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A KLOOSTERMAN SUM IN A RELATIVE TRACE FORMULA FOR GL_4

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Abstract. We study a Kloosterman sum for GL_4 and prove that it is equal to an exponential sum over a quadratic number field. This identity has applications in a relative trace formula for GL_4 which might be used to give a new proof of quadratic base change and characterize its image.

1. Introduction

A main result of this article is the following identity of exponential sums.

Theorem 1. *Let τ be a non-zero square-free integer which is not equal to 1. Let b be a non-zero integer and c a positive odd integer such that $(b,c)=(\tau,c)=1$. Then*

$$\sum_{\substack{1\le x_i\le c,\\ (x_i,c)=1\\ \text{for } i=1,\dots,4}} e^{2\pi i\left(x_1+x_2+\bar{x}_2\bar{x}_3+x_2x_3\bar{x}_4+b\bar{x}_1x_4+b\bar{x}_1x_3\right)/c}$$
$$(1)\qquad = \sum_{\substack{1\le n,x_1,x_2,y_1,y_2\le c,\\ (n,c)=(x_1^2-\tau y_1^2,c)=(x_2^2-\tau y_2^2,c)=1,\\ x_1^2-\tau y_1^2\equiv x_2^2-\tau y_2^2 \pmod c}} e^{2\pi i\left(n+bn(\overline{x_1^2-\tau y_1^2})+2\bar{n}(x_1+x_2)\right)/c}.$$

Here we denote by $\bar{x}$ the inverse of x modulo c.

The sum on the left side of the above identity can be regarded as a generalization of the classical Kloosterman sum

$$\sum_{\substack{1\le x\le c,\\ (x,c)=1}} e^{2\pi i(x+b\bar{x})/c}$$

where $(b,c)=1$ and $x\bar{x}\equiv 1 \pmod c$. We will see that it is indeed a Kloosterman sum for the group GL_4. The expression on the right side of (1) is an exponential sum taken over certain algebraic integers of the quadratic number field $\mathbb{Q}(\sqrt{\tau})$. Other identities of this kind have been studied by several authors including Zagier [24], Katz [12], Jacquet and Ye [9], Duke and Iwaniec [3], Ye [21] and [23], and Mao and Rallis [13]. Some of these known identities have applications in automorphic forms and representation theory. To look at similar applications of the identity in Theorem 1 let us consider its p-adic version.

Let F be a p-adic field of characteristic zero and $L=F(\sqrt{\tau})$ an unramified quadratic extension field of F with $\tau\in F$. Assume that $|2|_F=|\tau|_F=1$. Denote

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by A the group of diagonal matrices in GL_4, by N the group of upper triangular matrices with unit diagonal entries in GL_4, and by $K(F)$ the maximal compact subgroup of $GL_4(F)$ which consists of matrices with entries in R_F and determinants in $R_F^\times$. Here R_F is the ring of integers in F and $R_F^\times$ is the group of invertible elements in R_F. Let $w = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix}$ and $a = \mathrm{diag}(a_1, a_1, a_2, a_2)$ with $a_1, a_2 \in F^\times$. Denote by $U_w(F)$ the subgroup of $N(F)$ consisting of matrices whose entries at $(1, 2)$ and $(3, 4)$ positions are both zero, and by $N_w(F)$ the subgroup of $N(L)$ defined by ${}^t\bar{n}wn = w$.

Let ψ_F be a non-trivial character of F of order zero; hence ψ_F is trivial on R_F but non-trivial on $\varpi_F^{-1}R_F$, where ϖ_F is a prime element in F. Define a character θ_F on $N(F)$ by $\theta_F(n) = \psi_F(\sum_{1\le i<n} n_{i,i+1})$ for $n = (n_{ij}) \in N(F)$.

Theorem 2. *With the above assumptions, notation, and matrices w and a, for any $a_1, a_2 \in F^\times$ we have*

$$\int\limits_{\substack{u\in U_w(F),\, n\in N(F)\\ {}^tuwan\in K(F)}} \theta_F(un)\, du\, dn = \int\limits_{\substack{n\in N_w(F)\backslash N(L),\\ {}^t\bar{n}wan\in K(L)}} \theta_F(n\bar{n})\, dn. \tag{2}$$

The left side of (2) is a p-adic Kloosterman sum for GL_4 in the integral form as in Friedberg [4] and Stevens [14]. To see the significance of the identities in Theorems 1 and 2, let us introduce a relative trace formula for GL_n and look at its applications in representation theory.

Let E be an algebraic number field and $E' = E(\sqrt{\tau})$ a quadratic extension of E with $\tau \in E^\times$. Denote by $E_{\mathbf{A}}$ and $E'_{\mathbf{A}}$ the adele rings of E and E', respectively, and by $E_{\mathbf{A}}^\times$ and $E'^\times_{\mathbf{A}}$ the idele groups of E and E', respectively. Then $E_{\mathbf{A}}$ is the restricted product of local fields E_v over all places v of E and $E'_{\mathbf{A}}$ is the restricted product of local fields E'_w over all places w of E'. For $z = \prod_w z_w$ in $E'_{\mathbf{A}}$ the Galois conjugation $z \mapsto \bar{z}$ is defined using the Galois conjugation on E' over E. Denote by $\mathrm{N}_{E'_{\mathbf{A}}/E_{\mathbf{A}}}$ the global norm map. Then the global norm-one group $E'^1_{\mathbf{A}}$ is the kernel of $\mathrm{N}_{E'_{\mathbf{A}}/E_{\mathbf{A}}}$, and we have $E'^1_{\mathbf{A}} = \{z/\bar{z} | z \in E'^\times_{\mathbf{A}}\}$. If v is inert in E', we denote by E_v^+ the group of elements of $E_v^\times$ which are norms. Define $E_{\mathbf{A}}^+$ as the group of $z = \prod_v z_v \in E_{\mathbf{A}}^\times$ such that $z_v \in E_v^+$ for every inert place v.

From the exact sequence

$$1 \longrightarrow E'^1_{\mathbf{A}} \longrightarrow E'^\times_{\mathbf{A}} \xrightarrow{\mathrm{N}_{E'_{\mathbf{A}}/E_{\mathbf{A}}}} E_{\mathbf{A}}^+ \longrightarrow 1$$

we know that if an idele class character χ' of E' is trivial on $E'^1_{\mathbf{A}}$, then there is an idele class character χ of E such that $\chi' = \chi \circ \mathrm{N}_{E'_{\mathbf{A}}/E_{\mathbf{A}}}$. This character χ is uniquely determined up to a multiplication by the idele class character η of E attached to the quadratic extension field E'.

We note that $E^\times$ and $E'^\times$ are indeed $GL_1(E)$ and $GL_1(E')$, respectively. Also, E'^1 is actually the unitary group of one variable in E' over E. This suggests a possible generalization of the above example to GL_n.

Let $S(E)$ be the set of invertible Hermitian matrices in $GL_n(E')$. For any $s \in S(E)$ we denote by $H_s(E)$ the corresponding unitary group:

$$H_s(E) = \{h \in GL_n(E') |\, {}^t\bar{h}sh = s\}.$$

An automorphic irreducible cuspidal representation π' of $GL_n(E'_{\mathbf{A}})$ with central character ω' is said to be H_s-distinguished if the periodic integral

$$\mu(\phi) = \int_{H_s(E)\backslash H_s(E_{\mathbf{A}})} \phi(h)\, dh$$

is a non-zero linear form on the space of π'. Then the proposition below is a generalization of our example to GL_n.

Proposition 1. *Let π' be an H_s-distinguished representation of $GL_n(E'_{\mathbf{A}})$ with central character ω' for a unitary group H_s. Then π' is the quadratic base change of an automorphic irreducible cuspidal representation π of $GL_n(E_{\mathbf{A}})$ with a central character ω. The central characters satisfy the condition $\omega' = \omega \circ \mathrm{N}_{E'_{\mathbf{A}}/E_{\mathbf{A}}}$.*

In the case of GL_2 a representation π' is H_s-distinguished if its Asai L-function has a pole at $s = 1$ (Asai [2]). For $n > 2$ a similar situation is also true. Therefore, Proposition 1 characterizes quadratic base change by analytic behavior of L-functions.

A proof of this proposition can be found in Harder, Langlands, and Rapoport [6] and Jacquet [7]. The converse of this proposition is expected to be true also but the proof appears much more difficult. For GL_2 the converse is proved in Harder, Langlands, and Rapoport [6], Ye [16] and [17], and Jacquet and Ye [8]. For GL_3 it is proved in a series of papers by Jacquet and Ye ([7], [9], [10], [11], and [18]).

The main technique used in [7] through [11] and [16] through [18] is a relative trace formula which is indeed an equality of two trace formulas. On one side is a Kuznetsov trace formula and on the other side is a so-called relative Kloosterman integral.

First let us look at the Kuznetsov trace formula for GL_n. Let $f = \prod_v f_v$ be a smooth function of compact support on $GL_n(E_{\mathbf{A}})$. We want to assume that at any inert place v of E the local function f_v is supported on the group $GL_n^+(E_v) = \{g \in GL_n(E_v) \big| \det g \in E_v^+\}$. Let χ be an idele class character of E. Then we define the kernel function

$$K_f(g,h) = \int_{Z^+(E)\backslash Z^+(E_{\mathbf{A}})} \sum_{\xi \in GL_n(E)} f(zg^{-1}\xi h)\chi(z)\, d^\times z \tag{3}$$

where $Z^+(E)$ is the set of matrices $\mathrm{diag}(z, \dots, z)$ in the center $Z(E)$ of $GL_n(E)$ with z being in the subgroup E^+ of $E^\times$ consisting of norms. Let $\psi = \prod_v \psi_v$ be a non-trivial additive character of $E_{\mathbf{A}}$ trivial on E such that for almost all v the local character ψ_v has order zero, i.e., ψ_v is trivial on the ring of integer R_v of E_v but non-trivial on $\varpi_v^{-1}R_v$, where ϖ_v is a prime element in R_v. Denote by N the group of upper triangular matrices with unit diagonal entries. We define a character θ on N by $\theta(n) = \psi(\sum_{1\le i<n} n_{i,i+1})$ for $n = (n_{ij}) \in N$. Then the Kuznetsov trace formula is given by the integral

$$\int_{N(E)\backslash N(E_{\mathbf{A}})} \int_{N(E)\backslash N(E_{\mathbf{A}})} K_f({}^t n_1, n_2)\theta(n_1^{-1}n_2)\, dn_1\, dn_2. \tag{4}$$

