REPRESENTATION THEORY
An Electronic Journal of the American Mathematical Society
Volume 4, Pages 398–445 (September 12, 2000)
S 1088-4165(00)00076-5

AN ANALYTIC RIEMANN-HILBERT CORRESPONDENCE FOR SEMI-SIMPLE LIE GROUPS

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Abstract. Geometric Representation Theory for semi-simple Lie groups has two main sheaf theoretic models. Namely, through Beilinson-Bernstein localization theory, Harish-Chandra modules are related to holonomic sheaves of $\mathcal{D}$ modules on the flag variety. Then the (algebraic) Riemann-Hilbert correspondence relates these sheaves to constructible sheaves of complex vector spaces. On the other hand, there is a parallel localization theory for globalized Harish-Chandra modules—i.e., modules over the full semi-simple group which are completions of Harish-Chandra modules. In particular, Hecht-Taylor and Smithies have developed a localization theory relating minimal globalizations of Harish-Chandra modules to group equivariant sheaves of $\mathcal{D}$ modules on the flag variety. The main purpose of this paper is to develop an analytic Riemann-Hilbert correspondence relating these sheaves to constructible sheaves of complex vector spaces and to discuss the relationship between this "analytic" study of global modules and the preceding "algebraic" study of the underlying Harish-Chandra modules.

Contents

0. Introduction

Let $G_{\mathbb{R}}$ denote a real connected semisimple Lie group with finite center and let G be its complexification. Fix $K_{\mathbb{R}}$ a maximal compact subgroup in $G_{\mathbb{R}}$ and let K be the complexification of $K_{\mathbb{R}}$. Let $\mathfrak{g}$ be the Lie algebra of G and let X be the flag variety of $\mathfrak{g}$ (i.e., the complex manifold of all Borel subalgebras of $\mathfrak{g}$). Fix λ a regular weight and let $\mathcal{D}_\lambda^{\mathrm{alg}}$ (resp. $\mathcal{D}_\lambda$) be the sheaf of λ-twisted differential operators on X with algebraic (resp. holomorphic) coefficients. The weight λ (or, more precisely, its Weyl group orbit) parametrizes a character ϕ_λ of the center of the enveloping

Received by the editors July 21, 1999.
2000 *Mathematics Subject Classification.* Primary 22E46; Secondary 18D99, 55N91.
Key words and phrases. Localization, constructible sheaves, equivariant derived category.

algebra of $\mathfrak{g}$. Let $\mathcal{U}_\lambda(\mathfrak{g})$ be the quotient of $\mathcal{U}(\mathfrak{g})$ by the ideal generated by the kernel of ϕ_λ.

In [BB], Beilinson and Bernstein proved that, in the case of regular dominant λ, there is an equivalence from the category of Harish-Chandra $\mathcal{U}_\lambda(\mathfrak{g})$ modules (finitely generated $(\mathfrak{g}, K)$ modules with infinitesimal character determined by λ) to the category of K-equivariant coherent sheaves of $\mathcal{D}_\lambda^{\text{alg}}$ modules on X. This equivalence is the localization functor. Its inverse functor is the functor of global sections. This has turned out to be a very powerful tool in representation theory. Its most spectacular application is in a proof of the Kazhdan-Lusztig conjectures.

The localization of a Harish-Chandra module may be thought of as a realization of the module in terms of a more geometric object—a coherent K-equivariant sheaf of $\mathcal{D}_\lambda$ modules. There is a further reduction to purely geometric information called the Riemann-Hilbert correspondence (cf. [Bo], [BB2], [BL2], [K2],[M], [M2]). In our context, the Riemann-Hilbert correspondence assigns to each coherent K-equivariant sheaf of $\mathcal{D}_\lambda$ modules an object of a certain derived category based on $-\lambda$ twisted, K-equivariant sheaves of finite dimensional complex vector spaces (the so called "constructible sheaves").

Each Harish-Chandra module arises from taking the K-finite part of an admissible representation of $G_\mathbb{R}$. A given Harish-Chandra module M may, in general, be realized as the K finite part of many admissible representations. These represent the many ways in which the Harish-Chandra module may be given a topology and then completed in such a way that the $\mathfrak{g}$ module structure on M exponentiates to yield a global $G_\mathbb{R}$ module structure on the completion. Such a completion is called a globalization of M. Schmid [S] developed the essential properties of two canonical ways to globalize a Harish-Chandra module—the minimal globalization and the maximal globalization. We will be concerned here with the minimal globalization. The minimal globalization of a Harish-Chandra module M is a module which, as a topological vector space, is the dual of a nuclear Frechet space and which has the property that every vector is an analytic vector for the representation. In fact, it has the surprising property that it is naturally isomorphic to the space of analytic vectors in any Banach space globalization of M.

Given the results of Beilinson-Bernstein and Schmid it was natural to ask if there is a localization theory which realizes the minimal globalization of a Harish-Chandra module in terms of sheaves of $\mathcal{D}_\lambda$ modules on X in a fashion analogous to the Beilinson-Bernstein localization of the original module. This was accomplished in [HT] and [Sm]. A different, but related realization of the maximal globalization was developed in [SW]. Motivated by the results of [SW], Kashiwara, announced a series of conjectures concerning this situation in [K]. The following diagram represents an updated version of a diagram that appears in [K]. The Kashiwara conjectures essentially concern the existence of the equivalences indicated in the diagram and of their duals.

$$
\begin{array}{ccc}
D^b_{K,f}(\mathcal{U}_\lambda(\mathfrak{g})) & \xrightarrow{\hat{\gamma}} & D^b_{G_\mathbb{R},f}(\mathcal{U}^t_\lambda(\mathfrak{g})) \\
{\scriptstyle\Delta}\big\downarrow & & \big\downarrow{\scriptstyle\Delta} \\
D^b_{K,f}(\mathcal{D}_\lambda) & \xrightarrow{\bar{\gamma}} & D^b_{G_\mathbb{R},f}(\mathcal{D}_\lambda) \\
{\scriptstyle\mathrm{RH}}\big\downarrow & & \big\downarrow{\scriptstyle\mathrm{RH}} \\
D^b_{K,f}(\tilde{X})^{-\lambda} & \xrightarrow{\gamma} & D^b_{G_\mathbb{R},f}(\tilde{X})^{-\lambda}
\end{array}
\tag{0.1}
$$

Here each arrow represents an equivalence between appropriate equivariant derived categories (which we will describe more explicitly below) and the diagram is supposed to be commutative. It is necessary to pass to derived categories in formulating these results because not all of the functors involved are exact as functors on the underlying categories. This is apparent already for the Beilinson-Bernstein localization functor which is not exact for all regular λ, though it is exact for λ dominant.

Each node of (0.1) is a t-category (triangulated category with t-structure) having as heart the abelian category that is really of interest. The heart of a t-category is the full subcategory consisting of objects of pure degree 0. For example, on the left side of the diagram, the hearts of the indicated categories are, from top to bottom: the category of Harish-Chandra $\mathcal{U}_\lambda(\mathfrak{g})$ modules, the category of coherent K-equivariant $\mathcal{D}_\lambda$ modules and the category of $-\lambda$ twisted K-equivariant sheaves of finite dimensional vector spaces. On the right side of the diagram the hearts are, from top to bottom: the category of minimal globalizations of Harish-Chandra $\mathcal{U}_\lambda(\mathfrak{g})$ modules, the category of finite type $G_\mathbb{R}$-equivariant $\mathcal{D}_\lambda$ modules and the category of $-\lambda$ twisted $G_\mathbb{R}$-equivariant sheaves of finite dimensional vector spaces. Here, a finite type $\mathcal{D}_\lambda$ module is a $\mathcal{D}_\lambda$ module which has finite dimensional geometric fibers as an $\mathcal{O}$ module.

Each of the equivariant derived categories used here is defined using the method of Bernstein and Lunts from [BL]. Each of the categories in (0.1) is the full subcategory of finite type objects in a larger "parent" category. Thus, $D^b_{K,f}(\mathcal{U}_\lambda(\mathfrak{g}))$ is the full subcategory of finite type objects in $D^b_K(\mathcal{U}_\lambda(\mathfrak{g}))$, the equivariant derived category of $\mathcal{U}_\lambda(\mathfrak{g})$ modules. Here, a finite type object is one with finitely generated cohomologies. Similarly, $D^b_{K,f}(\mathcal{D}_\lambda)$ is the full subcategory of finite type objects in $D^b_K(\mathcal{D}_\lambda)$, the equivariant derived category of $\mathcal{D}_\lambda$ modules, where finite type objects are those whose cohomologies are coherent as $\mathcal{D}_\lambda$ modules. The category $D^b_{K,f}(\tilde{X})^{-\lambda}$ is the full subcategory of finite type objects in $D^b_K(\tilde{X})^{-\lambda}$, the equivariant derived category of $-\lambda$ monodromic sheaves of complex vector spaces for X, where finite type objects are those whose cohomologies have finite dimensional stalks. The categories on the right in (0.1) are defined similarly, except that $G_\mathbb{R}$ equivariance replaces K equivariance and the category $D^b_{G_\mathbb{R},f}(\mathcal{U}^t_\lambda(\mathfrak{g}))$ is necessarily defined using topological $\mathcal{U}_\lambda(\mathfrak{g})$ modules (hence the superscript "t"). The right side of (0.1) can be viewed as the "minimal globalization" of the left side.

The top functor, Δ on the left side of (0.1) is the equivariant derived category version of the Beilinson-Bernstein localization functor ([BB2], [BL2]). The bottom left functor RH is the equivariant Riemann-Hilbert correspondence ([Bo], [Bj], [K2], [M], [M2]). The top functor $\tilde{\gamma}$ is Schmid's minimal globalization functor [S] extended to an equivalence between the equivariant derived categories. These three functors and their properties were essentially understood at the time of Kashiwara's conjectures. In part, these conjectures essentially hypothesize the existence of the bottom equivalence and a functor from the bottom right category to the top right category which makes the diagram commutative. There are similar conjectures concerning a diagram dual to (0.1) in which the top arrow comes from Schmid's maximal globalization functor.

A big step toward proving Kashiwara's conjectures was provided by [MUV] in which the bottom horizontal equivalence was established. The full conjectures were established by Kashiwara and Schmid in [KSd]. Their formulation is somewhat different than the one stated here due to the fact that they do not use the equivariant derived category at the top right node and the functor from the bottom right node to the top right node is not formulated as an equivalence of categories. Also, the main focus of [KSd] is on the dual diagram, which involves the maximal rather than the minimal globalization.

The main purpose of this paper is to place $D^b_{G_{\mathbb{R}},f}(\mathcal{D}_\lambda)$ in its proper place in the middle of the right side of (0.1) and to establish the equivalences which connect it to the remainder of the diagram. Part of this is already accomplished in [Sm] where the equivariant analytic localization equivalence (the Δ on the right in (0.1)) is established. In this paper we define the Riemann-Hilbert correspondence $RH : D^b_{G_{\mathbb{R}},f}(\mathcal{D}_\lambda) \to D^b_{G_{\mathbb{R}},f}(\tilde{X})^{-\lambda}$ for the globalized setting and prove that it is an equivalence. We also establish the middle horizontal equivalence $\bar{\gamma}$ of (0.1) and prove that the diagram is commutative. A secondary purpose of this paper is to reformulate some of the results of [KSd] so that equivariant derived categories are used at each node in (0.1) and each functor is an equivalence.

The fact mentioned earlier, that the category $D^b_{G_{\mathbb{R}},f}(\mathcal{U}^t_\lambda(\mathfrak{g}))$ must be defined using topological modules, stems from the nature of the minimal globalization functor (a topological completion) and the analytic localization machinery of Hecht-Taylor and Smithies (which uses completed topological tensor product in an essential way). This leads to the main technical difficulty of this paper. The analytic localization functor of [HT] and [Sm] yields an equivalence between $D^b_{G_{\mathbb{R}},f}(\mathcal{U}^t_\lambda(\mathfrak{g}))$ and a category $D^b_{G_{\mathbb{R}},f}(\mathcal{D}^t_\lambda)$ defined in the same way as $D^b_{G_{\mathbb{R}},f}(\mathcal{D}_\lambda)$ but using a category of sheaves of topological $\mathcal{D}_\lambda$ modules. Unfortunately, this category does not have enough injectives and this fact creates a serious problem in attempting to prove the $G_{\mathbb{R}}$ equivariant Riemann-Hilbert correspondence. The solution to this problem is provided in Section 5 where we prove that $D^b_{G_{\mathbb{R}},f}(\mathcal{D}^t_\lambda)$ and $D^b_{G_{\mathbb{R}},f}(\mathcal{D}_\lambda)$ are actually equivalent.

Because $G_{\mathbb{R}}$ orbits on the flag variety have mixed real and complex analytic structure (CR structure), we will make extensive use of the theory of analytic CR manifolds. When we to refer a manifold Z as "an analytic CR manifold" we will mean that Z is an abstract real analytic CR manifold considered as a ringed space with structure sheaf the sheaf of complex valued real analytic CR functions $\mathcal{O}_Z$. In [Sm], the category of such spaces was called "the category $\mathcal{T}$". Morphisms in this category are analytic CR maps $f : Y \to Z$ between analytic CR manifolds Y and Z . We implicitly use the fact that such a map f is automatically a morphism of ringed spaces. The reader is advised to consult [Sm] for a development of $G_{\mathbb{R}}$ spaces, free $G_{\mathbb{R}}$ spaces, quotient spaces, fiber products and submanifolds in this category.

Because the top right category of (0.1) is constructed from topological modules and the middle right category has an equivalent formulation using sheaves of topological modules which is needed in the development of the analytic localization equivalence, we will make heavy use of homological algebra in the context of topological modules over topological algebras in the final 2 sections of this paper. The topological modules that are involved are all DNF; that is, as topological vector spaces, they belong to the category of strong duals of Nuclear Frechet spaces. This is a particularly stable category which has good properties relative to topological tensor product. These issues are discussed at length in Section 5 of this paper and in [HT] and [Sm].

The paper is organized as follows: The first three sections are devoted to establishing the globalized version of the Riemann-Hilbert correspondence

$$\mathrm{DR} : D^b_{G_{\mathbb{R}},f}(\mathcal{D}_\lambda) \to D^b_{G_{\mathbb{R}},f}(\tilde{X})^{-\lambda}. \tag{0.2}$$

A Riemann-Hilbert correspondence for sheaves of modules over a twisted sheaf of differential operators $\mathcal{D}_\lambda$ requires special machinery to deal with the twist. We use the machinery of monodromic sheaves on the enhanced flag variety. Section 1 is devoted to a discussion of this machinery in our context. In Section 2 we develop the properties of $G_{\mathbb{R}}$-equivariant sheaves of $\mathcal{D}_\lambda$ modules and prove a preliminary version of the analytic Riemann-Hilbert correspondence (Theorem 2.6). In Section 3 we complete the development of the analytic Riemann-Hilbert correspondence by extending Theorem 2.6 to the appropriate equivariant derived categories. The result is Corollary 3.6 which asserts the existence of the equivalence (0.2). In Section 4 we briefly discuss the horizontal functors that appear in (0.1) and the question of commutativity of the diagram. With the exception of fitting the middle horizontal arrow of (0.1) into the picture, most of the results discussed in Section 4 are reformulations of results that appear in [Sm], [MUV] and [KSd].

Section 5 is devoted to solving a technical problem mentioned earlier. Namely, the top right arrow in diagram (0.1) is established in [Sm] as an equivalence

$$\Delta : D^b_{G_{\mathbb{R}},f}(\mathcal{U}^t_\lambda(\mathfrak{g})) \to D^b_{G_{\mathbb{R}},f}(\mathcal{D}^t_\lambda)$$

between equivariant categories constructed using DNF topological modules and DNF topological sheaves. On the other hand, the analytic Riemann-Hilbert correspondence that we establish in Section 3 (equation (0.2)) is necessarily formulated as an equivalence between equivariant categories constructed from sheaves of modules and complex vector spaces with no topological vector space structure. We resolve this, and several similar issues in Theorem 5.13. In particular, we establish equivalences $D^b_{G_{\mathbb{R}},f}(\mathcal{D}^t_\lambda) \simeq D^b_{G_{\mathbb{R}},f}(\mathcal{D}_\lambda)$, $D^b_{K,f}(\mathcal{D}^t_\lambda) \simeq D^b_{K,f}(\mathcal{D}_\lambda)$ and $D^b_{K,f}(\mathcal{U}^t_\lambda(\mathfrak{g})) \simeq D^b_{K,f}(\mathcal{U}_\lambda(\mathfrak{g}))$, where the superscript "t" indicates the given derived category is based on the indicated category of DNF topological modules or sheaves of modules. Note that there is no similar equivalence $D^b_{G_{\mathbb{R}},f}(\mathcal{U}^t_\lambda(\mathfrak{g})) \simeq D^b_{G_{\mathbb{R}},f}(\mathcal{U}_\lambda(\mathfrak{g}))$ and so the need to use topological modules cannot be entirely eliminated from the theory.

1. Monodromic Sheaves

Let $G_{\mathbb{R}}$ be a real connected semisimple Lie group with finite center and let G denote its complexification. Denote by X the flag variety of G and by $\tilde{X}$ the enhanced flag variety of G. We make the identifications $X = G/B$ and $\tilde{X} = G/N$ where B is a Borel subgroup of G and N is the unipotent radical of B. The Cartan $H = B/N$ acts freely on $\tilde{X}$ from the right and the map $\pi : \tilde{X} \to X$ is the quotient map. In addition, the spaces X and $\tilde{X}$ are left $G_{\mathbb{R}}$-spaces and π is $G_{\mathbb{R}}$-equivariant. The Lie algebra $\mathfrak{h}$ of H acts on $\tilde{X}$ via the exponential map $e : \mathfrak{h} \to H$.

We shall have occasion to use a more general version of this setup. Throughout we will work in the category of analytic CR manifolds and analytic CR maps, as discussed in [Sm]. Let $\tilde{Y}$ be a free right H space in this category and let $\tilde{Y} \to Y$ be the quotient map modulo the H action. Then we shall call $\tilde{Y} \to Y$ a *monodromic system.* Suppose also that K is an analytic Lie group and $\tilde{Y}$ and Y are analytic CR manifolds which are both left K spaces and that the K and H actions on $\tilde{Y}$ commute. Then the map $\tilde{Y} \to Y$ is K equivariant. In this situation, we shall then refer to $\tilde{Y} \to Y$ as an K *equivariant monodromic system*. The projection $\tilde{X} \to X$ of the enhanced flag manifold onto the flag manifold is one example of a $G_{\mathbb{R}}$ equivariant monodromic system, but we shall need to consider others as well. Our discussion of monodromic systems and sheaves follows [BB2] but with some important differences.

For any analytic CR manifold Y, we denote by $\mathcal{O}_Y$ the sheaf of analytic CR functions on Y and by $\mathcal{D}_Y$ the sheaf of differential operators on $\mathcal{O}_Y$. When the manifold Y is understood, we shall simply denote these sheaves by $\mathcal{O}$ and $\mathcal{D}$. If $\pi : \tilde{Y} \to Y$ is a monodromic system, we shall denote the sheaf theoretic pullback $\pi^{-1}\mathcal{S}$ of a sheaf $\mathcal{S}$ on Y by $\tilde{\mathcal{S}}$ (or $\tilde{\mathcal{S}}_{\tilde{Y}}$ if it is necessary to exhibit the underlying manifold). Thus, the pullbacks of $\mathcal{O}$ and $\mathcal{D}$ to $\tilde{Y}$ are denoted $\tilde{\mathcal{O}}$ and $\tilde{\mathcal{D}}$ (or $\tilde{\mathcal{O}}_{\tilde{Y}}$ and $\tilde{\mathcal{D}}_{\tilde{Y}}$), respectively.

A monodromic system $\pi : \tilde{Y} \to Y$ is completely determined by the free H-space $\tilde{Y}$. In fact, Y is just the quotient of $\tilde{Y}$ modulo the H action and π is the quotient map. Thus, we shall denote such a monodromic system simply by $\tilde{Y}$ unless it is important to explicitly exhibit the quotient map π.

We will regard a monodromic system $\tilde{Y}$ as a ringed space with an H action. However, the ringed space structure we choose is rather nonstandard. That is, the structure sheaf for $\pi : \tilde{Y} \to Y$ will be the sheaf $\tilde{\mathcal{O}} = \tilde{\mathcal{O}}_{\tilde{Y}} = \pi^{-1}\mathcal{O}_Y$. Note that this sheaf is constant on each orbit of H and so is nonconstant only in the base (Y) direction. A morphism between monodromic systems $\tilde{W}$ and $\tilde{Y}$ is an H equivariant ringed space morphism $\tilde{f} : \tilde{W} \to \tilde{Y}$. Such a map will necessarily descend to a morphism of ringed spaces, $f : W \to Y$, such that the diagram

$$\begin{array}{ccc} \tilde{W} & \xrightarrow{\tilde{f}} & \tilde{Y} \\ \downarrow & & \downarrow \\ W & \xrightarrow{f} & Y \end{array}$$

commutes.

A morphism $\tilde{f} : \tilde{W} \to \tilde{Y}$ of monodromic systems induces an inverse image or *pullback* functor $\tilde{f}^*$ from sheaves of $\tilde{\mathcal{O}}_{\tilde{Y}}$ modules on $\tilde{Y}$ to sheaves of $\tilde{\mathcal{O}}_{\tilde{W}}$ modules on $\tilde{W}$. This is defined in the usual way using the structure sheaves $\tilde{\mathcal{O}}_{\tilde{Y}}$ and $\tilde{\mathcal{O}}_{\tilde{W}}$. That is, for $\mathcal{M}$ a sheaf of $\tilde{\mathcal{O}}_{\tilde{Y}}$ modules,

$$\tilde{f}^*\mathcal{M} = \tilde{\mathcal{O}}_{\tilde{W}} \otimes_{\tilde{f}^{-1}\tilde{\mathcal{O}}_{\tilde{Y}}} \mathcal{M}.$$

It is important to note that this is an exact functor when $\tilde{f} : \tilde{W} \to \tilde{Y}$ is a fibration of monodromic systems, that is when f locally has the form of a projection $Z \times U \to U$ with Z a CR manifold and U an H invariant open set in $\tilde{Y}$. This follows from the fact that $\tilde{\mathcal{O}}_{\tilde{W}}$ is, in this case, flat over $f^{-1}\tilde{\mathcal{O}}_{\tilde{Y}}$. Since this is a statement about the stalks of the sheaves involved, it follows from the analogous fact from complex analysis; that is, for complex spaces U and V, the projection $V \times U \to U$ is always a flat morphism (see [GPR]).

Along with the structure sheaf $\tilde{\mathcal{O}}$ on a monodromic system $\tilde{Y}$, we have the associated differential structure sheaf $\tilde{\mathcal{D}} = \mathrm{Diff}(\tilde{\mathcal{O}}) = \pi^{-1}\mathcal{D}$. If $\tilde{f} : \tilde{W} \to \tilde{Y}$ is a morphism of monodromic systems and $\mathcal{M}$ is a sheaf of $\tilde{\mathcal{D}}_{\tilde{Y}}$ modules on Y, then the pullback $\tilde{f}^*\mathcal{M}$ has a natural structure of a $\tilde{\mathcal{D}}_{\tilde{W}}$ module. This works just as it does in the non-monodromic case (see [Sm] or [Bo]). That is, we define a $(\tilde{\mathcal{D}}_{\tilde{W}}, \tilde{f}^{-1}\tilde{\mathcal{D}}_{\tilde{Y}})$-bimodule,

$$\tilde{\mathcal{D}}_{\tilde{W}\to\tilde{Y}} = \mathrm{Diff}(\tilde{f}^{-1}\tilde{\mathcal{O}}_{\tilde{Y}}, \tilde{\mathcal{O}}_{\tilde{W}})$$

and set

$$\tilde{f}^{+}\mathcal{M} = \tilde{\mathcal{D}}_{\tilde{W}\to\tilde{Y}} \otimes_{\tilde{f}^{-1}\tilde{\mathcal{D}}_{\tilde{Y}}} \mathcal{M}.$$

Then $\tilde{f}^{+}$ is the pullback functor for $\tilde{\mathcal{D}}$-modules. However, as an $(\tilde{\mathcal{O}}_{\tilde{W}}, \tilde{f}^{-1}\tilde{\mathcal{O}}_{\tilde{Y}})$-bimodule, $\tilde{\mathcal{D}}_{\tilde{W}\to\tilde{Y}} \simeq \tilde{\mathcal{O}}_{\tilde{W}} \otimes_{\tilde{f}^{-1}\tilde{\mathcal{O}}_{\tilde{Y}}} \tilde{f}^{-1}\tilde{\mathcal{D}}_{\tilde{Y}}$, and so it turns out that $\tilde{f}^{+}\mathcal{M}$ and $\tilde{f}^{*}\mathcal{M}$ are the same $\tilde{\mathcal{O}}_{\tilde{W}}$-module.

A sheaf $\tilde{\mathcal{S}}$ on $\tilde{Y}$ is the pullback of a sheaf $\mathcal{S}$ on Y if and only if $\tilde{\mathcal{S}}$ is H-equivariant. (For a discussion of equivariant sheaves see [BL], [Sm] or the next section of this paper.) In this case, $\mathcal{S} = \pi_*\tilde{\mathcal{S}}$. Through the exponential map $\mathfrak{h} \to H$, we may also consider $\tilde{Y}$ to be an $\mathfrak{h}$ space. The sheaves which are $\mathfrak{h}$ equivariant are called *monodromic sheaves* on $\tilde{Y}$. For each $y \in \tilde{Y}$ the isotropy group of y in $\mathfrak{h}$ is $\pi_1(H) = \ker\{\exp : \mathfrak{h} \to H\}$. Thus, for each $\mathfrak{h}$-equivariant sheaf $\mathcal{S}$, there is a homomorphism $\phi : \pi_1(H) \to \mathrm{Aut}(\mathcal{S})$. Of special interest to us is the case where ϕ is given by a character of $\pi_1(H)$. Using the identification $\mathrm{Hom}(\pi_1(H), \mathbb{C}^*) = \mathfrak{h}^*/\mathfrak{h}^*_{\mathbb{Z}}$, each such character has the form $\xi \to e^{\mu(\xi)}$ with $\mu \in \mathfrak{h}^*$. We say $\mathcal{S}$ has *monodromy* μ if ϕ has this form. Note that whether a sheaf has monodromy μ depends only on $\bar{\mu}$—the equivalence class of μ in $\mathfrak{h}^*/\mathfrak{h}^*_{\mathbb{Z}}$. If the sheaf $\mathcal{S}$ has monodromy μ, then it is actually H-equivariant and, hence, a pullback from Y, if and only if $\bar{\mu} = 0$.

For each $\lambda \in \mathfrak{h}^*$, let $\mathcal{O}_\lambda = \mathcal{O}_{\lambda,\tilde{Y}}$ denote the subsheaf of $\mathcal{O}_{\tilde{Y}}$ consisting of elements killed by the operators $\xi - \lambda(\xi)$ for $\xi \in \mathfrak{h}$, where ξ acts as a differential operator on $\mathcal{O}_{\tilde{Y}}$ via the differential of the H action on $\tilde{Y}$. The sheaf $\mathcal{O}_\lambda$ is $\mathfrak{h}$ equivariant and has monodromy λ. Each $\mathcal{O}_\lambda$ is a sheaf of modules over the sheaf $\tilde{\mathcal{O}} = \pi^{-1}\mathcal{O}_Y$. Furthermore, if $\mathcal{M}$ is any monodromic sheaf of $\tilde{\mathcal{O}}$ modules with monodromy μ, then $\mathcal{O}_\lambda \otimes_{\tilde{\mathcal{O}}} \mathcal{M}$ is a monodromic sheaf with monodromy $\mu + \lambda$.

For $\lambda \in \mathfrak{h}^*$ we define $\tilde{\mathcal{D}}_\lambda = \mathcal{D}_{\lambda,\tilde{Y}}$ to be the sheaf of differential operators on $\mathcal{O}_\lambda$. There is an obvious identification between this sheaf and the sheaf $\mathcal{O}_\lambda \otimes_{\tilde{\mathcal{O}}} \tilde{\mathcal{D}} \otimes_{\tilde{\mathcal{O}}} \mathcal{O}_{-\lambda}$. Thus, since $\tilde{\mathcal{D}} = \tilde{\mathcal{D}}_0$ has monodromy 0, the same thing is true of $\tilde{\mathcal{D}}_\lambda$ for any λ. Therefore, $\tilde{\mathcal{D}}_\lambda$ is the pullback to $\tilde{Y}$ of a sheaf of algebras $\mathcal{D}_\lambda$ on Y. This is the *sheaf of twisted differential operators* on Y with twist λ.

