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COMMUTATIVE QUANTUM CURRENT OPERATORS, SEMI-INFINITE CONSTRUCTION AND FUNCTIONAL MODELS

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ABSTRACT. We construct the commutative current operator $\bar{x}^+(z)$ inside $U_q(\hat{\mathfrak{sl}}(2))$. With this operator and the condition of quantum integrability on the quantum currents of $U_q(\hat{\mathfrak{sl}}(2))$, we derive the quantization of the semi-infinite construction of integrable modules of $\hat{\mathfrak{sl}}(2)$ which has been previously obtained by means of the current operator $e(z)$ of $\hat{\mathfrak{sl}}(2)$. The quantization of the functional models for $\hat{\mathfrak{sl}}(2)$ is also given.

1. Introduction

In this paper, we fix the notation that z, w, z_i are commuting formal variables. Given a current operator

$$\bar{a}(z) = \sum_{\mathbb{Z}} \bar{a}(n) z^{-n},$$

if

$$[\bar{a}(z), \bar{a}(w)] = 0,$$

which is equivalent to the condition that all the components $\bar{a}(n)$ commute with each other, then we call the current operator $\bar{a}(z)$ a commutative current operator. Here, we also assume that the current operator $\bar{a}(z)$ always acts on a space F in a truncated way such that, for any element $v \in F$, there exists an integer m such that

$$\bar{a}(n)v = 0$$

if $n > m$. In this case, if a current operator $\bar{a}(z)$ is commutative, then $\bar{a}(z)^n = \bar{a}(z) \times \bar{a}(z) \cdots \times \bar{a}(z)$, for $n \in \mathbb{Z}_{>0}$, is a well defined current operator.

For any integrable highest weight module of $\hat{\mathfrak{sl}}(2)$ of level k, the commutative current operators $e(z)$ and $f(z)$ of $\hat{\mathfrak{sl}}(2)$ satisfy the following relations:

$$e(z)^{k+1} = f(z)^{k+1} = 0,$$

which we call the condition of integrability [LP]. For any integrable highest weight module of $\hat{\mathfrak{sl}}(2)$, there is a natural grading such that the grade of any homogeneous element is always larger or equal to zero and the action of $x(n)$ changes the grade of a homogeneous element by $-n$. This ensures that the current operators from $\hat{\mathfrak{sl}}(2)$ always act in a truncated way. For the case of quantum affine algebras, Drinfeld

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presented a formulation of affine quantum groups with generators in the form of current operators [Dr3], which, for the case of $U_q(\hat{\mathfrak{sl}}(2))$, give us the quantized current operators corresponding to $e(z)$ and $f(z)$ of $\hat{\mathfrak{sl}}(2)$. In [DM], we derive the quantum integrability condition for $U_q(\hat{\mathfrak{sl}}(2))$. On any level k integrable highest weight module of $U_q(\hat{\mathfrak{sl}}(2))$, the matrix coefficients of $x^+(z_1)x^+(z_2)\cdots x^+(z_{k+1})$ are zero at $z_2/z_1 = z_3/z_2 = \cdots = z_{k+1}/z_k = q^2$, and those of $x^-(z_1)x^-(z_2)\cdots x^-(z_{k+1})$ are zero at $z_1/z_2 = z_2/z_3 = \cdots = z_k/z_{k+1} = q^2$, where $x^+(z)$ and $x^-(z)$ are the quantized current operators of $U_q(\hat{\mathfrak{sl}}(2))$ corresponding to $e(z)$ and $f(z)$ of $\hat{\mathfrak{sl}}(2)$ respectively. In the case of $\hat{\mathfrak{sl}}(2)$, the condition of integrability was used by Feigin and Stoyanovsky [FS1], [FS2] to construct a level k module from a semi-infinite tensor of the components of the current operator $e(z)$ of $\hat{\mathfrak{sl}}(2)$ and to use the function models to describe the dual spaces. With the condition of quantum integrability, still we cannot simply derive the quantization of the semi-infinite construction, because of the noncommutativity of the current operator $x^+(z)$, which is that

$$[x^+(z), x^+(w)] \neq 0.$$

Thus we have to modify the current operator $x^+(z)$ to "force" it to be a commutative current operator. We use the subalgebra coming from the Heisenberg algebra of $U_q(\hat{\mathfrak{sl}}(2))$ to construct a commutative current operator $\bar{x}^+(z) = \Sigma \bar{x}_i z^{-i}$ such that the condition

$$\bar{x}^+(z_1)\bar{x}^+(z_1q^2)\cdots\bar{x}^+(z_1q^{2k}) = 0$$

is satisfied as well. Then the quantization of the semi-infinite construction simply follows. Namely, the integrable modules of $U_q(\hat{\mathfrak{sl}}(2))$ can be identified with the space consisting of semi-infinite expressions $\bar{x}^+_{i_1}\cdots\bar{x}^+_{in}\cdots$, whose tails stabilize in a certain way and $\bar{x}^+_i$ acts by multiplication. Due to the introduction of the parameter q, we can describe the action of the operators explicitly, especially the action of the operator a_{-1} which corresponds to the operator h_{-1} of $\hat{\mathfrak{sl}}(2)$. As in the case of [FS2], the functional models for the dual spaces of the subspace generated by $\bar{x}^+(z)$ on the highest weight vector in any irreducible integrable module of level k are derived by using symmetric functions $f(t_1,\ldots,t_n)$, which are zero when $t_1 = t_2q^2 = \cdots = t_{k+1}q^{2k}$.

