REPRESENTATION THEORY
An Electronic Journal of the American Mathematical Society
Volume 4, Pages 64–96 (February 18, 2000)
S 1088-4165(00)00087-X

HARISH-CHANDRA MODULES FOR QUANTUM SYMMETRIC PAIRS

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ABSTRACT. Let U denote the quantized enveloping algebra associated to a semisimple Lie algebra. This paper studies Harish-Chandra modules for the recently constructed quantum symmetric pairs U,B in the maximally split case. Finite-dimensional U-modules are shown to be Harish-Chandra as well as the B-unitary socle of an arbitrary module. A classification of finite-dimensional spherical modules analogous to the classical case is obtained. A one-to-one correspondence between a large class of natural finite-dimensional simple B-modules and their classical counterparts is established up to the action of almost B-invariant elements.

Let $\mathbf{g}$ be a semisimple Lie algebra and let $\mathbf{g}^\theta$ be the Lie subalgebra fixed by an involution θ of $\mathbf{g}$. There is an extensive theory concerning the Harish-Chandra modules associated to the pair $\mathbf{g}$, $\mathbf{g}^\theta$. These are $\mathbf{g}$-modules which behave nicely with respect to the restriction of the action of $\mathbf{g}$ to $\mathbf{g}^\theta$. One of the main motivations in understanding such modules is their close connection and impact on the study of real Lie group representations. Harish-Chandra modules also are a class of infinite-dimensional $\mathbf{g}$-modules that have a reasonable amount of structure and thus are amenable to study. In this way they are similar to the other infinite-dimensional $\mathbf{g}$-modules that have been examined thoroughly, the so-called category $\mathcal{O}$ modules. These modules behave nicely with respect to the Cartan subalgebra of $\mathbf{g}$.

Let U denote the Drinfeld-Jimbo quantization of the enveloping algebra of $\mathbf{g}$. In the quantum case, there is already a well developed theory of category $\mathcal{O}$ modules (see for example [Jo]). However, much less is known about infinite-dimensional quantum modules that correspond to the classical Harish-Chandra modules. One of the main reasons for this difference is that there is an obvious quantum analog of the Cartan subalgebra of $\mathbf{g}$, while the analogs of the invariant Lie subalgebra are less apparent. In [L2], we introduced one-sided coideal algebras $B = B_\theta$ as quantum analogs of the enveloping algebra of the fixed Lie subalgebra under the maximally split form of an involution θ. These analogs generalize the known examples already in the literature in the maximally split case. Using these analogs in this paper, we lay the groundwork for the study of quantum Harish-Chandra modules.

In the first part of the paper, we prove elementary results about quantum Harish-Chandra modules associated to U, B. As in the classical case, every finite-dimensional simple U-module comes equipped with a positive definite conjugate linear form. One checks that B behaves nicely with respect to this form which allows us to decompose finite-dimensional U-modules into a direct sum of finite-dimen-

Received by the editors October 22, 1999 and, in revised form, November 19, 1999.
2000 *Mathematics Subject Classification.* Primary 17B37.
The author was supported by NSF grant no. DMS-9753211.

sional simple B-modules. This is exactly the condition necessary to make such a U-module into a quantum Harish-Chandra module with respect to B. Using this result applied to the locally finite part of U, we prove that the sum of all finite-dimensional unitary simple B-modules inside a U-module is a Harish-Chandra module. As in the classical case, this result should prove useful in finding examples of Harish-Chandra modules inside of infinite-dimensional U-modules.

In the next part of the paper, we study a special type of quantum Harish-Chandra module: the finite-dimensional spherical modules. We show that a finite-dimensional U-module is a spherical module for B exactly when the corresponding $\mathbf{g}^\theta$-module is spherical. Some of the proof here is similar to the classical case, while some new ideas are needed to avoid the use of Lie groups. These results extend the work of Noumi and Sugitani (see [N, Theorem 3.1] and [NS, Theorem 1]) in special cases and suggest that the quantum analogs of [L2] will yield a good quantum symmetric space theory for maximally split involutions (see [Di, end of Section 3]).

In order to have a reasonable Harish-Chandra module theory, one needs to understand the finite-dimensional simple B-modules. One obvious class of finite-dimensional simple unitary B-modules is the submodules of finite-dimensional U-modules. We extend this class when $\mathbf{g}^\theta$ has a nontrivial center. Unfortunately, in most cases, neither the enveloping algebra of the semisimple part nor the center of $\mathbf{g}^\theta$ lifts to a subalgebra of B. However, B is shown to be the direct sum of a polynomial ring whose generators correspond to a basis of $\mathbf{g}^\theta/[\mathbf{g}^\theta, \mathbf{g}^\theta]$ and a subalgebra of B. This decomposition is then used to produce analogs of the one-dimensional $U(\mathbf{g}^\theta)$ representations for B. The class of finite-dimensional unitary simple modules is extended by taking submodules of tensor products of these one-dimensional simple B-modules with finite-dimensional U-modules.

In order to gain further understanding of the finite-dimensional simple modules, we study those modules of B which can be specialized to $\mathbf{g}^\theta$-modules. One of the difficulties in understanding finite-dimensional B-modules is that unlike the quantized enveloping algebra, there is no obvious Cartan subalgebra inside of B. Thus one cannot analyze finite-dimensional B-modules using weights and highest weight vectors. In [GI] (see also [L1, Remark 2.4]), certain finite-dimensional modules are analyzed when B is the analog of $U(\mathbf{so}\ n)$ using Gelfand-Tsetlin basis; however, this approach does not generalize to other cases. Despite these problems, we show that every specializable simple B-module remains simple upon specialization and every finite-dimensional simple $U(\mathbf{g}^\theta)$-module is the image under specialization of some finite-dimensional simple B-module. Furthermore, every finite-dimensional unitary simple $U(\mathbf{g}^\theta)$-module can be lifted to a finite-dimensional unitary simple B-module. The picture is particularly nice when $\mathbf{g}^\theta$ is semisimple. Indeed in this case, there is a one-to-one correspondence between specializable finite-dimensional simple B-modules and finite-dimensional simple $U(\mathbf{g}^\theta)$-modules.

The main new idea in studying specialization is to embed finite-dimensional B-modules inside of certain U-modules coinduced from a subalgebra of B. The process is rather delicate since one must show that specializable B-modules embed in a way that is compatible with specialization of the larger U-modules. When $\mathbf{g}^\theta$ is not semisimple, we use the results on spherical modules to lift the center of $\mathbf{g}^\theta$ to elements of U (though not to elements of B). This is necessary to ensure that the center of $\mathbf{g}^\theta$ will acts as scalars on the specialization of the B-modules under question.

In the quantized enveloping algebra case, all finite-dimensional simple modules are specializable (up to a one-dimensional representation of the Cartan subalgebra). This is not true for the analogs of $U(\mathbf{g}^\theta)$. At the end of the paper, we give examples of one-dimensional nonspecializable modules for an example of B where $\mathbf{g}^\theta$ is semisimple.

There is more than one way to embed the analogs in [L2] of $U(\mathbf{g}^\theta)$ into U up to Hopf algebra automorphisms. These embeddings are divided into two classes, standard and nonstandard (see Section 2). All the results listed above are proved for the standard embedding. The same theorems are also true for a large class of nonstandard embeddings (see Section 3 and Section 4). This was somewhat surprising to the author and required a completely different argument for the theorem classifying finite-dimensional spherical modules (see the end of Section 4).

This paper is organized as follows. Section 1 sets notation and recalls basic facts about semisimple Lie algebras. In Section 2, we give the necessary background material on quantum symmetric pairs, and in particular, the construction of the analogs of $U(\mathbf{g}^\theta)$. Section 3 is devoted to basic results about quantum Harish-Chandra modules. Necessary and sufficient conditions for a finite-dimensional U-module to be spherical are established in Section 4. In Section 5, we produce a decomposition of B when $\mathbf{g}^\theta$ is not semisimple in order to obtain additional one-dimensional representations of B. Section 6 is an analysis of certain coinduced modules and their finite-dimensional B-submodules. In Section 7, we define and study specializable B-modules.

1. Background and Notation

Let $\mathbf{g} = \mathbf{n}^- \oplus \mathbf{h} \oplus \mathbf{n}^+$ be a semisimple Lie algebra of rank l over $\mathbf{C}$. Let (a_{ij}) denote the Cartan matrix of $\mathbf{g}$ and let Δ (resp. Δ^+) be the set of roots (resp. positive roots) for $\mathbf{g}$. Write $\pi = \{\alpha_1 \dots \alpha_l\}$ for the set of positive simple roots and set $Q = \sum_{1 \le i \le l} \mathbf{Z}\alpha_i$ and $Q^+ = \sum_{1 \le i \le l} \mathbf{N}\alpha_i$ (where $\mathbf{N}$ denotes the set of nonnegative integers). Let $(\ ,\ )$ denote the Cartan inner product on $\mathbf{h}^*$ and set $\langle \alpha, \beta \rangle = 2(\alpha, \beta)/(\beta, \beta)$. We use "$<$" for the usual order on Q. In particular, we say that $\lambda < \gamma$ whenever $\gamma - \lambda \in Q^+$.

Let q be a fixed indeterminate. We will be defining quantized enveloping algebras over an extension of $\mathbf{C}(q)$ and we will also need a notion of conjugation on this extended field. Note that the ring $\mathbf{C}[q]_{(q-1)}$ can be considered as a subring of the ring of formal power series $\mathbf{C}[[q-1]]$. Furthermore, every element in $\mathbf{C}[[q-1]]$ that is not in the ideal generated by $(q-1)$ has a square root in $\mathbf{C}[[q-1]]$. Let A be the smallest subring of the algebraic closure of $\mathbf{C}(q)$ such that every element in A which is not contained in the ideal generated by $(q-1)$ has a square root and is invertible. It follows that we can also embed A inside of $\mathbf{C}[[q-1]]$. Write K for the quotient field of A. Define $\mathcal{R}$ to be the intersection of K with the real closure of $\mathbf{R}(q)$. Using the theory of real ordered fields, we have that $K = \mathcal{R} + i\mathcal{R}$ (see [Ja, §11]). Conjugation $^-$ on $\mathbf{C}$ can be extended to conjugation on $\mathbf{C}(q)$ and hence on K by setting the conjugate of q^m equal to q^m for all integers m.

Given an indeterminate q, the Drinfeld-Jimbo quantized enveloping algebra $U = U_q(\mathbf{g})$ is the algebra over K generated by x_i, y_i, $t_i^{\pm 1}$, $1 \le i \le l$ subject to the relations in [L2] or [Jo, 3.2.9]. (We follow the conventions of the warning in the latter reference.) The algebra U is also a Hopf algebra with comultiplication $\varDelta$, counit ϵ, and antipode σ as in [Jo, 3.2.9].

The group $T = \langle t_1, \ldots, t_l\rangle$ is isomorphic to the additive group Q. Define the corresponding isomorphism τ by setting $\tau(\alpha_i) = t_i$ for $1 \leq i \leq l$. We will sometimes adjoin additional elements $\tau(\lambda)$ to U where $\lambda \in \mathbf{h}^*$ as in [JL2, §3.1].

Given an algebra S and subalgebras $S_i, 1 \leq i \leq r$, we call the map from $S_1 \otimes \cdots \otimes S_r$ to S which sends $s_1 \otimes \cdots \otimes s_r$ to $s_1 \ldots s_r$ the multiplication map.

Let U^+ (resp. U^-, resp. U^o) be the K subalgebra of U generated by x_i (resp. y_i, resp. $t_i^{\pm 1}$), for $1 \leq i \leq l$. There is a vector space isomorphism over K via the multiplication map ([R])

$$U \cong U^- \otimes U^o \otimes U^+.$$

We may replace U^+ (resp. U^-) by G^+ (resp. G^-) where G^+ (resp. G^-) is the subalgebra of U generated by $x_i t_i^{-1}$ (resp. $y_i t_i$) for $1 \leq i \leq l$.

Given a U-module M and a $\Lambda \in K[T]^*$, we write M_Λ for the Λ weight space of M. In particular, $M_\Lambda = \{v \in M | tv = \Lambda(t)v \text{ for } t \in T\}$. When Λ takes the form $\Lambda(\tau(\beta)) = q^{(\lambda,\alpha)}$ for some $\lambda \in \mathbf{h}^*$, we write $\Lambda = q^\lambda$ and M_λ instead of M_Λ. When M is a subset of U, the λ weight space of M is defined with respect to the adjoint action of T on U. If $v \in M$ has weight λ, then we set $\operatorname{wt} v = \lambda$.

Set $\hat{U}$ equal to the A subalgebra of U generated by $x_i, y_i, (t_i - 1)/(q - 1)$. By say [DK, Proposition 1.5] and [JL2, 6.11], we have that $U(\mathbf{g}) \cong \hat{U} \otimes_A \mathbf{C}$ where we identify $\mathbf{C}$ with $A/\langle q - 1\rangle$. Given a subalgebra S of U, set $\hat{S} = \hat{U} \cap S$. The specialization of S at $q = 1$ is the image of $\hat{S}$ in $U(\mathbf{g})$ using the above isomorphism.

Though we started with the field $\mathbf{C}$, the results in this paper also hold if we replace $\mathbf{C}$ by the algebraic closure of any real ordered field. Basic facts about quantized enveloping algebras which are not referenced explicitly can be found in [Jo].

2. Quantum Symmetric Pairs

Let θ be an involution of $\mathbf{g}$ and write $\mathbf{g}^\theta$ for the corresponding fixed subalgebra. Replacing θ by a conjugate using automorphisms of $\mathbf{g}$ (if necessary), we may assume that θ is maximally split (see [D, 1.13.8]). In particular, we may assume that θ satisfies the following three conditions:

(2.1) $\theta(\mathbf{h}) = \mathbf{h}$;

(2.2) $\theta(e_i) = e_i$ and $\theta(f_i) = f_i$ if $\theta(h_i) = h_i$;

(2.3) $\theta(e_i)$ (resp. $\theta(f_i)$) is a nonzero scalar multiple of a root vector in $\mathbf{n}^-$ (resp. $\mathbf{n}^+$) if $\theta(h_i) \neq h_i$.

The above conditions imply that θ induces an automorphism of the root system which we denote by Θ. Set $\pi_\Theta = \{\alpha_i \in \pi | \theta(h_i) = h_i\}$, $Q(\pi_\Theta) = \sum_{\alpha_i \in \pi_\Theta} \mathbf{Z}\alpha_i$, and $\Delta_\Theta = Q(\pi_\Theta) \cap \Delta$. We have that Δ_Θ is a root system with positive roots $\Delta^+_\Theta = \Delta^+ \cap \Delta_\Theta$, negative roots $\Delta^-_\Theta = \Delta^- \cap \Delta_\Theta$ and simple roots π_Θ. Let $\mathbf{m}_\Theta$ denote the reductive Lie subalgebra of $\mathbf{g}^\theta$ generated by $\mathbf{g}^\theta \cap \mathbf{h}$ and the semisimple Lie subalgebra whose root system is Δ_Θ. Write $\mathbf{n}^+_\Theta$ for the Lie subalgebra generated by the root vectors of weights in $\Delta^+ - \Delta^+_\Theta$ and define $\mathbf{n}^-_\Theta$ similarly using negative weights. (In what follows, we drop the Θ subscript from $\mathbf{m}_\Theta$ since Θ will be understood from context.)

As in say [L2, (3.7)], there exists a permutation p on the set $\{i | \alpha_i \in \pi - \pi_\Theta\}$ such that for each $\alpha_i \in \pi - \pi_\Theta$,

(2.4) $$\Theta(\alpha_i) + \alpha_{p(i)} \in Q(\pi_\Theta).$$

Quantum analogs of $U(\mathbf{g}^\theta)$ are constructed and studied in [L2]. These analogs are subalgebras of U which specialize to $U(\mathbf{g}^\theta)$ and are right coideals. Because of the convention of considering left modules more often than right modules, it will be convenient to replace these right coideals by left coideals. This is done by using the isomorphism of U which sends x_i to y_i, y_i to x_i, t_i to t_i^{-1}, and sends the Hopf structure of U to the opposite Hopf structure. Thus when referring to results in [L2], we will assume this isomorphism transforming the Hopf structure has been applied to U.

Let us briefly review the construction and properties of the quantum analogs of $U(\mathbf{g}^\theta)$ defined in [L2]. Let T_Θ denote the subgroup of T consisting of those $\tau(\lambda)$ such that $\lambda = \Theta(\lambda)$. Let R be the subalgebra of U generated by the x_i, y_i, with $\alpha_i \in \pi_\Theta$ and the group T_Θ. Alternatively, we can view R as the subalgebra of U generated by the quantized enveloping algebra $U_q([\mathbf{m},\mathbf{m}])$ and T_Θ.

The involution θ can be lifted to a $\mathbf{C}$-algebra involution $\tilde\theta$ of U such that (see [L2, Theorem 3.1])

(2.5) $\tilde\theta(q) = q^{-1}$;
(2.6) $\tilde\theta(\tau(\lambda)) = \tau(-\Theta(\lambda))$ for all $\tau(\lambda) \in T$;
(2.7) $\tilde\theta(x_i) = x_i$ and $\tilde\theta(y_i) = y_i$ for all $\alpha_i \in \pi_\Theta$;
(2.8) $\tilde\theta(x_i) \in G^-_{\Theta(\alpha_i)}$ and $\tilde\theta(y_i) \in G^+_{\Theta(-\alpha_i)}$ where $\alpha_i \notin \pi_\Theta$.

Let $B_{\tilde\theta}$ be the K-subalgebra of U generated by R and the elements

$$B_i^{\tilde\theta} = y_i t_i + \tilde\theta(y_i) t_i$$

for $\alpha_i \notin \pi_\Theta$. In what follows, we drop the $\tilde\theta$ subscript or superscript from $B_{\tilde\theta}$ and $B_i^{\tilde\theta}$ as long as the involution θ can be understood from context.

By [L2, Theorem 4.9], B satisfies the following properties:

(2.9) B specializes to $U(\mathbf{g}^\theta)$.
(2.10) $\Delta(B) \subset U \otimes B$.
(2.11) If $B \subset C$ and C is a subalgebra of U which also specializes to $U(\mathbf{g}^\theta)$, then $B = C$.

Moreover, if B' and B both satisfy (2.9), (2.10), (2.11), then B' is isomorphic as an algebra to B (see [L2, Theorem 5.8]). This isomorphism is not necessarily the restriction of a Hopf algebra automorphism of U. To classify such left coideal algebras up to Hopf algebra isomorphism, set $\mathcal{S}$ equal to the subset of $\pi - \pi_\Theta$ consisting of those α_i such that $(\lambda, \alpha_i) = 0$ for all $\tau(\lambda) \in T_\Theta$. Let $\mathbf{S}$ denote the set of l-tuples $(s_1, \dots, s_l)$ such that $s_i \in A$ and $s_i \neq 0$ implies that $\alpha_i \in \mathcal{S}$. Given $\mathbf{s} \in \mathbf{S}$, set

$$B_j^{\mathbf{s},\tilde\theta} = y_j t_j + \tilde\theta(y_j) t_j + s_j t_j$$

for $1 \leq j \leq l$. Let $B_{\mathbf{s},\tilde\theta}$ be the subalgebra of U generated by R and $B_j^{\mathbf{s},\tilde\theta}$ for $\alpha_j \notin \pi_\Theta$.

