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LARGE SCHUBERT VARIETIES

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ABSTRACT. For a semisimple adjoint algebraic group G and a Borel subgroup B, consider the double classes BwB in G and their closures in the canonical compactification of G; we call these closures large Schubert varieties. We show that these varieties are normal and Cohen-Macaulay; we describe their Picard group and the spaces of sections of their line bundles. As an application, we construct geometrically a filtration à la van der Kallen of the algebra of regular functions on B. We also construct a degeneration of the flag variety G/B embedded diagonally in $G/B \times G/B$, into a union of Schubert varieties. This yields formulae for the class of the diagonal of $G/B \times G/B$ in T-equivariant K-theory, where T is a maximal torus of B.

INTRODUCTION

Consider an adjoint semisimple algebraic group G and a Borel subgroup B. The Schubert varieties are the images in G/B of the closures in G of double classes BwB. These varieties are generally singular, but all of them are normal and Cohen-Macaulay [25]. The spaces of sections of line bundles over Schubert varieties play an important role in representation theory, see for example [14], [29].

The group G has a canonical smooth $G \times G$-equivariant compactification $\mathbf{X}$, constructed by De Concini and Procesi [9] in characteristic zero, and by Strickland [27] in arbitrary characteristics. In this paper, we study the closures of double classes BwB in $\mathbf{X}$; we call them *large Schubert varieties*.

These varieties are highly singular: by [4, 2.2], their singular locus has codimension two, apart from trivial exceptions. However, we show that large Schubert varieties are normal and Cohen-Macaulay (Corollary 3 and Theorem 20). Further, their Picard group is isomorphic to the weight lattice (Theorem 5).

Large Schubert varieties have an obvious relation to usual Schubert varieties: the latter are quotients by B of an open subset of the former. A more hidden connection arises by intersecting a large Schubert variety X with the unique closed $G \times G$-orbit Y in $\mathbf{X}$. As Y is isomorphic to $G/B \times G/B$ by [27], $X \cap Y$ is a union of Schubert varieties in $G/B \times G/B$.

The space $X \cap Y$ is generally reducible; its irreducible components were described in [4], e.g. those of $\overline{B} \cap Y$ are parametrized by the Weyl group. We show that the scheme-theoretic intersection $X \cap Y$ is reduced and Cohen-Macaulay (Corollaries 4, 21). Together with a construction of [4], this yields a degeneration of the diagonal

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in $G/B \times G/B$ into a union of Schubert varieties, and then formulae for the class of the diagonal in equivariant K-theory (Theorem 16 and Corollary 17).

Let $\tilde{B}$ be the preimage of B in the simply connected cover $\tilde{G}$ of G. Then the space of sections of each line bundle over a large Schubert variety X is a $\tilde{B} \times \tilde{B}$-module, endowed with a natural filtration by order of vanishing of sections along Y. We decompose the associated graded module into a direct sum of spaces of sections of line bundles over $X \cap Y$; the latter $\tilde{B} \times \tilde{B}$-modules are indecomposable (Theorem 7 and Corollary 22). In the case where $X = \overline{B}$, these modules can be seen as degenerations of induced $\tilde{G}$-modules; see Corollary 19.

As a consequence, we obtain in a geometric way a filtration of the affine algebra of $\tilde{B}$ (Theorem 11), similar to those constructed by van der Kallen in [28]. For this, consider regular functions on B as rational functions on its closure $\overline{B}$ with poles along the boundary. The factors of the filtration by order of poles are spaces of sections of line bundles on $\overline{B} \cap Y$. In particular, as a $B \times B$-module, the affine algebra of B, $k[B]$, admits a Schubert filtration in the sense of [24]. Filtering further by ordering the irreducible components of $\overline{B} \cap Y$ gives back a filtration à la van der Kallen of $k[B]$. This generalizes to regular functions over $\tilde{B}$, by decomposing them into sums of sections of line bundles over B.

Our proofs rely on the method of Frobenius splitting: following the approach of [21], we show that $\mathbf{X}$ is Frobenius split compatibly with all large Schubert varieties (Theorem 2). The normality of large Schubert varieties is a direct consequence of this fact: it is easy to see that they are smooth in codimension one, and that their depth is at least two (a regular sequence being provided by the "boundary divisors" of $\mathbf{X}$).

The proof that large Schubert varieties are Cohen-Macaulay is much more involved. As for usual Schubert varieties [25], we proceed by ascending induction on the dimension; but here the argument begins with $\overline{B}$ (instead of the point) which is handled through its intersection with Y. It would be interesting to obtain an equivariant desingularization of $\overline{B}$; then the classical construction of Bott-Samelson-Demazure would give equivariant resolutions of all large Schubert varieties. The present work raises many other questions, e.g. is there a standard monomial theory for large Schubert varieties?

One may also ask for extensions of our results to orbit closures of Borel subgroups in complete symmetric varieties [9], or, more generally, in regular embeddings of spherical homogeneous spaces [4]. In fact, it is shown in [5] that closures of orbits of maximal rank are normal, and that their intersection with any irreducible component of the boundary is reduced; further, these intersections can be described in terms of the Weyl group. But many orbit closures of smaller rank are neither normal, nor Cohen-Macaulay (see [18] for the notion of rank; all large Schubert varieties have maximal rank).

1. The canonical compactification of a semisimple adjoint group

We begin by introducing notation and recalling some properties of group compactifications.

Let G be a connected adjoint semisimple algebraic group over an algebraically closed field k of arbitrary characteristic. Let B and B^- be opposite Borel subgroups of G, with common torus T. Let $\mathcal{X}$ be the character group of T; we identify $\mathcal{X}$ with the character groups of B and B^-. Let W be the Weyl group of T, and let

Φ be the root system of (G,T) with subsets of positive (resp. negative) roots Φ^+, Φ^- defined by B, B^-. Let $\Delta = \{\alpha_1, \dots, \alpha_r\}$ be the set of simple roots, where r is the rank of G; let $s_1, \dots, s_r \in W$ be the simple reflections. The corresponding length function on W is denoted by ℓ. Let w_0 be the element of maximal length in W. Set $N := \ell(w_0)$, the number of positive roots.

We denote by $\tilde{G}$ the simply connected covering of G, and by $\tilde{B}$, $\tilde{T}$, $\dots$ the preimages of B, T, $\dots$ in $\tilde{G}$. The character group of $\tilde{T}$ is denoted by $\tilde{\mathcal{X}}$; it is the weight lattice of Φ with basis the set of fundamental weights $\omega_1, \dots, \omega_r$. The monoid generated by these weights is the set $\tilde{\mathcal{X}}^+$ of dominant weights. Let $\leq$ denote the usual partial order on $\tilde{\mathcal{X}}$ defined by $\lambda \leq \mu$ if there exist non-negative integers $n_1, \dots, n_r$ such that $\mu - \lambda = n_1\alpha_1 + \cdots + n_r\alpha_r$.

By [9] and [27], G admits a compactification $\mathbf{X}$ satisfying the following properties:

(i) $\mathbf{X}$ is a smooth projective variety, and the action of $G \times G$ on G by left and right multiplication extends to $\mathbf{X}$.

(ii) The boundary $\mathbf{X} - G$ is a union of r smooth irreducible divisors $D_1, \dots, D_r$ with normal crossings.

(iii) Each $G\times G$-orbit closure in $\mathbf{X}$ is the transversal intersection of the boundary divisors which contain it.

(iv) The intersection $Y := D_1 \cap \cdots \cap D_r$ is the unique closed $G \times G$-orbit in $\mathbf{X}$; it is isomorphic to $G/B \times G/B$.

Further, any compactification of G satisfying (i), (ii) and (iii) dominates $\mathbf{X}$, and any normal compactification of G with a unique closed orbit is dominated by $\mathbf{X}$ (this follows from embedding theory of homogeneous spaces; see [17]). We will call $\mathbf{X}$ the *canonical compactification* of G.

Note that in the above G is regarded as the homogeneous space $(G\times G)/\mathrm{diag}(G)$, where $\mathrm{diag}(G)$ denotes the diagonal subgroup, which is the fixed subgroup of the involution $(g_1, g_2) \mapsto (g_2, g_1)$ of $G \times G$. Thus, the compactification $\mathbf{X}$ is a special case of the compactification of symmetric spaces studied in [9] (for $k = \mathbb{C}$) and in [10] (for $char(k) \neq 2$). In fact, for those results from [10] that we shall need, it is easily seen that in this special case the hypothesis that $char(k) \neq 2$ is unnecessary.

For $w \in W$, consider the double class BwB in G, and its closure in $\mathbf{X}$. We denote this closure by $\mathbf{X}(w)$, and we call it a *large Schubert variety*. On the other hand, we denote by $S(w)$ the usual Schubert variety, that is, the closure in G/B of BwB/B. In other words, $S(w)$ is the image in G/B of the intersection $\mathbf{X}(w) \cap G$.

The intersections of large Schubert varieties with $G \times G$-orbits were studied in [4, §2]. In particular, we have the following decomposition of

$$Z(w) := \mathbf{X}(w) \cap Y$$

into irreducible components (which are Schubert varieties in $G/B \times G/B$):

$$Z(w) = \bigcup_{\substack{x \in W \\ \ell(wx) = \ell(w) + \ell(x)}} S(wx) \times S(xw_0).$$

For $w = 1$ (the identity of W), we denote $\mathbf{X}(w)$ by $\overline{B}$, and $Z(w)$ by Z. Then

$$Z = \bigcup_{x \in W} S(x) \times S(xw_0).$$

The large Schubert varieties of codimension one in $\mathbf{X}$ are $\mathbf{X}(w_0 s_1), \dots, \mathbf{X}(w_0 s_r)$. They are the irreducible $B \times B$-stable divisors in $\mathbf{X}$ which are not $G \times G$-stable, or, equivalently, which do not contain Y. By [10, Proposition 4.4], the divisor class group of $\mathbf{X}$ is freely generated by the classes of $\mathbf{X}(w_0s_1), \dots, \mathbf{X}(w_0s_r)$. On the other hand, the line bundles on $\mathbf{X}$ are described in [27, §2] (see also [10, §4]). We now recall this description; a generalization to large Schubert varieties will be obtained in Section 3.

For $\lambda, \mu \in \tilde{\mathcal{X}}$, we denote by $\mathcal{L}_{G/B}(\lambda)$ the $\tilde{G}$-linearized line bundle on G/B whose geometric fiber at the fixed point of B is the one-dimensional representation of $\tilde{B}$ corresponding to the character $-\lambda$, and by $\mathcal{L}_Y(\lambda,\mu)$, or simply $\mathcal{L}(\lambda,\mu)$, the line bundle $\mathcal{L}_{G/B}(\lambda) \boxtimes \mathcal{L}_{G/B}(\mu)$ on $Y = G/B \times G/B$. (With this convention, $H^0(G/B, \mathcal{L}_{G/B}(\lambda))$ is non-zero if and only if λ is dominant; this is the convention used in [27], it differs by a sign from the one used in [14].)

Then, the map $(\lambda,\mu) \mapsto \mathcal{L}_Y(\lambda,\mu)$ identifies the Picard group $\operatorname{Pic} Y$ with $\tilde{\mathcal{X}} \times \tilde{\mathcal{X}}$. Now the restriction $\operatorname{res}_Y : \operatorname{Pic} \mathbf{X} \to \operatorname{Pic} Y$ is injective, and its image consists of the $\mathcal{L}_Y(\lambda, -w_0\lambda)$, for $\lambda \in \tilde{\mathcal{X}}$.

We denote by $\mathcal{L}_{\mathbf{X}}(\lambda)$ the line bundle on $\mathbf{X}$ such that $\operatorname{res}_Y \mathcal{L}_{\mathbf{X}}(\lambda) = \mathcal{L}_Y(\lambda, -w_0\lambda)$. This identifies $\operatorname{Pic} \mathbf{X}$ with $\tilde{\mathcal{X}}$; we can index the boundary divisors $D_1, \dots, D_r$ so that the classes of the corresponding line bundles are $\mathcal{L}_{\mathbf{X}}(\alpha_1), \dots, \mathcal{L}_{\mathbf{X}}(\alpha_r)$. Then each $\mathcal{L}_{\mathbf{X}}(\alpha_i) = \mathcal{O}_{\mathbf{X}}(D_i)$ has a section σ_i with divisor D_i; this section is unique up to scalar multiplication.

Because $\tilde{G}$ is semisimple and simply connected, each line bundle $\mathcal{L}_{\mathbf{X}}(\lambda)$ has a unique $\tilde{G} \times \tilde{G}$-linearization. (See, for example, the collective volume [20]: since $k[\tilde{G}]$ is factorial (Prop. 4.6, p. 74), existence follows from the Remark on p. 67, and uniqueness from Prop. 2.3, p. 81.) Thus, each space $H^0(\mathbf{X}, \mathcal{L}_{\mathbf{X}}(\lambda))$ is a $\tilde{G} \times \tilde{G}$-module, which we denote by $H^0(\mathbf{X}, \lambda)$. Similarly, we denote $H^0(Y, \mathcal{L}_Y(\lambda,\mu))$ by $H^0(Y, \lambda, \mu)$. This $\tilde{G} \times \tilde{G}$-module is isomorphic to $H^0(G/B, \lambda) \boxtimes H^0(G/B, \mu)$.

Observe that the section σ_i of $\mathcal{L}_{\mathbf{X}}(\alpha_i)$ is $\tilde{G} \times \tilde{G}$-invariant. This is the starting point for an analysis of the $\tilde{G} \times \tilde{G}$-module $H^0(\mathbf{X}, \lambda)$ for arbitrary λ, see [27, §2]; the results will be generalized to large Schubert varieties in Section 3. Here we will need the following

Lemma 1. *Let λ be a dominant weight. The line bundle $\mathcal{L}_{\mathbf{X}}(\lambda)$ has a global section τ_λ, eigenvector of $\tilde{B} \times \tilde{B}$ of weight $(-w_0\lambda, \lambda)$. This section is unique up to scalar, and its divisor is*

$$\sum_{i=1}^{r} \langle \lambda, \alpha_i^\vee \rangle \mathbf{X}(w_0 s_i).$$

As a consequence, $\mathcal{L}_{\mathbf{X}}(\lambda)$ is generated by its global sections. Moreover, if λ is regular then $\mathcal{L}_{\mathbf{X}}(\lambda)$ is ample.

Proof. By [10, §4], the line bundle on $\mathbf{X}$ associated with the divisor $\mathbf{X}(w_0s_i)$ is $\mathcal{L}_{\mathbf{X}}(\omega_i)$. Let τ_i be the canonical section of this line bundle; then τ_i is an eigenvector of $\tilde{B} \times \tilde{B}$, because its divisor $\mathbf{X}(w_0s_i)$ is $B \times B$-stable. The closure in $\tilde{G}$ of $\tilde{B} w_0 s_i \tilde{B}$ is the divisor of a regular function on $\tilde{G}$, eigenvector of $\tilde{B} \times \tilde{B}$ of weight $(-w_0\omega_i, \omega_i)$, and unique up to scalar multiplication. Thus, the weight of τ_i is $(-w_0\omega_i, \omega_i)$. As the $\mathcal{L}_{\mathbf{X}}(\omega_i)$ generate the Picard group of $\mathbf{X}$, the existence of τ_λ and the formula for its divisor follow immediately. Finally, uniqueness of τ_λ up to scalar is a consequence of the fact that $B \times B$ has a dense orbit in $\mathbf{X}$. This proves the first assertion.

The second one follows immediately. Since τ_λ does not vanish identically on Y, the $\tilde{G} \times \tilde{G}$-translates of τ_λ generate $\mathcal{L}_{\mathbf{X}}(\lambda)$.

If, moreover, λ is regular, then $\mathcal{L}_{\mathbf{X}}(\lambda)$ is ample by [27, §2]. $\square$

2. Compatible Frobenius splitting and applications

In the beginning of this section, we assume that k has characteristic $p > 0$. For a scheme X over k, we denote by $F : X \to X$ the absolute Frobenius morphism. Recall that X is *Frobenius split* if the map $F : \mathcal{O}_X \to F_*\mathcal{O}_X$ is split, that is, if there exists $\sigma \in \mathrm{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)$ such that $\sigma \circ F$ is the identity. Let $Y \subseteq X$ be a closed subscheme with ideal sheaf $\mathcal{I}_Y$; then a splitting σ is *compatible with* Y if $\sigma(F_*\mathcal{I}_Y)$ is contained in $\mathcal{I}_Y$.