We may also describe $\tilde{\mathcal{D}}_\lambda$ in the following way: Let $\mathcal{D}_{\mathfrak{h}} = \pi^H_*\mathcal{D}_{\tilde{Y}}$ denote the H invariant direct image of $\mathcal{D}_{\tilde{Y}}$ and set $\tilde{\mathcal{D}}_{\mathfrak{h}} = \pi^{-1}\mathcal{D}_{\mathfrak{h}}$. This is the subsheaf of the sheaf of differential operators on $\tilde{Y}$ consisting of differential operators with coefficients constant in the H direction. Differentiation of the right H action on $\tilde{Y}$ yields an embedding of $\mathfrak{h}$ as a Lie algebra of central sections of $\tilde{\mathcal{D}}_{\mathfrak{h}}$. Each of the subsheaves $\mathcal{O}_\lambda$ of $\mathcal{O}_{\tilde{Y}}$ is invariant under the action of $\tilde{\mathcal{D}}_{\mathfrak{h}}$ and, in fact, is exactly the λ-eigenspace for the resulting action of $\mathfrak{h}$. The sheaf $\tilde{\mathcal{D}}_\lambda$ is just the quotient of $\tilde{\mathcal{D}}_{\mathfrak{h}}$ modulo the ideal of elements which kill $\mathcal{O}_\lambda$ and this, in turn, is the ideal in $\tilde{\mathcal{D}}_{\mathfrak{h}}$ generated by the maximal ideal of $\mathcal{U}(\mathfrak{h})$ determined by $\lambda \in \mathfrak{h}^*$.

For a monodromic system $\tilde{Y}$, the structure sheaf $\tilde{\mathcal{O}}$ and differential structure sheaves $\tilde{\mathcal{D}}_\lambda$ are sheaves of topological algebras. In fact, they are sheaves of DNF (dual nuclear Fréchet) algebras in the sense of [HT] and [Sm]. This will be important in Sections 4 and 5 where we will need to consider sheaves of DNF topological modules over these sheaves of topological algebras. However, in this section and in

Sections 2 and 3, we will ignore the topological vector space structure on $\tilde{\mathcal{O}}$ and $\tilde{\mathcal{D}}_\lambda$ and we will not require modules over these algebras to have a topological vector space structure.

If $\tilde{f} : \tilde{W} \to \tilde{Y}$ is a morphism of monodromic systems and $\mathcal{M}$ is a sheaf of $\tilde{\mathcal{D}}_{\lambda,\tilde{Y}}$ modules on $\tilde{Y}$, then the $\mathcal{O}$-module pullback $\tilde{f}^*\mathcal{M}$ has a natural structure of a $\tilde{\mathcal{D}}_{\lambda,\tilde{W}}$ module. We noted above that this is true when $\lambda = 0$ (so that $\tilde{\mathcal{D}}_{\lambda,\tilde{Y}} = \tilde{\mathcal{D}}_{\tilde{Y}}$). The general case is proved in the same way. That is, we define a $(\tilde{\mathcal{D}}_{\lambda,\tilde{W}}, \tilde{f}^{-1}\tilde{\mathcal{D}}_{\lambda,\tilde{Y}})$-bimodule,

$$\tilde{\mathcal{D}}^{\lambda}_{\tilde{W}\to\tilde{Y}} = \mathrm{Diff}(\tilde{f}^{-1}\tilde{\mathcal{O}}_{\lambda,\tilde{Y}}, \tilde{\mathcal{O}}_{\lambda,\tilde{W}})$$

and set

$$\tilde{f}^+\mathcal{M} = \tilde{\mathcal{D}}^{\lambda}_{\tilde{W}\to\tilde{Y}} \otimes_{\tilde{f}^{-1}\tilde{\mathcal{D}}_{\tilde{Y}}} \mathcal{M}.$$

This is a $\tilde{\mathcal{D}}_{\lambda,\tilde{W}}$ module and $\tilde{f}^+$, defined in this way, is the $\tilde{\mathcal{D}}_{\lambda,\tilde{W}}$ module pullback functor. As before, a calculation shows that $\tilde{f}^+\mathcal{M}$ is isomorphic to $\tilde{f}^*\mathcal{M}$ as an $\tilde{\mathcal{O}}_{\tilde{W}}$-module.

Clearly, $\tilde{\mathcal{D}}_\lambda$-module pullback preserves monodromy and, hence, sheaves with monodromy 0. It follows that it descends to a pullback functor from sheaves of $\mathcal{D}_{\lambda,Y}$-modules on Y to sheaves of $\mathcal{D}_{\lambda,W}$ modules on W.

We denote by $M^\mu(\tilde{\mathcal{D}}_\lambda)$ the category of $\mathfrak{h}$-equivariant sheaves of $\tilde{\mathcal{D}}_\lambda$ modules with monodromy μ. The morphisms in this category are $\mathfrak{h}$-equivariant module homomorphisms. If $M(\mathcal{D}_\lambda)$ denotes the category of sheaves of $\mathcal{D}_\lambda$ modules on Y, then $M(\mathcal{D}_\lambda)$ is equivalent to $M^0(\tilde{\mathcal{D}}_\lambda)$ under pullback. We shall also need the K equivariant versions of these categories in the case where $\tilde{Y}$ is a K equivariant monodromic system. These will be defined and characterized in the next section.

2. Equivariant Sheaves

The term *equivariant sheaf* will have one of several meanings in this paper, depending on the category of sheaves to which it is applied. Let K be a CR Lie group and let Y be an analytic CR K space. Let $p : K \times Y \to Y$ be the projection and $m : K \times Y \to Y$ the action map $(k, y) \to ky$. A sheaf $\mathcal{S}$ is K-equivariant if there is a sheaf isomorphism

$$\theta : p^{-1}\mathcal{S} \to m^{-1}\mathcal{S}$$

satisfying the associativity and identity conditions (cf. [Sm] or [BL]). Here, the pullbacks p^{-1} and m^{-1} are the ordinary sheaf theoretic pullbacks. However, we shall also need the notion of K-equivariance in the sense of sheaves of $\mathcal{D}_\lambda$ modules on Y and $\tilde{\mathcal{D}}_\lambda$-modules on $\tilde{Y}$ if $\tilde{Y} \to Y$ is a monodromic system. A K-equivariant sheaf of $\mathcal{D}_\lambda$-modules is a sheaf $\mathcal{M}$ of $\mathcal{D}_{\lambda,Y}$ modules with an isomorphism of $\mathcal{D}_{\lambda,K\times Y}$ modules

$$\theta : p^+\mathcal{M} \to m^+\mathcal{M}$$

with the usual associativity and identity properties. Here, the pullbacks p^+ and m^+ are $\mathcal{D}_\lambda$-module pullbacks. Similarly, if $\tilde{Y}$ is a K-equivariant monodromic system, then a sheaf $\mathcal{M}$ of monodromic $\tilde{\mathcal{D}}_\lambda$-modules on $\tilde{Y}$ will be called K-equivariant if there is an isomorphism of monodromic $\tilde{\mathcal{D}}_{\lambda,K\times\tilde{Y}}$-modules

$$\tilde{\theta} : \tilde{p}^+\mathcal{M} \to \tilde{m}^+\mathcal{M}$$

again satisfying the associativity and identity conditions, where $\tilde{p} : K \times \tilde{Y} \to \tilde{Y}$ is the projection and $\tilde{m} : K \times \tilde{Y} \to \tilde{Y}$ the action map. Here, the pullbacks $\tilde{m}^+$ and $\tilde{p}^+$ are those defined in the previous section for monodromic sheaves of $\tilde{\mathcal{D}}_\lambda$ modules on monodromic systems. The manifold $K \times \tilde{Y}$ is a monodromic system via the right action of H on the $\tilde{Y}$ factor and the fact that $\tilde{\theta}$ is an isomorphism of monodromic $\tilde{\mathcal{D}}_{\lambda, K\times\tilde{Y}}$-modules means, in particular, that it commutes with this action. We will denote by $M_K^\mu(\tilde{\mathcal{D}}_{\lambda,\tilde{Y}})$ the category of K-equivariant monodromic sheaves of $\tilde{\mathcal{D}}_\lambda$ modules on $\tilde{Y}$ with monodromy μ.

The category of K equivariant sheaves of $\mathcal{D}_\lambda$ modules appears in various guises in the literature (see, for example [BB2], [BL2], [K], [KSd], and [Sm]). The definition in the form given here is mentioned in [KSd] and, for the DNF $\mathcal{D}_\lambda$ module version, in [Sm].

There is an important characterization of K-equivariant sheaves of $\tilde{\mathcal{D}}_\lambda$-modules in the case of free K spaces. We shall state the version appropriate for monodromic systems. Let $\tilde{Y}$ be a free monodromic K space—that is, a K-equivariant monodromic system $\pi : \tilde{Y} \to Y$ for which each point of Y has a K invariant neighborhood U such that $\pi^{-1}(U)$ is isomorphic as a monodromic K space to $K \times V$ for some monodromic space V. Then the quotient $\tilde{W}$ of $\tilde{Y}$ modulo the K action is also a monodromic space and the quotient map $\tilde{q} : \tilde{Y} \to \tilde{W}$ is a map of monodromic systems. The group K acts trivially on $\tilde{W}$ and every monodromic sheaf of $\tilde{\mathcal{D}}_{\lambda,\tilde{W}}$-modules is K-equivariant under the trivial K action. The $\tilde{\mathcal{D}}_\lambda$-module pullback $\tilde{q}^+$ preserves K-equivariance (see [Sm]) and so it defines a functor $\tilde{q}^+ : M^\mu(\tilde{\mathcal{D}}_{\lambda,\tilde{W}}) \to M_K^\mu(\tilde{\mathcal{D}}_{\lambda,\tilde{Y}})$. There is also a functor $\tilde{q}_*^K : M_K^\mu(\tilde{\mathcal{D}}_{\lambda,\tilde{Y}}) \to M^\mu(\tilde{\mathcal{D}}_{\lambda,\tilde{W}})$. This is the functor of K-invariant direct image and is defined as follows: If $\mathcal{M}$ is a K-equivariant module, and U an open set in $\tilde{W}$, then $\tilde{q}_*\mathcal{M}(U) = \mathcal{M}(\tilde{q}^{-1}(U))$. The group K acts on this vector space of sections and the subspace of K-invariant sections is $\tilde{q}_*^K\mathcal{M}(U)$. It is easy to see that $\tilde{q}_*^K(\mathcal{M}) = \tilde{q}_+\mathcal{M}$ is a sheaf of $\tilde{\mathcal{D}}_{\lambda,\tilde{W}}$ modules. Furthermore, if $\mathcal{M}$ is monodromic of monodromy μ, then the same thing is true of $\tilde{q}_*^K\mathcal{M}$. The following result is proved in [Sm] based on the proof of an analogous result in [BL]:

Proposition 2.1. *If $\tilde{Y}$ is a free monodromic K space and $\tilde{q} : \tilde{Y} \to \tilde{W} = K\backslash\tilde{Y}$ the quotient map, then the pullback $\tilde{q}^+ : M^\mu(\tilde{\mathcal{D}}_{\lambda,\tilde{W}}) \to M_K^\mu(\tilde{\mathcal{D}}_{\lambda,\tilde{Y}})$ is an equivalence of categories with inverse the functor $\tilde{q}_*^K$ of K-invariant direct image.*

In what follows, it will be useful to have a very concrete description of what it means for a sheaf $\mathcal{M}$ on $\tilde{Y}$ to be K-equivariant as a $\tilde{\mathcal{D}}_\lambda$-module. More specifically, let $\tilde{p}, \tilde{m} : K \times \tilde{Y} \to \tilde{Y}$ be the projection and multiplication maps. Recall that $\mathcal{M}$ is K-equivariant as a $\tilde{\mathcal{D}}_{\lambda,\tilde{Y}}$-module if there is a $\tilde{\mathcal{D}}_{\lambda,K\times\tilde{Y}}$-module isomorphism

$$\tilde{\theta} : \tilde{p}^+\mathcal{M} \to \tilde{m}^+\mathcal{M}$$

satisfying the associativity and identity conditions. On the other hand, the module $\mathcal{M}$ is equivariant as an $\tilde{\mathcal{O}}_{\tilde{Y}}$ module if there is a $\tilde{\theta}$ as above which is an $\tilde{\mathcal{O}}_{K\times\tilde{Y}}$-module isomorphism. Here we will show that the additional conditions necessary for a $\tilde{\mathcal{O}}_{K\times\tilde{Y}}$-module isomorphism

$$\tilde{\theta} : \tilde{p}^+\mathcal{M} \to \tilde{m}^+\mathcal{M}$$

and a $\tilde{\mathcal{D}}_{\lambda,K\times\tilde{Y}}$-module isomorphism are exactly the sheaf theoretic versions of the Harish-Chandra compatibility conditions. This will show that our definition of equivariance for $\tilde{\mathcal{D}}_{\lambda,\tilde{Y}}$-modules is equivalent to the one used in [BL2] and [BB2].

If we represent $\tilde{p}^+\mathcal{M}$ as

$$\tilde{p}^+\mathcal{M} = \tilde{\mathcal{O}}_{K\times\tilde{Y}} \otimes_{\tilde{p}^{-1}\tilde{\mathcal{O}}_{\tilde{Y}}} \tilde{p}^{-1}\mathcal{M},$$

then the $\tilde{\mathcal{D}}_{\lambda,K\times\tilde{Y}}$ module structure is the obvious one. That is, functions from $\tilde{\mathcal{O}}_{K\times\tilde{Y}}$ act through multiplication on the left factor in the tensor product. Differential operators in the K and $\tilde{Y}$ directions operate as follows. If $\eta \in \mathcal{D}_K$ and $\xi \in \tilde{\mathcal{D}}_{\lambda,\tilde{Y}}$, then we denote by $\eta\boxtimes 1$ and $1\boxtimes\xi$ the corresponding operators in $\tilde{\mathcal{D}}_{\lambda,K\times\tilde{Y}}$. The collection of all such operators, together with $\tilde{\mathcal{O}}_{K\times\tilde{Y}}$, generate $\tilde{\mathcal{D}}_{\lambda,K\times\tilde{Y}}$. An operator of the form $\eta\boxtimes 1$ acts on $\tilde{\mathcal{O}}_{K\times\tilde{Y}} \otimes_{\tilde{p}^{-1}\tilde{\mathcal{O}}_{\tilde{Y}}} \tilde{p}^{-1}\mathcal{M}$ by its action on the first factor in the tensor product. A differential operator of the form $1\boxtimes\eta$ acts on $1\boxtimes\mathcal{M} = p^{-1}\mathcal{M}$ by

$$(1\boxtimes\eta)(1\otimes\mu) = 1\otimes\eta\mu$$

at a point $(k,y) \in K\times\tilde{Y}$, where μ belongs to the stalk of $\mathcal{M}$ at y. Its action on general elements of $\tilde{p}^+\mathcal{M} = \tilde{\mathcal{O}}_{K\times\tilde{Y}} \otimes_{\tilde{p}^{-1}\tilde{\mathcal{O}}_{\tilde{Y}}} \tilde{p}^{-1}\mathcal{M}$ is then determined by the commutation rules in $\tilde{\mathcal{D}}_{\lambda,K\times\tilde{Y}}$.

The $\tilde{\mathcal{D}}_{\lambda,K\times\tilde{Y}}$-module structure of $\tilde{m}^+\mathcal{M}$ is more complicated. We will describe it in terms of the shear transformation

$$s : K\times\tilde{Y} \to K\times\tilde{Y}, \quad \text{where} \quad s(k,\tilde{y}) = (k, k\tilde{y}).$$

Note that $\tilde{m} = \tilde{p}\circ s$ and so $\tilde{m}^+ = s^+\circ\tilde{p}^+$. Thus, we may describe $\tilde{m}^+\mathcal{M}$ as $s^+(p^+\mathcal{M})$.

The map s induces an isomorphism $s^* : s^{-1}\tilde{\mathcal{O}}_{K\times\tilde{Y}} \to \tilde{\mathcal{O}}_{K\times\tilde{Y}}$ by $s^*(f) = f\circ s$ and, in this way, an $s^{-1}\tilde{\mathcal{O}}_{K\times\tilde{Y}}$ module structure on $\tilde{\mathcal{O}}_{K\times\tilde{Y}}$. It follows that, as sheaves of vector spaces,

$$s^+(p^+\mathcal{M}) = \tilde{\mathcal{O}}_{K\times\tilde{Y}} \otimes_{s^{-1}\tilde{\mathcal{O}}_{K\times\tilde{Y}}} s^{-1}(p^+\mathcal{M}) \simeq s^{-1}(p^+\mathcal{M}).$$

Thus, a $\tilde{\mathcal{D}}_{\lambda,K\times\tilde{Y}}$ module structure on $s^+(p^+\mathcal{M})$ is defined by an isomorphism

$$\Phi : \tilde{\mathcal{D}}_{\lambda,K\times\tilde{Y}} \to s^{-1}\tilde{\mathcal{D}}_{\lambda,K\times\tilde{Y}}.$$

The obvious such isomorphism (and the one that is correct) is the isomorphism described by

$$\Phi(\gamma) = s^{*-1}\gamma s^*.$$

Thus, $\tilde{\theta}$ is a $\tilde{\mathcal{D}}_{\lambda,K\times\tilde{Y}}$-module isomorphism if (when it is expressed as a map from $\tilde{p}^+\mathcal{M}$ to $s^{-1}\tilde{p}^+\mathcal{M}$) it is a sheaf theoretic isomorphism which satisfies

$$\tilde{\theta}\circ\gamma\circ\tilde{\theta}^{-1} = \Phi(\gamma) = s^{*-1}\gamma s^*$$

for every local section γ of $\tilde{\mathcal{D}}_{\lambda,K\times\tilde{Y}}$. In fact, it suffices to establish this condition for a set of $\gamma's$ which generates $\tilde{\mathcal{D}}_\lambda$ as a sheaf of algebras. Below we will characterize $\Phi(\gamma)$ for each of two classes of operators γ which, taken together with $\tilde{\mathcal{O}}$, form a generating set for $\tilde{\mathcal{D}}_\lambda$.

First, note that the condition that $\tilde{\theta}$ be an $\tilde{\mathcal{O}}$ module morphism is

$$\tilde{\theta} \circ f \circ \tilde{\theta}^{-1} = s^{*-1} f$$

for each local section f of $\mathcal{O}_{K \times \tilde{Y}}$ considered as a section of $\tilde{\mathcal{D}}_{K \times \tilde{Y}}$. This implies, in particular, that $\tilde{\theta}$ commutes with multiplication by sections of $\mathcal{O}_K$.

Now let $\xi \in \mathfrak{k}$ be an element of the Lie algebra of K and consider the corresponding sections ξ_K of $\mathcal{D}_K$ and $\xi_{\tilde{Y}}$ of $\tilde{\mathcal{D}}_{\lambda,\tilde{Y}}$, where

$$\xi_K f(k) = \frac{d}{dt}|_{t=0} f(\mathrm{e}^{-t\xi} k), \quad f \in \mathcal{O}_K$$

and

$$\xi_{\tilde{Y}} f(\tilde{y}) = \frac{d}{dt}|_{t=0} f(\mathrm{e}^{-t\xi} \tilde{y}), \quad f \in \mathcal{O}_{\lambda,\tilde{Y}}. \tag{2.1}$$

Then, $\xi_K \boxtimes 1$ and $1 \boxtimes \xi_{\tilde{Y}}$ are sections of $\tilde{\mathcal{D}}_{\lambda,K\times\tilde{Y}}$ and

$$\begin{aligned}(\xi_K \boxtimes 1) \circ s^* f(k, \tilde{y}) &= \frac{d}{dt}|_{t=0} f(\mathrm{e}^{-t\xi} k, \mathrm{e}^{-t\xi} k\tilde{y}) \\ &= (\xi_K \boxtimes 1) f(k, k\tilde{y}) + (1 \boxtimes \xi_{\tilde{Y}}) f(k, k\tilde{y}).\end{aligned}$$

Thus,

$$\Phi(\xi_K \boxtimes 1) = s^{*-1} \xi_K s^* = \xi_K \boxtimes 1 + 1 \boxtimes \xi_{\tilde{Y}}.$$

Given $\xi \in \mathfrak{k}$, the condition

$$\tilde{\theta} \circ (\xi_K \boxtimes 1) \circ \tilde{\theta}^{-1} = \Phi(\xi_K \boxtimes 1)$$

will be satisfied on all of $p^+\mathcal{M}$ if it is satisfied on $p^{-1}\mathcal{M}$ since $\tilde{\theta}$ is an $\mathcal{O}_{K\times\tilde{Y}}$ module isomorphism. However, since $\xi_K \boxtimes 1$ vanishes on $p^{-1}\mathcal{M}$, it follows that this condition is equivalent to the statement

$$(\xi_K \boxtimes 1) \circ \tilde{\theta} + (1 \boxtimes \xi_{\tilde{Y}}) \circ \tilde{\theta} = 0 \quad \text{on} \quad p^{-1}\mathcal{M}.$$

Finally, let η be any local section of $\tilde{\mathcal{D}}_{\lambda,\tilde{Y}}$ and consider the local section $1 \boxtimes \eta$ of $\tilde{\mathcal{D}}_\lambda$. Then for a local section f of $\mathcal{O}_{\lambda,K\times\tilde{Y}}$ we have

$$\Phi(1 \boxtimes \eta) f(k, \tilde{y}) = (1 \boxtimes \mathrm{Ad}_k(\eta)) f(k, \tilde{y}).$$

Now let, $i_k : \tilde{Y} \to K \times \tilde{Y}$ and $t_k : \tilde{Y} \to \tilde{Y}$ be defined for each $k \in K$ by

$$i_k(\tilde{y}) = (k, \tilde{y}), \quad t_k(\tilde{y}) = k\tilde{y}.$$

Let i_k^+ denote the $\mathcal{D}$-module inverse image functor induced by i_k. A routine calculation shows that if i_k^+ is applied to the maps $\tilde{\theta} : \tilde{p}^+\mathcal{M} \to s^{-1}\tilde{p}^+\mathcal{M}$ and $\Phi : \tilde{\mathcal{D}}_{\lambda,K\times\tilde{Y}} \to s^{-1}\tilde{\mathcal{D}}_{\lambda,K\times\tilde{Y}}$, then it yields, for each k, an isomorphism

$$\tilde{\theta}_k = i_k^* \tilde{\theta} : \mathcal{M} \to t_k^* \mathcal{M}$$

and an action map

$$i_k^+ \Phi(1 \boxtimes \eta) = \mathrm{Ad}_k(\eta) : t_k^*\mathcal{M} \to t_k^*\mathcal{M}.$$

On the other hand, $i_k^+(1 \boxtimes \eta) = \eta$ for each k. Thus, applying i_k^+ to the identity $\tilde{\theta} \circ (1 \boxtimes \eta) \circ \tilde{\theta}^{-1} = \Phi(1 \boxtimes \eta)$ implies that

$$\tilde{\theta}_k \circ \eta \circ \tilde{\theta}_k^{-1} = \mathrm{Ad}_k(\eta)$$

for each $k \in K$. The converse is also true, since the identity in question is a statement about the equality of two differential operators with coefficients in $\tilde{\mathcal{O}}_{K\times\tilde{Y}}$. Thus, it is true if it is true pointwise. In summary,

Lemma 2.2. *If $\mathcal{M}$ is a $\tilde{\mathcal{D}}_\lambda$ module on $\tilde{Y}$ which is K-equivariant as a sheaf of $\mathcal{O}$ modules with action map $\tilde{\theta}$, then $\mathcal{M}$ is K-equivariant as a sheaf of $\tilde{\mathcal{D}}_\lambda$-modules under this action if and only if the action satisfies*

1. $\tilde{\theta}_k \circ \eta \circ \tilde{\theta}_k^{-1} = \mathrm{Ad}_k(\eta)$ *for each $k \in K$ and each $\eta \in \tilde{\mathcal{D}}_{\lambda,\tilde{Y}}$, and*
2. $(\xi_K \boxtimes 1) \circ \tilde{\theta} + (1 \boxtimes \xi_{\tilde{Y}}) \circ \tilde{\theta}$ *vanishes on $p^{-1}\mathcal{M}$ for each $\xi \in \mathfrak{k}$.*

Conditions (1) and (2) are the sheaf theoretic versions of the Harish-Chandra compatibility conditions for $(\mathfrak{g}, K)$-modules.

The main objective of this section is to show that, in the case where $\tilde{Y}$ is the augmented flag manifold $\tilde{X}$ for a semi-simple real Lie group $G_\mathbb{R}$, the finite type $G_\mathbb{R}$-equivariant sheaves of $\tilde{\mathcal{D}}_\lambda$ modules on $\tilde{Y}$ have a particularly simple structure.

First we show how to construct monodromic $\tilde{\mathcal{D}}_\lambda$ modules with monodromy μ using a μ-twisted version of the standard induction construction. Let $Y = G_\mathbb{R}$ and $\tilde{Y} = G_\mathbb{R} \times H$ be considered as monodromic systems through the action of H on the right factor. This is a trivial monodromic system in the sense that the bundle $G_\mathbb{R} \times H \to G_\mathbb{R}$ is trivial. For each λ the sheaf $\tilde{\mathcal{D}}_{\lambda,\tilde{Y}}$ is just the subsheaf $\mathcal{D}_{G_\mathbb{R}} \boxtimes 1 = \pi^{-1}\mathcal{D}_{G_\mathbb{R}}$ of $\mathcal{D}_{\tilde{Y}}$. In the obvious way, each of the sheaves $\mathcal{O}_{\mu,\tilde{Y}}$ is a module over this sheaf of algebras as is each of the sheaves $\mathcal{O}_{\mu,\tilde{Y}} \otimes E$ where E is a finite dimensional vector space. In fact, $\mathcal{O}_{\mu,\tilde{Y}} \otimes E$ is just $p^+(\mathcal{E}_\mu)$, where $p : G_\mathbb{R} \times H \to H$ is the projection on H and $\mathcal{E}_\mu$ is the locally constant sheaf with stalk E and monodromy μ considered as a $\tilde{\mathcal{D}}_{\lambda,H}$ module (each $\tilde{\mathcal{D}}_{\lambda,H}$ is just the constant sheaf with stalk $\mathbb{C}$). Theorem 2.1 implies that, for each λ and μ, $\mathcal{O}_{\mu,\tilde{Y}} \otimes E$ is, in fact, a $G_\mathbb{R}$ equivariant sheaf of monodromic $\tilde{\mathcal{D}}_{\lambda,\tilde{Y}}$ modules with monodromy μ.

Now suppose S is an orbit of the $G_\mathbb{R}$ action on X and $s \in S$. Let P be the isotropy group of s in $G_\mathbb{R}$, B the isotropy group of s in G and N the maximal nilpotent subgroup of B. We may then identify H with B/N and define a homomorphism $b \to \bar{b} : P \to H$ to be the composition of the embedding $P \to B$ and the quotient map $B \to H$. The differential of this homomorphism is a Lie algebra homomorphism $\xi \to \bar{\xi} : \mathfrak{p} \to \mathfrak{h}$. We set $\tilde{S} = \pi^{-1}(S)$ and choose $\tilde{s} \in \tilde{S}$ such that $\pi(\tilde{s}) = s$. Note that $\tilde{S}$ is an orbit of the $G_\mathbb{R} \times H$ action on $\tilde{X}$ determined by the left action of $G_\mathbb{R}$ and the right action of H. We define a map $q : G_\mathbb{R} \times H \to \tilde{S}$ by $q(g, h) = g\tilde{s}h$. Note that the isotropy group of $\tilde{s}$ in $G_\mathbb{R} \times H$ is the copy of P in $G_\mathbb{R} \times H$ given by the embedding $b \to (b, \bar{b}^{-1})$. Thus, $\tilde{S}$ is the quotient of the free P space $G_\mathbb{R} \times H$ by the P-action determined by this embedding.