2. $U_q(\hat{\mathfrak{sl}}(n))$ AND COMMUTATIVE QUANTUM CURRENT OPERATORS

For the case of affine quantum groups, Drinfeld gave a realization of those algebras in the form of current operators [Dr3]. We will first present such a realization for the case of $U_q(\hat{\mathfrak{sl}}(n))$.

Let $A = (a_{ij})$ be the Cartan matrix of type A_{n-1}.

Definition 1. The algebra $U_q(\mathfrak{sl}_n)$ is an associative algebra with unit 1 and the generators $\varphi_i(-m)$, $\psi_i(m)$, $x_i^{\pm}(l)$, for $i = 1,\ldots,n-1$, $l \in \mathbb{Z}$ and $m \in \mathbb{Z}_{\geq 0}$ and a central element c. Let $x_i^{\pm}(z) = \sum_{l\in\mathbb{Z}} x_i^{\pm}(l)z^{-l}$, $\varphi_i(z) = \sum_{m\in\mathbb{Z}_{\leq 0}} \varphi_i(m)z^{-m}$ and $\psi_i(z) = \sum_{m\in\mathbb{Z}_{\geq 0}} \psi_i(m)z^{-m}$. In terms of the formal variables, the defining relations are

$$
\begin{aligned}
&\varphi_i(0)\psi_i(0)=\psi_i(0)\varphi_i(0)=1,\\
&\varphi_i(z)\varphi_j(w)=\varphi_j(w)\varphi_i(z),\\
&\psi_i(z)\psi_j(w)=\psi_j(w)\psi_i(z),\\
&\varphi_i(z)\psi_j(w)\varphi_i(z)^{-1}\psi_j(w)^{-1}=\frac{g_{ij}(\frac{z}{w}q^{-c})}{g_{ij}(\frac{z}{w}q^{c})},\\
&\varphi_i(z)x_j^{\pm}(w)\varphi_i(z)^{-1}=g_{ij}(\frac{z}{w}q^{\mp\frac{1}{2}c})^{\pm1}x_j^{\pm}(w),\\
&\psi_i(z)x_j^{\pm}(w)\psi_i(z)^{-1}=g_{ij}(\frac{w}{z}q^{\mp\frac{1}{2}c})^{\mp1}x_j^{\pm}(w),\\
&[x_i^+(z),x_j^-(w)]=\frac{\delta_{i,j}}{q-q^{-1}}\left\{\delta(\frac{z}{w}q^{-c})\psi_i(wq^{\frac{1}{2}c})-\delta(\frac{z}{w}q^{c})\varphi_i(zq^{\frac{1}{2}c})\right\},\\
&(z-q^{\pm a_{ij}}w)x_i^{\pm}(z)x_j^{\pm}(w)=(q^{\pm a_{ij}}z-w)x_j^{\pm}(w)x_i^{\pm}(z),\\
&[x_i^{\pm}(z),x_j^{\pm}(w)]=0\quad \text{for } a_{ij}=0,\\
&x_i^{\pm}(z_1)x_i^{\pm}(z_2)x_j^{\pm}(w)-(q+q^{-1})x_i^{\pm}(z_1)x_j^{\pm}(w)x_i^{\pm}(z_2)\\
&\quad+x_j^{\pm}(w)x_i^{\pm}(z_1)x_i^{\pm}(z_2)+\{z_1\leftrightarrow z_2\}=0,\quad \text{for } a_{ij}=-1
\end{aligned}
$$

where

$$\delta(z)=\sum_{k\in\mathbb{Z}}z^k,\quad g_{ij}(z)=\frac{q^{a_{ij}}z-1}{z-q^{a_{ij}}}\quad \text{about } z=0.$$

We define a grading on this algebra such that $x_i^{\pm}(n)$, $\varphi_i(n)$ and $\psi_i(n)$ are of degree n. We also assume that q is a non-zero complex number, which is also not a root of unity; c always acts as a constant; and q^c is defined by the value of the analytic function $e^{x\log q}$ at $x=c$.

Clearly, we have that $x^+(z)$ is not a commutative current operator. In order to modify this operator, we have to rewrite the operators $\varphi_i(z)$ and $\psi_i(z)$ with new operators $a_{i,n}$ for $n\in\mathbb{Z}_{\neq0}$. From now on, we assume that our current operators act on a highest weight module, where $\varphi_i(0)$ and $\psi_i(0)$, with a suitable weighted basis of the module, act as invertible and diagonal operators.

For the case of $U_q(\mathfrak{sl}_2)$, where we will write $x_1^{\pm}(z)$ as $x^{\pm}(z)$, $a_{1,n}$ as a_n, $\psi_1(z)$ as $\psi(z)$ and $\varphi_1(z)$ as $\varphi(z)$. Because $\psi(0)$ and $\varphi(0)$ are invertible and diagonal, the new operators are defined as

$$
\begin{aligned}
-(q-q^{-1})\sum_{k>0}a_{-k}z^k&=\log(1+(\varphi(z)\varphi(0)^{-1}-1))\\
&=\sum_{n>0}(-\varphi(z)\varphi(0)^{-1}+1)^n/n,\\
(q-q^{-1})\sum_{k<0}a_{-k}z^k&=\log(1+(\psi(z)\psi(0)^{-1}-1))\\
&=\sum_{n>0}(-\psi(z)\psi(0)^{-1}+1)^n/n,
\end{aligned}
$$

where the right side of the first formula is understood as an infinite series over $n > 0$. We also have that

$$\varphi(z) = \varphi(0)\exp[-(q-q^{-1})\sum_{k>0} a_{-k}z^k],$$
$$\psi(z) = \psi(0)\exp[(q-q^{-1})\sum_{k<0} a_{-k}z^k].$$

Proposition 2.

$$[a_k, a_l] = \delta_{k+l,0}(q^{2k} - q^{-2k})(q^c - q^{-c})/(k(q-q^{-1})^2),$$

$$[a_k, x^{\pm}(l)] = (q^{2k} - q^{-2k})q^{\mp|c|/2}x^{\pm}(k+l)/(k(q-q^{-1})),$$

where k, l are not zero.