By [27, §3], the canonical compactification $\mathbf{X}$ is Frobenius split compatibly with all $G \times G$-orbit closures. We will need the following refinement of this result.

Theorem 2. *$\mathbf{X}$ is Frobenius split compatibly with all $G \times G$-orbit closures and all subvarieties $\mathbf{X}(w)$ and $(w_0, w_0)\mathbf{X}(w)$, for $w \in W$.*

Proof. Let $St = H^0(G/B, (p-1)\rho)$ be the Steinberg module for $\tilde{G}$; it is a simple, self-dual $\tilde{G}$-module [14, II.2.5, II.3.18]. On the other hand, the line bundle $\mathcal{L}_{\mathbf{X}}((p-1)\rho)$ is $\tilde{G}\times\tilde{G}$-linearized by construction of $\mathbf{X}$, and the $\tilde{G}\times\tilde{G}$-module $H^0(\mathbf{X}, (p-1)\rho)$ contains an eigenvector of $\tilde{B} \times \tilde{B}$ of weight $(p-1)(\rho, \rho)$, unique up to scalar, by Lemma 1. Further, the image of this eigenvector under restriction to Y is non-zero, since no $\mathbf{X}(w_0 s_i)$ contains Y. Using Frobenius reciprocity [14, I.3.4] and self-duality of St, we obtain a $\tilde{G} \times \tilde{G}$-homomorphism

$$f : St \boxtimes St \to H^0(\mathbf{X}, (p-1)\rho)$$

such that the composition

$$\mathrm{res}_Y \circ f : St \boxtimes St \to H^0(Y, (p-1)(\rho, \rho))$$

is non-zero. Since the $\tilde{G} \times \tilde{G}$-module $H^0(Y, (p-1)(\rho, \rho))$ is isomorphic to $St \boxtimes St$, hence simple, it follows that $\mathrm{res}_Y \circ f$ is an isomorphism.

We thus obtain a $\tilde{G} \times \tilde{G}$-homomorphism

$$f^2 : (St \boxtimes St)^{\otimes 2} \to H^0(\mathbf{X}, 2(p-1)\rho),$$

$$x_1 \boxtimes y_1 \otimes x_2 \boxtimes y_2 \mapsto f(x_1 \boxtimes y_1) f(x_2 \boxtimes y_2).$$

Moreover, the composition

$$\mathrm{res}_Y \circ f^2 : (St \boxtimes St)^{\otimes 2} \to H^0(Y, 2(p-1)(\rho, \rho))$$

is surjective, because the product map

$$H^0(Y, (p-1)(\rho, \rho))^{\otimes 2} \to H^0(Y, 2(p-1)(\rho, \rho))$$

is [14, II.14.20]. Now, by [21, 2.1,2.3], there is a natural $\tilde{G} \times \tilde{G}$-isomorphism

$$\mathrm{Hom}_{\mathcal{O}_Y}(F_*\mathcal{O}_Y, \mathcal{O}_Y) \xrightarrow{\cong} H^0(Y, 2(p-1)(\rho, \rho))$$

and there is a unique $\tilde{G} \times \tilde{G}$-homomorphism (up to a constant)

$$\varphi : (St \boxtimes St)^{\otimes 2} \to \mathrm{Hom}_{\mathcal{O}_Y}(F_*\mathcal{O}_Y, \mathcal{O}_Y).$$

Further, for a and b in $St \boxtimes St$, the map $\varphi(a \otimes b)$ is a splitting of Y (up to a constant) if and only if $\langle a, b \rangle \neq 0$ where $\langle\ ,\ \rangle$ is the $\tilde{G} \times \tilde{G}$-invariant bilinear form on $St \boxtimes St$. Finally, if $a = s^{p-1}$ and $b = t^{p-1}$ for sections s, t of $\mathcal{L}_Y(\rho, \rho)$, then the zero subschemes $Z(s)$, $Z(t)$ in Y are compatibly $\varphi(a \otimes b)$-split.

Because $\mathrm{res}_Y \circ f^2$ is a surjective $\tilde{G} \times \tilde{G}$-homomorphism, we can identify it with φ. Let v_+ (resp. v_-) be a highest (resp. lowest) weight vector in $H^0(G/B, \rho)$. Set

$s := v_+ \boxtimes v_+$, $t := v_- \boxtimes v_-$, $a := s^{p-1}$ and $b := t^{p-1}$. Then a, b are in $St \boxtimes St$ and they satisfy $\langle a, b\rangle \neq 0$. Thus, $\mathrm{res}_Y \circ f^2(a \otimes b)$ splits Y compatibly with $Z(s)$ and $Z(t)$.

Set $\tau := \varphi(a \otimes b)$ and consider

$$\sigma := \tau \prod_{i=1}^{r} \sigma_i^{p-1},$$

a global section of $\mathcal{L}_{\mathbf{X}}((p-1)(2\rho + \sum_{i=1}^{r} \alpha_i))$. Recall from [27, §3] that the dualizing sheaf of $\mathbf{X}$ is

$$\omega_{\mathbf{X}} = \mathcal{L}_{\mathbf{X}}(-2\rho - \sum_{i=1}^{r} \alpha_i).$$

Thus, $\sigma \in H^0(\mathbf{X}, \omega_{\mathbf{X}}^{1-p}) \cong \mathrm{Hom}_{\mathcal{O}_{\mathbf{X}}}(F_* \mathcal{O}_{\mathbf{X}}, \mathcal{O}_{\mathbf{X}})$. By [27, Th. 3.1], σ splits $\mathbf{X}$ compatibly with $D_1, \ldots, D_r$.

Set $\tau_+ = f(v_+^{p-1} \boxtimes v_+^{p-1})$ and $\tau_- = f(v_-^{p-1} \boxtimes v_-^{p-1})$. Then τ_+ and τ_- are in $H^0(\mathbf{X}, (p-1)\rho)$, and τ_+ (resp. τ_-) is an eigenvector of $\tilde{B} \times \tilde{B}$ (resp. $\tilde{B}^- \times \tilde{B}^-$) of weight $(p-1)(\rho, \rho)$ (resp. $-(p-1)(\rho, \rho)$). By Lemma 1, we have

$$\mathrm{div}(\tau_\pm) = (p-1)\mathbf{X}_\pm$$

where $\mathbf{X}_+$ is the sum of the classes of the $\mathbf{X}(w_0 s_i)$ over all simple reflections s_i, and $\mathbf{X}_- = (w_0, w_0)\mathbf{X}_+$. Thus, σ splits $\mathbf{X}$ compatibly with $\mathbf{X}_+$ and $\mathbf{X}_-$. This implies the theorem, as in the proof of [25, Th. 3.5(i)]. Namely, one uses [25, Lemma 1.11] and the fact that each $\mathbf{X}(w)$ is obtained from $\mathbf{X}_+$ by iterating the process of taking irreducible components and intersections. □

Corollary 3. *Let $char(k)$ be arbitrary.*

(i) *For any dominant weight λ and for any intersection X of large Schubert varieties and of boundary divisors, the restriction map*

$$\mathrm{res}_X : H^0(\mathbf{X}, \lambda) \to H^0(X, \lambda)$$

is surjective. Further, $H^i(X, \lambda) = 0$ for $i \geq 1$.

(ii) *Any intersection of large Schubert varieties and of boundary divisors is reduced.*

Proof. Let us prove (i). Note first that, by Lemma 1, $\mathbf{X}_+$ is ample (the associated line bundle is $\mathcal{L}_{\mathbf{X}}(\rho)$) and $\mathcal{L}_{\mathbf{X}}(\lambda)$ is generated by global sections. Therefore, (i) is a consequence of [26, Proposition 1.13(ii)] when $char(k) = p > 0$. Moreover, since G, B are defined over $\mathbb{Z}$, it follows from the construction of $\mathbf{X}$ ([27]) that $\mathbf{X}$, the boundary divisors D_i and the large Schubert varieties $\mathbf{X}(w)$ are all defined and flat over some open subset of Spec $\mathbb{Z}$ (in fact, they are defined over $\mathbb{Z}$ by [10]). Therefore, by the semicontinuity theorem, (i) holds in characteristic zero as well.

Moreover, by the proof of [25, Th. 3], (ii) follows (in arbitrary characteristic) from Theorem 2. □

For $1 \leq i \leq r$, multiplication by σ_i (a section of $\mathcal{L}_{\mathbf{X}}(\alpha_i)$ with divisor D_i) defines an exact sequence

$$0 \to \mathcal{O}_{\mathbf{X}}(-D_i) \to \mathcal{O}_{\mathbf{X}} \to \mathcal{O}_{D_i} \to 0.$$

Because Y is the transversal intersection of $D_1, \dots, D_r$, the image of the map

$$(\sigma_1, \dots, \sigma_r) : \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbf{X}}(-D_i) \to \mathcal{O}_{\mathbf{X}}$$

is the ideal sheaf $\mathcal{I}_Y$.

Corollary 4. *Again, let char(k) be arbitrary.*

(i) *The ideal sheaf of the set-theoretic intersection $Z(w) = \mathbf{X}(w) \cap Y$ in $\mathbf{X}(w)$ is generated by the image of $(\sigma_1, \dots, \sigma_r)$.*

(ii) *$\sigma_1, \dots, \sigma_r$ form a regular sequence in $\mathbf{X}(w)$.*

(iii) *$\mathbf{X}(w)$ is normal.*

Proof. By Corollary 3, the scheme-theoretic intersection $\mathbf{X}(w) \cap Y$ is reduced; this is equivalent to (i).

For (ii), we have to check that the image in $\mathcal{O}_{\mathbf{X}(w)}/(\sigma_1, \dots, \sigma_{j-1})$ of σ_j is not a zero divisor, for each j. But the scheme-theoretic intersection

$$\mathbf{X}(w)_{<j} := \mathbf{X}(w) \cap \bigcap_{i=1}^{j-1} D_i$$

is reduced. Further, by [4, Th. 2.1], each irreducible component of $\mathbf{X}(w)_{<j}$ has codimension $j-1$ in $\mathbf{X}(w)$, and is not contained in D_j. Thus, the restriction of σ_j to $\mathbf{X}(w)_{<j}$ does not vanish identically on any such component. It follows that σ_j is not a zero divisor in $\mathcal{O}_{\mathbf{X}(w)_{<j}} = \mathcal{O}_{\mathbf{X}(w)}/(\sigma_1, \dots, \sigma_{j-1})$.

For (iii), observe that $\mathbf{X}(w) \cap G$ is smooth in codimension one, as the preimage in G of a Schubert variety in G/B (this goes back to Chevalley [7, Cor., p. 10].) Further, the intersection $\mathbf{X}(w) \cap D_i$ is reduced for $1 \le i \le r$, so that each irreducible component of this intersection contains smooth points of $\mathbf{X}(w)$. Thus, $\mathbf{X}(w)$ is smooth in codimension one (this also follows from [4, Cor. 2.1]). By Serre's criterion, it is enough to prove that $\mathbf{X}(w)$ has depth at least two.

Because $B \times B$ acts on $\mathbf{X}(w)$ with finitely many orbits and a unique fixed point y (the base point of $Y = G/B \times G/B$), it suffices to prove that $\mathbf{X}(w)$ has depth at least two at y. This is clear if $r \ge 2$, because local equations of $D_1, \dots, D_r$ at y form a regular sequence in the local ring $\mathcal{O}_{\mathbf{X}(w),y}$. On the other hand, if $r = 1$, then each $\mathbf{X}(w)$ is smooth. We have indeed $\tilde{G} = \mathrm{SL}(2)$, $G = \mathrm{PGL}(2)$, and $\mathbf{X}$ is the projectivization of the space of 2×2 matrices where G acts by left and right multiplication. So $\mathbf{X}(w)$ is either $\mathbf{X}$ or the projectivization of the subspace of upper triangular matrices. □

3. Line bundles on large Schubert varieties

In this section, we describe the Picard group of large Schubert varieties, and the spaces of global sections of line bundles on these varieties.

Theorem 5. *For any $w \in W$, the restriction map*

$$\mathrm{res}_{\mathbf{X}(w)} : \mathrm{Pic}\, \mathbf{X} \to \mathrm{Pic}\, \mathbf{X}(w)$$

is bijective. Further, the line bundle $\mathcal{L}_{\mathbf{X}(w)}(\lambda)$ is generated by its global sections (resp. ample) if and only if λ is dominant (resp. dominant regular).

Remarks. 1) We will see in Corollary 9 that $\mathcal{L}_{\mathbf{X}(w)}(\lambda)$ admits nontrivial global sections if and only if λ is in the monoid generated by all simple roots and fundamental weights.

2) It is proved in [27, §2] that $\mathcal{L}_{\mathbf{X}}(2\lambda)$ is very ample for any regular dominant weight λ. In fact, one can check that $\mathcal{L}_{\mathbf{X}}(\lambda)$ is already very ample, using Corollary 3.

Proof. We will use the duality between line bundles and curves: each closed curve C in $\mathbf{X}(w)$ defines an additive map $\operatorname{Pic}\mathbf{X}(w) \to \mathbb{Z}, L \mapsto (L \cdot C)$ where $(L \cdot C)$ is the degree of the restriction of L to C. In fact, $(L \cdot C)$ only depends on the classes of L and C up to rational equivalence. Further, C is rationally equivalent to a positive integral combination of closed irreducible $B \times B$-stable curves [13].

Examples of such curves are the "Schubert curves" $C(\alpha_i) := S(s_i) \times S(1)$ and $C'(\alpha_i) := S(1) \times S(s_i)$ in $G/B \times G/B$. Note that

$$(\mathcal{L}_{\mathbf{X}(w)}(\lambda) \cdot C(\alpha_i)) = (\mathcal{L}_{\mathbf{X}(w)}(\lambda) \cdot C'(-w_0\alpha_i)) = \langle \lambda, \alpha_i^\vee \rangle$$

for all $\lambda \in \tilde{\mathcal{X}}$. We first show the following

Lemma 6. *The closed irreducible $B \times B$-stable curves in $\mathbf{X}$ are the $C(\alpha_i)$ and $C'(\alpha_i)$ for $1 \le i \le r$. They are contained in $\overline{B}$. Further, each $C(\alpha_i)$ is rationally equivalent in $\overline{B}$ to $C'(-w_0\alpha_i)$.*

Proof. The first assertion follows from the description of all $B \times B$-orbits in $\mathbf{X}$ given in [4, 2.1]. And as $\overline{B} \cap Y$ contains both $S(w_0) \times S(1)$ and $S(1) \times S(w_0)$, it contains the $C(\alpha_i)$ and $C'(\alpha_i)$.

For the latter assertion, we begin by the case where $G = \mathrm{PGL}(2)$. Then we saw that $\mathbf{X} = \mathbb{P}^3$ and $\overline{B} = \mathbb{P}^2$. Further, Y is a smooth quadric in $\mathbb{P}^3$, and both $C(\alpha)$ and $C'(-w_0\alpha)$ are embedded lines. Thus, they are rationally equivalent in $\mathbb{P}^2$.