Let $\mathcal{H}$ denote the set of the Hopf algebra automorphisms of U which fix elements of T and send x_i to $a_i y_i$, y_i to $a_i^{-1} y_i$, where a_i and a_i^{-1} are in $A - (q-1)A$ for $1 \leq i \leq l$. Let $[\Theta]$ denote the set of all involutions of $\mathbf{g}$ corresponding to the root system automorphism Θ. Replace condition (2.9) by

(2.12) B specializes to $U(\mathbf{g}^{\theta'})$ for some $\theta' \in [\Theta]$.

By [L2, §5 (the paragraph following Lemma 5.5, Theorem 5.8, and Remark 5.10)], we have the following classification of subalgebras of U satisfying (2.10), (2.11), and (2.12).

Theorem 2.1. *A subalgebra B of U satisfies conditions (2.10), (2.11), and (2.12) if and only if there is a Hopf algebra automorphism $\Upsilon \in \mathcal{H}$ such that $\Upsilon(B) = B_{\mathbf{s},\tilde{\theta}}$ for some $\mathbf{s} \in \mathcal{S}$. Moreover, B is isomorphic as an algebra to $B_{\tilde{\theta}}$.*

When $\mathbf{s}$ is identically zero, then $B_{\mathbf{s},\tilde{\theta}}$ is just the same as $B_{\tilde{\theta}}$. In this paper, we call B a standard analog of $U(\mathbf{g}^\theta)$ if B is the image of $B_{\tilde{\theta}}$ under a Hopf algebra automorphism of U in $\mathcal{H}$. Thus, any analog isomorphic to $B_{\mathbf{s},\tilde{\theta}}$ with $\mathbf{s} \neq 0$ via a Hopf algebra automorphism in $\mathcal{H}$ is nonstandard. We usually take the point of view of the standard analogs of $U(\mathbf{g}^\theta)$ in this paper. However, we show that the results of this paper are true as well for all nonstandard analogs of $U(\mathbf{g}^\theta)$ that are Hopf isomorphic to a subalgebra invariant under a certain real antiautomorphism of U.

Let N^+_Θ be the subalgebra of U^+ defined by

$$N^+_\Theta = \sum_{\gamma \in Q^+} (G^+ \cap \tilde{\theta}(U^-))_\gamma \tau(\gamma).$$

In the lemma below, we show that this algebra (defined somewhat differently than in [L2]) is a quantum analog of $U(\mathbf{n}^+_\Theta)$. Set $R^- = R \cap U^-$ and $R^+ = R \cap U^+$. Define a twisted adjoint action or, more precisely, a (right) skew derivation (see [Jo, 1.2.11]) of U^- on U by

$$(\tilde{\mathrm{ad}}\, y_i)b = y_i b t_i^{-1} - b y_i t_i^{-1}$$

for all $1 \leq i \leq l$. Similarly, we define an action of U^+ and U^o on U by

$$(\tilde{\mathrm{ad}}\, x_i)b = x_i b - t_i^{-1} b t_i x_i \qquad (\tilde{\mathrm{ad}}\, t_i)b = t_i^{-1} b t_i$$

for $1 \leq i \leq l$.

We can associate to the ordinary adjoint action a map $Ad : U \to U \otimes U^{op}$ such that if $Ad(a) = \sum a_{(1)} \otimes a_{(2)}$, then $(\mathrm{ad}\, a)b = \sum a_{(1)} b a_{(2)}$. Let ψ be the $\mathbf{C}$-algebra automorphism of U defined by $\psi(x_i) = x_i$, $\psi(y_i) = y_i$, $\psi(t_i) = t_i^{-1}$, and $\psi(q) = q^{-1}$ for $1 \leq i \leq l$. A map similar to Ad for $\tilde{\mathrm{ad}}$ can also be defined. In particular, the $\mathbf{C}$-algebra homomorphism $(\psi \otimes \psi) \circ Ad$ from U to $U \otimes U^{op}$ satisfies $(\psi \otimes \psi) \circ Ad\ (c) = \sum c_{(1)} \otimes c_{(2)}$ when $(\tilde{\mathrm{ad}}\, c)b = \sum c_{(1)} b c_{(2)}$. Thinking now of $\tilde{\mathrm{ad}}$ as a map from U into $\mathrm{End}\, U$, the above discussion implies that

(2.13)
$$(\tilde{\mathrm{ad}}\, x_i)(\tilde{\mathrm{ad}}\, y_j) = (\tilde{\mathrm{ad}}\, y_j)(\tilde{\mathrm{ad}}\, x_i) + \delta_{ij}(\tilde{\mathrm{ad}}\,(t_i - t_i^{-1}))/(q^{-1} - q)$$

for $1 \leq i \leq l$.

Proposition 2.2. *The algebra N^+_Θ is $(\tilde{\mathrm{ad}}\, R)$ invariant. Moreover, the algebra generated by N^+_Θ and R is isomorphic to $N^+_\Theta \otimes R$ as vector spaces via the multiplication map and the algebra generated by N^+_Θ and R^+ is equal to all of U^+.*

Proof. It is straightforward to check that $(\tilde{\mathrm{ad}}\, y_i)U^+$ is contained in $U^+ + U^+ t_i^{-2}$ for $1 \leq i \leq l$. Fix a nonnegative integer r and a sequence $i_1, \ldots, i_r$ of r elements in the set $\{i | \alpha_i \in \pi_\Theta\}$. Pick j such that $1 \leq j \leq l$ and $\alpha_j \notin \pi_\Theta$.

Set

$$Y_r = (\tilde{\mathrm{ad}}\, y_{i_r}) \cdots (\tilde{\mathrm{ad}}\, y_{i_1})(\tilde{\theta}(y_j)\tau(\Theta(-\alpha_j))).$$

We show by induction on r that

$$\Delta(Y_r) \subset t_{i_1}^{-1} \cdots t_{i_r}^{-1} \tau(\Theta(-\alpha_j)) \otimes Y_r + U \otimes R \tag{2.14}$$

and

$$Y_r \in U^+. \tag{2.15}$$

By [L2, Proposition 4.1], (2.14) and (2.15) hold for $r = 0$. Assume (2.14) holds for $r = n-1$. Set $\lambda = \mathrm{wt}\, Y_{n-1}$ and note that $\lambda = -\alpha_{i_1} \cdots - \alpha_{i_{n-1}} + \Theta(-\alpha_j)$. It follows that $y_{i_n}\tau(\lambda)t_{i_n}^{-1} \otimes t_{i_n}^{-1}Y_{n-1}t_{i_n}^{-1} = \tau(\lambda)y_{i_n}t_{i_n}^{-1} \otimes Y_{n-1}t_{i_n}^{-2}$. Using this fact, a straightforward computation shows that

$$\Delta(Y_n) = \Delta(y_{i_n}Y_{n-1}t_{i_n}^{-1} - Y_{n-1}y_{i_n}t_{i_n}^{-1}) = \tau(\lambda)t_{i_n}^{-1} \otimes (\tilde{\mathrm{ad}}\, y_{i_n})Y_{n-1} + U \otimes R.$$

Statement (2.14) thus follows by induction.

Now assume that (2.14) holds for Y_n and (2.15) holds for Y_{n-1}. Then, write $(\tilde{\mathrm{ad}}\, y_{i_n})Y_{n-1}$ as $b_1 + b_2t_{i_n}^{-2}$ where b_1 and b_2 are in $U^+_{\lambda-\alpha_{i_n}}$. We have

$$\Delta(b_1 + b_2t_{i_n}^{-2}) = \tau(\lambda) \otimes b_1 + \tau(\lambda)t_{i_n}^{-2} \otimes b_2t_{i_n}^{-2} + \sum_{\gamma<\lambda-\alpha_{i_n}} U \otimes U^+_\gamma U^o.$$

Since $\lambda - \alpha_{i_n} \notin Q(\pi_\Theta)$, we cannot have that $b_2t_{i_n}^{-2} \in R$. Thus by (2.14) for $r = n$, $b_2 = 0$ and hence (2.15) holds for $r = n$. Therefore (2.15) follows by induction on r.

Using induction again, we assume that $Y_{r-1} \in N^+_\Theta$. In particular, $Y_{r-1} = \tilde{\theta}(a_{r-1})\tau(\mathrm{wt}\, Y_{r-1})$ for some $a_{r-1} \in U^-$. It follows that $Y_r = (\tilde{\mathrm{ad}}\, y_{i_r})Y_{r-1} = \tilde{\theta}(a_r)\tau(\mathrm{wt}\, Y_{r-1})t_{i_r}^{-1}$ where $a_r = y_{i_r}a_{r-1} - q^{(\mathrm{wt}\, Y_{r-1},\alpha_{i_r})}a_{r-1}y_{i_r}$. Hence $Y_r \in N^+_\Theta$.

We have shown that $(\tilde{\mathrm{ad}}\, R^-)(\tilde{\theta}(y_j)) \in N^+_\Theta$ for all $\alpha_j \notin \pi_\Theta$. Thus N^+_Θ contains elements which specialize to the root vectors that form a basis of $\mathbf{n}^+_\Theta$. Since these generators are also in $\hat{U}$, we see that the dimension of each weight space of N^+_Θ is greater than or equal to the dimension of each weight space of $U(\mathbf{n}^+_\Theta)$. Furthermore, by definition, of N^+_Θ, the specialization of N^+_Θ is contained in $U(\mathbf{n}^+_\Theta)$. Hence N^+_Θ specializes to $U(\mathbf{n}^+_\Theta)$, there is an equality of character formulas $\mathrm{ch}\, N^+_\Theta = \mathrm{ch}\, U(\mathbf{n}^+_\Theta)$, and N^+_Θ must thus be generated by the set $\{(\tilde{\mathrm{ad}}\, R^-)(\tilde{\theta}(y_j)) | \alpha_j \notin \pi_\Theta\}$. This set is a vector space spanned by weight vectors, hence N^+_Θ is $(\tilde{\mathrm{ad}}\, R^o)$ invariant. It now follows from the definition of $\tilde{\mathrm{ad}}\, y_i$ that N^+_Θ is also $(\tilde{\mathrm{ad}}\, R^-)$ invariant.

To show that N^+_Θ is $(\tilde{\mathrm{ad}}\, R^+)$ invariant, it is sufficient to show that $(\tilde{\mathrm{ad}}\, x_i)a$ is in N^+_Θ for any of the generators of N^+_Θ and for any $x_i \in R$. In particular, assume that a takes the form $Y_r = (\tilde{\mathrm{ad}}\, y_{i_r}) \cdots (\tilde{\mathrm{ad}}\, y_{i_1})(\tilde{\theta}(y_j)\tau(\Theta(-\alpha_j)))$ for an appropriate sequence $i_r, \ldots, i_1$. By (2.13), we have that $(\tilde{\mathrm{ad}}\, x_i)Y_r \in N^+_\Theta + (\tilde{\mathrm{ad}}\, y_{i_r}) \cdots (\tilde{\mathrm{ad}}\, y_{i_1})(\tilde{\mathrm{ad}}\, x_i)(\tilde{\theta}(y_j)\tau(\Theta(-\alpha_j)))$. It is enough to show that zero equals $(\tilde{\mathrm{ad}}\, x_i)X$ where $X = \tilde{\theta}(y_j)\tau(\Theta(-\alpha_j))$. Now $(\tilde{\mathrm{ad}}\, x_i)X = x_iX - t_i^{-1}Xt_ix_i$, hence $\tilde{\theta}((\tilde{\mathrm{ad}}\, x_i)X) = x_iy_j\tau(\alpha_j) - t_iy_j\tau(\alpha_j)t_i^{-1}x_i$. Note that j cannot equal i because $\alpha_i \in \pi_\Theta$ while $\alpha_j \notin \pi_\Theta$. Thus $(\tilde{\mathrm{ad}}\, x_i)X = 0$ and N^+_Θ is $(\tilde{\mathrm{ad}}\, R)$ invariant.

Recall Sweedler's notation for Hopf algebras: $\Delta(r) = r_{(1)} \otimes r_{(2)}$. If $r \in R$ and $n \in N^+_\Theta$, then

$$rn = r_{(1)}n\epsilon(r_{(2)}) = r_{(1)}n\sigma(r_{(2)})r_{(3)} = ((\mathrm{ad}\, r_{(1)})n)r_{(2)}. \tag{2.16}$$

Since N^+_Θ is $(\tilde{\mathrm{ad}}\, R)$ invariant and R is a Hopf algebra, every element of the algebra generated by N^+_Θ and R is in the vector space $N^+_\Theta R = \mathrm{span}\{nr | n \in N^+_\Theta, r \in R\}$.

Let $\mathbf{m}^- \oplus \mathbf{m}^o \oplus \mathbf{m}^+$ be the triangle decomposition of the reductive Lie algebra $\mathbf{m}$. In the classical case, the subalgebra generated by $U(\mathbf{m}^+)$ and $U(\mathbf{n}_\Theta^+)$ is isomorphic as a vector space to $U(\mathbf{n}_\Theta^+) \otimes U(\mathbf{m}^+)$ via the multiplication map. Since $N_\Theta^+ R^+$ specializes to this algebra as q goes to 1, the subalgebra generated by N_Θ^+ and R^+ must be isomorphic to $N_\Theta^+ \otimes R^+$ as a vector space via the multiplication map.

Note that $(\operatorname{ch} N_\Theta^+)(\operatorname{ch} R^+) = (\operatorname{ch} U(\mathbf{n}_\Theta^+)(\operatorname{ch} U(\mathbf{m}^+)) = \operatorname{ch} U^+$. Hence the above tensor product decomposition shows that the subalgebra of U^+ generated by N_Θ^+ and R^+ is equal to all of U^+. The proposition now follows from the triangular decomposition of U and the triangular decomposition of R. $\square$

Let $\mathcal{A}$ be the subgroup of $\langle t_1^{1/2}, \ldots, t_l^{1/2} \rangle$ generated by $t_i^{1/2}\tau(\Theta(\alpha_i))^{-1/2}$ for $1 \leq i \leq l$. Let T' be the subgroup of $\langle t_1^{1/2}, \ldots, t_l^{1/2} \rangle$ generated by $T_\Theta^{1/2} = \langle t^{1/2} | t \in T_\Theta \rangle$ and $\mathcal{A}$. Note that T' contains the group T. Write U_Θ for the extension of U generated by U and T'. As in [L2, Theorem 4.5], there is a vector space isomorphism via the multiplication map called the quantum Iwasawa decomposition

$$U_\Theta \cong BT_\Theta^{1/2} \otimes K[\mathcal{A}] \otimes N_\Theta^+ \tag{2.17}$$

where B is any of the analogs of $U(\mathbf{g}^\theta)$ described in Theorem 2.1. As mentioned earlier, N_Θ^+ is defined somewhat differently from [L2]. This compensates for the fact that here we work inside U and not the larger algebra generated by U and $\langle t_1^{1/2}, \ldots, t_l^{1/2} \rangle$.

Define N_Θ^- to be the subalgebra $\tilde{\theta}(N_\Theta^+)$ of G^-. Note that

$$\tilde{N}_\Theta^- = \sum_{\gamma \in Q^+} (U^- \cap \tilde{\theta}(G^+))_{-\gamma}\tau(\gamma).$$

By Proposition 2.2 and the definition of $\tilde{\theta}$, it follows that $\tilde{N}_\Theta^-$ is $(\operatorname{ad} R)$ invariant. The proof of Proposition 2.2 also implies that $y_j t_j$ is contained in $\tilde{N}_\Theta^-$ for each $\alpha_j \notin \pi_\Theta$ and that $\tilde{N}_\Theta^-$ is generated by the sets $(\operatorname{ad} R^-)y_j t_j$ for $\alpha_j \notin \pi_\Theta$. Thus $\tilde{N}_\Theta^-$ is the subalgebra of G^- considered in [Ke]. Just as in (2.17), we have a vector space isomorphism via the multiplication map $U_\Theta \cong BT_\Theta^{1/2} \otimes K[\mathcal{A}] \otimes \tilde{N}_\Theta^-$.

Define a degree function on U by setting $\deg y_i = 1$ and $\deg x_i = \deg t_i^{\pm 1} = 0$ for $1 \leq i \leq l$. Set $B_i = y_i t_i$ whenever $\alpha_i \in \pi_\Theta$. Given an m-tuple $I = (i_1, \ldots, i_m)$, we set $B_I = B_{i_1} \ldots B_{i_m}$ and $y_I = y_{i_1} t_{i_1} \ldots y_{i_m} t_{i_m}$. Note that $B_I = y_I +$ lower degree terms. Also, y_I is a weight vector in U and $B_I = y_I +$ higher weight terms. By the discussion following [L2, Lemma 4.3], each element of B can be written as a sum of terms of the form $B_I r_I$ where I is an m tuple for some nonnegative integer m and $r_I \in R^+R^o$. Moreover, by [L2, Lemma 4.4], the highest degree term of any element b in B is contained in $G^-R^+R^o$. Let $\mathcal{L}$ be a set of tuples of different lengths such that $\{y_I | I \in \mathcal{L}\}$ is a basis for G^-. It follows that

$$B = \bigoplus_{I \in \mathcal{L}} B_I R^+ R^o.$$

3. Quantum Harish-Chandra modules

Let B satisfy the conditions of Theorem 2.1 for some fixed Θ. In this section we introduce the notion of quantum Harish-Chandra modules similar to [L1, §4]. We look at two important examples of Harish-Chandra modules: finite-dimensional U-modules and the locally finite part $F(U)$ of U (see [JL1]). Ultimately, $F(U)$ plays the same role as $\mathbf{g}$ in the classical case. In particular, for all standard analogs and some nonstandard analogs B, we show, using $F(U)$ and the coideal structure of B, that the sum of the finite-dimensional simple unitary B-modules inside a U-module is a Harish-Chandra module. Some of the results in this section generalize [L1]. One of the main tools will be a conjugate linear antiautomorphism which makes U into a Hopf $*$ algebra and B into a $*$ invariant subalgebra under a suitable embedding inside of U.

Consider the subalgebra of U generated by $F(U)$ and B. We claim that this subalgebra is equal to the set $F(U)B := \mathrm{span}\{ub | u \in U \text{ and } b \in B\}$. To see this, consider bu where $u \in F(U)$ and $b \in B$. As in (2.16), we have that $bu = \sum(\mathrm{ad}\, b_{(1)})ub_{(2)}$. The claim now follows from the fact that $F(U)$ is an $(\mathrm{ad}\, U)$-module and B is a left coideal.