To have the relative Kloosterman integral, we set $S^+(E)$ to be the set of $s \in S(E)$ such that $\det s \in E^+$. Let $\Phi = \prod_v \Phi_v$ be a smooth function of compact support on

$S^+(E_{\mathbf{A}})$. Now we define a kernel function

$$K_\Phi(g) = \int\limits_{Z^+(E)\backslash Z^+(E_{\mathbf{A}})} \sum_{\xi \in S(E)} \Phi(z^t\bar{g}\xi g)\chi(z)\,dz$$

and define the relative Kloosterman integral

$$\int\limits_{N(E')\backslash N(E'_{\mathbf{A}})} K_\Phi(n)\theta(n\bar{n})\,dn. \tag{5}$$

The relative trace formula is then

$$\int\limits_{N(E)\backslash N(E_{\mathbf{A}})} \int\limits_{N(E)\backslash N(E_{\mathbf{A}})} K_f({}^tn_1, n_2)\theta(n_1^{-1}n_2)\,dn_1\,dn_2 = \int\limits_{N(E')\backslash N(E'_{\mathbf{A}})} K_\Phi(n)\theta(n\bar{n})\,dn.$$

Here the equality means that for a given smooth function $f = \prod_v f_v$ of compact support on $GL(n, F_A)$ there exists a smooth function $\Phi = \prod_v \Phi_v$ of compact support on $S(F_A)$ or a finite sum of these Φ, and vice versa, such that the above relative trace formula holds. There are restrictions on the way in which one chooses these functions:

(i) The matching of f and Φ should be made through matching of local functions f_v and Φ_w.

(ii) At an inert unramified non-Archimedean place v of E the characteristic function f_0 of $K(E_v)$ should be matched with the characteristic function Φ_0 of $K(E'_w) \cap S(E_v)$.

(iii) At a non-Archimedean place v of E which splits into w_1 and w_2 the characteristic function of $K(E_v)$ should be matched with $\Phi_1 \otimes \Phi_2$ via convolution where Φ_i is the characteristic function of $K(E'_{w_i}) \cap S(E_v)$.

(iv) At an inert unramified non-Archimedean place v of E, a compactly supported bi-$K(E_v)$-invariant function f_v should be matched with a function Φ_w via the base change map of Hecke algebras (see Arthur and Clozel [1]).

(v) At a non-Archimedean place v of E which splits into w_1 and w_2 a compactly supported bi-$K(E_v)$-invariant function f_v should be matched with a function of the form $\Phi_1 \otimes \Phi_2$ via convolution. Here the convolution is used as the base change map of Hecke algebras in splitting cases.

The matchings in (iv) and (v) are called the fundamental lemma of the relative trace formula while the matchings in (ii) and (iii) are called the fundamental lemma for unit elements of Hecke algebras. The splitting cases (iii) and (v) are easy. Proving the matching in (ii) and hence the the fundamental lemma for unit elements is the first step in establishing the relative trace formula. Once this has been done, one might be able to prove the matchings in (iv) using the techniques discussed in Ye [20]. One might then be able to deduce certain matchings in (i) from the fundamental lemma using the Shalika germ expansions introduced in Jacquet and Ye [10] and [11] and exponential sum expansions in Ye [19] and [20]. To apply the relative trace formula to base change problems one needs to study continuous spectrum of the relative trace following the work of Jacquet [7].

The present work is a step toward a proof of the matchings in (ii) for GL_4, i.e., the fundamental lemma for unit elements of Hecke algebras. More precisely, Theorem 2 proves the matchings in (ii) for GL_4 for certain local orbital integrals which will be defined below.

By the Bruhat decomposition the group $GL_n(E)$ can be decomposed into the disjoint union of double cosets ${}^tN(E)wA(E)N(E)$, where w goes over the Weyl group W of the group A of diagonal matrices. Applying this decomposition to the sum in the kernel function $K_f(g,h)$ in (3) we can express the Kuznetsov trace formula in (4) as a sum of global orbital integrals

$$\sum_w \sum_a \int_{Z^+(E_{\mathbf{A}})} I(waz,f)\chi(z)\,dz$$

where

$$I(wa,f)=\int_{\substack{u\in U_w(E_{\mathbf{A}}),\\ n\in N(E_{\mathbf{A}})}} f({}^tuwan)\theta(un)\,du\,dn.$$

Here the sums are taken over w and a of the form

$$w=\begin{pmatrix} w_1 & & \\ & \ddots & \\ & & w_r\end{pmatrix},\qquad a=\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_r\end{pmatrix}, \tag{6}$$

where $w_i=\begin{pmatrix} & & 1\\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}\in GL_{n_i}(E)$, a_i is in the center of $GL_{n_i}(E)$, and $n_1+\cdots+n_r=n$. For such a w we denote by U_w the unipotent subgroup of GL_n consisting of matrices $\begin{pmatrix} I_1 & * & * \\ & \ddots & * \\ & & I_r\end{pmatrix}$ where I_i is the identity matrix in GL_{n_i}. Note that in our computation other Weyl matrices w yield zero orbital integrals.

Similarly, using a double coset decomposition of $GL_n(E')$ the relative Kloosterman integral in (5) can be written as a sum of global orbital integrals

$$\sum_w \sum_a \int_{Z^+(E_{\mathbf{A}})} J(waz,\Phi)\chi(z)\,dz$$

where

$$J(wa,\Phi)=\int_{N_w(E_{\mathbf{A}})\backslash N(E'_{\mathbf{A}})} \Phi({}^t\bar{n}wan)\theta(n\bar{n})\,dn.$$

Here the sums are taken over the same w and a as above and $N_w(E_{\mathbf{A}})$ is the subgroup of $N(E'_{\mathbf{A}})$ defined by ${}^t\bar{n}wn=w$.

Consequently, the relative trace formula can be reduced to identities of global orbital integrals

$$I(wa,f)=J(wa,\Phi)$$

for any w and a as in (6) with a_i being in the center of $GL_{n_i}(E_{\mathbf{A}})$. Expressing $I(wa,f)$ and $J(wa,\Phi)$ as products of local orbital integrals $I_v(wa,f_v)$ and $J_w(wa,\Phi_w)$, we need to prove that

$$I_v(wa,f_v)=J_w(wa,\Phi_w) \tag{7}$$

for all w and a of the form (6) but with a_i being in the center of $GL_{n_i}(E_w)$, when v is inert in E' with w lying above v. In the case of v being non-Archimedean and unramified, proving (7) for f_v being the characteristic function of $K(E_v)$ and Φ_w being the characteristic function of $K(E'_w) \cap S(E_v)$ for all w is equivalent to proving the fundamental lemma for the unit elements of Hecke algebras.

Back to GL_4, one needs to prove (7) for the following w:

$$w_1 = \begin{pmatrix} &&&1\\ &&1&\\ &1&&\\ 1&&& \end{pmatrix}, \qquad w_2 = \begin{pmatrix} &&1&\\ &1&&\\ 1&&&\\ &&&1 \end{pmatrix},$$

$$w_3 = \begin{pmatrix} 1&&&\\ &&&1\\ &&1&\\ &1&& \end{pmatrix}, \qquad w_4 = \begin{pmatrix} &1&&\\ 1&&&\\ &&&1\\ &&1& \end{pmatrix}, \qquad w_5 = I.$$

The case of w_1 is trivial. The cases of w_2 and w_3 were proved in Ye [22]. In this article we will prove (7) for w_4 with $E_v = F$, $E'_w = L$, and $\psi_v = \psi_F$. Thus, the only remaining unproved case for the fundamental lemma of unit elements of Hecke algebras is $w_5 = I$.

We want to point out that for the group GL_2, the non-trivial case for the fundamental lemma is similar to our case of w_2. For GL_3, the cases of $\begin{pmatrix} &1&\\ 1&&\\ &&1 \end{pmatrix}$ and $\begin{pmatrix} 1&&\\ &&1\\ &1& \end{pmatrix}$ are again similar to our cases of w_2 and w_3. The case of w_4 does not appear in GL_2 and GL_3.

Exponential sums corresponding to w_2 and w_3 are hyper-Kloosterman sums which are studied by Katz [12], Friedberg [4], and Stevens [14]. The Kloosterman sum of the form on the left side of (2) for w_4 in GL_4 has also been studied by Friedberg [4] and Stevens [14]. What is new in the present paper is its new expression given on the left side of (1). Also new in this paper is certainly the identity of exponential sums in Theorem 1. This identity can be regarded as a lifting of the exponential sum on the left side of (1) to a quadratic number field. Because of its connection with our relative trace formula, it might be a kind of manifestation of the underlying quadratic base change.

2. Proof of Theorem 1

We will use a local argument to prove the identity in Theorem 1. Let $\psi = \psi_{\mathbb{R}} \prod_{p<\infty} \psi_p$ be an additive character of $\mathbb{Q}_A$ which is trivial on $\mathbb{Q}$ such that its real component is given by $\psi_{\mathbb{R}}(x) = e^{-2\pi i x}$ and each p-adic local character ψ_p has order equal to 0. Since ψ is trivial on $\mathbb{Q}$, for any $x \in \mathbb{Q}$ we have $e^{2\pi i x} = \prod_{p<\infty} \psi_p(x)$. Recall that we assumed that c is odd and $(\tau, c) = 1$. For any prime divisor p of c if $\left(\frac{\tau}{p}\right) = 1$, then p splits in $E = \mathbb{Q}(\sqrt{\tau})$; if $\left(\frac{\tau}{p}\right) = -1$, then p is inert unramified in $E = \mathbb{Q}(\sqrt{\tau})$. In the former case $\mathbb{Q}_p \otimes_{\mathbb{Q}} E = E_{1p} \oplus E_{2p}$ is isomorphic to the direct sum of two copies of $\mathbb{Q}_p$ while in the latter case $\mathbb{Q}_p \otimes_{\mathbb{Q}} E = E_p$ is an unramified quadratic extension field of $\mathbb{Q}_p$. Now we can write the identity in (1) in terms of local products

$$\prod_{p|c} \sum_{\substack{x_i\in R_p^\times/(1+cR_p)\\ \text{for } i=1,\dots,4}} \psi_p\Big(\frac{1}{c}\Big(x_1+x_2+\frac{1}{x_2x_3}+\frac{x_2x_3}{x_4}+\frac{bx_4}{x_1}+\frac{bx_3}{x_1}\Big)\Big)$$
$$= \prod_{\substack{p|c,\\ \left(\frac{\tau}{p}\right)=-1}} \sum_{\substack{x_1,x_2\in R_{E_p}^\times/(1+cR_{E_p}),\\ m\in R_p^\times/(1+cR_p),\\ x_1\bar{x}_1\in x_2\bar{x}_2(1+cR_p)}} \psi_p\Big(\frac{m}{c}\Big(1+\frac{b}{x_1\bar{x}_1}\Big)\Big)\psi_p\circ\mathrm{tr}_{E_p/\mathbb{Q}_p}\Big(\frac{x_1+x_2}{cm}\Big)$$
$$\cdot \prod_{\substack{p|c,\\ \left(\frac{\tau}{p}\right)=1}} \sum_{\substack{m,x_1,x_2,y_1,y_2\in R_p^\times/(1+cR_p),\\ x_1y_1\in x_2y_2(1+cR_p)}} \psi_p\Big(\frac{m}{c}\Big(1+\frac{b}{x_1y_1}\Big)\Big)\psi_p\Big(\frac{x_1+x_2+y_1+y_2}{cm}\Big).$$

We can rewrite the sums on both sides of the above identity as local integrals. Theorem 1 is thus reduced to local identities in the following two lemmas.