The general case reduces to the previous one, as follows. Set $X_i := \bigcap_{j \ne i} D_j$, then X_i is the closure of a unique $G \times G$-orbit X_i^0 in $\mathbf{X}$. Let P_i be the parabolic subgroup generated by B and s_i; let Q_i be the opposite parabolic subgroup containing B^-, and let L_i be their common Levi subgroup. Then the $G \times G$-variety X_i fibers equivariantly over $G/P_i \times G/Q_i$, with fiber the canonical compactification of the adjoint group $L_i/Z(L_i)$ (this follows e.g. from [10, Th. 3.16]). This group is isomorphic to $\mathrm{PGL}(2)$. Set $P_j := w_0 Q_i w_0$, the parabolic subgroup generated by B and $w_0 s_i w_0$. Now X_i fibers equivariantly over $G/P_i \times G/P_j$ and the fiber over the base point is a closed $B \times B$-stable subvariety F_i of X_i, isomorphic to $\mathbb{P}^3$. Restricting this fibration to $Y \subset X_i$, we obtain the canonical map $G/B \times G/B \to G/P_i \times G/P_j$. Thus, F_i contains both $C(\alpha_i) = P_i/B \times B/B$ and $C'(-w_0\alpha_i) = B/B \times P_j/B$. Further, $B \times B$ has a unique closed orbit $\mathcal{O}_i$ in X_i^0; and $\mathcal{O}_i$ is contained in $F_i \cap \overline{B}$ (because $\overline{B}$ meets all $G \times G$-orbits). Thus, the closure of $\mathcal{O}_i$ in X_i is isomorphic to $\mathbb{P}^2$, and contains both $C(\alpha_i)$ and $C'(-w_0\alpha_i)$ as embedded lines. □

We return to the proof of Theorem 5. For injectivity, let λ be a weight such that the restriction of $\mathcal{L}_{\mathbf{X}}(\lambda)$ to $\mathbf{X}(w)$ is trivial. Then the restriction of $\mathcal{L}_{\mathbf{X}}(\lambda)$ to each $C(\alpha_i)$ is trivial. It follows that $\langle \lambda, \alpha_i^\vee \rangle = 0$ for $1 \le i \le r$, and that $\lambda = 0$.

For surjectivity, we first prove that the abelian group $\operatorname{Pic}\mathbf{X}(w)$ is free of finite rank. For this, we identify $\operatorname{Pic}\mathbf{X}(w)$ to the group of all Cartier divisors on $\mathbf{X}(w)$ up to rational equivalence (this holds because $\mathbf{X}(w)$ is normal). Let y be the $B \times B$-fixed point of Y. Let $\mathbf{X}_y$ be the set of all $x \in \mathbf{X}$ such that the orbit closure $\overline{(T \times T)x}$ contains y. Then $\mathbf{X}_y$ is an open affine $T \times T$-stable subset of $\mathbf{X}$, containing y as its unique closed $T \times T$-orbit (it is the image under $(1, w_0)$ of the affine chart $\mathcal{V}$ defined in [27, §2]). Because $y \in \mathbf{X}(w)$, the intersection $\mathbf{X}(w)_y := \mathbf{X}(w) \cap \mathbf{X}_y$ is a non-empty open affine $T \times T$-stable subset of $\mathbf{X}(w)$, containing y as its unique closed $T \times T$-orbit. By the equivariant Nakayama lemma [1, §6], it follows that any $T \times T$-linearizable line bundle over $\mathbf{X}(w)_y$ is trivial. But since $\mathbf{X}(w)_y$ is normal, any line bundle over $\mathbf{X}(w)_y$ is $T \times T$-linearizable [20, p. 67]. Thus, the Picard group of $\mathbf{X}(w)_y$ is trivial. On the other hand, any regular invertible function on $\mathbf{X}(w)_y$ is constant (for it is an eigenvector of $T \times T$ by [20, Prop. 1.3 (ii), p. 79], and all weights of non-constant regular functions on $\mathbf{X}(w)_y$ are contained in an open half space). Therefore, any Cartier divisor on $\mathbf{X}(w)$ is rationally equivalent to a unique Cartier divisor with support in the complement $\mathbf{X}(w) \setminus \mathbf{X}(w)_y$. Now the abelian group of Weil divisors with support in $\mathbf{X}(w) \setminus \mathbf{X}(w)_y$ is free of finite rank.

We now prove that any Cartier divisor D on $\mathbf{X}(w)$ which is numerically equivalent to zero (that is, $(D \cdot C) = 0$ for each closed curve C in $\mathbf{X}(w)$) is rationally equivalent to zero. Indeed, by [12, 19.3.3], there exists a positive integer m such that mD is algebraically equivalent to zero. But algebraic and rational equivalence coincide for Cartier divisors on $\mathbf{X}(w)$, by freeness of Pic $\mathbf{X}(w)$ and [12, 19.1.2]. Thus, the class of mD in Pic $\mathbf{X}(w)$ is zero, and we conclude by freeness of Pic $\mathbf{X}(w)$ again.

For a line bundle L on $\mathbf{X}(w)$, define a weight λ by

$$\lambda := (L \cdot C(\alpha_1))\omega_1 + \cdots + (L \cdot C(\alpha_r))\omega_r,$$

so that $(L \cdot C(\alpha_i)) = (\mathcal{L}_{\mathbf{X}}(\lambda) \cdot C(\alpha_i))$ for $1 \le i \le r$. By Lemma 6, it follows that $(L \cdot C) = (\mathcal{L}_{\mathbf{X}}(\lambda) \cdot C)$ for all closed curves C in $\mathbf{X}(w)$. By the previous step, L is isomorphic to $\operatorname{res}_{\mathbf{X}(w)} \mathcal{L}_{\mathbf{X}}(\lambda)$. This proves that $\operatorname{res}_{\mathbf{X}(w)}$ is bijective.

Finally, for the two remaining assertions of Theorem 5, the sufficiency was established in Lemma 1. Conversely, if $L = \mathcal{L}_{\mathbf{X}}(\lambda)$ is a line bundle on $\mathbf{X}$ such that $\operatorname{res}_{\mathbf{X}(w)}(L)$ is generated by its global sections (resp. ample), then $(L \cdot C) \ge 0$ (resp. > 0) for any closed curve C in $\mathbf{X}(w)$. Applying this to the curves $C(\alpha_i)$, one obtains that λ is dominant (resp. dominant regular). □

For any weight λ, the space

$$H^0(\mathbf{X}(w), \lambda) := H^0(\mathbf{X}(w), \mathcal{L}_{\mathbf{X}(w)}(\lambda))$$

is a finite-dimensional $\tilde{B} \times \tilde{B}$-module. Its $\tilde{B} \times \tilde{B}$-submodules

$$F_n H^0(\mathbf{X}(w), \lambda) := H^0(\mathbf{X}(w), \mathcal{L}_{\mathbf{X}}(\lambda) \otimes \mathcal{I}^n_{Z(w)})$$

(where $n \in \mathbb{N}$) form a decreasing filtration, which we call the *canonical filtration*.

Since $\bigcap_{n \ge 0} F_n H^0(\mathbf{X}(w), \lambda) = 0$ and since $H^0(\mathbf{X}(w), \lambda)$ is finite dimensional, this filtration is finite, that is, there exists an integer $n_0(\lambda)$ such that $F_n H^0(\mathbf{X}(w), \lambda) = 0$ for $n > n_0(\lambda)$.

Let $\mathbf{n} = (n_1, \dots, n_r) \in \mathbb{N}^r$ and let $|\mathbf{n}| = n_1 + \cdots + n_r$. Then multiplication by the section $\sigma_1^{n_1} \cdots \sigma_r^{n_r}$ defines a map

$$\sigma^{\mathbf{n}} : H^0(\mathbf{X}(w), \lambda - n_1\alpha_1 - \cdots - n_r\alpha_r) \to H^0(\mathbf{X}(w), \lambda).$$

Because each σ_i is $\tilde{G} \times \tilde{G}$-invariant and nonidentically zero on $\mathbf{X}(w)$, this map is injective and $\tilde{B} \times \tilde{B}$-equivariant. Let $F_{\mathbf{n}} H^0(\mathbf{X}(w), \lambda)$ be the image of $\sigma^{\mathbf{n}}$; it is a $\tilde{B} \times \tilde{B}$-submodule of $F_n H^0(\mathbf{X}(w), \lambda)$, where $n = |\mathbf{n}|$.

Theorem 7. *With notation as above, we have*

$$F_n H^0(\mathbf{X}(w),\lambda) = \sum_{\substack{\mathbf{n}\in\mathbb{N}^r \\ |\mathbf{n}|=n}} F_{\mathbf{n}} H^0(\mathbf{X}(w),\lambda)$$

Further, the n-th layer of the associated graded module satisfies

$$gr_n H^0(\mathbf{X}(w),\lambda) = \bigoplus_{(n_1,\dots,n_r)} H^0(Z(w),\lambda - n_1\alpha_1 - \cdots - n_r\alpha_r),$$

the sum being taken over all $(n_1,\dots,n_r)\in\mathbb{N}^r$ such that $n_1+\cdots+n_r=n$ and that $\lambda - n_1\alpha_1 - \cdots - n_r\alpha_r$ is dominant.

In particular,

$$gr H^0(\mathbf{X}(w),\lambda) = \bigoplus_{\substack{\mu\in\tilde{\mathcal{X}}^+ \\ \mu\le\lambda}} H^0(Z(w),\mu).$$

Proof. Set $\mathcal{I} = \mathcal{I}_{Z(w)}$. From the exact sequence of sheaves on $\mathbf{X}(w)$,

$$0 \to \mathcal{I}^{n+1}\otimes\mathcal{L}_{\mathbf{X}(w)}(\lambda) \to \mathcal{I}^n\otimes\mathcal{L}_{\mathbf{X}(w)}(\lambda) \to \mathcal{I}^n/\mathcal{I}^{n+1}\otimes\mathcal{L}_{\mathbf{X}(w)}(\lambda) \to 0,$$

we see that $gr_n H^0(\mathbf{X}(w),\lambda)$ injects into $H^0(Z(w),\mathcal{I}^n/\mathcal{I}^{n+1}\otimes\mathcal{L}_{\mathbf{X}(w)}(\lambda))$. The latter is equal to

$$\bigoplus_{n_1+\cdots+n_r=n} H^0(Z(w),\lambda - n_1\alpha_1 - \cdots - n_r\alpha_r).$$

We have indeed

$$\mathcal{I}^n/\mathcal{I}^{n+1} = \bigoplus_{n_1+\cdots+n_r=n} \sigma_1^{n_1}\cdots\sigma_r^{n_r}\mathcal{L}_{Z(w)}(-n_1\alpha_1 - \cdots - n_r\alpha_r)$$

because $\mathcal{I}$ is generated by the regular sequence $(\sigma_1,\dots,\sigma_r)$ (Corollary 4). We now need the following

Lemma 8. *For a weight μ, the following conditions are equivalent:*
(i) *μ is dominant.*
(ii) *$H^0(Z(w),\mu)$ is nonzero.*

Proof. (i)⇒(ii). If μ is dominant, then the restriction to Y of $\mathcal{L}_{\mathbf{X}}(\mu)$ is generated by its global sections.

(ii)⇒(i). Recall that

$$Z(w) = \bigcup_{\substack{x\in W \\ \ell(wx)=\ell(w)+\ell(x)}} S(wx)\times S(xw_0),$$

and that the restriction of $\mathcal{L}_{\mathbf{X}}(\mu)$ to Y is equal to $\mathcal{L}_Y(\mu,-w_0\mu)$. Thus, there exists $x\in W$ such that $H^0(S(wx),\mu)$ and $H^0(S(xw_0),-w_0\mu)$ are both nonzero. But $H^0(S(wx),\mu)\neq 0$ implies that $\langle\mu,\check{\alpha}\rangle\ge 0$ for each $\alpha\in\Delta$ such that $wx\alpha\in\Phi^-$; this follows from [24, Cor. 2.3]; see also [8]. Similarly, $H^0(S(xw_0),-w_0\mu)\neq 0$ implies that $\langle -w_0\mu,\check{\beta}\rangle\ge 0$ for each $\beta\in\Delta$ such that $xw_0\beta\in\Phi^-$. Since $-w_0$ permutes the simple roots, the latter is equivalent to $\langle\mu,\check{\alpha}\rangle\ge 0$ for each $\alpha\in\Delta$ such that $x\alpha\in\Phi^+$. Now, for each $\alpha\in\Delta$, we have either $x\alpha\in\Phi^+$ or $wx\alpha\in\Phi^-$, because $\ell(wx)=\ell(w)+\ell(x)$. □

Returning to the proof of Theorem 7, let $\mu = \lambda - n_1\alpha_1 - \cdots - n_r\alpha_r$ such that the space $H^0(Z(w),\mu)$ is nonzero. Then μ is dominant by Lemma 8. By Corollary 3, the restriction

$$H^0(\mathbf{X}(w),\mu) \to H^0(Z(w),\mu)$$

is surjective; therefore, the restriction

$$F_{\mathbf{n}}H^0(\mathbf{X}(w),\lambda) \to H^0(Z(w),\lambda - n_1\alpha_1 - \cdots - n_r\alpha_r)$$

is surjective. It follows that, first,

$$gr_n H^0(\mathbf{X}(w),\lambda) \cong \bigoplus_{(n_1,\dots,n_r)} H^0(Z(w),\lambda - n_1\alpha_1 - \cdots - n_r\alpha_r),$$

where the sum is taken over all $(n_1,\dots,n_r) \in \mathbb{N}^r$ such that $n_1 + \cdots + n_r = n$ and that $\lambda - n_1\alpha_1 - \cdots - n_r\alpha_r$ is dominant, and, second, that

$$F_n H^0(\mathbf{X}(w),\lambda) = \sum_{\substack{\mathbf{n}\in\mathbb{N}^r \\ |\mathbf{n}|=n}} F_{\mathbf{n}}H^0(\mathbf{X}(w),\lambda) + F_{n+1}H^0(\mathbf{X}(w),\lambda).$$

Let $t \geq 1$. Replacing in the above formula n by $n+1,\dots,n+t$, and using the fact that $F_{\mathbf{n}'}H^0(\mathbf{X}(w),\lambda) \subseteq F_{\mathbf{n}}H^0(\mathbf{X}(w),\lambda)$ if $\mathbf{n}'$ is greater than $\mathbf{n}$ for the product order on $\mathbb{N}^r$, one obtains that

$$F_n H^0(\mathbf{X}(w),\lambda) = \sum_{\substack{\mathbf{n}\in\mathbb{N}^r \\ |\mathbf{n}|=n}} F_{\mathbf{n}}H^0(\mathbf{X}(w),\lambda) + F_{n+t}H^0(\mathbf{X}(w),\lambda)$$

for all $t \geq 1$. Since $F_{n+t}H^0(\mathbf{X}(w),\lambda) = 0$ for $t \gg 0$, this implies our statements. $\square$

In particular, one obtains the following

Corollary 9. *$\mathcal{L}_{\mathbf{X}(w)}(\lambda)$ admits a nonzero global section if and only if λ belongs to the monoid generated by Δ and $\tilde{\mathcal{X}}^+$.*

Consider now the space

$$R(w) := \bigoplus_{\lambda\in\tilde{\mathcal{X}}} H^0(\mathbf{X}(w),\lambda).$$

Then $R(w)$ is a k-algebra, with a grading by $\tilde{\mathcal{X}}$. By Theorem 5, $R(w)$ can be seen as the multihomogeneous coordinate ring of $\mathbf{X}(w)$. Observe that $\sigma_1,\dots,\sigma_r$ are homogeneous elements of $R(w)$ of degrees $\alpha_1,\dots,\alpha_r$.

We define similarly,

$$A(w) := \bigoplus_{\mu} H^0(Z(w),\mu)$$

(sum over all weights μ, or, equivalently, over all dominant weights by Lemma 8). Then $A(w)$ is the multihomogeneous coordinate ring over $Z(w)$, a union of Schubert varieties in $G/B \times G/B$.