Let (σ, E) be a finite dimensional representation of P. As above, we consider $G_\mathbb{R} \times H$ to be a monodromic system, under the action of H on the right factor and $\mathcal{O}_\mu \otimes E$ to be a $G_\mathbb{R}$ equivariant sheaf of $\tilde{\mathcal{D}}_\lambda$-modules with monodromy μ. However, now we use the representation σ to give $\mathcal{O}_\mu \otimes E$ the structure of a P-equivariant sheaf of $\tilde{\mathcal{D}}_\lambda$-modules. Here, the action of P on $\mathcal{O}_\mu \otimes E$ is defined by

$$\tilde{\theta} f(b, (g, h)) = \sigma(b) f(b, (gb, \bar{b}^{-1}h)).$$

It is obvious that $\tilde{\theta}$ is an $\mathcal{O}$ module isomorphism and that condition (1) of the above lemma is satisfied (with P playing the role of K and $G_\mathbb{R} \times H$ playing the role of $\tilde{Y}$). Whether condition (2) of the lemma is also satisfied depends on the properties of σ, as we see below. In fact, sections of $p^{-1}\mathcal{M}$ are sections of $\mathcal{O}_{\mu,P\times(G_{\mathbb{R}}\times H)}\otimes E$ which are constant in the P variable. If f is such a section and $\xi\in\mathfrak{p}$, then

$$(\xi_P\boxtimes 1)\circ\tilde{\theta}f(b,(g,h))=\frac{d}{dt}|_{t=0}[\sigma(\mathrm{e}^{-t\xi}b)f(g\mathrm{e}^{-t\xi}b,\bar{b}^{-1}\mathrm{e}^{t\bar{\xi}}h)]$$
$$=\sigma(\xi)\sigma(b)f(gb,\bar{b}^{-1}h)+\frac{d}{dt}|_{t=0}\sigma(b)f(g\mathrm{e}^{-t\xi}b,\bar{b}^{-1}h)+\mu(\bar{\xi})\sigma(b)f(gb,\bar{b}^{-1}h)$$

while, when the embedding $\xi\to\xi_{G_{\mathbb{R}}\times H}:\mathfrak{p}\to\mathcal{D}_{\lambda,G_{\mathbb{R}}\times H}$ is combined with formula (2.1), we find

$$(1\boxtimes\xi_{G_{\mathbb{R}}\times H})\circ\tilde{\theta}f(b,(g,h))=\frac{d}{dt}|_{t=0}\sigma(b)f(g\mathrm{e}^{-t\xi}b,\bar{b}^{-1}h)-\lambda(\bar{\xi})\sigma(b)f(gb,\bar{b}^{-1}h).$$

Thus, condition (3) is satisfied if and only if $\sigma(\xi)$ is the scalar operator $\lambda(\bar{\xi})-\mu(\bar{\xi})$ for each $\xi\in\mathfrak{p}$.

Proposition 2.3. *Let (E,σ) be a finite dimensional representation of the isotropy group, P, in $G_{\mathbb{R}}$ of a point in X. The sheaf $\mathcal{O}_\mu\otimes E$ on $G_{\mathbb{R}}\times H$ is P-equivariant as a $\tilde{\mathcal{D}}_\lambda$-module under the action determined by σ, as above, if and only if the differential of σ satisfies $\sigma(\xi)=\lambda(\bar{\xi})-\mu(\bar{\xi})$ for all $\xi\in\mathfrak{p}$.*

Suppose σ satisfies the condition of Proposition 2.3, so that $\mathcal{O}_\mu\otimes E$ is a P-equivariant $\tilde{\mathcal{D}}_\lambda$-module. Then, since $G_{\mathbb{R}}\times H$ is a free P-space, Proposition 2.1 implies that $\mathcal{O}_\mu\otimes E$ is isomorphic to the pullback to $G_{\mathbb{R}}\times H$ of the monodromic sheaf $q_*^P(\mathcal{O}_\mu\otimes E)$ of $\tilde{\mathcal{D}}_\lambda$-modules on $\tilde{\mathcal{S}}$. We extend this sheaf by zero to all of $\tilde{X}$ (i.e., apply direct image with proper supports for the inclusion $\tilde{\mathcal{S}}\to\tilde{X}$) and denote the resulting sheaf by $\tilde{\mathcal{I}}^\mu_\lambda(\sigma)$. This is a monodromic sheaf on $\tilde{X}$ with monodromy μ. It is also a monodromic sheaf of $\tilde{\mathcal{D}}_{\lambda,\tilde{X}}$-modules, but to see why requires some comment. The embedding of a $G_{\mathbb{R}}$ orbit S into X has the property that at each point, the tangent space of S generates the tangent space of the complex manifold X over $\mathbb{C}$ (cf. [Sm]). This implies, in particular, that the structure sheaf $\mathcal{O}_S$ of S is just the sheaf theoretic restriction to S of the structure sheaf $\mathcal{O}_X$. The same thing is then true of the structure sheaf of the monodromic system $\tilde{\mathcal{S}}$ and of each of the sheaves $\mathcal{O}_{\lambda,\tilde{\mathcal{S}}}$—they are the sheaf theoretic restrictions to $\tilde{\mathcal{S}}$ of the corresponding sheaves on $\tilde{X}$. From this it follows that $\tilde{\mathcal{D}}_{\lambda,\tilde{\mathcal{S}}}$ is the sheaf theoretic restriction to $\tilde{\mathcal{S}}$ of the sheaf $\tilde{\mathcal{D}}_{\lambda,\tilde{X}}$. Hence, the extension by zero from $\tilde{\mathcal{S}}$ to $\tilde{X}$ of a sheaf of $\tilde{\mathcal{D}}_{\lambda,\tilde{\mathcal{S}}}$ modules on $\tilde{\mathcal{S}}$ will automatically have the structure of a sheaf of $\tilde{\mathcal{D}}_{\lambda,\tilde{X}}$ modules.

The sheaf $\tilde{\mathcal{I}}^\mu_\lambda(\sigma)$ will be called the *induced sheaf* for the data σ,λ and μ and the orbit $\tilde{\mathcal{S}}$. In concrete terms, for each open set $U\subset\tilde{X}$,

$$\tilde{\mathcal{I}}^\mu_\lambda(\sigma)(U)=\{f\in\mathcal{O}_\mu(q^{-1}(U)):f(gb,\bar{b}^{-1}h)=\sigma(b)^{-1}f(g,h)\quad\text{for}\quad b\in P\}.$$

This is a $G_{\mathbb{R}}$-equivariant sheaf of $\tilde{\mathcal{D}}_\lambda$-modules because $\mathcal{O}_\mu\otimes E$ is $G_{\mathbb{R}}$-equivariant and the $G_{\mathbb{R}}$ and P-actions commute.

It turns out that every $G_{\mathbb{R}}$-equivariant $\tilde{\mathcal{D}}_\lambda$-module which is monodromic with monodromy μ and which has finite dimensional geometric fibers as an $\tilde{\mathcal{O}}$-module has the above form when restricted to an orbit $\tilde{\mathcal{S}}$. Here, by the geometric fiber of a monodromic $\tilde{\mathcal{O}}$-module $\mathcal{M}$ at $s\in X$, we mean the $\mathcal{O}$-module pullback $i_s^+\mathcal{M}$, where i_s is the inclusion of the monodromic system $\pi^{-1}(s)\to s$ into the monodromic system $\tilde{X}\to X$. Thus, it is an $\mathfrak{h}$-equivariant sheaf on $\pi^{-1}(s)\simeq H$. If this sheaf has finite dimensional stalks, we say $\mathcal{M}$ has finite dimensional geometric fiber at s.

Note that the sheaf induced, as above, from a finite dimensional representation of P has this property at each point. A sheaf of $\tilde{\mathcal{D}}_\lambda$ modules which has finite dimensional geometric fiber at each point will be said to be of *finite type*.

Theorem 2.4. *If $\mathcal{M}$ is a $G_\mathbb{R}$-equivariant $\tilde{\mathcal{D}}_\lambda$-module on $\tilde{X}$ which is monodromic with monodromy μ and of finite type, then, for each $G_\mathbb{R} \times H$ orbit $\tilde{S}$, the restriction of $\mathcal{M}$ to $\tilde{\mathcal{S}}$ is isomorphic to the restriction of $\tilde{\mathcal{I}}^\mu_\lambda(\sigma)$ to $\tilde{\mathcal{S}}$ for some finite dimensional representation σ of the isotropy group P of a point of $\tilde{\mathcal{S}}$.*

Proof. Since $\mathcal{M}$ is $G_\mathbb{R}$-equivariant as a $\tilde{\mathcal{D}}_\lambda$-module, there is a $\tilde{\mathcal{D}}_\lambda$-module isomorphism

$$\tilde{\theta} : \tilde{p}^+\mathcal{M} \to \tilde{m}^+\mathcal{M}$$

satisfying the associativity and identity laws. We choose $\tilde{s} \in \tilde{S}$ and consider the map

$$j : G_\mathbb{R} \times H \to G_\mathbb{R} \times \tilde{X} \quad \text{where} \quad j(g,h) = (g, \tilde{s}h).$$

We have $m \circ j = q : G_\mathbb{R} \times H \to \tilde{X}$ and $p \circ j = i_s \circ k : G_\mathbb{R} \times H \to \tilde{X}$, where q is the map which appears in the above construction of $\tilde{\mathcal{I}}^\mu_\lambda(\sigma)$, $k : G_\mathbb{R} \times H \to H$ is the projection and $i_s : H \simeq \pi^{-1}(s) \to \tilde{X}$ is the inclusion. If we pull back the isomorphism $\tilde{\theta}$ using j^+ we obtain a $\mathcal{D}_\lambda$ module isomorphism

$$j^+(\tilde{\theta}) : k^+ \circ i_s^+\mathcal{M} \to q^+\mathcal{M}.$$

By hypothesis, $i_s^+\mathcal{M}$ is an $\mathfrak{h}$-equivariant sheaf on H with monodromy μ and with finite dimensional stalks; in other words, a locally constant sheaf on H with stalks isomorphic to a finite dimensional vector space F and with monodromy μ. We denote this sheaf by $\mathcal{F}$. Since $H \to pt$ and $G_\mathbb{R} \times H \to G_\mathbb{R}$ are trivial monodromic systems, $\tilde{\mathcal{D}}_\lambda$ and $\tilde{\mathcal{D}} = \tilde{\mathcal{D}}_0$ are the same for these systems. Thus, the $\mathcal{D}_\lambda$-module pullback $k^+\mathcal{F}$ of $\mathcal{F}$ to $G_\mathbb{R} \times H$ is $\mathcal{O}_\mu \otimes F$ with $\tilde{\mathcal{D}}_\lambda$-module structure determined by the first factor in the tensor product.

Therefore, we have a $\tilde{\mathcal{D}}_\lambda$-module isomorphism

$$j^+(\tilde{\theta}) : \mathcal{O}_\mu \otimes F \to q^+\mathcal{M}.$$

Since $G_\mathbb{R} \times H$ is a free P space and q is the quotient map, Proposition 2.1 implies that $q^+\mathcal{M}$ has the structure of a P-equivariant sheaf of $\tilde{\mathcal{D}}_\lambda$-modules. Via the isomorphism $j^+(\tilde{\theta})$, this imposes the structure of a P-equivariant sheaf of $\tilde{\mathcal{D}}_\lambda$-modules on $\mathcal{O}_\mu \otimes F$. Since $\mathcal{M}$ is isomorphic to the P-invariant direct image of this sheaf, it remains to show that the P action on $\mathcal{O}_\mu \otimes F$ has the form prescribed in the construction of $\tilde{\mathcal{I}}^\mu_\lambda(\sigma)$. However, it is easy to see that the action necessarily has the form

$$\tilde{\theta} f(b,(g,h)) = \sigma(b) f(b,(gb, \bar{b}^{-1}h))$$

for some representation σ of P on F. That σ has differential given by the character $\lambda - \mu$ then follows from Proposition 2.3. This completes the proof. ◻

The next proposition concerns the induced sheaf in the case where $\lambda = 0$. A *flat section* of a sheaf of $\tilde{\mathcal{D}}$ modules is one which is killed by all vector fields.

Proposition 2.5. *The sheaf $\tilde{\mathcal{I}}^\mu_0(\sigma)$ is isomorphic as a $\tilde{\mathcal{D}}$-module to $\tilde{\mathcal{O}}_{\tilde{X}} \otimes \mathcal{F}^\mu$, where $\mathcal{F}^\mu$ is the sheaf of flat sections of $\tilde{\mathcal{I}}^\mu_0(\sigma)$.*

Proof. This follows immediately from the description of the flat sections of the sheaf $\tilde{\mathcal{I}}_0^\mu(\sigma)$ on a neighborhood U as those which, when regarded as functions on $q^{-1}(U)$, are constant in the $G_\mathbb{R}$ variable. □

We are now in a position to prove the main theorem of this section which is a special case of our version of the Riemann-Hilbert correspondence. Let $M^\mu_{G_\mathbb{R}}(\tilde{\mathcal{D}}_{\tilde{X}}, f)$ denote the category of $G_\mathbb{R}$ equivariant $\tilde{\mathcal{D}}$-modules on $\tilde{X}$ with monodromy μ and with finite type. Similarly, we denote by $M^\mu_{G_\mathbb{R}}(\tilde{X}, f)$ the category of $G_\mathbb{R}$-equivariant sheaves on $\tilde{X}$ with monodromy μ and of finite type in the sense that they have finite dimensional stalks. These are the so-called *constructible sheaves* for the $G_\mathbb{R}$ orbit stratification, which are also $G_\mathbb{R}$-equivariant and monodromic with monodromy μ (see [KSd]).

If $\mathcal{M}$ is any $\tilde{\mathcal{D}}$-module on $\tilde{X}$, we denote by $F(\mathcal{M})$ its sheaf of flat sections. If $\mathcal{M}$ is $G_\mathbb{R}$-equivariant as a $\tilde{\mathcal{D}}$ module, then $F(\mathcal{M})$ is $G_\mathbb{R}$-equivariant as a sheaf. In fact, if $\theta : \tilde{p}^+\mathcal{M} \to \tilde{m}^+\mathcal{M}$ is the action map for $\mathcal{M}$, then the fact that θ is a $\tilde{\mathcal{D}}$-module isomorphism implies that θ and θ^{-1} preserve flat sections. Since $F(\tilde{p}^+\mathcal{M}) = \tilde{p}^{-1}F(\mathcal{M})$ and $F(\tilde{m}^+\mathcal{M}) = \tilde{m}^{-1}F(\mathcal{M})$, it follows that $F(\theta)$ defines an action which makes $F(\mathcal{M})$ a $G_\mathbb{R}$-equivariant sheaf. Clearly, F preserves monodromy and takes sheaves of finite type to sheaves with finite dimensional stalks. Thus, we have a functor

$$F : M^\mu_{G_\mathbb{R}}(\tilde{\mathcal{D}}_{\tilde{X}}, f) \to M^\mu_{G_\mathbb{R}}(\tilde{X}, f).$$

There is also a functor which goes the other way, namely

$$\tilde{\mathcal{O}}_{\tilde{X}} \otimes (\ \cdot\ ) : M^\mu_{G_\mathbb{R}}(\tilde{X}, f) \to M^\mu_{G_\mathbb{R}}(\tilde{\mathcal{D}}_{\tilde{X}}, f)$$

where, for $\mathcal{E}$ an object of $M^\mu_{G_\mathbb{R}}(\tilde{X}, f)$, $\tilde{\mathcal{O}}_{\tilde{X}} \hat{\otimes} \mathcal{E}$ is a $\tilde{\mathcal{D}}$ module through the action of $\tilde{\mathcal{D}}_{\tilde{X}}$ on the first factor.

Theorem 2.6. *The functor $F : M^\mu_{G_\mathbb{R}}(\tilde{\mathcal{D}}_{\tilde{X}}, f) \to M^\mu_{G_\mathbb{R}}(\tilde{X}, f)$ is an equivalence of categories with inverse $\tilde{\mathcal{O}}_{\tilde{X}} \otimes (\ \cdot\ )$.*

Proof. Since sections of $\tilde{\mathcal{D}}_{\tilde{X}}$ act just on the first factor of the tensor product, it is obvious that $F(\tilde{\mathcal{O}}_{\tilde{X}} \otimes \mathcal{E}) \simeq \mathcal{E}$ if $\mathcal{E}$ is a sheaf in $M^\mu_{G_\mathbb{R}}(\tilde{X}, f)$.

On the other hand, if $\mathcal{M}$ is a sheaf in $M^\mu_{G_\mathbb{R}}(\tilde{\mathcal{D}}_{\tilde{X}}, f)$, then the $\mathcal{O}$-module action map defines a morphism $\tilde{\mathcal{O}}_{\tilde{X}} \otimes F(\mathcal{M}) \to \mathcal{M}$. This is an isomorphism if it is an isomorphism on each stalk. That this is so, follows from Proposition 2.5 and the fact that, on each $G_\mathbb{R}$ orbit, $\mathcal{M}$ is isomorphic to an induced sheaf $\tilde{\mathcal{I}}_0^\mu(\sigma)$ (Theorem 2.4). □

3. The Riemann-Hilbert Correspondence

We are now prepared to prove the main theorem of this paper—the Riemann-Hilbert correspondence in the context of $\mathcal{D}_\lambda$-modules and $G_\mathbb{R}$-equivariance. This is just an extension to the appropriate equivariant derived categories of the equivalence of Theorem 2.6. Since the functors of Theorem 2.6 define an equivalence and, hence, are exact, this extension is relatively straightforward. However, there are still technical matters that must be handled carefully.

We will use the Bernstein-Lunts version of the equivariant derived category [BL]. For our purposes here, the best description of this category is the one in terms of fibered categories given in Section 2.4 of [BL]. We give a brief description of this in each of the four cases of interest to us here—sheaves of complex vector spaces, monodromic sheaves of complex vector spaces, sheaves of $\mathcal{D}_\lambda$ modules and monodromic sheaves of $\tilde{\mathcal{D}}_\lambda$ modules.

With $G_{\mathbb{R}}$ as before, let Y be a $G_{\mathbb{R}}$ space in the category of analytic CR manifolds. A *resolution of* Y is a fibration $P \to Y$ of analytic $G_{\mathbb{R}}$-equivariant CR spaces where P is a free $G_{\mathbb{R}}$ space. If we set $P_0 = G_{\mathbb{R}} \backslash P$, then the statement that P is a free $G_{\mathbb{R}}$ space means that P_0 is Hausdorff and the quotient map $P \to P_0$ is a fibration in the category of analytic CR manifolds and its fiber is $G_{\mathbb{R}}$. The resolutions of Y form the objects of a category in which the morphisms are $G_{\mathbb{R}}$-equivariant fibrations $P \to Q$ over Y for which the diagram

$$\begin{tikzcd} P \arrow[rr] \arrow[dr] & & Q \arrow[dl] \\ & Y & \end{tikzcd} \tag{3.1}$$

commutes. Thus, each such morphism $P \to Q$ induces a map $P_0 \to Q_0$.

Note that in [BL] arbitrary maps $P \to Q$ over Y are allowed as morphisms of resolutions. For technical reasons, we need to work only with fibrations in the category of analytic CR manifolds. This still provides a rich enough category of resolutions and morphisms of resolutions to define the equivariant derived category (cf. 2.4.4 of [BL]).

Let $D^b(Y)$ denote the bounded derived category of sheaves of vector spaces on Y. Then an object $\mathcal{S}$ of the bounded equivariant derived category $D^b_{G_{\mathbb{R}}}(Y)$ is a functor which assigns to each resolution $P \to Y$ an object $\mathcal{S}(P)$ of $D^b(P_0)$ and, to each morphism $f : P \to Q$ of resolutions of Y, an isomorphism $\alpha(f) : f^{-1}\mathcal{S}(Q) \to \mathcal{S}(P)$ in such a way that compositions and identities are preserved (see [BL] for details). Morphisms in $D^b_{G_{\mathbb{R}}}(Y)$ are defined in the obvious way as morphisms of functors. There are natural truncation functors on $D^b_{G_{\mathbb{R}}}(Y)$. For example, $\tau_{\leq b}(\mathcal{S})(P) = \tau_{\leq b}(\mathcal{S}(P))$, for each integer b. With these truncation functors, $D^b_{G_{\mathbb{R}}}(Y)$ is a triangulated category with t-structure or a *t-category* (see [KSc] p. 411 or [BL] p. 88). There is a forgetful functor For : $D^b_{G_{\mathbb{R}}}(Y) \to D^b(Y)$ which respects the t-structure and, thus, takes the heart of $D^b_{G_{\mathbb{R}}}(Y)$ to the heart of $D^b(Y)$. Namely, let $T = G_{\mathbb{R}} \times Y$ have the diagonal action of $G_{\mathbb{R}}$. The projection from T to Y is called the *trivial resolution* of Y. For each $\mathcal{S} \in D^b_{G_{\mathbb{R}}}(Y)$ let $\mathrm{For}(\mathcal{S}) = \mathcal{S}(T)$. The heart of $D^b(Y)$ is the category of sheaves on Y while For determines an equivalence between the heart of $D^b_{G_{\mathbb{R}}}(Y)$ and the category of $G_{\mathbb{R}}$ equivariant sheaves. The cohomology functors $\{H^n\}$ associated with the t-structure on $D^b_{G_{\mathbb{R}}}(Y)$ take values in the heart—i.e., take values which are $G_{\mathbb{R}}$-equivariant sheaves. Other functors usually associated with derived categories of sheaves, such as inverse image and direct image also have analogues in the equivariant derived category.

Next, we consider a $G_{\mathbb{R}}$ equivariant monodromic system $\tilde{Y} \to Y$. The bounded derived category of monodromic sheaves of complex vector spaces on $\tilde{Y}$ with monodromy μ will be denoted by $D^b(\tilde{Y})^\mu$. To define a corresponding equivariant derived category $D^b_{G_{\mathbb{R}}}(\tilde{Y})^\mu$ we must first define resolutions of monodromic $G_{\mathbb{R}}$ spaces. A resolution of $\tilde{Y}$ is a $G_{\mathbb{R}}$ equivariant map $\tilde{P} \to \tilde{Y}$ which is a fibration in the category of monodromic spaces and for which the space $\tilde{P}$ is a free monodromic $G_{\mathbb{R}}$ space as defined in the paragraph preceding Proposition 2.1. With morphisms between resolutions $\tilde{P} \to \tilde{Y}$ and $\tilde{Q} \to \tilde{Y}$ defined to be $G_{\mathbb{R}}$ equivariant fibrations $\tilde{P} \to \tilde{Q}$ of monodromic systems satisfying the analogue of (3.1), the resolutions of $\tilde{Y}$ form a category. Note that if $\tilde{Y} \to Y$ is a monodromic $G_{\mathbb{R}}$ space and $P \to Y$ is a resolution of Y by a free $G_{\mathbb{R}}$ space in the category of analytic CR manifolds, then $\tilde{P} = \tilde{Y} \times_Y P \to \tilde{Y}$ is a resolution of $\tilde{Y}$. Thus, for each resolution $P \to Y$, there is associated a natural monodromic system $\tilde{P} \to P$, which is, in fact, a $G_{\mathbb{R}}$-equivariant monodromic system. In particular, the sheaves $\mathcal{D}_{\lambda,P}$ on P and $\tilde{\mathcal{D}}_{\lambda,\tilde{P}}$ on $\tilde{P}$ are defined.

Now we may define an object $\mathcal{S}$ of the $G_{\mathbb{R}}$ equivariant derived category $D^b_{G_{\mathbb{R}}}(\tilde{Y})^\mu$ to be a functor which assigns to each resolution $P \to Y$ an object $\mathcal{S}(P)$ of $D^b(\tilde{P}_0)^\mu$ and to each morphism $f : \tilde{P} \to \tilde{Q}$ of resolutions an isomorphism $\alpha(f) : f^{-1}\mathcal{S}(Q) \to \mathcal{S}(P)$ in a composition and identity preserving fashion. Morphisms in $D^b_{G_{\mathbb{R}}}(\tilde{Y})^\mu$ are morphisms of functors. As before, there is a forgetful functor $For : D^b_{G_{\mathbb{R}}}(\tilde{Y})^\mu \to D^b(\tilde{Y})^\mu$. The heart of $D^b_{G_{\mathbb{R}}}(\tilde{Y})^\mu$ is equivalent under the forgetful functor to the category $M^\mu_{G_{\mathbb{R}}}(\tilde{Y})$ of $G_{\mathbb{R}}$ equivariant monodromic sheaves of vector spaces on $\tilde{Y}$ with monodromy μ (cf. [BB2], [MUV]).

In [Sm] a variant of the Bernstein-Lunts construction is used to construct an equivariant derived category of DNF $\mathcal{D}_\lambda$-modules. We briefly describe the definition of a similar category, but one which does not use DNF modules.

We will denote the bounded derived category of $\mathcal{D}_{\lambda,Y}$ modules by $D^b(\mathcal{D}_{\lambda,Y})$ and then define a corresponding equivariant derived category $D^b_{G_{\mathbb{R}}}(\mathcal{D}_{\lambda,Y})$. An object $\mathcal{M}$ of this category is a functor which assigns to each $G_{\mathbb{R}}$ resolution of Y an object $\mathcal{M}(P)$ of $D^b(\mathcal{D}_{\lambda,P_0})$ and to each morphism $f : P \to Q$ of resolutions an isomorphism $\alpha(f) : f^+\mathcal{M}(Q) \to \mathcal{M}(P)$ in a composition and identity preserving way. The category $D^b_{G_{\mathbb{R}}}(\mathcal{D}_{\lambda,Y})$ is a t-category with the truncation functors defined as above.

Note that in this definition we do not require $\mathcal{D}_\lambda$ modules to be DNF or to be topological modules at all. This is a departure from [Sm] where a similar equivariant derived category is constructed using DNF sheaves of $\mathcal{D}_\lambda$ modules. In the next section we shall need to make use of this category. However, in the development of the Riemann-Hilbert correspondence, the topological vector space structure is not useful and, in fact, creates serious technical difficulties. Furthermore, in Section 5 we shall show that, in the end, it doesn't matter which approach is used. Once one restricts to the subcategory of interest (the subcategory of objects of finite type), the category constructed using DNF modules is equivalent to the one constructed using general modules. When we do wish to refer to the equivariant derived category of DNF $D^b_{G_{\mathbb{R}}}(\mathcal{D}_{\lambda,Y})$ modules, as defined in [Sm], we will denote it by $D^b_{G_{\mathbb{R}}}(\mathcal{D}^t_{\lambda,Y})$.

Similarly, we will denote by $D^b(\tilde{\mathcal{D}}_{\lambda,\tilde{Y}})^\mu$ the bounded derived category of monodromic $\tilde{\mathcal{D}}_{\lambda,\tilde{Y}}$ modules with monodromy μ. We define, as above, an object $\mathcal{M}$ of the corresponding equivariant derived category $D^b_{G_{\mathbb{R}}}(\tilde{\mathcal{D}}_{\lambda,\tilde{Y}})^\mu$ to be a functor which assigns to each $G_{\mathbb{R}}$ resolution $\tilde{P} \to \tilde{Y}$ an object $\mathcal{M}(\tilde{P})$ of $D^b(\mathcal{D}_{\lambda,\tilde{P}_0})^\mu$ and to each morphism $f : \tilde{P} \to \tilde{Q}$ of resolutions an isomorphism $\alpha(f) : f^+\mathcal{M}(\tilde{Q}) \to \mathcal{M}(\tilde{P})$ in such a way that compositions and identities are preserved. Here, $f^+ : D^b(\mathcal{D}_{\lambda,\tilde{Q}_0})^\mu \to D^b(\mathcal{D}_{\lambda,\tilde{P}_0})^\mu$ is the functor induced on the derived category by the inverse image functor for $\tilde{\mathcal{D}}_\lambda$ modules associated to the map $f_0 : \tilde{P}_0 \to \tilde{Q}_0$ induced by f. As before, morphisms in the category $D^b_{G_{\mathbb{R}}}(\tilde{\mathcal{D}}_{\lambda,\tilde{Y}})^\mu$ are defined to be morphisms of functors. Like $D^b_{G_{\mathbb{R}}}(\mathcal{D}_{\lambda,Y})$, the category $D^b_{G_{\mathbb{R}}}(\tilde{\mathcal{D}}_{\lambda,\tilde{Y}})^\mu$ is a t-category and the forgetful functor, $For : D^b_{G_{\mathbb{R}}}(\tilde{\mathcal{D}}_{\lambda,\tilde{Y}})^\mu \to D^b(\tilde{\mathcal{D}}_{\lambda,\tilde{Y}})^\mu$, sends $D^b_{G_{\mathbb{R}}}(\tilde{\mathcal{D}}_{\lambda,\tilde{Y}})^\mu$ into objects in

$D^b(\tilde{\mathcal{D}}_{\lambda,\tilde{Y}})^\mu$ whose cohomologies are in $M^\mu_{G_\mathbb{R}}(\tilde{\mathcal{D}}_{\lambda,\tilde{Y}})$, the category of $G_\mathbb{R}$ equivariant $\tilde{\mathcal{D}}_{\lambda,\tilde{Y}}$ modules with monodromy μ.

