aligned}\tilde{T}_w &= -v^{-1}\tilde{T}_{w'} - (-\tilde{T}_{\sigma_{i_1}} - v^{-1})\tilde{T}_{w'} = -v^{-1}j_{w'*}(\xi_{w'}) - \mathbf{a}_i j_{w'*}(\xi_{w'}) \\ &= -v^{-1}j_{w*}j_*(\xi_{w'}) - \mathbf{a}_i j_{w*}j_*(\xi_{w'}) = -v^{-1}j_{w*}j_*(\xi_{w'}) - j_{w*}(r_i j_*(\xi_{w'})).\end{aligned}$$

This shows that (b) holds for w and that $\xi_w = -v^{-1}j_*(\xi_{w'}) - r_i j_*(\xi_{w'})$. Since $h^* j_* = 0$, we have $h^*\xi_w = -h^* r_i j_*(\xi_{w'}) = -f(\xi_{w'})$ where f is as in 8.4. Using the definition of $\tilde{f}$ in 8.7 and the induction hypothesis, we have

$$f(\xi_{w'}) = \tilde{f}((-v)^{-l(w')}\mathbf{C});$$

hence

$$h^*\xi_w = -\tilde{f}((-v)^{-l(w')}\mathbf{C}) = -v^{-1}(-v)^{-l(w')}\mathbf{C} = (-v)^{-l(w)}\mathbf{C},$$

where the second equality follows from 8.8. The lemma is proved.

Lemma 8.10. *Let $\mathcal{H}_0$ be the set of all ξ in $\mathcal{H} = K_{\mathcal{G}}(Z)$ such that $\tilde{T}_{\sigma_i}\xi = v\xi$ for all $i \in I$. Let $r : \mathcal{H}_0 \to K_{\mathcal{G}}(Z_{w_0})$ be the restriction of the map $K_{\mathcal{G}}(Z) \to K_{\mathcal{G}}(Z_{w_0})$ (inverse image under the open imbedding $Z_{w_0} \subset Z$). Then r is an isomorphism.*

Let $\xi = \sum_{w\in W} v^{l(w)}\tilde{T}_w \in \mathcal{H}$. It is easy to see, using 1.21, that $\{\xi\theta_x | x \in \mathcal{X}\}$ is an $\mathcal{A}$-basis of $\mathcal{H}_0$. From 8.9(c) we see that $r(\xi\theta_x) = (-1)^\nu(\mathbf{C} \boxtimes L_x)$. Since $\{\mathbf{C} \boxtimes L_x | x \in \mathcal{X}\}$ is an $\mathcal{A}$-basis of $K_{\mathcal{G}}(Z_{w_0})$, the lemma follows.

Lemma 8.11. *Let $j : \mathcal{B} \times \mathcal{B} \to Z$ be the map $(\mathfrak{b}, \mathfrak{b}') \mapsto (0, \mathfrak{b}, \mathfrak{b}')$. In $\mathcal{H} = K_{\mathcal{G}}(Z)$ we have*

(a) $\sum_{w\in W} v^{l(w)}\tilde{T}_w = (-1)^\nu j_*(L_{-\rho} \boxtimes L_{-\rho})$.

Let ξ (resp. ξ') be the left (resp. right) hand side of (a). Then $\xi \in \mathcal{H}_0$ and $r(\xi) = (-1)^\nu\mathbf{C}$. (See the proof of 8.10.) Using 7.22, 7.23, we have for any $i \in I$:

$$\tilde{T}_{\sigma_i}(L_{-\rho} \boxtimes L_{-\rho}) = vL_{-\rho} \boxtimes L_{-\rho}$$

in $K_{\mathcal{G}}(\mathcal{B} \times \mathcal{B})$. Since j_* is $\mathcal{H}$-linear, it follows that $\tilde{T}_{\sigma_i}\xi' = v\xi$ in $\mathcal{H} = K_{\mathcal{G}}(Z)$. Thus, $\xi' \in \mathcal{H}_0$. The restriction of ξ' to $Z_{w_0} = \mathcal{O}_{w_0}$ is just the restriction of $(-1)^\nu L_{-\rho} \boxtimes L_{-\rho}$ to $\mathcal{O}_{w_0}$ and by 7.4, this is

$$(-1)^\nu\mathbf{C} \boxtimes ({}^{w_0}L_{-\rho} \otimes L_{-\rho}) = (-1)^\nu\mathbf{C} \boxtimes L_{-w_0(\rho)-\rho} = (-1)^\nu\mathbf{C} \boxtimes \mathbf{C} = (-1)^\nu\mathbf{C}.$$

Thus, $r(\xi) = r(\xi')$. Using 10.1, we deduce that $\xi = \xi'$. The lemma is proved.

8.12. Let $x \in \mathcal{X}_+$. The space of sections of L_x is naturally a (finite dimensional irreducible) $G \times \mathbf{C}^*$-module, denoted by $\mathbf{E}_x$ (the action of $\mathbf{C}^*$ on $\mathbf{E}_x$ is trivial).

Let n_x be the unique element of $\hat{W}^a$ contained in the double coset $Wx^\epsilon W$, which has maximal length in that double coset.

The main result of [L3] was the discovery of a very close connection between the element $c'_{n_x} \in \mathcal{H}$ (see 1.5, 1.9) and $\mathbf{E}_x$. In particular, in [L3, 8.6, 6.12] it is shown that

(a) $$c'_{n_x} = (v^{-\nu}\sum_{w\in W} v^{l(w)}\tilde{T}_w)\theta_{\mathbf{E}_x} = \theta_{\mathbf{E}_x}(v^{-\nu}\sum_{w\in W} v^{l(w)}\tilde{T}_w).$$

Here, $\mathbf{E}_x$ is regarded as an element of $R_{\mathcal{G}} = \mathcal{A}\mathcal{X}^W$ (see 7.8) so that $\theta_{\mathbf{E}_x}$ is in the centre of $\mathcal{H}$.

Combining (a) with 8.11 we obtain a description of c'_{n_x} as $\pm$ a coherent sheaf on Z supported on $\mathcal{B} \times \mathcal{B}$.

Corollary 8.13. *In $K_{\mathcal{G}}(Z)$ we have*

$$c'_{n_x} = (-v)^{-\nu} j_*(\mathbf{E}_x \otimes (L_{-\rho} \boxtimes L_{-\rho})),$$

(j as in 8.11.)

9. K-theoretic description of the involution $^{-}$: $\mathcal{H} \to \mathcal{H}$

Lemma 9.1. *Let $p : Y \to Y'$ be a vector bundle with Y' irreducible. We regard Y with the $\mathbf{C}^*$-action $(\lambda, y) \mapsto \lambda^{-2}y$. Let L' be a line bundle on Y' and let $L = p^*L'$.*

The set of $\mathbf{C}^*$-equivariant structures on the line bundle L is in natural bijection with $\mathbf{Z}$: to $n \in \mathbf{Z}$ corresponds the equivariant structure $\lambda : L_y \to L_{\lambda^{-2}y}$ given by multiplication by λ^n. (Here $y \in Y$ and we identify $L_y = L_{\lambda^{-2}y} = L'_{p(y)}$.)

We fix a $\mathbf{C}^*$-equivariant structure on L. If $y' \in Y', \lambda \in \mathbf{C}^*$, then for any $y \in p^{-1}(y')$, $\lambda : L_y \to L_{\lambda^{-2}y}$ or equivalently $\lambda : L'_{y'} \to L'_{y'}$ is multiplication by $f(\lambda, y) \in \mathbf{C}^*$. Now $f : \mathbf{C}^* \times Y \to \mathbf{C}^*$ is a morphism of algebraic varieties; hence we have $f(\lambda, y) = \sum_{n \in \mathbf{Z}} f_n(y)\lambda^n$ where $f_n : Y \to \mathbf{C}$ are morphisms which are identically zero for all but finitely many n.

Assume that for some $n' \neq n''$, neither $f_{n'}$ or $f_{n''}$ is identically zero. Since Y is irreducible, we can find $y_0 \in Y$ such that $f_{n'}(y_0) \neq 0, f_{n''}(y_0) \neq 0$. This implies that $\sum_{n \in \mathbf{Z}} f_n(y_0)\lambda^n = 0$ for some $\lambda \neq 0$. This contradicts the fact that $f(\mathbf{C}^* \times \{y_0\}) \subset \mathbf{C}^*$.

We see that there exists $n \in \mathbf{Z}$ such that $f(\lambda, y) = f_n(y)\lambda^n$ for all λ, y. Since $f(1, y) = 1$, we must have $f_n(y) = 1$ and $f(\lambda, y) = \lambda^n$ for all λ, y. The lemma follows.

Lemma 9.2. *Let $i \in I$ and let $\mathcal{L}_i \in \mathrm{Vec}_{\mathcal{G}}(\bar{\mathcal{O}}_i)$ be as in 7.19. We have $\mathcal{L}_i \otimes \mathcal{L}_i = L_i^{-1} \boxtimes L_i^{-1}$ in $\mathrm{Vec}_{\mathcal{G}}(\bar{\mathcal{O}}_i)$.*

Let $L', L'' \in \mathbf{X}$ be as in 7.19. We have $L'L' = L_i^{-1}\mathcal{L}$, $L''L'' = L_i^{-1}\mathcal{L}^{-1}$, where $\mathcal{L} \in \mathbf{X}$ is such that $\check{\alpha}_i(\mathcal{L}) = 0$. Hence,

$$\begin{aligned}\mathcal{L}_i \otimes \mathcal{L}_i &= (L' \boxtimes L'') \otimes (L' \boxtimes L'') = (L'L') \boxtimes (L''L'') = (L_i^{-1} \otimes \mathcal{L}) \boxtimes (L_i^{-1} \otimes \mathcal{L}^{-1})\\ &= (L_i^{-1} \boxtimes L_i^{-1}) \otimes (\mathcal{L} \boxtimes \mathcal{L}^{-1}) = (L_i^{-1} \boxtimes L_i^{-1}).\end{aligned}$$

(The last equality uses 7.19(a).)

For any $\mathbf{C}$-vector space $\mathcal{V}$ of dimension n we denote by $\Omega(\mathcal{V})$ the dual of the nth exterior power of $\mathcal{V}$.

Lemma 9.3. *$v^{2\nu-2}\mathcal{L}_i \otimes \mathcal{L}_i$ and $\Omega_{\bar{Z}_i}$ are isomorphic as objects on $\mathrm{Vec}_{\mathcal{G}}(\bar{Z}_i)$. (See 7.19, 6.8.)*

We first show that

(a) $\mathcal{L}_i \otimes \mathcal{L}_i \cong \Omega_{\bar{Z}_i}$ are isomorphic as objects of $\mathrm{Vec}_G(\bar{Z}_i)$.

Let $(x, \mathfrak{b}, \mathfrak{b}') \in \bar{Z}_i$ and let $\mathfrak{p} = \pi_i(\mathfrak{b}) = \pi_i(\mathfrak{b}')$. By 9.2, the fibre of $\mathcal{L}_i \otimes \mathcal{L}_i$ at $(x, \mathfrak{b}, \mathfrak{b}')$ is $(\mathfrak{p}/\mathfrak{b})^* \otimes (\mathfrak{p}/\mathfrak{b}')^*$. The fibre of $\Omega_{\bar{Z}_i}$ at $(x, \mathfrak{b}, \mathfrak{b}')$ is

$$\Omega(\mathfrak{n}_{\mathfrak{p}}) \otimes (\mathfrak{p}/\mathfrak{b})^* \otimes (\mathfrak{p}/\mathfrak{b}')^* \otimes \Omega(\mathfrak{g}/\mathfrak{p}).$$

It remains to show that $\Omega(\mathfrak{n}_{\mathfrak{p}}) \otimes \Omega(\mathfrak{g}/\mathfrak{p}) = \mathbf{C}$ canonically. This follows from the fact that the vector spaces $\mathfrak{n}_{\mathfrak{p}}, \mathfrak{g}/\mathfrak{p}$ are naturally in duality (via the Killing form of $\mathfrak{g}$). Thus, (a) is proved.

Next we note that the action of $\lambda \in \mathbf{C}^*$ on the fibres of the two line bundles in the lemma at $(0, \mathfrak{b}, \mathfrak{b}')$ is by multiplication by $\lambda^{2\nu-2}$. (For $\Omega_{\bar{Z}_i}$ this comes from the way that $\mathbf{C}^*$ acts on the factor $\Omega(\mathfrak{n}_\mathfrak{p})$ which in turn comes from the action of $\mathbf{C}^*$ on $\mathfrak{n}_\mathfrak{p}$ given by multiplication by λ^{-2}.)

From 9.1 it follows that an isomorphism of G-equivariant line bundles

$$v^{2\nu-2}\mathcal{L}_i \otimes \mathcal{L}_i \to \Omega_{\bar{Z}_i}$$

(see (a)) is automatically an isomorphism of $\mathcal{G}$-equivariant line bundles. The lemma is proved.

Lemma 9.4. *We have $\Omega_\Lambda \cong v^{2\nu}$ in* $\mathrm{Vec}_\mathcal{G}(\Lambda)$.

As in the proof of 9.3, it is enough to show that $\Omega_\Lambda \cong \mathbf{C}$ in $\mathrm{Vec}_G(\Lambda)$. The fibre of Ω_Λ at $(y, \mathfrak{b})$ is $\Omega(\mathfrak{n}_\mathfrak{b}) \otimes \Omega(\mathfrak{g}/\mathfrak{b})$. It remains to show that $\Omega(\mathfrak{n}_\mathfrak{b}) \otimes \Omega(\mathfrak{g}/\mathfrak{b}) = \mathbf{C}$ canonically. This follows from the fact that the vector spaces $\mathfrak{n}_\mathfrak{b}, \mathfrak{g}/\mathfrak{b}$ are naturally in duality (via the Killing form of $\mathfrak{g}$). The lemma is proved.

Lemma 9.5. *For $F, F' \in K_\mathcal{G}(Z)$ we have $D_Z(F \star F') = v^{-2\nu} D_Z(F) \star D_Z(F')$. Here D_Z is as in 6.10.*

With the notation in 7.9, we have

$$\begin{aligned} D_Z(F \star F') &= D_Z(p_{13*}(p_{12}^*F \otimes^L_{\Lambda^3} p_{23}^*F')) = p_{13*}(D_\Theta(p_{12}^*F \otimes^L_{\Lambda^3} p_{23}^*F')) \\ &= v^{-6\nu} p_{13*}(D_{\Lambda^{aab}}(p_{12}^*F) \otimes^L_{\Lambda^3} D_{\Lambda^{abb}}(p_{23}^*F')) \\ &= v^{-6\nu} p_{13*}(v^{2\nu}(p_{12}^* D_Z F) \otimes^L_{\Lambda^3} v^{2\nu} p_{23}^* D_Z F')) = v^{-2\nu} D_Z(F) \star D_Z(F'). \end{aligned}$$

The first equality holds by definition. The second equality holds by 6.11. The third equality follows from 6.15 using the equality $\Omega_{\Lambda^3} = v^{6\nu}$ in $\mathrm{Vec}_\mathcal{G}(\Lambda^3)$ (a consequence of 9.4.) The fourth equality holds by 6.12 using the fact that $\Omega_{p_{12}}$ (see 6.12) is isomorphic to $v^{2\nu}$ in $\mathrm{Vec}_\mathcal{G}(\Lambda^{aab})$ (a consequence of 9.4) and the analogous fact for p_{23} instead of p_{12}. The lemma is proved.

Lemma 9.6. (a) *Let $L \in \mathbf{X}$. Then $D_Z({}^!L) = {}^!(L^{-1})v^{2\nu}$.*
(b) *Let $i \in I$. We have $D_Z(\mathbf{a}_i) = v^{2\nu}\mathbf{a}_i$.*

We prove (a). By 6.11, it suffices to show that, if we regard L as an object of $\mathrm{Vec}_\mathcal{G}(\Lambda)$, we have $D_\Lambda(L) = v^{2\nu}L^{-1}$. This follows from 6.13 since Λ is smooth and $\Omega_\Lambda = v^{2\nu}$.

We prove (b). By 6.11, it suffices to show that $D_{\bar{Z}_i}(v^{-1}\mathcal{L}_i) = v^{2\nu}v^{-1}\mathcal{L}_i$ in $K_\mathcal{G}(\bar{Z}_i)$. We have

$$D_{\bar{Z}_i}(v^{-1}\mathcal{L}_i) = v\mathcal{L}_i^* \otimes \Omega_{\bar{Z}_i} = v\mathcal{L}_i^{-1} \otimes v^{2\nu-2}\mathcal{L}_i \otimes \mathcal{L}_i = v^{2\nu}(v^{-1}\mathcal{L}_i).$$

The first equality holds by 6.13, 6.16, since $\bar{Z}_i$ is smooth, connected, of even dimension. The second equality holds by 9.3. The lemma is proved.

Lemma 9.7. *The map $D' : K_\mathcal{G}(Z) \to K_\mathcal{G}(Z)$ given by $D'(\xi 0 = v^{-2\nu} D_Z(\xi)$, corresponds under the isomorphism $\mathcal{H} \xrightarrow{\sim} K_\mathcal{G}(Z)$ in 8.6, to the involution of the ring $\mathcal{H}$ which takes $\tilde{T}_{\sigma_i} + v^{-1}$ to $\tilde{T}_{\sigma_i} + v^{-1}$ for all $i \in I$, θ_x to θ_{-x} for all $x \in \mathcal{X}$, v^n to v^{-n} for all $n \in \mathbf{Z}$.*

By 9.5, D' is a ring homomorphism. It remains to use 9.6.

9.8. A Lie algebra automorphism $\varpi : \mathfrak{g} \to \mathfrak{g}$ is said to be an *opposition* (of $\mathfrak{g}$) if there exists a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $\varpi(y) = -y$ for all $y \in \mathfrak{h}$. Then ϖ is the tangent map of a well defined automorphism of G denoted by $\varpi : G \to G$. It takes any element of the maximal torus corresponding to $\mathfrak{h}$ to its inverse.

It is well known that for any $\mathfrak{h}$ as above, there exists at least one opposition ϖ such that $\varpi(y) = -y$ for all $y \in \mathfrak{h}$.

If ϖ is an opposition, then for any $i \in I$ and any $\mathfrak{p} \in \mathcal{P}_i$ we have $\varpi(\mathfrak{p}) \in \mathcal{P}_{i^*}$ where $i^* \in I$ is as in 1.24. It follows that ϖ^2 is an inner automorphism. More precisely, ϖ^2 is the identity map (it is an inner automorphism which is the identity map on some Cartan subalgebra).

9.9. If ϖ is an opposition and $g \in G$, then $Ad(g)\varpi Ad(g^{-1})$ is an opposition. Conversely, if ϖ, ϖ' are oppositions, we have by a standard argument, $\varpi' = Ad(g)\varpi Ad(g^{-1})$ for some $g \in G$.

9.10. Let ϖ be an opposition of $\mathfrak{g}$. This induces an involution $Z \to Z$ (denoted again by ϖ) given by $(y, \mathfrak{b}, \mathfrak{b}') \mapsto (-\varpi(y), \varpi(\mathfrak{b}), \varpi(\mathfrak{b}'))$. If $F \in \mathrm{Coh}_{\mathcal{G}}(Z)$, then the coherent sheaf $\varpi^*(F)$ is naturally an object of $\mathrm{Coh}_{\mathcal{G}}(Z)$. We obtain an involution $F \mapsto \varpi^*(F)$ of $K_{\mathcal{G}}(Z)$ denoted by ϖ^*.

Lemma 9.11. *The involution $\varpi^* : K_{\mathcal{G}}(Z) \to K_{\mathcal{G}}(Z)$ corresponds under the isomorphism 8.6 to the involution $\chi \mapsto \chi^{\circledast}$ of $\mathcal{H}$ (see 1.24).*

From the definition it is clear that ϖ^* preserves the $\mathcal{A}$-algebra structure of $K_{\mathcal{G}}(Z)$, that it carries $\mathbf{a}_i$ to $\mathbf{a}_{i^*}$ for any $i \in I$, and that it carries ${}^!L_i$ to ${}^!L_{i^*}$ for any $i \in I$ (hence ${}^!L$ to ${}^!({}^{w_0}L^{-1})$ for any $L \in \mathbf{X}$). The lemma follows.