Let $k^-(z)$ be a current operator in $U_q(\hat{\mathfrak{sl}}(2))$ such that

$$\text{(I)} \quad k^-(z) = 1 + \sum_{n>0} k^-(n)z^{-n},$$
$$\text{(II)} \quad k^-(z)x^+(w) = \frac{z-wq^2}{z-w}x^+(w)k^-(z),$$

where $k^-(n)$ are operators of degree n, $k^-(n)k^-(m) = k^-(m)k^-(n)$ and $\frac{z-wq^2}{z-w}$ is expanded about zero.

Let $\bar{x}^+(w) = x^+(w)k^-(w)$, then we have

Proposition 3.

$$(z-w)\bar{x}^+(z)\bar{x}^+(w) = (z-w)\bar{x}^+(w)\bar{x}^+(z).$$

The proof comes from the following calculation:

$$\begin{aligned}
(z-w)\bar{x}^+(z)\bar{x}^+(w) &= (z-w)x^+(z)k^-(z)x^+(w)k^-(w) \\
&= (z-w)\frac{z-wq^2}{z-w}x^+(z)x^+(w)k^-(z)k^-(w) \\
&= (zq^2-w)x^+(w)x^+(z)k^-(w)k^-(z) \\
&= x^+(w)k^-(w)x^+(z)k^-(z)(zq^2-w)\left(\frac{w-zq^2}{w-z}\right)^{-1} \\
&= (z-w)\bar{x}^+(w)\bar{x}^+(z).
\end{aligned}$$

Theorem 4.

$$\bar{x}^+(z)\bar{x}^+(w) = \bar{x}^+(w)\bar{x}^+(z).$$

Proof. Let V_k be a highest weight module of $U_q(\hat{\mathfrak{sl}}(2))$ and V_k^* be its restricted dual. Let $v \in V_k$ and $v^* \in V_k^*$. First, we have that $\langle v^*, x^+(z)x^+(w)v\rangle$ is a formal infinite series in z, w, z^{-1}, w^{-1}. However, we know that this infinite series converges in the complex domain $0 < |w| << |z|$ in $\mathbb{C}^2$, which we can extend to $\mathbb{C}^2$ as a single-valued holomorphic function through analytic continuation. Now, we treat $\langle v^*, x^+(z)x^+(w)v\rangle$ as this complex function. We will denote it by $F(z,w)_{v,v^*}$, which we call the correlation function. From the commutation relation between $x^+(z)$ and $x^+(w)$, we know that the function $F(z,w)_{v,v^*}$ is zero when $z = w$, thus $F(z,w)_{v,v^*}$

always has a factor $z-w$. This implies that $F(z,w)_{v,v^*}$ does not have a pole at $z=w$, thus

$$\bar{x}^+(z)\bar{x}^+(w) = \bar{x}^+(w)\bar{x}^+(z)$$

follows from

$$(z-w)\bar{x}^+(z)\bar{x}^+(w) = (z-w)\bar{x}^+(w)\bar{x}^+(z).$$

$\square$

Proposition 5. *Let*

$$k^-(z) = \exp[(q-q^{-1})\sum_{n>0} -q^{n(2+c/2)}/(1+q^{2n})a_n z^{-n}].$$

Then $k^-(z)$ *satisfies* (I) *and* (II).

Now, we will denote $x^+(z)\exp[(q-q^{-1})\sum_{n>0} -q^{n(2+c/2)}/(1+q^{2n})a_n z^{-n}]$ by $\bar{x}^+(z)$ throughout this paper.

Let $k^+(z)$ be a current operator in $U_q(\hat{\mathfrak{sl}}(2))$ such that

$$\text{(I')}\quad k^+(z) = 1 + \Sigma_{n<0} k^+(n) z^{-n},$$

$$\text{(II')}\quad k^+(z)x^+(w) = \frac{z-w}{zq^2-w}x^+(w)k^+(z),$$

where $k^+(n)$ are operators of degree n, $k^+(n)k^+(m) = k^+(m)k^+(n)$ and $\frac{z-w}{zq^2-w}$ is expanded about zero. Let $\tilde{x}^+(w) = k^+(w)x^+(w)$. Then we have

Proposition 6.

$$(z-w)\tilde{x}^+(z)\tilde{x}^+(w) = (z-w)\tilde{x}^+(w)\tilde{x}^+(z).$$

The proof comes from the following calculation:

$$\begin{aligned}
(z-w)\tilde{x}^+(z)\tilde{x}^+(w) &= (z-w)k^+(z)x^+(z)k^+(w)x^+(w) \\
&= (z-w)\frac{z-wq^2}{z-w}k^+(z)k^+(w)x^+(z)x^+(w) \\
&= (zq^2-w)k^+(z)k^+(w)x^+(w)x^+(z) \\
&= k^+(w)x^+(w)k^+(z)x^+(z)(zq^2-w)\left(\frac{z-w}{zq^2-w}\right) \\
&= (z-w)\tilde{x}^+(w)\tilde{x}^+(z).
\end{aligned}$$

Theorem 7.

$$\tilde{x}^+(z)\tilde{x}^+(w) = \tilde{x}^+(w)\tilde{x}^+(z).$$

The proof is the same as that of Theorem 4 above.