We now specialize to the case of primary interest to us—the case where Y is the flag manifold X for $\mathfrak{g}$. Our goal is to prove a Riemann-Hilbert correspondence for $D^b_{G_\mathbb{R}}(\mathcal{D}_{\lambda,X})$. Specifically, we will construct a functor

$$D^b_{G_\mathbb{R}}(\mathcal{D}_{\lambda,X}) \to D^b_{G_\mathbb{R}}(\tilde{X})^{-\lambda}$$

which is an equivalence between the corresponding full subcategories consisting of objects of finite type (these will be defined in due course). The first step in the construction of this functor is the following:

Proposition 3.1. *The map $\pi : \tilde{X} \to X$ induces a natural equivalence of categories*

$$\pi^{-1} : D^b_{G_\mathbb{R}}(\mathcal{D}_{\lambda,X}) \to D^b_{G_\mathbb{R}}(\tilde{\mathcal{D}}_{\lambda,\tilde{X}})^0.$$

Proof. This equivalence is defined as follows: If $P \to X$ is a a resolution of X, let $\tilde{P} \to \tilde{X}$ be the corresponding resolution of $\tilde{X}$ ($\tilde{P} = \tilde{X} \times_X P$) and $\pi : \tilde{P} \to P$ the quotient map. Then $\pi^{-1} : M(\mathcal{D}_{\lambda,P}) \to M^0(\tilde{\mathcal{D}}_{\lambda,\tilde{P}})$ is an equivalence for every P (see Section 1) and this, in turn, defines an equivalence between $D^b_{G_\mathbb{R}}(\mathcal{D}_{\lambda,X})$ and $D^b_{G_\mathbb{R}}(\tilde{\mathcal{D}}_{\lambda,\tilde{X}})^0$. □

Recall from Section 1, that on any monodromic system, the functor $\mathcal{O}_\lambda \otimes_{\tilde{\mathcal{O}}} (\cdot)$ takes sheaves of $\tilde{\mathcal{D}}_\nu$-modules with monodromy μ to sheaves of $\tilde{\mathcal{D}}_{\nu+\lambda}$-modules with monodromy $\mu + \lambda$. Clearly this defines an equivalence of categories with inverse $\mathcal{O}_{-\lambda} \otimes_{\tilde{\mathcal{O}}} (\cdot)$. If we apply the analogous equivalences between derived categories

$$\mathcal{O}_\lambda \otimes_{\tilde{\mathcal{O}}} (\cdot) : D^b(\tilde{\mathcal{D}}_{\nu,\tilde{P}_0})^\mu \to D^b(\tilde{\mathcal{D}}_{\nu+\lambda,\tilde{P}_0})^{\mu+\lambda}$$

for each resolution $P \to Y$ of Y, we clearly define an equivalence between $D^b_{G_\mathbb{R}}(\tilde{\mathcal{D}}_\nu)^\mu$ and $D^b_{G_\mathbb{R}}(\tilde{\mathcal{D}}_{\nu+\lambda})^{\mu+\lambda}$. Thus,

Proposition 3.2. *For each triple (λ, ν, μ) the functor $\mathcal{O}_\lambda \otimes_{\tilde{\mathcal{O}}} (\cdot)$ defines an equivalence of categories from $D^b_{G_\mathbb{R}}(\tilde{\mathcal{D}}_\nu)^\mu$ to $D^b_{G_\mathbb{R}}(\tilde{\mathcal{D}}_{\nu+\lambda})^{\mu+\lambda}$.*

We define the equivariant DeRham functor

$$\tilde{DR} : D^b_{G_\mathbb{R}}(\tilde{\mathcal{D}}_{\tilde{X}})^\mu \to D^b_{G_\mathbb{R}}(\tilde{X})^\mu$$

in the following way. To begin with, for an analytic CR manifold Y, we set

$$\tilde{DR}_{\tilde{Y}} = R\mathcal{H}om_{\tilde{\mathcal{D}}}(\tilde{\mathcal{O}}_{\tilde{Y}},\ \cdot\ ) : D^b(\tilde{\mathcal{D}}_{\tilde{Y}})^\mu \to D^b(\tilde{Y})^\mu.$$

We give a more concrete description of this functor following [Bj] and [Bo]. Let Ω^p_Y denote the space of p-forms with coefficients in $\mathcal{O}_Y$. If $\pi : \tilde{Y} \to Y$ is a monodromic system, then $\tilde{\Omega}^p_{\tilde{Y}}$ will denote $\pi^{-1}\Omega^p_Y$. This may be regarded as the sheaf of analytic CR sections of the bundle of p forms on the vector bundle T^π on $\tilde{Y}$, where T^π is the quotient of the tangent bundle of $\tilde{Y}$ by the subbundle of vectors tangent to the fibers of π. Clearly, for each open set $U \subset \tilde{Y}$ sections of T^π over U act as differential operators on $\tilde{\mathcal{O}}(U)$ and, hence, determine elements of $\tilde{\mathcal{D}}(U)$. It follows that if $\mathcal{M}$ is a $\tilde{\mathcal{D}}$-module, then the differential

$$d : \tilde{\Omega}^p_{\tilde{Y}} \otimes_{\tilde{\mathcal{O}}_{\tilde{Y}}} \mathcal{M} \to \tilde{\Omega}^{p+1}_{\tilde{Y}} \otimes_{\tilde{\mathcal{O}}_{\tilde{Y}}} \mathcal{M}$$

may be defined in the usual way so as to yield a complex

$$\tilde{DR}_{\tilde{Y}}(\mathcal{M}) = \{\tilde{\Omega}^\bullet_{\tilde{Y}} \otimes_{\tilde{\mathcal{O}}_{\tilde{Y}}} \mathcal{M}, d\}.$$

This is the DeRham complex of a $\tilde{\mathcal{D}}_{\tilde{Y}}$ module $\mathcal{M}$. Its cohomology in degree zero yields the sheaf of flat sections $F(\mathcal{M})$ of $\mathcal{M}$. If $\mathcal{M}$ is $\mathfrak{h}$-equivariant, then the differentials in this complex are also $\mathfrak{h}$-equivariant and so it is a complex of monodromic modules. If $\mathcal{M}$ has monodromy μ, then it is a complex of modules with monodromy μ. When the DeRham functor is applied to a complex $\mathcal{M}$ of $\tilde{\mathcal{D}}$-modules with monodromy μ, the total complex of the resulting double complex is the DeRham complex of $\mathcal{M}$ and will be denoted $\tilde{DR}_{\tilde{Y}}(\mathcal{M})$.

We now define the DeRham functor for the equivariant derived category. Given an object $\mathcal{M} = \{\mathcal{M}(\tilde{P})\}$ of $D^b_{G_\mathbb{R}}(\tilde{\mathcal{D}}_{\tilde{X}})^\mu$ we set

$$\tilde{DR}(\mathcal{M})(\tilde{P}) = \tilde{DR}_{\tilde{P}_0}(\mathcal{M}(\tilde{P}))$$

for each resolution $\tilde{P} \to \tilde{X}$. To show that this defines an object of $D^b_{G_\mathbb{R}}(\tilde{X})$ we must show that the isomorphism $\alpha(f) : f^+\mathcal{M}(\tilde{Q}) \to \mathcal{M}(\tilde{P})$, associated to a morphism $f : \tilde{P} \to \tilde{Q}$, is taken by $\tilde{DR}_{\tilde{P}}$ to an isomorphism from $f^{-1}\tilde{DR}_{\tilde{Q}}(\mathcal{M}(\tilde{Q}))$ to $\tilde{DR}_{\tilde{P}}(\mathcal{M}(\tilde{P}))$. This amounts to showing that

$$\tilde{DR}_{\tilde{P}} \circ f^+ = f^{-1} \circ \tilde{DR}_{\tilde{Q}}.$$

To see this, note that, since $\tilde{P} \to \tilde{Q}$ is a fibration, the complex $f^{-1} \circ \tilde{DR}_{\tilde{Q}}(\mathcal{M})$ may be viewed as the subcomplex of $\tilde{DR}_{\tilde{P}} \circ f^+(\mathcal{M})$ consisting of forms which have coefficients constant in the fiber direction for f and which kill vector fields in the fiber direction. Thus, there is a morphism $\phi : f^{-1} \circ \tilde{DR}_{\tilde{Q}}(\mathcal{M}) \to \tilde{DR}_{\tilde{P}} \circ f^+(\mathcal{M})$. This will be a quasi-isomorphism of complexes if it is so locally. But, locally over $\tilde{Q}$, the map $f : \tilde{P} \to \tilde{Q}$ is a projection of the form $\tilde{U} \times W \to \tilde{U}$ with $\tilde{U}$ an H-invariant neighborhood in $\tilde{Q}$ and W a space in the category of analytic CR manifolds. That ϕ is a quasi-isomorphism of complexes and, hence an isomorphism in the derived category, follows from an application of the Poincaré Lemma in the W variable. This shows that $\tilde{DR}_{\tilde{X}}$ defines a functor from $D^b_{G_\mathbb{R}}(\tilde{\mathcal{D}}_{\tilde{X}})^\mu$ to $D^b_{G_\mathbb{R}}(\tilde{X})^\mu$.

In view of the above remarks and Proposition 3.2, we define the functor

$$\tilde{DR}_\lambda : D^b_{G_\mathbb{R}}(\tilde{\mathcal{D}}_{\lambda,\tilde{X}})^\mu \to D^b_{G_\mathbb{R}}(\tilde{X})^{\mu-\lambda}$$

to be the composition of

$$\mathcal{O}_{-\lambda} \otimes_{\tilde{\mathcal{O}}_{\tilde{X}}} (\cdot) : D^b_{G_\mathbb{R}}(\tilde{\mathcal{D}}_{\lambda,\tilde{X}})^\mu \to D^b_{G_\mathbb{R}}(\tilde{\mathcal{D}}_{0,\tilde{X}})^{\mu-\lambda}$$

and

$$\tilde{DR} : D^b_{G_\mathbb{R}}(\tilde{\mathcal{D}}_{0,\tilde{X}})^{\mu-\lambda} \to D^b_{G_\mathbb{R}}(\tilde{X})^{\mu-\lambda}.$$

This functor itself is not an equivalence; however, it is an equivalence between the two subcategories which we now describe.

Let $\mathcal{M}$ be a sheaf of $\tilde{\mathcal{D}}_{\lambda,\tilde{X}}$ modules. As in Section 2, we say that $\mathcal{M}$ is of *finite type* if the stalks of $\mathcal{M}$ have finite dimensional geometric fiber. We say an object in $D^b_{G_\mathbb{R}}(\tilde{\mathcal{D}}_{\lambda,\tilde{X}})^\mu$ has finite type if its cohomology modules all have finite type (recall that the heart of $D^b_{G_\mathbb{R}}(\tilde{\mathcal{D}}_{\lambda,\tilde{X}})^\mu$ is equivalent to the heart of $D^b(\tilde{\mathcal{D}}_{\lambda,\tilde{X}})^\mu$ under the forgetful functor, so, through this equivalence, the cohomologies of $D^b_{G_\mathbb{R}}(\tilde{\mathcal{D}}_{\lambda,\tilde{X}})^\mu$

may be regarded as sheaves of $\tilde{\mathcal{D}}_{\lambda,\tilde{X}}$ modules). We denote by $D^b_{G_{\mathbb{R}},f}(\tilde{\mathcal{D}}_{\lambda,\tilde{X}})^\mu$ the full subcategory of $D^b_{G_{\mathbb{R}}}(\tilde{\mathcal{D}}_{\lambda,\tilde{X}})^\mu$ consisting of objects of finite type.

Similarly, we shall say that a sheaf of vector spaces on X has *finite type* if it has finite dimensional stalks. We then let $D^b_{G_{\mathbb{R}},f}(\tilde{X})^\mu$ be the full subcategory of $D^b_{G_{\mathbb{R}}}(\tilde{X})^\mu$ consisting of objects whose cohomology sheaves have finite type (again, via the forgetful functor, cohomologies of objects in $D^b_{G_{\mathbb{R}}}(\tilde{X})^\mu$ are regarded as sheaves on $\tilde{X}$). The objects in $D^b_{G_{\mathbb{R}}}(\tilde{X})^\mu$ that belong to $D^b_{G_{\mathbb{R}},f}(\tilde{X})^\mu$ will be said to have finite type. Since the cohomologies of objects in $D^b_{G_{\mathbb{R}}}(\tilde{X})^\mu$ are $G_{\mathbb{R}}$ equivariant sheaves, the objects of finite type in $D^b_{G_{\mathbb{R}}}(\tilde{X})^\mu$ are those whose image under the forgetful functor are constructible with respect to the $G_{\mathbb{R}}$-orbit stratification of $\tilde{X}$.

Proposition 3.3. *The functor $\tilde{DR}_\lambda$ takes $D^b_{G_{\mathbb{R}},f}(\tilde{\mathcal{D}}_\lambda)^\mu$ into $D^b_{G_{\mathbb{R}},f}(\tilde{X})^{\mu-\lambda}$.*

Proof. Clearly, we have that $\mathcal{O}_{-\lambda}\otimes_{\tilde{\mathcal{O}}_{\tilde{X}}}(\cdot)$ maps $D^b_{G_{\mathbb{R}},f}(\tilde{\mathcal{D}}_\lambda)^\mu$ into $D^b_{G_{\mathbb{R}},f}(\tilde{\mathcal{D}}_0)^{\mu-\lambda}$, since $\mathcal{O}_{-\lambda}$ is locally isomorphic to $\tilde{\mathcal{O}}_{\tilde{X}}$. It remains only to show that $\tilde{DR}$ takes objects in $D^b_{G_{\mathbb{R}}}(\tilde{\mathcal{D}}_{\tilde{X}})^{\mu-\lambda}$ of finite type to objects in $D^b_{G_{\mathbb{R}}}(\tilde{X})^{\mu-\lambda}$ of finite type. We do this now, after replacing the parameter $\mu-\lambda$ by the equally general parameter μ. Indeed, if $\mathcal{M}$ is an object of $D^b_{G_{\mathbb{R}}}(\tilde{\mathcal{D}}_{\tilde{X}})^\mu$, then it follows easily from the definition of the forgetful functor (cf. [BL], 2.4) that

$$\mathrm{For}(\tilde{DR}(\mathcal{M}))=\tilde{DR}_{\tilde{X}}(\mathrm{For}(\mathcal{M})).$$

This is the total complex of the double complex L with $L^{i,j}=\tilde{\Omega}^i_{\tilde{X}}\otimes_{\tilde{\mathcal{O}}_{\tilde{X}}}M^j$ where $\mathcal{M}$ is a complex of $\tilde{\mathcal{D}}_{\tilde{X}}$ modules representing $\mathrm{For}(\mathcal{M})$. Since each $H^q(\mathcal{M})$ is of finite type, it has the form $\tilde{\mathcal{O}}_{\tilde{X}}\otimes F(H^q(\mathcal{M}))$ by Theorem 2.6, where $F(H^q(\mathcal{M}))=H^0(\tilde{DR}_{\tilde{X}}(H^q(\mathcal{M})))$ is the sheaf of flat sections of $H^q(\mathcal{M})$, which is a sheaf with finite dimensional stalks. Thus, the second spectral sequence of L converges at stage two with

$$E_2^{p,q}=H^p(\tilde{DR}_{\tilde{X}}(H^q(\mathcal{M})))=\begin{cases}0 & \text{if } p\neq 0,\\ F(H^q(\mathcal{M})) & \text{if } p=0.\end{cases}$$

Since each $F(H^q(\mathcal{M}))$ has finite dimensional stalks, this shows that $\tilde{DR}_{\tilde{X}}$ maps $D^b_{G_{\mathbb{R}},f}(\mathcal{D}_0)^\mu$ into $D^b_{G_{\mathbb{R}},f}(\tilde{X})^\mu$. □

There is a functor from $D^b_{G_{\mathbb{R}},f}(\tilde{X})^{\mu-\lambda}$ into $D^b_{G_{\mathbb{R}},f}(\tilde{\mathcal{D}}_\lambda)^\mu$ which turns out to be the inverse for $\tilde{DR}_\lambda$. Specifically, for an object $\mathcal{M}$ of $D^b_{G_{\mathbb{R}},f}(\tilde{X})^{\mu-\lambda}_t$, we set

$$\mathcal{O}_\lambda\otimes(\mathcal{M})(\tilde{P})=\mathcal{O}_{\lambda,\tilde{P}_0}\otimes\mathcal{M}(\tilde{P})$$

and note that this defines an object of $D^b_{G_{\mathbb{R}},f}(\tilde{\mathcal{D}}_\lambda)^\mu$. In this way, we define a functor from $D^b_{G_{\mathbb{R}},f}(\tilde{X})^{\mu-\lambda}$ to $D^b_{G_{\mathbb{R}},f}(\tilde{\mathcal{D}}_\lambda)^\mu$. We will denote this by $\mathcal{O}_\lambda\otimes(\cdot)$.

Proposition 3.4. *For each μ, functor*

$$\tilde{DR}:D^b_{G_{\mathbb{R}},f}(\tilde{\mathcal{D}}_{\tilde{X}})^\mu\to D^b_{G_{\mathbb{R}},f}(\tilde{X})^\mu$$

is an equivalence of categories with inverse functor $\mathcal{O}_{\tilde{X}}\otimes(\cdot)$.

Proof. Let $\tilde{Y}\to Y$ be a monodromic system and $\mathcal{M}$ be an object of $D^b(\tilde{\mathcal{D}}_{\tilde{Y}})^\mu$. We have $\tilde{DR}_{\tilde{Y}}(\mathcal{M})=R\mathcal{H}om_{\tilde{\mathcal{D}}}(\tilde{\mathcal{O}}_{\tilde{Y}},\mathcal{M})$. If the object $\mathcal{M}$ is represented by a complex of $\tilde{\mathcal{D}}$ modules which are acyclic for $\mathcal{H}om_{\tilde{\mathcal{D}}}(\tilde{\mathcal{O}}_{\tilde{Y}},\cdot)$ (for example, by a complex of injective modules), then $R\mathcal{H}om_{\tilde{\mathcal{D}}}(\tilde{\mathcal{O}}_{\tilde{Y}}, \mathcal{M}) = \mathcal{H}om_{\tilde{\mathcal{D}}}(\tilde{\mathcal{O}}_{\tilde{Y}}, \mathcal{M})$. But there is clearly an evaluation morphism

$$\tilde{\mathcal{O}}_{\tilde{Y}} \otimes \mathcal{H}om_{\tilde{\mathcal{D}}}(\tilde{\mathcal{O}}_{\tilde{Y}}, \mathcal{M}) \to \mathcal{M}.$$

This yields a well defined morphism

$$\beta_{\mathcal{M}} : \mathcal{O}_{\tilde{Y}} \otimes \tilde{DR}_{\tilde{Y}}(\mathcal{M}) \to \mathcal{M}$$

which is functorial in $\mathcal{M}$.

To show that β defines a morphism of objects in the equivariant derived category we must show that it commutes with inverse image for morphisms between resolutions of $\tilde{X}$. To this end, let $f : \tilde{Y} \to \tilde{Z}$ be a fibration of monodromic systems and $\mathcal{M}$ an object of $D^b(\tilde{\mathcal{D}}_{\tilde{Z}})$. Then it follows from the Poincaré Lemma in the fiber direction that

$$R\mathcal{H}om_{\tilde{\mathcal{D}}}(\tilde{\mathcal{O}}_{\tilde{Y}}, f^+\mathcal{M}) = \tilde{DR}_{\tilde{Y}}(f^+\mathcal{M}) = f^{-1}\tilde{DR}_{\tilde{Z}}(\mathcal{M}) = f^{-1}R\mathcal{H}om_{\tilde{\mathcal{D}}}(\tilde{\mathcal{O}}_{\tilde{Z}}, \mathcal{M}).$$

If $\mathcal{M}$ is represented by a complex of injectives, then it follows that $f^+\mathcal{M}$ is acyclic for $\mathcal{H}om_{\tilde{\mathcal{D}}}(\tilde{\mathcal{O}}_{\tilde{Y}}, \cdot)$ and

$$\mathcal{H}om_{\tilde{\mathcal{D}}}(\tilde{\mathcal{O}}_{\tilde{Y}}, f^+\mathcal{M}) = f^{-1}\mathcal{H}om_{\tilde{\mathcal{D}}}(\tilde{\mathcal{O}}_{\tilde{Z}}, \mathcal{M}).$$

Since $\tilde{\mathcal{O}}_{\tilde{Y}} \otimes f^{-1}\mathcal{H}om_{\tilde{\mathcal{D}}}(\tilde{\mathcal{O}}_{\tilde{Z}}, \mathcal{M}) = f^+(\tilde{\mathcal{O}}_{\tilde{Z}} \otimes \mathcal{H}om_{\tilde{\mathcal{D}}}(\tilde{\mathcal{O}}_{\tilde{Z}}, \mathcal{M}))$, this implies that

$$\beta_{f^+\mathcal{M}} \circ f^+ = f^+ \circ \beta_{\mathcal{M}}.$$

The fact that β commutes with inverse image under fibrations ensures that, by setting $\beta_{\mathcal{M}}(\tilde{P}) = \beta_{\mathcal{M}(\tilde{P})}$ for an object $\mathcal{M}$ of $D^b_{G_{\mathbb{R}}}(\tilde{\mathcal{D}}_{\tilde{X}})^\mu$ and a resolution $\tilde{P} \to \tilde{X}$ of $\tilde{X}$, we define a functor $\mathcal{M} \to \beta_{\mathcal{M}}$ from $D^b_{G_{\mathbb{R}}}(\tilde{\mathcal{D}}_{\tilde{X}})^\mu$ to morphisms

$$\beta_{\mathcal{M}} : \tilde{\mathcal{O}}_{\tilde{X}} \otimes \tilde{DR}(\mathcal{M}) \to \mathcal{M}.$$

We have that β is an isomorphism on the heart of $D^b_{G_{\mathbb{R}}}(\tilde{\mathcal{D}}_{\tilde{X}})^\mu$ by Theorem 2.6. Since the heart generates $D^b_{G_{\mathbb{R}}}(\tilde{\mathcal{D}}_{\tilde{X}})^\mu$ as a triangulated category, it follows that β is an isomorphism in general. This shows that $\tilde{DR}$ followed by $\mathcal{O} \otimes (\cdot)$ is isomorphic to the identity functor. That the same thing is true of the composition in the other direction is trivial. □

Theorem 3.5. *The functor*

$$\tilde{DR}_\lambda : D^b_{G_{\mathbb{R}},f}(\tilde{\mathcal{D}}_\lambda)^\mu \to D^b_{G_{\mathbb{R}},f}(\tilde{X})^{\mu-\lambda}$$

is an equivalence of categories with inverse $\mathcal{O}_\lambda \otimes (\cdot)$.

Proof. By Proposition 3.2, $\mathcal{O}_{-\lambda} \otimes_{\tilde{\mathcal{O}}_{\tilde{X}}} (\cdot)$ defines an equivalence from $D^b_{G_{\mathbb{R}}}(\tilde{\mathcal{D}}_\lambda)^\mu$ to $D^b_{G_{\mathbb{R}}}(\tilde{\mathcal{D}}_0)^{\mu-\lambda}$ with inverse $\mathcal{O}_\lambda \otimes_{\tilde{\mathcal{O}}_{\tilde{X}}} (\cdot)$. Furthermore, these functors take finite type objects to finite type objects. This follows from the fact that $\mathcal{O}_{-\lambda}$ and $\mathcal{O}_\lambda$ are locally isomorphic to $\tilde{\mathcal{O}}_{\tilde{X}}$.

Since $\mathcal{O}_\lambda \otimes (\cdot)$ is the composition of $\mathcal{O}_\lambda \otimes_{\tilde{\mathcal{O}}_{\tilde{X}}} (\cdot)$ with $\tilde{\mathcal{O}}_{\tilde{X}} \otimes (\cdot)$, the theorem follows from Proposition 3.4. □

In view of the above and Proposition 3.1, we have proved our version of the Riemann-Hilbert correspondence for $\mathcal{D}_{\lambda,X}$-modules.

Corollary 3.6. *The composition* $DR_\lambda = \tilde{DR}_\lambda \circ \pi^{-1}$ *is an equivalence of categories from* $D^b_{G_{\mathbb{R}},f}(\mathcal{D}_{\lambda,X})$ *to* $D^b_{G_{\mathbb{R}},f}(\tilde{X})^{-\lambda}$.

4. Globalization

In this section we complete our discussion of diagram (0.1) of the introduction by describing the horizontal arrows in that diagram and outlining the proof that the diagram is commutative.

The results of this section strongly overlap with results in [KSd]. The main differences being that we work in a setting dual to that of [KSd], we use the equivariant derived category at each node of our diagram, each arrow represents an equivalence and the middle horizontal arrow discussed here does not appear in [KSd].

As we shall see, since the top horizontal arrow of (0.1) is essentially the minimal globalization functor of Schmid [S2], the other two horizontal arrows are sheaf theoretic versions of the minimal globalization functor. The bottom horizontal arrow

$$\gamma : D^b_{K,f}(\tilde{X})^{-\lambda} \to D^b_{G_{\mathbb{R}},f}(\tilde{X})^{-\lambda}$$

was defined in [MUV] and may be described as follows:

The categories $D^b_K(\tilde{X})^\mu$ and $D^b_{K,f}(\tilde{X})^\mu$ are defined just as the corresponding categories $D^b_{G_{\mathbb{R}}}(\tilde{X})^\mu$ and $D^b_{G_{\mathbb{R}},f}(\tilde{X})^\mu$ were defined in Section 3 except that K replaces $G_{\mathbb{R}}$, where K is the complexification of a maximal compact subgroup $K_{\mathbb{R}}$ of $G_{\mathbb{R}}$. Thus, $D^b_K(\tilde{X})^\mu$ is the K-equivariant derived category of μ-monodromic sheaves of complex vector spaces on $\tilde{X}$ and $D^b_{K,f}(\tilde{X})^\mu$ is its full subcategory consisting of objects of finite type. Here an object of finite type is one which has cohomologies with finite dimensional stalks. We consider the diagram

$$\text{(4.1a)} \qquad \tilde{X} \xleftarrow{\ p\ } G_{\mathbb{R}} \times \tilde{X} \xrightarrow{\ r\ } K_{\mathbb{R}} \backslash (G_{\mathbb{R}} \times \tilde{X}) \xrightarrow{\ v\ } \tilde{X}$$

where p is the projection, r is the quotient by the $K_{\mathbb{R}}$ action $k \times (g,x) \to (gk^{-1}, kx)$ and v is the map from this quotient to X induced by the action map $m : (g,x) \to gx : G_{\mathbb{R}} \times \tilde{X} \to \tilde{X}$. This is not exactly the diagram considered in [MUV] but is equivalent to it under a shear transformation of $G_{\mathbb{R}} \times X$. Each of the spaces in this diagram is a $G_{\mathbb{R}} \times K_{\mathbb{R}}$ monodromic space and the maps are $G_{\mathbb{R}} \times K_{\mathbb{R}}$-equivariant monodromic maps. Here, the $G_{\mathbb{R}} \times K_{\mathbb{R}}$ actions are as follows: on the left-hand copy of $\tilde{X}$ the action is $(g,k) \times x \to kx$. On $G_{\mathbb{R}} \times \tilde{X}$ it is $(g,k) \times (g',x) \to (gg'k^{-1}, kx)$ and this induces the action on the quotient $K_{\mathbb{R}} \backslash (G_{\mathbb{R}} \times \tilde{X})$. On the right-hand copy of $\tilde{X}$ the action is $(g,k) \times x \to gx$.

For notational convenience, given any $K_{\mathbb{R}}$ space Y in the category of analytic CR manifolds, we denote by $S(Y)$ the quotient $K_{\mathbb{R}} \backslash (G_{\mathbb{R}} \times Y)$ of $G_{\mathbb{R}} \times Y$ by the $K_{\mathbb{R}}$ action $k \times (g,x) \to (gk^{-1}, kx)$ and by $r : G_{\mathbb{R}} \times Y \to S(Y)$ the quotient map. Then diagram (4.1a) becomes

$$\text{(4.1b)} \qquad \tilde{X} \xleftarrow{\ p\ } G_{\mathbb{R}} \times \tilde{X} \xrightarrow{\ r\ } S(\tilde{X}) \xrightarrow{\ v\ } \tilde{X}.$$

The functor γ of (0.1) is the composition of the forgetful functor

$$\mathcal{F}^K_{K_{\mathbb{R}}} : D^b_K(\tilde{X})^\mu \to D^b_{K_{\mathbb{R}}}(\tilde{X})^\mu$$

and a functor

$$\gamma^{G_{\mathbb{R}}}_{K_{\mathbb{R}}} : D^b_{K_{\mathbb{R}}}(\tilde{X})^\mu \to D^b_{G_{\mathbb{R}}}(\tilde{X})^\mu$$

called the integration functor. It will be defined below, but first we need a short discussion of one of the main ingredients—a form of the inverse image functor in a situation where the group changes.