Proposition 9.12. *Let $\tau : K_{\mathcal{G}}(Z) \to K_{\mathcal{G}}(Z)$ be the $\mathcal{A}$-algebra automorphism given by $\xi \mapsto \tilde{T}_{w_0}^{-1}\xi\tilde{T}_{w_0}$. Let $\varpi^* : K_{\mathcal{G}}(Z) \to K_{\mathcal{G}}(Z)$ be the $\mathcal{A}$-algebra involution defined in 9.10 in terms of an opposition ϖ. (ϖ^* is independent of the choice of ϖ by 9.9.) Let $D' : K_{\mathcal{G}}(Z) \to K_{\mathcal{G}}(Z)$ be the ring involution defined in 9.7.*

(a) *We have $D'\varpi^* = \varpi^*D', \tau\varpi^* = \varpi^*\tau, D'\tau = \tau^{-1}D'$.*
(b) *The ring automorphism $\tau\varpi^*D'$ of $K_{\mathcal{G}}(Z)$ corresponds under the isomorphism $\mathcal{H} \xrightarrow{\sim} K_{\mathcal{G}}(Z)$ to the involution ${}^- : \mathcal{H} \to \mathcal{H}$ in 1.8.*

Using 9.11 and 9.7, we see that automorphism of the ring $\mathcal{H}$ corresponding to $\tau\varpi^*D' : K_{\mathcal{G}}(Z) \to K_{\mathcal{G}}(Z)$ takes v to v^{-1},

$$\tilde{T}_{\sigma_i} + v^{-1} \text{ to } \tilde{T}_{w_0}^{-1}(\tilde{T}_{\sigma_{i^*}} + v^{-1})\tilde{T}_{w_0} = \tilde{T}_{\sigma_i} + v^{-1}$$

for all $i \in I$, and θ_x to $\tilde{T}_{w_0}^{-1}\theta_{w_0 x}\tilde{T}_{w_0} = \overline{\theta_x}$ for all $x \in \mathcal{X}$. (The last equality follows from 1.22.) (b) follows.

The first equality in (a) follows from the definitions using 6.12. Using 9.11, we see that the second equality in (a) follows from the fact that $\chi^{\circledast} = \chi$ for $\chi = \tilde{T}_{w_0}$ or $\chi = \tilde{T}_{w_0}^{-1}$ (where $\chi^{\circledast}$ is as in 1.24). The third equality in (a) follows from the first two equalities in (a) and the fact that $D', \varpi^*, \tau\varpi^*D'$ are involutions. (The fact that $\tau\varpi^*D'$ is an involution follows from (b).) The proposition is proved.

Corollary 9.13. *Let Y be the closure of a $\mathcal{G}$-orbit in $\mathfrak{g}_n$. Let*

$$Z_Y = \{(y, \mathfrak{b}, \mathfrak{b}') \in Z | y \in Y\}.$$

Consider the direct image map $j_*^Y : K_{\mathcal{G}}(Z_Y) \to K_{\mathcal{G}}(Z)$ induced by the inclusion $j_Y : Z_Y \subset Z$. Then the image $\mathrm{Im}(j_*^Y)$ of j_*^Y is a two-sided ideal of $K_{\mathcal{G}}(Z)$ stable under the involution ${}^-$ of $\mathcal{H} = K_{\mathcal{G}}(Z)$.

The fact that $\mathrm{Im}(j^Y_*)$ is a two-sided ideal is analogous to [KL2, 3.5]. It can be proved by the arguments in 10.1. To prove that it is stable under $^-$, it is enough to check that it is stable under the maps τ, ϖ^*, D_Z of $K_{\mathcal{G}}(Z)$ into itself. The stability under τ follows from the fact that $\mathrm{Im}(j^Y_*)$ is a two-sided ideal. The stability under D_Z follows from the commutative diagram

$$\begin{array}{ccc} K_{\mathcal{G}}(Z_Y) & \xrightarrow{j^Y_*} & K_{\mathcal{G}}(Z) \\ {\scriptstyle D_{Z_Y}}\downarrow & & {\scriptstyle D_Z}\downarrow \\ K_{\mathcal{G}}(Z_Y) & \xrightarrow{j^Y_*} & K_{\mathcal{G}}(Z) \end{array}$$

(a special case of 6.11). It remains to show that $\mathrm{Im}(j^Y_*)$ is stable under ϖ^*. It is enough to show that $\varpi : Z \to Z$ maps Z_Y into itself, or that $\varpi : \mathfrak{g} \to \mathfrak{g}$ maps Y into itself. This follows from the well known fact that any nilpotent orbit in $\mathfrak{g}$ is stable under any opposition. (A more precise result is proved in 12.1, 12.2.) The corollary is proved.

9.14. The Corollary is evidence for the conjecture that the "canonical" basis $\{c'_w | w \in \hat{W}^a\}$ (see 1.5, 1.9) of $\mathcal{H} = K_{\mathcal{G}}(Z)$ is compatible with each of the subspaces $\mathrm{Im}(j^Y_*)$ above.

10. The $\mathcal{H}$-modules $K_{\mathcal{T}}(\mathcal{B}), K_{\mathcal{T}}(\Lambda)$

10.1. Let H be a closed reductive subgroup of $\mathcal{G}$. Let Y be a closed H-stable subvariety of $\mathfrak{g}_n$. Then $\dot{Y} = \{(y, \mathfrak{b}) \in \Lambda | y \in Y\}$ is a closed H-stable subvariety of Λ.

We regard Z and $\Lambda \times \dot{Y}$ as closed subvarieties of Λ^2 in an obvious way; the intersection of these subvarieties is $Z \cap (\dot{Y} \times \dot{Y})$. Let $p' : \Lambda \times \dot{Y} \to \dot{Y}$ be the second projection and let $p'' : Z \cap (\dot{Y} \times \dot{Y}) \to \dot{Y}$ be the first projection. Then p' is a smooth morphism and p'' is a proper morphism. They are compatible with the natural actions of H. Let $F \in \mathrm{Coh}_{\mathcal{G}}(Z), F' \in \mathrm{Coh}_H(\dot{Y})$. We regard F as an object of $\mathrm{Coh}_H(Z)$ and we define

$$F \star_Y F' = p''_*(F \otimes^L_{\Lambda^2} p'^* F') \in K_H(\dot{Y}).$$

(The Tor-product is relative to the smooth H-variety Λ^2 and its closed subvarieties Z and $\Lambda \times \dot{Y}$.) This extends to a bilinear pairing $K_{\mathcal{G}}(Z) \times K_H(\dot{Y}) \to K_H(\dot{Y})$, denoted $\xi, \xi' \mapsto \xi \star_Y \xi'$, which may be regarded as a $K_{\mathcal{G}}(Z)$-module structure on $K_H(\dot{Y})$, as a routine argument using Λ^3 shows.

Now let Y, Y' be two closed H-stable subvarieties of $\mathfrak{g}_n$ such that $Y \subset Y'$. Then $\dot{Y} \subset \dot{Y}'$ and this induces a direct image map $K_H(\dot{Y}) \to K_H(\dot{Y}')$. This is in fact a homomorphism of $K_{\mathcal{G}}(Z)$-modules. (This follows from the definitions using 6.5.)

10.2. In the remainder of this section, we fix a Borel subgroup B_0 of G and a maximal torus T of B_0. Let $\mathcal{T} = T \times \mathbf{C}^*$. Let $\mathfrak{b}_0 \in \mathcal{B}$ be the Lie algebra of B_0 and let $\mathfrak{n}_0 = \mathfrak{n}_{\mathfrak{b}_0}$. We shall regard Λ as a subvariety of $\Lambda \times \mathcal{B}$ by $(y, \mathfrak{b}) \mapsto (y, \mathfrak{b}, \mathfrak{b}_0)$. It is a $(B_0 \times \mathbf{C}^*)$-stable subvariety. Then

$$Z_0 = \{(y, \mathfrak{b}) \in \Lambda | \mathfrak{b}' = \mathfrak{b}_0\}$$

(see 7.15) is a $(B_0 \times \mathbf{C}^*)$-stable subvariety of Λ and $\mathcal{B}$ is a $(B_0 \times \mathbf{C}^*)$-stable subvariety of Z (by $\mathfrak{b} \mapsto (0, \mathfrak{b})$).

If V is one of the $(B_0 \times \mathbf{C}^*)$-varieties $\Lambda, Z_0, \mathcal{B}, \{0\}$, we set

$$\tilde{V} = (B_0 \times \mathbf{C}^*)\backslash(\mathcal{G} \times V)$$

where $(B_0 \times \mathbf{C}^*)$ acts on $\mathcal{G} \times V$ by $\beta : (\gamma, \xi) \mapsto (\gamma\beta^{-1}, \beta\xi)$ for $\beta \in B_0 \times \mathbf{C}^*, \gamma \in \mathcal{G}, \xi \in V$. Note that $\tilde{V}$ is a $\mathcal{G}$-variety, where $\mathcal{G}$ acts by left multiplication on the first factor. By a general result in [T1] we have canonically

(a) $$K_{\mathcal{G}}(\tilde{V}) \xrightarrow{\sim} K_{\mathcal{T}}(V).$$

This is defined as follows. We consider $F \in \mathrm{Coh}_{\mathcal{G}}(\tilde{V})$. The inverse image of F under the orbit map $\mathcal{G} \times V \to \tilde{V}$ is naturally an object $\tilde{F} \in \mathrm{Coh}_{B_0 \times \mathbf{C}^*}(\mathcal{G} \times V)$. This is then the inverse image of a well defined object $F' \in \mathrm{Coh}_{B_0 \times \mathbf{C}^*}(V)$ under the second projection $\mathcal{G} \times V \to V$. Since $T \times \mathbf{C}^*$ is a subgroup of $B_0 \times \mathbf{C}^*$, we may regard F' as an object of $\mathrm{Coh}_{\mathcal{T}}(V)$. Then (a) is defined by $F \mapsto F'$.

(b) *For $V = \Lambda, Z_0, \mathcal{B}, \{0\}$, we have canonically $\tilde{V} = \Lambda \times \mathcal{B}, Z, \mathcal{B} \times \mathcal{B}, \mathcal{B}$ respectively (as $\mathcal{G}$-varieties).*

For $V = \Lambda$ or $V = Z$, this is $((g, \lambda), (y, \mathfrak{b})) \mapsto (\lambda^{-2}Ad(g)y, Ad(g)\mathfrak{b}, Ad(g)\mathfrak{b}_0)$.

For $V = \mathcal{B}$, this is $((g, \lambda), \mathfrak{b}) \mapsto (Ad(g)\mathfrak{b}, Ad(g)\mathfrak{b}_0)$.

For $V = \{0\}$, this is $((g, \lambda), \{0\}) \mapsto Ad(g)\mathfrak{b}_0$.

10.3. By 10.1 with $Y = \{0\}, \dot{Y} = \mathcal{B}$ (via $\mathfrak{b} \mapsto (0, \mathfrak{b})$) and $H = \mathcal{T}$, we have a natural $K_{\mathcal{G}}(Z)$-module structure on

(a) $K_{\mathcal{T}}(\mathcal{B}) = K_{\mathcal{G}}(\tilde{\mathcal{B}}) = K_{\mathcal{G}}(\mathcal{B} \times \mathcal{B})$.

(See 10.2(a),(b).) From the definitions, this coincides with the $K_{\mathcal{G}}(Z)$-module structure defined in 7.13.

By 10.1 with $Y = \mathfrak{g}_n, \dot{Y} = \Lambda$ and $H = \mathcal{T}$, we have a natural $K_{\mathcal{G}}(Z)$-module structure on

(b) $K_{\mathcal{T}}(\Lambda) = K_{\mathcal{G}}(\tilde{\Lambda}) = K_{\mathcal{G}}(\Lambda \times \mathcal{B})$.

(See 10.2(a),(b).) From the definitions, this coincides with the $K_{\mathcal{G}}(Z)$-module structure defined in 7.11.

By 10.1 with $Y = \mathfrak{n}_0, \dot{Y} = Z_0$ and $H = \mathcal{T}$, we have a natural $K_{\mathcal{G}}(Z)$-module structure on

(c) $K_{\mathcal{T}}(Z_0) = K_{\mathcal{G}}(\tilde{Z}_0) = K_{\mathcal{G}}(Z)$.

(See 10.2(a),(b).) From the definitions, this coincides with the left multiplication in the ring $K_{\mathcal{G}}(Z)$. From 10.2(a),(b) we have

(d) $R_{\mathcal{T}} = K_{\mathcal{T}}(\{0\}) = K_{\mathcal{G}}(\tilde{\{0\}}) = K_{\mathcal{G}}(\mathcal{B})$.

Composing with the identification $\mathcal{AX} = K_{\mathcal{G}}(\mathcal{B})$ (see 7.8(a)) we obtain an identification of rings

(e) $\mathcal{AX} = R_{\mathcal{T}} = K_{\mathcal{G}}(\mathcal{B})$.

10.4. The natural $R_{\mathcal{T}} = \mathcal{AX}$-module structure on $K_{\mathcal{T}}(\mathcal{B}), K_{\mathcal{T}}(Z_0), K_{\mathcal{T}}(\Lambda)$ is denoted by $[x], \xi \mapsto [x] \centerdot \xi$. (Here $x \in \mathcal{X}$.) We identify $K_{\mathcal{T}}(\mathcal{B}) = K_{\mathcal{T}}(\Lambda)$ as $\mathcal{AX}$-modules via the isomorphism $K_{\mathcal{T}}(\mathcal{B}) \mapsto K_{\mathcal{T}}(\Lambda)$ induced by inverse image under the second projection $\Lambda \to \mathcal{B}$.

For $L \in \mathbf{X}$, we can regard the $\mathcal{G}$-equivariant line bundle L as a $\mathcal{T}$-equivariant line bundle on $\mathcal{B}$; hence L may be regarded as an element of $K_{\mathcal{T}}(\mathcal{B})$. If $L \in \mathbf{X}$ and $x \in \mathcal{X}$, then $[x] \centerdot L \in K_{\mathcal{T}}(\mathcal{B}) = K_{\mathcal{T}}(\Lambda)$ corresponds under 10.3(a) or 10.3(b) to the element $L \boxtimes L_x$ of $K_{\mathcal{G}}(\mathcal{B} \times \mathcal{B}) = K_{\mathcal{G}}(\Lambda \times \mathcal{B})$.

The element of $K_{\mathcal{T}}(Z_0)$ corresponding to the unit element of the ring $K_{\mathcal{G}}(Z)$ is denoted by 1.

Lemma 10.5. *Let $j' : \mathcal{B} \to Z_0$ be the map $\mathfrak{b} \mapsto (0, \mathfrak{b})$. In $K_{\mathcal{T}}(Z_0)$ we have*
(a) $\sum_{w \in W} v^{l(w)} \tilde{T}_w \cdot 1 = (-1)^\nu j'_*([-\rho] \bullet L_{-\rho})$.

This is a reformulation of 8.11, using the identifications 10.3(a),(c).

10.6. Consider the diagram

$$\begin{tikzcd}
K_{\mathcal{T}}(\mathcal{B}) \arrow[r, "j'_*"] \arrow[d, "g"'] & K_{\mathcal{T}}(Z_0) \arrow[r, "\tilde{j}'_*"] \arrow[d, "g'"'] & K_{\mathcal{T}}(\Lambda) \arrow[d, "\tilde{g}"'] \\
\mathcal{A}\mathcal{X}^{\otimes 2}_{d'} \arrow[r, "u"] \arrow[d, "f"'] & \mathcal{H} \arrow[d, "f'"'] & \mathcal{A}\mathcal{X}^{\otimes 2}_{d} \arrow[d, "\tilde{f}"'] \\
\mathcal{M}_{d'} \arrow[r, "a"] & \mathcal{M}_c \arrow[r, "b"] & \mathcal{M}_d
\end{tikzcd}$$

where the following notation is used.

$j' : \mathcal{B} \to Z_0$ is as in 10.5 and $\tilde{j}' : Z_0 \to \Lambda$ is the obvious inclusion.

a, b are the obvious imbeddings. u is defined by

$$u([x] \otimes [x']) = (-1)^\nu v^{2\nu} \theta_{x+\rho} \Big(\sum_{w \in W} v^{-l(w)} \tilde{T}_w^{-1}\Big) \theta_{x'+\rho}$$

for $x, x' \in \mathcal{X}$.

$f, \tilde{f}$ are the isomorphisms defined in 4.4, 4.5. (They are $\mathcal{H}$-linear by definition; see 4.6, 4.7.)

f' is defined by $\chi \mapsto (-1)^\nu v^{-\nu} \chi \cdot {}_0A^+_\epsilon$.

g is the inverse of the composition $\mathcal{A}\mathcal{X}^{\otimes 2} \xrightarrow{\sim} K_{\mathcal{G}}(\mathcal{B} \times \mathcal{B}) \xrightarrow{\sim} K_{\mathcal{T}}(\mathcal{B})$ (the first map as in 7.8(c); the second map as in 10.3(a)).

$\tilde{g}$ is the inverse of the composition $\mathcal{A}\mathcal{X}^{\otimes 2} \xrightarrow{\sim} K_{\mathcal{G}}(\Lambda \times \mathcal{B}) \xrightarrow{\sim} K_{\mathcal{T}}(\Lambda)$ (the first map as in 7.8(c); the second map as in 10.3(b)).

g' is the inverse of the composition $\mathcal{H} \xrightarrow{\sim} K_{\mathcal{G}}(Z) \to K_{\mathcal{T}}(Z_0)$ (the first map as in 8.6; the second map as in 10.3(c)).

Here $K_{\mathcal{T}}(\mathcal{B}), K_{\mathcal{T}}(\Lambda), K_{\mathcal{T}}(Z_0)$ are regarded as $\mathcal{H} = K_{\mathcal{G}}(Z)$-modules as in 10.3.

Then $g, \tilde{g}$ are isomorphisms of $\mathcal{H}$-modules by the proof of 7.23, 7.24. Moreover, $j'_*, \tilde{j}'_*$ are $\mathcal{H}$-linear (a reformulation of 7.14).

Proposition 10.7. (a) *The diagram in 10.6 is commutative.*
(b) *All its maps are $\mathcal{H} = K_{\mathcal{G}}(Z)$-linear.*
(c) *All its vertical maps are isomorphisms.*

We prove (c). We only have to prove that f' is an isomorphism. Let $\mathrm{Im}(f')$ be the image of f'. The equality $\tilde{T}_{\underline{x}} \theta_x({}_0A^+_\epsilon) = {}_0A^+_{x+\epsilon}$ for $x \in \mathcal{X}$ shows that ${}_0A_{\epsilon'} \in \mathrm{Im}(f')$ for any $\epsilon' \in \mathfrak{E}$. Let $A \in X$. We can find $\epsilon' \in \mathfrak{E}$ so that ϵ' is in the closure of A. Then $A = \tilde{T}_w^{-1}(A^+_{\epsilon'})$ (equality in M_c) for some $w \in W_{\epsilon'}$. Hence, ${}_0A = \tilde{T}_w^{-1}({}_0A_{\epsilon'}) \in \mathrm{Im}(f')$. Now if $\iota \in \underline{\mathcal{X}}$, we have ${}_\iota A = \tilde{T}_\iota({}_0A) \in \mathrm{Im}(f')$. Thus, f' is surjective.

We can regard $\mathcal{H}$ as a left $\mathcal{A}\mathcal{X}$-module by $[x'] : \chi \mapsto \chi\theta_{x'}$. This module is free of rank $|W|$ by 1.21. We can regard $\mathcal{M}_c$ as a left $\mathcal{A}\mathcal{X}$-module by the $\bullet$-action. This module is also free of rank $|W|$. (See 3.1(c).) Now f' respects these $\mathcal{A}\mathcal{X}$-module structures. Being a surjective map between free modules of the same (finite) rank, it must be an isomorphism. (c) is proved.

We prove (a). We first show that the lower left square of the diagram is commutative. It is enough to show that, for any $x, x' \in \mathcal{X}$, we have

$$(-1)^\nu v^{-\nu}(-1)^\nu v^{2\nu}\theta_{x+\rho}(\sum_{w\in W} v^{-l(w)}\tilde{T}_w^{-1})\theta_{x'+\rho}({}_0A_\epsilon^+) = v^\nu\theta_{x+\rho}([x'+\rho]\,\mathbf{.}\,{}_0e_\epsilon).$$

The left hand side equals

$$\begin{aligned}&v^\nu\theta_{x+\rho}(\sum_{w\in W} v^{-l(w)}\tilde{T}_w^{-1})([x'+\rho]\,\mathbf{.}\,{}_0A_\epsilon^+)\\ &= v^\nu\theta_{x+\rho}([x'+\rho]\,\mathbf{.}\,(\sum_{w\in W} v^{-l(w)}\tilde{T}_w^{-1}{}_0A_\epsilon^+) = v^\nu\theta_{x+\rho}([x'+\rho]\,\mathbf{.}\,{}_0e_\epsilon,\end{aligned}$$

as desired.