Corollary 10. *$\sigma_1,\dots,\sigma_r$ form a regular sequence in $R(w)$, and the quotient $R(w)/(\sigma_1,\dots,\sigma_r)$ is isomorphic to $A(w)$. As a consequence, $R(w)$ is generated as a k-algebra by $\sigma_1,\dots,\sigma_r$ together with the subspaces $H^0(\mathbf{X}(w),\omega_i)$, for $1 \leq i \leq r$.*

Proof. The canonical filtrations of the $H^0(\mathbf{X}(w),\lambda)$ fit together into a filtration $(F_nR(w))$ of $R(w)$. Theorem 7 implies that

$$F_nR(w) = (\sigma_1,\dots,\sigma_r)^n$$

(the n-th power of the ideal generated by $\sigma_1,\dots,\sigma_r$) and that

$$F_nR(w)/F_{n+1}R(w) = \bigoplus_{\substack{n_1+\dots+n_r=n\\ \mu\in\tilde{\mathcal{X}}^+}} \sigma_1^{n_1}\cdots\sigma_r^{n_r}H^0(Z(w),\mu).$$

Thus, the associated graded ring is isomorphic to the polynomial ring $A(w)[t_1,\dots,t_r]$. By [6, 1.1.15], $\sigma_1,\dots,\sigma_r$ form a regular sequence in $R(w)$. Further, by [14, II.14.15, II.14.21], the graded ring $A(w)$ is a quotient of $\bigoplus_\mu H^0(Y,\mu)$, and the latter ring is generated by its subspaces $H^0(Y,\omega_i)$. So $R(w)$ is generated by $\sigma_1,\dots,\sigma_r$ and the $H^0(\mathbf{X}(w),\omega_i)$ (which lift the $H^0(Z(w),\omega_i)$). □

We will show in Section 7 that the rings $R(w)$ and $A(w)$ are Cohen-Macaulay.

4. A filtration of $k[\tilde{B}]$

In this section, we construct geometrically a filtration à la van der Kallen of the $\tilde{B}\times\tilde{B}$-module $k[\tilde{B}]$ (the ring of regular functions on $\tilde{B}$); see [28, Th. 1.13]. For this, we first obtain a coarser filtration whose layers are spaces of global sections of line bundles over $\overline{B}\cap Y = Z$. In particular, $k[\tilde{B}]$ admits a Schubert filtration as defined in [24, 2.8] (see also [29, 6.3.4]).

For $\mu\in\tilde{\mathcal{X}}^+$, we set

$$M(\mu) := H^0(Z,\mu).$$

Then $M(\mu)$ is a finite dimensional $\tilde{B}\times\tilde{B}$-module. By Theorem 7, each $\tilde{B}\times\tilde{B}$-module $H^0(\overline{B},\lambda)$ has a filtration with layers $M(\mu)$ where $\mu\in\tilde{\mathcal{X}}^+$ and $\mu\le\lambda$.

We will need more notation on $\tilde{B}$-modules, taken from [28]. Let ν be a weight; then there exist a unique $w = w_{min}\in W$ and a unique dominant weight μ such that $\nu = w\mu$ and that the length of w is minimal. Set

$$P(\nu) := H^0(S(w_{min}),\mu)$$

and

$$Q(\nu) := H^0(S(w_{min}),\mathcal{I}_{\partial S(w_{min})}\otimes\mathcal{L}_{G/B}(\mu)),$$

where $\partial S(w)$ denotes the boundary of $S(w)$, that is, the complement of its open B-orbit BwB/B. Then both $P(\nu)$ and $Q(\nu)$ are finite dimensional $\tilde{B}$-modules.

Theorem 11. (i) *The $\tilde{B}\times\tilde{B}$-module $k[\tilde{B}]$ has a canonical increasing filtration by finite dimensional submodules, with associated graded module*

$$\bigoplus_{\mu\in\tilde{\mathcal{X}}^+} M(\mu).$$

(ii) *For any $\mu\in\tilde{\mathcal{X}}^+$, the $\tilde{B}\times\tilde{B}$-module $M(\mu)$ has a filtration with associated graded module*

$$\bigoplus_{\nu\in W\mu} P(\nu)\boxtimes Q(-\nu).$$

Proof. Set $\Gamma = \tilde{\mathcal{X}}/\mathcal{X}$. For $\gamma \in \Gamma$, let $k[\tilde{B}]_\gamma$ be the sum of $\tilde{T}$-weight spaces in $k[\tilde{B}]$ (for the right $\tilde{T}$-action) over all weights in the coset γ. Then each $k[\tilde{B}]_\gamma$ is a $\tilde{B} \times \tilde{B}$-submodule of $k[\tilde{B}]$, and we have

$$k[\tilde{B}] = \bigoplus_{\gamma \in \Gamma} k[\tilde{B}]_\gamma.$$

Further, as a $k[B]$-module, $k[\tilde{B}]_\gamma$ is freely generated by any $\tilde{B}$-eigenvector.

Choose $\gamma \in \Gamma$. It is known (see [3, Chap.VI, Ex. §2.5]) that the set of dominant weights belonging to the coset γ admits a unique minimal element; denote it by λ. By Theorem 7, $H^0(\overline{B}, \lambda)$ is isomorphic to $M(\lambda)$. Thus, $H^0(\overline{B}, \lambda)$ contains a right $\tilde{B}$-eigenvector v_λ of weight λ, and one deduces that

$$H^0(B, \mathcal{L}_{\overline{B}}(\lambda)) \cong k[B]\, v_\lambda = k[\tilde{B}]_\gamma.$$

Now, let us filter $k[\tilde{B}]_\gamma$ by the order of pole along the boundary of $\overline{B}$. Specifically, consider the section $\sigma := \sigma_1 \cdots \sigma_r$ of $\mathcal{L}_{\overline{B}}(\beta)$, where $\beta = \alpha_1 + \cdots + \alpha_r$. Then σ is invariant by $\tilde{B} \times \tilde{B}$, and its zero set is the boundary $\overline{B} - B$. Therefore, the $\tilde{B} \times \tilde{B}$-module $k[\tilde{B}]_\gamma = H^0(B, \mathcal{L}_{\overline{B}}(\lambda))$ is the increasing union of its finite dimensional submodules

$$\sigma^{-n} H^0(\overline{B}, \lambda + n\beta)$$

for $n \geq 0$. The associated graded module of this filtration satisfies

$$gr^n k[\tilde{B}]_\gamma \cong H^0(\overline{B}, \lambda + n\beta)/\sigma H^0(\overline{B}, \lambda + (n-1)\beta).$$

Let R be the multihomogeneous coordinate ring on $\overline{B}$, then $\sigma \in R_\beta$ and

$$gr^n k[\tilde{B}]_\gamma \cong R_{\lambda+n\beta}/\sigma R_{\lambda+(n-1)\beta} = (R/\sigma R)_{\lambda+n\beta}.$$

Consider the decreasing filtration of $R/\sigma R$, image of the filtration of R by the ideals $(\sigma_1, \ldots, \sigma_r)^m R$. Let

$$\mathbb{N}^r(0) := \{(m_1, \ldots, m_r) \in \mathbb{N}^r \mid m_1 \cdots m_r = 0\}.$$

Then, as $\sigma_1, \ldots, \sigma_r$ form a homogeneous regular sequence in R, the associated graded ring of $R/\sigma R$ satisfies

$$gr_m R/\sigma R = \bigoplus_{\substack{(m_1,\ldots,m_r) \in \mathbb{N}^r(0) \\ m_1+\cdots+m_r=m}} \sigma_1^{m_1} \cdots \sigma_r^{m_r} A$$

where A is the multihomogeneous coordinate ring of Z. Taking homogeneous components of degree $\lambda + n\beta$, we see that each $gr^n k[\tilde{B}]_\gamma$ has a finite decreasing filtration with associated graded

$$\bigoplus_{\substack{(m_1,\ldots,m_r) \in \mathbb{N}^r(0) \\ \lambda+n\beta-m_1\alpha_1-m_r\alpha_r \in \tilde{\mathcal{X}}^+}} M(\lambda + n\beta - m_1\alpha_1 - m_r\alpha_r).$$

Reordering the indices, we obtain a canonical increasing filtration of $k[\tilde{B}]$ satisfying the requirements of (i).

For (ii), recall that the irreducible components of Z are exactly the $S(w) \times S(ww_0)$ for $w \in W$. We first construct an increasing filtration of Z by partial unions of these components, as follows. Choose an indexing $W = \{w_1, \ldots, w_M\}$

which is compatible with the Bruhat-Chevalley order, that is, $i \le j$ if $w_i \le w_j$. In particular, $w_1 = 1$ and $w_M = w_0$. Set

$$Z_i := S(w_i) \times S(w_i w_0), \quad Z_{\ge i} := \bigcup_{j \ge i} Z_j, \quad Z_{>i} = \bigcup_{j>i} Z_j.$$

Then we have the following

Lemma 12. *$Z_i \cap Z_{>i} = S(w_i) \times \partial S(w_i w_0)$.*

Proof. Let x, y in W such that $S(x) \times S(yw_0) \subseteq Z_i$, that is, $x \le w_i \le y$. If, moreover, $S(x) \times S(yw_0) \subseteq Z_{>i}$, then $w_j \le y$ for some $j > i$. It follows that $y \ne w_i$, so that $Z_i \cap Z_{>i} \subseteq S(w_i) \times \partial S(w_i w_0)$.

For the opposite inclusion, let $y \in W$ such that $S(yw_0) \subseteq \partial S(w_i w_0)$, that is, $y > w_i$. Then $y = w_j$ for some $j > i$. Thus, $S(w_i) \times S(yw_0) \subseteq S(y) \times S(yw_0)$ is contained in $Z_i \cap Z_{>i}$. □

Returning to the proof of Theorem 11, let $\mathcal{I}_i$ be the ideal sheaf of $Z_{>i}$ in $Z_{\ge i}$. Then $\mathcal{I}_i$ identifies to the ideal sheaf of $Z_i \cap Z_{>i}$ in Z_i. By definition, we have an exact sequence of sheaves of $\mathcal{O}_Z$-modules: $0 \to \mathcal{I}_i \to \mathcal{O}_{Z_{\ge i}} \to \mathcal{O}_{Z_{>i}} \to 0$. Thus, the sequence

$$0 \to \mathcal{I}_i \otimes \mathcal{L}_Z(\mu) \to \mathcal{L}_{Z_{\ge i}}(\mu) \to \mathcal{L}_{Z_{>i}}(\mu) \to 0$$

is exact. Further, $H^1(Z_{\ge i}, \mathcal{L}_Z(\mu)) = 0$ as $Z_{\ge i}$ is a union of Schubert varieties in Y, and μ is dominant. So we obtain an exact sequence

$$0 \to H^0(Z_{\ge i}, \mathcal{I}_i \otimes \mathcal{L}_Z(\mu)) \to H^0(Z_{\ge i}, \mu) \to H^0(Z_{>i}, \mu) \to 0.$$

Now Lemma 12 implies that

$$\begin{aligned} H^0(Z_{\ge i}, \mathcal{I}_i \otimes \mathcal{L}_Z(\mu)) &= H^0(Z_i, \mathcal{I}_i \otimes \mathcal{L}_Z(\mu)) \\ &= H^0(S(w_i), \mu) \boxtimes H^0(S(w_i w_0), \mathcal{I}_{\partial S(w_i w_0)} \otimes \mathcal{L}_{G/B}(-w_0\mu)). \end{aligned}$$

By induction on i, we thus obtain a filtration of $M(\mu)$ with associated graded module

$$\bigoplus_{x \in W} H^0(S(x), \mu) \boxtimes H^0(S(w_0 x), \mathcal{I}_{\partial S(w_0 x)} \otimes \mathcal{L}_{G/B}(-w_0\mu)).$$

Further, we have $H^0(S(x), \mu) = P(x\mu)$ by [29, Lemma 2.3.2]. And

$$H^0(S(x), \mathcal{I}_{\partial S(x)} \otimes \mathcal{L}_{G/B}(\mu)) = Q(x\mu)$$

if x is the element of minimal length in its coset xW_μ (where W_μ is the isotropy group of μ in W). Otherwise, $H^0(S(x), \mathcal{I}_{\partial S(x)} \otimes \mathcal{L}_{G/B}(\mu)) = 0$; indeed, the restriction map $H^0(S(x), \mu) \to H^0(S(x_{min}), \mu)$ is an isomorphism by [*loc.cit.*].

It follows that the $\tilde{B}$-module $H^0(S(xw_0), \mathcal{I}_{\partial S(xw_0)} \otimes \mathcal{L}_{G/B}(-w_0\mu))$ equals $Q(-x\mu)$ if x has maximal length in its coset xW_μ, and equals 0 otherwise. Thus, the $\tilde{B} \times \tilde{B}$-module $M(\mu)$ has a filtration with associated graded module $\bigoplus P(x\mu) \boxtimes Q(-x\mu)$, sum over all $x \in W$ such that x has maximal length in its W_μ-coset. This implies (ii). □

Remark. A similar argument shows that the $\tilde{G} \times \tilde{G}$-module $k[\tilde{G}]$ has an increasing filtration by finite dimensional submodules with associated graded module

$$\bigoplus_{\mu \in \tilde{\mathcal{X}}^+} H^0(G/B, \mu) \boxtimes H^0(G/B, -w_0\mu).$$

This gives a geometric proof of a result of Donkin and Koppinen [14, II.4.20].

For any dominant weight μ, we denote by $c_\mu = \operatorname{ch} M(\mu)$ the character of the finite dimensional $\tilde{T} \times \tilde{T}$-module $M(\mu)$. Then c_μ is a regular function on $\tilde{T} \times \tilde{T}$, and we have by Theorem 7, for $t, u \in \tilde{T}$,

$$c_\mu(t,u) = \sum_{\nu \in W\mu} \operatorname{ch} P(\nu)(t) \operatorname{ch} Q(-\nu)(u).$$

Further, $\operatorname{ch} P(\nu)$ is given by the Demazure character formula (see, for example, [14, II.14.18]), and $\operatorname{ch} Q(\nu)$ is given by a closely related formula [22, Th. 2.1].

By Corollary 3, we have $H^i(Z,\mu) = 0$ for $i \geq 1$; thus, we can extend the map $\mu \mapsto c_\mu$ to the group $\tilde{\mathcal{X}}$ by setting

$$c_\lambda = \chi(Z,\lambda) = \sum_{i \geq 0} (-1)^i \operatorname{ch} H^i(Z,\lambda)$$

for arbitrary $\lambda \in \tilde{\mathcal{X}}$.

We now establish two symmetry properties of the resulting map $\lambda \mapsto c_\lambda$; the second symmetry will be an essential ingredient of the proof that $\overline{B}$ is Gorenstein. We will determine the value of c_λ at (t, t^{-1}) and, in particular, the dimension of $M(\lambda)$ in Corollary 19 below.

Theorem 13. *We have $c_{-w_0\lambda}(t,u) = c_\lambda(u,t)$ and*

$$c_{-\lambda}(t^{-1},u^{-1}) = (-1)^N \rho(t)\rho(u) c_{\lambda-\rho}(t,u)$$

for all $\lambda \in \tilde{\mathcal{X}}$ and $t, u \in \tilde{T}$.