Let A be a closed normal subgroup of a Lie group B and let $f : W \to Z$ be a B map from a B space W to a B space Z on which A acts trivially (so that Z is actually a B/A space). In this situation, there is an inverse image functor

$$f^{-1} : D^b_{B/A}(Z) \to D^b_B(W)$$

with the property that the diagram

$$\begin{array}{ccc} D^b_{B/A}(Z) & \xrightarrow{f^{-1}} & D^b_B(W) \\ {\scriptstyle\text{For}}\downarrow & & \downarrow{\scriptstyle\text{For}} \\ D^b(Z) & \xrightarrow{f^{-1}} & D^b(W) \end{array} \tag{4.2}$$

commutes, where the bottom f^{-1} is the ordinary inverse image functor. This functor is defined in terms of the usual inverse image functor in the following way. As before, we use the definition of the equivariant derived category in terms of fibered categories as in 2.4 of [BL]. If $P \to W$ is a resolution of W (a B map with P a free B space), then there is an induced map $A\backslash P \to Z$ which is a resolution of Z so that the diagram

$$\begin{array}{ccc} P & \xrightarrow{\hat{f}} & A\backslash P \\ \downarrow & & \downarrow \\ W & \xrightarrow{f} & Z \end{array}$$

commutes, where $\hat{f}$ is the quotient map. There is then an induced map $f_0 : P_0 \to A\backslash P_0$. We define $f^{-1}(M)$ for an object M of $D^b_{B/A}(Z)$ by

$$f^{-1}(M)(P) = f_0^{-1} M(P)$$

where the map f_0^{-1} on the right is $f_0^{-1} : D^b(P_0) \to D^b(A\backslash P_0)$ and P_0 (resp. $A\backslash P_0$) is $B\backslash P$ (resp. $(B/A)\backslash(A\backslash P)$) (see [BL] 2.6.2).

Note that in the case where A acts freely on W, $Z = A\backslash W$ and f is the quotient map, the inverse image functor constructed in this this way is actually an equivalence of categories. In fact, f^{-1} is the inverse of the quotient equivalence $q_f : D^b_B(W) \to D^b_{B/A}(W/A)$ of 2.6.2 [BL] in this case. This is the case in both of the following uses of this functor. Note also, that the above construction can be adapted as in Section 3 to cover the situation where we are dealing with equivariant monodromic systems.

If we apply the inverse image functor defined as above in the case where $f = p$, $A = G_{\mathbb{R}}$ and $B = G_{\mathbb{R}} \times K_{\mathbb{R}}$ we obtain an equivalence of categories

$$p^{-1} : D^b_{K_{\mathbb{R}}}(\tilde{X})^{\mu} \to D^b_{G_{\mathbb{R}} \times K_{\mathbb{R}}}(G_{\mathbb{R}} \times \tilde{X})^{\mu}.$$

On the other hand, if we apply it in the case where $f = r$, $A = K_{\mathbb{R}}$ and $B = G_{\mathbb{R}} \times K_{\mathbb{R}}$ we obtain an equivalence of categories

$$r^{-1} : D^b_{G_{\mathbb{R}}}(S(\tilde{X}))^{\mu} \to D^b_{G_{\mathbb{R}} \times K_{\mathbb{R}}}(G_{\mathbb{R}} \times \tilde{X})^{\mu}$$

with inverse

$$q_r : D^b_{G_{\mathbb{R}} \times K_{\mathbb{R}}}(G_{\mathbb{R}} \times \tilde{X})^{\mu} \to D^b_{G_{\mathbb{R}}}(S(\tilde{X}))^{\mu}.$$

Next, we consider

$$v_! : D^b_{G_{\mathbb{R}}}(S(\tilde{X}))^{\mu} \to D^b_{G_{\mathbb{R}}}(\tilde{X})^{\mu}$$

where $v_!$ is the derived functor of direct image with proper supports. This functor also commutes with the forgetful functor—that is, the diagram analogous to (4.2) with f^{-1} replaced by $v_!$ commutes (3.4.1 of [BL]). Finally, we define

$$\gamma_{K_{\mathbb{R}}}^{G_{\mathbb{R}}} = v_![n] \circ q_r \circ p^{-1}$$

where n is the dimension of the fiber $S = G_{\mathbb{R}}/K_{\mathbb{R}}$ of v, $v_!$ is the functor of direct image with proper supports associated with v and $v_![n]$ is $v_!$ followed by the indicated degree shift—that is, $v_![n]\mathcal{S} = (v_!\mathcal{S})[n]$.

The composition $\gamma_{K_{\mathbb{R}}}^{G_{\mathbb{R}}} \circ \mathcal{F}_{K_{\mathbb{R}}}^{K}$ is a functor from $D_K^b(\tilde{X})^\mu$ to $D_{G_{\mathbb{R}}}^b(\tilde{X})^\mu$. It is not obvious that it takes objects of finite type to objects of finite type. However, in [MUV] it is proved that not only is this true but, in fact:

Theorem 4.1 ([MUV])**.** *The restriction of $\gamma_{K_{\mathbb{R}}}^{G_{\mathbb{R}}} \circ \mathcal{F}_{K_{\mathbb{R}}}^{K}$ to $D_{K,f}^b(\tilde{X})^\mu$ is an equivalence of categories*

$$\gamma : D_{K,f}^b(\tilde{X})^\mu \to D_{G_{\mathbb{R}},f}^b(\tilde{X})^\mu.$$

Next we describe the middle horizontal arrow from diagram (0.1):

$$\bar{\gamma} : D_{K,f}^b(\mathcal{D}_{\lambda,X}) \to D_{G_{\mathbb{R}},f}^b(\mathcal{D}_{\lambda,X}).$$

Here, we define a category $D_K^b(\mathcal{D}_{\lambda,X})$ in the same way that $D_{G_{\mathbb{R}}}^b(\mathcal{D}_{\lambda,X})$ was defined in Section 3 but with K replacing $G_{\mathbb{R}}$. There is a question as to whether we should do this in the holomorphic or the algebraic context. That is, in defining $D_K^b(\mathcal{D}_{\lambda,X})$, X is considered a complex manifold with the Euclidean topology, $\mathcal{D}_{\lambda,X}$ is the sheaf of λ twisted differential operators with holomorphic coefficients and G resolutions of X are resolutions by complex analytic free G spaces. However, we could also define an algebraic equivariant derived category, $D_K^b(\mathcal{D}_{\lambda,X}^{alg})$ by considering X to be a projective variety, letting $\mathcal{D}_{\lambda,X}^{alg}$ be the sheaf of λ twisted differential operators with regular coefficients and using resolutions of X by smooth algebraic free G spaces. This is essentially the category described in [BL2]. It turns out, as we shall see below, that it doesn't matter whether we use $D_K^b(\mathcal{D}_{\lambda,X}^{alg})$ or $D_K^b(\mathcal{D}_{\lambda,X})$, because their subcategories of finite type objects are equivalent.

We define $D_{K,f}^b(\mathcal{D}_{\lambda,X})$ to be the full subcategory of $D_K^b(\mathcal{D}_{\lambda,X})$ consisting of objects of finite type—which in this case means objects which have cohomologies which are coherent $\mathcal{D}_\lambda$-modules. We define $D_{K,f}^b(\mathcal{D}_{\lambda,X}^{alg})$ similarly. Note that the cohomologies of $D_K^b(\mathcal{D}_{\lambda,X}^{alg})$ are K equivariant and X has finitely many K orbits. Thus, for such a module, coherence as a $\mathcal{D}_{\lambda,X}^{alg}$ module is equivalent to regular holonomicity (cf. [Bo], VII, 12.11).

The category $D_{K,f}^b(\mathcal{D}_{\lambda,X}^{alg})$ is closely related to the one used in [BL2]; the difference is that in [BL2] modules with generalized infinitesimal character determined by λ replace modules with infinitesimal character. With minor modification, everything we do here could also be done in this context.

There is a λ twisted DeRham functor

$$DR_\lambda : D_K^b(\mathcal{D}_{\lambda,X}) \to D_K^b(\tilde{X})^{-\lambda}$$

defined the same way as in Section 3 but with K replacing $G_{\mathbb{R}}$. When restricted to objects of finite type, this yields an equivalence of categories

$$DR_\lambda : D_{K,f}^b(\mathcal{D}_{\lambda,X}) \to D_{K,f}^b(\tilde{X})^{-\lambda}$$

called the Riemann-Hilbert correspondence. The proof that this is an equivalence essentially amounts to lifting the problem to $\tilde{X}$ and and tensoring with $\mathcal{O}_{-\lambda}$, as in Section 3, and then applying the standard Riemann-Hilbert correspondence (as in [Bj]) on each $\tilde{P}_0$ for $\tilde{P} \to \tilde{X}$ a resolution of $\tilde{X}$. Our proof of the analogous result in the $G_{\mathbb{R}}$ equivariant case in Section 3 was somewhat simpler than the proof of the usual Riemann-Hilbert correspondence due to the fact that a $G_{\mathbb{R}}$ equivariant finite type $\mathcal{D}_\lambda$ module on X has the simple form described in Proposition 2.5 on each $G_{\mathbb{R}}$ orbit.

There is also an algebraic λ twisted DeRham functor

$$DR_\lambda : D^b_K(\mathcal{D}^{alg}_{\lambda,X}) \to D^b_K(\tilde{X})^{-\lambda}.$$

This is defined by first applying the GAGA functor which replaces the Zariski topology by the Euclidean topology and $\mathcal{O}^{alg}$ modules by $\mathcal{O}$ modules and then applying the analytic λ-twisted DeRham functor described above. When restricted to objects of finite type, this yields a functor

$$DR_\lambda : D^b_{K,f}(\mathcal{D}^{alg}_{\lambda,X}) \to D^b_{K,f}(\tilde{X})^{-\lambda}.$$

Using the standard algebraic Riemann-Hilbert correspondence as developed, for example in [K2, Bo] one can show that this is also an equivalence of categories. A corollary of this is that the GAGA functor

$$D^b_{K,f}(\mathcal{D}^{alg}_{\lambda,X}) \to D^b_{K,f}(\mathcal{D}_{\lambda,X})$$

is an equivalence of categories. Thus, one can use either the algebraic or the holomorphic approach. The resulting categories are equivalent. We will use the holomorphic approach.

The definition of the functor $\bar{\gamma}$ of diagram (0.1) is formally much the same as that of γ, as discussed above, but uses the inverse image and direct image with proper supports functors that are appropriate for $\mathcal{D}$-modules. We define $\bar{\gamma}$ as the composition of two functors

$$\bar{\gamma} = \bar{\gamma}^{G_{\mathbb{R}}}_{K_{\mathbb{R}}} \circ \bar{\mathcal{F}}^K_{K_{\mathbb{R}}}.$$

The first of these is the forgetful functor

$$\bar{\mathcal{F}}^K_{K_{\mathbb{R}}} : D^b_K(\mathcal{D}_{\lambda,X}) \to D^b_{K_{\mathbb{R}}}(\mathcal{D}_{\lambda,X})$$

and the second is the integration functor

$$\bar{\gamma}^{G_{\mathbb{R}}}_{K_{\mathbb{R}}} : D^b_{K_{\mathbb{R}}}(\mathcal{D}_{\lambda,X}) \to D^b_{G_{\mathbb{R}}}(\mathcal{D}_{\lambda,X}).$$

The integration functor is the composition $\bar{\gamma}^{G_{\mathbb{R}}}_{K_{\mathbb{R}}} = v_\dagger[n] \circ \bar{q}_r \circ p^+$, where

$$p^+ : D^b_{K_{\mathbb{R}}}(\mathcal{D}_{\lambda,X}) \to D^b_{G_{\mathbb{R}} \times K_{\mathbb{R}}}(\mathcal{D}_{\lambda,G_{\mathbb{R}} \times X})$$

and

$$r^+ : D^b_{G_{\mathbb{R}}}(\mathcal{D}_{\lambda,S(X)}) \to D^b_{G_{\mathbb{R}} \times K_{\mathbb{R}}}(\mathcal{D}_{\lambda,G_{\mathbb{R}} \times X})$$

are given by $\mathcal{D}$ module inverse image, $\bar{q}_r = (r^+)^{-1}$, while $v_\dagger$ is the $\mathcal{D}$-module version of the functor $v_!$ of direct image with proper supports

$$v_\dagger : D^b_{G_{\mathbb{R}}}(\mathcal{D}_{\lambda,G_{\mathbb{R}} \times X}) \to D^b_{G_{\mathbb{R}}}(\mathcal{D}_{\lambda,X})$$

and $v_\dagger[n]$ is $v_\dagger$ followed by a shift in degree by n. The functor $v_\dagger$ may be expressed as the composition, $v_! \circ DR^v$, of two derived functors: a functor DR^v, which is the DeRham functor along the fibers of v and the derived functor $v_!$ of sheaf theoretic direct image with proper supports.

Note that the DeRham functor along v has the following description. let $\mathcal{T}_v$ be the sheaf of sections of the subbundle of the tangent space of $S(X)$ consisting of vectors tangent to the fibers of v and let Ω_v^p be the sheaf of sections of the bundle of alternating p-forms on $\mathcal{T}_v$. Note that $\mathcal{T}_v$ is, in a natural way, embedded in $\mathcal{D}_{\lambda,S(X)}$. In fact, if we make the change of variables $(g,x) \to (g,gx) : G_{\mathbb{R}} \times X \to G_{\mathbb{R}} \times X$ and let $S = G_{\mathbb{R}}/K_{\mathbb{R}}$, then $S(X)$ becomes $S \times X$ and v becomes the projection of this space on X. In this picture, $\mathcal{T}_v$ is the pullback via the projection $S \times X \to S$ of the sheaf of vector fields on S, which is naturally contained in $\mathcal{D}_{\lambda,S\times X}$. It follows that if $\mathcal{M}$ is a $\mathcal{D}_{\lambda,S(X)}$ module, and $\Omega_v^p(\mathcal{M}) = \Omega_v^p \otimes_{\mathcal{O}} \mathcal{M}$, then there is a differential $d : \Omega_v^p(\mathcal{M}) \to \Omega_v^{p+1}(\mathcal{M})$, defined in the usual way, which makes $\{\Omega_v^p, d\}$ into a complex of sheaves. We denote this complex by $DR^v(\mathcal{M})$ and call it the relative DeRham complex of $\mathcal{M}$ along v. This is extended in the usual way to a functor DR^v from complexes of $\mathcal{D}_{\lambda,S(X)}$ modules to complexes of sheaves.

Recall that $v^+(\mathcal{D}_{\lambda,X})$ is naturally a $\mathcal{D}_{\lambda,S(X)} \times \nu^{-1}\mathcal{D}_{\lambda,X}$ bimodule which we denote by $\mathcal{D}_{\lambda,S(X)\to X}$. Following [Bo] (VI, 5), we set

$$\mathcal{D}_{\lambda,X\leftarrow S(X)} = \mathcal{D}_{\lambda,S(X)\to X} \otimes_{\mathcal{O}_{S(X)}} \omega_{S(X)/X}$$

and note that this is naturally a $\nu^{-1}\mathcal{D}_{\lambda,X} \times \mathcal{D}_{\lambda,S(X)}$ bimodule and that the relative DeRham complex shifted by n, $DR^v(\mathcal{D}_{\lambda,S(X)})[n]$, provides a locally $\mathcal{D}_{\lambda,S(X)}$-free bimodule resolution of $\mathcal{D}_{\lambda,X\leftarrow S(X)}$. Thus, for any object $\mathcal{M}$ of $D^b(\mathcal{D}_{\lambda,S(X)})$, we have that $DR^v(\mathcal{M})[n] = DR^v(\mathcal{D}_{\lambda,S(X)})[n] \otimes_{\mathcal{D}_{\lambda,S(X)}} \mathcal{M}$ represents the left derived tensor product $\mathcal{D}_{\lambda,X\leftarrow S(X)} \otimes^L_{\mathcal{D}_{\lambda,S(X)}} \mathcal{M}$ and

$$v_\dagger[n](\mathcal{M}) = v_!(\mathcal{D}_{\lambda,X\leftarrow S(X)} \otimes^L_{\mathcal{D}_{\lambda,S(X)}} \mathcal{M}). \tag{4.3}$$

Note that, since v is a $G_{\mathbb{R}}$ equivariant map, if $\mathcal{M}$ is a $G_{\mathbb{R}}$ equivariant complex of sheaves, then $v_\dagger[n](\mathcal{M})$ is a $G_{\mathbb{R}}$ equivariant complex of sheaves.

If we replace X by P_0 for any $G_{\mathbb{R}}$ resolution $P \to X$, then we may define $DR^v(\mathcal{M})$ and $v_\dagger[n](\mathcal{M})$ exactly as above for any complex $\mathcal{M}$ of sheaves of $\mathcal{D}_{\lambda,S(P)_0}$ modules on $S(P)_0$. Thus, we have a functor $v_\dagger[n] : D^b(\mathcal{D}_{\lambda,S(P)_0}) \to D^b(\mathcal{D}_{\lambda,P_0})$. As P ranges over the category of resolutions of X, this defines a functor $v_\dagger[n] : D^b_{G_{\mathbb{R}}}(\mathcal{D}_{\lambda,S(X)}) \to D^b_{G_{\mathbb{R}}}(\mathcal{D}_{\lambda,X})$.

With $\bar{\gamma}^{G_{\mathbb{R}}}_{K_{\mathbb{R}}} = v_\dagger[n] \circ \bar{q}_r \circ p^+$, we may define

$$\bar{\gamma} = \bar{\gamma}^{G_{\mathbb{R}}}_{K_{\mathbb{R}}} \circ \bar{\mathcal{F}}^K_{K_{\mathbb{R}}} : D^b_{K,f}(\mathcal{D}_{\lambda,X}) \to D^b_{G_{\mathbb{R}}}(\mathcal{D}_{\lambda,X}).$$

Recall that $D^b_{G_{\mathbb{R}},f}(\mathcal{D}_{\lambda,X})$ is the full subcategory of $D^b_{G_{\mathbb{R}}}(\mathcal{D}_{\lambda,X})$ consisting of objects of finite type, where, in this context, the objects of finite type are those whose cohomologies have stalks with finite geometric fiber.

Theorem 4.2. *The functor $\bar{\gamma} : D^b_{K,f}(\mathcal{D}_{\lambda,X}) \to D^b_{G_{\mathbb{R}}}(\mathcal{D}_{\lambda,X})$ has its range in the subcategory $D^b_{G_{\mathbb{R}},f}(\mathcal{D}_{\lambda,X})$; the diagram*

$$\begin{array}{ccc} D^b_{K,f}(\mathcal{D}_{\lambda,X}) & \xrightarrow{\bar{\gamma}} & D^b_{G_{\mathbb{R}},f}(\mathcal{D}_{\lambda,X}) \\ {\scriptstyle DR_\lambda}\downarrow & & \downarrow{\scriptstyle DR_\lambda} \\ D^b_{K,f}(\tilde{X})^{-\lambda} & \xrightarrow{\gamma} & D^b_{G_{\mathbb{R}},f}(\tilde{X})^{-\lambda} \end{array}$$

is commutative; and $\bar{\gamma} : D^b_{K,f}(\mathcal{D}_{\lambda,X}) \to D^b_{G_{\mathbb{R}},f}(\mathcal{D}_{\lambda,X})$ is an equivalence of categories.

Proof. Since $K/K_{\mathbb{R}}$ is contractible, the forgetful functor $\bar{\gamma}^K_{K_{\mathbb{R}}}$ is simply an inclusion of $D^b_{K,f}(\mathcal{D}_{\lambda,X})$ (resp. $D^b_{K,f}(\tilde{X})^{-\lambda}$) into $D^b_{K_{\mathbb{R}},f}(\mathcal{D}_{\lambda,X})$ (resp. $D^b_{K_{\mathbb{R}},f}(\tilde{X})^{-\lambda}$) as the full subcategory consisting of objects with K equivariant cohomology ([MUV], 4.5). Obviously this preserves cohomology and commutes with the DeRham functor.

The fact that the DeRham functor is defined on the equivariant derived category follows from the fact that the DeRham functor on the ordinary derived category commutes with inverse image (see [Bo], VIII 14.6 for a proof of this in the algebraic case—the analytic case is similar). This same fact implies the commutativity of the diagram

$$\begin{array}{ccccc} D^b_{K_{\mathbb{R}}}(\mathcal{D}_{\lambda,X}) & \xrightarrow{p^+} & D^b_{G_{\mathbb{R}}\times K_{\mathbb{R}}}(\mathcal{D}_{\lambda,G_{\mathbb{R}}\times X})) & \xleftarrow{r^+} & D^b_{G_{\mathbb{R}}}(\mathcal{D}_{\lambda,S(X)}) \\ {\scriptstyle DR_\lambda}\downarrow & & {\scriptstyle DR_\lambda}\downarrow & & \downarrow{\scriptstyle DR_\lambda} \\ D^b_{K_{\mathbb{R}}}(\tilde{X})^{-\lambda} & \xrightarrow{p^{-1}} & D^b_{G_{\mathbb{R}}\times K_{\mathbb{R}}}(G_{\mathbb{R}}\times\tilde{X})^{-\lambda} & \xleftarrow{r^{-1}} & D^b_{G_{\mathbb{R}}}(S(\tilde{X}))^{-\lambda}. \end{array} \tag{4.4}$$

Since q_r and $\bar{q}_r$ are the inverses of r^{-1} and r^+, respectively, this implies that

$$\begin{array}{ccc} D^b_{G_{\mathbb{R}}}(\mathcal{D}_{\lambda,S(X)}) & \xrightarrow{v_\dagger} & D^b_{G_{\mathbb{R}}}(\mathcal{D}_{\lambda,X}) \\ {\scriptstyle DR_\lambda}\downarrow & & \downarrow{\scriptstyle DR_\lambda} \\ D^b_{G_{\mathbb{R}}}(S(\tilde{X}))^{-\lambda} & \xrightarrow{v_!} & D^b_{G_{\mathbb{R}}}(\tilde{X})^{-\lambda} \end{array} \tag{4.5}$$

is commutative.

It remains to show that $\bar{\gamma}$ takes objects of finite type to objects of finite type—that is, takes objects with regular holonomic cohomology to objects with cohomology having finite dimensional geometric fiber.

We first observe that $\bar{q}_r \circ p^+ \circ \mathcal{F}^K_{K_{\mathbb{R}}}$ takes an object in $D^b_{K,f}(\mathcal{D}_{\lambda,X})$ with regular holonomic cohomology to an object in $D^b_{G_{\mathbb{R}},f}(\mathcal{D}_{\lambda,S(X)})$ which has cohomology modules each of which is the restriction to $S(X) = K_{\mathbb{R}}\backslash(G_{\mathbb{R}}\times K_{\mathbb{R}})$ of a regular holonomic G equivariant module on the complex space $K\backslash(G\times K)$. To see this, note that each of p^+, r^+ and $\mathcal{F}^K_{K_{\mathbb{R}}}$ is exact and commutes with the forgetful functor. Thus, if $\mathcal{L}$ is a cohomology module of an object in $D^b_{K,f}(\mathcal{D}_{\lambda,X})$ and $\mathcal{N}$ is the corresponding cohomology module of the image of this object in $D^b_{G_{\mathbb{R}},f}(\mathcal{D}_{\lambda,S(X)})$, then $p^+\mathcal{L}\simeq r^+\mathcal{N}$. However, there is also a G-equivariant regular holonomic module $\mathcal{N}_1$ on $K\backslash(G\times X)$ which satisfies the relation $p_1^+\mathcal{L} = r_1^+\mathcal{N}$, where $p_1 : G\times X\to X$ and $r_1 : G\times X\to K\backslash(G\times X)$ are the projections. Evidently, $\mathcal{N}$ is $\mathcal{N}_1$ restricted to $S(X) = K_{\mathbb{R}}\backslash(G_{\mathbb{R}}\times X)$.

Now let $\mathcal{M}$ be the image under $\bar{q}_r\circ p^+\circ\mathcal{F}^K_{K_{\mathbb{R}}}$ of an object in $D^b_{K,f}(\mathcal{D}_{\lambda,X})$. In order to compute the geometric fiber of $v_\dagger[n](\mathcal{M})$ at $x\in X$, we consider the following commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{u} & \{x\} \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle i} \\ S(X) & \xrightarrow{v} & X \end{array}$$

with u and i the obvious projection and inclusion and j induced by the map $g\to(g^{-1},gx) : G_{\mathbb{R}}\to G_{\mathbb{R}}\times X$. Let $j^+ : D^b_{G_{\mathbb{R}}}(\mathcal{D}_{\lambda,S(X)})\to D^b_{G_{\mathbb{R}}}(\mathcal{D}_S)$ and $i^+ : D^b_{G_{\mathbb{R}}}(S(X))\to D^b_{G_{\mathbb{R}}}(\{x\})$ denote the respective derived functors. Then

$$i^+\circ v_\dagger(\mathcal{M})\simeq v_\dagger\circ j^+\mathcal{M}.$$

The above diagram has a complex analytic version with the map j replaced by the corresponding holomorphic map $j_1 : K\backslash G\to K\backslash(G\times X)$. If B is the isotropy group of x for the G action on X, then j_1 is a B equivariant map. We observed in the preceding paragraph that the object $\mathcal{M}$ extends to an object on $K\backslash(G\times X)$ with regular holonomic G-equivariant cohomology. This implies that each cohomology module of the object $j^+\mathcal{M}$ is the restriction to S of a B equivariant regular holonomic module on $K\backslash G$. Thus, $DR[n]\circ j^+\mathcal{M}$ has constructible cohomology. Furthermore, it is constructible relative to a finite algebraic stratification of S—that determined by the B orbit stratification of $K\backslash G$. It follows that $v_!\circ DR[n]\circ j^+\mathcal{M}$ has finite dimensional cohomology and, hence, that $i^+\circ\nu_\dagger(\mathcal{M})$ has finite dimensional cohomology. We cannot yet conclude that each cohomology sheaf of $\nu_\dagger(\mathcal{M})$ has finite dimensional geometric fiber. The geometric fiber of $H^p(\nu_\dagger\mathcal{M})$ is $H^0(i^+(H^p(\nu_\dagger\mathcal{M})))$. There is a spectral sequence with E_2 termed $E_2^{p,q}=H^q(i^+(H^p(\nu_\dagger\mathcal{M})))$ that converges to the cohomology of $i^+\nu_\dagger(\mathcal{M})$. We use this in an induction argument to finish the proof.

The non-vanishing terms of $E_2^{p,q}$ lie in a bounded rectangle, that is independent of x, with the bottom edge being the row $H^0(i^+(H^p(\nu_\dagger\mathcal{M})))$. If there is one, let t be the smallest integer such that for all x and all $p>t$, $H^0(i^+(H^p(\nu_\dagger\mathcal{M})))$ is finite dimensional and $H^q(i^+(H^p(\nu_\dagger\mathcal{M})))=0$ for $q>0$. Then the differentials to and from E_2 in the E_2 term of this spectral sequence both vanish. It follows that $H^0(i^+(H^t(\nu_\dagger\mathcal{M})))$ is finite dimensional for each x. Then Theorem 2.4 and Proposition 2.5 imply that the sheaf $H^t(\nu_\dagger\mathcal{M})$ is locally free as an $\mathcal{O}_X$ module on each $G_{\mathbb{R}}$ orbit. This, in turn, implies that $H^q(i^+(H^t(\nu_\dagger\mathcal{M})))$ vanishes for all $q>0$ for all x. By induction, we conclude that there is no such t and, thus, that $H^0(i^+(H^p(\bar{\gamma}\mathcal{M})))$ is finite dimensional for all p and all choices of x. Hence, $\nu_\dagger\mathcal{M}$ is of finite type. □

Up to this point in the paper, we have not required a topological vector space structure on $\mathcal{D}_\lambda$ modules. However, as we now turn our attention to the top row of diagram (0.1) and its relationship to the middle row, we encounter constructions which depend heavily on a topological vector space structure. This is true, in particular, of the minimal globalization functor, which is a topological completion and the analytic localization functor, as developed in [HT] and [Sm], which requires the use of a completed topological tensor product.

Let $D^b_K(\mathcal{D}^t_{\lambda,Y})$ and $D^b_{G_{\mathbb{R}}}(\mathcal{D}^t_{\lambda,Y})$ denote the K equivariant and $G_{\mathbb{R}}$ equivariant derived categories of DNF topological $\mathcal{D}_{\lambda,Y}$ modules. These are defined just as $D^b_K(\mathcal{D}_{\lambda,Y})$ and $D^b_{G_{\mathbb{R}}}(\mathcal{D}_{\lambda,Y})$ were defined, except that sheaves of $\mathcal{D}_\lambda$ modules are required to be sheaves of DNF topological modules, morphisms are required to be continuous, and a completed topological tensor product, $\hat{\otimes}$, replaces the algebraic tensor product, $\otimes$. This, in turn, effects the definition of the inverse image functor f^+ and, hence, the definition of equivariance. Note that $D^b_{G_{\mathbb{R}}}(\mathcal{D}^t_{\lambda,Y})$, is the equivariant derived category developed in [Sm], where it was denoted simply $D^b_{G_{\mathbb{R}}}(\mathcal{D}_{\lambda,Y})$. We include a discussion of DNF topological modules and their equivariant derived categories in the next section. The full subcategories of finite type objects $D^b_{K,f}(\mathcal{D}^t_{\lambda,Y})$ and $D^b_{G_{\mathbb{R}},f}(\mathcal{D}^t_{\lambda,Y})$ are defined as before. There are functors which forget the topology

$$FT: D^b_{K,f}(\mathcal{D}^t_{\lambda,Y})\to D^b_{K,f}(\mathcal{D}_{\lambda,Y})$$

and

$$FT: D^b_{G_{\mathbb{R}},f}(\mathcal{D}^t_{\lambda,Y})\to D^b_{G_{\mathbb{R}},f}(\mathcal{D}_{\lambda,Y})$$

and, in the case where Y is the flag manifold X, we prove in Theorem 5.13 that these functors are equivalences of categories. This and the fact that the forget topology functor commutes with the DeRham functor and the inverse image functor (Lemma 5.1 and Proposition 5.2) implies that the construction of the equivalence $\bar{\gamma}$ of Theorem 4.2 can be carried out using the appropriate equivariant derived category of DNF $\mathcal{D}_\lambda$ modules at each step with equivalent results. We shall use this construction in what follows.