Note that any $F(U)B$-module is automatically a B-module using the restricted action.

Definition 3.1. A $F(U)B$-module M is called a Harish-Chandra module for the pair U, B if M can be written as a direct sum of finite-dimensional simple B-modules using the restriction of the action to B.

We could have defined a U, B Harish-Chandra module using a U-module instead of an $F(U)B$-module. However, we show that sum of the finite-dimensional simple unitary B-modules inside a U-module form a $F(U)B$-module (Theorem 3.7 below) but not necessarily a U-module. (See example 3.8.)

Let $U_{\mathcal{R}}$ be the $\mathcal{R}$ subalgebra of U generated by x_i, y_i, and T over $\mathcal{R}$. Let κ be the $\mathcal{R}$-algebra antiautomorphism of $U_{\mathcal{R}}$ defined by $\kappa(x_i) = y_i t_i$, $\kappa(y_i) = t_i^{-1} x_i$, and $\kappa(t) = t$ for all $t \in T$.

We may extend κ to U by insisting that the resulting antiautomorphism is linear as a K map. We will refer to this extension by κ as well. Note that κ is the composition of a Hopf algebra automorphism fixing the root system of $\mathbf{g}$ composed with the chevellay antiautomorphism defined in [Jo, 3.3.3].

There is an alternate extension which results in just a $\mathcal{R}$ linear map. Extend κ to an antiautomorphism $\bar{\kappa}$ of U over $\mathcal{R}$ by setting $\bar{\kappa}(aw) = \bar{a}\kappa(w)$ for $a \in K$ and $w \in U_{\mathcal{R}}$.

It is straightforward to check that $(\kappa \otimes \kappa) \circ \Delta = \Delta \circ \kappa$. The same identity also holds for $\bar{\kappa}$. Since $\bar{\kappa}$ also commutes with the counit, this conjugate linear antiautomorphism describes a Hopf $*$ algebra structure on U (see [CP, 4.1F]).

Note that κ (resp. $\bar{\kappa}$) is a K (resp. $\mathcal{R}$) Hopf algebra antiautomorphism. It follows that both $\kappa(B)$ and $\bar{\kappa}(B)$ are left coideals satisfying conditions (2.12), (2.10), and (2.11).

Lemma 3.2. *Assume that B is a standard analog of $U(\mathbf{g}^\theta)$. There exists a Hopf algebra automorphism Υ in $\mathcal{H}$ such that $\kappa(\Upsilon(B)) = \bar{\kappa}(\Upsilon(B)) = \Upsilon(B)$ and $\Upsilon(B_i) \in U_{\mathcal{R}}$ for all $\alpha_i \notin \pi_\Theta$.*

Proof. Without loss of generality, we may assume that $B = B_{\tilde{\theta}}$. Now the construction in [L2, Theorem 3.1] finds one particular involution corresponding to Θ. Other involutions of U corresponding to Θ can be obtained using an element of $\mathcal{H}$. Checking the maps used in [L2, Theorem 3.1] to construct this involution of U, we see that this particular involution restricts to a $\mathbf{R}$ algebra involution of $U_{\mathcal{R}}$. In particular, replacing $\tilde{\theta}$ by $\chi\tilde{\theta}\chi^{-1}$ for some $\chi \in \mathcal{H}$ if necessary, we may assume that $B_i \in U_{\mathcal{R}}$ for all $\alpha_i \notin \pi_\Theta$.

Consider the elements $B_i = y_it_i+\tilde\theta(y_i)t_i$ where $\alpha_i = -\Theta(\alpha_i)$. It follows that $\tilde\theta(y_i)$ is a nonzero scalar multiple, say $a_i \in A-(q-1)A$, of $x_it_i^{-1}$. Thus $a_i^{-1/2}y_it_i+a_i^{1/2}x_i$ is contained in B. Let Υ_1 be the element in $\mathcal{H}$ which sends x_i to $a_i^{-1/2}x_i$ and y_i to $a_i^{1/2}y_i$ for each i such that $\alpha_i = -\Theta(\alpha_i)$. It follows that $\bar\kappa(\Upsilon_1(B_i)) = \kappa(\Upsilon_1(B_i)) = \Upsilon_1(B_i) = y_it_i + x_i \in U_{\mathcal{R}}$ whenever $\alpha_i = -\Theta(\alpha_i)$.

Note that $\bar\kappa(\Upsilon_1(B))$ is an algebra satisfying (2.10),(2.11), and (2.12). Hence there exists an l-tuple $\mathbf{s} \in \mathbf{S}$ such that $\kappa(\Upsilon_1(B))$ is isomorphic to $B_{\mathbf{s},\tilde\theta}$ via a Hopf algebra automorphism in $\mathcal{H}$. By the previous paragraph, $B_i^{\mathbf{s},\tilde\theta} = y_it_i + b_ix_i$ for some b_i in A whenever $\Theta(\alpha_i) = -\alpha_i$. Hence $\mathbf{s} = 0$ and there is a Hopf algebra automorphism $\Upsilon_2 \in \mathcal{H}$ such that $\bar\kappa(\Upsilon_1(B)) = \Upsilon_2(B)$.

Since $B_i \in U_{\mathcal{R}}$ for each i, we can find scalars a_i in $A\cap\mathcal{R} - (q-1)A$ such that $\Upsilon_2(x_i) = a_ix_i$ and $\Upsilon_2(y_i) = a_i^{-1}y_i$ for $1 \le i \le l$. Since $\mathcal{R}$ is the intersection of K with a real closed field, there is a well defined notion of absolute value of an element and every positive element has a square root in $\mathcal{R}$. Also, by construction, every square root of an element in $A-(q-1)A$ is also in $A-(q-1)A$. Let Υ_3 be the element in $\mathcal{H}$ such that $\Upsilon_3(x_i) = |a_i|^{1/2}x_i$ for $1 \le i \le l$. Let Υ_4 be the element in $\mathcal{H}$ such that $\Upsilon_3^2\Upsilon_4 = \Upsilon_2$. By the choice of Υ_3 and Υ_4, we must have that $\Upsilon_4(x_i) = \epsilon_ix_i$ for $1 \le i \le l$ where each $\epsilon_i \in \{1,-1\}$. Furthermore,

$$\bar\kappa(\Upsilon_3(B)) = \Upsilon_4(\Upsilon_3(B)). \tag{3.1}$$

Note that by definition of Υ_3, we have that $\Upsilon_3(B_i) \in U_{\mathcal{R}}$ for each $\alpha_i \notin \pi_\Theta$.

Without loss of generality, we may assume that $a_i = \epsilon_i = 1$ for all $\alpha_i \in \pi_\Theta$. Set $B' = \Upsilon_3(B)$. The proof of the lemma is finished if we can show that $\Upsilon_4(B') = B'$. This will be accomplished by showing that the generators of B' are contained in $\Upsilon_4(B')$. Note that the algebra R is contained in $\Upsilon_4(B')$.

Consider i where $\alpha_i \notin \pi_\Theta$. There exists $b_i \in A\cap\mathcal{R}$ such that $y_it_i+b_i\tilde\theta(y_i)t_i \in B'$. Set $j = p(i)$. If $\epsilon_j = \epsilon_i$, then $\Upsilon_4(y_it_i + b_i\tilde\theta(y_i)t_i) = \epsilon_i(y_it_i + b_i\tilde\theta(y_i)t_i)$ and hence $\Upsilon_4(B')$ contains the generator $y_it_i + b_i\tilde\theta(y_i)t_i$.

Now assume that $\epsilon_j \neq \epsilon_i$. We can find $y \in U^-U^o$ of weight $\Theta(\alpha_i)$ such that $x_i + y$ is also in B'. It follows that $\Upsilon_4(B')$ contains $x_i - y = \pm\epsilon_i(x_i - y)$ which in turn must equal $\bar\kappa(y_it_i + b_i\tilde\theta(y_i)t_i)$ by (3.1). By Theorem 2.1, B' specializes to $U(\mathbf{g}^{\theta'})$ for some involution θ' corresponding to Θ. Upon specialization, we see that both $e_i + f$ and $f_i + e$ are elements of $\mathbf{g}^{\theta'}$. Here e_i (resp. f_i) is the root vector in $\mathbf{g}$ corresponding to α_i (resp. $-\alpha_i$). Similarly, f (resp. e) is a root vector of weight $\Theta(\alpha_i)$ (resp. $\Theta(-\alpha_i)$). Let $\mathbf{c}$ denote the classical chevellay antiautomorphism which is just the specialization of $\bar\kappa$. Since $\bar\kappa(y_it_i + b_i\tilde\theta(y_i)t_i) = x_i - y$, it follows that $\mathbf{c}(f_i + e) = e_i - f$. Let $\mathbf{K}$ denote the Killing form. By say [D, 1.13.1], $\mathbf{K}(f_i + e, e_i - f) = 0$ since $f_i + e \in \mathbf{g}^{\theta'}$ while $\theta'(e_i - f) = -(e_i - f)$. However, by [Jo2, Lemma 2.5], $\mathbf{K}(f_i + e, e_i - f) \neq 0$ because $e_i - f = \mathbf{c}(f_i + e)$. This contradiction forces $\epsilon_j = \epsilon_i$. In particular, $\Upsilon_4(B')$ contains the generators of B'. Thus $\Upsilon_4(\Upsilon_3(B)) = \Upsilon_3(B)$. Setting $\Upsilon_3 = \Upsilon$ proves the lemma. □

Now consider a nonstandard analog B of $U(\mathbf{g}^\theta)$. Modifying the arguments of Lemma 3.2 slightly, one can show that there is a Hopf algebra automorphism Υ' such that $\kappa(\Upsilon'(B)) = \Upsilon'(B)$. However, Lemma 3.2 in full, does not quite work for an arbitrary $B_{\mathbf{s},\tilde\theta}$. Consider $\Upsilon(B_{\mathbf{s},\tilde\theta})$ where Υ is as in the proof of Lemma 3.2. Note that $\bar\kappa(\Upsilon(B_{\mathbf{s},\tilde\theta}))$ contains R and the generators $\Upsilon(B_i^{\mathbf{s},\tilde\theta})$ where $\Theta(\alpha_i) \neq -\alpha_i$.

Moreover, when $\Theta(\alpha_i) = -\alpha_i$, we have that $\Upsilon(B_i^{\mathbf{s},\tilde{\theta}})$ is of the form $y_i t_i + x_i + a_i t_i$ for some $a_i \in A$. We say that $B_{\mathbf{s},\tilde{\theta}}$ is a *real nonstandard analog* of $U(\mathbf{g}^\theta)$ if a_i is in $\mathcal{R} \cap A$ for all choices of i such that $\alpha_i = -\Theta(\alpha_i)$. More generally, call a nonstandard analog B of $U(\mathbf{g}^\theta)$ a *real nonstandard analog* if there is a Hopf algebra automorphism $\psi \in \mathcal{H}$ such that $\psi(B) = B_{\mathbf{s},\tilde{\theta}}$ and $B_{\mathbf{s},\tilde{\theta}}$ is a real nonstandard analog. Note that the assumptions on a_i imply that Lemma 3.2 is true for all real nonstandard analogs.

Now let Υ be as in Lemma 3.2. Note that $\Upsilon\tilde{\theta}\Upsilon^{-1}$ also satisfies the conditions (2.5), (2.6), (2.7), and (2.8). Thus we may replace $\tilde{\theta}$ by $\Upsilon\tilde{\theta}\Upsilon^{-1}$. For the remainder of the paper (unless stated otherwise), we will assume that B is a standard analog or real nonstandard analog such that

(3.2) $B = B_{\tilde{\theta}}$ or $B = B_{\mathbf{s},\tilde{\theta}}$ for some $\mathbf{s} \in \mathbf{S}$;

(3.3) $\bar{\kappa}(B) = B = \kappa(B)$;

(3.4) the generators of B, (i.e. the B_i for $\alpha_i \notin \pi_\Theta$ and the generators of R) all sit inside of $U_{\mathcal{R}}$.

Condition (3.3) ensures that B is invariant under the $*$ structure of the Hopf $*$ algebra U. (Of course, equivalently, we could have replaced $\bar{\kappa}$ by $\Upsilon^{-1}\bar{\kappa}\Upsilon$ instead of changing B.)

Let Λ be an element of $K[T]^*$ such that the simple highest weight U-module $L(\Lambda)$ generated by a highest weight vector of weight Λ is finite-dimensional.

Let U_+^+ (resp. U_+^-) denote the augmentation ideal of U^+ (resp. U^-). Define the Harish-Chandra map φ as the projection of U onto U^o using the direct sum decomposition $U = (UU_+^+ + U_+^- U) \oplus U^o$ of U. Set $S = S_\Lambda$ equal to the conjugate linear form on $L(\Lambda)$ which satisfies

$$S(v_\Lambda, v_\Lambda) = 1 \tag{3.5}$$

and

$$S(fv_\Lambda, gv_\Lambda) = \Lambda(\varphi(\bar{\kappa}(f)g)) \tag{3.6}$$

for all f and g in U.

Note that (3.6) implies that $S(fv, v) = S(v, \bar{\kappa}(f)v)$ for all $v \in L(\Lambda)$ and $f \in B$. Let v_Λ denote the highest weight vector of $L(\Lambda)$. By [L1, Lemma 4.2], we have that

$$S(w, w) \neq 0 \text{ for any nonzero } w \in L(\Lambda).$$

Theorem 3.3. *Each finite-dimensional U-module is completely reducible as a B-module.*

Proof. The conjugate linear form S can now be used as in [L1, Lemma 4.3] to break down a finite-dimensional U-module into a direct sum of its simple B-submodules. □

Theorem 3.3 implies that any finite-dimensional U-module is a Harish-Chandra module for the pair U, B. It follows that $F(U)$ is a Harish-Chandra module for the pair U, B using the adjoint action. As in [L1, Section 4], we can show that $F(U)$ is the maximal Harish-Chandra module inside of U with respect to the adjoint action. The following key lemma is an adaptation of [L1, Lemma 4.4].

Lemma 3.4. *Let M be a U-module which admits a semisimple T-action. If W is a finite-dimensional simple B-submodule of M, then W generates a locally finite semisimple U-submodule of M.*

Proof. Note that we may restrict the action of B on W to R and thus W is a finite-dimensional R-module. Since R is generated by the quantized enveloping algebra $U_q([\mathbf{m},\mathbf{m}])$, and the group T_Θ, it follows that x_i and y_i act locally nilpotent on W for $\alpha_i \in \pi_\Theta$. For $\alpha_i \notin \pi_\Theta$, we may apply the argument in the proof of [L1, Lemma 4.4] to show that $x_i^m W = y_i^m W = 0$ for some large m. Using the arguments in [L1, Lemma 4.4] one can show that for all $1 \leq i \leq l$, x_i and y_i act locally nilpotent on the U-submodule generated by W and thus the lemma follows by [JL1, Theorem 5.9]. □

Note that $F(U)$ is a semisimple $(\operatorname{ad} T)$-module and, furthermore, $F(U)$ can be written as the direct sum of all the finite-dimensional simple U-modules inside U. Thus Theorem 3.3 and Lemma 3.4 imply the following result as in [L1, Corollary 4.6].

Theorem 3.5. *The* $(\operatorname{ad} U)$*-module* $F(U)$ *is the maximal Harish-Chandra module for the pair* U, B *inside of* U*.*

Consider again a finite-dimensional U-module $L(\Lambda)$ with conjugate linear form S and highest weight generating vector v_Λ as above. Define a real lattice $L_{\mathcal{R}}$ by $L_{\mathcal{R}} = U_{\mathcal{R}} v_\Lambda$. The vector space $L_{\mathcal{R}}$ is a $\mathcal{R}$ subspace of $L(\Lambda)$ such that $L(\Lambda) = L_{\mathcal{R}} \oplus iL_{\mathcal{R}}$, $U_{\mathcal{R}} L_{\mathcal{R}} \subset L_{\mathcal{R}}$, and $S_\Lambda(v,v) > 0$ for all $0 \neq v \in L_{\mathcal{R}}$.

Set $B_{\mathcal{R}}$ equal to $B \cap U_{\mathcal{R}}$. By (3.3) and (3.4), all the generators of B are contained in $B_{\mathcal{R}}$ and $\bar{\kappa}(B_{\mathcal{R}}) = B_{\mathcal{R}}$. Note that a simple $B_{\mathcal{R}}$-module tensored with K is automatically a simple B-module. Let V be a simple $B_{\mathcal{R}}$-module inside of $L_{\mathcal{R}}$. Set W equal to the perpendicular of V inside of $L_{\mathcal{R}}$ with respect to S. The perpendicular of V in $L(\Lambda)$ with respect to S is just $V \otimes K$. In particular, the decomposition of $L(\Lambda)$ as a B-module corresponds to the decomposition of $L_{\mathcal{R}}$ as a $B_{\mathcal{R}}$-module. Thus a simple B-submodule W which is contained in $L(\Lambda)$ has the following properties:

(3.7) W has a $\mathcal{R}$ vector subspace $W_{\mathcal{R}}$ such that $W_{\mathcal{R}} \otimes K = W$ and $W_{\mathcal{R}}$ is a simple $B_{\mathcal{R}}$-module of the same dimension as W.

(3.8) W has a conjugate linear form S_W such that $S_W(v, bw) = S_W(\bar{\kappa}(b)v, w)$ for all $b \in B$ and $S_W(v,v)$ is a positive element of $\mathcal{R}$ for each nonzero $v \in W_{\mathcal{R}}$.

Of course, condition (3.8) implies that $S_W(v,v) \neq 0$ for all $0 \neq v \in W$. In general, we call a simple B-module W a *unitary* B-module if W satisfies conditions (3.7) and (3.8). Given a finite-dimensional simple U-module $L(\Lambda)$, we can form the finite-dimensional B-module $L(\Lambda) \otimes W$.