Next we show that the upper left square of the diagram is commutative. Since g, g' are isomorphisms, it suffices to check that $j_*g^{-1} = g'^{-1}u$. Thus, it is enough to show that, for any $x, x' \in \mathcal{X}$, we have

$$j'_*([x']\,\mathbf{.}\,L_x) = (-1)^\nu v^{2\nu}\theta_{x+\rho}(\sum_{w\in W} v^{-l(w)}\tilde{T}_w^{-1})\theta_{x'+\rho}\cdot 1$$

in $K_\mathcal{T}(Z_0)$. Since j'_* is $\mathcal{H}$-linear and $\mathcal{AX}$-linear, we may assume that $x = x' = -\rho$. Then the desired formula follows from 10.1, using that

$$v^\nu\sum_{w\in W} v^{-l(w)}\tilde{T}_w^{-1} = v^{-\nu}\sum_{w\in W} v^{l(w)}\tilde{T}_w.$$

It remains to show that the right rectangle in our diagram is commutative. We will deduce this from the commutativity of the squares already considered, together with the commutativity of the diagram 4.11. Since all our maps are $R_\mathcal{T} = \mathcal{AX}$-linear, and all the $R_\mathcal{T} = \mathcal{AX}$-modules in the diagram are free of finite rank, it suffices to show the commutativity of the rectangle after tensoring each module in the diagram over $R_\mathcal{T}$ with the quotient field of $R_\mathcal{T}$. After this tensoring, the maps a, b in the diagram become isomorphisms (by 4.10); hence j'_* becomes an isomorphism (by the commutativity of the two left squares in the diagram). Hence it suffices to show that $bf'g'j'_* = \tilde{f}\tilde{g}\tilde{j}'_*j'_*$ holds after tensoring. It is also enough to prove that this holds before tensoring. Since $f'g'j'_* = f'ug = afg$ (by the earlier part of the proof), it is enough to show that $bafg = \tilde{f}\tilde{g}\tilde{j}'_*j'_*$ or that $baf = \tilde{f}\tilde{g}\tilde{j}'_*j'_*g^{-1}$.

Now $\tilde{j}'j' : \mathcal{B} \to \Lambda$ is the imbedding of the zero section of the vector bundle $\Lambda \to \mathcal{B}$ (second projection). It follows that

$$\tilde{j}'_*j'_* = (\tilde{j}'j')_* : K_\mathcal{T}(\mathcal{B}) \to K_\mathcal{T}(\Lambda) = K_\mathcal{T}(\mathcal{B})$$

is just multiplication by $\prod_{\alpha\in\mathcal{R}^+}(1 - v^2L_\alpha) \in K_\mathcal{T}(\mathcal{B})$. Equivalently, $\tilde{g}\tilde{j}'_*j'_*g^{-1}(\xi_1) = \theta_{\bar{\Delta}}(\xi_1)$ for all $\xi_1 \in \mathcal{AX}_{d'}^{\otimes 2}$. Thus we are reduced to showing that $baf(\xi_1) = \tilde{f}(\theta_{\bar{\Delta}}(\xi_1))$. This follows from 4.11. This proves (a).

We prove (b). Note that all maps in our diagram are already known to be $\mathcal{H}$-linear except possibly for u. But then u is automatically $\mathcal{H}$-linear since $u = f'^{-1}af$ (by (a)) and f'^{-1}, a, f are $\mathcal{H}$-linear. The proposition is proved.

10.8. We now fix an opposition $\varpi : \mathfrak{g} \to \mathfrak{g}$ such that $\varpi = -1$ on the Lie algebra of T. (Note that ϖ is uniquely determined up to composition with $Ad(t)$ for some $t \in T$.)

If $F \in \mathrm{Coh}_{\mathcal{T}}(\mathcal{B})$ (resp. $F \in \mathrm{Coh}_{\mathcal{T}}(\Lambda)$), then the inverse image $\varpi^* F$ under the involution $\varpi : \mathcal{B} \to \mathcal{B}$ (resp. $\varpi : \Lambda \to \Lambda$) given by $\mathfrak{b} \mapsto \varpi(\mathfrak{b})$ (resp. by $(y, \mathfrak{b}) \mapsto (\varpi(y), \varpi(\mathfrak{b}))$) is naturally an object of $\mathrm{Coh}_{\mathcal{T}}(\mathcal{B})$ (resp. $\mathrm{Coh}_{\mathcal{T}}(\Lambda)$). We obtain an involution $F \mapsto \varpi^*(F)$ of $K_{\mathcal{T}}(\mathcal{B})$ (resp. of $K_{\mathcal{T}}(\Lambda)$) denoted by ϖ^*. The involutions ϖ^* on $K_{\mathcal{T}}(\mathcal{B})$ and $K_{\mathcal{T}}(\Lambda)$ correspond to each other under the identification $K_{\mathcal{T}}(\mathcal{B}) = K_{\mathcal{T}}(\Lambda)$ as $\mathcal{A}$-modules in 10.4. From the definitions we see that in the $K_{\mathcal{G}}(Z)$-module structures of $K_{\mathcal{T}}(\mathcal{B})$ and $K_{\mathcal{T}}(\Lambda)$ (in 10.3) we have $\varpi^*(\chi\xi) = \varpi^*(\chi)\varpi^*(\xi)$ for $\chi \in K_{\mathcal{G}}(Z)$ and ξ in $K_{\mathcal{T}}(\mathcal{B})$ or $K_{\mathcal{T}}(\Lambda)$. Moreover, from the definitions, we see that in the natural $R_{\mathcal{T}} = \mathcal{A}\mathcal{X}$-module structures of $K_{\mathcal{T}}(\mathcal{B})$ and $K_{\mathcal{T}}(\Lambda)$ (in 10.4) we have $\varpi^*([x] \centerdot \xi) = [-x] \centerdot \varpi^*(\xi)$ for $x \in \mathcal{X}$ and ξ in $K_{\mathcal{T}}(\mathcal{B})$ or $K_{\mathcal{T}}(\Lambda)$. (This comes from the fact that $\varpi = -1$ on the Lie algebra of T.) It follows that the involution ϖ^* on $K_{\mathcal{T}}(\mathcal{B})$ or $K_{\mathcal{T}}(\Lambda)$ takes

(a) $[x] \centerdot L$ to $[-x] \centerdot {}^{w_0}L^{-1}$ for any $x \in \mathcal{X}, L \in \mathbf{X}$.

(It suffices to check this for $x = 0, L = \mathbf{C}$ where it is obvious.) Hence, this involution corresponds to the involution $\diamond : \mathcal{A}\mathcal{X}^{\otimes 2} \to \mathcal{A}\mathcal{X}^{\otimes 2}$ in 5.4, under the isomorphism $g : \mathcal{A}\mathcal{X}^{\otimes 2} \xrightarrow{\sim} K_{\mathcal{T}}(\mathcal{B})$ or $\tilde{g} : \mathcal{A}\mathcal{X}^{\otimes 2} \xrightarrow{\sim} K_{\mathcal{T}}(\Lambda)$ in 10.6.

10.9. Consider the $R_{\mathcal{T}}$-bilinear pairing $K_{\mathcal{T}}(\mathcal{B}) \times K_{\mathcal{T}}(\Lambda) \to R_{\mathcal{T}}$ given by

$$(F : F') = \pi_*(F \otimes^L_\Lambda F').$$

(The Tor-product is relative to the smooth $\mathcal{T}$-variety Λ and its closed subvarieties $\mathcal{B}, \Lambda$ with intersection $\mathcal{B}$; π is the map from $\mathcal{B}$ to the point.)

Lemma 10.10. *Under the isomorphisms $g : \mathcal{A}\mathcal{X}^{\otimes 2} \xrightarrow{\sim} K_{\mathcal{T}}(\mathcal{B}), \tilde{g} : \mathcal{A}\mathcal{X}^{\otimes 2} \xrightarrow{\sim} K_{\mathcal{T}}(\Lambda)$ in 10.6 (that is, $[x] \otimes [x'] \mapsto [x'] \centerdot L_{x'}$) and $R_{\mathcal{T}} \xrightarrow{\sim} \mathcal{A}\mathcal{X}$ in 10.3*(e)*, the pairing in 10.9 corresponds to the pairing $(:) : \mathcal{A}\mathcal{X}^{\otimes 2} \times \mathcal{A}\mathcal{X}^{\otimes 2} \to \mathcal{A}\mathcal{X}$ in 5.1.*

Since both pairings are $\mathcal{A}\mathcal{X} = R_{\mathcal{T}}$-bilinear, it suffices to show that, for any $x, x' \in \mathcal{X}$, we have

$$(L_x : L_{x'}) = (x, x')$$

where (x, x') is as in 5.1. Using 6.7, we have $(L_x : L_{x'}) = \pi_*(L_x \otimes L_{x'})$ where L_x is regarded as a line bundle on $\mathcal{B}$ in both sides, while $L_{x'}$ is regarded as a line bundle on Λ in the left hand side and as a line bundle on $\mathcal{B}$, in the right hand side. Setting $\tilde{x} = x + x'$, we see that it is enough to show that

$$\pi_*(L_{\tilde{x}}) = \delta^{-1} \sum_{w \in W} \mathrm{sgn}_w {}^w[\tilde{x} + \rho]$$

for any $\tilde{x} \in \mathcal{X}$. (The left hand side is in $R_{\mathcal{T}} = \mathcal{A}\mathcal{X}$, the right hand side is in $\mathcal{A}\mathcal{X}^W$.) This follows from Weyl's character formula. The lemma is proved.

10.11. Let $k : \mathcal{B} \to \Lambda$ be the imbedding $\mathfrak{b} \mapsto (0, \mathfrak{b})$. As we have remarked earlier, the map $k_* : K_{\mathcal{T}}(\mathcal{B}) \to K_{\mathcal{T}}(\Lambda)$ corresponds, under the isomorphisms $g : \mathcal{A}\mathcal{X}^{\otimes 2} \xrightarrow{\sim} K_{\mathcal{T}}(\mathcal{B}), \tilde{g} : \mathcal{A}\mathcal{X}^{\otimes 2} \xrightarrow{\sim} K_{\mathcal{T}}(\Lambda)$ in 10.6 to the map $\mathcal{A}\mathcal{X}^{\otimes 2} \to \mathcal{A}\mathcal{X}^{\otimes 2}$ given by multiplication by $(\bar{\Delta} \otimes 1)$.

Consider the pairing $(,)_{\mathcal{B}} : K_{\mathcal{T}}(\mathcal{B}) \times K_{\mathcal{T}}(\mathcal{B}) \to R_{\mathcal{T}}$ defined by

(a) $$(F, F')_{\mathcal{B}} = (-v)^{-\nu} \pi_*(F \otimes^L_\Lambda (\tilde{T}_{w_0} \varpi^*(F'))).$$

(The Tor-product is relative to the smooth $\mathcal{T}$-variety Λ and its closed subvarieties $\mathcal{B},\mathcal{B}$ with intersection $\mathcal{B}$.)

Let $\mathfrak{R}$ be the quotient field of $R_{\mathcal{T}}$. The pairing (a) is $R_{\mathcal{T}}$-linear in the first variable and is $R_{\mathcal{T}}$-semilinear in the second variable, with respect to the involution † of $R_{\mathcal{T}} = \mathcal{A}\mathcal{X}$. Hence, that pairing extends naturally to a pairing

$$(\mathfrak{R} \otimes_{R_{\mathcal{T}}} K_{\mathcal{T}}(\mathcal{B})) \times (\mathfrak{R} \otimes_{R_{\mathcal{T}}} K_{\mathcal{T}}(\mathcal{B})) \to \mathfrak{R}.$$

Composing this with the map

$$K_{\mathcal{T}}(\Lambda) \times K_{\mathcal{T}}(\Lambda) \to (\mathfrak{R} \otimes_{R_{\mathcal{T}}} K_{\mathcal{T}}(\mathcal{B})) \times (\mathfrak{R} \otimes_{R_{\mathcal{T}}} K_{\mathcal{T}}(\mathcal{B}))$$

given by $k_*^{-1} \times k_*^{-1}$ (recall that k_* is an isomorphism over $\mathfrak{R}$) we obtain a pairing

(b) $$(,)_\Lambda : K_{\mathcal{T}}(\Lambda) \times K_{\mathcal{T}}(\Lambda) \to \mathfrak{R}.$$

Let $\tilde{\mathfrak{U}}$ be the set of all elements $\tilde{p} \in \mathfrak{U}$ (see 3.8) such that $p\tilde{p} \in R_{\mathcal{T}}$ for some $p \in R_{\mathcal{T}} - \{0\}$. Then $\tilde{\mathfrak{U}}$ may be regarded as a subring of either $\mathfrak{U}$ or of the field $\mathfrak{R}$. From the definitions we see that $(,)_\Lambda$ takes values in the subring $\tilde{\mathfrak{U}}$ of $\mathfrak{R}$.

Let $(,)_{Z_0} : K_{\mathcal{T}}(Z_0) \times K_{\mathcal{T}}(Z_0) \to \mathfrak{R}$ be the composition $K_{\mathcal{T}}(Z_0) \times K_{\mathcal{T}}(Z_0) \xrightarrow{\tilde{j}'_* \times \tilde{j}'_*} K_{\mathcal{T}}(\Lambda) \times K_{\mathcal{T}}(\Lambda) \xrightarrow{(,)_\Lambda} \mathfrak{R}$.

Lemma 10.12. (a) *Under the isomorphisms $g : \mathcal{A}\mathcal{X}^{\otimes 2} \xrightarrow{\sim} K_{\mathcal{T}}(\mathcal{B})$ in 10.6 and $R_{\mathcal{T}} \xrightarrow{\sim} \mathcal{A}\mathcal{X}$ in 10.3*(e)*, the pairing 10.11*(a) *corresponds to the pairing $(|)_{d'} : \mathcal{A}\mathcal{X}^{\otimes 2} \times \mathcal{A}\mathcal{X}^{\otimes 2} \to \mathcal{A}\mathcal{X}$ in 5.8.*

(b) *Under the isomorphisms $\tilde{g} : \mathcal{A}\mathcal{X}^{\otimes 2} \xrightarrow{\sim} K_{\mathcal{T}}(\Lambda)$ in 10.6 and $\mathfrak{R} \xrightarrow{\sim} \mathcal{A}\mathcal{X}_q$ induced by 10.3*(e)*, the pairing 10.11*(b) *corresponds to the pairing $(|)_d : \mathcal{A}\mathcal{X}^{\otimes 2} \times \mathcal{A}\mathcal{X}^{\otimes 2} \to \mathcal{A}\mathcal{X}_q$ in 5.8.*

(c) *If $\xi, \xi' \in K_{\mathcal{T}}(Z_0)$, then $\partial(\xi,\xi')_{Z_0} \in \mathcal{A}$.*

Note that $F \otimes^L_\Lambda (\tilde{T}_{w_0}\varpi^*(F'))$ in 10.11(a) is equal to $F \otimes^L_\Lambda k_*(\tilde{T}_{w_0}\varpi^*(F'))$ where the last $\otimes^L_\Lambda$ is relative to the smooth variety Λ and its closed subvarieties $\mathcal{B},\Lambda$. Hence, (a) follows from the identifications done earlier in this section. Now (b) follows from (a) using 5.9. Now (c) follows from (a) and the commutative diagram 10.6, since $\partial(m|m') \in \mathcal{A}$ for any $m,m' \in \mathcal{M}_c$ (see 3.4). The lemma is proved.

Lemma 10.13. *Let $L \in \mathbf{X}, x \in \mathcal{X}, n \in \mathbf{Z}$. We have*

$$D_{\mathcal{B}}(v^n[x] \centerdot L) = (-1)^\nu v^{-n}[-x] \centerdot L^{-1}L_{-2\rho} \in K_{\mathcal{T}}(\mathcal{B}),$$

$$D_\Lambda(v^n[x] \centerdot L) = v^{2\nu} v^{-n}[-x] \centerdot L^{-1} \in K_{\mathcal{T}}(\Lambda).$$

This follows from 6.13, 6.16 since $\mathcal{B},\Lambda$ are smooth. (We also use 9.4 and the equality $\Omega_{\mathcal{B}} = L_{-2\rho}$.)

Proposition 10.14. *Let ϖ^* be the involution of $K_{\mathcal{T}}(\mathcal{B})$ or $K_{\mathcal{T}}(\Lambda)$ described in 10.8.*

(a) *$\varpi^* : K_{\mathcal{T}}(\mathcal{B}) \to K_{\mathcal{T}}(\mathcal{B})$ commutes with $\tilde{T}_{w_0} : K_{\mathcal{T}}(\mathcal{B}) \to K_{\mathcal{T}}(\mathcal{B})$ and with $D_{\mathcal{B}} : K_{\mathcal{T}}(\mathcal{B}) \to K_{\mathcal{T}}(\mathcal{B})$. Moreover, $D_{\mathcal{B}}$ is $\mathcal{H}$-antilinear.*

(b) *$\varpi^* : K_{\mathcal{T}}(\Lambda) \to K_{\mathcal{T}}(\Lambda)$ commutes with $\tilde{T}_{w_0} : K_{\mathcal{T}}(\Lambda) \to K_{\mathcal{T}}(\Lambda)$ and with $D_\Lambda : K_{\mathcal{T}}(\Lambda) \to K_{\mathcal{T}}(\Lambda)$. Moreover, D_Λ is $\mathcal{H}$-antilinear.*

(c) *The map $(-v)^{-\nu}\tilde{T}_{w_0}^{-1}\varpi^* D_{\mathcal{B}} : K_{\mathcal{T}}(\mathcal{B}) \to K_{\mathcal{T}}(\mathcal{B})$ corresponds under $fg : K_{\mathcal{T}}(\mathcal{B}) \xrightarrow{\sim} \mathcal{M}_{d'}$ (see 10.6) to $\hat{b} : \mathcal{M}_{d'} \to \mathcal{M}_{d'}$.*

(d) *The map* $(-v)^{\nu}\tilde{T}_{w_0}^{-1}\varpi^* D_\Lambda : K_{\mathcal{T}}(\Lambda) \to K_{\mathcal{T}}(\Lambda)$ *corresponds under* $\tilde{f}\tilde{g} : K_{\mathcal{T}}(\Lambda) \xrightarrow{\sim} \mathcal{M}_d$ *(see 10.6) to* $\hat{\hat{b}} : \mathcal{M}_d \to \mathcal{M}_d$.

Let $L \in \mathbf{X}, x \in \mathcal{X}, n \in \mathbf{Z}$. We have (in $K_{\mathcal{T}}(\mathcal{B})$):

$$(-v)^{-\nu}\tilde{T}_{w_0}^{-1}\varpi^* D_{\mathcal{B}}(v^n[x] \centerdot L) = (-v)^{-\nu}(-1)^{\nu} v^{-n}\tilde{T}_{w_0}^{-1}\varpi^*([-x] \centerdot L^{-1}L_{-2\rho})$$
$$= v^{-\nu}v^{-n}\tilde{T}_{w_0}^{-1}([x] \centerdot ({}^{w_0}LL_{-2\rho})).$$

The first equality holds by 10.13. The second equality holds by 10.8(a). This shows that our map corresponds under $g : K_{\mathcal{T}}(\mathcal{B}) \xrightarrow{\sim} \mathcal{A}\mathcal{X}_{d'}^{\otimes 2}$ to the map $\hat{b}$ in 4.9. Hence, (c) follows from 4.9. We have (in $K_{\mathcal{T}}(\Lambda)$):

$$(-v)^{\nu}\tilde{T}_{w_0}^{-1}\varpi^* D_\Lambda(v^n[x] \centerdot L) = (-v)^{\nu}v^{2\nu}v^{-n}\tilde{T}_{w_0}^{-1}\varpi^*([-x] \centerdot L^{-1})$$
$$= (-1)^{\nu}v^{3\nu}v^{-n}\tilde{T}_{w_0}^{-1}([x] \centerdot {}^{w_0}L).$$

The first equality holds by 10.13. The second equality holds by 10.8(a). This shows that our map corresponds under $\tilde{g} : K_{\mathcal{T}}(\Lambda) \xrightarrow{\sim} \mathcal{A}\mathcal{X}_d^{\otimes 2}$ to the map $\hat{\hat{b}}$ in 4.9. Hence, (d) follows from 4.9.

We prove (a). The first assertion of (a) follows from 5.4(a) (using 10.8). The second assertion of (a) follows from the definitions using 6.12. The third assertion of (a) follows from (c), using the fact that $\hat{b}$ is $\mathcal{H}$-antilinear.

The proof of (b) is entirely similar to that of (a). The proposition is proved.

10.15. Let

$$\mathbf{B}_{\mathcal{B}}^{\pm} = \{\xi \in K_{\mathcal{T}}(\mathcal{B}) | (-v)^{-\nu}\tilde{T}_{w_0}^{-1}\varpi^* D_{\mathcal{B}}(\xi) = \xi, \quad \partial(\xi,\xi)_{\mathcal{B}} \in 1 + v^{-1}\mathbf{Z}[v^{-1}]\},$$
$$\mathbf{B}_{\Lambda}^{\pm} = \{\xi \in K_{\mathcal{T}}(\Lambda) | (-v)^{\nu}\tilde{T}_{w_0}^{-1}\varpi^* D_{\Lambda}(\xi) = \xi, \quad \partial(\xi,\xi)_{\Lambda} \in 1 + v^{-1}\mathbf{Z}[[v^{-1}]]\},$$
$$\mathbf{B}_{Z_0}^{\pm} = \{\xi \in K_{\mathcal{T}}(Z_0) | \partial(\xi,\xi)_{Z_0} = 1\}.$$

Here $(,)_{\mathcal{B}}, (,)_{\Lambda}, ()_{Z_0}$ are as in 10.11 and ∂ is as in 3.13. (Note that $\partial(\xi,\xi)_\Lambda$ is well defined since $(\xi,\xi)_\Lambda \in \tilde{\mathfrak{U}} \subset \mathfrak{U}$, see 10.11.)