Proof. With notation as in Lemma 12, we have for any $\lambda \in \tilde{\mathcal{X}}$ and any index i,

$$\begin{aligned} \chi(Z_{\geq i},\lambda) &= \chi(Z_{>i},\lambda) + \chi(Z_{\geq i}, \mathcal{I}_i \otimes \mathcal{L}_Z(\lambda)) \\ &= \chi(Z_{>i},\lambda) + \chi(S(w_i),\lambda)\, \chi(S(w_i w_0), \mathcal{I}_{\partial S(w_i w_0)} \otimes \mathcal{L}_{G/B}(-w_0\lambda)), \end{aligned}$$

by Lemma 12 and the argument thereafter. Since $Z = Z_{\geq 1}$, it follows that

$$(*) \qquad \chi(Z,\lambda) = \sum_{x \in W} \chi(S(x),\lambda)\, \chi(S(xw_0), \mathcal{I}_{\partial S(xw_0)} \otimes \mathcal{L}_{G/B}(-w_0\lambda))$$

for all $\lambda \in \tilde{\mathcal{X}}$. Thus, one has

$$c_{-\lambda}(t^{-1},u^{-1}) = \sum_{x \in W} \chi(S(x),-\lambda)(t^{-1})\ \chi(S(xw_0), \mathcal{I}_{\partial S(xw_0)} \otimes \mathcal{L}_{G/B}(w_0\lambda))(u^{-1}).$$

Let $x \in W$. Recall that, by [25, Prop. 4 and Th. 4], the Schubert variety $S(x)$ is Cohen-Macaulay. Denote by $\omega_{S(x)}$ its dualizing sheaf, a B-linearized sheaf. In fact, [25, Th. 4] shows more precisely that any standard resolution $\phi : V(x) \to S(x)$ satisfies $\phi_* \omega_{V(x)} = \omega_{S(x)}$. Combining this with the results in [29, A.4], one obtains an isomorphism of B-linearized sheaves

$$\omega_{S(x)} \cong \mathcal{I}_{\partial S(x)} \otimes \mathcal{L}_{G/B}(-\rho)[\rho]$$

where $[\rho]$ denotes the shift by the character ρ in the B-linearization. Thus, using Serre duality on each $S(x)$, one obtains that

$$\begin{aligned}
&(-1)^N \rho(t^{-1})\rho(u^{-1})\, c_{-\lambda}(t^{-1}, u^{-1}) \\
&\quad = \sum_{x\in W} \chi(S(x), \mathcal{I}_{\partial S(x)} \otimes \mathcal{L}_{G/B}(-\rho+\lambda))(t)\; \chi(S(xw_0), -\rho - w_0\lambda)(u) \\
&\quad = \sum_{y\in W} \chi(S(y), -\rho - w_0\lambda)(u)\, \chi(S(yw_0), \mathcal{I}_{\partial S(yw_0)} \otimes \mathcal{L}_{G/B}(-\rho+\lambda))(t) \\
&\quad = c_{-w_0\lambda-\rho}(u, t).
\end{aligned}$$

On the other hand, set $Z^i := S(w_i w_0) \times S(w_i)$ and define similarly $Z^{\geq i}$, $Z^{>i}$. Then we obtain as in Lemma 12 that $Z^i \cap Z^{>i} = \partial S(w_i w_0) \times S(w_i)$. As above, it follows that

$$\chi(Z, \lambda) = \sum_{x\in W} \chi(S(xw_0), \mathcal{I}_{\partial S(xw_0)} \otimes \mathcal{L}_{G/B}(\lambda))\; \chi(S(x), -w_0\lambda)$$

for any λ. Thus, $\chi(Z, -w_0\lambda)(u, t) = \chi(Z, \lambda)(t, u)$ and the first identity is proved. In particular, $c_{-w_0\lambda-\rho}(u, t) = c_{\lambda-\rho}(t, u)$ which completes the proof of the second identity. $\square$

5. Closures of Borel subgroups are Gorenstein

Let R (resp. A) be the multihomogeneous coordinate ring of $\overline{B}$ (resp. $Z = \overline{B} \cap Y$) as defined in Section 2. We show that both R and A are Gorenstein; as a consequence, $\overline{B}$ and Z are Gorenstein as well.

It will be convenient to set

$$\beta := \alpha_1 + \cdots + \alpha_r.$$

Then the dualizing sheaf of $\mathbf{X}$ is $\mathcal{L}_{\mathbf{X}}(-2\rho - \beta)$ by [27, §3].

Theorem 14. (i) $\overline{B}$ *(resp. Z) is Gorenstein and its dualizing sheaf is isomorphic to $\mathcal{L}_{\overline{B}}(-\rho-\beta)[\rho,\rho]$ (resp. $\mathcal{L}_Z(-\rho)[\rho,\rho]$) as a $\tilde{B}\times\tilde{B}$-linearized sheaf.*

(ii) *The graded ring R (resp. A) is Gorenstein and its dualizing module is generated by a homogeneous element of degree $\rho+\beta$ (resp. ρ), eigenvector of $\tilde{B}\times\tilde{B}$ of weight (ρ,ρ).*

Proof. We begin by proving that Z is Cohen-Macaulay. For this, we use the notation introduced in the proof of Theorem 11. We check by decreasing induction on i that each $Z_{\geq i}$ is Cohen-Macaulay. If $i = M$, then $Z_{\geq M} = S(w_0) \times S(1) \cong G/B$, a nonsingular variety. For arbitrary i, we have an exact sequence

$$0 \to \mathcal{O}_{Z_{\geq i}} \to \mathcal{O}_{Z_i} \oplus \mathcal{O}_{Z_{>i}} \to \mathcal{O}_{Z_i \cap Z_{>i}} \to 0.$$

Further, we know that $Z_i = S(w_i) \times S(w_i w_0)$ is Cohen-Macaulay; and, by the induction hypothesis, the same holds for $Z_{>i}$. On the other hand, $Z_i \cap Z_{>i} = S(w_i) \times \partial S(w_i w_0)$ by Lemma 12. The dualizing sheaf of $S(w_i w_0)$ is the tensor product of the ideal sheaf of $\partial S(w_i w_0)$ with the invertible sheaf $\mathcal{L}_{G/B}(-\rho)$. By [6, Proposition 3.3.18], it follows that $\partial S(w_i w_0)$ is Cohen-Macaulay, of depth $\ell(w_i w_0) - 1$. Thus, the depth of $Z_i \cap Z_{>i}$ is $\ell(w_i) + \ell(w_i w_0) - 1 = \ell(w_0) - 1 = \dim Z_{\geq i} - 1$. Together with the exact sequence above, this implies easily that $Z_{\geq i}$ is Cohen-Macaulay; see [6, Proposition 1.2.9].

We now prove that the ring A is Cohen-Macaulay. For this, let $C = \operatorname{Spec} A$ be the corresponding affine scheme. Then C is the multicone over Z in the sense of [16]; we now recall some constructions from that article. Let $\mathcal{S}$ denote the sheaf of graded algebras

$$\mathcal{S} := Sym_{\mathcal{O}_Z}\big(\bigoplus_{i=1}^{r} \mathcal{L}_Z(\omega_i)\big),$$

let $E = Spec(\mathcal{S})$ denote the corresponding vector bundle, with projection $q : E \to Z$. Then

$$q_*\mathcal{O}_E = Sym_{\mathcal{O}_Z}\big(\bigoplus_{i=1}^{r} \mathcal{L}_Z(\omega_i)\big) = \bigoplus_{\mu \in \tilde{\mathcal{X}}^+} \mathcal{L}_Z(\mu).$$

In particular, $H^0(E, \mathcal{O}_E) = A$ so that we have a morphism $p : E \to C$. The torus $\tilde{T}$ acts on E and on C (because A is graded by the character group of $\tilde{T}$), compatibly with the action of $\tilde{B}\times\tilde{B}$. Clearly, p and q are equivariant for the action of $\tilde{B}\times\tilde{B}\times\tilde{T}$.

As the line bundles $\mathcal{L}_Z(\omega_1), \ldots, \mathcal{L}_Z(\omega_r)$ are generated by their global sections, p is proper. Further, we have $p_*\mathcal{O}_E = \mathcal{O}_C$, as C is affine and $H^0(E, \mathcal{O}_E) = H^0(C, \mathcal{O}_C)$. In particular, p is surjective.

Let E^0 be the total space of E minus the union of all sub-bundles $\bigoplus_{j\neq i} \mathcal{L}_Z(\omega_j)$ for $1 \le i \le r$; let $p^0 : E^0 \to C$ and $q^0 : E^0 \to Z$ be the restrictions of p and q. Then p^0 is an isomorphism onto an open subset C^0 of C, and q^0 is a principal $\tilde{T}$-bundle over Z. As a consequence, the restriction of p to each irreducible component of E (that is, to each $q^{-1}(S(w) \times S(ww_0))$ is birational. Thus, C is equidimensional of dimension $\dim(B) = N + r$.

We claim that $R^i p_*\mathcal{O}_E = 0$ for $i \ge 1$. Indeed, as C and q are affine, this amounts to

$$0 = H^i(E, \mathcal{O}_E) = H^i(Z, q_*\mathcal{O}_E) = \bigoplus_{\mu \in \tilde{\mathcal{X}}^+} H^i(Z, \mu),$$

which follows from Corollary 3.

Because Z is Cohen-Macaulay, the same holds for E, and we have

$$\omega_E = q^*\omega_Z \otimes \omega_{E/Z} = q^*(\omega_Z \otimes \mathcal{L}_Z(\omega_1) \otimes \cdots \otimes \mathcal{L}_Z(\omega_r)) = q^*(\omega_Z \otimes \mathcal{L}_Z(\rho))$$

as a $\tilde{B} \times \tilde{B}$-linearized sheaf. Both ω_E and $q^*(\omega_Z \otimes \mathcal{L}_Z(\rho))$ have canonical $\tilde{T}$-linearizations, since $\tilde{T}$ acts on E and fixes Z pointwise; but

$$\omega_E = q^*(\omega_Z \otimes \mathcal{L}_Z(\rho))[-\rho]$$

as a $\tilde{B} \times \tilde{B} \times \tilde{T}$-linearized sheaf, where $[\chi]$ denotes the shift of the $\tilde{T}$-linearization by the character χ.

We claim that $R^i p_*\omega_E = 0$ for $i \ge 1$, that is, $H^i(E, \omega_E) = 0$ for $i \ge 1$. Indeed, we have

$$\begin{aligned} H^i(E, \omega_E) = H^i(Z, q_*\omega_E) &= \bigoplus_{\mu \in \tilde{\mathcal{X}}^+} H^i(Z, \omega_Z \otimes \mathcal{L}_Z(\rho + \mu)) \\ &= \bigoplus_{\mu \in \tilde{\mathcal{X}}^+} H^{N-i}(Z, -\rho - \mu)^*, \end{aligned}$$

as Z is equidimensional of dimension N. For $\mu \in \tilde{\mathcal{X}}^+$, the line bundle $\mathcal{L}_Z(\rho+\mu)$ is ample. Because Z is Cohen-Macaulay, we therefore have

$$H^j(Z, -n(\rho+\mu)) = 0$$

for $j < N$ and large n. But Z, being a union of Schubert varieties in Y, is Frobenius split. Thus, $H^j(Z, -\rho-\mu) = 0$ for $j < N$ by [29, Proposition A.2.1]. This proves our claim.

We now recall a version of a result of Kempf; see e.g. [15, p. 49-51].

Lemma 15. *Let $p : \hat{X} \to X$ be a proper morphism of algebraic schemes. Assume that $\hat{X}$ is Cohen-Macaulay, X is equidimensional of the same dimension as $\hat{X}$, $p_*\mathcal{O}_{\hat{X}} = \mathcal{O}_X$ and $R^ip_*\mathcal{O}_{\hat{X}} = R^ip_*\omega_{\hat{X}} = 0$ for $i \geq 1$. Then X is Cohen-Macaulay with dualizing sheaf $p_*\omega_{\hat{X}}$.*

Proof. The statement is local in X, so that we may assume that X is a closed subscheme of a smooth affine scheme S. Denote by $\iota : X \to S$ the inclusion and set $\pi := \iota \circ p$. Then $\pi_*\mathcal{O}_{\hat{X}} = \iota_*\mathcal{O}_X$ and $R^i\pi_*\mathcal{O}_{\hat{X}} = 0$ for $i \geq 1$. Applying the duality theorem to the proper morphism $\pi : \hat{X} \to S$ and the sheaves $\mathcal{O}_{\hat{X}}$ and ω_S, we obtain

$$\begin{aligned} RHom(\iota_*\mathcal{O}_X, \omega_S) &= RHom(R\pi_*\mathcal{O}_{\hat{X}}, \omega_S) = R\pi_*\, RHom(\mathcal{O}_{\hat{X}}, \pi^!\omega_S) \\ &= R\pi_*\pi^!\omega_S = R\pi_*\omega_{\hat{X}}[\dim X - \dim S] = \pi_*\omega_{\hat{X}}[\dim X - \dim S], \end{aligned}$$

that is, $Ext^i_S(\iota_*\mathcal{O}_X, \omega_S) = 0$ for $i \neq \dim S - \dim X$, and

$$Ext_S^{\dim X - \dim S}(\iota_*\mathcal{O}_X, \omega_S) = \pi_*\omega_{\hat{X}}.$$

This means that X is Cohen-Macaulay with dualizing sheaf $p_*\omega_{\hat{X}}$. □

Lemma 15 implies that the graded ring A is Cohen-Macaulay with dualizing module

$$\begin{aligned} \omega_A = H^0(C, p_*\omega_E) &= \bigoplus_{\mu \in \tilde{\mathcal{X}}^+} H^0(Z, \omega_Z \otimes \mathcal{L}_Z(\mu+\rho))[-\rho] \\ &= \bigoplus_{\mu \in \tilde{\mathcal{X}}^+} H^N(Z, -\mu-\rho)^*[-\rho]. \end{aligned}$$

Further, $H^j(Z, -\mu-\rho) = 0$ for $j \neq N$. The module ω_A is $\tilde{\mathcal{X}}$-graded and each homogeneous component is a finite-dimensional $\tilde{B} \times \tilde{B}$-module. Thus, we can consider the Hilbert series

$$H_{\omega_A}(t, u, z) := \sum_{\lambda \in \tilde{\mathcal{X}}} \operatorname{ch} \omega_{A,\lambda}(t, u)\, z^\lambda$$

where t, u are in $\tilde{T}$, and the z^λ are the canonical basis elements of the group algebra $\mathbb{Z}[\tilde{\mathcal{X}}]$. Now we have

$$\begin{aligned} H_{\omega_A}(t, u, z) &= \sum_{\mu \in \tilde{\mathcal{X}}^+} \operatorname{ch} H^N(Z, -\mu-\rho)(t^{-1}, u^{-1})\, z^{\mu+\rho} \\ &= (-1)^N \sum_{\mu \in \tilde{\mathcal{X}}^+} c_{-\mu-\rho}(t^{-1}, u^{-1}) z^{\mu+\rho}. \end{aligned}$$

Together with Theorem 13, it follows that

$$H_{\omega_A}(t, u, z) = \rho(t)\rho(u) z^\rho H_A(t, u, z).$$

Using [6, Cor. 4.3.8.a)], we therefore have

$$H_A(t^{-1}, u^{-1}, z^{-1}) = (-1)^{\dim(A)} \rho(t)\rho(u) z^{\rho} H_A(t, u, z).$$

Now a result of Stanley [6, Cor. 4.3.8.c)] would imply that A is Gorenstein if A were a domain. This is not the case, but A is the quotient of the domain R by the ideal generated by the regular sequence $(\sigma_1, \ldots, \sigma_r)$. It follows that R is Cohen-Macaulay with Hilbert series

$$H_R(t, u, z) = \frac{H_A(t, u, z)}{\prod_{i=1}^r (1 - z^{\alpha_i})}$$

because each σ_i is the restriction of a $\tilde{G} \times \tilde{G}$-invariant section. As a consequence, we obtain

$$H_R(t^{-1}, u^{-1}, z^{-1}) = (-1)^{\dim(R)} \rho(t)\rho(u) z^{\rho+\beta} H_R(t, u, z).$$

Thus, by the result of Stanley quoted above, R is Gorenstein and its dualizing module is generated by a homogeneous element of degree $\rho + \beta$, eigenvector of $\tilde{B} \times \tilde{B}$ of weight (ρ, ρ). It follows that A is Gorenstein as well and that its dualizing module is generated in degree ρ and weight (ρ, ρ).

It remains to prove that $\overline{B}$ and Z are Gorenstein and to determine their dualizing sheaves. For this, consider the isomorphism $p^0 : E^0 \to C^0$ where C^0 is an open subset of C, and the principal $\tilde{T}$-bundle $q^0 : E^0 \to Z$. Then $\omega_{E^0} = \mathcal{O}_{E^0}[\rho, \rho]$ as a $\tilde{B} \times \tilde{B}$-linearized sheaf, because the same holds for C. Further, $\omega_{E^0} = q^{0*}(\omega_Z \otimes \mathcal{L}_Z(\rho))$ so that

$$\omega_Z \otimes \mathcal{L}_Z(\rho) \otimes q^0_* \mathcal{O}_{E^0} = q^0_* \mathcal{O}_{E^0}[\rho, \rho].$$

Taking invariants of $\tilde{T}$, we obtain

$$\omega_Z \otimes \mathcal{L}_Z(\rho) = \mathcal{O}_Z[\rho, \rho],$$

that is, Z is Gorenstein with dualizing sheaf $\mathcal{L}_Z(-\rho)[\rho, \rho]$. The argument for $\overline{B}$ is similar. ☐

6. The class of the diagonal for flag varieties

We will construct a degeneration of the diagonal in $G/B \times G/B$ into a union of Schubert varieties. For this, we recall a special case of a construction of [4, 1.6].