Lemma 3.6. *If* W *is a finite-dimensional simple unitary* B*-module, then* $L(\Lambda) \otimes W$ *is also unitary.*

Proof. Note that the vector space $L_{\mathcal{R}} \otimes W_{\mathcal{R}}$ satisfies (3.7). Set $S(v \otimes w, v' \otimes w') = S_\Lambda(v,v') S_W(w,w')$. The fact that S is conjugate linear and positive definite on $L_{\mathcal{R}} \otimes W_{\mathcal{R}}$ follows from the properties of S_Λ and S_W. Now $S(a(v_1 \otimes v_2), (w_1 \otimes w_2)) = S(a_{(1)} v_1 \otimes a_{(2)} v_2, w_1 \otimes w_2)$ Since B is a left coideal, we have that $a_{(2)} \in B$. Thus $S(a_{(1)} v_1 \otimes a_{(2)} v_2, w_1 \otimes w_2) = S(v_1 \otimes v_2, \bar{\kappa}(a_{(1)}) w_1 \otimes \bar{\kappa}(a_{(2)}) w_2)$ again by properties of S_Λ and S_W. Since $\bar{\kappa}$ commutes with the coproduct, this equals $S(v_1 \otimes v_2, \bar{\kappa}(a)(w_1 \otimes w_2))$. □

Let W be a finite-dimensional U-module and let V be a finite-dimensional simple unitary B-module. Just as in the proof of Theorem 3.3, Lemma 3.6 implies that

$W \otimes V$ can be written as a direct sum of finite-dimensional simple unitary B-modules. Thus the finite-dimensional unitary modules make good building blocks for Harish-Chandra modules (see Theorem 3.7 below). Now assume that V is a B-submodule of a finite-dimensional U-module W'. Then the B-module $W \otimes V$ is contained in the U-module $W \otimes W'$. Thus the only simple unitary B-modules we have seen so far are submodules of finite-dimensional U-modules. We will produce more examples of unitary modules in Section 5.

Theorem 3.7. *Let M be a U-module and let W be the sum inside of M of all the finite-dimensional simple unitary B-submodules. Then W is a Harish-Chandra module for the pair U, B.*

Proof. By the definition of Harish-Chandra modules, it is sufficient to show that W is a $F(U)$-submodule of M. Make $F(U) \otimes W$ into a B-module as follows:

$$c \cdot (F(U) \otimes W) = (\operatorname{ad} c_{(1)})F(U) \otimes c_{(2)}W$$

for all $c \in B$. Note that $c_{(2)} \in B$ for $c \in B$ and hence $c_{(2)}W \subset W$. Also, since $F(U)$ is an $(\operatorname{ad} U)$-module, it follows that $(\operatorname{ad} c_{(1)})F(U) \subset F(U)$. Furthermore, since $F(U)$ is a direct sum of finite-dimensional $(\operatorname{ad} U)$-modules and W is unitary, the discussion preceding the theorem implies that $F(U) \otimes W$ is a direct sum of finite-dimensional simple unitary B-modules.

Consider the map $F(U) \otimes W$ to the $F(U)$-submodule $F(U)W$ of M defined by the multiplication map. We have

$$\begin{aligned}(\operatorname{ad} c_{(1)})F(U)c_{(2)}W &= c_{(1)}F(U)\sigma(c_{(2)})c_{(3)}W\\ &= c_{(1)}F(U)\epsilon(c_{(2)})W\\ &= cF(U)W.\end{aligned}$$

Hence the map from $F(U) \otimes W$ to $F(U)W$ is a B-module map. Note that this map restricts to a map of B-modules $F(U) \otimes W$ onto $F(U)W$. Thus $F(U)W$ is a locally finite semisimple B-submodule of M. By definition of W, $F(U)W = W$. □

In the example below we return to the issue of why quantum Harish-Chandra modules are defined using an $F(U)B$ action instead of the action of U.

Example 3.8. Keep the notation of Theorem 3.7. In special cases, W is actually a U-module such as when M is a finite-dimensional U-module or when $M = F(U)$. Unfortunately we cannot expect $W = F(U)W$ to be a U-module in general. For example, consider the case where $\mathbf{g} = \mathbf{sl}\,2$ and B is just the polynomial ring in one variable, $yt + x$. Let v be the generator of a one-dimensional B-module such that $(yt + x)v = \lambda v$ where $\lambda \in \mathcal{R}$. Note that Kv is a unitary B-module with conjugate linear form S such that $S(v, v) = 1$. Hence $F(U) \otimes Kv$ is a Harish-Chandra module for the pair U, B by Theorem 3.7. Now $F(U) \otimes Kv$ is a proper submodule of the U-module $U \otimes Kv$. Consider $u \in U - F(U)$. Then either $(\operatorname{ad} x)^m u \neq 0$ for all m or $(\operatorname{ad} y)^m u \neq 0$ for all m. Thus by weight space considerations, $(yt + x)^m(u \otimes v) = (\operatorname{ad} yt + x)^m u \otimes v + (\operatorname{ad} t)u \otimes \lambda v$ is never zero. It follows that B does not act locally finite on $u \otimes v$ for any $u \in U - F(U)$. Hence $F(U) \otimes v$ is the sum of all the finite-dimensional B-modules inside of $U \otimes v$. In particular, the appropriate choice for a Harish-Chandra module contained in $U \otimes v$ is a $F(U)B$-module but not a U-module.

4. Spherical Modules

We continue our assumptions on B, namely that B is either a standard or a real nonstandard analog of $U(\mathbf{g}^\theta)$ satisfying (3.2), (3.3), and (3.4). In this section we consider spherical modules associated to the subalgebra B. We show that U, B is a Gelfand pair (see the definition below). We further prove that the finite-dimensional spherical modules can be classified in exactly the same way as in the classical case. This generalizes work of Noumi and Sugitani who explicitly find the spherical vectors for specific families of quantum analogs (see [N, Theorem 3.1] and [N, Theorem 1]). (See also [L2, §6] which shows that the examples in [NS] fit into the framework of Theorem 2.1.) Our proof is general and translates to a new proof even in the classical case using just enveloping algebras and no Lie groups.

Definition 4.1. A U-module V is called spherical if the set $V^B = \{v \in V | bv = \epsilon(b)v \text{ for all } b \in B\}$ has dimension 1. If $\dim V^B \leq 1$ for all finite-dimensional simple U-modules V then U, B is called a Gelfand pair.

Let $\mathbf{a}$ be the classical counterpart to $\mathcal{A}$ appearing in the classical Iwasawa decomposition. Let $\Delta(\mathbf{a}, \mathbf{g})$ denote the corresponding set of restricted roots with positive roots $\Delta^+(\mathbf{a}, \mathbf{g})$. Write $Q(\mathbf{a}, \mathbf{g})$ (resp. $Q^+(\mathbf{a}, \mathbf{g})$) for the integral (resp. nonnegative integral) linear combinations of elements of $\Delta(\mathbf{a}, \mathbf{g})$. There is a standard partial order on $Q(\mathbf{a}, \mathbf{g})$ defined by $\lambda > \lambda'$ if $\lambda - \lambda'$ is in $Q^+(\mathbf{a}, \mathbf{g})$. Note that $Q(\mathbf{a}, \mathbf{g})$ can be identified with the subset of $Q(\pi)$ consisting of those weights orthogonal to the set $\{\alpha_i + \Theta(\alpha_i) | 1 \leq i \leq l\}$ under the Cartan inner product. In particular, $\Delta(\mathbf{a}, \mathbf{g})$ can be taken to be the nonzero elements of the set $\{\alpha_i - \Theta(\alpha_i) | 1 \leq i \leq l\}$. (For more information about restricted roots, see for example [Kn].)

Let $\mathcal{Z}$ denote the subgroup of the nonzero complex numbers consisting of all fourth roots of unity. One can associate a group homomorphism of $\langle t_1^{1/2}, \ldots, t_l^{1/2} \rangle$ into $\mathcal{Z}$ to each l-tuple $\zeta = (\zeta_1, \ldots, \zeta_l) \in \mathcal{Z}^l$ by setting $\zeta(t_i^{1/2}) = \zeta_i$. Given $\lambda \in Q(\mathbf{a}, \mathbf{g})$ and $\zeta \in \mathcal{Z}^l$, define elements $\zeta q^\lambda \in K[\mathcal{A}]^*$ by $\zeta q^\lambda (t_i^{1/2} \tau(\Theta(\alpha_i))^{-1/2}) = \zeta(t_i^{1/2} \tau(\Theta(\alpha_i))^{-1/2}) q^{(\lambda, (\alpha_i - \Theta(\alpha_i))/2)}$ for each $1 \leq i \leq l$.

Now let V be a finite-dimensional simple U-module. By say [R2], there exists a fixed $\zeta \in \mathcal{Z}^l$ such that all weights of V are of the form ζq^β where $\beta \in Q(\pi)$. (Here ζ is restricted to its action on T; elements of T are sent to elements of the group $\{1, -1\}$.) We can also break V up into weight spaces for the action of $\mathcal{A}$. In particular, given $\Gamma \in K[\mathcal{A}]^*$, the Γ weight space of V consists of those vectors v such that $av = \Gamma(a)v$ for all $a \in \mathcal{A}$. It is straightforward to see that the possible weights of V considered as an $\mathcal{A}$-module take the form ζq^λ where $\lambda \in Q(\mathbf{a}, \mathbf{g})$. We write $\zeta q^\lambda < \zeta q^{\lambda'}$ if $\lambda < \lambda'$. Since V is finite-dimensional, it makes sense to talk about a maximal $\mathcal{A}$-weight of V, say ζq^γ. Let V^γ be the corresponding weight space.

Theorem 4.2. *If V is a finite-dimensional simple U-module, then $\dim V^B \leq 1$. In particular, U, B is a Gelfand pair.*

Proof. By (3.3) $\kappa(B) = B$. Set B^+ equal to the intersection of the augmentation ideal of U with B.

Let V be a finite-dimensional simple U-module. Write U' for the extension of U by the elements in $\langle t_1^{1/2}, \ldots, t_l^{1/2} \rangle$. It is straightforward to extend the action of U on V to an action of U' on V so that the highest weight of V takes the form

ζq^λ where $\zeta \in Z^l$ and $\lambda \in \mathbf{h}^*$ is dominant integral. Note that U_Θ (see Section 2) is contained in U' and thus V is also a U_Θ-module.

The proof here uses the quantum Iwasawa decomposition (2.17) and follows the argument of Kostant in [K1, §2]. Let N_Θ^{++} denote the intersection of N_Θ^+ with the augmentation ideal of U. By Proposition 2.2 and its proof, we have that $RN_\Theta^{++} = N_\Theta^{++}R$. Since N_Θ^+ is generated by weight vectors, we also have $\mathcal{A}N_\Theta^{++} = N_\Theta^{++}\mathcal{A}$.

Let γ be chosen so that ζq^γ is the maximal $\mathcal{A}$ weight of V where $\zeta \in Z^l$ and $\gamma \in Q(\mathbf{a}, \mathbf{g})$. Set $V^n = \{v \in V | N_\Theta^{++}v = 0\}$. Using the argument in [K1, Proposition 1.2.3], it follows that $V^\gamma = V^n$ and that V^n is irreducible as a $\mathcal{A}R$-module.

Let V' denote the dual of V which becomes a U-module using the antiautomorphism κ. Assume that V^B is nonzero. By definition, $\langle v, x \cdot w\rangle = \langle \kappa(x)v, w\rangle$ for all $v \in V$, $w \in V'$, and $x \in B$. Hence if $v \in V^B$ and $\langle v, w\rangle \neq 0$, then $w \notin B^+V'$. Now V'/B^+V' is isomorphic to a direct sum of one-dimensional trivial B-modules. By Theorem 3.3, there exists a trivial B-submodule W of V' such that $W \cong V'/B^+V'$. Of course W is just equal to V'^B. Thus $V^B \neq 0$ implies that V'^B is nonzero. Similarly, if $V'^B \neq 0$, then $V^B \neq 0$.

We continue the assumption that $V^B \neq 0$. Using the quantum Iwasawa decomposition (2.17) as in [K1, Lemma 1.2.4], we can now show that V^n, V^B, V'^B, and V'^n are each one-dimensional. In particular, choose $0 \neq w \in V'^B$ and $0 \neq v \in V^n$ and assume that $0 = \langle v, w\rangle$. Then $0 = \langle v, Bw\rangle = \langle Bv, w\rangle = \langle BT_\Theta^{1/2}\mathcal{A}N_\Theta^+ v, w\rangle = \langle U_\Theta v, w\rangle = \langle V, w\rangle$, a contradiction. It follows that V'^B and V^n are dual to each other under the bilinear form $\langle\ ,\ \rangle$. Now V^n is an irreducible R-module and V^B is a direct sum of trivial R-modules. Since R is the algebra generated by the quantized enveloping algebra $U_q([\mathbf{m}, \mathbf{m}])$ and T_Θ, finite-dimensional simple R-modules are self dual. In particular, V^n must be a trivial one-dimensional R-module and hence V'^B is one-dimensional. A similar argument works for V^B and V'^n. □

In the next theorem, we classify the finite-dimensional simple spherical modules for analogs B of $U(\mathbf{g}^\theta)$. The classification parallels the classical case; see for example [Kn, Theorem 8.49].

Theorem 4.3. *Assume that B is a standard or real nonstandard analog of $U(\mathbf{g}^\theta)$ satisfying (3.2), (3.3), and (3.4). A finite-dimensional simple U-module V is spherical (with respect to B) if and only if the highest T weight ζq^λ of V vanishes on T_Θ and its restriction $\tilde{\lambda}$ to $\mathcal{A}$ satisfies $(\tilde{\lambda}, \beta)/(\beta, \beta)$ is a positive integer for every positive restricted root β.*

Before proving the theorem, we replace the last condition on restricted roots with one on simple (nonrestricted) roots.

Lemma 4.4. *Let λ be an element of $\mathbf{h}^*$ such that λ is zero on γ whenever $\tau(\gamma) \in T_\Theta$. The restriction $\tilde{\lambda}$ of λ to $\mathcal{A}$ satisfies $(\tilde{\lambda}, \beta)/(\beta, \beta)$ is a nonnegative integer for every positive restricted root β if and only if*

(4.1) $2(\lambda, \alpha)/(\alpha, \alpha)$ *is a nonnegative integer for all positive simple roots α such that $\alpha \neq -\Theta(\alpha)$*

and

(4.2) $(\lambda, \alpha)/(\alpha, \alpha)$ *is a nonnegative integer for all positive simple roots α such that $\alpha = -\Theta(\alpha)$.*

Proof. Let λ be a weight and write $\tilde{\lambda}$ for its restriction to $\mathbf{a}$. By the proof of [Kn, Theorem 8.49], we have $(\tilde{\lambda},\beta)/(\beta,\beta)$ is a nonnegative integer for every positive restricted root β if and only if

(4.3) $2(\lambda,\alpha)/(\alpha,\alpha)$ is a nonnegative integer for all positive roots α such that $\alpha \neq -\Theta(\alpha)$

and

(4.4) $(\lambda,\alpha)/(\alpha,\alpha)$ is a nonnegative integer for all positive roots α such that $\alpha = -\Theta(\alpha)$.

So it is enough to show that (4.3) and (4.4) are equivalent to (4.1) and (4.2).

Clearly (4.3) (resp. (4.4)) implies (4.1) (resp. (4.2)). To see the other direction, we modify the proof of [Kn, Proposition 4.62]. Indeed if $\Theta(\alpha) \neq -\alpha$, then the induction argument of [Kn, Proposition 4.62] works here as well. On the other hand, if $\Theta(\alpha) = -\alpha$, then one adapts the argument by replacing a reflection corresponding to a simple root by the reflection corresponding to some $\alpha_i - \Theta(\alpha_i)$. □

Proof of Theorem 4.3. We assume first that B is a standard analog. Let V be a finite-dimensional simple U-module with highest weight ζq^{λ} and highest weight generating vector v_λ. Set $m_i = 2(\lambda,\alpha_i)/(\alpha_i,\alpha_i)$. Since V is finite-dimensional, the m_i, $1 \leq i \leq l$ are all nonnegative integers. Hence it is sufficient to show that V is spherical if and only if λ acts trivially on T_Θ and m_i is even whenever $\alpha_i = -\Theta(\alpha_i)$.

We argue first that dim V/B^+v_λ is less than or equal to 1. To see this, we can find a basis of V consisting of elements of the form $y_\eta v_\lambda$ where y_η is in G^- of weight η. Without loss of generality, we may assume that y_η is a monomial, say y_I. Since B is a standard analog, the elements B_i, $1 \leq i \leq l$ are all contained in B^+. Thus, $B_I v_\lambda = y_I v_\lambda$ + higher weight terms and is contained in B^+v_λ. Induction on weight implies the dimension claim. Moreover, $B^+v_\lambda = V$ if and only if $v_\lambda \in B^+v_\lambda$.

If dim $V/B^+v_\lambda = 1$, then, by Theorem 3.3, V contains a one-dimensional trivial B-module. On the other hand, if B^+ acts trivially on the nonzero vector w, then, again by Theorem 3.3, we cannot have $w \in B^+V$. Thus V is spherical if and only if

$$v_\lambda \notin B^+v_\lambda. \tag{4.5}$$

To complete the proof of the theorem we argue that (4.5) happens if and only if λ has the nice form of Theorem 4.3 as interpreted by Lemma 4.4.

Note that if $v_\lambda \notin B^+v_\lambda$, then $v_\lambda \notin (R)^+v_\lambda$ where $(R)^+$ denotes the augmentation ideal of R. Recall that R is the algebra generated by the quantized enveloping algebra $U_q([\mathbf{m},\mathbf{m}])$ and T_Θ. Hence $v_\lambda \notin (R)^+v_\lambda$ if and only if $(R)^+v_\lambda = 0$. Moreover, $(R)^+v_\lambda = 0$ if and only if λ is trivial when restricted to T_Θ if and only if v_λ generates a one-dimensional trivial R-module. Furthermore, if Kv_λ is a one-dimensional R-module, then $m_i = 0$ for all $\alpha_i \in \pi_\Theta$.

Now a typical element of B is a sum of elements in $B_I R^+ R^o$ (see Section 2). Hence we only need to determine when v_λ cannot be written as a linear combination over A using terms of the form $B_I v_\lambda$.

It is well known that the annihilator of v_λ in U is

$$U\ \mathrm{Ann}_{U^o} v_\lambda + \sum_{1\leq i\leq l} Ux_i + \sum_{1\leq i\leq l} U(y_it_i)^{m_i+1}. \tag{4.6}$$

We have that $B_I v_\lambda = y_I v_\lambda$ + higher weight terms. Suppose that $B_I \in U(B_i)^{m_i+1}$ for some i. If $\alpha_i \in \pi_\Theta$, then $m_i = 0$, $B_i = y_it_i$ and $B_I v_\lambda = 0$.

Assume that $\alpha_i \notin \pi_\Theta$ and assume further that $\Theta(\alpha_i) \neq -\alpha_i$. It follows that $t\Theta(\alpha_i) + s\alpha_i$ is never contained in the set of positive weights where t and s are positive integers. In particular, if we rewrite $(B_i)^m$ as a sum of elements in the form ace where $a \in G^-$, $c \in U^o$, and $e \in U^+$, then $(B_i)^m = (y_i t_i)^m + UU^{++}$ where U^{++} is the augmentation ideal of U^+. Hence $B_i^m v_\lambda = (y_i t_i)^m v_\lambda$ for all m. Thus $(B_i)^{m_i+1} v_\lambda = 0$.