Throughout the remainder of this section, we work exclusively with DNF modules and topological sheaves of modules, with continuous morphisms between such objects, and with completed topological tensor product. In many cases, a completed topological tensor product or completed relative tensor product agrees with its algebraic counterpart as a module or sheaf of modules. In particular, for DNF $\mathcal{D}_\lambda$ modules $\mathcal{M}$ and $\mathcal{N}$ there is a natural map $\mathcal{M} \otimes_{\mathcal{D}_\lambda} \mathcal{N} \to \mathcal{M} \,\hat{\otimes}_{\mathcal{D}_\lambda} \mathcal{N}$ which is an isomorphism in the case where $\mathcal{M}$ is finitely generated (see Lemma 5.1(a)). However, even when this holds, it is important to use the notation "$\mathcal{M} \,\hat{\otimes}_{\mathcal{D}_\lambda} \mathcal{N}$", since this conveys the fact that a topological vector space structure on the modules of sections of this sheaf is part of the structure.

The top row of (0.1) involves the equivariant derived categories, $D^b_K(\mathcal{U}_\lambda(\mathfrak{g}))$ and $D^b_{G_\mathbb{R}}(\mathcal{U}^t_\lambda(\mathfrak{g}))$ and their full subcategories $D^b_{K,f}(\mathcal{U}_\lambda(\mathfrak{g}))$ and $D^b_{G_\mathbb{R},f}(\mathcal{U}^t_\lambda(\mathfrak{g}))$ consisting of objects of finite type. These are defined as in [Sm] and will be described below. As with $\mathcal{D}_\lambda$ modules, there is a question as to whether one should use arbitrary $\mathcal{U}_\lambda(\mathfrak{g})$ modules in the construction of these categories or DNF topological modules. The answer is clear for $D^b_{G_\mathbb{R},f}(\mathcal{U}^t_\lambda(\mathfrak{g}))$. If one does not use topological modules and the completed topological tensor product in the definition of the inverse image functor, then the class of $G_\mathbb{R}$ equivariant modules is too small and does not contain the interesting infinite dimensional examples. Thus, in the case of the $G_\mathbb{R}$ equivariant derived category, the only sensible choice is to use a class of topological modules, and good results are obtained if one uses the class of DNF modules. Thus, $D^b_{G_\mathbb{R}}(\mathcal{U}^t_\lambda(\mathfrak{g}))$ and $D^b_{G_\mathbb{R},f}(\mathcal{U}^t_\lambda(\mathfrak{g}))$ will, in what follows, denote the $G_\mathbb{R}$ equivariant derived categories of DNF $\mathcal{U}_\lambda(\mathfrak{g})$ modules as defined in [Sm]. As before, the fact that DNF topological modules are used is indicated by the use of the superscript "t". In the K equivariant case, both the purely algebraic category $D^b_K(\mathcal{U}_\lambda(\mathfrak{g}))$ and the DNF module version $D^b_K(\mathcal{U}^t_\lambda(\mathfrak{g}))$ make sense. Their full subcategories of finite type objects, $D^b_{K,f}(\mathcal{U}_\lambda(\mathfrak{g}))$ and $D^b_{K,f}(\mathcal{U}^t_\lambda(\mathfrak{g}))$, are, in fact, equivalent, by Theorem 5.13. For technical reasons that will be explained below, we will use the DNF module version in what follows.

We now give a brief description of $D^b_{K,f}(\mathcal{U}_\lambda(\mathfrak{g}))$, $D^b_{K,f}(\mathcal{U}^t_\lambda(\mathfrak{g}))$ and $D^b_{G_\mathbb{R},f}(\mathcal{U}^t_\lambda(\mathfrak{g}))$ following [Sm]. We consider $\mathcal{U}_\lambda(\mathfrak{g})$ modules to be sheaves on a point, pt, equipped with a *descended* differential structure sheaf $\mathcal{U}_\lambda(\mathfrak{g}) = \pi_*\mathcal{D}_{\lambda,X}$ where $\pi : X \to pt$ is the projection. More generally, if $\pi : Y \to Z$ is any X fibered map (i.e., a fibration in the category of analytic CR manifolds which has X as fiber), then we define a descended differential structure sheaf on Z by $\hat{\mathcal{D}}_{\lambda,Z} = \pi_*\mathcal{D}_{\lambda,Y}$. Since π is a proper map, the sheaf of algebras $\hat{\mathcal{D}}_{\lambda,Z}$ is, like $\mathcal{D}_{\lambda,Y}$, a DNF sheaf of algebras (see Section 5). If the fibration is a map of K spaces, then we define what it means for a DNF $\hat{\mathcal{D}}_{\lambda,Z}$ module to be K equivariant in formally the same way as before but using the descended differential structure sheaves in place of the usual ones.

Following Bernstein-Lunts [BL] we consider free K space resolutions $P \to Z$ of Z. For each such resolution, the fiber product $Q = Y \times_Z P$ yields a resolution

$Q \to Y$ which is compatible with $P \to Z$ in the sense that the square

$$\begin{array}{ccc} Q & \longrightarrow & Y \\ \downarrow & & \downarrow \\ P & \longrightarrow & Z \end{array}$$

commutes. The map $Q \to P$ is X-fibered, as is the corresponding map $Q_0 \to P_0$ between the quotients modulo the group action, and so there are descended differential structure sheaves $\hat{\mathcal{D}}_{\lambda,P}$ and $\hat{\mathcal{D}}_{\lambda,P_0}$. As before, $D^b(\hat{\mathcal{D}}^t_{\lambda,P_0})$ is defined to be the bounded derived category of sheaves of DNF $\hat{\mathcal{D}}_{\lambda,P_0}$ modules. We define $D^b_K(\hat{\mathcal{D}}^t_{\lambda,Z})$ (resp. $D^b_{G_\mathbb{R}}(\hat{\mathcal{D}}^t_{\lambda,Z})$) to be the category of functors from the category of free K (resp. $G_\mathbb{R}$) space resolutions $P \to Z$ to the fibered category consisting of pairs $(P, D^b(\hat{\mathcal{D}}^t_{\lambda,P_0}))$. Of course, $D^b_K(\hat{\mathcal{D}}_{\lambda,Z})$ is defined in the same way, but using general modules rather than DNF modules. In the case of greatest interest, $Z = pt$ we have $\hat{\mathcal{D}}_{\lambda,Z} = \Gamma(\mathcal{D}_{\lambda,X}) = \mathcal{U}_\lambda(\mathfrak{g})$ and so $D^b_K(\hat{\mathcal{D}}^t_{\lambda,pt}) = D^b_K(\mathcal{U}^t_\lambda(\mathfrak{g}))$, $D^b_K(\hat{\mathcal{D}}_{\lambda,pt}) = D^b_K(\mathcal{U}_\lambda(\mathfrak{g}))$ and $D^b_{G_\mathbb{R}}(\hat{\mathcal{D}}^t_{\lambda,pt}) = D^b_{G_\mathbb{R}}(\mathcal{U}^t_\lambda(\mathfrak{g}))$.

What are objects of finite type in this context? An object in $D^b_K(\mathcal{U}^t_\lambda(\mathfrak{g}))$ will have cohomology modules which, if Hausdorff, are K equivariant $\mathcal{U}^t_\lambda(\mathfrak{g})$ modules—that is, they are $(\mathfrak{g}, K)$ modules. The finite type objects in $D^b_K(\mathcal{U}^t_\lambda(\mathfrak{g}))$ are those with Hausdorff cohomologies which are finitely generated as $\mathcal{U}_\lambda(\mathfrak{g})$ modules, that is those which are actually Harish-Chandra modules. Similarly, the objects of $D^b_{G_\mathbb{R}}(\mathcal{U}^t_\lambda(\mathfrak{g}))$ have cohomologies which, if Hausdorff, are $G_\mathbb{R}$-equivariant $\mathcal{U}_\lambda(\mathfrak{g})$ modules. The finite type objects are those whose cohomologies are minimal globalizations of Harish-Chandra modules (cf. [Sm]). The full subcategories of $D^b_K(\mathcal{U}^t_\lambda(\mathfrak{g}))$ and $D^b_{G_\mathbb{R}}(\mathcal{U}^t_\lambda(\mathfrak{g}))$ consisting of objects of finite type will be denoted $D^b_{K,f}(\mathcal{U}^t_\lambda(\mathfrak{g}))$ and $D^b_{G_\mathbb{R},f}(\mathcal{U}^t_\lambda(\mathfrak{g}))$, respectively. The full subcategory $D^b_{K,f}(\mathcal{U}_\lambda(\mathfrak{g}))$ of finite type objects in $D^b_K(\mathcal{U}_\lambda(\mathfrak{g}))$ is defined similarly, but without the Hausdorff condition.

In [Sm] the Bernstein-Beilinson localization functor is adapted to yield an equivalence of equivariant derived categories $\Delta : D^b_{G_\mathbb{R},f}(\mathcal{U}^t_\lambda(\mathfrak{g})) \to D^b_{G_\mathbb{R},f}(\mathcal{D}^t_{\lambda,X})$ with inverse the derived global sections functor. The same approach, applied in the K-equivariant case yields an equivalence $\Delta : D^b_{K,f}(\mathcal{U}^t_\lambda(\mathfrak{g})) \to D^b_{K,f}(\mathcal{D}^t_{\lambda,X})$. This is essentially the derived Beilinson-Bernstein localization functor, as discussed in [BB2] but adapted to the K-equivariant derived category. Its inverse is the derived global sections functor $\pi_* : D^b_{K,f}(\mathcal{D}^t_{\lambda,X}) \to D^b_{K,f}(\mathcal{U}^t_\lambda(\mathfrak{g}))$, where π is the map $\pi : X \to pt$. Note that the proofs of these results use the DNF structure of modules and the completed topological tensor product in essential ways. In the $G_\mathbb{R}$ equivariant case, the localization functor does not yield a reasonable functor unless DNF modules and the completed topological tensor product are used in its definition. Furthermore, the base change Lemma 5.7 in [Sm] is used in an essential way in these results and it is is not true unless the completed topological tensor product is used in the definition of the inverse image functor.

We define $\hat{\gamma} : D^b_{K,f}(\mathcal{U}^t_\lambda(\mathfrak{g})) \to D^b_{G_\mathbb{R},f}(\mathcal{U}^t_\lambda(\mathfrak{g}))$, as before, as the composition of the forgetful functor

$$\hat{\mathcal{F}}^K_{K_\mathbb{R}} : D^b_{K,f}(\mathcal{U}^t_\lambda(\mathfrak{g})) \to D^b_{K_\mathbb{R},f}(\mathcal{U}^t_\lambda(\mathfrak{g}))$$

with an integration functor

$$\hat{\gamma}^{G_\mathbb{R}}_{K_\mathbb{R}} : D^b_{K_\mathbb{R}}(\mathcal{U}^t_\lambda(\mathfrak{g})) \to D^b_{G_\mathbb{R}}(\mathcal{U}^t_\lambda(\mathfrak{g})).$$

The latter is defined by essentially calculating what each of the ingredients in the definition of $\bar{\gamma}_{K_{\mathbb{R}}}^{G_{\mathbb{R}}}$ descends to under π_*.

First, there is a descended inverse image functor

$$\hat{p}^+ : D^b_{K_{\mathbb{R}}}(\mathcal{U}^t_\lambda(\mathfrak{g})) \to D^b_{G_{\mathbb{R}} \times K_{\mathbb{R}}}(\hat{\mathcal{D}}^t_{\lambda, G_{\mathbb{R}}})$$

which makes commutative the diagram

$$\begin{array}{ccc} D^b_{K_{\mathbb{R}}}(\mathcal{D}^t_{\lambda,X}) & \xrightarrow{p^+} & D^b_{G_{\mathbb{R}} \times K_{\mathbb{R}}}(\mathcal{D}^t_{\lambda, G_{\mathbb{R}} \times X}) \\ {\scriptstyle \pi_*}\big\downarrow & & \big\downarrow{\scriptstyle \pi_*} \\ D^b_{K_{\mathbb{R}}}(\mathcal{U}^t_\lambda(\mathfrak{g})) & \xrightarrow{\hat{p}^+} & D^b_{G_{\mathbb{R}} \times K_{\mathbb{R}}}(\hat{\mathcal{D}}^t_{\lambda, G_{\mathbb{R}}}). \end{array} \tag{4.6}$$

In fact, $\hat{p}^+$ is defined in formally the same way as p^+ for DNF $\mathcal{D}_\lambda$ modules, the only difference being the structure sheaves defining the relevant categories of modules. As before, this functor commutes with the forgetful functor in the sense that it is the top arrow of a diagram in which the vertical arrows are the forgetful functors and the bottom arrow is the descended inverse image functor. This, in turn, is just the functor induced on the derived category by the functor which assigns to a $\mathcal{U}^t_\lambda(\mathfrak{g})$-module M the sheaf of $\mathcal{D}_{G_{\mathbb{R}}} \hat{\otimes} \mathcal{U}^t_\lambda(\mathfrak{g})$ modules $\hat{p}^+ M = \mathcal{O}_{G_{\mathbb{R}}} \hat{\otimes} M$.

Similarly, if we set $S = G_{\mathbb{R}}/K_{\mathbb{R}}$, there is a descended inverse image functor

$$\hat{r}^+ : D^b_{G_{\mathbb{R}}}(\hat{\mathcal{D}}^t_{\lambda,S}) \to D^b_{G_{\mathbb{R}} \times K_{\mathbb{R}}}(\hat{\mathcal{D}}^t_{\lambda, G_{\mathbb{R}}})$$

which also commutes with the forgetful functor in the above sense and which makes commutative the diagram

$$\begin{array}{ccc} D^b_{G_{\mathbb{R}}}(\mathcal{D}^t_{\lambda,S(X)}) & \xrightarrow{r^+} & D^b_{G_{\mathbb{R}} \times K_{\mathbb{R}}}(\mathcal{D}^t_{\lambda, G_{\mathbb{R}} \times X}) \\ {\scriptstyle \pi_*}\big\downarrow & & \big\downarrow{\scriptstyle \pi_*} \\ D^b_{G_{\mathbb{R}}}(\hat{\mathcal{D}}^t_{\lambda,S}) & \xrightarrow{\hat{r}^+} & D^b_{G_{\mathbb{R}} \times K_{\mathbb{R}}}(\hat{\mathcal{D}}^t_{\lambda, G_{\mathbb{R}}}). \end{array} \tag{4.7}$$

This is an equivalence of categories and we will use its inverse,

$$\hat{q}_r : D^b_{G_{\mathbb{R}} \times K_{\mathbb{R}}}(\hat{\mathcal{D}}^t_{\lambda, G_{\mathbb{R}}}) \to D^b_{G_{\mathbb{R}}}(\hat{\mathcal{D}}^t_{\lambda,S}).$$

Finally, there is a descended $\mathcal{D}$ module direct image with proper supports functor

$$\begin{array}{ccc} D^b_{G_{\mathbb{R}}}(\mathcal{D}^t_{\lambda,S(X)}) & \xrightarrow{v_\dagger} & D^b_{G_{\mathbb{R}}}(\mathcal{D}^t_{\lambda,X}) \\ {\scriptstyle \pi_*}\big\downarrow & & \big\downarrow{\scriptstyle \pi_*} \\ D^b_{G_{\mathbb{R}}}(\hat{\mathcal{D}}^t_{\lambda,S}) & \xrightarrow{\hat{v}_\dagger} & D^b_{G_{\mathbb{R}}}(\mathcal{U}^t_\lambda(\mathfrak{g})). \end{array} \tag{4.8}$$

This is defined as follows: If $\mathcal{M}$ is an object of $D^b(\mathcal{D}^t_{\lambda,S(X)})$, then, as in the discussion preceding (4.3), the shifted relative DeRham complex $DR_v[n](\mathcal{M})$ along v may be defined as the left derived tensor product $\mathcal{D}_{\lambda,S(X)\leftarrow X} \hat{\otimes}^L_{\mathcal{D}_{\lambda,S(X)}} \mathcal{M}$. That is,

$$DR_v(\mathcal{D}_{\lambda,S(X)}) \to \mathcal{D}_{\lambda,S(X)\leftarrow X} \to 0 \tag{4.9}$$

is a locally $\mathcal{D}_{\lambda,S(X)}$-free bimodule resolution of $\mathcal{D}_{\lambda,S(X)\leftarrow X}$ and

$$DR_v[n](\mathcal{M}) = DR_v(\mathcal{D}_{\lambda,S(X)}) \hat{\otimes}_{\mathcal{D}_{\lambda,S(X)}} \mathcal{M}.$$

We define a descended version of this DeRham complex as follows: Set

$$\hat{D}R_v(\hat{\mathcal{D}}_{\lambda,S}) = \pi_* DR_v(\mathcal{D}_{\lambda,S(X)})$$

and note that this complex yields a locally $\hat{\mathcal{D}}_{\lambda,S} = \pi_*\mathcal{D}_{\lambda,S(X)}$-free resolution of $\hat{\mathcal{D}}_{\lambda,S\leftarrow pt} = \pi_*\mathcal{D}_{\lambda,S(X)\leftarrow X}$. This follows from the fact that $\mathcal{D}_{\lambda,X}$ is acyclic and (4.9) is an exact sequence of sheaves which are free $\mathcal{D}_{\lambda,S(X)}$ modules.

Having defined our descended version of the relative DeRham complex, we define $\hat{v}_\dagger\mathcal{M}$ for an object $\mathcal{M}$ of $D^b(\hat{\mathcal{D}}^t_{\lambda,S})$ by $v_!(\hat{D}R_{v_1}(\hat{\mathcal{D}}_{\lambda,S})\hat{\otimes}_{\hat{\mathcal{D}}_{\lambda,S}}\mathcal{M})$.

To define the $G_\mathbb{R}$ equivariant version of $\hat{v}_\dagger$ is now routine. If P is a free $G_\mathbb{R}$ space—that is, a free $G_\mathbb{R}$ resolution of a point, there are corresponding free $G_\mathbb{R}$ resolutions $X \times P \to X$, $S(X) \times P \to S(X)$ and $S \times P \to S$ of X, $S(X)$, and $S = S(pt)$, respectively, where the product spaces are given the diagonal $G_\mathbb{R}$ actions. Furthermore, we have a commutative diagram

$$\begin{array}{ccc} S(X)\times P & \xrightarrow{v_1} & X \times P \\ \pi_* \downarrow & & \downarrow \pi_* \\ S \times P & \xrightarrow{v_1} & P \end{array}$$

in which the maps $v_1 = v \times id$ are morphisms of resolutions and the maps π are X fibered $G_\mathbb{R}$ maps. On passing to the quotient modulo the $G_\mathbb{R}$ action, this leads to the commutative diagram

$$\begin{array}{ccc} (S(X)\times P)_0 & \xrightarrow{v_0} & (X \times P)_0 \\ \pi_0 \downarrow & & \downarrow \pi_0 \\ (S \times P)_0 & \xrightarrow{v_0} & P_0 \end{array}$$

in which the vertical arrows are still X fibered maps. We define

$$\hat{v}_\dagger : D^b(\hat{\mathcal{D}}^t_{\lambda,(S\times P)_0}) \to D^b(\hat{\mathcal{D}}^t_{\lambda,P_0})$$

as in the previous paragraph but with v_0, P_0, $(S\times P)_0$, $(X \times P)_0$ and $(S(X)\times P)_0$ replacing v, pt, S, X and $S(X)$, respectively. As P runs through the free $G_\mathbb{R}$ resolutions of pt, this defines a functor

$$\hat{v}_\dagger : D^b_{G_\mathbb{R}}(\hat{\mathcal{D}}^t_{\lambda,S}) \to D^b_{G_\mathbb{R}}(\hat{\mathcal{D}}^t_{\lambda,pt}) = D^b_{G_\mathbb{R}}(\mathcal{U}^t_\lambda(\mathfrak{g})).$$

It follows easily from the construction that, with this definition of $\hat{v}_\dagger$, diagram (4.8) is commutative.

Now with the integration functor $\hat{\gamma}^{G_\mathbb{R}}_{K_\mathbb{R}} : D^b_{K_\mathbb{R}}(\mathcal{U}^t_\lambda(\mathfrak{g})) \to D^b_{G_\mathbb{R}}(\mathcal{U}^t_\lambda(\mathfrak{g}))$ defined by

$$\hat{\gamma}^{G_\mathbb{R}}_{K_\mathbb{R}} = \hat{v}_\dagger[n] \circ \hat{q}_r \circ \hat{p}^+$$

and $\hat{\gamma}$ its composition with the forgetful functor:

$$\hat{\gamma} = \hat{\gamma}^{G_\mathbb{R}}_{K_\mathbb{R}} \circ \hat{\mathcal{F}}^K_{K_\mathbb{R}}$$

the commutativity of diagrams (4.6), (4.7), and (4.8), allows us to conclude:

Theorem 4.3. *The functor $\hat{\gamma}$ takes objects of finite type to objects of finite type and the diagram*

$$\begin{array}{ccc} D^b_{K,f}(\mathcal{D}^t_{\lambda,X}) & \xrightarrow{\bar{\gamma}} & D^b_{G_\mathbb{R},f}(\mathcal{D}^t_{\lambda,X}) \\ \pi \downarrow & & \downarrow \pi \\ D^b_{K,f}(\mathcal{U}^t_\lambda(\mathfrak{g})) & \xrightarrow{\hat{\gamma}} & D^b_{G_\mathbb{R},f}(\mathcal{U}^t_\lambda(\mathfrak{g})) \end{array}$$

is commutative. Hence, $\hat{\gamma} : D^b_{K,f}(\mathcal{U}^t_\lambda(\mathfrak{g})) \to D^b_{G_\mathbb{R},f}(\mathcal{U}^t_\lambda(\mathfrak{g}))$ is an equivalence of categories.

Now if we use Theorem 5.13 to replace the categories $D^b_{K,f}(\mathcal{D}^t_{\lambda,X})$,$D^b_{G_{\mathbb{R}},f}(\mathcal{D}^t_{\lambda,X})$ and $D^b_{K,f}(\mathcal{U}^t_\lambda(\mathfrak{g}))$ by the non-topologized versions $D^b_{K,f}(\mathcal{D}_{\lambda,X})$, $D^b_{G_{\mathbb{R}},f}(\mathcal{D}_{\lambda,X})$ and $D^b_{K,f}(\mathcal{U}_\lambda(\mathfrak{g}))$ we obtain the commutative square of equivalences comprising the upper square of diagram (0.1).

Note that the algebra $\mathcal{U}_\lambda(\mathfrak{g})$ and the functor $\hat{\gamma} : D^b_{K,f}(\mathcal{U}_\lambda(\mathfrak{g})) \to D^b_{G_{\mathbb{R}},f}(\mathcal{U}^t_\lambda(\mathfrak{g}))$ depend only on the equivalence class of λ under the equivalence relation: $\lambda \sim \lambda'$ if $\lambda+\rho$ and $\lambda'+\rho$ are in the same Weyl group orbit, where ρ is the half sum of the positive roots. On the other hand, the sheaf of algebras $\mathcal{D}_{\lambda,X}$ and the functor $\bar{\gamma} : D^b_{K,f}(\mathcal{D}_{\lambda,X}) \to D^b_{G_{\mathbb{R}},f}(\mathcal{D}_{\lambda,X})$ definitely depend on λ.

It remains to relate the functor $\hat{\gamma}$ to Schmid's minimal globalization functor. Specifically, we will show that $\hat{\gamma}$ is an exact extension to the equivariant derived category of the minimal globalization functor. For this we need to be able to calculate $\hat{\gamma}$ applied to an object in the heart of $D^b_{K,f}(\mathcal{U}_\lambda(\mathfrak{g}))$.

Recall that the heart of the t-category $D^b_{K,f}(\mathcal{U}_\lambda(\mathfrak{g}))$ is the category of finite type K equivariant $\mathcal{U}_\lambda(\mathfrak{g})$ modules (i.e., the category of Harish-Chandra modules). Also, the heart of the t-category $D^b_{G_{\mathbb{R}},f}(\mathcal{U}_\lambda(\mathfrak{g}))$ is the category of finite type $G_{\mathbb{R}}$ equivariant DNF $\mathcal{U}_\lambda(\mathfrak{g})$ modules. We will show that $\hat{\gamma}$ takes a Harish-Chandra module M to the minimal globalization of M. This is the content of the following theorem:

Theorem 4.4. *If M is a Harish-Chandra module, then $H^j(\hat{\gamma}M) = 0$ for $j \neq 0$ and*

$$H^0(\hat{\gamma}M) \simeq \mathcal{C}^\infty_c(G_{\mathbb{R}})\hat{\otimes}_{\mathcal{U}(\mathfrak{g})}M.$$

The later space is the minimal globalization of M.

Proof. Basically, this is just a reformulation of Theorem 6.13 of [KSd]. We will outline the proof but refer to [KSd] for key details.

Let M be a Harish-Chandra module. Without effecting $\mathcal{U}_\lambda(\mathfrak{g})$, we may choose λ from the dominant Weyl chamber (we adopt the same convention as [KSd] regarding the order relation on weights λ, so that dominant weights correspond to positive bundles). With this choice of λ, [BB] implies that M is the module of global sections of a unique holonomic K equivariant $\mathcal{D}_{\lambda,X}$ module $\mathcal{M}$.

We then have $p^+(\mathcal{M}) = \mathcal{O}_{G_{\mathbb{R}}} \hat{\boxtimes} \mathcal{M}$, where $\mathcal{O}_{G_{\mathbb{R}}} \hat{\boxtimes} \mathcal{M}$ denotes the completed topological tensor product $p_1^{-1}\mathcal{O}_{G_{\mathbb{R}}} \hat{\otimes} p_2^{-1}\mathcal{M}$ and $p_1 : G_{\mathbb{R}}\times X \to G_{\mathbb{R}}$ and $p_2 : G_{\mathbb{R}}\times X \to X$ are the projections. This sheaf is $K_{\mathbb{R}}$ equivariant for the given $K_{\mathbb{R}}$ action on $G_{\mathbb{R}}\times X$. When applied to such a sheaf, $\hat{q}_r = (r^+)^{-1}$ is just $r_*^{K_{\mathbb{R}}}$, that is, $K_{\mathbb{R}}$ invariant direct image under r. This is because $r : G_{\mathbb{R}} \times X \to S(X)$ is the quotient map modulo the $K_{\mathbb{R}}$ action. Thus, $\hat{q}_r \circ p^+(\mathcal{M}) = r_*^{K_{\mathbb{R}}}(\mathcal{O}_{G_{\mathbb{R}}} \hat{\boxtimes} \mathcal{M})$. Since $v_\dagger$ is the composition of the shifted DeRham complex $DR_v[n]$ along v with $v_!$, we have

$$\bar{\gamma}\mathcal{M} = v_! \circ DR_v[n] \circ r_*^{K_{\mathbb{R}}}(\mathcal{O}_{G_{\mathbb{R}}} \hat{\boxtimes} \mathcal{M}).$$

Let us examine the functor $DR_v[n]\circ r_*^{K_{\mathbb{R}}}$. The fibers of the map $v\circ r$ are the orbits of the diagonal $G_{\mathbb{R}}$ action $g\times(g_1,x) \to (g_1g^{-1}, gx)$ on $G_{\mathbb{R}}\times X$. The differential of this action determines an embedding of the Lie algebra $\mathfrak{g}$ in $\mathcal{D}_{\lambda,G_{\mathbb{R}}\times X}$ and, hence, an action of $\mathfrak{g}$ on the sheaf $r_*(\mathcal{O}_{G_{\mathbb{R}}} \hat{\boxtimes} \mathcal{M})$ via the $\mathcal{D}_{\lambda,G_{\mathbb{R}}\times X}$ module structure of $\mathcal{O}_{G_{\mathbb{R}}} \hat{\boxtimes} \mathcal{M}$. There is also a $K_{\mathbb{R}}$ action on $r_*(\mathcal{O}_{G_{\mathbb{R}}} \hat{\boxtimes} \mathcal{M})$ which arises from the $K_{\mathbb{R}}$ equivariance of $\mathcal{O}_{G_{\mathbb{R}}} \hat{\boxtimes} \mathcal{M}$. It is immediate from the definitions of these two actions that they satisfy the compatibility conditions for $(\mathfrak{g}, K_{\mathbb{R}})$-modules. Thus, $r_*(\mathcal{O}_{G_{\mathbb{R}}} \hat{\boxtimes} \mathcal{M})$ is a sheaf of $(\mathfrak{g}, K_{\mathbb{R}})$-modules. We claim that the complex $DR^v[n] \circ r_*^{K_{\mathbb{R}}}(\mathcal{O}_{G_{\mathbb{R}}} \hat{\boxtimes} \mathcal{M})$ is isomorphic to the $(\mathfrak{g}, K_{\mathbb{R}})$ relative Koszul complex for $r_*(\mathcal{O}_{G_{\mathbb{R}}} \hat{\boxtimes} \mathcal{M})$. Below, we give a brief description of this complex and a justification for the claim.