Proposition 10.16. (a) $\mathbf{B}_{\mathcal{B}}^{\pm}$ *is a signed basis of the* $\mathcal{A}$*-module* $K_{\mathcal{T}}(\mathcal{B})$*. Under the isomorphism* $fg : K_{\mathcal{T}}(\mathcal{B}) \xrightarrow{\sim} \mathcal{M}_{d'}$ *(see 10.6),* $\mathbf{B}_{\mathcal{B}}^{\pm}$ *corresponds to the signed basis* $\{\pm_\iota B^\flat | \iota \in \underline{\mathcal{X}}, B \in X\}$ *of the* $\mathcal{A}$*-module* $\mathcal{M}_{d'}$.

(b) $\mathbf{B}_{\Lambda}^{\pm}$ *is a signed basis of the* $\mathcal{A}$*-module* $K_{\mathcal{T}}(\Lambda)$*. Under the isomorphism* $\tilde{f}\tilde{g} : K_{\mathcal{T}}(\Lambda) \xrightarrow{\sim} \mathcal{M}_d$ *(see 10.6),* $\mathbf{B}_{\Lambda}^{\pm}$ *corresponds to the signed basis* $\{\pm_\iota B^\sharp | \iota \in \underline{\mathcal{X}}, B \in X\}$ *of the* $\mathcal{A}$*-module* $\mathcal{M}_d$.

(c) $\mathbf{B}_{Z_0}^{\pm}$ *is a signed basis of the* $\mathcal{A}$*-module* $K_{\mathcal{T}}(Z_0)$*. Under the isomorphism* $f'g' : K_{\mathcal{T}}(Z_0) \xrightarrow{\sim} \mathcal{M}_c$ *(see 10.6),* $\mathbf{B}_{Z_0}^{\pm}$ *corresponds to the signed basis* $\{\pm_\iota B | \iota \in \underline{\mathcal{X}}, B \in X\}$ *of the* $\mathcal{A}$*-module* $\mathcal{M}_c$.

Recall that in 3.14, the signed basis $\{\pm_\iota B^\flat | \iota \in \underline{\mathcal{X}}, B \in X\}$ of the $\mathcal{A}$-module $\mathcal{M}_{d'}$ has been characterized in terms of an inner product $(|)$ on $\mathcal{M}_{d'}$ and an antilinear map $\hat{b} : \mathcal{M}_{d'} \to \mathcal{M}_{d'}$. Under the isomorphism $fg : K_{\mathcal{T}}(\mathcal{B}) \xrightarrow{\sim} \mathcal{M}_{d'}$, the inner product $(|)$ on $\mathcal{M}_{d'}$ corresponds to the inner product $(,)_{\mathcal{B}}$ on $K_{\mathcal{T}}(\mathcal{B})$ (by 10.12, 5.15) and the antilinear map $\hat{b} : \mathcal{M}_{d'} \to \mathcal{M}_{d'}$ corresponds to $(-v)^{-\nu}\tilde{T}_{w_0}^{-1}\varpi^* D_{\mathcal{B}} : K_{\mathcal{T}}(\mathcal{B}) \to K_{\mathcal{T}}(\mathcal{B})$ (by 10.14(c)). Hence, the second assertion of (a) is proved. The first assertion of (a) clearly follows from the second assertion of (a). Thus, (a) is proved. The proof of (b) and (c) is entirely similar. The proposition is proved.

10.17. Let $X_{\min}$ be the set of all $a \in \mathcal{X}$ such that $a+\epsilon$ is contained in the closure of A_ϵ^+. Note that $x \mapsto \underline{x}$ is a bijection $X_{\min} \xrightarrow{\sim} \underline{\mathcal{X}}$.

Lemma 10.18. *Let $a \in X_{\min}$. Let $\tau \in W^a$ be such that $\tau A_\epsilon^+ = A_{a+\epsilon}^+$. Let $l = l(\tau)$. We have $\theta_{-a}({}_0e_\epsilon) = v^{-l}{}_ae_\epsilon, \quad \theta_{-a}({}_0\tilde{e}_\epsilon) = (-v)^l{}_a\tilde{e}_\epsilon$.*

From the definitions, for any $\tau \in W^a$, any $x \in \mathcal{X}$ and any $A \in X$, we have ${}^{\underline{x}}\tau(x+A) = x+\tau(B)$.

We apply this with $\tau \in W^a$ such that $\tau A_\epsilon^+ = A_{a+\epsilon}^+$, $x=-a$ and $A = A_\epsilon^+$. We see that

(a) ${}^{-\underline{a}}\tau(A_{-a+\epsilon}^+) = A_\epsilon^+$.

Our assumption on a implies that ϵ is in the closure of $A_{-a+\epsilon}$. This, together with (a) implies that

(b) ${}^{-\underline{a}}\tau \in W_\epsilon$.

By definition we have $\theta_{-a} = \tilde{T}^{-1}_{(-\underline{a})\tau} = \tilde{T}_\tau^{-1}\tilde{T}_{\underline{a}}$. Hence, if m is either e_ϵ or $\tilde{e}_\epsilon$, we have $\theta_{-a}({}_0m) = \tilde{T}_\tau^{-1}\tilde{T}_{\underline{a}}({}_0m) = \tilde{T}_\tau^{-1}({}_am) = {}_a((\tilde{T}_{{}^{-\underline{a}}\tau})m)$. We now use (b) and the fact that $\tilde{T}_we_\epsilon = v^{l(w)}e_\epsilon$, $\tilde{T}_w\tilde{e}_\epsilon = (-v)^{-l(w)}\tilde{e}_\epsilon$ for any $w \in W_\epsilon^a$. The lemma follows.

Lemma 10.19. *We preserve the notation in 10.18. Let $x' \in \mathcal{X}$. We have*

(a) $v^{l-\nu}[x'] \centerdot L_{-\rho-a} \in \mathbf{B}_{\mathcal{B}}^\pm$;
(b) $(-1)^{\nu+l}v^{2\nu-l}[x'] \centerdot L_{-a} \in \mathbf{B}_\Lambda^\pm$.

We prove (a). Let f be as in 4.4. We have

$$f(v^{l-\nu}[-\rho-a]\otimes[x']) = v^l\theta_{-a}([x'+\rho] \centerdot {}_0e_\epsilon) = [x'+\rho] \centerdot {}_ae_\epsilon.$$

We prove (b). Let $\tilde{f}$ be as in 4.5. We have

$$\begin{aligned}&\tilde{f}((-1)^{\nu+l}v^{2\nu-l}[-a] \centerdot [x'])\\ &\quad = (-1)^\nu v^{-2\nu}(-1)^{l+\nu}v^{2\nu-l}\theta_{-a}([x'+2\rho] \centerdot {}_0\tilde{e}_\epsilon) = [x'+2\rho] \centerdot {}_a\tilde{e}_\epsilon.\end{aligned}$$

The lemma follows.

10.20. In this and the next subsection we assume that $G = SL_3$ with $I = \{1,2\}$. Consider ${}_0e_{a+\epsilon}$ where $\check{\alpha}_1(a) = 0, \check{\alpha}_2(a) = 1$. Let $B = \sigma_2(A_{a+\epsilon}^+)$. We have ${}_0B^\flat = (\tilde{T}_{\sigma_2} + v^{-1})({}_0e_{a+\epsilon}^+)$. We want to describe the corresponding element of $\mathbf{B}_{\mathcal{B}}^\pm$.

By 10.19, $[a] \centerdot {}_ae_\epsilon = {}_0e_{a+\epsilon}$ corresponds to $v^{2-3}[-\rho-a]\otimes[a-\rho] \in \mathcal{A}\mathcal{X}_{d'}^{\otimes 2}$. Hence, ${}_0B^\flat$ corresponds to

(a) $v^{-1}(\tilde{T}_{\sigma_2}+v^{-1})[-\rho-a]\otimes[a-\rho]$.

We have

$$\begin{aligned}&\frac{v^{-1}[-\rho+\alpha_2-a+\alpha_2]-v^{-1}[-\rho-a+\alpha_2]+v[-\rho-a+\alpha_2]-v[-\rho+\alpha_2-a]}{[\alpha_2]-1}\\ &= v^{-1}[-\rho+\alpha_2-a].\end{aligned}$$

Hence, (a) equals $v^{-2}[-\rho+\alpha_2-a]\otimes[a-\rho]+v^{-2}[\rho-a]\otimes[a-\rho]$. This corresponds to a direct sum of two line bundles on $\mathcal{B}$.

10.21. Let s_0 be the unique element of $S - S_\epsilon$. Let $B = s_0(A^+_\epsilon)$. Then ${}_0B^\flat = (\tilde{T}_{s_0} + v^{-1})({}_0e_\epsilon)$. We want to describe the corresponding element of $\mathbf{B}^\pm_{\mathcal{B}}$.

We have $\theta_\rho = \tilde{T}_{\sigma_1}\tilde{T}_{\sigma_2}\tilde{T}_{\sigma_2}\tilde{T}_{s_0}$. Hence, $\tilde{T}_{s_0} = \tilde{T}^{-1}_{\sigma_1}\tilde{T}^{-1}_{\sigma_2}\tilde{T}^{-1}_{\sigma_2}\theta_\rho$. We compute (in M):

$$(\tilde{T}_{s_0} + v^{-1})v^{-\nu}[-\rho] \otimes [-\rho] = \tilde{T}^{-1}_{\sigma_1}\tilde{T}^{-1}_{\sigma_2}\tilde{T}^{-1}_{\sigma_2}\theta_{2\rho}v^{-\nu}[-\rho] \otimes [-\rho] + v^{-1}v^{-\nu}[-\rho] \otimes [-\rho]$$
$$= \tilde{T}^{-1}_{\sigma_1}\tilde{T}^{-1}_{\sigma_2}\tilde{T}^{-1}_{\sigma_2}v^{-3}[0] \otimes [-\rho] + v^{-1}v^{-3}[-\rho] \otimes [-\rho]$$
$$= -v^{-1}[-\rho] \otimes [-\rho]v^{-3} + v[-\rho] \otimes [-\rho]v^{-3} + v^3[-2\rho] \otimes [-\rho]v^{-3}$$
$$+ v^{-1}v^{-3}[-\rho] \otimes [-\rho] = v^{-2}[-\rho] \otimes [-\rho] + [-2\rho] \otimes [-\rho].$$

This corresponds to a direct sum of two line bundles on $\mathcal{B}$.

10.22. Let H be as in 10.1. Let $\mathfrak{Z}$ be a closed H-stable subvariety of Λ. Let $i \in I$. We say that $\mathfrak{Z}$ is *i-saturated* if the following holds:

$$(y, \mathfrak{b}') \in \mathfrak{Z}, (y, \mathfrak{b}) \in \Lambda, (\mathfrak{b}, \mathfrak{b}') \in \mathcal{O}_{\sigma_i} \implies (y, \mathfrak{b}) \in \mathfrak{Z}.$$

(Compare 8.1.) Assume that $y \in \mathfrak{g}_n$, that H is contained in the stabilizer of y in $\mathcal{G}$ and that $\mathfrak{Z}$ is an i-saturated closed H-stable subvariety of $\{y\}^{\cdot}$ (see 10.1). As in 8.1 we see that

(a) the operator $\tilde{T}_{\sigma_i} : K_H(\{y\}^{\cdot}) \to K_H(\{y\}^{\cdot})$ (see 10.1) maps into itself the image of $K_H(\mathfrak{Z}) \to K_H(\{y\}^{\cdot})$ (direct image under the imbedding $\mathfrak{Z} \to \{y\}^{\cdot}$).

11. Study of $K_H(\mathcal{B}_e), K_H(\Lambda_e)$

11.1. In this section we fix an $\mathfrak{sl}_2$-triple (e, f, h) in $\mathfrak{g}$, that is, three elements (e, f, h) of $\mathfrak{g}$ such that $[h, e] = 2e, [h, f] = -2f, [e, f] = h$. Let $\zeta : SL_2 \to G$ be the homomorphism of algebraic groups whose tangent map at 1 carries $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ to e, $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ to f, $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ to h. Let

$$\mathfrak{S} = \{g \in G | Ad(g)(e) = e, Ad(g)(f) = f, Ad(g)(h) = h\},$$
$$\tilde{\mathfrak{S}} = \{(g, \lambda) \in \mathcal{G} | Ad(g)(e) = \lambda^2 e, Ad(g)(f) = \lambda^{-2} f, Ad(g)(h) = h\}.$$

These are closed, reductive subgroups of $G, \mathcal{G}$ respectively. The map

$$(g, \lambda) \mapsto (g\zeta\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right), \lambda)$$

is an isomorphism of algebraic groups $\mathfrak{S} \times \mathbf{C}^* \xrightarrow{\sim} \tilde{\mathfrak{S}}$. Let C be a maximal torus of $\mathfrak{S}$ and let $H = C \times \mathbf{C}^*$. We will identify H with a subgroup of $\tilde{\mathfrak{S}}$ (a maximal torus) via

$$(c, \lambda) \mapsto (c\zeta\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right), \lambda).$$

Let $\mathfrak{s} = \{y \in \mathfrak{g} | [y, e] = [y, f] = [y, h] = 0\}$. This is a reductive Lie algebra. Let $\mathfrak{c}$ be the Lie algebra of C (a Cartan subalgebra of $\mathfrak{s}$). Let $\mathfrak{l}$ be the centralizer of $\mathfrak{c}$ in $\mathfrak{g}$. Clearly,

(a) $e \in \mathfrak{l}, f \in \mathfrak{l}, h \in \mathfrak{l}$.

11.2. Let

$$\mathcal{B}_e = \{\mathfrak{b} \in \mathcal{B} | e \in \mathfrak{b}\}.$$

Note that $\mathcal{B}_e$ may be identified with $\{e\}^{\cdot}$ (see 10.1) by $\mathfrak{b} \mapsto (e, \mathfrak{b})$.

Let $\mathfrak{z}(f)$ the centralizer of f in $\mathfrak{g}$ and let

$$\Sigma_e = \{y \in \mathfrak{g}_n | y - e \in \mathfrak{z}(f)\}, \quad \Lambda_e = \dot{\Sigma}_e = \{(y, \mathfrak{b}) \in \Lambda | y \in \Sigma_e\}.$$

According to Slodowy (see [Sl]), Λ_e is irreducible, smooth, of dimension $2 \dim \mathcal{B}_e$.

Note that $\{e\}$ and Σ_e are $\tilde{\mathfrak{S}}$-stable subvarieties of $\mathfrak{g}_n$. Hence, $\mathcal{B}_e$ and Λ_e are $\tilde{\mathfrak{S}}$-stable subvarieties of Λ. In particular, $\mathcal{B}_e$ and Λ_e are H-stable subvarieties of Λ.

11.3. Consider the action of $\mathbf{C}^*$ (a subgroup of H via $\lambda \mapsto (1, \lambda)$) on $\mathcal{B}_e$ and Λ_e. These actions on $\mathcal{B}_e, \Lambda_e$ have the same fixed point set:

$$\mathcal{B}_e^{\mathbf{C}^*} = \Lambda_e^{\mathbf{C}^*} = \{(0, \mathfrak{b}) \in \Lambda | e \in \mathfrak{b}, h \in \mathfrak{b}\}.$$

This fixed point set is smooth, since Λ_e is smooth and is a projective variety, since $\mathcal{B}_e$ is a projective variety.

For any connected component μ of $\mathcal{B}_e^{\mathbf{C}^*}$, let $\mathcal{B}_{e,\mu}$ be the set of all $\mathfrak{b} \in \mathcal{B}_e$ such that $\lim_{t \mapsto 0} Ad\zeta \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right) \mathfrak{b}$ belongs to μ. The limit above is denoted by $\pi_\mu(\mathfrak{b})$.

For any connected component μ of $\mathcal{B}_e^{\mathbf{C}^*} = \Lambda_e^{\mathbf{C}^*}$, let $\Lambda_{e,\mu}$ be the set of all $(y, \mathfrak{b}) \in \Lambda_e$ such that $\lim_{t \mapsto \infty}(\lambda^{-2} Ad\zeta \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right) y, Ad\zeta \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right) \mathfrak{b})$ is defined and belongs to μ. The limit above is denoted by $\pi'_\mu(y, \mathfrak{b})$.

Lemma 11.4. *Let μ be a connected component μ of $\mathcal{B}_e^{\mathbf{C}^*}$.*

(a) *$\mathcal{B}_{e,\mu}$ is a smooth subvariety of $\mathcal{B}_e$ and $\pi_\mu : \mathcal{B}_{e,\mu}^{\mathbf{C}^*} \to \mu$ is naturally a vector bundle.*
(b) *$\Lambda_{e,\mu}$ is a smooth subvariety of Λ_e and $\pi'_\mu : \Lambda_{e,\mu}^{\mathbf{C}^*} \to \mu$ is naturally a vector bundle.*
(c) *The subvarieties $\mathcal{B}_{e,\mu}$ (resp. $\Lambda_{e,\mu}$) for various μ as above form an α-partition (see* [DLP, 1.3]*) of $\mathcal{B}_e$ (resp. Λ_e).*

The assertions relative to $\mathcal{B}_{e,\mu}$ are proved in [DLP]. The assertions relative to $\Lambda_{e,\mu}$ are proved in [KL2, 4.6] using Hironaka's theorem on resolutions of singularities.

Lemma 11.5. *The R_H-modules $K_H(\mu), K_H(\mathcal{B}_{e,\mu}), K_H(\Lambda_{e,\mu})$ have finite rank.*

From 11.4 we see that it suffices to prove the assertion concerning $K_H(\mu)$. Since, μ is smooth, projective, it can be partitioned (Bialynicky-Birula) into locally closed H-stable pieces which are vector bundles over the various components of the fixed point set of H. It is then enough to show that for each of these pieces, the corresponding K_H has finite rank over R_H, or that, for any connected component of the fixed point set, the corresponding K_H has finite rank over R_H. Since H acts trivially on that component, it is enough to show that the non-equivariant K group of the component is a finitely generated abelian group. This follows from [DLP, 3.9]. The lemma is proved.

Lemma 11.6. *The R_H-modules $K_H(\mathcal{B}_e), K_H(\Lambda_e)$ have finite rank.*

This follows from 11.5 using the α-partitions $\mathcal{B}_e = \bigcup_\mu \mathcal{B}_{e,\mu}, \Lambda_e = \bigcup_\mu \Lambda_{e,\mu}$.

11.7. Assume now that e is a regular nilpotent element of $\mathfrak{l}$. Then $\mathcal{B}_e^{C \times \{1\}}$ is a finite set; hence the fixed point set of H in the proof of 11.5 is finite. In this case, the argument in 11.5 shows that the R_H-modules $K_H(\mu), K_H(\mathcal{B}_{e,\mu}), K_H(\Lambda_{e,\mu})$ are free (of finite rank) and then the argument in 11.6 shows that the R_H-modules $K_H(\mathcal{B}_e), K_H(\Lambda_e)$ are free (of finite rank).

The same holds in the case where G is a classical group and e is arbitrary since in this case, the fixed point sets in the proof of 11.5 admit algebraic cell decompositions (see [DLP, 3.9]).

We return to the general case. It is likely that the R_H-modules above are again free. Moreover, it should be possible to deduce this from the analysis in [DLP, 3.9].

Lemma 11.8. *Let $\mathfrak{R}$ be the quotient field of R_H. The direct image maps*

$$K_H(\mathcal{B}_e^{\mathbf{C}^*}) \to K_H(\mathcal{B}_e), \quad K_H(\Lambda_e^{\mathbf{C}^*}) \to K_H(\Lambda_e), \quad K_H(\mathcal{B}_e) \mapsto K_H(\Lambda_e)$$

induced by the inclusions $\mathcal{B}_e^{\mathbf{C}^*} \subset \mathcal{B}_e, \quad \Lambda_e^{\mathbf{C}^*} \subset \Lambda_e, \quad \mathcal{B}_e \subset \Lambda_e$, become isomorphisms after tensoring with $\mathfrak{R}$ over R_H.

It is enough to consider the first two maps in the lemma (since $\mathcal{B}_e^{\mathbf{C}^*} = \Lambda_e^{\mathbf{C}^*}$). This follows from the concentration theorem [T2] applied to the $\mathbf{C}^*$-action on $\mathcal{B}_e, \Lambda_e$.

11.9. In the special cases mentioned in 11.7, we deduce from 11.8 that the map $K_H(\mathcal{B}_e) \mapsto K_H(\Lambda_e)$ is injective. Again, this should be true in general.