Now assume that $\alpha_i = -\Theta(\alpha_i)$. In this case, $B_i = y_i t_i + x_i$. We drop the subscript i from y, t, and x in the argument that follows. Note that $(yt + x)^n v_\lambda = (yt+x)^{n-1} ytv_\lambda$ for any positive integer n. Now $(yt+x)^{n-1} ytv_\lambda$ is a sum of terms of the form $(yt)^{n-2j} v_\lambda$ where $n - 2j \geq 0$ and j is a nonnegative integer. Furthermore, the coefficient of $(yt)^n v_\lambda$ in this sum is one. Choosing $n = 1, 2, 3, \ldots$, we see that $B^+ v_\lambda$ contains $(yt)^n v_\lambda$ for all odd values of n.

Assume first that m_i is even. It follows that

$$(yt + x)^{m_i+1} v_\lambda \in \sum_{j<m_i, j \text{ odd}} K(yt + x)^j v_\lambda \subset B^+ v_\lambda.$$

Hence

$$(yt + x)^n v_\lambda \in \sum_{1 \leq j \leq m_i} K(yt + x)^j v_\lambda$$

for all $n \geq 1$. Thus $v_\lambda \notin (B^+ \cap K[yt + x]) v_\lambda$ in this case.

Now assume that m_i is odd. By the above, $(yt)^n v_\lambda \in B^+ v_\lambda$ for all odd n. Hence $(yt + x)(yt)^n v_\lambda \in B^+ v_\lambda$ whenever n is odd. Note that if $n < m_i + 1$, then $x(yt)^n v_\lambda$ is a nonzero multiple, say a_n, of $(yt)^{n-1} v_\lambda$. Thus $B^+ v_\lambda$ contains $(yt)^{n+1} v_\lambda + a_n (yt)^{n-1} v_\lambda$. Consider $(yt+x)(yt)^{m_i} v_\lambda$. Since $(yt)^{m_i+1} v_\lambda = 0$, we must have that $(yt)^{m_i-1} v_\lambda \in B^+ v_\lambda$. It now follows by induction that $(yt)^{m_i-1-2j} v_\lambda \in B^+ v_\lambda$ for $(m_i - 1)/2 \geq j \geq 0$. Hence $v_\lambda \in B^+ v_\lambda$ in this case.

We have shown that when $\alpha_i = -\Theta(\alpha_i)$, $v_\lambda \in (K[yt + x] \cap B^+) v_\lambda$ if and only if m_i is odd. Also, by the previous two paragraphs, there exists a polynomial p_i of degree $m_i + 1$ such that $p_i(yt + x) v_\lambda = 0$. Moreover, this polynomial p_i has a nonzero constant coefficient if and only if m_i is odd. Set a_i equal to the constant coefficient of p_i.

Recall (see the end of Section 2) that

$$B = \bigoplus_{J \in \mathcal{L}} B_J R^o R^+ \tag{4.7}$$

where $\{y_J | J \in \mathcal{L}\}$ is a basis for G^-. Let $\mathcal{I}$ be a subset $\mathcal{L}$ such that $\{v_\lambda\} \cup \{y_I v_\lambda | I \in \mathcal{I}\}$ is a basis for V. Furthermore, by (4.6), a typical element y in the augmentation ideal of G^- satisfies

$$y = \sum_{I \in \mathcal{I}} a_I y_I + \sum_i G^-(y_i t_i)^{m_i+1}. \tag{4.8}$$

Since $B_I = y_I +$ lower degree terms, we also have that $\{v_\lambda\} \cup \{B_I v_\lambda | I \in \mathcal{I}\}$ is a basis for V. Moreover, given a typical element $b \in B^+$, using (4.7) and (4.8), we may find scalars $a_I \in K$ such that

$$\begin{aligned} b = &\sum_{I \in \mathcal{I}} a_I B_I + \sum_{\{j | \Theta(\alpha_j) \neq -\alpha_j\}} B(B_j)^{m_j+1} \\ &+ \sum_{\{j | \Theta(\alpha_j) = -\alpha_j\}} B^+ p_j(B_j) + \sum_{\{j | \Theta(\alpha_j) = -\alpha_j\}} K(p_j(B_j) - a_j) + B(R)^+. \end{aligned}$$

Thus $bv_\lambda \in \sum_{I\in\mathcal{I}} KB_I v_\lambda + \sum_j Ka_j v_\lambda$. In particular, V is spherical if and only if R acts trivially on v_λ and each $a_j = 0$. This in turn is equivalent to λ acts trivially on T_Θ and m_j is even whenever $\alpha_j = -\Theta(\alpha_j)$. This completes the proof in the standard case.

Now consider the case where B is a real nonstandard analog of $U(\mathbf{g}^\theta)$. The proof above applies to B as well, except for the part concerning the B_i with $\alpha_i = -\Theta(\alpha_i)$. Without loss of generality, we may assume that B contains elements $y_it_i + x_i + s_it_i$ where $s_i \in A\cap\mathcal{R}$ for all α_i such that $\alpha_i = -\Theta(\alpha_i)$. (Note that $y_it_i + x_i + s_it_i$ is not in B^+ when $s_i \neq 0$. However, $y_it_i + x_i + (s_it_i - s_i)$ is in B^+. Thus we want to use $y_it_i + x_i + s_it_i - s_i$ instead of B_i in the preceding arguments.) Let V_i be the $K[y_it_i + x_i + s_it_i - s_i]$-submodule of V generated by v_λ. Theorem 4.3 will hold for real nonstandard B, if we can show that V_i admits a spherical vector if and only if m_i is even for each i such that $s_i \neq 0$.

Dropping the subscript i from x, y, and t, we are reduced to the following question about $U_q(\mathbf{sl}\ 2)$. Let $X = yt + x + a(t-1)$ where $a \in A\cap\mathcal{R}$. Note that B^+ is the polynomial ring generated by X. We must show that an n dimensional simple $U_q(\mathbf{sl}\ 2)$-module is spherical if and only if n is odd. (Here we may think of m_i as equal to $n+1$.)

Let w be a nonzero vector which generates a one-dimensional simple $U_q(\mathbf{sl}\ 2)$-module with $tw = -w$. Note that $\Delta(X) \in at\otimes X + U^+\otimes 1$. Thus, if $L(q^{n\alpha})$ admits a spherical vector v, then $w\otimes v$ is a spherical vector of $Kw\otimes L(q^{n\alpha}) \cong L(-q^{n\alpha})$. Hence, we reduce to the case of n dimensional simple modules L_n generated by a highest weight vector of weight $q^{(n-1)\alpha/2}$ for some nonnegative integer n.

Now, if n is even and L_n contains a vector v annihilated by X, then the specialization of L_n is a spherical $(\mathbf{sl}\ 2)^\theta$-module, which is a not possible by the classical theory. On the other hand, let $n = 3$, and let v_1 be the highest weight generating vector of L_3. One checks that

$$v_1 + a(q^{-2}-1)(q+q^{-1})^{-1}ytv_1 - (q+q^{-1})^{-1}(yt)^2v_1$$

is annihilated by X and hence L_3 is a spherical module.

To show that the other L_n are spherical for n odd, we pass to the locally finite part $F = F(U_q(\mathbf{sl}\ 2))$ of $U_q(\mathbf{sl}\ 2)$. We will use the following lemma later, so it will be proved for all U and not just $U_q(\mathbf{sl}\ 2)$. Set $F(U)^B = \{a|a \in F(U)$ and $(\operatorname{ad} b)a = \epsilon(b)a$ for all $b \in B\}$. By Theorem 3.5, $F(U)^B = U^B$ where U^B is defined in a similar fashion.

Lemma 4.5. *The set $F(U)^B$ equals $\{z \in F(U)|\sigma^{-1}(z)b = b\sigma^{-1}(z)$ for all $b\in B\}$. In particular, $F(U)^B$ is a subalgebra of $F(U)$.*

Proof. Note that the second sentence follows from the first. To prove the first statement, we follow the argument in [JL1, Corollary 2.4]. Set $Z_B = \{z\in F(U)|\sigma^{-1}(z)b = b\sigma^{-1}(z)$ for all $b\in B\}$. If $z\in Z_B$, then

$$(\operatorname{ad} b)z = b_{(1)}z\sigma(b_{(2)}) = \sigma(b_{(2)}\sigma^{-1}(z)\sigma^{-1}(b_{(1)})).$$

Since $b_{(2)}\in B$, the above equals

$$\sigma(\sigma^{-1}(z)b_{(2)}\sigma^{-1}(b_{(1)})) = b_{(1)}\sigma(b_{(2)})z = \epsilon(b)z.$$

On the other hand, if $z\in F(U)^B$, then

$$\sigma^{-1}(z)b = \sigma^{-1}(z)b_{(1)}\epsilon(b_{(2)}) = \epsilon(b_{(2)})\sigma^{-1}(z)b_{(1)} = \sigma^{-1}(\epsilon(b_{(2)})z)b_{(1)}.$$

Since $b_{(2)} \in B$ and $z \in F(U)^B$, $\sigma^{-1}(z)b$ equals

$$\sigma^{-1}(b_{(2)}z\sigma(b_{(3)}))b_{(1)} = b_{(3)}\sigma^{-1}(z)\sigma^{-1}(b_{(2)})b_{(1)} = b_{(2)}\sigma^{-1}(z)\epsilon(b_{(1)}) = b\sigma^{-1}(z). \quad \square$$

Using the Separation of Variables theorem for $U_q(\mathbf{sl}\ 2)$, we have that the harmonics are a direct sum of the modules $(\operatorname{ad} U_q(\mathbf{sl}\ 2))x^n \cong L_{2n+1}$ (see [JL2] and [JL1, §3.11]). Let $v \in (\operatorname{ad} U_q(\mathbf{sl}\ 2))x$ be a nonzero spherical vector. By weight space considerations, $v = x+$ lower weight terms. Lemma 4.5 implies that v^n is in $F(U)^B$ for all $n \geq 0$. Now $v^n = x^n+$ lower weight terms. One sees by the degree arguments using the filtration defined in [JL2, §2.2] and the decomposition in [JL1, Lemma 3.11], that v^n is not in FZ^+ where Z^+ is the augmentation ideal of the center Z. Hence by degree considerations, there is a nonzero spherical vector w in L_{2n+1} such that $v \in w + FZ^+ + \sum_{1\leq i\leq m} L_{2n-1}Z$. In particular, each L_{2n+1} is a spherical module for B which completes the proof of Theorem 4.3 in the nonstandard case. $\square$

5. Semisimple part of B

In the classical case, g^θ is not necessarily semisimple though it is reductive. In order to analyze finite-dimensional simple B-modules, we want to split off the "semisimple" part of B. We do this to a certain extent in this section and thus produce a large class of one-dimensional nontrivial B-modules

Write $\mathbf{g}^\theta$ as a direct sum of the semisimple Lie algebra $[\mathbf{g}^\theta, \mathbf{g}^\theta]$ and the center $Z(\mathbf{g}^\theta)$. Let $\{z_1, \dots, z_r\}$ be a basis for $Z(\mathbf{g}^\theta)$. (Of course, if $\mathbf{g}$ is simple, the classical version of Theorem 4.2 implies that $r \leq 1$.) Note that this decomposition of $\mathbf{g}^\theta$ implies that no z_i can been written as a sum of commutators of elements in $\mathbf{g}^\theta$.

Let $e_i, f_i, h_i, 1 \leq i \leq l$ be a set of generators for $\mathbf{g}$. Here e_i is the root vector corresponding to α_i, and x_i specializes to e_i. Similarly, f_i is the image of y_i and h_i is the image of $(t_i - t_i^{-1})/(q-q^{-1})$ under specialization. The generators for $\mathbf{g}^\theta$ are say $e_i + \theta(e_i)$ and $h_i + \theta(h_i)$ for i such that $\alpha_i \notin \pi_\Theta$ and e_i, f_i, h_i for i such that $\alpha_i \in \pi_\Theta$.

Assume that there exists an n such that $e_n + \theta(e_n)$ cannot be written as a sum of commutators. If $n \neq p(n)$, then $h_n - h_{p(n)}$ is a nonzero element of $\mathbf{g}^\theta$ and $[h_n - h_{p(n)}, e_n + \theta(e_n)] \neq 0$ which is not possible. On the other hand, if $\alpha_n \neq -\Theta(\alpha_n)$ and $n = p(n)$, then there must exist $\alpha_i \in \pi_\Theta$ such that $(\alpha_i, \alpha_n) \neq 0$. So $e_n + \theta(e_n)$ is a nonzero multiple of $[h_i, e_n + \theta(e_n)]$, a contradiction. It follows that $-\Theta(\alpha_n) = \alpha_n$ and so, without loss of generality, $e_n + \theta(e_n) = e_n + f_n$.

Choose $c_1, \dots c_r$ so that $z_i \in \mathbf{C}^* c_i + [\mathbf{g}^\theta, \mathbf{g}^\theta]$ and c_i is either equal to $h_n + \theta(h_n)$ or $e_n + f_n$ for some n. If $c_i = h_n + \theta(h_n)$, then set $C_i = t_n \tau(\Theta(\alpha_n))$ while if $c_i = e_n + f_n$, set $C_i = B_n$.

We may write $\mathbf{g}$ as a direct sum of simple Lie algebras $\mathbf{g}^j$ such that the root system of $\mathbf{g}$ is a disjoint union of the root systems Δ_j of the $\mathbf{g}^j$. By (2.1), (2.2), and (2.3), it follows that $\theta(\mathbf{g}^j)$ is equal to another summand $\mathbf{g}^k$ where $\mathbf{g}^k$ is isomorphic to $\mathbf{g}^j$. Suppose that $c_i = e_n + f_n$ for some i, $1 \leq i \leq r$ and some n, $1 \leq n \leq l$. Let Δ_j be the subset of Δ containing α_n. It follows that $\theta(\mathbf{g}^j) = \mathbf{g}^j$. Since $\mathbf{g}^j$ is simple, by the classical version of Theorem 4.2, z_i is the only element up to a scalar of $Z(\mathbf{g}^\theta)$ contained in $\mathbf{g}^j$. Thus we may write $\mathbf{g} = \bigoplus_{1\leq j\leq s} \mathbf{g}_j$ where $\mathbf{g}_1, \dots, \mathbf{g}_{s-1}$ are simple and θ restricts to an involution of each $\mathbf{g}_j$. Furthermore, we may assume that c_i is in $\mathbf{g}_s$ if and only if c_i is contained in $\mathbf{h}$.

The discussion in the previous paragraph implies that each $c_i \notin \mathbf{h}$ lie in a different simple summand of $\mathbf{g}_j$ of $\mathbf{g}$ where $1 \leq j \leq s-1$. Furthermore, all $c_i \in \mathbf{h}$ are contained in the last summand $\mathbf{g}_s$. It follows that $\mathbf{C}[c_1, \ldots, c_r]$ is a commutative polynomial ring. The same argument can be made in the quantum case. In particular, the ring over K generated by the C_i, $1 \leq i \leq r$ is a polynomial ring in r variables. Let Z be the ring generated by $\{C_i | 1 \leq i \leq r\}$ and $\{C_i^{-1} | C_i \in T$ and $1 \leq i \leq r\}$.

Let B' be the subalgebra of B generated by the sets

$$\{B_i | \alpha_i \notin \pi_\Theta \text{ and } B_i \neq C_j \text{ for any } 1 \leq j \leq r\},$$

$$\{t_i \tau(\Theta(\alpha_i)) | \alpha_i \notin \pi_\Theta \text{ and } t_i \tau(\Theta(\alpha_i)) \neq C_j \text{ for any } 1 \leq j \leq r\},$$

and

$$\{x_i, y_i, t_i, t_i^{-1} | \alpha_i \in \pi_\Theta\}.$$

Set $B'^+ = U^+ \cap B'$.

Theorem 5.1. *$B = BB'^+B \oplus Z$.*

Proof. Since the theorem takes place entirely inside of B, we may assume without loss of generality that B is a standard analog of $U(\mathbf{g}^\theta)$. By the above discussion, we may reduce to the following two cases.

Case 1: $r = 1$ and $C = B_n$ for some $1 \leq n \leq l$.

Case 2: Each $C_i \in T_\Theta$ for $1 \leq i \leq r$.

Let R_1 be the subalgebra $R \cap B'$ of B and write $(R_1)^+$ for the augmentation ideal of R_1. Let $\mathcal{I}$ denote the set of all m-tuples, where m is any positive integer, which satisfies the condition that at least one entry is contained in the set $\{i | B_i \in B'\}$. By [L2, discussion following Lemma 4.3], we have that $R^+ R^o B_i = B_i R^+ R^o$ for each $\alpha_i \notin \pi_\Theta$. Thus we can write $B(B')^+B = \sum_{I \in \mathcal{I}} B_I R^o R^+ + (R_1)^+ Z$.

Let $\mathcal{I}_1$ be a maximal subset of $\mathcal{I}$ such that the set $\{y_I | I \in \mathcal{I}_1\}$ is linearly independent over K. Recall the degree function on U defined at the end of Section 2. Note that $B_I = y_I +$ lower degree terms. Hence $\{B_I | I \in \mathcal{I}_1\}$ is also linearly independent over K.

Set $\mathcal{J}$ equal to the set of m-tuples such that $\{y_J | J \in \mathcal{J}\} = \{y_n^m | m \geq 1\}$ in Case 1 and equal to the empty set otherwise. By the choice of $\mathcal{I}_1$, $\{y_I | I \in \mathcal{I}_1 \cup \mathcal{J}\}$ is a basis for G^- over K. The proof of the theorem now follows from the following lemma. □

Lemma 5.2. *$B(B')^+B = \sum_{I \in \mathcal{I}_1} B_I R^o R^+ + (R_1)^+ Z$.*

Proof. It is sufficient to show that whenever we have a relation of the form

$$\sum_{I \in \mathcal{I}} a_I B_I r_I = \sum_{I \in \mathcal{I}_1 \cup \mathcal{J}} b_I B_I r'_I + c$$

where $r_I, r'_I \in R^o R^+$ and $c \in R_1 Z$, then $b_I = 0$ whenever $I \in \mathcal{J}$ and $c \in (R_1)^+ Z$. Now the relations among the $B_I R^o R^+$ come from the quantum Serre relations on G^- (and relations of R) (see the discussion at the end of Section 2). Hence without loss of generality we may assume that there is some positive integer m such that whenever a_I is nonzero, the degree of y_I is m and whenever $b_J \neq 0$, y_J has degree strictly less than m.

Recall that the quantum Serre relations each involve two elements $y_i t_i$ and $y_j t_j$ for some i and j. Thus, we can reduce to the case where the $a_I \neq 0$ if and only if I has one entry equal to j and $m_{ij} = -\langle\alpha_i, \alpha_j\rangle + 1$ entries equal to i for some $1 \leq i, j \leq l$. Furthermore, either B_i or B_j (or both) must be in B'. We are thus interested in relations of the form

$$\sum_k A_k B_i^{m_{ij}-k} B_j B_i^k = X \tag{5.1}$$

where $A_k \in K$ and X has degree strictly lower than $m_{ij}+1$. (Here the A_k are chosen from the quantum Serre relations. In particular, $\sum_k A_k y_i^{m_{ij}-k} y_j y_i^k = 0$.) We argue in the following cases that X is in $\sum_{I\in\mathcal{I}_1} B_I R^o R^+ + (R_1)^+ Z$. Set $m = m_{ij}$.