Consider the action of $B \times B$ on $\overline{B}$ and the associated fiber bundle

$$p : G \times G \times^{B \times B} \overline{B} \to G/B \times G/B,$$

a locally trivial fibration with fiber $\overline{B}$. The action map

$$G \times G \times \overline{B} \to \mathbf{X} = (G \times G)\overline{B} : \ (g, h, x) \mapsto (g, h)x$$

defines a $G \times G$-equivariant map

$$\pi : G \times G \times^{B \times B} \overline{B} \to \mathbf{X}.$$

Observe that π factors through the closed embedding

$$G \times G \times^{B \times B} \overline{B} \to G/B \times G/B \times \mathbf{X} : (g, h, x)(B \times B) \mapsto (gB, hB, (g, h)x)$$

followed by the projection $G/B \times G/B \times \mathbf{X} \to \mathbf{X}$. Thus, π is proper and its scheme-theoretic fibers identify to closed subschemes of $G/B \times G/B$ via p_*. Further, the fiber $\pi^{-1}(1)$ at the identity is the diagonal $\mathrm{diag}(G/B)$; and the reduced fiber $\pi^{-1}(y)_{red}$ at the base point y of $Y = G/B \times G/B$ is

$$\bigcup_{x \in W} S(x) \times S(w_0 x).$$

Consider now the closure $\overline{T}$ of T in $\mathbf{X}$, then $\overline{T}$ is a $T \times T$-stable subvariety fixed pointwise by $\mathrm{diag}(T)$. By [27], $\overline{T}$ is smooth and meets Y transversally at the points $(w, ww_0)y$ for $w \in W$. Further, each $(w, ww_0)y$ admits a $T \times T$-stable affine neighborhood in $\overline{T}$, isomorphic to affine r-space where $T \times T$ acts linearly with weights $(-w(\alpha_i), w(\alpha_i))$, $1 \le i \le r$. Since the character group of T is the root lattice, there exists a unique one-parameter subgroup λ of T such that $\langle \lambda, \alpha_i \rangle = 1$ for $1 \le i \le r$. Then the closure of λ in $\overline{T}$ is a curve γ containing 1 and isomorphic to projective line, with λ-fixed points $z := (1, w_0)y$ and $z' := (w_0, 1)y$. In particular, $\gamma \setminus \{z, z'\}$ is contained in T. Further, $\pi^{-1}(\gamma)$ is a $\mathrm{diag}(T)$-stable subvariety of $G \times G \times^{B \times B} \overline{B}$ and we have a $\mathrm{diag}(T)$-equivariant isomorphism

$$\pi^{-1}(\gamma) \simeq \{(gB, hB, x) \in G/B \times G/B \times \gamma \mid (g^{-1}, h^{-1})x \in \overline{B}\}$$

identifying $\pi^{-1}(\gamma) \to \gamma$ to the restriction of the projection $G/B \times G/B \times \gamma \to \gamma$.

Theorem 16. *The morphism $\pi : G \times G \times^{B \times B} \overline{B} \to \mathbf{X}$ is flat, with reduced fibers. Its restriction $\pi^{-1}(\gamma) \to \gamma$ is flat and $\mathrm{diag}(T)$-invariant, with fibers over $\gamma \setminus \{z, z'\}$ isomorphic to $\mathrm{diag}(G/B)$, and with fiber at z equal to*

$$\bigcup_{x \in W} S(x) \times w_0 S(w_0 x).$$

Proof. By [4, Proposition 1.6], π is equidimensional. Further, $G \times G \times^{B \times B} \overline{B}$ is Cohen-Macaulay, as $\overline{B}$ is. Because $\mathbf{X}$ is smooth, it follows that π is flat (see, for example, [23, Cor. of Th. 23.1]).

For $x \in \mathbf{X}$, the scheme-theoretic fiber $\pi^{-1}(x)$ identifies to

$$\{(gB, hB) \in G/B \times G/B \mid (g^{-1}, h^{-1})x \in \overline{B}\}.$$

Set $F := \{(g, h) \in G \times G \mid (g^{-1}, h^{-1})x \in \overline{B}\}$. Then F is stable under right multiplication by $B \times B$, and left multiplication by $(G \times G)_x$ (the isotropy group of x in $G \times G$). The quotient of F by the right $B \times B$-action is $\pi^{-1}(x)$, whereas the quotient by the left $(G \times G)_x$-action is isomorphic to the scheme-theoretic intersection of $\overline{B}$ with the orbit $(G \times G)x$. This intersection is reduced by Corollary 3; thus, F and $\pi^{-1}(x)$ are reduced, too.

The remaining assertions are direct consequences of these facts. $\square$

We now deduce from Theorem 16 a formula for the class of the diagonal of the flag variety in equivariant K-theory.

For a variety X with an action of T, we denote by $K^T(X)$ the Grothendieck group of T-linearized coherent sheaves on X. The character group of T acts on $K^T(X)$ by shifting the linearization; this endows $K^T(X)$ with the structure of an $R(T)$-module. If the T-action on X extends to an action of the normalizer $N_G(T)$, then $N_G(T)$ acts on $K^T(X)$ by $n \cdot [\mathcal{F}] = [n^* \mathcal{F}]$. Clearly, this descends to an action of W on $K^T(X)$, compatible with the $R(T)$-module structure.

We apply this to $X = G/B \times G/B$ where T acts diagonally. For a T-stable subvariety S of $G/B \times G/B$, the class in $K^T(G/B \times G/B)$ of the structure sheaf $\mathcal{O}_S$ will be denoted by $[S]$. In particular, we have the classes of Schubert varieties and of their translates by $W\times W$; we also have the class of the diagonal, $\mathrm{diag}(G/B)$. We will express the latter in terms of the former.

This will imply a formula for the class of the diagonal in the Grothendieck group $K(G/B\times G/B)$ of coherent sheaves on that space, by applying the forgetful map

$$K^T(G/B\times G/B)\to K(G/B\times G/B).$$

Observe that the action of $G\times G$ on $K(G/B\times G/B)$ is trivial, because G is generated by subgroups isomorphic to the additive group. Thus, the forgetful map is W-invariant.

To simplify our statements, for $x\in W$ we set

$$S^-(x) := \overline{B^-xB}/B = w_0S(w_0x)$$

and

$$[S^-(x)]^0 := [S^-(x)] - [\partial S^-(x)] = w_0[S(w_0x)] - w_0[\partial S(w_0x)].$$

Corollary 17. *With notation as above, we have in* $K^T(G/B\times G/B)$,

$$\begin{aligned}[diag(G/B)] &= [\bigcup_{x\in W} S(x)\times S^-(x)]\\ &= \sum_{x\in W}[S(x)]\times[S^-(x)]^0 = \sum_{x,y\in W,\, x\le y}(-1)^{\ell(y)-\ell(x)}[S(x)]\times[S^-(y)].\end{aligned}$$

As a consequence, in $K(G/B\times G/B)$ *we have*

$$\begin{aligned}[diag(G/B)] &= [\bigcup_{x\in W} S(x)\times S(w_0x)]\\ &= \sum_{x\in W}[S(x)]\times[S(w_0x)]^0 = \sum_{x,y\in W,\, x\le y}(-1)^{\ell(y)-\ell(x)}[S(x)]\times[S(w_0y)].\end{aligned}$$

Proof. Since γ is a projective line fixed pointwise by $\mathrm{diag}(T)$, then $[z]=[1]$ in $K^{\mathrm{diag}(T)}(\gamma)$. As the map $\pi^{-1}(\gamma)\to\gamma$ is flat and $\mathrm{diag}(T)$-invariant, the fibers $\pi^{-1}(1)$ and $\pi^{-1}(z)$ have the same class in $K^T(\pi^{-1}(\gamma))$ and hence in $K^T(G\times G\times^{B\times B}\overline{B})$. Thus, the direct images of these fibers under p are equal in $K^T(G/B\times G/B)$. This proves the first equality.

For the second one, we use the notation introduced in the proof of Theorem 11. Set

$$\Delta_i := S(w_i)\times S^-(w_i),\quad \Delta_{\ge i} := \bigcup_{j\ge i}\Delta_i,\quad \Delta_{>i} := \bigcup_{j>i}\Delta_i.$$

Then $\Delta_{\ge M} = S(w_0)\times w_0S(1)$, $\Delta_{\ge 1} = \bigcup_{x\in W} S(x)\times S^-(x)$, and we obtain as in Lemma 12 that

$$\Delta_i\cap\Delta_{>i} = S(w_i)\times\partial S^-(w_i).$$

As a consequence, we have an exact sequence of sheaves

$$0\to I_i\to\mathcal{O}_{\Delta_{\ge i}}\to\mathcal{O}_{\Delta_{>i}}\to 0$$

where I_i fits into an exact sequence

$$0\to I_i\to\mathcal{O}_{\Delta_i}\to\mathcal{O}_{S(w_i)\times\partial S^-(w_i)}\to 0.$$

It follows that

$$\begin{aligned}[\Delta_{\geq i}] &= [\Delta_{>i}] + [I_i] = [\Delta_{>i}] + [S(w_i) \times S^-(w_i)] - [S(w_i) \times \partial S^-(w_i)] \\ &= [\Delta_{>i}] + [S(w_i)] \times [S^-(w_i)]^0.\end{aligned}$$

By decreasing induction on i, we thus have

$$\Delta_{\geq i} = \sum_{j \geq i} [S(w_j)] \times [S^-(w_j)]^0.$$

For the third equality, it suffices to prove that

$$[S(x)]^0 = \sum_{y \leq x} (-1)^{\ell(x)-\ell(y)} [S(y)]$$

in $K^T(G/B)$. But the definition of $[S(x)]^0$ implies that $[S(x)] = \sum_{y \in W,\, y \leq x} [S(y)]^0$. Further, the Möbius function of the partially ordered set $(W, \leq)$ is given by $\mu(y,x) = (-1)^{\ell(x)-\ell(y)}$ if $y \leq x$ and $\mu(y,x) = 0$ otherwise; see [11]. $\square$

Consider now the Grothendieck group $K^T(G/B)$ of T-linearized coherent sheaves on G/B. Since G/B is smooth and projective, $K^T(G/B)$ is isomorphic to the Grothendieck group of T-linearized vector bundles, see e.g. [2, 2.6–2.7]; as a consequence, it has the structure of a $R(T)$-algebra. Further, the $R(T)$-module $K^T(G/B)$ is free, with basis the $[S(x)]$ $(x \in W)$; and the $R(T)$-bilinear map

$$K^T(G/B) \times K^T(G/B) \to R(T),\ (u,v) \mapsto \chi(G/B, u \cdot v)$$

is a perfect pairing, where $u \cdot v$ denotes the product in $K^T(G/B)$, and $\chi(G/B, -)$ denotes the equivariant Euler characteristic; see [19, 3.39, 4.9].

Corollary 18. *The classes*

$$(x \in W) \qquad [S^-(x)]^0 = \sum_{y \in W,\, y \geq x} (-1)^{\ell(y)-\ell(x)} [S^-(y)]$$

form the dual basis of the basis of the $[S(x)]$ $(x \in W)$.

Proof. Observe that

$$\begin{aligned}\chi(G/B, u \cdot v) &= \chi(G/B \times G/B, [\mathrm{diag}(G/B)] \cdot (u \times v)) \\ &= \sum_{x \in W} \chi(G/B, u \cdot [S(x)])\, \chi(G/B, v \cdot [S^-(x)]^0)\end{aligned}$$

where the second equality follows from Corollary 17. In particular, for $y \in W$ we have

$$\chi(G/B, u \cdot [S(y)]) = \sum_{x \in W} \chi(G/B, u \cdot [S(x)])\, \chi(G/B, [S(y)] \cdot [S^-(x)]^0).$$

As the $R(T)$-linear forms $u \mapsto \chi(G/B, u \cdot [S(x)])$ are linearly independent, we obtain

$$\chi(G/B, [S(y)] \cdot [S^-(x)]^0) = \delta_{x,y}.$$

$\square$

As another consequence of the determination of the class of the diagonal, we recover a formula of Mathieu for the character of the G-module $H^0(G/B, \lambda + \mu)$ as a function of the dominant weights λ and μ ([22, Cor. 7.7]). Further, we determine the dimension of the modules $M(\lambda)$ introduced in Section 4.

Corollary 19. *For any weights λ and μ, and for any $t \in T$, we have*

$$\chi(G/B, \lambda+\mu)(t) = \sum_{x \in W} \chi(S(x), \lambda)(t)\ \chi(S(xw_0), \mathcal{I}_{\partial S(xw_0)} \otimes \mathcal{L}_{G/B}(-w_0\mu))(t^{-1}).$$

Therefore, $\chi(G/B, 2\lambda)(t) = c_\lambda(t, t^{-1})$ for all $t \in T$. Further, for $\lambda \in \tilde{\mathcal{X}}^+$, one has

$$\operatorname{ch} M(\lambda)(t, t^{-1}) = \operatorname{ch} H^0(G/B, 2\lambda)(t)$$

and

$$\dim M(\lambda) = \prod_{\alpha \in \Phi^+} \frac{\langle 2\lambda + \rho, \check{\alpha} \rangle}{\langle \rho, \check{\alpha} \rangle}.$$

Proof. Let $[\mathcal{L}(\lambda, \mu)]$ denote the class of the $\tilde{G} \times \tilde{G}$-linearized line bundle $\mathcal{L}(\lambda, \mu)$ in $K^T(G/B \times G/B)$. As the restriction of this line bundle to the diagonal is $\mathcal{L}_{G/B}(\lambda + \mu)$, we have

$$\chi(G/B, \lambda + \mu) = \chi(G/B \times G/B, [\operatorname{diag}(G/B)] \cdot [\mathcal{L}(\lambda, \mu)]).$$

By Corollary 17, the latter is equal to

$$\sum_{w \in W} \chi(S(w), \lambda)\ w_0 \chi(S(w_0 w), \mathcal{I}_{\partial S(w_0 w)} \otimes \mathcal{L}_{G/B}(\mu)).$$

To complete the proof of the first equality, it suffices to check that

$$\chi(S(w_0 w), \mathcal{I}_{\partial S(w_0 w)} \otimes \mathcal{L}_{G/B}(\mu))(w_0 t) = \chi(S(ww_0), \mathcal{I}_{\partial S(ww_0)} \otimes \mathcal{L}_{G/B}(-w_0\mu))(t^{-1}).$$

For this, using Serre duality as in the proof of Theorem 13, we obtain

$$\chi(S(w_0 w), \mathcal{I}_{\partial S(w_0 w)} \otimes \mathcal{L}_{G/B}(\mu))(w_0 t) = (-1)^{N - \ell(w)} \rho(t) \chi(S(w_0 w), -\rho - \mu)(-w_0 t).$$

Further, the Demazure character formula implies that

$$\chi(S(w_0 w), \nu)(-w_0 t) = \chi(S(ww_0), -w_0 \nu)(t)$$

for all weights ν. It follows that

$$\begin{aligned} \chi(S(w_0 w), \mathcal{I}_{\partial S(w_0 w)} \otimes \mathcal{L}_{G/B}(\mu))(w_0 t) &= (-1)^{N - \ell(w)} \rho(t)\ \chi(S(ww_0), -\rho + w_0\mu)(t) \\ &= \chi(S(ww_0), \mathcal{I}_{\partial S(ww_0)} \otimes \mathcal{L}_{G/B}(-w_0\mu))(-t), \end{aligned}$$

by Serre duality once more.

Now the second equality follows from formula $(*)$ in the proof of Theorem 13. For $\lambda \in \tilde{\mathcal{X}}^+$, the third equality follows from the vanishing of the $H^i(Z, \lambda)$ ([25, Th. 2]), and the fourth one from Weyl's dimension formula. □

7. Large Schubert varieties are Cohen-Macaulay

In this section, we prove the statement of the title and we give some applications. We begin by constructing a partial desingularization of $\mathbf{X}(w)$, by the total space of a fibration with fiber $\overline{B}$ over the usual Schubert variety $S(w)$.