Proposition 8. *Let*

$$k^+(z) = \exp[-(q-q^{-1})\sum_{n<0} -q^{n(2+c/2)}/(1+q^{2n})a_n z^{-n}].$$

Then $k^+(z)$ *satisfies the condition* (I') *and* (II').

Now, we will denote the operator

$$\exp[-(q-q^{-1})\sum_{n<0} -q^{n(2+c/2)}/(1+q^{2n})a_n w^{-n}]x^+(w)$$

by $\tilde{x}^+(w)$.

For the case of $U_q(\mathfrak{sl}_n)$, the new operators are defined as

$$\begin{aligned}
-(q-q^{-1})\sum_{k>0} a_{i,-k}z^k &= \log(1+(\varphi_i(z)\varphi_i(0)^{-1}-1)) \\
&= \sum_{n>0}(-\varphi_i(z)\varphi_i(0)^{-1}+1)^n/n, \\
(q-q^{-1})\sum_{k<0} a_{i,-k}z^k &= \log(1+(\psi_i(z)\psi_i(0)^{-1}-1)) \\
&= \sum_{n>0}(-\psi_i(z)\psi_i(0)^{-1}+1)^n/n.
\end{aligned}$$

We also have that

$$\begin{aligned}
\varphi_i(z) &= \varphi_i(0)\exp[-(q-q^{-1})\sum_{k>0} a_{i,-k}z^k], \\
\psi_i(z) &= \psi_i(0)\exp[(q-q^{-1})\sum_{k<0} a_{i,-k}z^k].
\end{aligned}$$

Let

$$k_i^+(z) = \exp[-(q-q^{-1})\sum_{n<0} -q^{n(2+c/2)}/(1+q^{2n})a_{i,n}z^{-n}],$$

and

$$k_i^-(z) = \exp[(q-q^{-1})\sum_{n>0} -q^{n(2+c/2)}/(1+q^{2n})a_{i,n}z^{-n}].$$

Let

$$\bar{x}_i^+(z) = x_i^+(z)k_i^-(z),$$

and

$$\tilde{x}_i^+(z) = k_i^+(z)x_i^+(z).$$

Theorem 9.

$$\begin{aligned}
\bar{x}_i^+(z)\bar{x}_i^+(w) &= \bar{x}_i^+(w)\bar{x}_i^+(z), \\
\tilde{x}_i^+(z)\tilde{x}_i^+(w) &= \tilde{x}_i^+(w)\tilde{x}_i^+(z).
\end{aligned}$$

It is obvious that both the set of current operators $\varphi_i(z)$, $\psi_i(z)$, $\tilde{x}_i^+(z)$ and $x_i^-(z)$ and the set of the current operators $\varphi_i(z)$, $\psi_i(z)$, $\bar{x}_i^+(z)$ and $x_i^-(z)$ generate the quantum affine algebra $U_q(\hat{\mathfrak{sl}}(n))$. The reformulation of the quantum affine algebra $U_q(\hat{\mathfrak{sl}}(n))$ with current operators $\varphi_i(z)$, $\psi_i(z)$, $\bar{x}_i^+(z)$ and $x_i^-(z)$ is the key for the quantized semi-infinite construction in the next section; namely, we need to use the kernel coming from the current operator $\bar{x}_i^+(z)$ to define the semi-infinite space. Now, we will restrict ourselves to the case of $U_q(\hat{\mathfrak{sl}}(2))$. The case for $U_q(\hat{\mathfrak{sl}}(n))$ can be dealt with in a similar way [FS1].

For the case of $U_q(\hat{\mathfrak{sl}}(2))$, the relations between $\psi(z)$, $\bar{x}^+(z)$ is the same as that of $\psi(z)$, $x^+(z)$, however, the rest are changed, which we write below.

Proposition 10.

$$\begin{aligned}
&\varphi(z)\bar{x}^+(w)\varphi(z)^{-1} = f_1(\frac{z}{w})g(\frac{z}{w}q^{-\frac{1}{2}c})\bar{x}^+(w),\\
&f_2(w/z)\bar{x}^+(z)x^-(w) - x^-(w)\bar{x}^+(z)\\
&\qquad = \frac{\delta_{i,j}}{q-q^{-1}}\left\{\delta(\frac{z}{w}q^{-c})\psi(wq^{\frac{1}{2}c})k^-(z) - \delta(\frac{z}{w}q^{c})\varphi(zq^{\frac{1}{2}c})k^-(z)\right\},\\
&\bar{x}^+(z)\bar{x}^+(w) = \bar{x}^+(w)\bar{x}^+(z),
\end{aligned}$$

where

$$f_1(\frac{z}{w}) = \left(\frac{(1-\frac{z}{w}q^2q^{c/2})(1-\frac{z}{w}q^{c/2})}{(1-\frac{z}{w}q^2q^{3c/2})(1-\frac{z}{w}q^{3c/2})}\right)^{-1}$$

and

$$f_2(w/z) = (q^2q^cw/z-1)(w/zq^c-1).$$

Similarly, one can write down the relations between $\varphi(z)$, $\psi(z)$, $\tilde{x}^+(z)$ and $x^-(z)$, which we omit here. In the next section, we will use $\bar{x}^+(z)$ instead of $x^+(z)$ as the current operator for our semi-infinite construction of representations of $U_q(\hat{\mathfrak{sl}}(2))$.