Lemma 1. *Let F be a non-Archimedean local field of characteristic 0 with $|2|_F = 1$. Let $L = F(\sqrt{\tau})$ be an unramified quadratic extension field of F with $\tau \in R_F^\times$. Denote by ψ_F a non-trivial character of F of order zero. For any $b \in R_F^\times$ and $c \in \varpi_F^C R_F^\times$ with $C > 0$ we have*

(8)
$$\int_{(R_F^\times)^4} \psi_F\Big(\frac{1}{c}\Big(x_1+x_2+\frac{1}{x_2x_3}+\frac{x_2x_3}{x_4}+\frac{bx_4}{x_1}+\frac{bx_3}{x_1}\Big)\Big)\,dx_1\,dx_2\,dx_3\,dx_4$$
$$= q_F^C \int_{\substack{m\in R_F^\times,\\ x_1,x_2\in R_L^\times,\\ x_1\bar{x}_1\in x_2\bar{x}_2(1+\varpi_F^C R_F)}} \psi_F\Big(\frac{m}{c}\Big(1+\frac{b}{x_1\bar{x}_1}\Big)\Big)\psi_F\circ\mathrm{tr}_{L/F}\Big(\frac{x_1+x_2}{cm}\Big)\,dm\,dx_1\,dx_2.$$

Lemma 2. *Let F be a non-Archimedean local field of characteristic 0 with $|2|_F = 1$. Denote by ψ_F a non-trivial character of F of order 0. For any $b \in R_F^\times$ and $c \in \varpi_F^C R_F^\times$ with $C > 0$ we have*

$$\int_{(R_F^\times)^4} \psi_F\Big(\frac{1}{c}\Big(x_1+x_2+\frac{1}{x_2x_3}+\frac{x_2x_3}{x_4}+\frac{bx_4}{x_1}+\frac{bx_3}{x_1}\Big)\Big)\,dx_1\,dx_2\,dx_3\,dx_4$$
$$= q_F^C \int_{\substack{m,x_1,x_2,y_1,y_2\in R_F^\times,\\ x_1y_1\in x_2y_2(1+\varpi_F^C R_F)}} \psi_F\Big(\frac{m}{c}\Big(1+\frac{b}{x_1y_1}\Big)\Big)$$
$$\cdot\,\psi_F\Big(\frac{x_1+x_2+y_1+y_2}{cm}\Big)\,dm\,dx_1\,dx_2\,dy_1\,dy_2.$$

Lemma 2 is trivial; its proof is by changing variables. We will devote the rest of this section to Lemma 1.

Proof of Lemma 1. We first consider the integral on the left side of (8). We note that for any $b_1, b_2 \notin R_F$ an integral of the form

$$\int_{R_F^\times} \psi_F\Big(xb_1+\frac{b_2}{x}\Big)\,dx \tag{9}$$

is non-zero only if $b_1 \in b_2 R_F^\times$. If one of b_1 and b_2 is in R_F but the other is not, the integral in (9) is non-zero only if the latter is in $\varpi_F^{-1} R_F^\times$. Applying these results to the integral with respect to x_1 on the left side of (8) we get two non-vanishing cases:

(i) $C = 1$ and $x_3 + x_4 \in \varpi_F R_F$, and
(ii) $C \geq 1$ and $x_3 + x_4 \in R_F^\times$.

In case (i) the integral can be computed directly. Namely, the integrand becomes $\psi_F(\frac{x_1}{c} + \frac{1}{cx_2x_3})$ because the order of ψ_F is zero. The integral with respect to $x_4 \in x_3 + \varpi_F R_F$ equals q_F^{-1} and the integrals with respect to x_1 and x_2 are both equal to $-q_F^{-1}$. With the integral with respect to x_3 being $1 - q_F^{-1}$ we conclude that case (i) yields $q_F^{-3}(1 - q_F^{-1})$.

To compute case (ii) we use a Mellin transform. Let χ be a multiplicative character of F. If χ is ramified, we denote its conductor exponent by $a(\chi)$ which is the smallest positive integer a such that χ is trivial on $1 + \varpi_F^a R_F$. We integrate the expression on the left side of (8) against $\chi^{-1}(b)$ with respect to $b \in R_F^\times$:

$$\int\limits_{\substack{(R_F^\times)^5 \\ x_3+x_4 \in R_F^\times}} \chi^{-1}(b)\psi_F\Big(\frac{1}{c}\Big(x_1 + x_2 + \frac{1}{x_2x_3} + \frac{x_2x_3}{x_4} + \frac{bx_4}{x_1} + \frac{bx_3}{x_1}\Big)\Big)\, db\, dx_1\, dx_2\, dx_3\, dx_4.$$

Now we change variables successively from b to $y_0 = b(x_3+x_4)/(cx_1) \in \varpi_F^{-C} R_F^\times$, from x_1 to $y_1 = x_1/c \in \varpi_F^{-C} R_F^\times$, from x_2 to $y_2 = 1/(cx_2x_3) \in \varpi_F^{-C} R_F^\times$, from x_4 to $x = x_4/x_3 \in R_F^\times$ with $x + 1 \in R_F^\times$, and finally from x_3 to $y_3 = (x+1)/(c^2 x x_3 y_2) \in \varpi_F^{-C} R_F^\times$. Then the above integral becomes

$$q_F^{-4C}\chi^{-4}(c)\Big(\int\limits_{\varpi_F^{-C} R_F^\times} \chi^{-1}(y)\psi_F(y)\, dy\Big)^4 \int\limits_{\substack{x \in R_F^\times, \\ x+1 \in R_F^\times}} \chi^{-1}\Big(\frac{x}{(x+1)^2}\Big)\, dx. \tag{10}$$

If the character χ is unramified, the first integral in (10) vanishes unless $C = 1$. When $C = 1$, we get $q_F^{-4}(1 - 2q_F^{-1})$. If χ is ramified, then the same integral vanishes unless $a(\chi) = C$; in this case

$$\int\limits_{\varpi_F^{-C} R_F^\times} \chi^{-1}(y)\psi_F(y)\, dy = \varepsilon(\chi, \psi)$$

where the local ε-factor is defined as in Tate [15]. Together with our results for case (i) we conclude that the Mellin transform of the integral on the left side of (8) equals

$$\begin{cases} q_F^{-3}(1 - q_F^{-1} - q_F^{-2}) & \text{if } \chi \text{ is unramified and } C = 1; \\ q_F^{-4C}\chi^{-4}(c)\Big(\varepsilon(\chi, \psi_F)\Big)^4 & \displaystyle\int\limits_{R_F - (\pm 1 + \varpi_F R_F)} \chi^{-1}\Big(\frac{1 - z^2}{4}\Big) \\ & \text{if } \chi \text{ is ramified and } a(\chi) = C; \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

Here we rewrote the last integral in (10) by using a new variable $z = (1-x)/(1+x) \in R_F - (\pm 1 + \varpi_F R_F)$.

Now we turn to the integral on the right side of (8). Again we integrate it against $\chi^{-1}(b)$ with respect to $b \in R_F^\times$ to get its Mellin transform

$$q_F^C \int_{\substack{b,m\in R_F^\times,\\ x_1,x_2\in R_L^\times,\\ x_1\bar{x}_1\in x_2\bar{x}_2(1+\varpi_F^C R_F)}} \chi^{-1}(b)\psi_F\Big(\frac{m}{c}\Big(1+\frac{b}{x_1\bar{x}_1}\Big)\Big) \cdot \psi_F\circ\mathrm{tr}_{L/F}\Big(\frac{x_1+x_2}{cm}\Big)\, db\, dm\, dx_1\, dx_2.$$

Changing variables successively from b to $y_0 = bm/(cx_1\bar{x}_1) \in \varpi_F^{-C}R_F^\times$, from x_1 to $y_1 = x_1/(cm) \in \varpi_L^{-C}R_L^\times$, from x_2 to $y_2 = x_2/(cm) \in \varpi_L^{-C}R_L^\times$ with $y_2\bar{y}_2 \in y_1\bar{y}_1(1+\varpi_F^C R_F)$, from m to $y_3 = m/c \in \varpi_F^{-C}R_F^\times$, and from y_2 to $\varepsilon \in R_L^\times$ with $\varepsilon\bar{\varepsilon} \in 1+\varpi_F^C R_F$ by $y_2 = y_1\varepsilon$, we get

$$q_F^{-3C}\chi^{-1}(c^4)\Big(\int_{\varpi_F^{-C}R_F^\times} \chi^{-1}(y)\psi_F(y)\, dy\Big)^2 \cdot \int_{\substack{y_1\in\varpi_L^{-C}R_L^\times,\\ \varepsilon\in R_L^\times,\\ \varepsilon\bar{\varepsilon}\in 1+\varpi_F^C R_F}} \chi^{-1}\circ\mathrm{N}_{L/F}(y_1)\psi_F\circ\mathrm{tr}_{L/F}(y_1(1+\varepsilon))\, d\varepsilon\, dy_1.$$