Proposition 11.10. *Let $b(e) = \dim \mathcal{B}_e$. We have $\Omega_{\Lambda_e} = v^{2b(e)} \in \mathrm{Vec}_{\Lambda_e}$.*

We have a cartesian diagram

$$\begin{array}{ccc} G \times \Lambda_e & \longrightarrow & G \times (e + \mathfrak{z}(f)) \\ {\scriptstyle a}\Big\downarrow & & {\scriptstyle b}\Big\downarrow \\ \Lambda & \longrightarrow & \mathfrak{g} \end{array}$$

where a, b are given by $(g; y, \mathfrak{b}) \mapsto (Ad(g)y, Ad(g)\mathfrak{b}), \quad (g, y) \mapsto Ad(g)y$ and the lower horizontal map is $((y, \mathfrak{b}) \mapsto y$. According to [Sl], b is smooth; hence a must be also smooth. Moreover, if $(y, \mathfrak{b}) \in \Lambda_e$, the tangent space along the fibres of a at $(1; y, \mathfrak{b})$ can be identified with the tangent space along the fibres of b at $(1, y)$. Hence, $(\Omega_{G\times\Lambda_e})_{1,y,\mathfrak{b}} \otimes (\Omega_\Lambda)^*_{(y,\mathfrak{b})} = (\Omega_{G\times(e+\mathfrak{z}(f))})_{1,y} \otimes \Omega(\mathfrak{g})^*$ where * denotes a dual space. Since $\Omega_\Lambda = v^{2\nu}$ canonically, it follows that $\Omega(\mathfrak{g}) \otimes (\Omega_{\Lambda_e})_{y,\mathfrak{b}} = v^{2\nu} \otimes \Omega(\mathfrak{g}) \otimes (\Omega_{e+\mathfrak{z}(f)})_y \otimes \Omega(\mathfrak{g})^*$. We have $(\Omega_{e+\mathfrak{z}(f)})_y = \Omega(\mathfrak{z}(f))$. Hence, $\Omega_{\Lambda_e} = v^{2\nu} \otimes \Omega(\mathfrak{z}(f)) \otimes \Omega(\mathfrak{g})^* = v^{2\nu} \otimes \Omega(\mathfrak{g}/\mathfrak{z}(f))^*$. Now $\mathfrak{g}/\mathfrak{z}(f)$ has a non-degenerate symplectic form (Kirillov) $x, y \mapsto \langle x, [f, y]\rangle$. Here $\langle,\rangle$ is the Killing form on $\mathfrak{g}$. Hence, $\Omega(\mathfrak{g}/\mathfrak{z}(f)) = v^{2\dim \mathfrak{g}/\mathfrak{z}(f)}$ and $\Omega_{\Lambda_e} = v^{2\nu - 2\dim \mathfrak{g}/\mathfrak{z}(f)} = v^{2b(e)}$. The proposition is proved.

12. The involution $^-$ and inner product on $K_H(\mathcal{B}_e), K_H(\Lambda_e)$

12.1. Let $A(\mathfrak{g})$ be the group of automorphisms of the Lie algebra $\mathfrak{g}$. Let $O(\mathfrak{g})$ be the set of oppositions of $\mathfrak{g}$. Let $A^0(\mathfrak{g})$ be the identity component of $A(\mathfrak{g})$ and let $A^1(\mathfrak{g})$ be the connected component of $A(\mathfrak{g})$ that contains $O(\mathfrak{g})$.

We fix an $\mathfrak{sl}_2$-triple (e, f, h) in $\mathfrak{g}$ such that

$$y \in \mathfrak{g}, [y, e] = [y, f] = [y, h] = 0 \implies y = 0.$$

(That is, (e, f, h) is distinguished in $\mathfrak{g}$.) Let

$$R = \{\zeta \in A(\mathfrak{g}) | \zeta(e) = e, \zeta(f) = f, \zeta(h) = h\},$$
$$R' = \{\zeta \in A(\mathfrak{g}) | \zeta(e) = -e, \zeta(f) = -f, \zeta(h) = h\}.$$

For any $j \in \mathbf{Z}$ we set $\mathfrak{g}_j = \{y \in \mathfrak{g} | [h, y] = jy\}$. It is well known (Bala-Carter) that $\mathfrak{g} = \bigoplus_{j \in 2\mathbf{Z}} \mathfrak{g}_j$.

We attach to (e, f, h) a linear map $\kappa' : \mathfrak{g} \to \mathfrak{g}$ by $\kappa'(y) = (-1)^j y$ for $y \in \mathfrak{g}_j$. It is clear that $\kappa' \in R' \cap A^0(\mathfrak{g})$.

We also attach to (e, f, h) an automorphism $\kappa \in R \cap A^1(\mathfrak{g})$, as follows. Assume that this has been done when $\mathfrak{g}$ is simple. In the general case we write $\mathfrak{g}$ canonically as a direct sum of simple Lie algebra and (e, f, h) as a corresponding direct sum of $\mathfrak{sl}_2$-triples. We then take the direct sum of the automorphisms κ attached to the various simple components; this will be κ for $\mathfrak{g}$. We now assume that $\mathfrak{g}$ is simple.

In the case where w_0 is in the centre of W that is, when $A^1(\mathfrak{g}) = A^0(\mathfrak{g})$, or equivalently, in type $B_n, C_n, D_{2n}, G_2, F_4, E_7, E_8$, we define κ to be the identity automorphism.

In the case where $\mathfrak{g}$ is of type $D_{2n+1}, n \geq 2$, we consider the standard $4n+2$ dimensional representation E of $\mathfrak{g}$. As an $\mathfrak{sl}_2$-module it is canonically a direct sum of irreducible modules of distinct, odd dimensions. Consider the automorphism of E which equals -1 on the smallest of these irreducible $\mathfrak{sl}_2$-submodules and is $+1$ on all the others. This gives rise to an automorphism of $\mathfrak{g}$ that is denoted by κ. Note that $\kappa \in R \cap A^1(\mathfrak{g})$.

In the case where $\mathfrak{g}$ is of type $A_n, n \geq 2$, (so that e is regular nilpotent) we note that, as an $\mathfrak{sl}_2$-module, $\mathfrak{g}$ is a direct sum of irreducible modules of distinct dimensions: $3, 5, 7, \dots, 2n+1$. We define a linear map $\kappa : \mathfrak{g} \to \mathfrak{g}$ to be $+1$ on the submodules of dimension $3, 7, 11, \dots$ and to be -1 on the submodules of dimension $5, 9, 13, \dots$. Note that $\kappa \in R \cap A^1(\mathfrak{g})$.

Assume now that $\mathfrak{g}$ is of type E_6. If e is regular nilpotent, we note that, as an $\mathfrak{sl}_2$-module, $\mathfrak{g}$ is a direct sum of irreducible modules of distinct dimensions: $3, 9, 11, 15, 17, 23$. We define a linear map $\kappa : \mathfrak{g} \to \mathfrak{g}$ to be $+1$ on the submodules of dimension $3, 11, 15, 23$ and to be -1 on the submodules of dimension $9, 17$. If e is subregular nilpotent, we define κ to be the unique element of $R \cap A^1(\mathfrak{g})$. If e is neither regular nor subregular, then $R \cap A(\mathfrak{g})$ consists of four elements, all involutions. The three non-identity elements γ have fixed point set $\mathfrak{g}^\gamma$ of type $A_5 \times A_1, C_4, F_4$ respectively. (The first one is in $A^0(\mathfrak{g})$, the other two are in $A^1(\mathfrak{g})$.) We define κ to be that γ for which $\mathfrak{g}^\gamma$ is of type F_4.

This completes the definition of κ in all cases.

In all cases, κ can be characterized as the involution in $R \cap A^1(\mathfrak{g})$ whose fixed point set on $\mathfrak{g}_0$ has maximum possible dimension.

12.2. For a general $\mathfrak{g}$ and a distinguished $\mathfrak{sl}_2$-triple (e, f, h) of $\mathfrak{g}$, we define $\varpi : \mathfrak{g} \to \mathfrak{g}$ as the composition $\kappa\kappa' = \kappa'\kappa \in R'$. One can check that ϖ is an opposition of $\mathfrak{g}$. Note that ϖ is canonically attached to (e, f, h). One can characterize ϖ as the unique opposition in R' whose fixed point set on $\mathfrak{g}_0$ has maximum possible dimension (or equivalently, whose -1 eigenspace on $\mathfrak{g}_2$ has maximum possible dimension).

12.3. In the remainder of this section we fix $(e, f, h), C, \mathfrak{c}$ as in 11.1. As in 11.1, we set $H = C \times \mathbf{C}^*$.

Let $\mathfrak{l}$ be the centralizer of $\mathfrak{c}$ in $\mathfrak{g}$. This is a Levi subalgebra of a parabolic subalgebra of $\mathfrak{g}$. Let $\mathfrak{z}$ be the centre of $\mathfrak{l}$. Clearly, $\mathfrak{c} \subset \mathfrak{z}$. (The reverse inclusion is also true.) Let $\mathfrak{l}'$ be the derived subalgebra of $\mathfrak{l}$. From 11.1(a) and the relations of $\mathfrak{sl}_2$, we see that e, f, h are contained in $\mathfrak{l}'$. If $y \in \mathfrak{l}'$ satisfies $[y, e] = [y, f] = [y, h] = 0$, then $y \in \mathfrak{l}' \cap \mathfrak{s}$. Thus, y is in the centralizer of $\mathfrak{c}$ in $\mathfrak{s}$, that is, $y \in \mathfrak{c}$. Thus, $y \in \mathfrak{z}$ and $y \in \mathfrak{l}'$; hence $y \in \mathfrak{z} \cap \mathfrak{l}' = 0$. Thus, (e, f, h) is distinguished in $\mathfrak{l}'$. Let $\varpi_0 : \mathfrak{l}' \to \mathfrak{l}'$ be the opposition of $\mathfrak{l}'$ attached in 12.2 to (e, f, h), relative to $\mathfrak{l}'$. In particular, we have $\varpi_0(e) = -e, \varpi_0(f) = -f, \varpi_0(h) = h$.

By a standard argument we can find an opposition ϖ of $\mathfrak{g}$ such that $\varpi|_{\mathfrak{l}'} = \varpi_0$ and $\varpi(y) = -y$ for all $y \in \mathfrak{z}$. We show that ϖ is uniquely determined up to replacing $Ad(c)\varpi Ad(c^{-1})$ with $c \in C$ (this is an opposition with the same property as ϖ). Indeed, if ϖ' is another opposition of $\mathfrak{g}$ with the same property as ϖ, then ϖ' is of the form $Ad(g)\varpi$ where g is in the centralizer of L in G. Hence, $g = zc$ where $c \in C$ and z is in the centre of G. Replacing g by $z^{-1}g$, we see that we may assume that

$g = c$. Let $c_1 \in C$ be such that $c_1^2 = c$. Then $\varpi' = Ad(c_1^2)\varpi = Ad(c_1)\varpi Ad(g_1^{-1})$, as claimed.

12.4. Thus, to $(e, f, h), C, \mathfrak{c}$ we have associated in an essentially canonical way (that is up to conjugation by C) an opposition ϖ of $\mathfrak{g}$ such that $\varpi(e) = -e, \varpi(f) = -f, \varpi(h) = h$ and $\varpi(y) = -y$ for all $y \in \mathfrak{z} = \mathfrak{c}$. In the remainder of this section we fix such a ϖ.

12.5. Let $k : \mathcal{B}_e \to \Lambda_e$ be the imbedding $\mathfrak{b} \mapsto (e, \mathfrak{b})$. Applying 10.1 with Y equal to $\{e\}$ or Σ_e, we obtain $K_{\mathcal{G}}(Z)$-module structures on $K_H(\mathcal{B}_e)$ and $K_H(\Lambda_e)$. By 10.1, the R_H-linear map $k_* : K_H(\mathcal{B}_e) \to K_H(\Lambda_e)$ is $\mathcal{H}$-linear.

12.6. The involution $\Lambda \to \Lambda$ given by $(y, \mathfrak{b}) \mapsto (-\varpi(y), \varpi(\mathfrak{b}))$ maps $\mathcal{B}_e = \{e\}^{\cdot}$ into itself and Λ_e into itself; the resulting involutions of $\Lambda, \mathcal{B}_e, \Lambda_e$ are denoted again by ϖ. The involution on Λ (hence its restriction to $\mathcal{B}_e$ or Λ_e) is compatible with the action of $C \times \mathbf{C}^* = H$ in the following way:

$$\varpi((c, \lambda)(y, \mathfrak{b})) = (c^{-1}, \lambda)\varpi(y, \mathfrak{b}) \quad \text{for } (c, \lambda) \in C \times \mathbf{C}^*, (y, \mathfrak{b}) \in \Lambda.$$

(We use that $\varpi(c) = c^{-1}$ for $c \in C$ and $\varpi(\zeta \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)) = \zeta \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$ which follows from $\varpi(h) = h$.)

Hence, if $F \in \mathrm{Coh}_H(\mathcal{B}_e)$ (resp. $F \in \mathrm{Coh}_H(\Lambda_e)$), then we have naturally $\varpi^* F \in \mathrm{Coh}_H(\mathcal{B}_e)$ (resp. $\varpi^* F \in \mathrm{Coh}_H(\Lambda_e)$) and $F \mapsto \varpi^* F$ defines an involution $\varpi^* : K_H(\mathcal{B}_e) \to K_H(\mathcal{B}_e)$ (resp. $\varpi^* : K_H(\Lambda_e) \to K_H(\Lambda_e)$).

This involution is semilinear with respect to the involution of $R_H = R_{C \times \mathbf{C}^*}$ induced by the involution $(c, \lambda) \mapsto (c^{-1}, \lambda)$ of $C \times \mathbf{C}^*$.

Using the definitions, we see that the involutions ϖ^* of $K_{\mathcal{G}}(Z)$ (see 9.10) and ϖ^* of $K_H(\mathcal{B}_e)$ (resp. $K_H(\Lambda_e)$) are compatible with the $K_{\mathcal{G}}(Z)$-module structures. Thus, if $\xi \in K_{\mathcal{G}}(Z)$ and $\xi' \in K_H(\mathcal{B}_e)$ (resp. $\xi' \in K_H(\Lambda_e)$), then

(a) $$\varpi^*(\xi\xi') = \varpi^*(\xi)\varpi^*(\xi').$$

12.7. Using 6.12, we see that

$$D_{\mathcal{B}_e}\varpi^* = \varpi^* D_{\mathcal{B}_e} : K_H(\mathcal{B}_e) \to K_H(\mathcal{B}_e),$$
$$D_{\Lambda_e}\varpi^* = \varpi^* D_{\Lambda_e} : K_H(\Lambda_e) \to K_H(\Lambda_e).$$

Lemma 12.8. *Let Y be either $\{e\}$ or Σ_e. For any $F \in K_{\mathcal{G}}(Z)$ and $F' \in K_H(\dot{Y})$ we have*

$$D_{\dot{Y}}(F \star_Y F') = v^{-2\nu} D_Z(F) \star_Y D_{\dot{Y}}(F').$$

With the notation in 10.1, we have by arguments similar to those in 9.5:

$$\begin{aligned}
&D_{\dot{Y}}(F \star_Y F') = D_{\dot{Y}}(p''_*(F \otimes^L_{\Lambda^2} p'^* F')) = p''_*(D_{Z \cap \dot{Y} \times \dot{Y}}(F \otimes^L_{\Lambda^2} p'^* F')) \\
&= v^{-4\nu} p''_*(D_Z(F) \otimes^L_{\Lambda^2} D_{\Lambda \times \dot{Y}}(p'^* F')) = v^{-4\nu} p''_*(D_Z(F) \otimes^L_{\Lambda^2} v^{2\nu} p'^*(D_{\dot{Y}} F')) \\
&= v^{-2\nu} D_Z(F) \star_Y D_{\dot{Y}}(F').
\end{aligned}$$

The lemma is proved.

12.9. Let L be the closed subgroup of G whose Lie algebra is $\mathfrak{l}$. Let $Ad(L)e \subset \mathfrak{l}$ be the orbit of e under $Ad(L)$ and let $d(e) = (1/2)\dim Ad(L)e$. We define a map $^- : K_H(\mathcal{B}_e) \to K_H(\mathcal{B}_e)$ by

$$\xi' \mapsto (-v)^{-\nu}\tilde{T}_{w_0}^{-1}\varpi^* D_{\mathcal{B}_e}(\xi').$$

We define a map $^- : K_H(\Lambda_e) \to K_H(\Lambda_e)$ by

$$\xi' \mapsto (-v)^{\nu-2d(e)}\tilde{T}_{w_0}^{-1}\varpi^* D_{\Lambda_e}(\xi').$$

Proposition 12.10. *Let Y be either $\{e\}$ or Σ_e. The map $^- : K_H(\dot{Y}) \to K_H(\dot{Y})$ in 12.9 is an $\mathcal{H}$-antilinear involution.*

In the following proof we write D instead of $D_{\dot{Y}}$, t instead of $\tilde{T}_{w_0}$, and will omit the $\star$ signs. We show that our map is $\mathcal{H}$-antilinear. Using 9.12, we see that it is enough to verify that for any $\xi \in K_{\mathcal{G}}(Z), \xi' \in K_H(\dot{Y})$ we have

$$t^{-1}\varpi^* D(\xi\xi') = v^{-2\nu}t^{-1}\varpi^*(D_Z(\xi))tt^{-1}\varpi^*(D(\xi'))$$

or equivalently (see 12.8), $\varpi^*(D_Z(\xi)D(\xi') = \varpi^*(D_Z(\xi))\varpi^*(D(\xi'))$. But this follows from 12.6(a).

We show that our map is an involution. It is enough to verify that for any $\xi' \in K_H(\dot{Y})$ and any integer n, we have

$$(-v)^n t^{-1}\varpi^* D((-v)^n t^{-1}\varpi^* D(\xi')) = \xi'.$$

By the $\mathcal{H}$-antilinearity of our map, the left hand side is equal to

$$(-v)^{-n}t(-v)^n t^{-1}\varpi^* D(\varpi^* D(\xi')) = \varpi^* D(\varpi^* D(\xi')).$$

By 12.7, this equals $\varpi^*\varpi^* DD(\xi') = DD(\xi') = \xi'$ (see 6.16). The proposition is proved.

12.11. Consider the R_H-bilinear pairing $K_H(\mathcal{B}_e) \times K_H(\Lambda_e) \to R_H$ given by $(F : F') = \pi_*(F \otimes^L_{\Lambda_e} F')$. (The Tor-product is relative to the smooth H-variety Λ_e and its closed subvarieties $\mathcal{B}_e, \Lambda_e$ with intersection $\mathcal{B}_e$; π is the map from $\mathcal{B}_e$ to the point.)

Let $p \mapsto p^\dagger$ be the involution of R_H induced by the automorphism of $H = C \times \mathbf{C}^*$ given by $(c, \lambda) \mapsto (c^{-1}, \lambda)$.

Lemma 12.12. *For $\xi \in K_H(\mathcal{B}_e), \xi' \in K_H(\Lambda_e), \chi \in \mathcal{H}$, we have*

(a) $$(\xi : \varpi^*(\xi')) = (\varpi^*(\xi) : \xi')^\dagger,$$

(b) $$(\chi\xi : \xi') = (\xi : \hat{\chi}\xi'),$$

where $\hat{\chi}$ is as in 1.24.

(a) follows directly from the definitions. To prove (b), we note that $\chi \mapsto \hat{\chi}$ can be interpreted geometrically as the map $K_{\mathcal{G}}(Z) \to K_{\mathcal{G}}(Z)$ induced by the involution $(y, \mathfrak{b}, \mathfrak{b}') \mapsto (y, \mathfrak{b}', \mathfrak{b})$ of Z. With this interpretation, the proof of (b) is routine.

Lemma 12.13. *For $\xi, \xi' \in K_H(\mathcal{B}_e)$, we have $(\xi : k_*(\xi')) = (\xi' : k_*(\xi))$.*

Using 6.5, we see that both sides are equal to $\pi_*(\xi \otimes^L_{\Lambda_e} \xi')$ where the Tor-product is relative to the smooth H-variety Λ_e and its closed subvarieties $\mathcal{B}_e, \mathcal{B}_e$. The lemma is proved.

12.14. We define a pairing $(||) : K_H(\mathcal{B}_e) \times K_H(\Lambda_e) \to R_H$ by

(a) $$(\xi||\xi') = (-v)^{d(e)-b(e)}(\xi : \tilde{T}_{w_0}\varpi^*(\xi')).$$

Lemma 12.15. *Let $\xi \in K_H(\mathcal{B}_e), \xi' \in K_H(\Lambda_e)$. We have $(\bar{\xi}||\xi') = \overline{(\xi||\bar{\xi}')}$.*

This is equivalent to

$$(-v)^{d(e)-b(e)}((-v)^{-\nu}\tilde{T}_{w_0}^{-1}\varpi^* D_{\mathcal{B}_e}(\xi) : \tilde{T}_{w_0}\varpi^*(\xi'))$$
$$= \overline{(-v)^{d(e)-b(e)}(\xi : (-v)^{\nu-2d(e)}\tilde{T}_{w_0}\varpi^*\tilde{T}_{w_0}^{-1}\varpi^* D_{\Lambda_e}(\xi'))}$$

or to

$$(-v)^{d(e)-b(e)}((-v)^{-\nu}\varpi^* D_{\mathcal{B}_e}(\xi) : \varpi^*(\xi'))$$
$$= \overline{(-v)^{d(e)-b(e)}(\xi : (-v)^{\nu-2d(e)} D_{\Lambda_e}(\xi'))},$$

or to $(-v)^{d(e)-b(e)}((-v)^{-\nu} D_{\mathcal{B}_e}(\xi) : (\xi')) = \overline{(-v)^{d(e)-b(e)}(\xi : (-v)^{\nu-2d(e)} D_{\Lambda_e}(\xi'))}^{\dagger}$, or to $(-v)^{-2b(e)}(D_{\mathcal{B}_e}(\xi) : (\xi')) = \overline{(\xi : D_{\Lambda_e}(\xi'))}^{\dagger}$. We have

$$\overline{(\xi : D_{\Lambda_e}(\xi'))}^{\dagger} = D_{\mathrm{point}}(\pi_*(\xi \otimes^L_{\Lambda_e} D_{\Lambda_e}(\xi')))$$
$$= \pi_*(D_{\mathcal{B}_e}(\xi \otimes^L_{\Lambda_e} D_{\Lambda_e}(\xi'))) = \pi_*((D_{\mathcal{B}_e}\xi) \otimes^L_{\Lambda_e} (D_{\Lambda_e} D_{\Lambda_e}(\xi'))(-v)^{-2b(e)})$$
$$= \pi_*((D_{\mathcal{B}_e}\xi) \otimes^L_{\Lambda_e} \xi')(-v)^{-2b(e)} = (-v)^{-2b(e)}(D_{\mathcal{B}_e}(\xi) : (\xi')).$$

The lemma is proved.