3. Quantum integrability condition and semi-infinite construction

The integrability condition of the current operator $e(z)$ induces the semi-infinite construction for the unquantized case. The quantum integrability condition was studied in [DM], which is stated as the following:

Theorem 11. *For any level $k \geq 1$ integrable highest weight module of $U_q(\hat{\mathfrak{sl}}(2))$, the correlation function of $x^+(z_1)x^+(z_2)\cdots x^+(z_k)x^+(z_{k+1})$ is zero if $z_2/z_1 = z_3/z_2 = \cdots = z_{k+1}/z_k = q^2$, the correlation function of $x^-(z_1)x^-(z_2)\cdots x^-(z_k)x^-(z_{k+1})$ is zero if $z_1/z_2 = z_2/z_3 = \cdots = z_k/z_{k+1} = q^2$.*

However, this condition cannot be directly used for the semi-infinite construction, because of the noncommutativity of the current operator $x^+(z)$ of $U_q(\hat{\mathfrak{sl}}(2))$. However, the theorem above implies:

Corollary 12. *For any level $k \geq 1$ integrable highest weight module of $U_q(\hat{\mathfrak{sl}}(2))$,*

$$\bar{x}^+(z_1)\bar{x}^+(z_2)\cdots\bar{x}^+(z_k)\bar{x}^+(z_{k+1}) = 0$$

if $z_2/z_1 = z_3/z_2 = \cdots = z_{k+1}/z_k = q^2$.

Proof. Let $\bar{F}(z_1,\ldots,z_n)$ be the correlation function of a vector v in any level $k \geq 1$ integrable module of $U_q(\hat{\mathfrak{sl}}(2))$ and v^* in the dual space of this level k module, $\langle v^*, \bar{x}^+(z_1)\bar{x}^+(z_2)\cdots\bar{x}^+(z_k)\bar{x}^+(z_{k+1})v\rangle$. Then we have

$$\begin{aligned}
&\langle v^*, \bar{x}^+(z_1)\bar{x}^+(z_2)\cdots\bar{x}^+(z_k)\bar{x}^+(z_{k+1})v\rangle\\
&\qquad = \langle v^*, \prod_{i<j}\frac{(z_i-z_jq^2)}{z_i-z_j}x^+(z_1)x^+(z_2)\cdots x^+(z_k)x^+(z_{k+1})\\
&\qquad\qquad\qquad \times k^+(z_1)k^+(z_2)\cdots k^+(z_k)k^+(z_{k+1})v\rangle.
\end{aligned}$$

Because $\langle v^*, x^+(z_1)x^+(z_2)\cdots x^+(z_k)x^+(z_{k+1})v_1\rangle$ for a vector v_1 is zero in any level $k \geq 1$ integrable module of $U_q(\hat{\mathfrak{sl}}(2))$ if $z_2/z_1 = z_3/z_2 = \cdots = z_{k+1}/z_k = q^2$, and the function $(\prod_{i<j} \frac{(z_i - z_j q^2)}{z_i - z_j})^{-1}$ is not zero if $z_2/z_1 = z_3/z_2 = \cdots = z_{k+1}/z_k = q^2$, we have

$$\bar{F}(z_1, \dots, z_n) = 0.$$

With the preparation above, in this section, we will describe a quantized semi-infinite construction along the line of [FS1], [FS2]. Their starting point for the case of $\hat{\mathfrak{sl}}(2)$ is the integrability condition for level k integrable modules, namely, any level k highest weight module is an integrable module if and only if $e(z)^{k+1}$ is zero. □

Similarly, we can make the following claim:

Theorem 13. *Any level k module of $U_q(\hat{\mathfrak{sl}}(2))$ from the category of representations with highest weight is a sum of irreducible integrable representations if and only if $\bar{x}^+(z)\bar{x}^+(zq^2)\cdots\bar{x}^+(zq^{2k})$ is zero.*

Proof. The theorem above already gives the proof for half of the theorem. The other half comes from the fact that if we quotient by the relation $q = 1$, the condition that $\bar{x}^+(z)\bar{x}^+(zq^2)\cdots x^+(zq^{2k})$ is zero simply degenerates into the condition that $e(z)^{k+1}$ is zero. Thus, it is integrable as a module of $\hat{\mathfrak{sl}}(2)$. From the theory of Lusztig, we know that all the integrable highest weight modules must come from the corresponding quantized module. Thus the module is also an integrable module when q is generic.

We will start our semi-infinite construction with the irreducible integrable module $V_{0,1}$ with the highest weight vector $v_{0,1}$ such that the weight of the highest weight vector is 0 and the central element c acts as 1.

Let $\bar{x}^+(z) = \Sigma \bar{x}^+_i z^{-i}$ and $U(\bar{x})$ be the subalgebra generated by $\bar{x}^+_i$. We denote by $U(\bar{x})^-$ the subalgebra generated by $\bar{x}^+_n, n \geq 0$ and by $U(\bar{x})^+$ the subalgebra generated by $\bar{x}^+_n, n < 0$. Let $W = U(\bar{x})v_{0,1}$. Because $U(\bar{x})^+ v_{0,1} = 0$, we have that W is equivalent to $U(\bar{x})^+/Iv_{0,1}$, where I is an ideal. □

Lemma 14. *The ideal I is generated by $S^1_k = \sum_{i \leq k-i} \bar{x}_i \bar{x}_{k-i}(q^{2i} + q^{2k-2i})$, for $k < -1$.*