If χ is unramified, then the integral with respect to y above indicates that it is non-zero only when $C = 1$. In this non-zero case, we get

$$q_F^{-3}\Big((q_F^2-1)\int_{\substack{\varepsilon\in -1+\varpi_L R_L,\\ \varepsilon\bar{\varepsilon}\in 1+\varpi_F R_F}} d\varepsilon - \int_{\substack{\varepsilon\in R_L^\times,\\ 1+\varepsilon\in R_L^\times,\\ \varepsilon\bar{\varepsilon}\in 1+\varpi_F R_F}} d\varepsilon\Big) = q_F^{-3}\Big(1-q_F^{-1}-q_F^{-2}\Big).$$

If χ is ramified, then the integral with respect to y vanishes unless $a(\chi) = C$. Since we assumed that the quadratic extension L is unramified and $|2|_F = 1$, we know that the conductor exponent $a(\chi\circ\mathrm{N}_{L/F}) = C$ when $a(\chi) = C$ and the order of $\psi_F\circ\mathrm{tr}_{L/F}$ is again zero. Consequently, the integral with respect to y_1 is non-zero only if $1+\varepsilon \in R_L^\times$ when $a(\chi) = C$. Changing variables from y_1 to $x = y_1(1+\varepsilon) \in \varpi_L^{-C}R_L^\times$ we get the local ε-factor $\varepsilon(\chi\circ\mathrm{N}_{L/F}, \psi_F\circ\mathrm{tr}_{L/F})$ multiplied by the integral

$$\int_{\substack{\varepsilon\in R_L^\times,\\ 1+\varepsilon\in R_L^\times,\\ \varepsilon\bar{\varepsilon}\in 1+\varpi_F^C R_F}} \chi\circ\mathrm{N}_{L/F}(1+\varepsilon)\, d\varepsilon = q_F^{-2C} \sum_{\substack{\varepsilon,1+\varepsilon\in R_L^\times/(1+\varpi_L^C R_L),\\ \varepsilon\bar{\varepsilon}\in 1+\varpi_F^C R_F}} \chi(2+\varepsilon+\bar{\varepsilon});$$

here we wrote the integral in terms of a finite sum. Now we can set $\varepsilon = (1+z\sqrt{\tau})/(1-z\sqrt{\tau})$ with $z \in R_F/\varpi_F^C R_F$. Then $\chi(2+\varepsilon+\bar{\varepsilon}) = \chi^{-1}\Big((1-\tau z^2)/4\Big)$. Using an integral again we get

$$\int_{\substack{\varepsilon\in R_L^\times,\\ 1+\varepsilon\in R_L^\times,\\ \varepsilon\bar{\varepsilon}\in 1+\varpi_F^C R_F}} \chi\circ\mathrm{N}_{L/F}(1+\varepsilon)\, d\varepsilon = q_F^{-C}\int_{R_F} \chi^{-1}\Big(\frac{1-\tau z^2}{4}\Big)\, dz.$$

Therefore, the Mellin transform of the right side of (8) becomes

$$\begin{cases} q_F^{-3}(1-q_F^{-1}-q_F^{-2}) & \text{if } \chi \text{ is unramified and } C=1; \\ q_F^{-4C}\chi^{-4}(c)\Big(\varepsilon(\chi,\psi_F)\Big)^2\varepsilon(\chi\circ \mathrm{N}_{L/F},\psi_F\circ \mathrm{tr}_{L/F})\displaystyle\int\limits_{R_F}\chi^{-1}\Big(\frac{1-\tau z^2}{4}\Big)\,dz & \\ & \text{if } \chi \text{ is ramified and } a(\chi)=C; \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

To compare the expressions in (11) and (12) we recall a well-known identity between local ε-factors (see, e.g., Gérardin and Labesse [5])

$$\varepsilon(\chi,\psi_F)\varepsilon(\chi\eta,\psi_F)=\varepsilon(\chi\circ \mathrm{N}_{L/F},\psi_F\circ \mathrm{tr}_{L/F})$$

where η is the quadratic multiplicative character of F attached to the extension field L. Since L is assumed to be unramified over F, we have $\eta(c)=(-1)^C$ for any $c\in\varpi_F^C R_F^\times$; hence $\varepsilon(\chi\eta,\psi)=(-1)^C\varepsilon(\chi,\psi_F)$ when $a(\chi)=C$. Now we need a lemma. □

Lemma 3. *Let F be a non-Archimedean local field of characteristic 0 with $|2|_F=1$. Let $L=F(\sqrt{\tau})$ be an unramified quadratic extension field of F with $\tau\in R_F^\times$. Denote by χ a ramified character of $F^\times$ whose conductor exponent is $a(\chi)=C$. Then*

$$\int\limits_{R_F-(\pm1+\varpi_F R_F)}\chi^{-1}(1-z^2)\,dz=(-1)^C\int\limits_{R_F}\chi^{-1}(1-\tau z^2)\,dz. \tag{13}$$

Together, with the above remark, Lemma 3 implies that the corresponding expressions in (11) and (12) are equal. That is to say, the Mellin transforms of the two sides of (8) are the same for any multiplicative character χ. By Fourier's inversion formula, we conclude that the two sides of (8) are equal.

To complete the proof of Lemma 1, we still have to prove Lemma 3. When $C=1$, the left side of (13) equals $q_F^{-1}(1+2\sum\chi^{-1}(1-x))$ where the sum is taken over all squares $x\neq1$ in $R_F^\times/(1+\varpi_F R_F)$, and the right side of (13) equals $-q_F^{-1}(1+2\sum\chi^{-1}(1-x))$ where x goes over all non-squares in $R_F^\times/(1+\varpi_F R_F)$. Then (13) follows from

$$\begin{aligned} q_F^{-1}\Big(1+2&\sum_{\substack{x\in R_F^\times/(1+\varpi_F R_F)\\ \text{is a square,}\\ x\neq1}}\chi^{-1}(1-x)\Big)\\ &+q_F^{-1}\Big(1+2\sum_{\substack{x\in R_F^\times/(1+\varpi_F R_F)\\ \text{is a non-square}}}\chi^{-1}(1-x)\Big)\\ &=2q_F^{-1}\sum_{a\in R_F^\times/(1+\varpi_F R_F)}\chi^{-1}(a)\\ &=0 \end{aligned}$$

where $a=1-x$.

When $C>1$, the integrals on both side of (13) can be taken over $z\in\varpi_F^{[C/2]}R_F$. Indeed, if $z\notin\varpi_F^{[C/2]}R_F$, we can set $z=u(1+v)$ and express an integral above as a finite sum with respect to u of integrals with respect to $v\in\varpi_F^{[(C+1)/2]}R_F$. Since

$u \notin \varpi_F^{[C/2]} R_F$, we can conclude that the integrals with respect to v vanish. This way the identity in (13) is reduced to

$$\int_{\varpi_F^{[C/2]} R_F} \chi^{-1}(1 - z^2)\, dz = (-1)^C \int_{\varpi_F^{[C/2]} R_F} \chi^{-1}(1 - \tau z^2)\, dz.$$

If C is even, then the integrands above are both equal to 1 and hence the equality. If C is odd, this equality can then be proved in the same way as what we did for the case of $C = 1$.

This completes the proof of Theorem 1.

3. The orbital integral $I_F(wa, f_0)$

To prove Theorem 2 we have to compute the integrals on the two sides of (2). Recall that the integral on the left side of (2) is the local orbital integral

$$I_F(wa, f_0) = \int_{\substack{u \in U_w(F), \\ n \in N(F)}} f_0({}^t uwan)\theta_F(un)\, du\, dn$$

where f_0 is the characteristic function of $K(F)$ and $w = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix}$. Similarly, the right side of (2) is the local orbital integral

$$J_F(wa, \Phi_0) = \int_{N_w(F) \backslash N(L)} \Phi_0({}^t \bar{n} wan)\theta_F(n\bar{n})\, dn$$

where Φ_0 is the characteristic function of $K(L) \cap S(F)$ with the same w. We will compute $I_F(wa, f_0)$ in this section and then $J_F(wa, \Phi_0)$ in the next section in order to show that they are equal.

Let us denote $w = \begin{pmatrix} w_1 & \\ & w_1 \end{pmatrix}$, $a = \begin{pmatrix} b_1 & \\ & b_2 \end{pmatrix}$, $u = \begin{pmatrix} I & y \\ & I \end{pmatrix} \in U_w(F)$, and $n = \begin{pmatrix} n_1 & x \\ & n_2^{-1} \end{pmatrix} \in N(F)$ in 2×2 blocks, where b_i is a scale matrix with diagonal entries equal to a_i and $w_1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$. Then the matrix condition ${}^t uwan \in K(F)$ for the integral defining $I_F(wa, f_0)$ becomes

$$\begin{pmatrix} w_1 b_1 n_1 & w_1 b_1 x \\ {}^t y w_1 b_1 n_1 & {}^t y w_1 b_1 x + w_1 b_2 n_2^{-1} \end{pmatrix} \in K(F). \tag{14}$$

We first conclude from this matrix condition that $a_1 a_2 \in R_F^\times$ and $a_1 \in R_F$. If $a_1, a_2 \in R_F^\times$ we can see that $I_F(wa, f_0) = 1$. Thus, from now on we assume that $a_1 \in \varpi_F^A R_F^\times$ and $a_2 \in \varpi_F^{-A} R_F^\times$ with $A > 0$. Then $n_1 = \begin{pmatrix} 1 & m_1 \\ & 1 \end{pmatrix}$ with $m_1 \in \varpi_F^{-A} R_F$. We can also apply the automorphism $g \mapsto w_G {}^t g^{-1} w_G$ to the matrix condition in (14), where $w_G = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}$. This way we can get $n_2 = \begin{pmatrix} 1 & m_2 \\ & 1 \end{pmatrix}$ with $m_2 \in \varpi_F^{-A} R_F$. Back to (14) we know that $x, w_1 {}^t y w_1 n_1 \in M_{2\times 2}(\varpi_F^{-A} R_F)$. Setting $z = w_1 {}^t y w_1 b_1 n_1 \in M_{2\times 2}(R_F)$ and changing from x to $x b_2$ we can rewrite the last condition in (14) and get

$$I_F(wa, f_0) = q_F^{8A} \int\limits_{\substack{m_1, m_2 \in \varpi_F^{-A} R_F,\\ x,z \in M_{2\times 2}(R_F),\\ zn_1^{-1}x \in -n_2^{-1} + M_{2\times 2}(\varpi_F^A R_F)}} \theta_F \begin{pmatrix} I & zn_1^{-1}b_1^{-1} \\ & I \end{pmatrix} \cdot \theta_F \begin{pmatrix} n_1 & xb_2 \\ & n_2^{-1} \end{pmatrix} dm_1\, dm_2\, dx\, dz.$$

Denote $x = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$ and $z = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}$ with $x_i, z_i \in R_F$. There are three cases.