Let N be a DNF $(\mathfrak{g}, K_{\mathbb{R}})$ module and let $\mathcal{K}_{\mathfrak{g}}(N)$ denote the Koszul complex for the $\mathfrak{g}$ action on N:

$$\cdots \wedge^{-p} \mathfrak{g} \otimes N \xrightarrow{d} \wedge^{-p-1} \mathfrak{g} \otimes N \xrightarrow{d} \cdots \xrightarrow{d} \mathfrak{g} \otimes N \xrightarrow{d} N \to 0.$$

Here $p \leq 0$ and $\wedge^{-p}\mathfrak{g} \otimes N$ is the degree p term of the complex. The relative $(\mathfrak{g}, K_{\mathbb{R}})$ Koszul complex is a quotient of this complex and is constructed as follows. We first replace the exterior algebra over $\mathfrak{g}$ by its quotient modulo the ideal generated by $\mathrm{Lie}(K) = \mathfrak{k}$. That is, we replace each $\wedge^{-p}\mathfrak{g} \otimes N$ by $\wedge^{-p}(\mathfrak{g}/\mathfrak{k}) \otimes N$. The latter space enjoys a $K_{\mathbb{R}}$ action (the tensor product of the adjoint action on $\wedge^{-p}(\mathfrak{g}/\mathfrak{k})$ with the action on N). By integrating with respect to this $K_{\mathbb{R}}$ action, we construct a projection of $\wedge^{-p}(\mathfrak{g}/\mathfrak{k}) \otimes N$ onto its subspace of $K_{\mathbb{R}}$ invariants, $(\wedge^{-p}(\mathfrak{g}/\mathfrak{k}) \otimes N)^{K_{\mathbb{R}}}$. Via the composition of the maps

$$\wedge^{-p}\mathfrak{g} \otimes N \to \wedge^{-p}(\mathfrak{g}/\mathfrak{k}) \otimes N \to (\wedge^{-p}(\mathfrak{g}/\mathfrak{k}) \otimes N)^{K_{\mathbb{R}}}.$$

The differential on $\mathcal{K}_{\mathfrak{g}}(N)$ induces a differential

$$d : (\wedge^{-p}(\mathfrak{g}/\mathfrak{k}) \otimes N)^{K_{\mathbb{R}}} \to (\wedge^{-p-1}(\mathfrak{g}/\mathfrak{k}) \otimes N)^{K_{\mathbb{R}}}$$

so that the resulting complex is a quotient complex of $\mathcal{K}_{\mathfrak{g}}(N)$. This is the relative Koszul complex $\mathcal{K}_{(\mathfrak{g},K_{\mathbb{R}})}(N)$. Its cohomology in degree p is $H_{-p}^{(\mathfrak{g},K_{\mathbb{R}})}(N)$ – the $(g, K_{\mathbb{R}})$ homology of N in degree $-p$. Note that this construction is dual to the construction of $(\mathfrak{g}, K_{\mathbb{R}})$ cohomology as carried out in Section 6.1 of [V].

As noted above, $r_*(\mathcal{O}_{G_{\mathbb{R}}} \hat{\boxtimes} \mathcal{M})$ is a sheaf of $(\mathfrak{g}, K_{\mathbb{R}})$ modules on $S(X)$. Thus, we may take the Koszul complex, stalkwise, of this sheaf and obtain a complex $\mathcal{K}_{(\mathfrak{g},K_{\mathbb{R}})}(r_*(\mathcal{O}_{G_{\mathbb{R}}} \hat{\boxtimes} \mathcal{M}))$ of sheaves on $S(X)$. The fibers of the map $v \circ r : G_{\mathbb{R}} \times X \to X$ are the orbits of the $G_{\mathbb{R}}$ action $g \times (g_1, x) \to (g_1 g^{-1}, gx)$ and, via this action, the subbundle of the tangent space of $G_{\mathbb{R}} \times X$ consisting of vectors tangent to fibers of $v \circ r$ is isomorphic to the trivial bundle with fiber $\mathfrak{g}$. We consider the corresponding quotient bundle with fiber $\mathfrak{g}/\mathfrak{k}$. If s is a point of $S(X)$ and U is a sufficiently small neighborhood of s, then we may choose a finite set of $K_{\mathbb{R}}$ invariant sections of this bundle over $r^{-1}(U)$ which form a basis for the bundle over $r^{-1}(U)$. Using this basis, we may identify $(\wedge^{-p}(\mathfrak{g}/\mathfrak{k}) \otimes (\mathcal{O}_{G_{\mathbb{R}}} \hat{\boxtimes} \mathcal{M}))^{K_{\mathbb{R}}}(r^{-1}(U))$ with $\wedge^{-p}\mathcal{T}_v \hat{\otimes}_{\mathcal{O}_{S(X)}} r_*^{K_{\mathbb{R}}}(\mathcal{O}_{G_{\mathbb{R}}} \hat{\boxtimes} \mathcal{M})(U)$, where $\mathcal{T}_v$ is the sheaf of sections of the bundle of tangents to the fibers of v on $S(X)$. Thus, the term of degree p of the Koszul complex $\mathcal{K}_{(\mathfrak{g},K_{\mathbb{R}})}(r_*(\mathcal{O}_{G_{\mathbb{R}}} \hat{\boxtimes} \mathcal{M}))$ may be identified with $\wedge^{-p}\mathcal{T}_v \hat{\otimes}_{\mathcal{O}_{S(X)}} r_*^{K_{\mathbb{R}}}(\mathcal{O}_{G_{\mathbb{R}}} \hat{\boxtimes} \mathcal{M})$. After tensoring with the sheaf of highest degree forms along the fibers of v, we obtain a complex whose terms, at least, are the terms of the DeRham complex $DR_v[n] \circ r_*^{K_{\mathbb{R}}}(\mathcal{O}_{G_{\mathbb{R}}} \hat{\boxtimes} \mathcal{M})$. A calculation shows that the differentials are correct as well and this establishes the claim that $DR^v[n] \circ r_*^{K_{\mathbb{R}}}(\mathcal{O}_{G_{\mathbb{R}}} \hat{\boxtimes} \mathcal{M})$ is isomorphic to $\mathcal{K}_{(\mathfrak{g},K_{\mathbb{R}})}(r_*(\mathcal{O}_{G_{\mathbb{R}}} \hat{\boxtimes} \mathcal{M}))$. From this, it follows that

$$\bar{\gamma}\mathcal{M} \simeq v_! \mathcal{K}_{(\mathfrak{g},K_{\mathbb{R}})}(r_*(\mathcal{O}_{G_{\mathbb{R}}} \hat{\boxtimes} \mathcal{M})). \tag{4.10}$$

Then, since

$$\tilde{\gamma} M = \tilde{\pi}_* \mathcal{M} = \pi_* \bar{\gamma} \mathcal{M} = \pi_! \bar{\gamma} \mathcal{M}$$

and $\pi \circ v$ is projection of $S(X)$ to a point, we have

$$\begin{aligned} \tilde{\gamma} M &= \pi_! v_! \mathcal{K}_{(\mathfrak{g},K_\mathbb{R})}(r_*(\mathcal{O}_{G_\mathbb{R}} \hat{\boxtimes} \mathcal{M})) \\ &= R\Gamma_c \mathcal{K}_{(\mathfrak{g},K_\mathbb{R})}(r_*(\mathcal{O}_{G_\mathbb{R}} \hat{\boxtimes} \mathcal{M})). \end{aligned} \tag{4.11}$$

The Koszul complex $\mathcal{K}_{(\mathfrak{g},K_\mathbb{R})}(r_*(\mathcal{O}_{G_\mathbb{R}} \hat{\boxtimes} \mathcal{M}))$ is a complex of sheaves of sections of analytic vector bundles and differential operators which act in directions along the fibers of $v : S(X) \to X$. This complex of differential operators satisfies a certain ellipticity condition which ensures that analytic solutions and $\mathcal{C}^\infty$ solutions agree, that is, that the inclusion of $\mathcal{K}_{(\mathfrak{g},K_\mathbb{R})}(r_*(\mathcal{O}_{G_\mathbb{R}} \hat{\boxtimes} \mathcal{M}))$ in $\mathcal{K}_{(\mathfrak{g},K_\mathbb{R})}(r_*(\mathcal{C}^\infty_{G_\mathbb{R}} \hat{\boxtimes} \mathcal{M}))$ is a quasi-isomorphism (see [KSd] 4.18 and 6.4). Now the degree p term of the complex $\mathcal{K}_{(\mathfrak{g},K_\mathbb{R})}(r_*(\mathcal{O}_{G_\mathbb{R}} \hat{\boxtimes} \mathcal{M}))$ is $(\wedge^{-p}\mathfrak{g}/\mathfrak{k} \otimes r_*(\mathcal{C}^\infty_{G_\mathbb{R}} \hat{\boxtimes} \mathcal{M}))^{K_\mathbb{R}}$ and, via the projection determined by integration with respect to $K_\mathbb{R}$, this is a direct summand (as a sheaf of vector spaces) of $\wedge^{-p}\mathfrak{g}/\mathfrak{k} \otimes r_*(\mathcal{C}^\infty_{G_\mathbb{R}} \hat{\boxtimes} \mathcal{M})$. The latter sheaf is Γ_c-acyclic on $S(X)$ due to the fact that the sheaf $\mathcal{C}^\infty_{G_\mathbb{R}} \hat{\boxtimes} \mathcal{M}$ is both Γ_c-acyclic and r_* acyclic on $G_\mathbb{R} \times X$ (since $\mathcal{M}$ is acyclic on X and $\mathcal{C}^\infty_{G_\mathbb{R}}$ is fine). Since it is a direct summand of a Γ_c-acyclic sheaf, it follows that $(\wedge^{-p}\mathfrak{g}/\mathfrak{k} \otimes r_*(\mathcal{C}^\infty_{G_\mathbb{R}} \hat{\boxtimes} \mathcal{M}))^{K_\mathbb{R}}$ is also Γ_c acyclic on $S(X)$, so that the derived functor $R\Gamma_c$ may be replaced by its ordinary counterpart Γ_c in (4.11). This, together with the obvious fact that taking $K_\mathbb{R}$ invariants commutes with Γ_c, yields

$$\begin{aligned} \tilde{\gamma} M &= \Gamma_c \mathcal{K}_{(\mathfrak{g},K_\mathbb{R})}(r_*(\mathcal{O}_{G_\mathbb{R}} \hat{\boxtimes} \mathcal{M})) \\ &= \mathcal{K}_{(\mathfrak{g},K_\mathbb{R})}(\mathcal{C}^\infty_c(G_\mathbb{R}) \hat{\otimes} M) \\ &= \mathcal{C}^\infty_c(G_\mathbb{R}) \hat{\otimes}^L_{(\mathfrak{g},K_\mathbb{R})} M. \end{aligned}$$

With this description of $\hat{\gamma}(M)$, it is clear that

$$H^0(\hat{\gamma} M) \simeq \mathcal{C}^\infty_c(G_\mathbb{R}) \hat{\otimes}_{(\mathfrak{g},K_\mathbb{R})} M = C^\infty_c(G_\mathbb{R}) \hat{\otimes}_{\mathcal{U}(\mathfrak{g})} M.$$

The vanishing of the higher cohomologies is proved by direct calculation in the case where M is in the principal series—that is, when M is given as the space of sections of an algebraic K equivariant line bundle on the open K orbit of X. The proof for irreducible M then follows by downward induction on j using the fact that every irreducible Harish-Chandra module may be embedded in a principal series module. Once the vanishing is proved for irreducibles the general result follows from the long exact sequence of cohomology (see 6.13 of [KSd]). □

As noted in [KSd], this theorem has as corollaries the following earlier results of Schmid:

The module $\mathcal{C}^\infty_c(G_\mathbb{R}) \hat{\otimes}_{\mathcal{U}(\mathfrak{g})} M$ is an, a priori, possibly non-Hausdorff quotient of the topological tensor product $\mathcal{C}^\infty_c(G_\mathbb{R}) \hat{\otimes} M$. In fact, the minimal globalization functor is initially defined in [S2] as the quotient of $\mathcal{C}^\infty_c(G_\mathbb{R}) \hat{\otimes}_{\mathcal{U}(\mathfrak{g})} M$ by the closure of its zero subspace. However, due to Theorem 4.4 and the fact that objects in $D^b_{G_\mathbb{R},f}(\mathcal{U}_\lambda(\mathfrak{g}))$ have Hausdorff cohomology, we have

Corollary 4.5. *The module $\mathcal{C}^\infty_c(G_\mathbb{R}) \hat{\otimes}_{\mathcal{U}(\mathfrak{g})} M$ is Hausdorff.*

Also, from the vanishing of higher cohomologies in Theorem 4.4 we conclude that

Corollary 4.6. *The minimal globalization functor is exact.*

5. Topology

It was essential in the preceding pages that we be able to construct certain equivariant derived categories of sheaves of $\mathcal{D}_\lambda$ modules in either of two ways with equivalent results: the first uses complexes of DNF sheaves of topological $\mathcal{D}_\lambda$ modules, i.e., $\mathcal{D}_\lambda^t$ modules; the second uses general complexes of $\mathcal{D}_\lambda$ modules with no topological module structure assumed. Both are essential. The first provides the correct framework for the machinery of globalization and analytic localization. The minimal globalization functor is a topological completion which yields a DNF topological module when applied to a Harish-Chandra module. Also, if we failed to take into account the topological module structure of globalizations of $\mathcal{U}_\lambda(\mathfrak{g})$ modules and make use of the completed topological tensor product we simply wouldn't get the right (or even a reasonable) notion of localization. This means that the target category for the analytic localization equivalence is naturally the one constructed using DNF sheaves. Furthermore, this DNF structure on sheaves is needed when we apply the inverse functor (derived global sections) in order to recreate the DNF structure on the original modules. On the other hand, the category of DNF sheaves of $\mathcal{D}_\lambda$ modules does not have enough injectives or projectives. This flaw creates serious obstacles to proving the analytic Riemann-Hilbert correspondence which relates the middle and bottom rows of diagram (0.1). For example, the lack of resolutions implies that the functor $R\mathcal{H}om$, which is crucial to the Riemann-Hilbert correspondence, is not defined on this category. Thus, we use the second (topology free) construction of our equivariant derived categories in proving this correspondence. Fortunately, for the subcategories of interest to us (those consisting of objects of finite type) the two constructions yield equivalent categories. This section is devoted to proving this and several similar results.

To simplify the exposition, we will drop the λ twist and hence refer to $\mathcal{D}$ or $\mathcal{D}^t$ modules rather than $\mathcal{D}_\lambda$ and $\mathcal{D}_\lambda^t$ modules. Similarly, we will focus on $\mathcal{U}_0(\mathfrak{g}) = \Gamma(X, \mathcal{D})$ and $\mathcal{U}_0^t(\mathfrak{g}) = \Gamma(X, \mathcal{D}_\lambda^t)$ instead of $\mathcal{U}_\lambda(\mathfrak{g})$ and $\mathcal{U}_\lambda^t(\mathfrak{g})$. However, all of the proofs given below were specifically formulated so that they remain true in the λ-twisted case. We begin with some general remarks concerning DNF sheaves and then develop the machinery needed to prove the specific equivalences that are used in the preceding sections. We refer the reader to [Sch] for a general treatment of topological vector space theory and to [T] and [HT] for background on homological algebra in the context of topological modules over topological algebras.

A complex vector space V can generally be given a topology which makes it a locally convex topological vector space in many different ways. However, there is always one locally convex topological vector space structure on V which is canonical—the strongest locally convex topology. This is the topology in which every non-empty convex balanced absorbing set is a neighborhood of zero. It is also the locally convex inductive limit topology determined by representing V as the inductive limit of its finite dimensional subspaces. Recall that a DNF space is the strong dual of a Nuclear Frechet space. Since the inductive limit, if Hausdorff, of a countable system of DNF spaces is DNF, the strongest locally convex topology on V is a DNF topology provided V has countable dimension. In particular, the algebras $\mathcal{U}(\mathfrak{g})$ and $\mathcal{U}_0(\mathfrak{g})$, being of countable dimension, both have a natural DNF space structure, as do all of the algebras $\mathcal{U}_\lambda(\mathfrak{g})$.

It is obvious from the definition that a linear transformation $\phi : V \to W$ between two locally convex topological vector spaces is automatically continuous if V has the strongest locally convex topology.

The category of DNF topological vector spaces has a number of very nice properties. The open mapping theorem and closed graph theorem hold for linear maps between DNF spaces. Furthermore, two ordinarily quite different ways of topologizing the tensor product of two topological vector spaces, the projective topology and the topology of uniform convergence on bi-equicontinuous sets, turn out to be equivalent if the spaces are DNF. This has a number of consequences. If $V \hat{\otimes} W$ denotes the completed topological tensor product of two DNF spaces using one of these equivalent topologies, then $V \hat{\otimes} W$ is an exact functor of each argument. Furthermore, this tensor product commutes with countable separated inductive limits in either argument. It follows that if either V or W is a vector space of at most countable dimension and the other space is a DNF space, then $V \otimes W$ is already complete under its tensor product topology. In particular, if both V and W are of at most countable dimension, then $V \hat{\otimes} W$ is just $V \otimes W$ with the strongest locally convex topology.

A complete topological algebra A is a complete locally convex topological vector space with an associative algebra structure determined by a continuous linear map $A \hat{\otimes} A \to A$. Similarly, a complete topological module over a topological algebra A is a complete locally convex topological vector space M with an A-module structure given by a continuous linear map $A \hat{\otimes} M \to M$. Topological algebras and modules will be complete (and, hence, Hausdorff) throughout this discussion.

It follows from the above considerations that the functor which assigns to a complex vector space that space with its strongest locally convex topology, takes any algebra of countable dimension to a DNF topological algebra and is an equivalence between the category of countably generated modules over such an algebra and the category of countably generated DNF topological modules over the resulting DNF topological algebra.

Thus, polynomial algebras, algebras of regular functions on an affine set, enveloping algebras of finite dimensional complex Lie algebras and algebras of algebraic differential operators on an affine set may all be considered to be DNF topological algebras. Then countably generated modules over such an algebra may be considered DNF topological modules. In particular, finitely generated modules may be so considered.

Less trivial examples include the algebra of holomorphic functions on a compact subset of a complex manifold (a DNF topological algebra) the module of sections over a compact set of a coherent analytic sheaf (a DNF module), the algebra of differential operators with holomorphic coefficients over a compact set (a DNF topological algebra) and the sections over a compact set of a coherent sheaf of modules over the sheaf of differential operators with holomorphic coefficients (a DNF topological module). Since the space of sections over a compact subset of the structure sheaf $\mathcal{O}$ of a CR manifold may also be regarded as the space of sections of the sheaf of holomorphic functions on a compact subset of a complex manifold (via a local embedding of the CR manifold into a complex manifold), this algebra is also a DNF algebra as is the corresponding algebra of differential operators.

A DNF sheaf is a sheaf of complex vector spaces such that the space of sections over each compact set is equipped with a DNF topology such that the restriction maps are all continuous. A sheaf morphism between DNF sheaves is continuous and, hence, is a morphism of DNF sheaves, if and only if it is continuous on each stalk ([HT] 3.2). DNF sheaves of algebras and DNF modules over a DNF sheaf of algebras are defined similarly. Obviously, a sheaf of complex vector spaces with the property that the space of sections over any compact subset is of countable dimension will automatically be a DNF sheaf if the space of sections over each compact set is given the strongest locally convex topology. The corresponding statements are true of sheaves of complex algebras and modules. Less trivially, the sheaf of germs of holomorphic (or CR) functions is a DNF sheaf of algebras, as is the sheaf of germs of holomorphic (or CR) differential operators and its twisted versions $\mathcal{D}_\lambda$. Coherent sheaves over either of these sheaves of algebras are DNF sheaves of modules.

Let Y be an analytic CR manifold and let $M(\mathcal{D})$ denote the category of sheaves of $\mathcal{D}$ modules on Y. Let $M(\mathcal{D}^t)$ denote the category of DNF sheaves of $\mathcal{D}$ modules on Y. Let $D^b(\mathcal{D})$ and $D^b(\mathcal{D}^t)$ denote the corresponding bounded derived categories. All the usual functors associated with sheaves of algebras (tensor product, direct and inverse image, etc.) are defined for $M(\mathcal{D})$ and $D^b(\mathcal{D})$. However, one has to be very careful working with $M(\mathcal{D}^t)$ and $D^b(\mathcal{D}^t)$ because of the need to preserve the DNF topology. Some of the standard functors are defined, some are not and some are defined only in certain circumstances. We briefly discuss these issues below.

The appropriate notion of tensor product for DNF sheaves is completed topological tensor product $\hat{\otimes}$. There is also a relative topological tensor product $\mathcal{M}\,\hat{\otimes}_{\mathcal{D}}\mathcal{N}$ for sheaves $\mathcal{M}$ and $\mathcal{N}$ of right (resp. left) $\mathcal{D}$ modules which, when defined, is the cokernel of the map

$$m \otimes \xi \otimes n \to m\xi \otimes n - m \otimes \xi n : \mathcal{M}\,\hat{\otimes}\,\mathcal{D}\,\hat{\otimes}\,\mathcal{N} \to \mathcal{M}\,\hat{\otimes}\,\mathcal{N}.$$

The problem is that this map does not always have a DNF module as cokernel. It does so if and only if the stalks of the cokernel are Hausdorff, that is, if the above map has closed image on each stalk. To know that this is so requires some special information about $\mathcal{M}$ or $\mathcal{N}$. It is so, for example, if $\mathcal{N}$ has the form $\mathcal{N} = \mathcal{D}\,\hat{\otimes}\,\mathcal{L}$, where $\mathcal{L}$ is a sheaf of DNF spaces. This is because $\mathcal{M}\,\hat{\otimes}_{\mathcal{D}}\mathcal{D}\,\hat{\otimes}\,\mathcal{L} = \mathcal{M}\,\hat{\otimes}\,\mathcal{L}$, (cf. [EP]). This, and the fact that $\hat{\otimes}$ is exact in each argument on the category of DNF sheaves, shows that modules of the form $\mathcal{D}\,\hat{\otimes}\,\mathcal{L}$ are flat as DNF $\mathcal{D}$ modules. Here, by a flat DNF $\mathcal{D}$ module we mean a module $\mathcal{N}$ so that $\mathcal{M}\,\hat{\otimes}_{\mathcal{D}}\mathcal{N}$ exists as a DNF sheaf for each DNF module $\mathcal{M}$ and is an exact functor of $\mathcal{M}$. Since multiplication provides a surjection $\mathcal{D}\,\hat{\otimes}\,\mathcal{M} \to \mathcal{M}$, it follows that every DNF $\mathcal{D}$ module $\mathcal{M}$ has a left resolution by flat DNF $\mathcal{D}$ modules. The fact that $\mathcal{D}$ has finite homological dimension implies that there is a finite length with such resolution. This leads to the conclusion that, although the relative tensor product does not always exist, its derived version $\mathcal{M}\,\hat{\otimes}^{L}_{\mathcal{D}}\mathcal{N}$ does exist for every object $\mathcal{M}$ in $D^b(\mathcal{D}^t)$ and every object $\mathcal{N}$ in the analogous category based on right modules.

If $f : Y \to Z$ is a map of CR manifolds and $\mathcal{M}$ a sheaf of DNF topological vector spaces on Y, then the sheaf theoretic direct image $f_*\mathcal{M}$ is not defined in general, although the direct image with proper supports, $f_!\mathcal{M}$, is always defined since the space of sections of $f_!\mathcal{M}$ over a compact subset of Z is an inductive limit of spaces of sections of $\mathcal{M}$ supported on compact subsets of Y.

Sheaf theoretic inverse image for DNF sheaves is always defined because continuous maps take compact sets to compact sets. However, if f is a map of CR manifolds, then the $\mathcal{O}$ module and $\mathcal{D}$ module inverse image $f^+\mathcal{M}$, since it is defined in terms of a relative topological tensor product, is only defined in situations where one can show that it leads to a Hausdorff topology on the stalks of the resulting sheaf. Fortunately, this happens in a wide variety of situations—in particular, when f is a fibration (see [Sm]). The derived inverse image functor, on the other hand, is always defined, since derived relative tensor product is always defined.

We denote by $FT : M(\mathcal{D}^t) \to M(\mathcal{D})$ the functor which forgets the topological vector space structure. This functor commutes with the sheaf theoretic operations f^{-1} and $f_!$ but not, in general, with $\mathcal{D}$ module inverse image f^+. The problem is that the notion of inverse image for $\mathcal{D}$ modules involves tensor product and the notions of tensor product used in $M(\mathcal{D}^t)$ and $M(\mathcal{D})$ are different. In particular, if $\mathcal{M}$ is a $\mathcal{D}$ module, and $f : Y \to Z$ a fibration, then the inverse image of $\mathcal{M}$ under f as an $\mathcal{O}$ module is

$$f^+\mathcal{M} = \mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_Z} f^{-1}\mathcal{M}.$$

This $\mathcal{O}$ module also has a natural $\mathcal{D}$ module structure, since it can also be described as (see section 1):

$$f^+\mathcal{M} = \mathcal{D}_{Y\to Z} \otimes_{f^{-1}\mathcal{D}_Z} f^{-1}\mathcal{M}.$$

With this structure, it is the $\mathcal{D}$ module inverse image of $\mathcal{M}$ under f. On the other hand, if $\mathcal{M}$ is a DNF $\mathcal{D}$ module, then its inverse image under f, if it exists, is

$$f^+\mathcal{M} = \mathcal{O}_Y \hat{\otimes}_{f^{-1}\mathcal{O}_Z} f^{-1}\mathcal{M} = \mathcal{D}_{Y\to Z} \hat{\otimes}_{f^{-1}\mathcal{D}_Z} f^{-1}\mathcal{M}$$

where $\hat{\otimes}$ denotes completed topological tensor product. For $f^+\mathcal{M}$ to exist this must be a DNF sheaf (the issue is whether the stalks are Hausdorff). This holds if f is a fibration.

The inclusion of the ordinary tensor product in the completed topological tensor product induces a morphism of $\mathcal{D}$ modules

$$\mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_Z} f^{-1}\mathcal{M} \to \mathcal{O}_Y \hat{\otimes}_{f^{-1}\mathcal{O}_Z} f^{-1}\mathcal{M}$$

and this defines a morphism of functors from $M(\mathcal{D}_Z^t)$ to $M(\mathcal{D}_Y)$:

$$\iota : f^+ \circ FT \to FT \circ f^+.$$

Since inverse image plays a central role in the definition of equivariant sheaves of modules and in the definition of the equivariant derived category, it is important to know that ι is an isomorphism for certain subcategories of modules. Fortunately, in the situations where we need this, f is a fibration and, hence, f^+ always exists and is exact (cf. [Sm]).

Lemma 5.1. *(a) If A is a DNF algebra which is Noetherian as an algebra and if M (resp. N) is a right (resp. left) DNF A module with N finitely generated, then the natural map $M \otimes_A N \to M \hat{\otimes}_A N$ is an isomorphism of vector spaces;*

(b) If $\mathcal{M}$ is a DNF $\mathcal{D}$ module on Z with finitely generated stalks and if $f : Y \to Z$ is a fibration of CR manifolds, then $\iota : f^+ \circ FT(\mathcal{M}) \to FT \circ f^+(\mathcal{M})$ is an isomorphism.

Proof. If $N = A \otimes F$ is a free finite rank A module, then $M \otimes_A N \to M \hat{\otimes}_A N$ is clearly an isomorphism since both objects are then naturally equal to $M \otimes F$. Since A is Noetherian, every finitely generated A module has a finite rank free resolution. Since both $M \otimes_A (\cdot)$ and $M \hat{\otimes}_A (\cdot)$ are right exact (on DNF modules), part (a) follows. Note that $M \hat{\otimes}_A N$ may not exist as a DNF topological vector space because it may not be Hausdorff. However, it does exist as a vector space and the conclusion of the lemma is still valid.