Proof. From the quantum integrability condition above, we know that the elements S^1_k for $k < -1$ are inside the ideal I. We will denote the ideal generated by those elements by I'. The proof follows from that fact that $U(\bar{x})^+/Iv_{0,1}$ has the same character as the case when we quotient by the relation $q - 1 = 0$. Thus $I = I'$. □

Definition 15. $\bar{V}_{0,1}$ is a vector space with the basis of infinite monomials M of $x_{i_1} x_{i_2} \cdots x_{i_n} \cdots$, where $\{i_1, i_2, \dots\}$ is an infinite sequence of indices such that, for some n, i_n is odd and $i_{p+1} = i_p + 2$, if $p > n$. Let $V_{0,1}$ be a quotient space of $\bar{V}_{0,1}$, the quotient is given by the following relations:

(1) $\bar{x}_i$ and $\bar{x}_j$ commute, if $i \neq j$.
(2) If an element $m \in \bar{V}_{0,1}$ contains a segment $\bar{x}_u \bar{x}_{2N+1} \bar{x}_{2N+3} \bar{x}_{2N+5} \cdots$ and $u > 2N - 1$, then $m = 0$.
(3) The operator $S_k = \sum_{a+b=k, a \leq b} \bar{x}_a \bar{x}_b (q^{2b} + q^{2a})$ acts on $\bar{V}_{0,1}$ by $S_k v = 0$ for $v \in \bar{V}_{0,1}$.

We define the action of $\bar{x}_i$ simply by multiplication. The action of a_i for $i > 0$ is given by

$$\begin{aligned} &a_i \bar{x}_{i_1} \bar{x}_{i_2} \cdots \\ &\quad = [a_i, \bar{x}_{i_1}] \bar{x}_{i_2} \cdots + \bar{x}_{i_1} [a_i, \bar{x}_{i_2}] \bar{x}_{i_3} \cdots \\ &\quad\quad + \bar{x}_{i_1} \bar{x}_{i_2} \cdots [a_i, \bar{x}_{i_n}] \bar{x}_{i_{n+1}} \cdots . \end{aligned}$$

This is a finite expression. We define the action of a_0 by

$$a_0(\bar{x}_{2N+1}\bar{x}_{2N+3}\bar{x}_{2N+5}\cdots) = -2N\bar{x}_{2N+1}\bar{x}_{2N+3}\bar{x}_{2N+5}\cdots .$$

The action of a_{-1} is defined as

$$\begin{aligned} &a_{-1}\bar{x}_1\bar{x}_3\bar{x}_5\cdots \\ &\quad = \bar{x}_0\bar{x}_3\bar{x}_5\cdots + \bar{x}_1\bar{x}_2\bar{x}_5\cdots + \bar{x}_1\bar{x}_3\bar{x}_4\bar{x}_7\bar{x}_9\cdots + \bar{x}_1\bar{x}_3\bar{x}_5\bar{x}_6\bar{x}_9 + \cdots \\ &\quad = \bar{x}_0\bar{x}_3\bar{x}_5\cdots - \frac{(q^6+1)}{(q^4+q^2)}\bar{x}_0\bar{x}_3\bar{x}_5\cdots + \frac{(q^6+1)}{(q^4+q^2)}\frac{(q^{10}+q^4)}{(q^6+q^8)}\bar{x}_0\bar{x}_3\bar{x}_5 + \cdots \\ &\quad = \bar{x}_0\bar{x}_3\bar{x}_5\cdots\frac{1}{1+\frac{(q^6+1)}{(q^4+q^2)}}. \end{aligned}$$

Thus, it converges if $|\frac{(q^6+1)}{(q^4+q^2)}| < 1$.

We would like to define the action of a_{-2} as

$$\begin{aligned} &a_{-2}\bar{x}_1\bar{x}_3\bar{x}_5\cdots \\ &\quad = (q^2-q^{-2})(\bar{x}_{-1}\bar{x}_3\bar{x}_5\cdots + \bar{x}_1\bar{x}_1\bar{x}_5\cdots + \bar{x}_1\bar{x}_3\bar{x}_3\bar{x}_7\bar{x}_9\cdots + \bar{x}_1\bar{x}_3\bar{x}_5\bar{x}_5\bar{x}_9 + \cdots) \\ &\quad = (q^2-q^{-2})(\bar{x}_{-1}\bar{x}_3\bar{x}_5\cdots - \left(\frac{(q^4+1)}{(2q)}\bar{x}_0\bar{x}_2\bar{x}_5\cdots + \frac{(q^6+q^{-2})}{(2q)}\bar{x}_{-1}\bar{x}_3\bar{x}_5\cdots\right) \\ &\quad\quad - \left(\frac{(q^{10}+q^2)}{(q^6+q^6)}\bar{x}_1\bar{x}_1\bar{x}_5\bar{x}_7\cdots + \frac{(q^8+q^4)}{(q^6+q^6)}\bar{x}_1\bar{x}_2\bar{x}_4\bar{x}_7\cdots\right)\cdots . \end{aligned}$$

To use the relations (1), (2), (3) to reduce this expression to prove the convergence of the expression is very complicated. Similar problems appears in defining the action of $a_{-n}, n > 2$.

Thus, we will use the same trick played in [FS1]. Let $\bar{V}_{0,1}(r)$ be the subspace of $\bar{V}_{0,1}$, which consists of the elements $\bar{x}_{i_1}\cdots\bar{x}_{i_n}\cdots$ and $i_j > r$ for any j.