(i) $m_2 \in R_F$. Then the condition

$$zn_1^{-1}x \in -n_2^{-1} + M_{2\times 2}(\varpi_F^A R_F) \tag{15}$$

implies that $zn_1^{-1}x \in K_{2\times 2}(F)$, $\det(zx), \det(x), \det(z) \in R_F^\times$, $x, z, n_1 \in K_{2\times 2}(F)$, and $m_1 \in R_F$.

(ii) $m_2 \in \varpi_F^M R_F^\times$ with $0 > M > -A$. Then (15) implies that $zn_1^{-1}x \in GL_2(F)$, $\det(zx), \det(x), \det(z) \in R_F^\times$, $x, z \in K_{2\times 2}(F)$, and $m_1 \in \varpi_F^M R_F^\times$.

(iii) $m_2 \in \varpi_F^{-A} R_F^\times$. Then from (15) we have $m_1 \in \varpi_F^{-A} R_F^\times$, $x_4, z_1 \in R_F^\times$, $x_3, z_3 \in \varpi_F^A R_F$, and $x_1, x_2, z_2, z_4 \in R_F$. We will denote the integrals corresponding to these three cases by I_1, I_2, and I_3 so that $I_F(wa, f_0) = I_1 + I_2 + I_3$.

First we compute I_1:

$$I_1 = q_F^{8A} \int\limits_{\substack{m_1, m_2 \in R_F,\\ x,z \in K_{2\times 2}(F),\\ zn_1^{-1}x \in -n_2^{-1} + M_{2\times 2}(\varpi_F^A R_F)}} \psi_F\Big(\frac{z_3}{a_1} + a_2 x_3\Big)\, dm_1\, dm_2\, dx\, dz.$$

By changing variables from x to $n_1 x$ and integrating with respect to m_1, m_2, x_2, x_4, z_1, and z_2 successively we arrive at

$$I_1 = q_F^{5A} \int\limits_{\substack{x_1, x_3, z_3, z_4 \in R_F\\ \text{with } x_1 \text{ or } x_3 \in R_F^\times\\ \text{and } z_3 \text{ or } z_4 \in R_F^\times,\\ x_1 z_3 + x_3 z_4 \in \varpi_F^A R_F}} \psi_F\Big(\frac{z_3}{a_1} + a_2 x_3\Big)\, dx_1\, dx_3\, dz_3\, dz_4.$$

If $x_1 \in R_F^\times$ we have $z_3 \in -x_3 z_4/x_1 + \varpi_F^A R_F$. If $x_1 \in \varpi_F R_F$, then $x_3 \in R_F^\times$ and $z_4 \in -x_1 z_3/x_3 + \varpi_F^A R_F$. By integrating z_3 in the first case and z_4 in the second case we can further compute I_1 and conclude that

$$I_1 = q_F^3\Big(1 - q_F^{-1} + q_F^{-2}\Big) \quad \text{if } A = 1; \tag{16}$$

$$ = q_F^{3A}\Big(1 - q_F^{-1}\Big) \quad \text{if } A > 1. \tag{17}$$

Next, let us turn to I_2. After integrating with respect to m_2 we get

$$I_2 = \sum_{-A<M<0} q_F^{7A} \int \psi_F\Big(\frac{z_3}{a_1} + a_2 x_3 + m_1 - m_1 x_4 z_1\Big)\, dm_1\, dx\, dz$$

where the integral is taken over

$$\begin{aligned} m_1 &\in \varpi_F^M R_F^\times, \\ x_1, x_4, z_1, z_4 &\in R_F^\times, \\ x_2, z_2 &\in R_F, \\ x_3, z_3 &\in \varpi_F^{-M} R_F, \\ x_1 z_1 - m_1 x_3 z_1 + x_3 z_2 &\in -1 + \varpi_F^A R_F, \\ x_2 z_3 - m_1 x_4 z_3 + x_4 z_4 &\in -1 + \varpi_F^A R_F, \\ x_1 z_3 - m_1 x_3 z_3 + x_3 z_4 &\in \varpi_F^A R_F. \end{aligned}$$

We will consider two cases:

(i) $x_3 \in \varpi_F^A R_F$ (then $z_3 \in \varpi_F^A R_F$) and

(ii) $x_3 \notin \varpi_F^A R_F$ (then $z_3 \in x_3 R_F^\times$).

In case (i) the integrand simplifies to $\psi_F(m_1 - m_1 x_4 z_1)$ and we have $x_4 \in -1/(z_4 - m_1 z_3) + \varpi_F^A R_F$ and $z_1 \in -1/(x_1 - m_1 x_3) + \varpi_F^A R_F$ with $x_1 - m_1 x_3 \in R_F^\times$ and $z_4 - m_1 z_3 \in R_F^\times$. After integrating the integrals with respect to x_4 and z_1, changing variables from x_1 to $x = x_1 - m_1 x_3$ and from z_4 to $z = z_4 - m_1 z_3$, and integrating with respect to x_3 and z_3, we get

$$\sum_{-A<M<0} q_F^{3A} \int_{\substack{m_1 \in \varpi_F^M R_F^\times, \\ x, z \in R_F^\times}} \psi_F\Big(m_1 - \frac{m_1}{xz}\Big)\, dm_1\, dx\, dz.$$

By integrating with respect to x_1 we see that this integral vanishes unless $M = -1$. Computing the case of $M = -1$, we get $q_F^{3A-1}(1 - q_F^{-1})$ for case (i).

In case (ii) we set $x_3, z_3 \in \varpi_F^X R_F^\times$ with $-M \le X < A$. After integrating with respect to x_2 and z_2 and setting $x_4 \in -1/(z_4 - m_1 z_3) + \varpi_F^X R_F$ and $z_1 \in -1/(x_1 - m_1 x_3) + \varpi_F^X R_F$ with $x_1 - m_1 x_3 \in R_F^\times$ and $z_4 - m_1 z_3 \in R_F^\times$ we get

$$\sum_{\substack{-A<M<0, \\ -M \le X < A}} q_F^{5A} \int_{\substack{m_1 \in \varpi_F^M R_F^\times, \\ x_1, z_4 \in R_F^\times, \\ x_3, z_3 \in \varpi_F^X R_F^\times, \\ (x_1 - m_1 x_3)(z_4 - m_1 z_3) \in x_1 z_4 + \varpi_F^{A+M} R_F}} \psi_F\Big(\frac{z_3}{a_1} + a_2 x_3 + m_1 - \frac{m_1}{(x_1 - m_1 x_3)(z_4 - m_1 z_3)}\Big)\, dm_1\, dx\, dz.$$

From the last condition attached to the integral we know that $z_3 \in -x_3 z_4 / (x_1 - m_1 x_3) + \varpi_F^A R_F$. Hence, the integral with respect to z_3 vanishes unless

$A+2M \geq 0$. Then we get for case (ii)

$$\sum_{-A/2\leq M<0} q_F^{4A} \int\limits_{\substack{m_1\in\varpi_F^M R_F^\times,\\ x_1,z_4\in R_F^\times,\\ x_3\in\varpi_F^{-M}R_F,\\ x_1-m_1x_3\in R_F^\times}} \psi_F\Big(-\frac{x_3z_4}{a_1(x_1-m_1x_3)}+a_2x_3+m_1-\frac{m_1}{x_1z_4}\Big)\,dm_1\,dx\,dz$$

$$-\sum_{-A/2\leq M<0} q_F^{4A} \int\limits_{\substack{m_1\in\varpi_F^M R_F^\times,\\ x_1,z_4\in R_F^\times,\\ x_3\in\varpi_F^{A}R_F}} \psi_F\Big(-\frac{x_3z_4}{a_1(x_1-m_1x_3)}+a_2x_3+m_1-\frac{m_1}{x_1z_4}\Big)\,dm_1\,dx\,dz.$$

The second integral above is the same as the integral in case (i) because the integrand is actually equal to $\psi_F(m_1-m_1/(x_1z_4))$; we thus get the same $q_F^{3A-1}(1-q_F^{-1})$.

For the first integral we change variables from x_3 to $x=m_1x_3\in R_F$ with $x-x_1\in R_F^\times$ and from z_4 to $z=z_4/m_1\in\varpi_F^{-M}R_F^\times$ to get

$$\sum_{-A/2\leq M<0} q_F^{4A} \int\limits_{\substack{m_1\in\varpi_F^M R_F^\times,\\ x\in R_F,\\ x_1,x-x_1\in R_F^\times,\\ z\in\varpi_F^{-M}R_F^\times}} \psi_F\Big(-\frac{xz}{a_1(x-x_1)}+\frac{a_2x}{m_1}+m_1-\frac{1}{x_1z}\Big)\,dm_1\,dx\,dx_1\,dz.$$

Applying our results on (9) to the integral with respect to z, we conclude that $x\in\varpi_F^{A+2M}R_F^\times$ if $M<-1$ and $x\in\varpi_F^{A+2M}R_F$ if $M=-1$. We claim that the integral vanishes when $M<-1$ and $A+2M>0$. Indeed, we change variables from x to a_1x and set $b=a_1a_2$. Integrating the integral with respect to $b\in R_F^\times$ against $\chi^{-1}(b)$ we get the Mellin transform

$$\sum_{-A/2<M<-1} q_F^{3A}\int\limits_{R_F^\times}\chi^{-1}(b)\,db$$

$$\cdot\int\limits_{\substack{m_1\in\varpi_F^M R_F^\times,\\ x\in\varpi_F^{2M}R_F^\times,\\ x_1\in R_F^\times,\\ z\in\varpi_F^{-M}R_F^\times}} \psi_F\Big(\frac{xz}{(x_1-a_1x)}+\frac{bx}{m_1}+m_1-\frac{1}{x_1z}\Big)\,dm_1\,dx\,dx_1\,dz.$$

If χ is unramified, the integral with respect to b vanishes, because $M<-1$. Assume now that χ is ramified. We change variables successively from b to $y_1=bx/m_1\in\varpi_F^M R_F^\times$, from x to $y=x_1/x\in\varpi_F^{-2M}R_F^\times$, from x_1 to $y_2=-1/(x_1z)\in\varpi_F^M R_F^\times$, and from z to $y_3=z/(y-a_1)\in\varpi_F^M R_F^\times$. Then the Mellin transform becomes

$$\sum_{-A/2<M<-1} q_F^{3A}\Big(\int\limits_{\varpi_F^M R_F^\times}\chi^{-1}(y_1)\psi_F(y_1)\,dy_1\Big)^4\int\limits_{\varpi_F^{-2M}R_F^\times}\chi^{-1}\Big(y(a_1-y)\Big)\,dy.$$

The integral with respect to y_1 vanishes if $a(\chi)\neq -M$. If $a(\chi)=-M\geq 2$, the integral with respect to y equals zero because we can set y as $y(1+c)$ with $c\in\varpi_F^{-[M/2]}R_F$ and integrate with respect to c. Since the Mellin transform of our integral vanishes for any character χ when $M<-1$ and $A+2M>0$, we prove the claim.