Case (i): α_i or α_j is in π_Θ: Assume that $\alpha_i \in \pi_\Theta$ and $\alpha_j \notin \pi_\Theta$. We may write $B_j = y_j t_j + \theta(y_j) t_j$. In this case $0 = \sum_k A_k (y_i t_i)^{m-k} (y_j t_j)(y_i t_i)^k = \sum_k q^s A_k y_i^{m-k} y_j y_i^k t_i^m t_j$ for some s independent of k. Hence

$$\begin{aligned} 0 &= \tilde{\theta}(\sum_k q^s A_k y_i^{m-k} y_j y_i^k) t_i^m t_j = \sum_k q^s A_k y_i^{m-k} \tilde{\theta}(y_j) y_i^k t_i^m t_j \\ &= \sum_k A_k (y_i t_i)^{m-k} \tilde{\theta}(y_j) t_j (y_i t_i)^k. \end{aligned}$$

It follows that $\sum_k A_k B_k^{m-k} B_j B_i^k = X = 0$. The same argument works when $\alpha_j \in \pi_\Theta$ and $\alpha_i \notin \pi_\Theta$. Furthermore, if both α_i and α_j are in π_Θ, then we must also have that $X = 0$.

For the remaining cases, we assume that $\alpha_i \notin \pi_\Theta$ and $\alpha_j \notin \pi_\Theta$.

Case (ii): $i \neq p(i)$, $i \neq p(j)$: It follows immediately that $j \neq p(i)$. Recall (Section 2) that the highest degree term of X must be contained in $G^- R^o R^+$. Furthermore, the assumptions in this case say that any monomial in $\tilde{\theta}(y_i)t_i$, $\tilde{\theta}(y_j)t_j$, $y_i t_i$, and $y_j t_j$ is not in $G^- R^o R^+$ unless it is actually a monomial in $y_i t_i$ and $y_j t_j$. In particular, $X = 0$ in this case.

Case (iii): $i = p(i)$: One checks that $i \neq p(j)$ and $j \neq p(i)$ in this case. Moreover, it will not matter whether or not j is equal to $p(j)$. Now the highest degree term of X can be written as a sum of monomials of the form $y_i^{m-2k-k'} y_j y_i^{k'} R^o R^+$. Hence X can be written as a sum of terms of the form $B_i^{m-2k-k'} B_j B_i^{k'} R^o R^+$ where $k \geq 1$ and $m - 2k \geq 0$ and $m - 2k \geq k' \geq 0$. Since $m - 2k + 1$ is strictly less than $m + 1$ for $k \geq 1$, we may assume the elements $B_i^{m-2k-k'} B_j B_i^{k'}$ are contained in the set $\{B_I | I \in \mathcal{I}_1 \cup \mathcal{J}\}$.

Now if $\mathcal{J} = 0$ or $Z = K[B_n]$ with $j \neq n$, then it follows that $X \in \sum_{I\in\mathcal{I}_1} B_I R^o R^+$ as desired. So assume that $Z = K[B_j]$. Now if $\mathbf{g}$ is of type G_2, we must have that $\mathbf{g}^\theta$ is semisimple (see [Kn, p. 543]). Hence, we may assume that $m = 2$ or $m = 3$. Using the above, if $m = 3$, then $X \in \sum_{I\in\mathcal{I}_1} B_I R^o R^+$ since each B_I contains at least one B_i term. Hence we are reduced to the case where $Z = K[B_j]$ and $m = 2$.

If $\alpha_i = -\Theta(\alpha_i)$, then by the classical version of the identity in [L1, Lemma 2.2], we have that $e_n + f_n \in [\mathbf{g}^\theta, \mathbf{g}^\theta]$, a contradiction. If $\alpha_i \neq -\Theta(\alpha_i)$, then one checks by looking at the possible monomials on the left hand side of (5.1) that X is in $\sum_{I\in\mathcal{I}_1} B_I R^o R^+ + (R_1)^+ Z$ as desired.

Case 4: $i = p(j)$: It follows that $j = p(i)$, $j \neq p(j)$, and $i \neq p(i)$. In this case, X will be a sum of terms of the form $B_i^{m-1} R^o R^+$ where B_i appears exactly $m - 1$ times. By assumption, it is impossible for $B_i = B_n$, and thus X has the desired form. $\square$

Using Theorem 5.2, we can now construct a family of nontrivial one-dimensional B-modules.

Corollary 5.3. *For each $x_1, \dots, x_r \in K$ such that x_i is invertible whenever C_i is in T, there exists a (unique) one-dimensional B-module Kv such that $C_i v = x_i v$ and $B'^{+}v = 0$.*

Unfortunately, we cannot expect to actually lift $U([\mathbf{g}^\theta, \mathbf{g}^\theta])$ to a subalgebra of B. To see this, consider the case when $\mathbf{g} = \mathbf{sl}\ 3$ and $\Theta(\alpha_1) = -\alpha_2, \Theta(\alpha_2) = -\alpha_1$. In this case, B is generated by $B_1 = y_1 t_1 + x_2 t_2^{-1} t_1, B_2 = y_2 t_2 + x_1 t_1 t_2^{-1}$ and k_1, k_2 where $k_1 = t_1 t_2^{-1}$ and $k_2 = k_1^{-1}$. Relations for B are given in [L1, Lemma 2.2]. (Note that we must apply a suitable automorphism to U and take into account the difference in the definition of the generators of the quantized enveloping algebra so that the subalgebra here agrees with the one in [L1].) In particular, for $i \neq j$, we have that

$$B_i^2 B_j - (q + q^{-1}) B_i B_j B_i + B_j B_i^2 = (q + q^{-1}) B_i (q^3 k_i + k_i^{-1}). \tag{5.2}$$

In addition, each B_i is a weight vector for the action of $(\mathrm{ad}\, k_i)$, $i = 1, 2$, in the obvious way.

We may set $C = k_1$ and $Z = K[C, C^{-1}]$. Suppose that we have a subalgebra B'' of B which specializes to $U([\mathbf{g}^\theta, \mathbf{g}^\theta])$ and furthermore, such that $B \cong B'' \otimes K[t, t^{-1}]$. We should have that B'' contains B_1 and B_2. By (5.2), B'' contains $B_i(q^3 k_i + k_i^{-1})$. Hence $B'' \cap \hat{U}$ contains the element $(q-1)^{-1}(B_i(q^3 k_i + k_i^{-1}) - 2B_i)$ which does not specialize to an element of $U([\mathbf{g}^\theta, \mathbf{g}^\theta])$. Note that the same problem occurs if we replace B_i by $B_i t^m$ for any integer m or even any rational number m. Thus, such a subalgebra B'' does not exist. Problems also arise in trying to construct such a B'' when C is not in T_Θ.

Set $Z_{\mathcal{R}}$ equal to the $\mathcal{R}$ subring of Z generated by all $C_i \in Z$ and all of the C_i^{-1} which are in Z for $1 \leq i \leq r$. Let x be an element in the set of algebra homomorphisms Z to K which restricts to an algebra homomorphism from $Z_{\mathcal{R}}$ to $\mathcal{R}$. Set V_x to be the one-dimensional B-module Kv_x where $B'v_x = 0$ and $sv_x = x(s)v_x$ for all $s \in Z$. Since $x(s) \in \mathcal{R}$ for all $s \in Z_{\mathcal{R}}$, we can define a conjugate linear form on V_x which satisfies (3.7) and (3.8). In particular, we have the following additional unitary B-modules.

Theorem 5.4. *Let W be a finite-dimensional U-module and let x be an element of Hom (Z, K) which restricts to an element of* $\mathrm{Hom}(Z_{\mathcal{R}}, \mathcal{R})$. *The B-module $W \otimes V_x$ can be written as a direct sum of finite-dimensional simple unitary B-modules.*

6. Coinduced Modules

Let M denote the subalgebra of R corresponding to $U_q([\mathbf{m}, \mathbf{m}])$. In this section we use M to construct coinduced modules. Such modules play an important role in the classical theory. Here we use them in a novel way. In particular, in the next section, these modules are used to study and specialize certain finite-dimensional simple B-modules.

In this section we only need to assume that B is a standard or nonstandard analog of $U(\mathbf{g}^\theta)$ such that $\kappa(B) = B$ as in Lemma 3.2. (Thus the results in this section hold for all nonstandard analogs.) Let W be a finite-dimensional simple M-module. We make M into a T-module by assuming that the highest weight generating vector of W is also a weight vector for T of some chosen weight. In particular, let $\mathbf{h}_1^*$ equal

$\sum_{1\leq i\leq l}\mathbf{Z}[1/2]\omega_i$ where ω_i is the fundamental weight corresponding to the root α_i. It follows that q^λ is a well defined weight in the quantum case whenever $\lambda\in\mathbf{h}_1^*$. Let $\mathbf{h}_W^*$ be the subset of $\mathbf{h}_1^*$ so that for each $\lambda\in\mathbf{h}_W^*$, ζq^λ restricts on $M\cap T$ to the highest weight of W (where ζ is some fixed homorphism on $T\cap M$ sending each $t_i\in T\cap M$ to the set $\{1,-1\}$). The argument below is independent of the choice of weight in $\mathbf{h}_W^*$.

Make W^* into a M-module using the antiautomorphism κ. Note that κ preserves both M and T. Set $N_\Theta^-=c(\tilde{N}_\Theta^-)$ where c is the automorphism of G^- fixing each y_it_i and sending q to q^{-1}.

Lemma 6.1. $\kappa(N_\Theta^+)=N_\Theta^-$.

Proof. By Proposition 2.2, N_Θ^+ has generators X_β where X_β is an element in U_β^+ and β is a weight of $\mathbf{n}_\Theta^+$. Furthermore, using (2.14), we have that

$$\Delta(X_\beta)\subset T\otimes X_\beta+U\otimes R.$$

Applying the same argument as in [L2, Proposition 4.1], we have that X_β is the unique element up to scalar of U_β^+ satisfying this condition on the coproduct. We can similarly show that N_Θ^- is generated by elements Y_β in G_β^- where $-\beta$ is a weight of $\mathbf{n}_\Theta^+$. Once again we have that Y_β is the unique element up to nonzero scalar of G_β^- satisfying the coproduct condition

$$\Delta(Y_\beta)\subset T\otimes Y_\beta+U\otimes R.$$

Since κ sends U^+ to G^- and preserves the Hopf structure of U, it follows that κ sends the generators of N_Θ^- to the generators of N_Θ^+. Hence $\kappa(N_\Theta^-)=N_\Theta^+$. $\square$

We can make W into a MTN_Θ^--module by letting N_Θ^- act as zero on the lowest weight generating vector of W. It follows that W^* becomes a MTN_Θ^+-module where N_Θ^+ acts as zero on the highest weight generating vector of W^*.

Let V be the right U-module $Hom_{MTN_\Theta^-}(U,W)$ coinduced from W. Here the U action is defined as follows. Let $f\in V$ and $u\in U$. Then $u\cdot f(v)=f(vu)$. We have $V\cong(U\otimes_{MTN_\Theta^+}W^*)^*$ as U-modules where f_χ is mapped to χ and

$$f_\chi(\kappa(m))(w^*)=\chi(m\otimes w^*).$$

Recall the definition of $T_\Theta^{1/2}$ (Section 2) and note that there is an obvious way to make a finite-dimensional B-module into a $BT_\Theta^{1/2}$-module. In particular, let $\psi\in\{1,-1\}$ and let E be a set of generators of T_Θ. If $tv=\psi q^mv$ for some v inside a finite-dimensional B-module and $t\in E$, then we may set $t^{1/2}v=q^{m/2}v$ if $\psi=1$ and $t^{1/2}v=iq^{m/2}v$ if $\psi=-1$.

Theorem 6.2. *Let N be a B-module. There is a vector space isomorphism*

$$Hom_{BT_\Theta^{1/2}}(N,Hom_{MTN_\Theta^-}(U_\Theta,W))\cong Hom_M(N,W). \tag{6.1}$$

Proof. This is a standard argument (see [D, §5.5]) which we include for completeness. Let ϵ be the map from $Hom_{MTN_\Theta^-}(U_\Theta,W)$ to W defined by $\epsilon(f)=f(1)$ for f in the space $Hom_{MTN_\Theta^-}(U_\Theta,W)$. Define a map χ from the left hand side of (6.1) to the right hand side of (6.1) by $\chi:g\to\epsilon\circ g$. The map χ will be the desired isomorphism.

We first check that $\epsilon\circ g$ is indeed an element of $Hom_M(N,W)$. Given $n\in N$ and $r\in M$, we have that $\epsilon\circ g(n)=g(n)(1)$ which is in W since g is in the left hand side of (6.1). Furthermore, $\epsilon \circ g(rn) = g(rn)(1) = rg(n)(1)$ since g is a $BT_\Theta^{1/2}$-module map. Hence $\epsilon \circ g(rn) = r\epsilon \circ g(n)$.

To see that χ is one-to-one, suppose that $\epsilon \circ g = 0$. Then $g(n)(1) = 0$ for all $n \in N$. Hence $\kappa(u)g(n)(1) = g(n)(u) = 0$ for all $u \in U$ and $n \in N$. Thus $g(n) = 0$ for all $n \in N$ and so g is identically equal to zero.

We now check that χ is onto. First we argue that

$$Hom_{MTN_\Theta^-}(U_\Theta, W) \cong Hom_M(BT_\Theta^{1/2}, W)$$

as vector spaces using the restriction map. Let f, g be two elements of $Hom_{MTN_\Theta^-}(U_\Theta, W)$ and assume that f and g are equal when restricted to $BT_\Theta^{1/2}$. Let $u \in U$ and write $u = \sum_i m_i b_i$ using the quantum Iwasawa decomposition (2.17) where $m_i \in N_\Theta^- \mathcal{A}$ and $b_i \in BT_\Theta^{1/2}$. It follows that $g(u) = \sum m_i g(b_i) = \sum m_i f(b_i) = f(u)$ for all $u \in U$. Hence $f = g$. Thus the restriction map gives a one-to-one map from $Hom_{MTN_\Theta^-}(U, W)$ into $Hom_M(BT_\Theta^{1/2}, W)$. To see onto, one notes that any $f \in Hom_M(BT_\Theta^{1/2}, W)$ can be lifted to a map $f \in Hom_{MTN_\Theta^-}(U, W)$ in a well defined unique way using the quantum Iwasawa decomposition. In particular $f(u)$ is defined to equal $\sum m_i f(b_i)$ where $u = \sum m_i b_i$ as above.

We are thus reduced to showing that $Hom_{BT_\Theta^{1/2}}(N, Hom_M(BT_\Theta^{1/2}, W))$ maps onto $Hom_M(N, W)$ using χ. Let $f \in Hom_M(N, W)$. We define the function g in $Hom(N, Hom_M(BT_\Theta^{1/2}, W))$ by setting $g(n)(1) = f(n)$ and $g(n)(b) = g(bn)(1)$. A straightforward check shows that g is an element of the set $Hom_{BT_\Theta^{1/2}}(N, Hom_M(BT_\Theta^{1/2}, W))$ and g maps to f. □

Now assume that N is a simple B-module which is locally finite as a M-module (where the action of M comes from restricting the action of B). Since M is just a quantized enveloping algebra of the semisimple Lie algebra $\mathbf{m}$, we can decompose N into a direct sum of finite-dimensional simple M-modules. Hence the dimension of the right hand side of (6.1) is just the index $[N : W]$. Let $X(W)$ denote the sum of all the finite-dimensional simple B-modules contained in $(U \otimes_{MTN_\Theta^+} W^*)^*$. Then the left hand side of (6.1) just equals $[X(W) : N]$. Thus we have the following.

Corollary 6.3. *If N is a simple B-module and is locally finite as a M-module under the restriction of the action to M, then*

$$[N : W] = [X(W) : N].$$

Note that $U \otimes_{RAN_\Theta^+} W^*$ is just a quantum version of a generalized Verma module corresponding to the reductive algebra $\mathbf{m} + \mathbf{h}$ (resp. parabolic subalgebra $\mathbf{p}$) whose enveloping algebra is the specialization of MT (resp. MTN_Θ^+.) In particular, let Λ be the highest weight of W^* considered as a T-module. Then W^* is the finite-dimensional simple MT module generated by a vector v_Λ of highest weight Λ. Write $M'(\Lambda)$ for the U-module $U \otimes_{MTN_\Theta^+} W^*$. We have by the proof of Proposition 2.2 that $\mathrm{ch}\, M'(\Lambda) = \mathrm{ch}\, N_\Theta^- \mathrm{ch}\, W^* = \mathrm{ch}\, U(\mathbf{n}_\Theta^-) \mathrm{ch}\, W^*$. Furthermore, $\hat{G}^- v_\Lambda = \hat{N}_\Theta^- \hat{R} v_\lambda$ is a free A-module with weight vectors as a basis which form a basis for $M'(\Lambda)$. Set $\hat{M}'(\Lambda) := \hat{G}^- v_\Lambda$.

We say that the algebra homomorphism Λ from U^o to K is linear and specializes to the weight $\lambda \in \mathbf{h}_W^*$ if $\Lambda(\tau(\lambda)) = q^{(\lambda,\alpha)}$ for all $\tau(\alpha) \in T$. If Λ is a linear weight, then the specialization of $\hat{M}'(\Lambda)$ at $q = 1$, $\hat{M}'(\Lambda) \otimes \mathbf{C}$, is a $U(\mathbf{g})$-module. Furthermore, the above equality of character formulas ensures that this specialization is equal to $\bar{M}'(\lambda) := U(\mathbf{g}) \otimes_{U(\mathbf{p})} \bar{W}^*$ where $\bar{W}^*$ is the finite-dimensional $\mathbf{m}+\mathbf{h}$ module of highest weight λ.

Using [J], we may choose $\lambda \in \mathbf{h}^*_W$ so that $\bar{M}'(\lambda)$ is simple. Again, set $\Lambda = q^\lambda$. Suppose that w is a nonzero highest weight vector in $M'(\Lambda)$. Rescaling if necessary, we may assume that w is in $\hat{M}'(\Lambda)$ and $w \notin (q-1)\hat{M}'(\Lambda)$. Thus w specializes to a nonzero highest weight vector of $\bar{M}'(\lambda)$. It follows that $M'(\Lambda)$ is also simple.