For this, consider the action of B on $\overline{B}$ by left multiplication, and the associated fiber bundle $G \times^B \overline{B}$ over G/B. The map $G \times \overline{B} \to \mathbf{X} : (g, x) \mapsto gx$ defines a birational, $G \times B$-equivariant morphism

$$\varphi : G \times^B \overline{B} \to \mathbf{X}$$

where the action of $G \times B$ on $G \times^B \overline{B}$ is defined by $(g,b)(g',x) = (gg', xb^{-1})$. On the other hand, the projection

$$\psi : G \times^B \overline{B} \to G/B$$

is a locally trivial fibration with fiber $\overline{B}$. Observe that (φ, ψ), being the composition of

$$G \times^B \overline{B} \hookrightarrow G \times^B \mathbf{X} \cong G/B \times \mathbf{X},$$

is a closed embedding.

Let $\mathbf{X}'(w)$ be the preimage of $S(w)$ under ψ; then $\mathbf{X}'(w)$ is stable by the subgroup $B \times B$ of $G \times B$. Observe that $\mathbf{X}'(w)$ is the closure of $BwB \times^B B \simeq BwB$ in $G \times^B \overline{B}$. As a consequence, φ restricts to a $B \times B$-equivariant morphism

$$f : \mathbf{X}'(w) \to \mathbf{X}(w)$$

which is an isomorphism over BwB. Denote by $\partial\mathbf{X}(w)$ the complement of BwB in $\mathbf{X}(w)$, and by $\partial\mathbf{X}'(w)$ its preimage under f. Finally, let

$$g : \mathbf{X}'(w) \to S(w)$$

be the restriction of ψ. Then g is a locally trivial fibration with fiber $\overline{B}$, also.

Theorem 20. *With notation as above, we have*

(i) $\mathbf{X}'(w)$ *is Cohen-Macaulay and its dualizing sheaf is isomorphic to* $\mathcal{I}_{\partial\mathbf{X}'(w)} \otimes f^*\mathcal{L}_{\mathbf{X}(w)}(-\rho)[\rho,\rho]$ *as a* $\tilde{B} \times \tilde{B}$*-linearized sheaf.*

(ii) $f_*\mathcal{O}_{\mathbf{X}'(w)} = \mathcal{O}_{\mathbf{X}(w)}$ *and* $R^i f_*\mathcal{O}_{\mathbf{X}'(w)} = 0 = R^i f_*\omega_{\mathbf{X}'(w)}$ *for* $i \geq 1$.

(iii) $\mathbf{X}(w)$ *is Cohen-Macaulay and its dualizing sheaf is isomorphic to* $f_*\omega_{\mathbf{X}'(w)} = \mathcal{I}_{\partial\mathbf{X}(w)} \otimes \mathcal{L}_{\mathbf{X}(w)}(-\rho)[\rho,\rho]$ *as a* $\tilde{B} \times \tilde{B}$*-linearized sheaf.*

(iv) *The graded ring* $R(w) = \bigoplus_{\lambda \in \tilde{\mathcal{X}}} H^0(\mathbf{X}(w), \lambda)$ *is Cohen-Macaulay.*

Proof. Because $S(w)$ and $\overline{B}$ are Cohen-Macaulay, the same holds for $\mathbf{X}'(w)$. And because f is birational and $\mathbf{X}(w)$ is normal, we have $f_*\mathcal{O}_{\mathbf{X}'(w)} = \mathcal{O}_{\mathbf{X}(w)}$.

We now show that $R^i f_*\mathcal{O}_{\mathbf{X}'(w)} = 0$ for $i \geq 1$. For this, it suffices, by a lemma of Kempf (see e.g. [14, II.14.13]), to show that $H^i(\mathbf{X}'(w), f^*\mathcal{L}_{\mathbf{X}(w)}(\lambda)) = 0$ for $i \geq 1$ and for any regular dominant weight λ. Consider the line bundle $\varphi^*\mathcal{L}_{\mathbf{X}}(\lambda)$ and its higher direct images $R^j\psi_*(\varphi^*\mathcal{L}_{\mathbf{X}}(\lambda))$ for $j \geq 0$. Then $R^j\psi_*(\varphi^*\mathcal{L}_{\mathbf{X}}(\lambda))$ is the $\tilde{G}$-linearized sheaf on $G/B = \tilde{G}/\tilde{B}$ associated with the $\tilde{B}$-module $H^j(\overline{B}, \lambda)$, and $R^j g_*(f^*\mathcal{L}_{\mathbf{X}(w)}(\lambda))$ is the restriction to $S(w)$ of this $\tilde{G}$-linearized sheaf. As $H^j(\overline{B}, \lambda) = 0$ for all $j \geq 1$ by Corollary 3, we have

$$R^j g_*(f^*\mathcal{L}_{\mathbf{X}(w)}(\lambda)) = 0$$

for $j \geq 1$.

For a $\tilde{B}$-module M, denote by $\underline{M}$ the corresponding homogeneous vector bundle on G/B. Then we obtain from the Leray spectral sequence for g that

$$H^i(\mathbf{X}'(w), f^*\mathcal{L}_{\mathbf{X}(w)}(\lambda)) \cong H^i(S(w), g_*f^*\mathcal{L}_{\mathbf{X}(w)}(\lambda)) \cong H^i(S(w), \underline{H^0(\overline{B}, \lambda)}).$$

By Theorems 7 and 11, the left $\tilde{B}$-module $H^0(\overline{B}, \lambda)$ has a filtration with associated graded a direct sum of $P(\mu)$'s for certain dominant weights μ. Further, we have for $i \geq 1$,

$$H^i(S(w), \underline{P(\mu)}) = 0,$$

as follows from [24, Prop. 1.4.2] or [29, Lemma 3.1.12]. Thus, for each $i \geq 1$, we have $H^i(\mathbf{X}'(w), f^*\mathcal{L}_{\mathbf{X}(w)}(\lambda)) = 0$ and, therefore, $R^i f_* \mathcal{O}_{\mathbf{X}'(w)} = 0$.

We now determine the dualizing sheaf $\omega_{\mathbf{X}'(w)}$; we begin with the relative dualizing sheaf ω_g of $g : \mathbf{X}'(w) \to S(w)$. Observe that the relative dualizing sheaf of $\psi : G \times^B \overline{B} \to G/B$ equals $\varphi^*\mathcal{L}_{\mathbf{X}}(-\beta-\rho) \otimes \psi^*\mathcal{L}_{G/B}(\rho)[\rho]$ as a $\tilde{G} \times \tilde{B}$-linearized sheaf, where $[\rho]$ denotes the shift by ρ of the $\tilde{B}$-linearization. Indeed, ω_ψ is the $(\tilde{G} \times \tilde{B})$-linearized sheaf on $G \times^B \overline{B}$ associated with the $(\tilde{B} \times \tilde{B})$-linearized sheaf $\omega_{\overline{B}}$ on $\overline{B}$. On the other hand, the sheaf $\varphi^*\mathcal{L}_{\mathbf{X}}(-\beta-\rho) \otimes \psi^*\mathcal{L}_{G/B}(\rho)[\rho]$ is $\tilde{G} \times \tilde{B}$-linearized, and the associated $\tilde{B} \times \tilde{B}$-linearized sheaf on $\overline{B}$ is $\omega_{\overline{B}}$ by Theorem 14. As $g : \mathbf{X}'(w) \to S(w)$ is the pull-back of ψ under the inclusion $S(w) \to G/B$, it follows that $\omega_g = f^*\mathcal{L}_{\mathbf{X}(w)}(-\beta-\rho) \otimes g^*\mathcal{L}_{S(w)}(\rho)[\rho]$. In particular, ω_g is invertible. Thus,

$$\omega_{\mathbf{X}'(w)} = g^*\omega_{S(w)} \otimes \omega_g = g^*\mathcal{I}_{\partial S(w)} \otimes f^*\mathcal{L}_{\mathbf{X}(w)}(-\beta-\rho)[\rho,\rho].$$

We now claim that

$$g^*\mathcal{I}_{\partial S(w)} \otimes f^*\mathcal{L}_{\mathbf{X}(w)}(-\beta) = \mathcal{I}_{\partial \mathbf{X}'(w)}. \tag{1}$$

For this, observe that $\partial\mathbf{X}'(w)$ contains the preimage under f of $\mathbf{X}(w) \cap (\mathbf{X} \setminus G) = \mathbf{X}(w) \cap (D_1 \cup \cdots \cup D_r)$, a Cartier divisor. Further, the complement $\partial\mathbf{X}'(w) \cap f^{-1}(G)$ of that divisor is equal to $g^{-1}(\partial S(w)) \cap f^{-1}(G)$. As the line bundle associated with $\mathbf{X}(w) \cap (D_1 \cup \cdots \cup D_r)$ is $\mathcal{L}_{\mathbf{X}(w)}(-\beta)$, it follows that

$$\mathcal{I}_{\partial\mathbf{X}'(w)} = g^*\mathcal{I}_{\partial S(w)} \otimes f^*\mathcal{I}_{\mathbf{X}(w)\cap(D_1\cup\cdots\cup D_r)} = g^*\mathcal{I}_{\partial S(w)} \otimes f^*\mathcal{L}_{\mathbf{X}(w)}(-\beta),$$

which proves the claim. We conclude that

$$\omega_{\mathbf{X}'(w)} = \mathcal{I}_{\partial\mathbf{X}'(w)} \otimes f^*\mathcal{L}_{\mathbf{X}(w)}(-\rho)[\rho,\rho]. \tag{2}$$

Further, since $f_*\mathcal{O}_{\mathbf{X}'(w)} = \mathcal{O}_{\mathbf{X}(w)}$ and $f(\partial\mathbf{X}'(w)) = \partial\mathbf{X}(w)$, then

$$f_*\mathcal{I}_{\partial\mathbf{X}'(w)} = \mathcal{I}_{\partial\mathbf{X}(w)}, \tag{3}$$

and, therefore,

$$f_*\omega_{\mathbf{X}'(w)} = \mathcal{I}_{\partial\mathbf{X}(w)} \otimes \mathcal{L}_{\mathbf{X}(w)}(-\rho)[\rho,\rho]. \tag{4}$$

We now prove that $R^i f_*\omega_{\mathbf{X}'(w)} = 0$ for $i \geq 1$. Using Kempf's lemma, again, it suffices to prove that

$$H^i(\mathbf{X}'(w), g^*\mathcal{I}_{\partial S(w)} \otimes f^*\mathcal{L}_{\mathbf{X}(w)}(\lambda-\beta)) = 0 \tag{5}$$

for $i \geq 1$ and for $\lambda \in \tilde{\mathcal{X}}^+$ big enough (we consider here $\omega_{\mathbf{X}'(w)} \otimes f^*\mathcal{L}_{\mathbf{X}(w)}(\lambda+\rho)$). We argue by induction over $\ell(w)$, the case where $\ell(w) = 0$ being obvious.

In the general case, choose a decomposition $w = sx$ where s is a simple reflection, and $\ell(x) = \ell(w) - 1$. Let P_s be the parabolic subgroup generated by B and s. This defines the variety

$$\hat{S}(w) := P_s \times^B S(x)$$

together with the map $\sigma : \hat{S}(w) \to S(w)$. Let

$$\hat{\mathbf{X}}(w) = \hat{S}(w) \times_{S(w)} \mathbf{X}'(w) = P_s \times^B \mathbf{X}'(x)$$

with projections $\tau : \hat{\mathbf{X}}(w) \to \mathbf{X}'(w)$ and $q : \hat{\mathbf{X}}(w) \to \hat{S}(w)$. Let $p : \hat{\mathbf{X}}(w) \to \mathbf{X}(w)$ be the composition of f and τ, then p is an isomorphism above BwB. Further, q is a locally trivial fibration with fiber $\overline{B}$. The $B \times B$-action on $\mathbf{X}'(w)$ lifts to $\hat{\mathbf{X}}(w)$, where $1 \times B$ acts trivially on $S(w)$ and $\hat{S}(w)$. Let $\partial \hat{S}(w)$ denote the complement of the open B-orbit in $\hat{S}(w)$.

We claim that $\sigma_* \mathcal{O}_{\hat{S}(w)} = \mathcal{O}_{S(w)}$, $\sigma_* \mathcal{I}_{\partial \hat{S}(w)} = \mathcal{I}_{\partial S(w)}$ and $R^i \sigma_* \mathcal{I}_{\partial \hat{S}(w)} = 0$ for $i \geq 1$. This follows from [25]. In more detail, consider a reduced expression for x and let $\phi : V(w) \to S(w)$ denote the standard resolution associated to the corresponding reduced expression of $w = sx$. Observe that ϕ factors through σ, say $\phi = \sigma\theta$. Since ϕ, θ, σ are proper and birational and $S(w)$, $\hat{S}(w)$ are normal, then $\phi_* \mathcal{O}_{V(w)} = \mathcal{O}_{S(w)}$, $\theta_* \mathcal{O}_{V(w)} = \mathcal{O}_{\hat{S}(w)}$ and $\sigma_* \mathcal{O}_{\hat{S}(w)} = \mathcal{O}_{S(w)}$. Let $\partial V(w)$ denote the complement of the open B-orbit in $V(w)$, then $\theta(\partial V(w)) = \partial \hat{S}(w)$ and $\sigma\theta(\partial V(w)) = \partial S(w)$, so that $\sigma_* \mathcal{I}_{\partial \hat{S}(w)} = \mathcal{I}_{\partial S(w)}$. Further, by [25, Prop. 2, Th. 4], one has

$$\omega_{V(w)} \cong \mathcal{I}_{\partial V(w)} \otimes \phi^* \mathcal{L}_{S(w)}(-\rho), \quad \omega_{S(w)} \cong \phi_* \omega_{V(w)} \cong \mathcal{I}_{\partial S(w)} \otimes \mathcal{L}_{S(w)}(-\rho),$$

and $R^i \phi_* \mathcal{O}_{V(w)} = 0 = R^i \phi_* \omega_{V(w)}$ for $i \geq 1$. Since σ is proper with fibres being points or projective lines, then $R^i \sigma_* \mathcal{I}_{\partial \hat{S}(w)} = 0$ for $i \geq 2$ and, therefore, one obtains, by using the projection formula, that

$$R^1 \sigma_* (\mathcal{I}_{\partial \hat{S}(w)}) \otimes \mathcal{L}_{S(w)}(-\rho)) \cong R^1 \sigma_* (\theta_* \omega_{V(w)}) \hookrightarrow R^1 \phi_* \omega_{V(w)} = 0.$$

This proves the claim.

Define a sheaf $\mathcal{F}(w)$ on $\hat{\mathbf{X}}(w)$ by

$$\mathcal{F}(w) := q^* \mathcal{I}_{\partial \hat{S}(w)} \otimes p^* \mathcal{L}_{\mathbf{X}(w)}(\lambda - \beta).$$

Since τ is the pull-back of σ under the locally trivial fibration g, then, using again the projection formula and the fact that cohomology commutes with flat base extension, it follows that

$$\begin{aligned} R^i \tau_* \mathcal{F}(w) &\cong (R^i \tau_* q^* \mathcal{I}_{\partial \hat{S}(w)}) \otimes f^* \mathcal{L}_{\mathbf{X}(w)}(\lambda - \beta), \\ &\cong \begin{cases} g^* \mathcal{I}_{\partial S(w)} \otimes f^* \mathcal{L}_{\mathbf{X}(w)}(\lambda - \beta) & \text{if } i = 0, \\ 0 & \text{if } i \geq 1. \end{cases} \end{aligned}$$

This yields

$$H^i(\mathbf{X}'(w), g^* \mathcal{I}_{\partial S(w)} \otimes f^* \mathcal{L}_{\mathbf{X}(w)}(\lambda - \beta)) \cong H^i(\hat{\mathbf{X}}(w), \mathcal{F}(w)), \tag{6}$$

and it suffices to prove that the right-hand side vanishes for $i \geq 1$ and for $\lambda \in \tilde{\mathcal{X}}^+$ big enough.