Now we compute the integral when $M = -1$ and $A + 2M > 0$, i.e., $A > 2$. Since the integrand becomes

$$\psi_F\Big(\frac{xz}{a_1x_1} + \frac{a_2x}{m_1} + m_1 - \frac{1}{x_1z}\Big),$$

we can integrate with respect to $x \in \varpi_F^{A+2M}R_F$. The integral is non-zero only if $z/(a_1x_1) + a_2/m_1 \in \varpi_F^{2-A}R_F$. Consequently, we get

$$q_F^{3A+2} \int\limits_{\substack{m_1\in\varpi_F^{-1}R_F^\times,\\ x_1\in R_F^\times,\\ z\in\varpi_F R_F^\times,\\ z/(a_1x_1)+a_2/m_1\in\varpi_F^{2-A}R_F}} \psi_F\Big(m_1 - \frac{1}{x_1z}\Big)\, dm_1\, dx_1\, dz.$$

The last condition attached to the integral implies that $z \in -a_1a_2x_1/m_1 + \varpi_F^2R_F$. After integrating with respect to z we get

$$\begin{aligned} q_F^{3A} \int\limits_{\substack{m_1\in\varpi_F^{-1}R_F^\times,\\ x_1\in R_F^\times}} &\psi_F\Big(m_1\Big(1 + \frac{1}{a_1a_2x_1^2}\Big)\Big)\, dm_1\, dx_1 \\ &= q_F^{3A}\Big(q_F \int\limits_{\substack{x_1\in R_F^\times,\\ x_1^2\in -1/(a_1a_2)+\varpi_F R_F}} dx_1 - \int\limits_{x_1\in R_F^\times} dx_1\Big) \end{aligned}$$

which equals $q_F^{3A}(1 + q_F^{-1})$ if $-a_1a_2$ is a square in $R_F^\times/(1 + \varpi_F R_F)$ and equals $q_F^{3A}(-1 + q_F^{-1})$ if $-a_1a_2$ is not a square in $R_F^\times/(1 + \varpi_F R_F)$.

For the case of $A + 2M = 0$, with A being even, we have

$$\int\limits_{\substack{m_1\in\varpi_F^{-A/2}R_F^\times,\\ x\in R_F,\\ x_1,x-x_1\in R_F^\times,\\ z\in\varpi_F^{A/2}R_F^\times}} \psi_F\Big(-\frac{xz}{a_1(x-x_1)} + \frac{a_2x}{m_1} + m_1 - \frac{1}{x_1z}\Big)\, dm_1\, dx\, dx_1\, dz.$$

For the above integral, now we choose $c \in \varpi_F^{A/2}R_F^\times$ and set $b = a_2c^4/a_1 \in R_F^\times$. We then change variables from m_1 to $y_1 = cm_1 \in R_F^\times$, from z to $y_2 = -cz/a_1 \in R_F^\times$, from x to $y_4 = a_1(x - x_1)/c^2 \in R_F^\times$, and from x_1 to $y_3 = a_1x_1/c^2 \in R_F^\times$. After collecting all of these results on I_2 we get

$$\begin{aligned} I_2 = q_F^{4A} \int\limits_{(R_F^\times)^4} &\psi_F\Big(\frac{1}{c}\Big(y_1 + y_2 + \frac{1}{y_2y_3} + \frac{y_2y_3}{y_4} \\ &\qquad + \frac{by_4}{y_1} + \frac{by_3}{y_1}\Big)\Big)\, dy_1\, dy_2\, dy_3\, dy_4 \qquad \text{if } A \geq 2 \text{ is even} \\ &+ \begin{cases} q_F^{3A}(q_F^{-1} + 1) & \text{if } -a_1a_2 \text{ is a square in } R_F^\times/(1 + \varpi_F R_F) \\ & \text{and } A > 2 \\ q_F^{3A}(q_F^{-1} - 1) & \text{if } -a_1a_2 \text{ is not a square in } R_F^\times/(1 + \varpi_F R_F) \\ & \text{and } A > 2. \end{cases} \end{aligned} \tag{18}$$

Now we compute

$$I_3 = q_F^{8A} \int \psi_F(m_1 - m_2)\, dm_1\, dm_2\, dx\, dz$$

where the integral is taken over

$$\begin{aligned} m_1, m_2 &\in \varpi_F^{-A} R_F^\times, \\ x_1, x_2, z_2, z_4 &\in R_F, \\ x_3, z_3 &\in \varpi_F^A R_F, \\ x_4, z_1 &\in R_F^\times, \\ x_1 z_1 - m_1 x_3 z_1 + x_3 z_2 &\in -1 + \varpi_F^A R_F, \\ x_2 z_3 - m_1 x_4 z_3 + x_4 z_4 &\in -1 + \varpi_F^A R_F, \\ x_2 z_1 - m_1 x_4 z_1 + x_4 z_2 &\in m_2 + \varpi_F^A R_F. \end{aligned}$$

If we integrate with respect to m_2 first, we will get q_F^{-A} and the above integrand becomes $\psi_F(m_1 + m_1 x_4 z_1)$. Having integrated with respect to x_1, x_2, x_3, z_2, z_3, and z_4 we get

$$\begin{aligned} I_3 &= q_F^{3A} \int\limits_{\substack{m_1 \in \varpi_F^{-A} R_F^\times, \\ x_4, z_1 \in R_F^\times}} \psi_F\Big(m_1(1 + x_4 z_1)\Big)\, dm_1\, dx_4\, dz_1 \\ &= q_F^2(1 - q_F^{-1}) \quad \text{if } A = 1; \qquad (19) \\ &= 0 \quad \text{if } A > 1. \qquad (20) \end{aligned}$$

Collecting our results in (16) through (20) we get the following expression of $I_F(wa, f_0)$.

Lemma 4. *Under the assumption of Lemma 1 we have*

$$\begin{aligned} I_F(wa, f_0) &= 1 && \text{if } A = 0; \\ &= q_F^3 && \text{if } A = 1; \\ &= 2q_F^{3A} && \text{if } A \geq 3 \text{ is odd and } -a_1 a_2 \text{ is a square in } F; \\ &= q_F^{4A} \int\limits_{(R_F^\times)^4} \psi_F\Big(\frac{1}{c}\Big(x_1 + x_2 + \frac{1}{x_2 x_3} \\ &\qquad + \frac{x_2 x_3}{x_4} + \frac{b x_4}{x_1} + \frac{b x_3}{x_1}\Big)\Big)\, dx_1\, dx_2\, dx_3\, dx_4 \\ &&& \text{if } A \geq 2 \text{ is even;} \\ &\quad + q_F^{3A}(1 - q_F^{-1}) && \text{if } A = 2; \\ &\quad + 2q_F^{3A} && \text{if } A \geq 4 \text{ is even and } -a_1 a_2 \text{ is a square in } F; \\ &= 0 && \text{otherwise.} \end{aligned}$$

4. The orbital integral $J_F(wa, \Phi_0)$

We write any element in $N_w(F)\backslash N(L)$ as $n = \begin{pmatrix} n_1^{-1} & \\ & n_2 \end{pmatrix} \begin{pmatrix} I & u \\ & I \end{pmatrix}$ where $n_i = \begin{pmatrix} 1 & m_i \\ & 1 \end{pmatrix}$ with $m_1, m_2 \in F$ and $u \in M_{2\times 2}(L)$. Then the matrix condition attached to the integral defining $J_F(wa, \Phi_0)$ on the right side of (2) becomes

$$\begin{pmatrix} {}^t n_1^{-1} w_1 b_1 n_1^{-1} & {}^t n_1^{-1} w_1 b_1 n_1^{-1} u \\ {}^t \bar{u}\, {}^t n_1^{-1} w_1 b_1 n_1^{-1} & {}^t \bar{u}\, {}^t n_1^{-1} w_1 b_1 n_1^{-1} u + {}^t n_2 w_1 b_2 n_2 \end{pmatrix} \in K(L). \qquad (21)$$

First, we conclude from (21) that $a_1 \in R_F$ and $a_1 a_2 \in R_F^\times$ if the integral is non-zero. If $a_1 \in R_F^\times$, then $J_F(wa, \Phi_0) = 1$. We will assume from now on that $a_1 \in \varpi_F^A R_F^\times$ and $a_2 \in \varpi_F^{-A} R_F^\times$ with $A > 0$. Then from (21) we get $m_1, m_2 \in \varpi_F^{-A} R_F$ following the arguments used in Section 3. Changing variables from u to $z = \left(\begin{smallmatrix} z_1 & z_2 \\ z_3 & z_4 \end{smallmatrix}\right) = {}^t n_1^{-1} w_1 b_1 n_1^{-1} u \in M_{2\times 2}(R_L)$ and from m_i to $m_i/2$ we get

$$J_F(wa, \Phi_0) = q_F^{8A} \cdot \int_{\substack{m_1, m_2 \in \varpi_F^{-A} R_F, \\ z \in M_{2\times 2}(R_L), \\ {}^t\bar{z}\left(\begin{smallmatrix} m_1 & 1 \\ 1 & 0 \end{smallmatrix}\right) z \in -a_1 a_2 \left(\begin{smallmatrix} 0 & 1 \\ 1 & m_2 \end{smallmatrix}\right) + M_{2\times 2}(\varpi_L^A R_L)}} \psi_F(m_2 - m_1) \psi_F \circ \mathrm{tr}_{L/F}\left(\frac{z_1}{a_1}\right) dm_1\, dm_2\, dz.$$

As in the last section there are three cases:

(i) $m_1, m_2 \in R_F$, $z \in K_{2\times 2}(L)$;
(ii) $m_1, m_2 \in \varpi_F^M R_F^\times$, $z \in K_{2\times 2}(L)$ with $-A < M < 0$; and
(iii) $m_1, m_2 \in \varpi_F^{-A} R_F^\times$, $z \in M_{2\times 2}(R_L)$.