Write $\delta M'(\Lambda)$ for the locally finite T dual of $M'(\Lambda)$ (see [Jo, 4.1.4]) where Λ is chosen as in the previous paragraph. Since $M'(\Lambda)$ is simple, we have that $M'(\Lambda) \cong \delta M'(\Lambda)$. Now $(U \otimes_{R\mathcal{A}N^+_\Theta} W^*)^*$ can be considered as a T completion of $\delta M'(\Lambda)$ or equivalently $M'(\Lambda)$ in the following sense. Elements of $(U \otimes_{MTN^+_\Theta} W^*)^*$ can be written as possibly infinite sums of weight vectors of distinct weight in $M'(\Lambda)$. Let $\hat{M}'(\Lambda)^*$ denote the A-submodule of $M'(\Lambda)^*$ which consists of infinite sums of weight vectors of distinct weight in $\hat{M}'(\Lambda)$ with coefficients in A. We may further assume that these weight vectors have been chosen so that $\hat{M}'(\Lambda) \otimes_A \mathbf{C} = \bar{M}'(\Lambda)$.

7. Specializable modules.

In this section, we study the modules of B which can be specialized to $U(\mathbf{g}^\theta)$-modules. In particular, let V be a finite-dimensional B-module. We call V specializable if there exists a basis $\mathcal{B}$ of V such that $\hat{B}\sum_{v\in\mathcal{B}} Av \subset \sum_{v\in\mathcal{B}} Av$. Of course there could be more than one choice of basis which makes V specializable. Any such basis of V will be called a specializable basis. Given such a basis $\mathcal{B}$ we can form the specialization $\bar{V} := \sum_{v\in\mathcal{B}} Av \otimes_A \mathbf{C}$ which inherits a $U(\mathbf{g}^\theta)$-module structure. Note that $\dim_{\mathbf{C}} \bar{V} = \dim_K V$. Of course such a specialization seems dependent on the choice of specializable basis. We show below that this is not so for simple modules.

Let V be a finite-dimensional B-module with specializable basis $\mathcal{B}$. Let C be an invertible matrix with entries in A. If D is the matrix whose rows are the vectors in $\mathcal{B}$, then it is straightforward to see that the rows of CD also form a specializable basis and moreover the specialization of V using $\mathcal{B}$ and this second basis are isomorphic as $U(\mathbf{g}^\theta)$-modules. Now assume that $C \in M_{n\times n}(A)$ and det $C \in \mathbf{C}^*$. Elementary matrices that exchange rows or add a scalar (in A) multiple of one row to another are invertible with inverse also in $M_{n\times n}(A)$. Hence C is equivalent to a diagonal matrix D with all entries in $\mathbf{C}$. In particular, C is invertible with inverse also in $M_{n\times n}(A)$.

Though two specializable basis might produce different specializations, the following two lemmas show that these specializations at least have the same composition series.

Lemma 7.1. *Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a specializable basis for the B-module V. Let V_1 denote the specialization of V using this basis. If $\{v_1, \ldots, v_s\}$ specializes to a basis of a submodule W of V_1, and $\mathcal{B}_2 = \{v_1, \ldots, v_s, (q-1)^{-1}v_{s+1}, \ldots, (q-1)^{-1}v_n\}$ is also a specializable basis of V with specialization V_2, then V_1 and V_2 have the same composition series as B-modules.*

Proof. Let $b \in \hat{B}$. Write $bv_j = \sum_{1\le i\le n} b_{ij}v_i$ where $b_{ij} \in A$. Since both $\mathcal{B}_1$ and $\mathcal{B}_2$ are specializable basis, we must have that $b_{ij} \in (q-1)A$ whenever $1 \le i \le s$ and $s+1 \le j \le n$. In particular, $v_{s+1}, \ldots, v_n$ also specializes to a basis of a submodule, say W' of V_1. Moreover, $W \oplus W' \cong V_1$. Now we cannot say that

$(q-1)^{-1}v_{s+1},\dots,(q-1)^{-1}v_n$ specializes to a basis of a submodule of V_2. However, let N be the A-module generated by $v_1,\dots,v_s$. Then $(q-1)^{-1}v_{s+1}+N,\dots,(q-1)^{-1}v_n+N$ specializes to a basis of a module isomorphic to W'. In particular, $V_1/W \cong V_2/W$. The lemma now follows. □

Lemma 7.2. *Let V be a finite-dimensional specializable B-module. Let $\mathcal{B}_1,\mathcal{B}_2$ be two specializable basis with corresponding specializations V_1 and V_2. Then V_1 and V_2 have the same composition series as $U(\mathbf{g}^\theta)$-modules.*

Proof. Write $\mathcal{B}_1=\{v_1,\dots,v_n\}$ and $\mathcal{B}_2=\{w_1,\dots,w_n\}$. Note that if we multiply all elements of a specializable basis by the same nonzero scalar, we get another specializable basis with isomorphic specialization to the first specialization. Write $v_i=\sum a_{ij}w_j$. Multiplying $\mathcal{B}_1$ by a nonzero scalar if necessary, we may assume that all the $a_{ij}\in A$ and not all the $a_{ij}\in(q-1)A$. Let C be the matrix with entries a_{ij}. Note that we must have det $C\neq 0$ since it transforms one basis of V into another. Using standard P.I.D. theory, we can find elementary invertible matrices E and F with entries in A so that ECF is a diagonal matrix D with diagonal entries $d_1,\dots d_n$ in A. Moreover d_i divides d_{i+1} for each i. Note that both E and F are invertible matrices with inverses in $M_n(A)$. Hence $EC\mathcal{B}_2$ specializes to the same module as does $\mathcal{B}_1$.

We can also find a diagonal matrix H with entries a power of $(q-1)$ so that HD is a diagonal matrix with all entries in $\mathbf{C}$. Since $\det HD\neq 0$ and F is invertible in $M_{n\times n}(A)$, it follows that $HDF^{-1}\mathcal{B}_2$ has the same specialization as $\mathcal{B}_2$. By Lemma 7.1 and induction, the specializations of $D\mathcal{B}_2$ and $HDF^{-1}\mathcal{B}_2$ have the same composition series. □

Let V be a $\hat{U}$-module such that each element of V is a (possibly infinite) sum of T-weight vectors of distinct weight and of weight less than or equal to some fixed Λ. Assume further that there is an A-module V_A inside of V such that $V=V_A\otimes K$, $(\hat{U})V_A\subset V_A$ and that there is a well ordered set of linearly independent weight vectors in V_A such that each element in V_A is a possibly infinite linear combination of these vectors over A. Suppose that N is a finite-dimensional B-submodule of V such that there exists a K basis for N inside of V_A.

The following lemma is critical to showing that N embeds in a nice way inside of certain coinduced modules from Section 6. This nice embedding will allow us to specialize the quantum picture to the classical case.

Lemma 7.3. *$N\cap V_A$ is a free A-submodule of V_A.*

Proof. Think of A as a subalgebra of the formal power series $\mathbf{C}[[q-1]]$. Let $\{v_i|i\geq 0\}$ be the set of linearly independent weight vectors of V_A such that each element of V_A is a possibly infinite linear combination of these vectors over A. Let $\{w_i|1\leq i\leq r\}$ be a basis for N over K where each w_i is in V_A. We may write each w_i as $\sum_{j\geq 0}p_{ij}v_j$ for some $p_{ij}\in A$. Set $p(w_i)=j$ where p_{ij} is the first nonzero polynomial in the sequence $p_{1j},p_{2j},\dots$. Using the fact that A is a P.I.D., reordering the basis of N and taking linear combinations if necessary allows us to assume that $p(w_1)>p(w_2)>\cdots>p(w_r)$. Let $m(w_i)$ be the highest power of $(q-1)$ dividing the coefficient $p_{ip(w_i)}$ of $v_{p(w_i)}$ in the expression of w_i above.

Now write $w_i=\sum_{m\geq 0}(q-1)^m w_{im}$ where each w_{im} is in the A span of the set $\{v_i|i\geq 0\}$. Let j be the smallest integer such that $\sum_{1\leq k\leq j}Aw_{k0}$ does not equal $\sum_{1\leq k\leq j}Kw_{k0}\cap V_A$. It follows that there exist elements $a_{ji}\in A$ such that w_j-

$\sum_{1\le i<j} a_{ji}w_i$ is divisible by $(q-1)$ in V_A. Set $w'_j = (q-1)^{-1}(w_j - \sum_{1\le i<j} a_{ji}w_i)$. Note that the set $w_1, \dots, w_{j-1}, w'_j, w_{j+1}, \dots, w_r$ is still in V_A and is a basis for N over K. Also, $p(w'_j) = p(w_j)$ and $m(w'_j) = m(w_j) - 1$. Repeating this process, and applying induction on $m(w_j)$, we can find a vector w such that the intersection of the K span of $\{w_1, \dots, w_j\}$ with V_A is a free A-module with basis $\{w_1, \dots, w_{j-1}, w\}$. The lemma now follows by induction on j. □

We recall some of the notation from the previous section. First note that any M-submodule of a finite-dimensional specializable B-module must be a specializable M-module. Let W be a finite-dimensional (specializable) simple M-module. Choose $\lambda \in \mathbf{h}^*_W$ and set $\Lambda = q^\lambda$ such that both $M'(\Lambda)$ and $\bar{M}'(\lambda)$ are simple. Recall that $(U \otimes_{MTN^+_\Theta} W^*)^*$ embeds in a T completion of $M'(\Lambda)$ and that $\hat{M}'(\Lambda)^*$ is an A-submodule of this completion. Let N be a finite-dimensional specializable simple B-module with $[N : W] \neq 0$. Let $\{n_1, n_2, \dots, n_r\}$ be a specializable basis of N. Let $\tilde{N}$ be a copy of N contained in $(U \otimes_{MTN^+_\Theta} W^*)^*$ and write $\tilde{n}_i$ for the image of n_i in $(U \otimes_{MTN^+_\Theta} W^*)^*$.

Lemma 7.4. *For each i there exists a smallest integer s_i such that $(q-1)^{s_i}\tilde{n}_i \in \hat{M}'(\Lambda)^*$. Thus $N \cap \hat{M}'(\Lambda)^*$ is a free A-submodule of $\hat{M}'(\Lambda)^*$.*

Proof. Let Ω denote the set of weights of $M'(\Lambda)^*$. We may write $\tilde{n}_i = \sum_{\eta\in\Omega} c^i_\eta w^i_\eta$ where $c^i_\eta \in \mathbf{C}(q)$ and w^i_η is a weight vector of weight η in $\hat{M}'(\Lambda)^*$ and $w^i_\eta \notin (q-1)\hat{M}'(\Lambda)^*$. We can extend the standard partial order (the Bruhat order) on Ω to a complete ordering. Multiplying each n_i by the same nonzero scalar does not change the specialization of $\{n_1, \dots, n_r\}$. So without loss of generality, we may assume for each i, that if η_i is the highest weight with $c^i_{\eta_i} \neq 0$, then $c^i_{\eta_i} \in A$.

Let ζ be the highest weight such that $c^i_\zeta \notin A$ for some choice of i. Since both $M'(\Lambda)$ and its specialization is simple, there exists x_j such that $x_j w_\zeta \notin (q-1)\hat{M}'(\Lambda)$. Consider $\kappa(B_j)\tilde{n}_i = \sum_{\eta\in\Omega} c'_\eta w'_\eta$. The $w'_{\zeta+\alpha_j}$ term comes from $x_j c^i_\zeta w_\zeta$ and terms $\hat{G}^- c^i_\beta w_\beta$ where $\beta > \zeta$. In particular, $c'_{\zeta+\alpha_i} \notin A$. This contradicts the choice of ζ and the fact that $\{n_1, \dots n_r\}$ is a specializable basis of N. The second statement follows immediately from Lemma 7.3. □

If $\mathbf{g}^\theta$ is semisimple, then we are ready to show that the finite-dimensional simple modules of B remain simple upon specialization. However, when $\mathbf{g}^\theta$ has nonzero central elements then it is necessary to understand how these nonzero central elements behave. We use material from Section 4 to do this.

It is well known that U specializes to $U(\mathbf{g})$ at $q = 1$. Let $\check{U}$ denote the simply connected quantized enveloping algebra as defined in [JL2]. (Note that all our U-modules can be given an obvious $\check{U}$-module structure. Furthermore, one can define the A subalgebra $\check{U}^\wedge$ in a manner similar to $\hat{U}$.) We also have that $\check{U}$ specializes to $U(\mathbf{g})$ since the extra terms in the extension $\check{T}$ of T do not contribute anything new to the specialization. Let $F(\check{U})$ denote the locally finite part of $\check{U}$. Write $T_<$ for $\check{T} \cap F(\check{U})$. Recall [JL1] that $\check{U}$ is a free module with basis say $r_1, \dots, r_f$ in $\check{T}$ over $F(\check{U})T_<^{-1}$. Since the specialization of $(ts-1)/(q-1)$ is the same as the specialization of $(t-1)/(q-1) + (s-1)/(q-1)$ for all $s, t \in T$, we have that both $F(\check{U})T_<^{-1}$ and $F(\check{U})$ specialize to $U(\mathbf{g})$.

One can define a notion of specializable modules for U just as we did for B. By [JL2], the center $\mathbf{Z}$ of $\check{U}$ specializes to the center of $U(\mathbf{g})$. Also by [JL2], $F(\check{U})$

is free over $\mathbf{Z}$ with basis $\mathbf{H}$. Furthermore, $\mathbf{H}$ is a direct sum of finite-dimensional simple (specializable) U-modules with the same multiplicities as in the classical case. Thus the simple U-submodules of $\mathbf{H}$ specialize to their classical counterparts in the harmonics of $U(\mathbf{g})$. Recall by Lemma 4.5 that $\sigma^{-1}(F(\check{U})^B)$ is the set of elements in $F(\check{U})$ which commute with all b in B.

Lemma 7.5. *For each y in $U(\mathbf{g})^{\mathbf{g}^\theta}$, there exists a $Y \in \sigma^{-1}(F(\check{U})^B)$ such that Y specializes to y.*

Proof. If y is in the center of $U(\mathbf{g})$, then by say [JL2, Theorem 6.17], we can lift y to an element of the center of U. By [JL2, Theorem 7.4] and the classical version [K2], we thus reduce to the case where y is an element of the classical harmonics. Since $\mathbf{g}^\theta$ is a Lie subalgebra of $\mathbf{g}$, it is invariant under the antipode, $\bar{\sigma}$, of $\mathbf{g}$. Hence $\bar{\sigma}(y)$ is also in $U(\mathbf{g})^{\mathbf{g}^\theta}$. Furthermore, without loss of generality, we may assume that $\bar{\sigma}(y)$ lies in some simple summand say V of the harmonics of $U(\mathbf{g})$. There exists a simple summand $\tilde{V}$ of $\mathbf{H}$ which specializes to V. Since finite-dimensional (specializable) U-modules are spherical if and only if their specializations are (see Theorem 4.5), we can find an element $\tilde{Y}$ in $F(U)^B \cap \tilde{V}$ such that $\tilde{Y}$ specializes to $\bar{\sigma}(y)$. Now set Y equal to $\sigma^{-1}(\tilde{Y})$. By properties of σ and $\bar{\sigma}$ it follows that Y specializes to y. □

Recall the definition of the basis $\{z_1, \dots, z_r\}$ for $Z(\mathbf{g}^\theta)$ and the elements $\{c_1, \dots, c_r\}$ in $\mathbf{g}^\theta$ from Section 5. Using Lemma 7.5, find $Z_i \in \sigma^{-1}(F(\check{U})^B)$ for each i such that Z_i specializes to z_i.

We return now to specializing simple B-modules.

Theorem 7.6. *Let N be a finite-dimensional simple specializable B-module. Then N specializes to a finite-dimensional simple $U(g^\theta)$-module of the same dimension which is independent of the choice of specializable basis of N.*

Proof. The independence of the choice of basis follows from Lemma 7.2 after we prove the rest of the theorem.

Let $\{n_1, \dots, n_s\}$ be a specializable basis for N. Let $N_1, \dots, N_s$ be linear independent copies of N inside of $\hat{M}'(\Lambda)^*$. Set n_{ij} equal to the image of n_j in N_i.

Reordering and taking linear combinations if necessary, we may assume that the span of the first r vectors $\{n_1, \dots, n_r\}$ specialize to a finite-dimensional simple $U([\mathbf{g}^\theta, \mathbf{g}^\theta])$-module Y. Without loss of generality, we may assume that n_1 specializes to the highest weight generating vector of Y under a suitable choice of the Cartan subalgebra of $U([\mathbf{g}^\theta, \mathbf{g}^\theta])$. By Lemma 7.3, we can replace $N_1, \dots, N_s$ by suitable linear combinations of the N_j so that $\{n_{i1} | 1 \leq i \leq s\}$ is an A basis for $\sum_{1 \leq i \leq s} K n_{i1} \cap \hat{M}'(\Lambda)^*$.

Thus we may assume that the image of the $\mathbf{C}$ span of the set $\{n_{ij} | 1 \leq i, j \leq s\}$ includes s linearly independent copies Y_j of Y. By the choice of the Z_i and Lemma 4.5, the Z_j commute with each element of B. Also, since B specializes to $U(\mathbf{g}^\theta)$ we can find for each $1 \leq j \leq s$, a $S_j \in \hat{B}$ such that $Z_j = S_j + (q-1)\hat{U}$. Since each N_i is a B-module, it follows that $S_j N_i \subset N_i$ for each choice of j. Furthermore, the action of S_j is the same on each N_i since the N_i are isomorphic as B-modules. Hence there exists a scalar b such that $Z_j n = bn + (q-1)\hat{M}'(\Lambda)^*$ for all $n \in \bigoplus_{1 \leq i \leq s}(N_i \cap \hat{M}'(\Lambda)^*)$. Thus z_j acts by the same scalar on each Y_i. In particular, the Y_i are actually isomorphic as $U(\mathbf{g}^\theta)$-modules.

Let $\bar{W}$ denote the specialization of W. Recall that W is an M-module and M is the quantized enveloping algebra of the semisimple Lie algebra $[\mathbf{m},\mathbf{m}]$. By Corollary 6.3 and its classical version, we have that

$$[Y:\bar{W}] = [\bar{X}:Y] = s = [N:W].$$

Note that this argument works for each finite-dimensional (specializable) simple M-module W. By the quantized enveloping algebra theory, the finite-dimensional specializable M-modules correspond to their classical counterparts. Hence we must have that $\dim Y = \dim N$. The result now follows. ☐

The next theorem goes the other direction lifting simple $U(\mathbf{g}^\theta)$-modules to the quantum case.

Theorem 7.7. *For each finite-dimensional simple $U(\mathbf{g}^\theta)$-module V, there exists a specializable simple B-module $\tilde{V}$ such that the specialization of $\tilde{V}$ is V.*

Proof. Let V be a finite-dimensional simple $U(\mathbf{g}^\theta)$-module. Choose a finite-dimensional M-module W and a T-weight $\Lambda = q^\lambda$ so that the highest weight of W is the restriction of Λ to $T \cap M$ and $\bar{M}'(\lambda)$ is simple. We may further assume that $[V:\bar{W}] \neq 0$ where $\bar{W}$ is the specialization of W. Hence V appears in the composition series of $\bar{M}'(\lambda)$ considered as a $U(\mathbf{g}^\theta)$-module.