Lemma 16. *$\bar{V}_{0,1}(r)$ spans the whole space $V_{0,1}$.*

Proof. The proof is the same as that of Lemma 2.5.1 in [FS1] by using the relation (3) to express any element in $V_{0,1}$ with linear combination of elements in $V_{0,1}(r)$. □

For any element expressed in a linear combination of elements in $\bar{V}_{0,1}(r)$, we define the action of $x^-(k)$ from $x^-(z) = \sum x^-(k)z^{-k}$, for $k + r > 0$, as that of a_{-1} by using the commutation relations between $\bar{x}^+(z)$ and $\bar{x}^-(z)$. Because $k + r > 0$, we know that it is well defined. As in [FS1], this is a well defined action, namely, if we express an element in two different ways in $\bar{V}_{0,1}(r)$, the actions of $x^-(k)$ defined above coincide. Again, with the commutation relation between $\bar{x}^+(z)$ and $x^-(z)$, we can define the action of $a_n, n < -1$, because $\bar{x}^+(z)$ and $x^-(z)$ generate the whole algebra. Thus, we have

Theorem 17. *There exists an action of $U_q(\hat{\mathfrak{sl}}(2))$ on the space $\bar{V}_{0,1}$, such that $\bar{V}_{0,1}$ is equivalent to $V_{0,1}$ as a representation of $U_q(\hat{\mathfrak{sl}}(2))$ and the action of $\bar{x}_i$ acts by multiplication.*

Let $\bar{W}$ be the set of the elements $\bar{x}_{i_1}\cdots\bar{x}_{i_n}\cdots$ in $\bar{V}_{0,1}$, such that $i_{j+1}-i_j>1$.

Proposition 18. *$\bar{W}$ forms a basis of the space $\bar{V}_{0,1}$.*

The proof is the same as in [FS1] which gives the character of the representation.

Similarly, as in [FS2], a functional model for the description of $\bar{W}^*$, the dual space of $\bar{W}$, can be derived from the lemma above.

As a commutative algebra, $U(\bar{x})$ can be identified with the space $\mathbb{C}[t,t^{-1}]$. Let $U(\bar{x})^+=\bigoplus U(\bar{x})^+(n)$, where $U(\bar{x})^+(n)$ consists of the elements $\bar{x}^+_{i_1}\bar{x}^+_{i_2}\cdots\bar{x}^+_{i_n}$. We identify any element $\bar{x}^+_{i_1}\bar{x}^+_{i_2}\cdots\bar{x}^+_{i_n}$ in $U(\bar{x})^+(n)$ as $t_1^{i_1}\cdots t_n^{i_n}$, where t_i are variables. Similarly, we can express any element $\bar{x}^+_{i_1}\bar{x}^+_{i_2}\cdots\bar{x}^+_{i_m}$ in $\bar{W}$ as $t_1^{i_1}\cdots t_m^{i_m}$. Let $S^n(\Omega^1\mathbb{C})$ be the space spanned by the expressions $f(t_1,\dots,t_n)dt_1\cdots dt_n$, such that $f(t_1,\dots,t_n)$ is a symmetric function. $S^n(\Omega^1\mathbb{C})$ is also called the space of n particles. We can pair $S^n(\Omega^1\mathbb{C})$ with $U(\bar{x})^+(n)$ by

$$\begin{aligned}&\langle f(t_1,\dots,t_n)dt_1\cdots dt_n, t_1^{i_1}\cdots t_n^{i_n}\rangle\\ &\qquad=\mathrm{Residue}_{t_1=\cdots=t_n=0}(f(t_1,\dots,t_n)t_1^{i_1}\cdots t_n^{i_n}dt_1\cdots dt_n).\end{aligned}$$

Thus, $\bar{W}^*=\bigoplus\bar{W}^*\bigcap S^n(\Omega^1\mathbb{C})$.

Theorem 19.

$$\bar{W}^*\bigcap S^n(\Omega^1\mathbb{C})=\{f(t_1,\dots,t_n)dt_1\cdots dt_n : f=0,\quad \text{if } t_1=t_2q^2\}.$$

Similarly, we can present the semi-infinite constructions for the higher level cases.

Let $V_{l,k}$ be the irreducible highest weight representation of $U_q(\hat{\mathfrak{sl}}(2))$ with the action of c as k, the highest weight as l times the fundamental weight and $v_{l,k}$ its highest weight vector. Let $W_{l,k}=U(\bar{x})v_{l,k}$. Because $U(\bar{x})^+v_{l,k}=0$, we have that $W_{l,k}$ is equivalent to $U(\bar{x})^+/I_{l,k}v_{l,k}$, where $I_{l,k}$ is an ideal.

Lemma 20. *The ideal $I_{l,k}$ is generated by*

$$S_i^{k+1}=\sum_{\sum a_i=-i, a_i\le a_j}\bar{x}_{a_1}\bar{x}_{a_2}\cdots\bar{x}_{a_{k+1}}\Big(\sum_{\sigma\in S_{k+1}}(q^{\sum_{\sigma(i)=2,K+1}2(\sigma(i)-1)a_{\sigma i}})\Big),$$

$i<-k$ and $\bar{x}_{-1}^{k-l+1}$, if $k-l+1>0$.