12.16. We define a pairing $(|)_{\mathcal{B}_e} : K_H(\mathcal{B}_e) \times K_H(\mathcal{B}_e) \to R_H$ by

$$(\xi|\xi')_{\mathcal{B}_e} = (\xi||k_*(\xi')).$$

We define a pairing $(|)_{\Lambda_e} : K_H(\Lambda_e) \times K_H(\Lambda_e) \to \mathfrak{R}$ ($\mathfrak{R}$ as in 11.8) by

$$(\xi|\xi')_{\Lambda_e} = (k_*^{-1}\xi||\xi').$$

Here $k_*^{-1}\xi$ and ξ' are regarded as elements of $\mathfrak{R} \otimes_{R_H} K_H(\mathcal{B}_e)$ (see 11.8) and the pairing 12.14(a) is extended in an obvious way to a pairing $(||) : \mathfrak{R} \otimes_{R_H} K_H(\mathcal{B}_e) \times \mathfrak{R} \otimes_{R_H} K_H(\Lambda_e) \to \mathfrak{R}$.

Lemma 12.17. *For $\xi, \xi' \in K_H(\mathcal{B}_e)$ and $\chi \in \mathcal{H}$, we have*

(a) $$(\xi|\xi')_{\mathcal{B}_e} = (\xi'|\xi)^{\dagger}_{\mathcal{B}_e},$$

(b) $$(\chi\xi|\xi')_{\mathcal{B}_e} = (\xi|\chi^{\blacktriangle}\xi')_{\mathcal{B}_e}.$$

This follows from 12.12, 12.13, by arguments similar to those in 5.10, 5.12.

12.18. Let $\mathfrak{U}_H$ be the ring of power series in v^{-1} with coefficients in the ring R_C. Then $R_H = R_C[v, v^{-1}]$ is naturally a subring of $\mathfrak{U}_H$. Let $\tilde{\mathfrak{U}}_H$ be the set of all elements $\tilde{p} \in \mathfrak{U}_H$ such that $p\tilde{p} \in R_H$ for some $p \in R_H - \{0\}$. Then $\tilde{\mathfrak{U}}_H$ may be regarded as a subring of either $\mathfrak{U}_H$ or of the field $\mathfrak{R}$. Let $p \mapsto p^{(0)}$ be the group homomorphism $R_C \to \mathbf{Z}$ which sends a non-trivial irreducible representation of C to 0 and sends the unit representation of C to 1.

Let $\partial : \mathfrak{U}_H \to \mathbf{Z}((v^{-1}))$ be the group homomorphism defined by $\sum_{n\in\mathbf{Z}} p_n v^n \mapsto \sum_{n\in\mathbf{Z}} p_n^{(0)} v^n$. Let

$$\mathbf{B}^{\pm}_{\mathcal{B}_e} = \{\xi \in K_H(\mathcal{B}_e) | \bar{\xi} = \xi, \quad \partial(\xi|\xi)_{\mathcal{B}_e} \in 1 + v^{-1}\mathbf{Z}[v^{-1}]\},$$
$$\mathbf{B}^{\pm}_{\Lambda_e} = \{\xi \in K_H(\Lambda_e) | \bar{\xi} = \xi, \quad (\xi|\xi)_{\Lambda_e} \in \tilde{\mathfrak{U}}_H, \quad \partial(\xi|\xi)_{\Lambda_e} \in 1 + v^{-1}\mathbf{Z}[[v^{-1}]]\}.$$

Conjecture 12.19. (a) $\mathbf{B}^{\pm}_{\mathcal{B}_e}$ *is a signed basis of the $\mathcal{A}$-module $K_H(\mathcal{B}_e)$.*

(b) *If $\xi, \xi' \in \mathbf{B}^{\pm}_{\mathcal{B}_e}$ and $\xi' \neq \pm\xi$, then $\partial(\xi|\xi')_{\mathcal{B}_e} \in v^{-1}\mathbf{Z}[v^{-1}]$.*

(c) $\mathbf{B}^{\pm}_{\Lambda_e}$ *is a signed basis of the $\mathcal{A}$-module $K_H(\Lambda_e)$.*

(d) *If $\xi, \xi' \in \Lambda_{\mathcal{B}_e}$ and $\xi' \neq \pm\xi$, then $(\xi|\xi')_{\Lambda_e} \in \tilde{\mathfrak{U}}_H$ and $\partial(\xi|\xi')_{\Lambda_e} \in v^{-1}\mathbf{Z}[[v^{-1}]]$.*

(e) *For any $\xi \in \mathbf{B}^{\pm}_{\mathcal{B}_e}$ there exists $\tilde{\xi} \in \mathbf{B}^{\pm}_{\Lambda_e}$ such that $\partial(\xi||\tilde{\xi}) = 1$ and $\partial(\xi'||\tilde{\xi}) = 0$ for all $\xi' \in \mathbf{B}^{\pm}_{\mathcal{B}_e} - \{\pm\xi\}$.*

(f) *$\tilde{\xi}$ in* (e) *is unique and $\xi \mapsto \tilde{\xi}$ is a bijection $\mathbf{B}^{\pm}_{\mathcal{B}_e} \xrightarrow{\sim} \mathbf{B}^{\pm}_{\Lambda_e}$.*

For $e = 0$, this holds by 10.16 and 3.14.

Now R_H has an obvious $\mathcal{A}$-basis (λ) where λ runs over the one-dimensional representations of C. For λ as above we have from the definition:

$$\xi \in \mathbf{B}^{\pm}_{\mathcal{B}_e} \implies \lambda\xi \in \mathbf{B}^{\pm}_{\mathcal{B}_e},$$
$$\xi \in \mathbf{B}^{\pm}_{\Lambda_e} \implies \lambda\xi \in \mathbf{B}^{\pm}_{\Lambda_e}.$$

12.20. Assuming the conjecture, we see that $\{\xi \in K_H(\mathcal{B}_e)|\partial(\xi|\xi)_{\mathcal{B}_e} \in \mathbf{Z}[v^{-1}]\}$ is a $\mathbf{Z}[v^{-1}]$-submodule of $K_H(\mathcal{B}_e)$ and that $\mathbf{B}^{\pm}_{\mathcal{B}_e}$ is a signed basis for it.

On the other hand, assuming only that 12.19(b) holds and that there exists a basis β of the $\mathcal{A}$-module $K_H(\mathcal{B}_e)$ such that $\beta \subset \mathbf{B}^{\pm}_{\mathcal{B}_e}$, we can show that $\mathbf{B}^{\pm}_{\mathcal{B}_e}$ is equal to the signed basis $\pm\beta$ (so that 12.19(a) holds).

Indeed, let $\xi \in \mathbf{B}^{\pm}_{\mathcal{B}_e}$. By our assumption, we have $\xi = \sum_{b\in\beta} c_b b$ where $c_b \in \mathcal{A}$ are zero for all but finitely many b. Since not all c_b are 0, we can find $n \in \mathbf{Z}$ such that $c_b \in v^n\mathbf{Z}[v^{-1}]$ for all b and $c_b \notin v^{n-1}\mathbf{Z}[v]$ for some b. Let $c_{b,n} \in \mathbf{Z}$ be such that $c_b = c_{b,n}v^n \mod v^{n-1}\mathbf{Z}[v^{-1}]$ for all b. We have $c_{b,n} \neq 0$ for some b. Hence, $\sum_b c_{b,n}^2 > 0$. Using our assumptions we see that

$$\partial(\xi|\xi)_{\mathcal{B}_e} = v^{2n}\sum_b c_{b,n}^2 \mod v^{2n-1}\mathbf{Z}[v^{-1}].$$

On the other hand, $\partial(\xi|\xi)_{\mathcal{B}_e} = 1 \mod v^{-1}\mathbf{Z}[v^{-1}]$. It follows that $n = 0$ and $\sum_b c_{b,n}^2 = 1$. In particular, we have $c_b \in \mathbf{Z}[v^{-1}]$ for all b. Since $\bar{\xi} = \xi$, we must have $\bar{c}_b = c_b$; hence $c_b \in \mathbf{Z}$ for all b. We then have $\sum_b c_b^2 = 1$; hence $c_b = 0$ for all b but one for which $c_b = \pm 1$. Thus, $\xi \in \pm\beta$ as claimed.

12.21. We consider the example where e is a regular nilpotent element. Let $\mathfrak{b}_0$ be the unique Borel subalgebra that contains e. In this case $\mathcal{B}_e = \{\mathfrak{b}_0\}$ and $\Lambda_e = \{(e, \mathfrak{b}_0)\}$ are points. We have $C = 1$ and $H = \mathbf{C}^*$. Hence, $K_H(\mathcal{B}_e) = \mathcal{A}$. If $L \in \mathrm{Vec}_{\mathcal{G}}(\mathcal{B})$, then the restriction of L to $\{\mathfrak{b}_0\}$ is $v^{n(L)} \in \mathrm{Vec}_{\mathbf{C}^*}(\text{point})$ where $n(L) \in \mathbf{Z}$. Note that $L \mapsto n(L)$ is a group homomorphism $\mathbf{X} \to \mathbf{Z}$. For any $\lambda \in \mathbf{C}^*$, $\zeta\left(\begin{smallmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{smallmatrix}\right)$ acts on the fibre $L_{\mathfrak{b}_0}$ by $\lambda^{n(L)}$. If $L = L_i$, then $L_{\mathfrak{b}_0} = \mathfrak{p}/\mathfrak{b}_0$ where $\mathfrak{p} \in \mathcal{P}_i$ contains $\mathfrak{b}_0$. Since $Ad\zeta\left(\begin{smallmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{smallmatrix}\right)(f) = \lambda^{-2}f$, we must have $n(L_i) = -2$ for any i. This condition determines completely the homomorphism n. If $i \in I$, then $\tilde{T}_{\sigma_i}$ acts on $K_H(\mathcal{B}_e)$ as multiplication by some element $a \in \mathcal{A}$ (necessarily v or $-v^{-1}$). From the relation 1.18(b) we see that we must have $v^{n(L_xL_i^{-1})} = a^{-1}v^{n(L_x)}a^{-1}$ if $\check{\alpha}_i(x) = 1$. Thus, $a^{-2} = v^{n(L_i^{-1})} = v^2$; hence $a = -v^{-1}$. It follows that $\tilde{T}_{w_0}^{-1}$ acts on $K_H(\mathcal{B}_e)$ as multiplication by $(-v)^{\nu}$. Note that ϖ^* is the identity map on $K_H(\mathcal{B}_e)$ and $D_{\mathcal{B}_e} : \mathcal{A} \to \mathcal{A}$ takes $v^n \to v^{-n}$ for all n. It follows that in this case $^- : K_H(\mathcal{B}_e) \to K_H(\mathcal{B}_e)$ is the homomorphism $\mathcal{A} \to \mathcal{A}$ which takes $v^n \to v^{-n}$ for all n. In this case, Conjecture 12.19 holds trivially; $\mathbf{B}^{\pm}_{\mathcal{B}_e} = \mathbf{B}^{\pm}_{\Lambda_e}$ consists of $\pm 1 \in \mathcal{A}$.

12.22. Let $\bar{C}$ be the image of C in the adjoint group of G and let $\bar{H} = \bar{C} \times \mathbf{C}^*$. The action of H on $\mathcal{B}_e, \Lambda_e$ factors through the quotient $\bar{H}$ of H.

Let $\mathbf{B}^{\pm}_{\mathcal{B}_e,\mathrm{ad}}$ be the set of all $\xi \in \mathbf{B}^{\pm}_{\mathcal{B}_e}$ which are in the image of the obvious homomorphism $K_{\bar{H}}(\mathcal{B}_e) \to K_H(\mathcal{B}_e)$. Let $\mathbf{B}^{\pm}_{\Lambda_e,\mathrm{ad}}$ be the set of all $\xi \in \mathbf{B}^{\pm}_{\Lambda_e}$ which are in the image of the obvious homomorphism $K_{\bar{H}}(\Lambda_e) \to K_H(\Lambda_e)$. We now state a conjecture that complements 12.19.

Conjecture 12.23. (a) $\mathbf{B}^{\pm}_{\mathcal{B}_e,\mathrm{ad}}$ *is a signed basis of the $\mathcal{A}$-module* $\mathrm{Im}(K_{\bar{H}}(\mathcal{B}_e) \to K_H(\mathcal{B}_e))$*;* $\mathbf{B}^{\pm}_{\Lambda_e,\mathrm{ad}}$ *is a signed basis of the $\mathcal{A}$-module* $\mathrm{Im}(K_{\bar{H}}(\Lambda_e) \to K_H(\Lambda_e))$.

(b) *From* $\mathbf{B}^{\pm}_{\mathcal{B}_e,\mathrm{ad}}$ *one can extract uniquely a basis* $\mathbf{B}_{\mathcal{B}_e,\mathrm{ad}}$ *of the $\mathcal{A}$-module* $\mathrm{Im}(K_{\bar{H}}(\mathcal{B}_e) \to K_H(\mathcal{B}_e))$ *and from* $\mathbf{B}^{\pm}_{\Lambda_e,\mathrm{ad}}$ *one can extract uniquely a basis* $\mathbf{B}_{\Lambda_e,\mathrm{ad}}$ *of the $\mathcal{A}$-module* $\mathrm{Im}(K_{\bar{H}}(\Lambda_e) \to K_H(\Lambda_e))$ *so that the following hold:*

$$(-1)^{b(e)} v^{2b(e)} \mathbf{C} \in \mathbf{B}_{\Lambda_e,\mathrm{ad}};$$

$\mathbf{B}_{\mathcal{B}_e,\mathrm{ad}}, \mathbf{B}_{\Lambda_e,\mathrm{ad}}$ *correspond to each other under the bijection 12.19*(f)*; for any β, β' in $\mathbf{B}_{\mathcal{B}_e,\mathrm{ad}}$, we have $\partial(\beta,\beta')_{\mathcal{B}_e} \in \mathbf{N}[v, v^{-1}]$.*

Again, this holds for $e = 0$.

13. An example in D_4

13.1. In this section we assume that G is almost simple of type D_4. The elements of I are denoted $0, 1, 2, 3$ where $\sigma_1, \sigma_2, \sigma_3$ commute with each other.

We fix $(e, f, h), C, \mathfrak{c}$ as in 11.1; we assume that e is a subregular nilpotent element of $\mathfrak{g}$. Hence, (e, f, h) is distinguished, $C = \{1\}$, $\mathfrak{c} = 0$ and $H = \mathbf{C}^*$. In this case, $\mathcal{B}_e$ has irreducible components V_i (indexed by $i \in I$). Moreover, V_i is a single fibre of $\pi_i : \mathcal{B} \to \mathcal{P}_i$ (hence is a P^1). If $i \in I - \{0\}$, then V_0, V_i intersect in a unique point p_{0i}. The H-action on $\mathcal{B}_e$ is trivial on V_0. For each $i \in I - \{0\}$, the H-action on V_i has exactly two fixed points. These are p_{i0}, q_i. We have $\mathcal{B}_e^H = V_0 \cup \{q_1, q_2, q_3\}$. Consider the homomorphism

$$\bigoplus_{i \neq 0} K_H(p_{i0}) \xrightarrow{a} \oplus_i K_H(V_i)$$

with components $K_H(p_{i0}) \to K_H(V_i)$ for any $i \neq 0$ and $K_H(p_{i0}) \to K_H(V_0)$. (One is the direct image map, the other is minus the direct image map.)

Lemma 13.2. *a is injective and $K_H(\mathcal{B}_e) = \mathrm{coker}(a)$.*

This is easily checked.

13.3. For any $i \in I$, we define a homomorphism $n_i : \mathcal{X} \to \mathbf{Z}$ and a connected component μ_i of $\mathcal{B}_e^H$ as follows.

$$n_0(\alpha_j) = -2 \text{ if } j \neq 0; \quad n_0(\alpha_0) = 0; \quad \mu_0 = V_0.$$
$$n_i(x) = n_0({}^{\sigma_i}x); \quad \mu_i = q_i \text{ for } i \neq 0.$$

Lemma 13.4. *Let $L = L_x \in \mathbf{X}$ and let $\mathfrak{b} \in \mu_i \subset \mathcal{B}_e^H$. We have $L_{\mathfrak{b}} = v^{n_i(x)}$.*

This is easily checked.

13.5. Let $\mathcal{C}$ be a curve isomorphic to P^1 with an action of $\mathbf{C}^*$. Let $p \neq p'$ be fixed points for this action. Assume that the weight of the tangent action at p (resp. p') is $n \geq 0$ (resp. $-n$).

For $m \geq 0$, let O^m be the line bundle on $\mathcal{C}$ whose space of sections has dimension $m+1$. For $m < 0$ let O^m be the dual of O_i^{-m}.

If mn is even, we can regard O^m has an object of $\mathrm{Vec}_{\mathbf{C}^*}(\mathcal{C})$ such that the weight of $\mathbf{C}^*$ on the fibre O^m_p (resp. on the fibre $O^m_{p'}$) is $nm/2$ (resp. $-nm/2$).

Let us identify $R_{\mathbf{C}^*} = \mathcal{A}$ in such a way that v^m corresponds to the one-dimensional representation of weight m of $\mathbf{C}^*$.

Let $j^p : \{p\} \to \mathcal{C}, j^{p'} : \{p'\} \to \mathcal{C}$ be the inclusions. If n is even, we have exact sequences of coherent sheaves

$$0 \to v^{-n/2} O^{-1} \to O^0 \to j^p_*(\mathbf{C}) \to 0, \quad 0 \to v^{n/2} O^{-1} \to O^0 \to j^{p'}_*(\mathbf{C}) \to 0,$$

Hence,

$$j^p_*(\mathbf{C}) = O^0 - v^{-n/2} O^{-1} \in K_{\mathbf{C}^*}(\mathcal{C}), \quad j^{p'}_*(\mathbf{C}) = O^0 - v^{n/2} O^{-1} \in K_{\mathbf{C}^*}(\mathcal{C}).$$

We have $O^s + O^{-s} = v^{ns/2} + v^{-ns/2}$.

13.6. The discussion in 13.5 can be applied to $\mathcal{C} = V_i$. In this case we write O_i^m instead of O^m.

Assume that $i \neq 0$. We have $n = 2$; hence $O_i^s + O_i^{-s} = v^s + v^{-s}$. The weight of the H-action on the tangent space at q_i is 2 and on the tangent space at p_{i0} is -2. Hence,

$$j^{q_i}_*(\mathbf{C}) = O_i^0 - v^{-1} O_i^{-1} \in K_{\mathbf{C}^*}(V_i), \quad j^{p_{i0}}_*(\mathbf{C}) = O_i^0 - v O_i^{-1} \in K_{\mathbf{C}^*}(V_i).$$

Now assume that $i = 0$. Then $j^{p_{i0}}_*(\mathbf{C}) = O_0^0 - O_0^{-1} \in K_{\mathbf{C}^*}(V_0)$.

13.7. Let $o_i^m \in \mathrm{Coh}_H(\mathcal{B}_e)$ be the image of O_i^m under the direct image map induced by the inclusion $V_i \subset \mathcal{B}_e$. From 13.2 we see that $K_H(\mathcal{B}_e)$ is the $\mathcal{A}$-module with generators o_i^m $(i \in I, m \in \mathbf{Z})$ and relations:

$$o_0^0 - o_0^{-1} = o_i^0 - v o_i^{-1} \quad \text{for } i = 1,2,3,$$
$$o_i^{m+1} + o_i^{m-1} = (v + v^{-1}) o_i^m \quad \text{for } i = 1,2,3,$$
$$o_0^{m+1} + o_0^{m-1} = 2 o_0^m.$$

In fact, an $\mathcal{A}$-basis is given by o_i^{-1}, $i = 0,1,2,3$ and $p = o_0^0 - o_0^{-1}$.