Recall that $M'(\Lambda) = G^- v_\Lambda$ where v_Λ is the highest weight generating vector for W and that $M'(\Lambda)$ is also simple. Given a weight vector n in G^- one can find a $t \in T$ such that $nt \in F(U)$. (This follows from the embeddings of finite-dimensional simple U-modules $L(\lambda)$ in $G^-\tau(-1/2\lambda) \cap F(\check{U})$ found in [JL2, §4].) Thus $M'(\Lambda) = F(U)v_\Lambda = F(U)Mv_\Lambda$. By Theorem 3.7 applied to M instead of B, it follows that $M'(\Lambda)$ is isomorphic to a direct sum of finite-dimensional simple M-modules.

Let Y_1 and Y_2 be submodules of $\bar{M}'(\lambda)$ such that Y_1/Y_2 is isomorphic to V. Choose $v \in \hat{M}'(\Lambda)$ such that the image of v under specialization is contained in Y_1 but not in Y_2. Assume further that the image of the specialization of v in the simple $U(\mathbf{g}^\theta)$-module Y_1/Y_2 corresponds to a highest weight vector under a suitable choice of Cartan subalgebra of $U([\mathbf{g}^\theta,\mathbf{g}^\theta])$. Let Y be the B-submodule of $M'(\Lambda)$ generated by v and let Y' be its maximal proper B-submodule. Note that v cannot be contained in Y'. Given $w \in \hat{M}'(\Lambda) - (q-1)\hat{M}'(\Lambda)$, let $\tilde{w}$ be the weight vector of $\hat{M}'(\Lambda) - (q-1)\hat{M}'(\Lambda)$ such that $w = \tilde{w} +$ weight vectors of weight higher than the weight of w + an element in $(q-1)\hat{M}'(\Lambda)$. Using induction on weight, we can find a basis $\mathcal{B}'$ (resp. $\mathcal{B}$) of Y' (resp. Y) such that the set $\{\tilde{w} | w \in \mathcal{B}'\}$ (resp. $\{\tilde{w} | w \in \mathcal{B}\}$) remains linearly independent upon specialization of $\hat{M}'(\Lambda)$. Moreover, one can choose the basis for Y' first and then enlarge this to a basis of Y. Note that these bases make Y and Y' into specializable modules.

Set $\tilde{V} = Y/Y'$ and note that this is a simple B-module. The set $\{y + Y' | y \in \mathcal{B} - \mathcal{B}'\}$ of $\tilde{V}$ is a specializable basis of Y/Y' by the previous paragraph. Furthermore, the element $v + Y'$ will be nonzero under the specialization of Y/Y' with respect to the above basis. It follows that the specialization of $\tilde{V}$ has V in its composition series.

We have constructed a simple specializable B-module $\tilde{V}$ whose specialization includes V in its composition series. The proof now follows as in Theorem 7.6. Note that we have not assumed that $\tilde{V}$ is finite-dimensional. However, $\tilde{V}$ is a locally finite semisimple M-module and we have assumed that V is finite-dimensional. Thus the arguments of Theorem 7.6 are applicable in this case as well. $\square$

A natural question is: What is the relationship between B-modules with identical specializations? This is addressed in the next theorem. First, we need finer information about elements of U which commute with elements of B.

Lemma 7.8. *Let y be an element of $Z(\mathbf{g}^\theta)$. For each positive integer n, there exists an element $b_n \in \hat{B}$ such that $b_n \in \sigma^{-1}(F(\check{U})^B) \cap \check{U}\hat{} + (q-1)^n\check{U}\hat{}$ and b_n specializes to y.*

Proof. Consider y in $Z(\mathbf{g}^\theta)$ and let Y_0 be an element in $\sigma^{-1}(F(\check{U})^B)\cap\check{U}\hat{}$ such that the specialization of Y_0 is y_0 as in Lemma 7.5. Since $y_0 \in U(\mathbf{g}^\theta)$, we can find $X_0 \in \hat{B}$ which also specializes to y_0. It follows that $Y_0 - X_0 \in (q-1)\check{U}\hat{}$. Furthermore, $b(Y_0 - X_0) - (Y_0 - X_0)b \in (q-1)\hat{B}$ for all $b \in \hat{B}$. Therefore, $(q-1)^{-1}(Y_0 - X_0)$ specializes to an element of $U(\mathbf{g}^\theta) + U(\mathbf{g})^{\mathbf{g}^\theta}$ because $\mathbf{g}^\theta$ acts locally finite and semisimple on $U(\mathbf{g})$. By Lemma 7.5 and (2.9), we can find $Y_1 \in \sigma^{-1}(F(\check{U})^B) \cap \hat{U}$ and $X_1 \in \hat{B}$ such that $(q-1)^{-1}(Y_0 - X_0) = (X_1 - Y_1) + (q-1)\check{U}\hat{}$. In particular, the element $Y_0 + (q-1)Y_1$ commutes with all elements of B, specializes to y_0 and is in $\hat{B} + (q-1)^2\check{U}\hat{}$. The lemma follows by induction. $\square$

Consider again the basis $\{z_1, \ldots, z_r\}$ of $Z(\mathbf{g}^\theta)$. By the above lemma, we can find two sequences $\{y_{in}\}_{n\geq 1}$ and $\{z_{in}\}_{n\geq 1}$ for each i such that for all $n \geq 1$, $z_{in} \in \sigma^{-1}(F(\check{U})^B) \cap \check{U}\hat{}$, $y_{in} \in \hat{B}$, $y_{in} - z_{in} \in (q-1)^n\check{U}\hat{}$, and z_{in} specializes to z_i. Since B is $\bar{\kappa}$ invariant and each c_i is invariant under the classical chevellay antiautomorphism, we may assume without loss of generality that each z_{in} and each y_{in} is $\bar{\kappa}$ invariant. (This can be easily accomplished by replacing each z_{in} (resp. y_{in}) by $1/2(\bar{\kappa}(z_{in}) + z_{in})$ (resp. $1/2(\bar{\kappa}(y_{in}) + y_{in})$).) Note that this invariance implies that we can further assume that each $z_{in} \in \check{U}_{\mathcal{R}}$ and each $y_{in} \in B_{\mathcal{R}}$.

Given a finite-dimensional simple specializable B-module V, we can embed it inside a U-module using Theorem 6.2 and Corollary 6.3. For any positve integer n, the arguments of Theorem 7.6 ensure that this embedding can be chosen so that z_{in} acts as a scalar on V. Thus y_{in} must act like a scalar modulo $(q-1)^n$ on V for each n (i.e. y_{in} acts as a matrix which is the sum of a scalar matrix and a matrix with entries all in $(q-1)^nA$). Call two B-modules compatible if for each i and each n, y_{in} acts by the same scalar a_{in} modulo $(q-1)^n$ on both modules.

Theorem 7.9. *Let V_1 and V_2 be two specializable finite-dimensional simple B-modules. Then V_1 and V_2 are isomorphic if and only if they are compatible and there specializations are isomorphic. Moreover, each finite-dimensional simple B-module is self dual.*

Proof. Note that the M-module structure of V_1 and V_2 must be the same since their specializations are isomorphic. Let W be a finite-dimensional simple M-module and $\Lambda = q^\lambda$ such that $\lambda \in \mathbf{h}_W^*$ and both $\bar{M}'(\lambda)$ and $M'(\Lambda)^*$ are simple. Set $s = [V_1 : W] = [V_2 : W]$. Let $\{V_{ij} | 1 \leq j \leq s\}$ be linearly independent copies of V_i in $M'(\Lambda)^*$ for $i = 1$ and $i = 2$ and let $v_{ij}^1, \ldots v_{ij}^m$ be a basis for the j^{th} copy of V_i. Using the arguments in Theorem 7.6 we can find a new basis such that using this basis the $\sum_{ij} V_{ij}$ specializes under the specialization of $M'(\Lambda)^*$ to a finite-dimensional $U(\mathbf{g}^\theta)$-module N of the same dimension. The process of finding this basis consists of multiplying the basis $\{v_{ij}^k\}$ by a suitable matrix with entries in K. Let n be the smallest nonnegative integer n such that $(q-1)^n$ times this matrix forces all entries to be in A. By assumption, $z_{i,n+1}$ acts on the original basis by a scalar matrix $C_{i,n+1}$ plus a matrix with entries in $(q-1)^{n+1}A$. Hence for each i, $z_{i,n+1}$ acts on the new basis by the same scalar matrix $C_{i,n+1}$ plus a matrix with entries in $(q-1)A$. Since $z_{i,n+1}$ specializes to z_i, each z_i acts as a scalar on the specialization N. Thus $U(\mathbf{g}^\theta)$ acts semisimply on N and N is a direct sum of $2s$ copies of $\bar{V}$. This contradicts the fact that $[\bar{V} : \bar{W}] = [V_i : W] = s$.

Now let V be a finite-dimensional simple specializable B-module and let V^* be the dual using κ. Since each z_{in} and y_{in} is κ invariant, the action is the same on V and V^*. The theorem now follows from the previous paragraph and the fact that the specializations of V and V^* are isomorphic. □

The picture is particularly nice when $\mathbf{g}^\theta$ is semisimple.

Corollary 7.10. *If $\mathbf{g}^\theta$ is semisimple, then there is a one-to-one correspondence between finite-dimensional specializable simple B-modules and finite-dimensional simple $\mathbf{g}^\theta$-modules.*

We now turn our attention to unitary modules.

Corollary 7.11. *For each finite-dimensional simple unitary $U(\mathbf{g}^\theta)$-module V, there exists a finite-dimensional specializable simple unitary B-module $\tilde{V}$ such that the specialization of $\tilde{V}$ is V.*

Proof. Let $\bar{V}$ be a finite-dimensional simple unitary $U(\mathbf{g}^\theta)$-module. So $\bar{V}$ has a real vector subspace $\bar{V}_{\mathbf{R}}$ such that $\dim_{\mathbf{R}} \bar{V}_{\mathbf{R}} = \dim_{\mathbf{C}} \bar{V}$. The proofs of this section work if the base field is $\mathcal{R}$ instead of K. In particular, there is a finite-dimensional specializable simple $B_{\mathcal{R}}$-module $V_{\mathcal{R}}$ which specializes to $\bar{V}_{\mathbf{R}}$. Set $V = V_{\mathcal{R}} \otimes K$ and note that V specializes to $\bar{V}$ and satisfies condition (3.7). By the previous corollary, it follows that V is isomorphic to its dual V^*. Rescaling if necessary, we may assume that the bilinear pairing between $V_{\mathcal{R}}$ and $V_{\mathcal{R}}^*$ specializes to the bilinear pairing between $\bar{V}_{\mathbf{R}}$ and its dual. This bilinear pairing becomes the conjugate linear form on V which satisfies (3.8) because its specialization is positive definite. □

Unfortunately, the methods in this section are not constructive. The modules considered in Section 5 are more computable. We show below that many of the finite-dimensional simple B-modules considered earlier in the paper are specializable B-modules. Note that any finite-dimensional simple U-module with highest weight of the form q^λ where λ is dominant integral is a specializable B-module. Let x denote the r-tuple $x_1, \dots, x_r$ such that each $x_i \in A$. Write V_x for the one-dimensional B-module as defined in Corollary 5.2. Note that for such a choice of x, V_x is a specializable B-module.

Proposition 7.12. *If W is a B-submodule of $L(q^\lambda) \otimes V_x$, then W is a specializable B-module.*

Proof. Let v_λ be the highest weight generating vector of $L(\Lambda)$ and let v be a basis vector for V_x. Let $u_1 v_\lambda \otimes v, \dots, u_s v_\lambda \otimes v$ be a basis for W. Rescaling if necessary, we may assume that $u_i \in \hat{U}$. Furthermore, using linear combinations and the fact that A is a principle ideal domain, we may assume that the set $\{u_1 v_\lambda \otimes v, \dots, u_s v_\lambda \otimes v\}$ remains linearly independent upon specialization. Let $b \in \hat{B}$ and find $b_{ij} \in K$ such that $b u_i v_\lambda \otimes v = \sum_j b_{ij} u_j v_\lambda \otimes v$. Fix i. If b_{ij} is not in A for all j, then there exists a smallest positive integer s such that $(q-1)^s b_{ij} \in A$ for all j. Now $(q-1)^s b \in (q-1)\hat{U}$, so upon specialization of $(q-1)^s \sum_j b_{ij} u_j v_\lambda$ we get a nontrivial linear combination of linearly independent vectors is zero. This contradiction ensures that $b_{ij} \in A$ for all i, j. Hence W is specializable. □

Unlike the quantized enveloping algebra, not all finite-dimensional B-modules are specializable. Of course when $\mathbf{g}^\theta$ is not semisimple, it is easy to construct finite-dimensional modules which are not specializable. Indeed, any one-dimensional module V_x where x takes some C_i to an element not in A is such an example. The surprising thing is that there are non specializable modules even when $\mathbf{g}^\theta$ is semisimple. We consider such an example below.

Example 7.13. Let $\mathbf{g} = \mathbf{sl}\ 3$ and define Θ by $\Theta(\alpha_i) = -\alpha_i$ for $1 \leq i \leq 2$. It follows that $\mathbf{g}^\theta$ is isomorphic to the simple Lie algebra $\mathbf{so}\ 2 = \mathbf{sl}\ 2$. By [L2, Lemma 2.2] the generators $B_j = y_j t_j + x_j$, $j = 1, 2$ satisfy the following relations.

$$B_k^2 B_j - (q + q^{-1}) B_k B_j B_k + B_j B_k^2 = q B_j \tag{7.1}$$

for $\{k, j\} = \{1, 2\}$, $j \neq k$. Let λ be an algebra homomorphism from B to the scalars K. Assume that v_λ generates a one-dimensional B-module such that $B_j v_\lambda = \lambda(B_j) v_\lambda$. Note that by (7.1), $\lambda(B_1) = 0$ if and only if $\lambda(B_2) = 0$. For $j = 1, 2$, set $\lambda_j = \lambda(B_j)$. The only specializable one-dimensional B-module is the one where both λ_1 and λ_2 are zero. Now assume that neither of them are zero. Substituting λ_i for B_i in (7.1) yields four additional one-dimensional nonspecializable B-modules. In particular, solving for λ_j, we get the following two choices for each j:

$$\lambda_j = \pm iq/(q-1).$$

Note that since $\bar{\kappa}(B) = B$, these one-dimensional nonspecializable B-modules are not unitary.

References

[CP] V. Chari and A. Pressley, *A guide to quantum groups*, Cambridge University Press, Cambridge, 1995. MR **96h:**17014

[DK] C. DeConcini and V.G. Kac, Representations of quantum groups at roots of 1, In: *Operator Algebras, Unitary Representations, Envelping Algebras, and Invariant Theory*, Prog. Math. **92** (1990), 471-506. MR **92g:**17012

[Di] M.S. Dijkhuizen, Some remarks on the construction of quantum symmetric spaces, In: *Representations of Lie groups, Lie algebras and their quantum analogues,* Acta Appl. Math. **44** (1996), no. 1-2, 59-80. MR **98c:**33020

[D] J. Dixmier, *Algebres Enveloppantes*, Cahiers Scientifiques, XXXVII, Gauthier-Villars, Paris (1974). MR **58:**16803a

[GI] A.M. Gavrilik and N.Z. Iorgov, q-deformed inhomogeneous algebras $U_q(so_n)$ and their representations, In: *Symmetry in nonlinear mathematical physics, Vol. 1, 2*, Natl. Acad. Sci. Ukraine, Inst. Math., Kiev (1997), 384-392. MR **99c:**81087

[Ja] N. Jacobson, *Basic Algebra.* II, W. H. Freeman and Co., San Francisco, CA, 1980. MR **81g:**00001

[J] J.C. Jantzen, Kontravariante Formen auf induzierten Darstellungen halbeinfacher Lie-Algebren, *Math. Ann.* **226** (1977), 53-65. MR **55:**12783

[JL1] A. Joseph, and G. Letzter, Local finiteness of the adjoint action for quantized enveloping algebras, *Journal of Algebra*, **153** (1992), 289-318. MR **94b:**17023

[JL2] A. Joseph, and G. Letzter, Separation of variables for quantized enveloping algebras, *American Journal of Mathematics*, **116**, (1994), 127-177. MR **95e:**17017

[Jo] A. Joseph, *Quantum Groups and their Primitive Ideals*, Springer-Verlag, New York (1995). MR **96d:**17015

[Jo2] A. Joseph, On a Harish-Chandra homomorphism, *C. R. Acad. Sci. Paris Sér. I Math.* **324** (1997), no. 7, 759-764. MR **98d:**17017

[Ke] M.S. Kebe, $\mathcal{O}$-algebres quantiques, *C. R. Acad. Sci. Paris Sér. I Math.* **322** (1996), no. 1, 1-4. MR **97c:**17022

[Kn] A.W. Knapp, *Lie groups beyond an introduction*, Progress in Mathematics, Birkhäuser, Boston, MA **140** (1996). MR **98b:**22002

[K1] B. Kostant, On the existence and Irreducibility of Certain Series of Representations. In: *Lie groups an their representations, Summer School Conference, Budapest, 1971*, Halsted press, New York (1975), 231-331. MR **53:**3206

[K2] B. Kostant, Lie group representations on polynomial rings, *American Journal of Mathematics* **85** (1963), 327-404. MR **28:**1252

[L1] G. Letzter, Subalgebras which appear in quantum Iwasawa decompositions, *Canadian Journal of Mathematics* **49** (1997), no. 6, 1206-1223. MR **99g:**17022

[L2] G. Letzter, Symmetric Pairs for Quantized Enveloping algebras *Journal of Algebra,* **220**, (1999), 729-767. CMP 2000:03

[N] M. Noumi, Macdonald's symmetric polynomials as zonal spherical functions on some quantum homogeneous spaces, *Advances in Mathematics* **123** (1996) no. 1, 16-77. MR **98a:**33004

[NS] M. Noumi and T. Sugitani, Quantum symmetric spaces and related q-orthogonal polynomials, *Group Theoretical Methods in Physics (ICGTMP),* (Toyonaka, Japan, 1994) World Sci. Publishing, River Edge, N.J. (1995) 28-40. MR **97h:**33033

[R1] M. Rosso, Groupes Quantiques, Representations Lineaires et Applications, Thesis Paris 7, (1990).

[R2] M. Rosso, Finite-dimensional representations of the quantum analog of the enveloping algebra of a complex simple Lie algebra, Comm. Math. Phys. **117** (1988) 581-593. MR **90c:**17019

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