Definition 21. Let $\bar{V}_{l,k}$ be the space spanned by the elements of

$$\bar{x}_{i_1}\cdots\bar{x}_{i_n}\bar{x}^l_{2N}\bar{x}^{k-l}_{2N+1}\bar{x}^l_{2N+2}\bar{x}^{k-l}_{2N+3}\cdots,$$

such that

(1) $\bar{x}_i$ commutes with $\bar{x}_j$;
(2) if an element $m\in\bar{V}$ contains a part $\bar{x}_i\bar{x}^l_{2N}\bar{x}^{k-l}_{2N+1}\bar{x}^l_{2N+2}\bar{x}^{k-l}_{2N+3}\cdots$, $i>2N-1$ or $\bar{x}_i\bar{x}^{k-l}_{2N+1}\bar{x}^l_{2N+2}\bar{x}^{k-l}_{2N+3}\cdots$, $i>2N$, then $m=0$;
(3) the operator

$$S_k=\sum_{\sum a_i=n}\bar{x}_{a_1}\bar{x}_{a_2}\cdots\bar{x}_{a_{k+1}}\Big(\sum_{\sigma\in S_{k+1}}(q^{\sum_{\sigma(i)=2,K+1}2(\sigma(i)-1)a_{\sigma i}})\Big)$$

acts on $\bar{V}_{l,k}$ by $S_kv=0$ for $v\in\bar{V}_{l,k}$, where S_{k+1} is the permutation group on $k+1$ numbers.

Theorem 22. *On the space $\bar{V}_{l,k}$, there is an action of $U_q(\hat{\mathfrak{sl}}(2))$, such that the action of $\bar{x}^+(n)$ is given by comultiplication. This representation is the irreducible highest weight representation of $U_q(\hat{\mathfrak{sl}}(2))$ with the action of c as k and the highest weight is l times the fundamental weight.*

Let $\bar{W}_{(l,k)}$ be the set of the elements $\bar{x}_{i_1} \cdots \bar{x}_{i_n} \cdots$ in $\bar{V}_{l,k}$, such that $i_{j+k} - i_j > 1$.

Proposition 23. *$\bar{W}_{(l,k)}$ forms a basis of the space $\bar{V}_{l,k}$.*

As in the case of $\bar{V}_{0,1}$, let $U(\bar{x})^+ = \bigoplus U(\bar{x})^+(n)$, where $U(\bar{x})^+(n)$ consists of the elements $\bar{x}^+_{i_1}\bar{x}^+_{i_2} \cdots \bar{x}^+_{i_n}$. We identify any element $\bar{x}^+_{i_1}\bar{x}^+_{i_2} \cdots \bar{x}^+_{i_n}$ in $U(\bar{x})^+(n)$ as $t_1^{i_1} \cdots t_n^{i_n}$, where t_i are variables. Similarly, we can express any element $\bar{x}^+_{i_1}\bar{x}^+_{i_2} \cdots \bar{x}^+_{i_m}$ in $W_{l,k}$ as $t_1^{i_1} \cdots t_m^{i_m}$. Let $S^n(\Omega^1\mathbb{C})$ be the space of expressions $f(t_1, \dots, t_n)dt_1 \cdots dt_n$, such that $f(t_1, \dots, t_n)$ is a symmetric function and different dt_i commute. $S^n(\Omega^1\mathbb{C})$ is also called the space of n particles. We can pair $S^n(\Omega^1\mathbb{C})$ with $U(\bar{x})^+(n)$ by

$$\begin{aligned} &\langle f(t_1, \dots, t_n)dt_1 \cdots dt_n, t_1^{i_1} \cdots t_n^{i_n} \rangle \\ &\qquad = \mathrm{Residue}_{t_1 = \cdots = t_n = 0}(f(t_1, \dots, t_n)t_1^{i_1} \cdots t_n^{i_n} dt_1 \cdots dt_n). \end{aligned}$$

Thus $\bar{W}^*_{l,k} = \bigoplus \bar{W}^*_{l,k} \bigcap S^n(\Omega^1\mathbb{C})$.

Theorem 24.

$$\begin{aligned} \bar{W}^*_{l,k} \bigcap S^n(\Omega^1\mathbb{C}) = \{ &f(t_1, \dots, t_n)dt_1 \cdots dt_n : f = 0 \\ &\text{if } t_1 = t_2q^2 = \cdots = t_{k+1}q^{2(k)} \text{ or } t_1 = \cdots t_{k-l+1} = 0 \} \end{aligned}$$

if $k - l + 1 > 0$.

In Section 2, we define the operator $\bar{x}^+_i(z)$. It is clear that this can also be applied to other cases. Our next step is to extend the semi-infinite construction to the cases of $U_q(\hat{\mathfrak{g}})$, where $\mathfrak{g}$ is a simply-laced simple Lie algebra, for which we need to define the proper $\bar{x}^+_\alpha(z)$ associated to the roots of $\mathfrak{g}$. The simplest case $\mathfrak{g} = \mathfrak{sl}(3)$ will be the subject of a subsequent paper. On the other hand, this paper follows the algebraic theory developed in [FS1], [FS2], [FS3]. The semi-infinite constructions can be geometrically understood according to the structure of the corresponding infinite dimensional flag manifold and the infinite Schubert cells. The geometric interpretation of the quantized semi-infinite construction is still an open problem. This is related to another immediate problem to extend the explicit construction of the modular functors [FS2] to the quantized case. This should lead us toward the quantization of conformal field theory. It is also possible to extend such a construction to even more general cases. From the point of view of the functional realization of the dual space, one generalization is to substitute the condition $x_1 = x_2q^2$, which is a generalization of the classical condition $x_1 = x_2$, with more general conditions, for example, $x_1 = x_2q_1 = x_3q_2$. One would like to ask the following question: what kind of structures are behind the corresponding generalized spaces? We believe it is related to the recent work about the generalization of the quantum affine algebras [DI], where these kind of new conditions should be satisfied for the quantum current operators. We hope our construction can help us to understand the structures of those new algebras in [DI], for which we have not yet been able to give any concrete realization of the non-trivial integrable representations.

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