We will denote the corresponding integrals by J_1, J_2, and J_3.

For J_1 we change variables from z to $y = \left(\begin{smallmatrix} y_1 & y_2 \\ y_3 & y_4 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ m_1/2 & 1 \end{smallmatrix}\right) z$. Then the integrand becomes $\psi_F \circ \mathrm{tr}_{L/F}(y_1/a_1)$ and the matrix condition attached to J_1 becomes ${}^t\bar{y} w_1 y \in -a_1 a_2 \left(\begin{smallmatrix} 0 & 1 \\ 1 & m_2 \end{smallmatrix}\right) + M_{2\times 2}(\varpi_L^A R_L)$. By integrating the integral with respect to m_1, m_2, y_2, and y_4 we get

$$J_1 = q_F^{5A} \int_{\substack{y_1, y_3 \in R_L, \\ y_1 \text{ or } y_3 \in R_L^\times, \\ y_1 \bar{y}_3 + \bar{y}_1 y_3 \in \varpi_F^A R_F}} \psi_F \circ \mathrm{tr}_{L/F}\left(\frac{y_1}{a_1}\right) dy_1\, dy_3.$$

If $y_1 \in R_L^\times$, the integral with respect to y_3 yields q_F^{-A} and we get

$$q_F^{4A} \int_{y_1 \in R_L^\times} \psi_F \circ \mathrm{tr}_{L/F}\left(\frac{y_1}{a_1}\right) dy_1$$

which equals $-q_F^2$ when $A = 1$ and vanishes when $A > 1$. If $y_1 \in \varpi_L R_L$, then $y_3 \in R_L^\times$. When $A = 1$ the integrand equals 1 and we get $q_F^3(1 - q_F^{-2})$. When $A > 1$ we change variables from y_1 to $y_0 = y_1 \bar{y}_3 \in \varpi_L R_L$ and get

$$q_F^{5A} \int_{\substack{y_0 \in \varpi_L R_L \text{ with } y_0 + \bar{y}_0 \in \varpi_F^A R_F, \\ y_3 \in R_L^\times}} \psi_F \circ \mathrm{tr}_{L/F}\left(\frac{y_0}{a_1 \bar{y}_3}\right) dy_0\, dy_3$$

which equals $q_F^{3A}(1 - q_F^{-1})$. Adding these results together, we finally have

$$\begin{aligned} J_1 &= q_F^3(1 - q_F^{-1} - q_F^{-2}) && \text{if } A = 1; \\ &= q_F^{3A}(1 - q_F^{-1}) && \text{if } A > 1. \end{aligned}$$

To compute J_2, we first integrate with respect to m_2. Then

$$J_2 = q_F^{7A} \sum_{-A < M < 0} \int \psi_F\left(-m_1 - \frac{m_1 z_2 \bar{z}_2}{a_1 a_2}\right) \psi_F \circ \mathrm{tr}_{L/F}\left(\frac{z_1}{a_1}\right) dm_1\, dz_1\, dz_2\, dz_3\, dz_4$$

where the integral is taken over

$$\begin{aligned} m_1 &\in \varpi_F^M R_F^\times, \\ z_1 &\in \varpi_L^{-M} R_L, \\ z_2, z_3 &\in R_L^\times, \\ z_4 &\in R_L, \\ m_1 z_1 \bar{z}_1 + \bar{z}_1 z_3 + z_1 \bar{z}_3 &\in \varpi_F^A R_F, \\ m_1 \bar{z}_1 z_2 + \bar{z}_1 z_4 + z_2 \bar{z}_3 &\in -a_1 a_2 + \varpi_L^A R_L. \end{aligned}$$

Now we consider two cases:

(i) $z_1 \in \varpi_L^A R_L$ and

(ii) $z_1 \in \varpi_L^Z R_L^\times$ with $-M \le Z < A$.

In case (i) we can first integrate with respect to z_4, z_3, and z_1 to get

$$q_F^{3A} \sum_{-A<M<0} \int_{\substack{m_1 \in \varpi_F^M R_F^\times, \\ z_2 \in R_L^\times}} \psi_F\Big(-m_1 - \frac{m_1 z_2 \bar{z}_2}{a_1 a_2}\Big)\, dm_1\, dz_2.$$

Writing the above sum as an integral taken over $m_1 \in \varpi_F^{1-A} R_F$ minus the same integral over $m_1 \in R_F$, we get $q_F^{3A-1}(1 + q_F^{-1})$. Note that for J_2 we always have $A > 1$.

In case (ii) we have $z_2 \in -a_1 a_2/(m_1 \bar{z}_1 + \bar{z}_3) + \varpi_L^Z R_L$. Integrating with respect to z_4 and z_2 we arrive at

$$q_F^{5A} \sum_{0<-M\le Z<A} \int_{\substack{m_1 \in \varpi_F^M R_F^\times, \\ z_1 \in \varpi_L^Z R_L^\times, \\ z_3 \in R_L^\times, \\ m_1 z_1 + z_3 \in R_L^\times, \\ m_1 z_1 \bar{z}_1 + z_1 \bar{z}_3 + \bar{z}_1 z_3 \in \varpi_F^A R_F}} \psi_F\Big(-m_1 - \frac{a_1 a_2 m_1}{(m_1 z_1 + z_3)(m_1 \bar{z}_1 + \bar{z}_3)}\Big) \psi_F \circ \mathrm{tr}_{L/F}\Big(\frac{z_1}{a_1}\Big)\, dm_1\, dz_1\, dz_3.$$

Note that the last two conditions above are equivalent to $(m_1 z_1 + z_3)(m_1 \bar{z}_1 + \bar{z}_3) \in z_3 \bar{z}_3 + \varpi_F^{A+M} R_F$. If we change variables from z_3 to $z_0 = z_3/m_1 \in \varpi_L^{-M} R_L^\times$ with $(z_1 + z_0)(\bar{z}_1 + \bar{z}_0) \in z_0 \bar{z}_0 (1 + \varpi_F^{A+M} R_F)$, we get an integral of

$$\psi_F\Big(-m_1 - \frac{a_1 a_2}{m_1 (z_1 + z_0)(\bar{z}_1 + \bar{z}_0)}\Big)$$

integrated with respect to $m_1 \in \varpi_F^M R_F^\times$. This integral is of the same kind as the integral in (9) and hence is non-zero only if $1 + a_1 a_2/(z_1 + z_0)(\bar{z}_1 + \bar{z}_0) \in R_L^\times$ when $M < -1$. Using three new variables $m = -m_1 \in \varpi_F^M R_F^\times$, $u = m_1 z_1 / z_3 \in \varpi_L^{M+Z} R_L^\times$ with $(1+u)(1+\bar{u}) \in 1 + \varpi_F^{A+M} R_F$, and $z = -(1+u) z_3 \in R_L^\times$ (with $1 + a_1 a_2/(z\bar{z}) \in R_F^\times$ when $M < -1$) we can rewrite the integral in case (ii) as

(22)
$$q_F^{5A} \sum_{0<-M\le Z<A} q_F^{2M} \int \psi_F\Big(m\Big(1 + \frac{a_1 a_2}{z\bar{z}}\Big)\Big) \psi_F \circ \mathrm{tr}_{L/F}\Big(\frac{uz}{a_1 m (1+u)}\Big)\, dm\, du\, dz.$$

Let us first consider the case of $M < -1$. By an argument similar to the one for integrals like (9), the integral with respect to m vanishes unless the orders of

$m(1+a_1a_2/(z\bar{z}))$ and $c_1/(a_1m)$ are the same, where $uz/(1+u) = c_1 + c_2\sqrt{\tau}$. Since $c_1 \in \varpi_F^{M+Z}R_F$, we have $M \geq Z-A$; hence the sum in (22) in this case is taken over $1 < -M \leq Z \leq A+M$ which implies that $A+2M \geq 0$. Now by changing variables from z to $z/(1+u)$ we can simply erase the factor $(1+u)$ from the integrand in (22). We claim that we must have $M = -A/2$ if the above integral is non-zero. To prove this claim, we assume $M > -A/2$ and apply the Mellin transform to the integral:

$$q_F^{5A} \sum_{\substack{-A/2<M<-1,\\ -M\leq Z\leq A+M}} q_F^{2M} \int_{R_F^\times} \chi^{-1}(b)\,db \cdot \int_{\substack{m\in\varpi_F^M R_F^\times,\\ z\in R_L^\times,\\ u\in\varpi_L^{M+Z}R_L^\times,\\ (1+u)(1+\bar{u})\in 1+\varpi_F^{A+M}R_F}} \psi_F\Big(m\Big(1+\frac{b}{z\bar{z}}\Big)\Big)\psi_F\circ\mathrm{tr}_{L/F}\Big(\frac{uz}{a_1m}\Big)\,dm\,du\,dz$$

where we set $b = a_1a_2$. If χ is unramified, the integral with respect to b vanishes because $M < -1$. Assume that χ is ramified. After changing variables from b to $c = bm/(z\bar{z}) \in \varpi_F^M R_F^\times$ and from z to $y = uz/(a_1m) \in \varpi_L^{Z-A}R_L^\times$ we see that the conductor exponent $a(\chi)$ must equal $-M$ if the integral is non-zero. Since $a(\chi\circ N_{L/F}) = a(\chi)$ we also have $Z - A = M$, i.e., $Z = A+M$ if the integral is non-zero. Therefore, when $a(\chi) = -M$ and $Z = A+M$ we get the integral

$$\int_{\substack{u\in\varpi_L^{A+2M}R_L^\times,\\ (1+u)(1+\bar{u})\in 1+\varpi_F^{A+M}R_F}} \chi(u\bar{u})\,du.$$

Since $A+2M > 0$ we can set $1+u = (1+w)(1+v\sqrt{\tau})/(1-v\sqrt{\tau})$ with $w \in \varpi_L^{A+M}R_L$ and $v \in \varpi_F^{A+2M}\Big(R_F^\times/(1+\varpi_F^{-M}R_F)\Big)$. Then $\chi(u\bar{u}) = \chi(-4\tau v^2/(1-\tau v^2))$ and the sum with respect to v vanishes because we can set $v = v_0(1+c)$ with $c \in \varpi_F^{-[M/2]}R_F$ and sum over c. Since the Mellin transform for any character χ is zero, we prove our claim. Back to the case of $A+2M = 0$, we point out that then $Z = -M = A+M$ and $u \in R_L^\times$. Since the integral with respect to m vanishes if $u \in \varpi_L R_L$ when $M < -1$, we can take the integral with respect to u over R_L.