Embed $S(x) = B \times^B S(x)$ into $\hat{S}(w)$, as a Cartier divisor; then $\mathbf{X}'(x)$ embeds into $\hat{\mathbf{X}}(w)$. Observe that $\partial \hat{S}(w) = S(x) \cup (P_s \times^B \partial S(x))$ whereas $S(x) \cap (P_s \times^B \partial S(x)) = \partial S(x)$. Therefore, we have an exact sequence

$$0 \to \mathcal{I}_{\partial \hat{S}(w)} \to \mathcal{I}_{P_s \times^B \partial S(x)} \to \mathcal{I}_{\partial S(x)} \otimes_{\mathcal{O}_{\hat{S}(w)}} \mathcal{O}_{S(x)} \to 0.$$

Denote by $\iota : \mathbf{X}'(x) \to \hat{\mathbf{X}}(w)$ the inclusion and set

$$\mathcal{F}(w, x) := q^* \mathcal{I}_{P_s \times^B \partial S(x)} \otimes p^* \mathcal{L}_{\mathbf{X}(w)}(\lambda - \beta),$$

$$\mathcal{F}(x) := \iota^* \mathcal{F}(w, x).$$

Then $\mathcal{F}(x)$ is the sheaf $g^* \mathcal{I}_{\partial S(x)} \otimes f^* \mathcal{L}_{\mathbf{X}(x)}(\lambda - \beta)$ on $\mathbf{X}'(x)$, and we have an exact sequence

$$0 \to \mathcal{F}(w) \to \mathcal{F}(w, x) \to \iota_* \mathcal{F}(x) \to 0.$$

The corresponding long exact sequence of cohomology groups, together with the induction hypothesis applied to x, yields an exact sequence

$$\begin{aligned} H^0(\hat{\mathbf{X}}(w), \mathcal{F}(w,x)) &\to H^0(\mathbf{X}'(x), \mathcal{F}(x)) \\ &\to H^1(\hat{\mathbf{X}}(w), \mathcal{F}(w)) \to H^1(\hat{\mathbf{X}}(w), \mathcal{F}(w,x)) \end{aligned} \tag{7}$$

and isomorphisms for $i \geq 2$,

$$H^i(\hat{\mathbf{X}}(w), \mathcal{F}(w)) \cong H^i(\hat{\mathbf{X}}(w), \mathcal{F}(w,x)). \tag{8}$$

Further, consider the projection

$$\pi : \hat{\mathbf{X}}(w) = P_s \times^B \mathbf{X}'(x) \to P_s/B.$$

Then the sheaf $R^j\pi_*\mathcal{F}(w,x)$ is the homogeneous vector bundle on P_s/B associated with the B-module $H^j(\mathbf{X}'(x), \mathcal{F}(x))$. The latter vanishes for $j \geq 1$ and large λ by the induction hypothesis applied to x. As P_s/B is a projective line, it follows that $H^i(\hat{\mathbf{X}}(w), \mathcal{F}(w,x)) = 0$ for $i \geq 2$. Therefore, by (8), we obtain that

$$H^i(\hat{\mathbf{X}}(w), \mathcal{F}(w)) = 0$$

for $i \geq 2$. Moreover, setting

$$M := H^0(\mathbf{X}'(x), \mathcal{F}(x)),$$

we have that $\pi_*\mathcal{F}(w,x) = \underline{M}$. Thus, (7) gives the exact sequence

$$H^0(P_s/B, \underline{M}) \to M \to H^1(\hat{\mathbf{X}}(w), \mathcal{F}(w)) \to H^1(P_s/B, \underline{M}).$$

To complete the proof, it remains to show that $H^1(\hat{\mathbf{X}}(w), \mathcal{F}(w)) = 0$. For this, it is enough to check that $\underline{M}$ is generated by its global sections, that is, that M is the quotient of a P_s-module. Now, using (1) and (3), applied to x instead of w, observe that

$$M \cong H^0(\mathbf{X}(x), \mathcal{I}_{\partial\mathbf{X}(x)} \otimes \mathcal{L}_{\mathbf{X}(x)}(\lambda)).$$

Further, $\partial\mathbf{X}(x) = \mathbf{X}(x) \cap P_s\partial\mathbf{X}(x)$ (indeed, $\partial\mathbf{X}(x)$ is obviously contained in $P_s\partial\mathbf{X}(x) \cap \mathbf{X}(x)$; and $\mathbf{X}(x)$ is not contained in $P_s\partial\mathbf{X}(x)$, because $\mathbf{X}(x)$ is not stable by P_s), and this intersection is reduced as large Schubert varieties are compatibly split in $\mathbf{X}$. Therefore, $\mathcal{I}_{\partial\mathbf{X}(x)} = \mathcal{I}_{P_s\partial\mathbf{X}(x)} \otimes_{\mathcal{O}_{\mathbf{X}(w)}} \mathcal{O}_{\mathbf{X}(x)}$, and the restriction map

$$H^0(\mathbf{X}(w), \mathcal{I}_{P_s\partial\mathbf{X}(x)} \otimes \mathcal{L}_{\mathbf{X}(w)}(\lambda)) \to H^0(\mathbf{X}(x), \mathcal{I}_{\partial\mathbf{X}(x)} \otimes \mathcal{L}_{\mathbf{X}(x)}(\lambda)) = M$$

is surjective for λ big enough, by Serre's theorem. Thus, M is a quotient of a P_s-module. This completes the proof of (ii).

Now the previous arguments and Lemma 15 imply assertion (iii). Then (iv) follows by arguing as in the proof of Theorem 14. □

In particular, the closure in $\mathbf{X}$ of any parabolic subgroup P is Cohen-Macaulay. As in Section 6, this yields a degeneration of the diagonal in G/P into a union of Schubert varieties, and formulae for the class of the diagonal in $K^T(G/P \times G/P)$.

Consider now the subvariety $Z(w) = \mathbf{X}(w) \cap Y$ of $\mathbf{X}(w)$, and its preimage $Z'(w)$ under $f : \mathbf{X}'(w) \to \mathbf{X}(w)$. We still denote by $f : Z'(w) \to Z(w)$ and $g : Z'(w) \to S(w)$ the restrictions of f and g; then g is a locally trivial fibration with fiber Z.

As $Z'(w) = (\overline{BwB} \cap G) \times^B Z$, one has $g^{-1}\partial S(w) = \bigcup_{x<w} (\overline{BxB} \cap G) \times^B Z$, and hence $f(g^{-1}\partial S(w)) = \bigcup_{x<w} \overline{BxZ}$. Therefore,

$$f(g^{-1}\partial S(w)) = \bigcup_{x<w} Z(x) = \bigcup_{\substack{x,y\in W \\ x<w,\, \ell(xy)=\ell(x)+\ell(y)}} S(xy) \times S(yw_0).$$

We shall denote this subvariety of $Z(w)$ by $\delta Z(w)$.

Corollary 21. *With notation as above, we have:*

(i) *$Z'(w)$ is Cohen-Macaulay and its dualizing sheaf is isomorphic to $g^*\mathcal{I}_{\partial S(w)} \otimes f^*\mathcal{L}_{Z(w)}(-\rho)[\rho,\rho]$ as a $\tilde{B} \times \tilde{B}$-linearized sheaf.*

(ii) *$f_*\mathcal{O}_{Z'(w)} = \mathcal{O}_{Z(w)}$ and $R^i f_*\mathcal{O}_{Z'(w)} = R^i f_*\omega_{Z'(w)} = 0$ for $i \geq 1$.*

(iii) *$Z(w)$ is Cohen-Macaulay and its dualizing sheaf is isomorphic to $f_*\omega_{Z'(w)} = \mathcal{I}_{\delta Z(w)} \otimes \mathcal{L}_{Z(w)}(-\rho)[\rho,\rho]$ as a $\tilde{B} \times \tilde{B}$-linearized sheaf.*

(iv) *The graded ring $A(w) = \bigoplus_{\mu\in\tilde{\mathcal{X}}^+} H^0(Z(w),\mu)$ is Cohen-Macaulay.*

Proof. Since $Z(w)$ is the complete intersection in $\mathbf{X}(w)$ of the Cartier divisors $\mathbf{X}(w) \cap D_1, \ldots, \mathbf{X}(w) \cap D_r$, by Corollary 4, it follows that $Z(w)$ is Cohen-Macaulay. Similarly, $Z'(w)$ is Cohen-Macaulay and its dualizing sheaf is the restriction to $Z'(w)$ of

$$\omega_{\mathbf{X}'(w)} \otimes f^*\mathcal{L}_{\mathbf{X}(w)}(\alpha_1) \otimes \cdots \otimes f^*\mathcal{L}_{\mathbf{X}(w)}(\alpha_r) = \omega_{\mathbf{X}(w')} \otimes f^*\mathcal{L}_{\mathbf{X}(w)}(\beta).$$

The latter is equal to $g^*\mathcal{I}_{\partial S(w)} \otimes f^*\mathcal{L}_{\mathbf{X}(w)}(-\rho)[\rho,\rho]$, as we saw in the proof of Theorem 20. This proves (i).

The multiplication by σ_1 defines exact sequences

$$0 \to \mathcal{L}_{\mathbf{X}(w)}(-\alpha_1) \to \mathcal{O}_{\mathbf{X}(w)} \to \mathcal{O}_{\mathbf{X}(w)\cap D_1} \to 0$$

and

$$0 \to f^*\mathcal{L}_{\mathbf{X}(w)}(-\alpha_1) \to \mathcal{O}_{\mathbf{X}'(w)} \to \mathcal{O}_{\mathbf{X}'(w)\cap f^{-1}(D_1)} \to 0.$$

By Theorem 20(ii), it follows that

$$f_*\mathcal{O}_{\mathbf{X}'(w)\cap f^{-1}(D_1)} = \mathcal{O}_{\mathbf{X}(w)\cap D_1} \quad \text{and} \quad R^i f_*\mathcal{O}_{\mathbf{X}'(w)\cap D_1} = 0 \quad \text{for } i \geq 1.$$

Iterating this argument, we obtain $f_*\mathcal{O}_{Z'(w)} = \mathcal{O}_{Z(w)}$ and $R^i f_*\mathcal{O}_{Z'(w)} = 0$ for $i \geq 1$. The vanishing of $R^i f_*\omega_{Z'(w)}$ and the equality $f_*\omega_{Z'(w)} = \omega_{Z(w)}$ follow similarly from the exact sequences

$$0 \to \omega_{\mathbf{X}(w)} \to \omega_{\mathbf{X}(w)} \otimes \mathcal{L}_{\mathbf{X}(w)}(\alpha_1) \to \omega_{\mathbf{X}(w)\cap D_1} \to 0$$

and

$$0 \to \omega_{\mathbf{X}'(w)} \to \omega_{\mathbf{X}'(w)} \otimes f^*\mathcal{L}_{\mathbf{X}(w)}(\alpha_1) \to \omega_{\mathbf{X}'(w)\cap f^{-1}(D_1)} \to 0$$

together with Theorem 20(ii). This proves (ii).

It also follows, using Lemma 15, that

$$\omega_{Z(w)} = f_*\omega_{Z'(w)} = f_*g^*\mathcal{I}_{\partial S(w)} \otimes \mathcal{L}_{Z(w)}(-\rho)[\rho,\rho].$$

Furthermore, $g^*\mathcal{I}_{\partial S(w)} = \mathcal{I}_{g^{-1}(\partial S(w))}$, since g is a locally trivial fibration, and $f_*\mathcal{I}_{g^{-1}(\partial S(w))} = \mathcal{I}_{fg^{-1}(\partial S(w))}$, since $f_*\mathcal{O}_{Z'(w)} = \mathcal{O}_{Z(w)}$. This completes the proof of (iii).

Finally, (iv) is checked as in the proof of Theorem 14. $\square$

We now apply these geometric results to the structure of the $\tilde{B} \times \tilde{B}$-modules $H^0(\mathbf{X}(w), \lambda)$ and $H^0(Z(w), \lambda)$. For this, we recall the definition of the Joseph functors; see [24, 1.4] and [29, 2.2]. Let $y, z \in W$ and let N (resp. M) be a $\tilde{B}$-module (resp. $\tilde{B} \times \tilde{B}$-module), then

$$H_y N := H^0(S(y), \underline{N}) \quad \text{and} \quad H_{y,z} M := H^0(S(y) \times S(z), \underline{M}),$$

where $\underline{N}$ (resp. $\underline{M}$) is the corresponding $\tilde{G}$ (resp. $\tilde{G} \times \tilde{G}$) linearized vector bundle on G/B (resp. $G/B \times G/B$). Observe that $H_y M$, where M is regarded as a $\tilde{B} \times 1$-module, has a natural structure of $\tilde{B} \times \tilde{B}$-module and, furthermore, there is an isomorphism of $\tilde{B} \times \tilde{B}$-modules $H_y M \cong H_{y,1} M$.

Corollary 22. *For any weight λ, we have*

$$H^0(\mathbf{X}(w), \lambda) \cong H_{w,1} H^0(\overline{B}, \lambda) \quad \textit{and} \quad H^0(Z(w), \lambda) \cong H_{w,1} M(\lambda).$$

Further, each endomorphism of the $\tilde{B} \times \tilde{B}$-module $H^0(Z(w), \lambda)$ is scalar. In particular, this module is indecomposable.

Proof. Recall that $H^0(\mathbf{X}(w), \lambda) = H^0(\mathbf{X}(w), \mathcal{L}_{\mathbf{X}(w)}(\lambda))$. By Theorem 20, the latter is isomorphic to

$$\begin{aligned} H^0(\mathbf{X}'(w), f^* \mathcal{L}_{\mathbf{X}(w)}(\lambda)) &\cong H^0(S(w), g_* f^* \mathcal{L}_{\mathbf{X}(w)}(\lambda)) \\ &\cong H^0(S(w), \underline{H^0(\overline{B}, \lambda)}) \cong H_{w,1} H^0(\overline{B}, \lambda). \end{aligned}$$

Using Corollary 21, we obtain similarly that $H^0(Z(w), \lambda) \cong H_{w,1} M(\lambda)$.

We prove that $\operatorname{End}_{\tilde{B} \times \tilde{B}} H^0(Z(w), \lambda) = k$ by descending induction on $\ell(w)$. If $w = w_0$, then $Z(w) = Y$ and $H^0(Z(w), \lambda) = H^0(G/B, \lambda) \boxtimes H^0(G/B, -w_0 \lambda)$. In this case, the assertion follows from [14, II.2.8, II.4.7].

In the general case, let s be a simple reflection such that $\ell(sw) = \ell(w) + 1$; let $\tilde{P}_s$ be the parabolic subgroup of $\tilde{G}$ generated by $\tilde{B}$ and s. Then, using [29, 2.2.5], we obtain that

$$H^0(Z(sw), \lambda) \cong H_{sw} M(\lambda) \cong \operatorname{ind}_{\tilde{B}}^{\tilde{P}_s} H_w M(\lambda) \cong \operatorname{ind}_{\tilde{B}}^{\tilde{P}_s} H^0(Z(w), \lambda).$$

Further, the natural map

$$\operatorname{ind}_{\tilde{B}}^{\tilde{P}_s} H^0(Z(w), \lambda) \to H^0(Z(w), \lambda)$$

is surjective by Corollary 3. Thus, $\operatorname{End}_{\tilde{B} \times \tilde{B}} H^0(Z(w), \lambda)$ embeds into

$$\operatorname{Hom}_{\tilde{B} \times \tilde{B}} \big(\operatorname{ind}_{\tilde{B}}^{\tilde{P}_s} H^0(Z(w), \lambda),\ H^0(Z(w), \lambda)\big) \cong \operatorname{End}_{\tilde{P}_s \times \tilde{B}} \big(\operatorname{ind}_{\tilde{B}}^{\tilde{P}_s}\ H^0(Z(w), \lambda)\big).$$

The latter equals $\operatorname{End}_{\tilde{B} \times \tilde{B}} H^0(Z(sw), \lambda)$ by [14, II.2.1.(7)], and we conclude by the induction hypothesis. $\square$

Remark. By looking at right actions, one can also prove that

$$H^0(\mathbf{X}(w), \lambda) \cong H_{1,w^{-1}} H^0(\overline{B}, \lambda) \text{ and } H^0(Z(w), \lambda) \cong H_{1,w^{-1}} M(\lambda).$$

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