13.8. For $x \in \mathcal{X}$ the restriction of L_x to V_i is $v^s O_i^{\check{\alpha}_i(x)}$ where s is determined as follows.

If $i \neq 0$, then $(L_x)_{p_{i0}} = v^{n_0(x)} = v^s v^{-\check{\alpha}_i(x)}$; hence $s = n_0(x) + \check{\alpha}_i(x)$.

If $i = 0$, then $(L_x)_{p_{10}} = v^{n_0(x)} = v^s$; hence $s = n_0(x)$.

Thus, the restriction of L_x on V_i is

$$v^{n_0(x) + \check{\alpha}_i(x)} O_i^{\check{\alpha}_i(x)} \quad \text{if } i = 1,2,3,$$
$$v^{n_0(x)} O_0^{\check{\alpha}_0(x)} \quad \text{if } i = 0.$$

Lemma 13.9. (a) *$\theta_x p = v^{n_0(x)} p$.*

(b) *Assume that $i \neq 0$ and $\check{\alpha}_i(x) = 1$. We have*

$$\theta_x o_i^{-1} = v^{n_0(x)+1}(v o_i^{-1} + p), \quad \theta_{x - \alpha_i} o_i^{-1} = v^{n_0(x)+1}(v^{-1} o_i^{-1} - p).$$

(c) *Assume that $i=0$ and $\check{\alpha}_0(x)=1$. We have*

$$\theta_x o_0^{-1} = v^{n_0(x)}(o_0^{-1}+p), \quad \theta_{x-\alpha_0} o_0^{-1} = v^{n_0(x)}(o_0^{-1}-p).$$

(a) follows from the fact that $p = j_*\mathbf{C}$ where $j : \{p_{i0}\} \to \mathcal{B}_e$ is the inclusion. In the case (b) we have

$$\begin{aligned}
&\theta_x o_i^{-1} = v^{n_0(x)+\check{\alpha}_i(x)} o_i^{-1+\check{\alpha}_i(x)} = v^{n_0(x)+1} o_i^0 = v^{n_0(x)+1}(v o_i^{-1}+p),\\
&\theta_{x-\alpha_i} o_i^{-1} = v^{n_0(x-\alpha_i)+\check{\alpha}_i(x-\alpha_i)} o_i^{-1+\check{\alpha}_i(x-\alpha_i)}\\
&\ = v^{n_0(x)+1} o_i^{-2} = v^{n_0(x)+1}(-o_i^0 + (v+v^{-1})o_i^{-1}) = v^{n_0(x)+1}(v^{-1}o_i^{-1}-p).
\end{aligned}$$

In the case (c) we have

$$\begin{aligned}
&\theta_x o_i^{-1} = v^{n_0(x)} o_0^{-1+\check{\alpha}_0(x)} = v^{n_0(x)} o_0^0 = v^{n_0(x)}(o_i^{-1}+p),\\
&\theta_{x-\alpha_0} o_i^{-1} = v^{n_0(x-\alpha_0)} o_0^{-1+\check{\alpha}_0(x-\alpha_0)} = v^{n_0(x)} o_0^{-2} = v^{n_0(x)}(o_0^{-1}-p).
\end{aligned}$$

Lemma 13.10. *$\tilde{T}_{\sigma_i} o_i^{-1} = v o_i^{-1}$ for $i = 0,1,2,3$.*

This can be deduced from the knowledge (Section 10) of the action of $\tilde{T}_{\sigma_i}$ on $K_{\mathcal{G}}(\mathcal{B})$.

Lemma 13.11. *For $i \neq 0$ we have $\tilde{T}_{\sigma_i} p = -v^{-1}p$.*

By 10.22(a), the image of $K_{\mathbf{C}^*}(V_i) \to K_{\mathbf{C}^*}(\mathcal{B}_e)$ (direct image) is stable under $\tilde{T}_{\sigma_i}$. Hence, $\tilde{T}_{\sigma_i} p = ap + bo_i^{-1}$ for some $a, b \in \mathcal{A}$. Let x be such that $\check{\alpha}_i(x) = 1$. We have $\theta_{x-\alpha_i}\tilde{T}_{\sigma_i} p = (\tilde{T}_{\sigma_i} + v^{-1} - v)\theta_x p$. Hence,

$$\begin{gathered}
a\theta_{x-\alpha_i} p + b\theta_{x-\alpha_i} o_i^{-1} = (\tilde{T}_{\sigma_i} + v^{-1} - v) v^{n_0(x)} p,\\
a v^{n_0(x-\alpha_i)} p + b v^{n_0(x)+1}(v^{-1}o_i^{-1} - p) = v^{n_0(x)}(ap + bo_i^{-1} + v^{-1}p - vp),\\
av^2 p + bv(v^{-1}o_i^{-1} - p) = ap + bo_i^{-1} + v^{-1}p - vp,\\
av^2 - bv = a + v^{-1} - v.
\end{gathered}$$

From 13.10 we see that $\tilde{T}_{\sigma_i}$ acts as a triangular matrix with respect to p, o_i^{-1}. This forces $a = -v^{-1}$ (hence $b = 0$) or $a = v$ (hence $b = v^2 - v^{-2}$). This last case is impossible since it would imply that $\tilde{T}_{\sigma_i}$ acts as a non-semisimple 2×2 matrix. Hence, $a = -v^{-1}$.

Lemma 13.12. *We have $\tilde{T}_{\sigma_0} p = -v^{-1}p + (v - v^{-1})o_0^{-1}$.*

By 10.22(a), the image of $K_{\mathbf{C}^*}(V_0) \to K_{\mathbf{C}^*}(\mathcal{B}_e)$ (direct image) is stable under $\tilde{T}_{\sigma_0}$. Hence, $\tilde{T}_{\sigma_0} p = ap + bo_0^{-1}$ for some $a, b \in \mathcal{A}$. Let x be such that $\check{\alpha}_0(x) = 1$. We have $\theta_{x-\alpha_0}\tilde{T}_{\sigma_0} p = (\tilde{T}_{\sigma_0} + v^{-1} - v)\theta_x p$. Hence,

$$\begin{gathered}
a\theta_{x-\alpha_0} p + b\theta_{x-\alpha_0} o_0^{-1} = (\tilde{T}_{\sigma_i} + v^{-1} - v) v^{n_0(x)} p,\\
a v^{n_0(x-\alpha_0)} p + b v^{n_0(x)}(o_0^{-1} - p) = v^{n_0(x)}(ap + bo_0^{-1} + v^{-1}p - vp),\\
ap + b(o_0^{-1} - p) = ap + bo_0^{-1} + v^{-1}p - vp.
\end{gathered}$$

Hence, $b = v - v^{-1}$. As in the proof of 13.11, we must have $a = v$ or $a = -v^{-1}$. If $a = v$, then $\tilde{T}_{\sigma_0}$ would act as a non-semisimple 2×2 matrix (since $b \neq 0$). This is impossible. Thus $a = -v^{-1}$.

Lemma 13.13. *For $i \neq 0$ we have $\tilde{T}_{\sigma_0} o_i^{-1} = -v^{-1}o_i^{-1} - o_0^{-1}$.*

By 10.22(a), the image of $K_{\mathbf{C}^*}(V_0 \cup V_i) \to K_{\mathbf{C}^*}(\mathcal{B}_e)$ (direct image) is stable under $\tilde{T}_{\sigma_0}$. Hence, $\tilde{T}_{\sigma_0} o_i^{-1} = a o_i^{-1} + bp + c o_0^{-1}$ for some $a, b, c \in \mathcal{A}$. Let x be such that $\check{\alpha}_0(x) = 1, \check{\alpha}_i(x) = 1$. We have

$$\theta_{x-\alpha_0} \tilde{T}_{\sigma_0} o_i^{-1} = (\tilde{T}_{\sigma_0} + v^{-1} - v)\theta_x o_i^{-1},$$
$$\theta_{x-\alpha_0}(a o_i^{-1} + bp + c o_0^{-1}) = v^{n_0(x)+1}(\tilde{T}_{\sigma_0} + v^{-1} - v)(v o_i^{-1} + p).$$

Now

$$\begin{aligned}
&\theta_{x-\alpha_0} o_i^{-1} = \theta_x \theta_{-\alpha_0} o_i^{-1} = \theta_x v^{n_0(-\alpha_0)+1}(v o_i^{-1} + p)\\
&= v(v^{n_0(x)+1}(v^2 o_i^{-1} + vp) + v^{n_0(x)} p) = v^{n_0(x)}(v^4 o_i^{-1} + v^3 p + vp),\\
&a v^{n_0(x)}(v^4 o_i^{-1} + v^3 p + vp) + b v^{n_0(x-\alpha_0)} p + c v^{n_0(x)}(o_0^{-1} - p) = v^{n_0(x)+1}\\
&\times (a v o_i^{-1} + bvp + c v o_0^{-1} + o_i^{-1} - v^2 o_i^{-1} - v^{-1}p + (v - v^{-1}) o_0^{-1} + v^{-1}p - vp),\\
&a(v^4 o_i^{-1} + v^3 p + vp) + bp + c(o_0^{-1} - p)\\
&= v(a v o_i^{-1} + bvp + c v o_0^{-1} + o_i^{-1} - v^2 o_i^{-1} - v^{-1}p + (v - v^{-1}) o_0^{-1} + v^{-1}p - vp).
\end{aligned}$$

The lemma follows.

Lemma 13.14. *For $i \neq 0$ we have $\tilde{T}_{\sigma_i} o_0^{-1} = -v^{-1} o_0^{-1} - o_i^{-1}$.*

By 10.22(a), the image of $K_{\mathbf{C}^*}(V_0 \cup V_i) \to K_{\mathbf{C}^*}(\mathcal{B}_e)$ (direct image) is stable under $\tilde{T}_{\sigma_i}$. Hence, $\tilde{T}_{\sigma_i} o_0^{-1} = a o_0^{-1} + bp + c o_i^{-1}$ for some $a, b, c \in \mathcal{A}$. Let x be such that $\check{\alpha}_0(x) = 1, \check{\alpha}_i(x) = 1$. We have

$$\theta_{x-\alpha_i} \tilde{T}_{\sigma_i} o_0^{-1} = (\tilde{T}_{\sigma_i} + v^{-1} - v)\theta_x o_0^{-1},$$
$$\theta_{x-\alpha_i}(a o_0^{-1} + bp + c o_i^{-1}) = v^{n_0(x)}(\tilde{T}_{\sigma_i} + v^{-1} - v)(o_0^{-1} + p).$$

Now

$$\begin{aligned}
&\theta_{x-\alpha_i} o_0^{-1} = \theta_x \theta_{-\alpha_i} o_0^{-1} = \theta_x v^{n_0(-\alpha_i)}(o_0^{-1} + p)\\
&= v^2(v^{n_0(x)}(o_0^{-1} + p) + v^{n_0(x)} p) = v^{n_0(x)}(v^2 o_0^{-1} + 2v^2 p),\\
&a v^{n_0(x)}(v^2 o_0^{-1} + 2v^2 p) + b v^{n_0(x-\alpha_i)} p + c v^{n_0(x)+1}(v^{-1} o_i^{-1} - p)\\
&= v^{n_0(x)}(a o_0^{-1} + bp + c o_i^{-1} + v^{-1} o_0^{-1} - v o_0^{-1} - v^{-1}p + v^{-1}p - vp),\\
&a(v^2 o_0^{-1} + 2v^2 p) + b v^2 p + cv(v^{-1} o_i^{-1} - p)\\
&= a o_0^{-1} + bp + c o_i^{-1} + v^{-1} o_0^{-1} - v o_0^{-1} - v^{-1}p + v^{-1}p - vp.
\end{aligned}$$

It follows that $a = -v^{-1}$ and $c = b(v - v^{-1}) - 1$.

The endomorphism $\tilde{T}_{\sigma_i}$ of our module of rank 3 modulo the span of o_i^{-1} is of the form $p \mapsto -v^{-1}p, o_0^{-1} \mapsto -v^{-1} o_0^{-1} + bp$. But it must be given by a semisimple 2×2-matrix. Hence, $b = 0$. It follows that $c = -1$. The lemma is proved.

Lemma 13.15. *If $i, j, 0$ are distinct, we have $\tilde{T}_{\sigma_i} o_j^{-1} = -v^{-1} o_j^{-1}$.*

By 10.22(a), the image of $K_{\mathbf{C}^*}(V_i \cup V_j) \to K_{\mathbf{C}^*}(\mathcal{B}_e)$ (direct image) is stable under $\tilde{T}_{\sigma_i}$. Hence, $\tilde{T}_{\sigma_i} o_j^{-1} = a o_j^{-1} + bp + c o_i^{-1}$ for some $a, b, c \in \mathcal{A}$. Let x be such that

$\check\alpha_i(x)=1,\check\alpha_j(x)=1$. We have

$$\begin{aligned}
&\theta_{x-\alpha_i}\tilde T_{\sigma_i}o_j^{-1}=(\tilde T_{\sigma_i}+v^{-1}-v)\theta_x o_j^{-1},\\
&\theta_{x-\alpha_i}(ao_j^{-1}+bp+co_i^{-1})=v^{n_0(x)+1}(\tilde T_{\sigma_i}+v^{-1}-v)(vo_j^{-1}+p),\\
&av^{n_0(x-\alpha_i)+1}(vo_j^{-1}+p)+bv^{n_0(x-\alpha_i)}p+cv^{n_0(x)+1}(v^{-1}o_i^{-1}-p)\\
&\ =v^{n_0(x)+1}(vao_j^{-1}+vbp+vco_i^{-1}+o_j^{-1}-v^2o_j^{-1}-v^{-1}p+v^{-1}p-vp),\\
&av^2(vo_j^{-1}+p)+bvp+c(v^{-1}o_i^{-1}-p)\\
&\ =vao_j^{-1}+vbp+vco_i^{-1}+o_j^{-1}-v^2o_j^{-1}-v^{-1}p+v^{-1}p-vp.
\end{aligned}$$

It follows that $a=-v^{-1},c=0$. The endomorphism $\tilde T_{\sigma_i}$ of our module of rank 3 modulo the span of o_i^{-1} is of the form $p\mapsto -v^{-1}p, o_j^{-1}\mapsto -v^{-1}o_j^{-1}+bp$. But it must be given by a semisimple 2×2-matrix. Hence, $b=0$. The lemma is proved.

Lemma 13.16. *The action of $\tilde T_{w_0}^{-1}$ on $K_H(\mathcal{B}_e)$ is given as follows.*

(a) $\tilde T_{w_0}^{-1}(o_i^{-1})=-v^6o_i^{-1}$ *for $i=0,1,2,3$,*

(b) $\tilde T_{w_0}^{-1}(p)=v^{12}p+v^6(v^2+1)(1-v^4)o_0^{-1}+v^6(v^2+1)(v-v^3)(o_1^{-1}+o_2^{-1}+o_3^{-1})$.

The $\mathcal{A}$-submodule (of rank 4) spanned by $o_i^{-1}(i\in I)$ is stable under $\tilde T_{\sigma_i}$ for $i=0,1,2,3$. Under the specialization $v=1$ this becomes the reflection representation of W tensor the sign representation. On this module, $\tilde T_{w_0}^{-1}$ must act as a scalar and we clearly have $\det\tilde T_{w_0}=(-v^{-2})^{12}=v^{-24}$; hence $\tilde T_{w_0}^{-1}=\pm v^6$. Setting $v=1$ we see that $\pm=-1$. Hence, (a) follows.

We can find uniquely $a,b\in\mathbf{Q}(v)$ so that the vector

$$\xi=p+ao_0^{-1}+b(o_1^{-1}+o_2^{-1}+o_3^{-1})$$

satisfies $\tilde T_{\sigma_i}\xi=-v^{-1}\xi$ for $i\in I$ (after extending the scalars to $\mathbf{Q}(v)$). Indeed, the condition that

$$\begin{aligned}
&\tilde T_{\sigma_0}(p+ao_0^{-1}+bo_1^{-1}+bo_2^{-1}+bo_3^{-1})\\
&\quad=-v^{-1}p+(v-v^{-1})o_0^{-1}+avo_0^{-1}\\
&\qquad+b(-v^{-1}o_1^{-1}-o_0^{-1}-v^{-1}o_2^{-1}-o_0^{-1}-v^{-1}o_3^{-1}-o_0^{-1})\\
&\quad=-v^{-1}(p+ao_0^{-1}+bo_1^{-1}+bo_2^{-1}+bo_3^{-1})
\end{aligned}$$

and

$$\begin{aligned}
&\tilde T_{\sigma_1}(p+ao_0^{-1}+bo_1^{-1}+bo_2^{-1}+bo_3^{-1})\\
&=-v^{-1}p-av^{-1}o_0^{-1}-ao_1^{-1}+bvo_1^{-1}-v^{-1}bo_2^{-1}-v^{-1}bo_3^{-1}\\
&=-v^{-1}p-v^{-1}ao_0^{-1}-v^{-1}bo_1^{-1}-v^{-1}bo_2^{-1}-v^{-1}bo_3^{-1}
\end{aligned}$$

is that $(v-v^{-1})+av-3b=-v^{-1}a$ and $-a+bv=-v^{-1}b$, so that

$$a=\frac{1-v^4}{v^4-v^2+1},\quad b=\frac{v-v^3}{v^4-v^2+1}.$$

We have $\tilde T_{w_0}^{-1}\xi=v^{12}\xi$. Hence,

$$\begin{aligned}
&\tilde T_{w_0}^{-1}(p+ao_0^{-1}+b(o_1^{-1}+o_2^{-1}+o_3^{-1}))=v^{12}(p+ao_0^{-1}+b(o_1^{-1}+o_2^{-1}+o_3^{-1}))\\
&=\tilde T_{w_0}^{-1}(p)-v^6(ao_0^{-1}+b(o_1^{-1}+o_2^{-1}+o_3^{-1})),\\
&\tilde T_{w_0}^{-1}(p)=v^{12}p+(v^{12}+v^6)ao_0^{-1}+(v^{12}+v^6)b(o_1^{-1}+o_2^{-1}+o_3^{-1}).
\end{aligned}$$

The lemma follows.

13.17. If a,b are as in 13.16, we have

$$\tilde T_{w_0}(p+ao_0^{-1}+b(o_1^{-1}+o_2^{-1}+o_3^{-1}))=v^{-12}(p+ao_0^{-1}+b(o_1^{-1}+o_2^{-1}+o_3^{-1}))$$
$$=\tilde T_{w_0}(p)-v^{-6}(ao_0^{-1}+b(o_1^{-1}+o_2^{-1}+o_3^{-1})),$$
$$\tilde T_{w_0}(p)=v^{-12}p+(v^{-12}+v^{-6})ao_0^{-1}+(v^{-12}+v^{-6})b(o_1^{-1}+o_2^{-1}+o_3^{-1}),$$
$$\tilde T_{w_0}(p)=v^{-12}p+v^{-12}(v^2+1)(1-v^4)o_0^{-1}$$
$$+v^{-12}(v^2+1)(v-v^3)(o_1^{-1}+o_2^{-1}+o_3^{-1}).$$

Lemma 13.18. *Consider an R_H-bilinear inner product $(,)$ on $K_H(\mathcal{B}_e)$ such that $(\chi\xi,\xi')=(\xi,\chi^\blacktriangle\xi')$ and $(\xi,\xi')=(\xi',\xi)$ for $\xi,\xi'\in K_H(\mathcal{B}_e),\chi\in\mathcal{H}$. There exists $c\in\mathcal{A}$ such that*

(a) $(o_0^{-1},o_i^{-1})=c$ *for* $i=1,2,3$,
(b) $(o_i^{-1},o_i^{-1})=-c(v+v^{-1})$ *for* $i=0,1,2,3$,
(c) $(o_j^{-1},o_i^{-1})=0$ *for* $i,j,0$ *distinct,*
(d) $(p,o_i^{-1})=0$ *for* $i=1,2,3$,
(e) $(p,o_0^{-1})=-c(v-v^{-1})$,
(f) $(p,p)=cv^{-6}(v^2+1)(1-v^4)(v-v^{-1})$.

For $i\ne 0$ we have $(\tilde T_{\sigma_i}o_0^{-1},o_i^{-1})=(o_0^{-1},\tilde T_{\sigma_i}o_i^{-1})$; hence

$$(-v^{-1}o_0^{-1}-o_i^{-1},o_i^{-1})=(o_0^{-1},vo_i^{-1}),\quad (o_i^{-1},o_i^{-1})=-(v+v^{-1})(o_0^{-1},o_i^{-1}).$$

Similarly, $(o_0^{-1},o_0^{-1})=-(v+v^{-1})(o_0^{-1},o_i^{-1})$; (a),(b) follow.