Now let us turn to the case of $M = -1$. Since $m(1+a_1a_2/(z\bar{z})) \in \varpi_F^{-1}R_F$ and $(1+u)(1+\bar{u}) \in 1+\varpi_F^{A-1}R_F$, we can again change variables from z to $z(1+u)$ and thus erase the factor $(1+u)$ from the integrand in (22). We can also set $z = z_0(1+v)$ with $v \in \varpi_L R_L$. Integrating with respect to v we know that Z must be equal to $A-1$ in order to have a non-zero integral. Consequently, we get

$$q_F^{5A-2} \int_{\substack{m\in\varpi_F^{-1}R_F^\times,\\ z\in R_L^\times,\\ u\in\varpi_L^{A-2}R_L^\times,\\ (1+u)(1+\bar{u})\in 1+\varpi_F^{A-1}R_F}} \psi_F\Big(m\Big(1+\frac{a_1a_2}{z\bar{z}}\Big)\Big)\psi_F\circ\mathrm{tr}_{L/F}\Big(\frac{uz}{a_1m}\Big)\,dm\,du\,dz.$$

Recall that from case (i) we got $q_F^{3A-1}(1+q_F^{-1})$. This is indeed equal to the above integral with u being taken over $\varpi_L^{A-1}R_L$. Adding this expression to the above integral we thus can set $u \in \varpi_L^{A-2} R_L$ with $(1+u)(1+\bar{u}) \in 1+\varpi_F^{A-1} R_F$. If $A > 2$, we can further compute the integral with respect to u

$$\begin{aligned}
&\int_{\substack{u\in\varpi_L^{A-2}R_L,\\ (1+u)(1+\bar{u})\in 1+\varpi_F^{A-1}R_F}} \psi_F\circ\mathrm{tr}_{L/F}\Big(\frac{uz}{a_1 m}\Big)\,du \\
&= \int_{\substack{u\in\varpi_L^{A-2}R_L,\\ u+\bar{u}\in\varpi_F^{A-1}R_F}} \psi_F\circ\mathrm{tr}_{L/F}\Big(\frac{uz}{a_1 m}\Big)\,du \\
&= \int_{\substack{w\in\varpi_F^{A-2}R_F,\\ v\in\varpi_F^{A-1}R_F}} \psi_F\circ\mathrm{tr}_{L/F}\Big(\frac{w\sqrt{\tau}z}{a_1 m}\Big)\,dv\,dw
\end{aligned}$$

with $u = v + w\sqrt{\tau}$. Since the order of ψ_F is zero, the last integral above equals q_F^{3-2A} if $y \in \varpi_F R_F$ and vanishes if $y \in R_F^\times$, where $z = x + y\sqrt{\tau}$. Back to (22), when $A > 2$ we have

$$\begin{aligned}
&q_F^{3A+1} \int_{\substack{m\in\varpi_F^{-1}R_F^\times,\\ z\in R_L^\times \text{ with } y\in\varpi_F R_F}} \psi_F\Big(m\Big(1+\frac{a_1a_2}{z\bar{z}}\Big)\Big)\,dm\,dz \\
&= q_F^{3A} \int_{\substack{m\in\varpi_F^{-1}R_F^\times,\\ x\in R_F^\times}} \psi_F\Big(m\Big(1+\frac{a_1a_2}{x^2}\Big)\Big)\,dm\,dx \\
&= \begin{cases} q_F^{3A}(q_F^{-1}+1) & \text{if } -a_1a_2 \text{ is a square;} \\ q_F^{3A}(q_F^{-1}-1) & \text{if } -a_1a_2 \text{ is not a square.} \end{cases}
\end{aligned}$$

To summarize our computation we have

$$\begin{aligned}
J_2 = q_F^{4A} &\int_{\substack{m\in\varpi_F^{-A/2}R_F^\times,\\ z\in R_L^\times,\\ u\in R_L,\\ (1+u)(1+\bar{u})\in 1+\varpi_F^{A/2}R_F}} \psi_F\Big(m\Big(1+\frac{a_1a_2}{z\bar{z}}\Big)\Big) \\
&\cdot \psi_F\circ\mathrm{tr}_{L/F}\Big(\frac{uz}{a_1 m}\Big)\,dm\,du\,dz \quad \text{if } A \geq 2 \text{ is even} \\
&+ \begin{cases} q_F^{3A}(q_F^{-1}+1) & \text{if } -a_1a_2 \text{ is a square in } R_F^\times/(1+\varpi_F R_F) \\ & \text{and } A \geq 3 \\ q_F^{3A}(q_F^{-1}-1) & \text{if } -a_1a_2 \text{ is not a square in } R_F^\times/(1+\varpi_F R_F) \\ & \text{and } A \geq 3. \end{cases}
\end{aligned}$$

After changing variables from m to $n = cm \in R_F^\times$ where $c \in \varpi_F^{A/2} R_F^\times$, from z to $x_1 = -c^2 z/a_1 \in R_L^\times$, and from u to $x_2 = -x_1(1+u) \in R_L^\times$, we can write the above integral as

$$J_2 = q_F^{9A/2} \int_{\substack{n \in R_F^\times,\\ x_1, x_2 \in R_L^\times,\\ x_1\bar{x}_1 \in x_2\bar{x}_2 + \varpi_F^{A/2} R_F}} \psi_F\Big(\frac{n}{c}\Big(1 + \frac{b}{x_1\bar{x}_1}\Big)\Big)$$
$$\cdot\, \psi_F \circ \mathrm{tr}_{L/F}\Big(\frac{x_1 + x_2}{cn}\Big)\, dn\, dx_1\, dx_2 \quad \text{if } A \geq 2 \text{ is even}$$
$$+ \begin{cases} q_F^{3A}(q_F^{-1} + 1) & \text{if } -a_1a_2 \text{ is a square in } R_F^\times/(1 + \varpi_F R_F) \\ & \text{and } A \geq 3 \\ q_F^{3A}(q_F^{-1} - 1) & \text{if } -a_1a_2 \text{ is not a square in } R_F^\times/(1 + \varpi_F R_F) \\ & \text{and } A \geq 3 \end{cases}$$

where as before $b = a_2c^4/a_1 \in R_F^\times$.

Finally, we compute J_3:

$$J_3 = q_F^{8A}$$
$$\cdot \int_{\substack{m_1, m_2 \in \varpi_F^{-A} R_F^\times,\\ z \in M_{2\times 2}(R_L),\\ {}^t\bar{z}\left(\begin{smallmatrix} m_1 & 1 \\ 1 & 0 \end{smallmatrix}\right) z \in -a_1a_2\left(\begin{smallmatrix} 0 & 1 \\ 1 & m_2 \end{smallmatrix}\right) + M_{2\times 2}(\varpi_L^A R_L)}} \psi_F(m_2 - m_1)\psi_F \circ \mathrm{tr}_{L/F}\Big(\frac{z_1}{a_1}\Big)\, dm_1\, dm_2\, dz$$

where the matrix condition is equivalent to

$$\begin{aligned} &m_1 z_1\bar{z}_1 + \bar{z}_1 z_3 + z_1\bar{z}_3 \in \varpi_F^A R_F, \\ &m_1\bar{z}_1 z_2 + \bar{z}_1 z_4 + z_2\bar{z}_3 \in -a_1a_2 + \varpi_L^A R_L \\ &m_1 z_2\bar{z}_2 + \bar{z}_2 z_4 + z_2\bar{z}_4 \in -a_1a_2m_2 + \varpi_F^A R_F. \end{aligned}$$

The last condition above implies that $z_2 \in R_L^\times$; hence from the second condition we get $z_1 \in \varpi_L^A R_L$. Then we can integrate with respect to m_2, z_4, z_3, and z_1 to get

$$J_3 = q_F^{3A} \int_{\substack{m_1 \in \varpi_F^{-A} R_F^\times,\\ z_2 \in R_L^\times}} \psi_F\Big(-m_1\Big(1 + \frac{z_2\bar{z}_2}{a_1a_2}\Big)\Big)\, dm_1\, dz_2.$$

A similar integral has been computed and this one equals

$$\begin{aligned} J_3 &= q^2(1 + q_F^{-1}) \quad \text{if } A = 1; \\ &= 0 \quad \text{if } A > 1. \end{aligned}$$

Collecting our results on J_1, J_2, and J_3 we arrive at

Lemma 5. *Under the assumption of Lemma 1 we have*

$$
\begin{aligned}
J_F(wa, \Phi_0) &= 1 && \text{if } A = 0;\\
&= q_F^3 && \text{if } A = 1;\\
&= 2q_F^{3A} && \text{if } A \geq 3 \text{ is odd and } -a_1a_2 \text{ is a square in } F;\\
&= q_F^{9A/2} \int\limits_{\substack{n \in R_F^\times,\\ x_1, x_2 \in R_L^\times,\\ x_1\bar{x}_1 \in x_2\bar{x}_2 + \varpi_F^{A/2} R_F}} \psi_F\Big(\frac{n}{c}\Big(1 + \frac{b}{x_1\bar{x}_1}\Big)\Big)\\
&\qquad \cdot \psi_F \circ \mathrm{tr}_{L/F}\Big(\frac{x_1 + x_2}{cn}\Big)\, dn\, dx_1\, dx_2\\
& && \text{if } A \geq 2 \text{ is even;}\\
&+ q_F^{3A}(1 - q_F^{-1}) && \text{if } A = 2;\\
&+ 2q_F^{3A} && \text{if } A \geq 4 \text{ is even and } -a_1a_2 \text{ is a square in } F;\\
&= 0 && \text{otherwise.}
\end{aligned}
$$

Comparing Lemmas 4 and 5 and using the local identity in Lemma 1, we prove Theorem 2.

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