For $i,j,0$ distinct we have

$$(\tilde T_{\sigma_i}o_j^{-1},o_i^{-1})=(o_j^{-1},\tilde T_{\sigma_i}o_i^{-1}),\quad (-v^{-1}o_j^{-1},o_i^{-1})=(o_j^{-1},vo_i^{-1}).$$

Hence, $(v+v^{-1})(o_j^{-1},o_i^{-1})=0$ and (c) follows. For $i=1,2,3$, we have

$$(\tilde T_{\sigma_i}p,o_i^{-1})=(p,\tilde T_{\sigma_i}o_i^{-1}),\quad (-v^{-1}p,o_i^{-1})=(p,vo_i^{-1})$$

and (d) follows. We have

$$(\tilde T_{\sigma_0}p,o_0^{-1})=(p,\tilde T_{\sigma_0}o_0^{-1}),\quad (-v^{-1}p+(v-v^{-1})o_0^{-1},o_0^{-1})=(p,vo_0^{-1}).$$

Hence,

$$(v+v^{-1})(p,o_0^{-1})=(v-v^{-1})(o_0^{-1},o_0^{-1})=-c(v-v^{-1})(v+v^{-1})$$

and (e) follows.

Assume that $\check\alpha_0(x)=1$. Then

$$(\theta_xp,o_0^{-1})=(p,\tilde T_{w_0}^{-1}\theta_x\tilde T_{w_0}o_0^{-1})=(\tilde T_{w_0}^{-1}p,-\theta_xv^{-6}o_0^{-1}),$$
$$v^{n_0(x)}(p,o_0^{-1})=(v^{12}p+v^6(v^2+1)(1-v^4)o_0^{-1}$$
$$+v^6(v^2+1)(v-v^3)(o_1^{-1}+o_2^{-1}+o_3^{-1}),-v^{-6}v^{n_0(x)}(o_0^{-1}+p)),$$
$$-c(v-v^{-1})=cv^6(v-v^{-1})+c(v^2+1)(1-v^4)(v-v^{-1})-v^6(p,p)$$
$$+c(v^2+1)(1-v^4)(v+v^{-1})-3c(v^2+1)(v-v^3)$$

and (f) follows. The lemma is proved.

Lemma 13.19. *We have $(,)_{\mathcal{B}_e}=(,)$ with $c=-v^5$.*

By 13.18, it is enough to show that $(o_0^{-1}, o_i^{-1})_{\mathcal{B}_e} = -v^5$ for $i = 1,2,3$. Since V_0, V_i intersect transversally (at p_{i0}) in Λ_e, we have $(o_0^{-1} : k_*(o_i^{-1})) = v^N$ where N is the weight of the $\mathbf{C}^*$-action on $(o_0^{-1})_{p_{i0}} \otimes (o_i^{-1})_{p_{i0}}$, that is $N = 0 + 1 = 1$. We have $\tilde{T}_{w_0} o_i^{-1} = -v^{-6} o_i^{-1}$; hence

$$(o_0^{-1}, o_i^{-1})_{\mathcal{B}_e} = (-v)^{11-1}(o_0^{-1} : (-v^{-6})o_i^{-1}) = -v^4 v^N = -v^5,$$

since $d(e) = 11, b(e) = 1$.

Proposition 13.20. $\mathbf{B}^{\pm}_{\mathcal{B}_e}$ *is the signed basis of the* $\mathcal{A}$*-module* $K_H(\mathcal{B}_e)$ *consisting of* $\pm$ *the elements*

$$v^{-3}o_i^{-1} \quad \textit{for } i \in I, \quad \textit{and } p - (1 + v^{-2})o_0^{-1} - v^{-1}(o_1^{-1} + o_2^{-1} + o_3^{-1}).$$

In this proof we write $(,)$ instead of $(,)_{\mathcal{B}_e}$. We have

$$\overline{o_i^{-1}} = (-v)^{-12}\tilde{T}_{w_0}^{-1}(-o_i^{-1}) = (-v)^{-12}(-v^6)(-o_i^{-1}) = v^{-6}o_i^{-1};$$

hence $\overline{v^{-3}o_i^{-1}} = v^{-3}o_i^{-1}$. We have $(v^{-3}o_i^{-1}, v^{-3}o_i^{-1}) = v^{-6}v^5(v + v^{-1}) = 1 + v^{-2}$. Hence, $v^{-3}o_i^{-1} \in \mathbf{B}^{\pm}_{\mathcal{B}_e}$. We have

$$\begin{aligned}
\bar{p} &= (-v)^{-12}\tilde{T}_{w_0}^{-1}p \\
&= p + v^{-6}(v^2+1)(1-v^4)o_0^{-1} + v^{-6}(v^2+1)(v-v^3)(o_1^{-1} + o_2^{-1} + o_3^{-1}) \\
&= p + (v^{-4} + v^{-6} - 1 - v^{-2})o_0^{-1} + (v^{-5} - v^{-1})(o_1^{-1} + o_2^{-1} + o_3^{-1}).
\end{aligned}$$

Hence, $p-(1+v^{-2})o_0^{-1} - v^{-1}(o_1^{-1} + o_2^{-1} + o_3^{-1})$ is fixed by $^-$. The self-inner product of $p - (1+v^{-2})o_0^{-1} - v^{-1}(o_1^{-1} + o_2^{-1} + o_3^{-1})$ is again $1 + v^{-2}$ (by calculation). Thus the elements described in the proposition belong to $\mathbf{B}^{\pm}_{\mathcal{B}_e}$. They form an $\mathcal{A}$-basis of $K_{\mathbf{C}^*}(\mathcal{B}_e)$. Next note that

$(v^{-3}o_i^{-1}, v^{-3}o_j^{-1}) = 0$ for $i, j, 0$ distinct,
$(v^{-3}o_i^{-1}, v^{-3}o_0^{-1}) = -v^{-1}$ for $i \neq 0$,
$(v^{-3}o_i^{-1}, p - (1+v^{-2})o_0^{-1} - v^{-1}(o_1^{-1} + o_2^{-1} + o_3^{-1})) = 0$ for $i \neq 0$,
$(v^{-3}o_0^{-1}, p - (1+v^{-2})o_0^{-1} - v^{-1}(o_1^{-1} + o_2^{-1} + o_3^{-1})) = -v^{-1}$.

Now the proposition follows from the argument in 12.20.

13.21. Let $V'_0 = \Lambda_{e,\mu_0}$. (See 13.3, 11.3.) Then V'_0 is naturally a line bundle over V_0 (see 11.4(b)). For $i = 1,2,3$, let V'_i be the $\mathbf{C}^*$-stable line in Λ_e that flows to q_i for $\lambda \to \infty$ (see 11.4(b)). Let $j'_i : V'_i \subset \Lambda_e$ be the inclusion. We define a line bundle $D_i \in \mathrm{Vec}_H(\Lambda_e)$ such that there is an exact sequence in $\mathrm{Coh}_H(\Lambda_e)$

$$0 \to D_i \to \mathbf{C} \to j'_{i*}\mathbf{C} \to 0.$$

If $x \in \mathcal{X}$, we can regard L_x as an object of $\mathrm{Vec}_H(\Lambda_e)$ via inverse image under $\Lambda_e \to \mathcal{B}$, $(y, \mathfrak{b}) \mapsto \mathfrak{b}$.

Let $x_0 \in \mathcal{X}$ be such that $\check{\alpha}_0(x_0) = -1, \check{\alpha}_i(x_0) = 0$ for $i \neq 0$. Then $x_0 = -2\alpha_0 - \alpha_1 - \alpha_2 - \alpha_3$; hence $n_0(x_0) = 6$.

Lemma 13.22. $K_H(\Lambda_e)$ *has an* $\mathcal{A}$*-basis consisting of* D_i $(i = 1,2,3), \mathbf{C}, L_{x_0}$.

This follows easily from 11.4(c).

13.23. Let $d_i = j'_{i*}\mathbf{C} \in K_H(\Lambda_e)$. We have

$(p : d_i) = 0$,
$(o_j^{-1} : d_i) = 0$ for $j \in I - \{i\}$ (since supports are disjoint),
$(o_i^{-1} : d_i) = v^{-1}$, (since V_i, V'_i intersect transversally at q_i in Λ_e),
$(p : \mathbf{C}) = 1$,
$(o_i^{-1} : \mathbf{C}) = 0$,

(we use 6.7 and the fact that the cohomology of P^1 with coefficients in

O^{-1} is zero.) We have
$(p : L_{x_0}) = v^{n_0(x_0)} = v^6$,
$(o_i^{-1} : L_{x_0}) = 0$ for $i \neq 0$,
$(o_0^{-1} : L_{x_0}) = -v^6$;

(we use 6.7 and the fact that the cohomology of P^1 with coefficients in O^{-2} is $-\mathbf{C}$).

13.24. We have
$(p||\mathbf{C}) = v^{10}(\tilde{T}_{w_0}p : \mathbf{C}) = v^{-2}$,
$(o_i^{-1}||\mathbf{C}) = 0$, for $i \in I$,
$(p - (1 + v^{-2})o_0^{-1} - v^{-1}(o_1^{-1} + o_2^{-1} + o_3^{-1})||\mathbf{C}) = v^{-2}$.

13.25. For $i = 1, 2, 3$, we compute (using 13.16, 13.17)

$$(p||d_i) = v^{10}(\tilde{T}_{w_0}p : d_i) = v^{10}(v^{-12}p + v^{-12}(v^2+1)(1-v^4)o_0^{-1} + v^{-12}(v^2+1)(v-v^3)(o_1^{-1} + o_2^{-1} + o_3^{-1}) : d_i) = v^{-3}(v^2+1)(v-v^3),$$

$$(o_j^{-1}||d_i) = v^{10}(\tilde{T}_{w_0}o_j^{-1}||d_i) = v^{10}(-v^{-6}o_j^{-1}||d_i) = 0, \quad \text{if } j \neq i,$$
$$(o_i^{-1}||d_i) = v^{10}(\tilde{T}_{w_0}o_i^{-1}||d_i) = v^{10}(-v^{-6}o_i^{-1}||d_i) = -v^{10-6-1} = -v^3,$$
$$(p - (1+v^{-2})o_0^{-1} - v^{-1}(o_1^{-1} + o_2^{-1} + o_3^{-1})||d_i) = v^{-3}(v^2+1)(v-v^3) + v^2 = v^{-2}.$$

Since $D_i = \mathbf{C} - d_i$, we have
$(p - (1+v^{-2})o_0^{-1} - v^{-1}(o_1^{-1} + o_2^{-1} + o_3^{-1})||D_i) = 0$,
$(o_j^{-1}||D_i) = 0$ if $j \neq i$,
$(o_i^{-1}||D_i) = v^3$.

13.26. We have

$$\begin{aligned}(p||L_{x_0}) &= v^{10}(\tilde{T}_{w_0}p : L_{x_0}) = v^{10}(v^{-12}p + v^{-12}(v^2+1)(1-v^4)o_0^{-1} \\ &\qquad + v^{-12}(v^2+1)(v-v^3)(o_1^{-1} + o_2^{-1} + o_3^{-1}) : L_{x_0}) \\ &= v^6v^{-2}(1-(v^2+1)(1-v^4)) = v^4(v^6+v^4-v^2) = v^{10}+v^8-v^6,\end{aligned}$$

$(o_i^{-1}||L_{x_0}) = 0$ for $i \neq 0$,
$(o_0^{-1}||L_{x_0}) = v^{10}(\tilde{T}_{w_0}o_0^{-1} : L_{x_0}) = v^{10}(-v^{-6})(-v^6) = v^{10}$,

$$\begin{aligned}&(p - (1+v^{-2})o_0^{-1} - v^{-1}(o_1^{-1} + o_2^{-1} + o_3^{-1})||L_{x_0}) \\ &= v^{10}+v^8-v^6-(1+v^{-2})v^{10} = v^{10}+v^8-v^6-v^{10}-v^8 = -v^6.\end{aligned}$$

Hence,
$(p - (1+v^{-2})o_0^{-1} - v^{-1}(o_1^{-1} + o_2^{-1} + o_3^{-1})||v^{-7}L_{x_0} + v\mathbf{C}) = 0$,
$(o_0^{-1}||v^{-7}L_{x_0} + v\mathbf{C}) = v^3$,
$(o_i^{-1}||v^{-7}L_{x_0} + v\mathbf{C}) = 0$ for $i \neq 0$.

Proposition 13.27. $\mathbf{B}^{\pm}_{\Lambda_e}$ *is the signed basis of the $\mathcal{A}$-module $K_H(\Lambda_e)$ consisting of $\pm$ the line bundles D_i $(i = 1, 2, 3)$, $v^2\mathbf{C}$ and the two-dimensional vector bundle $v^{-7}L_{x_0} + v\mathbf{C}$.*

We identify $K_H(\Lambda_e)$ with an $\mathcal{A}$-lattice in the $\mathbf{Q}(v)$-vector space $E = \mathbf{Q}(v) \otimes_{\mathcal{A}} K_H(\Lambda_e)$ (see 13.22) and $K_H(\mathcal{B}_e)$ with a sublattice of $K_H(\Lambda_e)$ via k_* (see 12.5, 11.8). There is a well defined symmetric $\mathbf{Q}(v)$-bilinear form $(,)$ on E whose restriction to $K_H(\Lambda_e)$ (resp. $K_H(\mathcal{B}_e)$) is $(,)_{\Lambda_e}$ (resp. $(,)_{\mathcal{B}_e}$). Using 13.24, 13.25, 13.26, we see that the elements

(a) $D_i(i = 1, 2, 3), v^2\mathbf{C}, v^{-7}L_{x_0} + v\mathbf{C}$

form the basis of E dual to the basis

(b) $v^{-3}o_i^{-1}(i = 1, 2, 3), p - (1 + v^{-2})o_0^{-1} - v^{-1}(o_1^{-1} + o_2^{-1} + o_3^{-1}), v^{-3}o_0^{-1}$

with respect to $(,)$. (In particular, $(,)$ is non-singular on E.) Since the matrix of $(,)$ with respect to the basis (b) is congruent to the identity matrix modulo $v^{-1}\mathbf{Z}[v^{-1}]$ (by 13.18, 13.19), its inverse, that is the matrix of $(,)$ with respect to the basis (a), is congruent to the identity matrix modulo $v^{-1}\mathbf{Z}[[v^{-1}]]$.

In particular, $(\xi, \xi) \in 1 + v^{-1}\mathbf{Z}[[v^{-1}]]$ for all ξ in (a).

Using 12.14, we see that the elements $\bar{\xi}$ with ξ in (a) form again the basis of E dual to the basis (b) with respect to $(,)$. Since $(,)$ is non-singular on E, we must have $\bar{\xi} = \xi$ for ξ in (a).

We see therefore that $\xi \in \mathbf{B}^{\pm}_{\Lambda_e}$ for all ξ in (a). The elements (a) form a basis of the $\mathcal{A}$-module $K_H(\Lambda_e)$, by 13.22. By an argument similar to that in 12.20, we see that any element in $\mathbf{B}^{\pm}_{\Lambda_e}$ is, up to sign, as in (a). The proposition is proved.

13.28. We see that in our case, Conjecture 12.19 holds. Note that Conjecture 12.23 also holds in our case: $\mathbf{B}_{\mathcal{B}_e,\mathrm{ad}}$ consists of

$$\xi = -v^{-3}o_i^{-1}(i = 1, 2, 3), -p + (1 + v^{-2})o_0^{-1} + v^{-1}(o_1^{-1} + o_2^{-1} + o_3^{-1}), v^{-3}o_0^{-1};$$

$\mathbf{B}_{\Lambda_e,\mathrm{ad}}$ consists of $-D_i(i = 1, 2, 3), -v^2\mathbf{C}, v^{-7}L_{x_0} + v\mathbf{C}$.

13.29. The results of this section can be generalized to the case where G is of type D_n or E_n and (e, f, h) is subregular. This will be discussed elsewhere.

14. Comments

14.1. This section contains a (non-rigorous) discussion of possible connections with the theory [J1],[J2],[J3] of unrestricted representations of Lie algebras over $\mathbf{k}$, an algebraic closure of the field with p elements.

We assume that p is large enough. Let $\mathfrak{g}'$ be the Lie algebra of a semisimple simply connected algebraic group G' over $\mathbf{k}$ of the same type as G. For any linear form χ on $\mathfrak{g}'$, let U_χ be the quotient of the enveloping algebra of $\mathfrak{g}'$ by the ideal generated by the elements $x^p - x^{[p]} - \chi(x)^p$ with $x \in \mathfrak{g}'$. (Here $x \mapsto x^{[p]}$ is the pth power map of $\mathfrak{g}'$ into itself.) Then U_χ is a finite dimensional algebra and any simple $\mathfrak{g}'$-module can be regarded as a module over U_χ for a unique χ as above (Kac-Weisfeiler). We fix χ and identify it with an element of $\mathfrak{g}'$ via the Killing form. We assume that χ is nilpotent. Let C' be a maximal torus of the centralizer of χ in G' and let $\bar{C}'$ be the image of C' in the adjoint group of G'. Let $\mathcal{C}_\chi$ be the category of U_χ-modules (of finite dimension over $\mathbf{k}$) which are also $\bar{C}'$-modules in a compatible way (as explained in [J1]). We fix a "generic block" of this category. Let $\mathbf{I}$ be an indexing set for the simple objects in this block. For $\mathbf{i} \in \mathbf{I}$, let $L_{\mathbf{i}}$ be the corresponding simple object of $\mathcal{C}_\chi$ and let $Q_{\mathbf{i}}$ be the corresponding indecomposable projective object. For $\mathbf{i} \in \mathbf{I}$ we have $Q_{\mathbf{i}} = \sum_{\mathbf{i}' \in \mathbf{I}} n_{\mathbf{i},\mathbf{i}'} L_{\mathbf{i}'}$ in the appropriate Grothendieck group, where $n_{\mathbf{i},\mathbf{i}'} \in \mathbf{N}$ are zero for all but finitely many $\mathbf{i}'$.

14.2. Let (e, f, h) be an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$. Let C and $H = C \times \mathbf{C}^*$ be as in 11.1. We assume that the nilpotent element $e \in \mathfrak{g}$ is of the same type as the nilpotent element $\chi \in \mathfrak{g}'$. We assume that Conjectures 12.19, 12.23 hold in our case. Let $\mathfrak{B}$ be an indexing set for $\mathbf{B}_{\mathcal{B}_e,\mathrm{ad}}$ and for $\mathbf{B}_{\Lambda_e,\mathrm{ad}}$. For $\mathbf{b} \in \mathfrak{B}$ let $\beta_{\mathbf{b}}$ be the corresponding element of $\mathbf{B}_{\mathcal{B}_e,\mathrm{ad}}$ and let $\tilde{\beta}_{\mathbf{b}}$ be the corresponding element of $\mathbf{B}_{\Lambda_e,\mathrm{ad}}$. For $\mathbf{b} \in \mathfrak{B}$ we can write $k_*(\beta_{\mathbf{b}}) = \sum_{\mathbf{b}' \in \mathfrak{B}} N_{\mathbf{b},\mathbf{b}'} \tilde{\beta}_{\mathbf{b}'}$ where k_* is as in 12.5 and $N_{\mathbf{b},\mathbf{b}'} \in \mathcal{A}$ are zero for all but finitely many $\mathbf{b}'$. The sum is taken in $K_H(\Lambda_e)$. From 12.23(b) (which we assume) it follows that $N_{\mathbf{b},\mathbf{b}'} \in \mathbf{N}[v, v^{-1}]$.

14.3. In the case where $\chi = 0$, one can combine the known results on restricted representations of $\mathfrak{g}'$ with the K-theory constructions in this paper to deduce that

(a) *there exists a natural bijection* $\zeta : \mathbf{I} \xrightarrow{\sim} \mathfrak{B}$ *such that* $n_{\mathbf{i},\mathbf{i}'} = N_{\zeta(\mathbf{i}),\zeta(\mathbf{i}')}|_{v=1}$ *for all* $\mathbf{i}, \mathbf{i}' \in \mathbf{I}$.

14.4. Let us now assume that χ is a nilpotent element of $\mathfrak{g}'$ which is regular inside some Levi subalgebra $\mathfrak{l}$ of a parabolic algebra of $\mathfrak{g}'$. Let $W_{\mathfrak{l}}$ be the Weyl group of $\mathfrak{l}$. It seems likely that the following generalization of 14.3(a) continues to hold:

(a) *there exists a natural bijection* $\zeta : \mathbf{I} \xrightarrow{\sim} \mathfrak{B}$ *such that* $n_{\mathbf{i},\mathbf{i}'} = |W_{\mathfrak{l}}| N_{\zeta(\mathbf{i}),\zeta(\mathbf{i}')}|_{v=1}$ *for all* $\mathbf{i}, \mathbf{i}' \in \mathbf{I}$.

The factor $|W_{\mathfrak{l}}|$ is needed in view of [J3, 11.18].

(Note that both $\mathbf{I}$ and $\mathfrak{B}$ have natural actions of a free abelian group of rank $\dim C'$ (with finitely many orbits) and the bijection ζ should be compatible with these actions.)

14.5. One could hope that the statement 14.4(a) remains true when χ is any nilpotent element of $\mathfrak{g}'$ which is distinguished inside $\mathfrak{l}$ (with $\mathfrak{l}, |W_{\mathfrak{l}}|$ as in 14.4).

This would predict for example that, if $\mathfrak{g}'$ is of type D_4 and χ is subregular, then $|\mathbf{I}| = 5$.

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