Clearly ι is an isomorphism for $\mathcal{M}$ if and only if it is an isomorphism stalkwise. Since both tensor product and completed topological tensor product commute with countable direct limits, the map ι on the stalk at $y \in Y$ of $\mathcal{M}$ is

$$(\mathcal{D}_{X\to Y})_y \otimes_{\mathcal{D}_z} \mathcal{M}_z \to (\mathcal{D}_{X\to Y})_y \hat{\otimes}_{\mathcal{D}_z} \mathcal{M}_z$$

where $z = f(y)$. Thus, (b) follows from (a) and the fact that $\mathcal{D}_z$ is Noetherian. $\square$

Since the definition of equivariant sheaf of $\mathcal{D}$ modules uses the inverse image functor for the two maps p and m from Section 2, this notion differs fundamentally in $M(\mathcal{D}^t)$ and $M(\mathcal{D})$. A module which is K or $G_{\mathbb{R}}$ equivariant as an object in $M(\mathcal{D}^t)$ may not be equivariant when considered an object in $M(\mathcal{D})$. The above lemma shows that the two notions of equivariance do agree for modules in $M(\mathcal{D}^t)$ with finitely generated stalks. That is, FT preserves equivariance for sheaves of modules with finitely generated stalks.

There are similar considerations for the equivariant derived categories. Recall that an object of $D^b_{G_{\mathbb{R}}}(\mathcal{D}^t_Y)$ is a functor which assigns to each $G_{\mathbb{R}}$ resolution $P \to Y$ of Y an object $\mathcal{M}(P)$ of $D^b(\mathcal{D}_{P_0})$ and to each morphism $f : P \to Q$ of resolutions an isomorphism $\alpha(f) : f^+\mathcal{M}(Q) \to \mathcal{M}(P)$ in a composition and identity preserving way. This definition is, of course, strongly dependent on the use of the inverse image functor and so objects will not be preserved, in general, by the functor FT. However,

Proposition 5.2. *Let $D^b_{K,fg}(\mathcal{D}^t_Y)$ (resp. $D^b_{K,fg}(\mathcal{D}_Y)$) be the full subcategory of $D^b_K(\mathcal{D}^t_Y)$ (resp. $D^b_K(\mathcal{D}_Y)$) of objects whose cohomology sheaves have finitely generated stalks. Then the forget topology functor defines a functor*

$$FT : D^b_{K,fg}(\mathcal{D}^t_Y) \to D^b_{K,fg}(\mathcal{D}_Y).$$

The analogous result holds with K replaced by $G_{\mathbb{R}}$.

Proof. Let $\mathcal{M}$ be an object of $D^b_{K,fg}(\mathcal{D}^t_X)$. In defining $\mathcal{M}$, we may restrict attention to K free resolutions $p : P \to X$ of X which are fibrations. Then, $\mathcal{M}$ assigns to each such resolution an object $\mathcal{M}(P)$ of $D^b(\mathcal{D}^t_{P_0})$. Now, the cohomology modules of $\mathcal{M}$ have finitely generated stalks. This condition is preserved by p^+ and so $p^+For(\mathcal{M})$ is an object of $D^b(\mathcal{D}^t_P)$ with cohomology sheaves with finitely generated stalks. However, if $q : P \to P_0$ is the projection, then $q^+\mathcal{M}(P)$ is isomorphic in $D^b(\mathcal{D}^t_P)$ to $p^+For(\mathcal{M})$ (see [BL], 2.4.3) and, hence, it also has cohomology sheaves with finitely generated stalks. It follows that the same is true of $\mathcal{M}(P)$, since it is the K equivariant direct image $q^K_* q^+\mathcal{M}(P)$ of $q^+\mathcal{M}(P)$ under q. That is, an object in $D^b_{K,fg}(\mathcal{D}^t_Y)$ may be represented by objects in the categories $D^b(\mathcal{D}^t_{P_0})$ which have cohomologies with finitely generated stalks. For such an object $\mathcal{M}$, and a morphism of resolutions $f : P \to Q$, Lemma 5.1 implies that $\iota : f_0^+ \circ FT(\mathcal{M}(Q)) \to FT \circ f_0^+(\mathcal{M}(Q))$ is a quasi-isomorphism of complexes of sheaves and, hence, an isomorphism in $D^b(\mathcal{D}_{P_0})$. It follows that $P \to FT\mathcal{M}(P)$ defines an object of $D^b_{K,fg}(\mathcal{D}_Y)$. The same argument applies if K is replaced by $G_{\mathbb{R}}$. $\square$

Generally, categories of sheaves do not have enough projectives but in most circumstances the existence of enough injectives makes up for this. However, the category of DNF sheaves over a DNF topological algebra has neither enough projectives nor enough injectives. In what follows we will get around this difficulty by the simultaneous use of two categories of sheaves of modules—one whose objects have some of the properties of projectives and another whose objects have some of the properties of injectives.

In what follows, $\mathcal{A}$ will be a DNF sheaf of algebras on a space Y. Sheaves of modules will be DNF sheaves of $\mathcal{A}$ modules. We a use $\mathcal{H}om_{\mathcal{A}}(\mathcal{M},\mathcal{N})$ to denote the sheaf of local homomorphisms from $\mathcal{M}$ to $\mathcal{N}$. We make extensive use of the properties of c-soft sheaves. The reader is referred to [KSc] Section 2.5 for a discussion of these properties.

Definition 5.3. *A sheaf of $\mathcal{A}$ modules $\mathcal{P}$ is called semi-projective if, whenever*

$$0 \to \mathcal{S} \to \mathcal{B} \to \mathcal{C} \to 0$$

is an exact sequence of sheaves of modules with $\mathcal{S}$ c-soft, then $\mathcal{H}om_{\mathcal{A}}(\mathcal{P},\cdot)$ preserves the exactness of the sequence.

Recall that, for a sheaf $\mathcal{M}$ on Y and Z a locally closed subset of Y, the sheaf $\mathcal{M}_Z$ is the unique sheaf which is $\mathcal{M}|_Z$ on Z and has stalk 0 at every point of the complement of Z.

Proposition 5.4. *If $\mathcal{P}$ is semi-projective and $\mathcal{S}$ is c-soft, then $\mathcal{H}om_{\mathcal{A}}(\mathcal{P},\mathcal{S})$ is c-soft.*

Proof. Let K be a compact set, set $U = Y - K$ and consider the exact sequence

$$0 \to \mathcal{S}_U \to \mathcal{S} \to \mathcal{S}_K \to 0.$$

Since $\mathcal{P}$ is semi-projective and $\mathcal{S}_U$ is also c-soft, the sequence

$$0 \to Hom_{\mathcal{A}}(\mathcal{P},\mathcal{S}_U) \to Hom_{\mathcal{A}}(\mathcal{P},\mathcal{S}) \to Hom_{\mathcal{A}}(\mathcal{P},\mathcal{S}_K) \to 0$$

is exact. Since

$$Hom_{\mathcal{A}}(\mathcal{P},\mathcal{S}) = \Gamma(Y,\mathcal{H}om_{\mathcal{A}}(\mathcal{P},\mathcal{S})) \quad \text{and} \quad Hom_{\mathcal{A}}(\mathcal{P},\mathcal{S}_K) = \Gamma(Y,\mathcal{H}om_{\mathcal{A}}(\mathcal{P},\mathcal{S}_K)),$$

each section of $\mathcal{H}om_{\mathcal{A}}(\mathcal{P},\mathcal{S}_K)$, that is each section of $\mathcal{H}om_{\mathcal{A}}(\mathcal{P},\mathcal{S})$ on K, extends to a section of $\mathcal{H}om_{\mathcal{A}}(\mathcal{P},\mathcal{S})$ on X. Thus, $\mathcal{H}om_{\mathcal{A}}(\mathcal{P},\mathcal{S})$ is c-soft. $\square$

Proposition 5.5. *If $\mathcal{P}$ is a sheaf of $\mathcal{A}$ modules which is locally a direct summand of a free sheaf of $\mathcal{A}$ modules, then $\mathcal{P}$ is semi-projective.*

Proof. Suppose $\mathcal{P}$ is locally free and let U be an open set on which $\mathcal{P}$ is free. Thus, $\mathcal{P} = \mathcal{A} \otimes F$ on U for some vector space F. Then $\mathcal{H}om_{\mathcal{A}}(\mathcal{P}|_U,\mathcal{S}|_U) = \mathcal{S}|_U \otimes F^*$. It follows that $\mathcal{H}om_{\mathcal{A}}(\mathcal{P},\cdot)$ preserves exactness of any short exact sequence of sheaves of $\mathcal{A}$ modules. Obviously we may draw the same conclusion if $\mathcal{P}$ is locally a direct summand of a free sheaf of $\mathcal{A}$ modules. This proves that $\mathcal{P}$ is semi-projective. $\square$

We will shortly show that, in the two situations where we need to prove the equivalence between equivariant derived categories defined with and without the DNF structure, objects in the heart of the derived category have semi-projective resolutions. The relevance of this is shown in the next result.

Recall that, for sheaves of $\mathcal{A}$ modules $\mathcal{M}$ and $\mathcal{N}$ the nth Ext group is defined in terms of Hom in the derived category by

$$Ext^n_{\mathcal{A}}(\mathcal{M},\mathcal{N}) = Hom_{D^b(\mathcal{A})}(\mathcal{M},\mathcal{N}[n]).$$

Proposition 5.6. *If $\mathcal{P}$ is a semi-projective $\mathcal{A}$ module and $\mathcal{S}$ is a c-soft $\mathcal{A}$ module, then $Ext^n_{\mathcal{A}}(\mathcal{P},\mathcal{S}) = 0$ for $n \neq 0$.*

Proof. We identify $\mathcal{P}$ and $\mathcal{S}$ with complexes, non-zero only in degree zero, representing the corresponding images in $D^b(\mathcal{A})$. Then an element of $Ext^n_{D^b(\mathcal{A})}(\mathcal{P},\mathcal{S})$ is a morphism $\mathbf{f}$ from $\mathcal{P}$ to $\mathcal{S}[n]$ in $D^b(\mathcal{A})$ and, as such, is represented by a pair

$$\mathcal{P} \xleftarrow{\ s\ } \mathcal{G} \xrightarrow{\ f\ } \mathcal{S}[n]$$

of morphisms in $K^b(\mathcal{A})$ with s a quasi-isomorphism. Since $\mathcal{G}$ is quasi-isomorphic to $\mathcal{P}$, it has non-vanishing cohomology only in degree zero and that is isomorphic to the DNF sheaf $\mathcal{P}$. Thus, both truncation functors are defined for $\mathcal{G}$ (in general, for a complex of DNF sheaves, the left truncation $\tau \geq c$ may not be defined since, because it involves a quotient which may fail to be Hausdorff, it may fail to yield a complex of DNF sheaves). Using truncation, we can show that the morphism $\mathbf{f}$ is equivalent to one in which $\mathcal{G}$ vanishes except in degrees between 0 and $-n$ inclusively.

First suppose n is a negative integer. Then the fact that $H^{-n}(\mathcal{G}) = 0$ implies that

$$\mathcal{G}^{-n} = \ker \delta^{-n} = \operatorname{im} \delta^{-n-1}.$$

But $\operatorname{im}\delta^{-n-1} \subset \ker(f)$ and so f must be zero. Thus, $Ext^n_{D^b(\mathcal{A})}(\mathcal{P},\mathcal{S})$ is also zero.

Now suppose n is positive. Then f maps the left most term $\mathcal{G}^{-n}$ of the complex $\mathcal{G}$ into $\mathcal{S}$ and is zero on the other terms, while δ^{-n} is a topological isomorphism of $\mathcal{G}^{-n}$ onto the closed subspace $\ker \delta^{-n+1}$ of $\mathcal{G}^{-n+1}$. We may factor the closed subspace $\ker f$ out of $\mathcal{G}^{-n}$ and the closed subspace $\delta^{-n} \ker f$ out of $\mathcal{G}^{-n+1}$ and obtain an equivalent representative of $\mathbf{f}$ in which f is injective. Thus, we may assume this was true already of f. Similarly, $\mathcal{G}^{-n}$ may be replaced by $\mathcal{S}$, $\mathcal{G}^{-n+1}$ by $(\mathcal{G}^{-n+1} \oplus \mathcal{S})/K$ where $K = \{(-\delta^{-n}(g), f(g)) : g \in \mathcal{G}^{-n}\}$, δ^{-n} by the natural inclusion of $\mathcal{S}$ in $(\mathcal{G}^{-n+1} \oplus \mathcal{S})/K$ and δ^{-n+1} by the natural map $(\mathcal{G}^{-n+1} \oplus \mathcal{S})/K \to \mathcal{G}^{-n+1}$ followed by δ^{-n+1} (note that K is closed because δ^{-n} is a topological isomorphism onto $\ker \delta^{-n+1}$). In this way, we obtain an equivalent representative of $\mathbf{f}$ in which f is an isomorphism of $\mathcal{G}^{-n}$ onto $\mathcal{S}$. We conclude that, without loss of generality, we may assume that $\mathcal{G}$ is a finite complex beginning with $\mathcal{S}$ and ending with $\mathcal{G}^0$, which is exact except at $\mathcal{G}^0$ where the cohomology is $\mathcal{P}$. Thus, the augmented sequence $\mathcal{G} \to P \to 0$ is an exact sequence. This is the classical description of an element of $Ext^n_{\mathcal{A}}(\mathcal{P},\mathcal{S})$ in terms of extensions.

With this description of $\mathcal{G}$, we now make a further reduction. We replace each $\mathcal{G}^k$ for $k > -n$ by a c-soft resolution ($\mathcal{G}^{-n} = \mathcal{S}$ is already c-soft). This can be done in a functorial way (using the Čech resolution of [HT]) so as to yield a double complex of c-soft sheaves whose total complex still has $\mathcal{S}$ in degree $-n$ as its initial non-zero term and which still has $\mathcal{P}$ in degree zero as its only non-zero cohomology. By truncating at zero, we obtain a complex $\mathcal{H}$ which is c-soft in every degree except zero, has $\mathcal{S}$ as initial term in degree $-n$, and has final term $\mathcal{H}^0$ which has $\mathcal{P}$ as quotient modulo the image of δ^{-1}. This complex $\mathcal{H}$, with the obvious maps to $\mathcal{S}$ and $\mathcal{P}$ yields an equivalent description of the morphism $\mathbf{f}$.

Note that the third term in a short exact sequence is c-soft if the other two are. Thus, because $\mathcal{P}$ is semi-projective and all but the last two terms of the augmented complex $\mathcal{H} \to \mathcal{P} \to 0$ are c-soft, applying $\mathcal{H}om_{\mathcal{A}}(\mathcal{P}, \cdot)$ to this complex preserves its exactness. But each of the sheaves $\mathcal{H}om_{\mathcal{A}}(\mathcal{P},\mathcal{H}^k)$ for $k > 0$ is also c-soft by Proposition 5.4. It follows by a standard induction argument on the length n of the complex $\mathcal{H}$ that the sequence remains exact when we pass to global sections.

Thus, $Hom_{\mathcal{A}}(\mathcal{P},\mathcal{H}^0) \to Hom_{\mathcal{A}}(\mathcal{P},\mathcal{P})$ is surjective. That is, $\mathcal{H}^0 \to P$ splits and $\mathbf{f}$ is the trivial element of $Ext^n_{\mathcal{A}}(\mathcal{P},\mathcal{S})$. ☐

We now return to the situation where $\mathcal{A} = \mathcal{D}$. That is, let Y be a quasi-projective smooth complex algebraic variety considered as a complex manifold and let $\mathcal{D}$ be the algebra of differential operators on Y with holomorphic coefficients. Let $D^b_{coh}(\mathcal{D})$ (resp. $D^b_{coh}(\mathcal{D}^t)$) denote the full subcategory of $D^b(\mathcal{D})$ (resp. $D^b(\mathcal{D}^t)$) consisting of objects whose cohomology sheaves are algebraically coherent. Here, by an *algebraically coherent* sheaf of $\mathcal{D}$ modules we mean the image under GAGA of a coherent $\mathcal{D}^{alg}$ module. By [Bo], VI 2.5 such a module has a finite left resolution by modules, all of which are locally free of finite rank except the last one and it is locally a direct summand of a free finite rank module. By Proposition 5.5, this is a finite resolution by semi-projective modules.

Proposition 5.7. *The forget topology functor*

$$FT : D^b_{coh}(\mathcal{D}^t) \to D^b_{coh}(\mathcal{D})$$

is an equivalence of categories.

Proof. Since each algebraically coherent $\mathcal{D}$-module has a well defined DNF topology under which it is a $\mathcal{D}^t$ module and since a $\mathcal{D}$ module morphism between coherent modules is automatically continuous, the hearts of the two derived categories of the proposition are the same. For each $n \in \mathbb{Z}$, the forgetful functor induces a natural morphism of bi-δ-functors

$$\phi_n : Ext^n_{\mathcal{D}^t}(\mathcal{M},\mathcal{N}) \to Ext^n_{\mathcal{D}}(\mathcal{M},\mathcal{N}). \tag{5.1}$$

We claim each ϕ_n is an isomorphism of functors on $M_{coh}(\mathcal{D}^t) \times M(\mathcal{D}^t)$. First suppose that $\mathcal{P}$ is a semi-projective module in $M(\mathcal{D}^t)$ with finitely generated stalks and $\mathcal{S}$ is a c-soft module in $M(\mathcal{D}^t)$. Then $Ext^n_{\mathcal{D}^t}(\mathcal{P},\mathcal{S}) = 0$ for $n \neq 0$ by Proposition 5.6. A similar but much simpler argument using the existence of enough injectives in $M(\mathcal{D})$ (and, hence, the existence of $RHom_{\mathcal{D}}$) shows that we also have $Ext^n_{\mathcal{D}}(\mathcal{P},\mathcal{S}) = 0$ for $n \neq 0$.

In the case $n = 0$, $Hom_{\mathcal{D}^t}(\mathcal{P},\mathcal{S}) \to Hom_{\mathcal{D}}(\mathcal{P},\mathcal{S})$ is an isomorphism because a $\mathcal{D}$ module morphism between from $\mathcal{P}$ to $\mathcal{S}$ is automatically continuous. The latter statement follows from the fact that the stalks of $\mathcal{P}$ are finitely generated and it is only necessary to check continuity on each stalk ([HT], Prop. 3.2). This proves that (5.1) is an isomorphism for each n in the case where $\mathcal{M}$ is semi-projective with finitely generated stalks and $\mathcal{N}$ is c-soft.

Now each algebraically coherent $\mathcal{D}$ module $\mathcal{M}$ has a finite length left resolution by modules which are both semi-projective and have finitely generated stalks. Also, each $\mathcal{D}^t$ module has a finite right resolution by c-soft DNF sheaves of modules ([HT], Prop. 3.5). Thus, an induction on the length of a semi-projective resolution of $\mathcal{M}$, using the long exact sequence for Ext, shows that (5.1) is also an isomorphism for each n if $\mathcal{M}$ is any algebraically coherent module and $\mathcal{N}$ is c-soft. Another induction, this time on the length of a c-soft resolution of $\mathcal{N}$, shows that (5.1) is an isomorphism if $\mathcal{M}$ is algebraically coherent and $\mathcal{N}$ is any $\mathcal{D}^t$ module.

The fact that (5.1) is an isomorphism for all n implies that FT is fully faithful on the triangulated subcategory of $D^b(\mathcal{D}^t)$ generated by the algebraically coherent modules. This subcategory is $D^b_{coh}(\mathcal{D}^t)$. Since $FT : D^b_{coh}(\mathcal{D}^t) \to D^b_{coh}(\mathcal{D})$ is also essentially onto, the proof is complete. ☐

The above result will lead to the equivalence we need in the case of the K equivariant derived category. For the $G_{\mathbb{R}}$ equivariant derived category, we need a similar result but with the algebraically coherent modules replaced by modules of the type that occur in the heart of the finite type $G_{\mathbb{R}}$ equivariant derived category. These are modules which are locally free of finite rank as $\mathcal{O}$ modules on each strata in the orbit stratification of X. Actually, we need to work with a fibered category which assigns to each free $G_{\mathbb{R}}$ resolution $Y \to X$ a category of modules of this type on Y. This leads to the following definition:

Given an analytic CR manifold Y, we denote by $\mathcal{F}_Y$ the space of $\mathcal{D}_Y$ modules which are locally free of finite rank as $\mathcal{O}_Y$ modules on each set in some finite stratification of Y by CR submanifolds.

Proposition 5.8. *If Z is a locally closed CR submanifold of a CR manifold Y, then a $\mathcal{D}_Y$ module $\mathcal{M}$ on Z which is locally free of finite rank as an $\mathcal{O}_Y$ module on Z is the restriction to Z of a $\mathcal{D}_Y$ module, locally free of finite rank as an $\mathcal{O}_Y$ module, on a neighborhood of Z in Y.*

Proof. We choose a locally finite cover $\{V_i\}$ of Z by open sets in Z such that $\mathcal{M}$ is trivial on a neighborhood of the compact closure $\bar{V}_i$ of V_i. For each i, we choose a basis $\{e^i_\nu\}_\nu$ for $\mathcal{M}$ over a neighborhood of $\bar{V}_i$ in Z and set $e^i = (e^i_1, \cdots, e^i_\nu)$. Then, for each pair (i,j) we have a matrix ϕ_{ij}, with entries that are sections of $\mathcal{O}_Y$ defined over a neighborhood in Z of $\bar{V}_i \cap \bar{V}_j$ by the equations

$$e^i = \phi_{ij} e^j.$$

Each ϕ_{ij} is a non-singular matrix valued function in a neighborhood of $\bar{V}_i \cap \bar{V}_j$ in Z. Since they are entries which are sections of $\mathcal{O}_Y$, the ϕ_{ij} are actually defined on a neighborhood W_{ij} of $\bar{V}_i \cap \bar{V}_j$ in Y and they may be assumed to form a non-singular matrix valued function over W_{ij} as well. These functions satisfy the cocycle condition

$$\phi_{ik} = \phi_{ij} \circ \phi_{jk}$$

as sections of $Gl_n(\mathcal{O}_Y)$ on a neighborhood of $\bar{V}_i \cap \bar{V}_j \cap \bar{V}_k$ in Z for each triple (i,j,k). Since, this is an identity between sections of $Gl_n(\mathcal{O}_Y)$, it remains true in a neighborhood Ω_{ijk} of $\bar{V}_i \cap \bar{V}_j \cap \bar{V}_k$ in Y.

We can now choose a neighborhood U_i of each $\bar{V}_i$ in such a way that $U_i \cap U_j \subset W_{ij}$. That this can be done for a fixed pair (i,j) follows from the fact that $\bar{V}_i \cap \bar{V}_j$ is compact and the intersection of the sets $U_i \cap U_j$ over all possible choices of U_i and U_j is $\bar{V}_i \cap \bar{V}_j$. However, this gives a possibly different choice of U_i for each j. But, because the cover $\bar{V}_i$ is locally finite, we can first choose a collection of neighborhoods U'_i of $\bar{V}_i$ with the property that, for each i there are at most finitely many $j's$ for which $U'_i \cap U'_j \neq \emptyset$. We then need only look at the possible choices of U_i and U_j for finitely many indices j for each i. We then take for U_i the intersection of the choices that work for each such j. It follows in the same way that we can modify our choice of U_i for each i in such a way that $U_i \cap U_j \cap U_k \subset \Omega_{ijk}$ for each triple (i,j,k). With this choice of the sets U_i, the cocycle condition is satisfied on each triple intersection $U_i \cap U_j \cap U_k$. The data $(\{U_i\}, \{\phi_{ij}\})$ defines a CR vector bundle on the union U of the sets U_i whose sheaf of local sections is an extension to U of $\mathcal{M}$ as a $\mathcal{O}_Y$ module.

If ξ is a section of $\mathcal{D}_Y$ over a neighborhood V in Z on which $\mathcal{M}$ is free with basis $\{e_i\}$, then each ξe_i is a section of $\mathcal{O}_Y$ over V and, hence, it extends to a neighborhood of V in Y. In this way, we see that the action of ξ on $\mathcal{M}|_V$ extends to a neighborhood of V in Y. Furthermore, any two such extensions must agree in a neighborhood of V in Y. Since, $\mathcal{O}_Y$ and finitely many vector fields generate $\mathcal{D}$, the entire action of $\mathcal{D}$ extends to a neighborhood of V and any two extensions agree on a neighborhood. Now, arguing as in the preceding paragraphs, using a locally finite cover by open sets on which $\mathcal{M}$ is free, we conclude that the action of $\mathcal{D}_Y$ on $\mathcal{M}$ extends to a neighborhood of Z in Y.

Note that this proof extends easily to cover the case of sheaves of modules over the twisted sheaf of differential operators $\mathcal{D}_\lambda$. The theorem, as stated, has simpler proofs (for example, use the flat sections functor to reduce to the problem of extending a locally constant sheaf and then use the fact that Z is a neighborhood retract to extend this locally constant sheaf), however, it is not clear that they extend to the twisted case. □

Proposition 5.9. *A $\mathcal{D}$ module on an open set U, which is locally free of finite rank as an $\mathcal{O}$ module, has a finite resolution on U consisting of locally free finite rank $\mathcal{D}$ modules.*

Proof. If $\mathcal{M}$ is such a module, then $\mathrm{Diff}(\mathcal{O},\mathcal{M})$ is a left $\mathcal{D}$ module and a right $\mathcal{D}$-module. Evaluation at the identity gives a surjective map $\mathrm{Diff}(\mathcal{O},\mathcal{M}) \to \mathcal{M}$ with kernel equal to the image of $\delta : \mathrm{Diff}(\mathcal{O},\mathcal{M}) \otimes_{\mathcal{O}} \mathcal{T} \to \mathrm{Diff}(\mathcal{O},\mathcal{M})$ where $\mathcal{T}$ is the sheaf of vector fields and $\delta(\eta \otimes \xi) = \eta \circ \xi$. In fact, this extends in the usual way to a complex (the Koszul complex for the right action of vector fields on $\mathrm{Diff}(\mathcal{O},\mathcal{M})$) which gives a finite resolution of $\mathcal{M}$. It is not difficult to see that this is a complex of locally free finite rank left $\mathcal{D}$ modules. □

Proposition 5.10. *If P is a semi-projective sheaf of $\mathcal{A}$ modules, then so is P_V for any open set V.*

Proof. Let P be semi-projective and let V be an open set. Suppose

$$0 \to F_1 \to F_2 \to F_3 \to 0$$

is a short exact sequence with F_1 c-soft. Then

$$0 \to \mathcal{H}om_{\mathcal{A}}(P,F_1) \to \mathcal{H}om_{\mathcal{A}}(P,F_2) \to \mathcal{H}om_{\mathcal{A}}(P,F_3) \to 0$$

is exact. This implies that, for any open set W we have

$$0 \to \Gamma(W,\mathcal{H}om_{\mathcal{A}}(P,F_1)) \to \Gamma(W,\mathcal{H}om_{\mathcal{A}}(P,F_2)) \to \Gamma(W,\mathcal{H}om_{\mathcal{A}}(P,F_3)) \to 0$$

is exact because $\mathcal{H}om_{\mathcal{A}}(P,F_1)$ is c-soft by Proposition 5.4. We use this in the case where $W = U \cap V$ for an open set U and observe that there are natural identifications $\Gamma(U \cap V,\mathcal{H}om_{\mathcal{A}}(P,F_i)) \simeq \Gamma(U,\mathcal{H}om_{\mathcal{A}}(P_V,F_i))$. Hence,

$$0 \to \Gamma(U,\mathcal{H}om_{\mathcal{A}}(P_V,F_1)) \to \Gamma(U,\mathcal{H}om_{\mathcal{A}}(P_V,F_2)) \to \Gamma(U,\mathcal{H}om_{\mathcal{A}}(P_V,F_3)) \to 0$$

is exact. This proves that P_V is semi-projective. □

Proposition 5.11. *Let Z be as in Proposition 5.8 and let $\mathcal{M}$ be a $\mathcal{D}_Y$ module which is supported on Z and is locally free of finite rank as an $\mathcal{O}_Y$ module on Z. Then $\mathcal{M}$ has a finite resolution by semi-projective modules with finitely generated stalks.*

Proof. We extend the module $\mathcal{M}$ to a similar module $\mathcal{N}$ on open set U using Proposition 5.8 and then choose a finite resolution of $\mathcal{N}$ by locally free $\mathcal{D}$ modules on U using Proposition 5.9. We extend this to all of Y by extension by zero ($i_!$ for

$i : U \to Y$ the inclusion). The result is still semi-projective (argue as in Proposition 5.10). If $\mathcal{P} \to i_! \mathcal{N}$ is the resulting resolution of $i_! \mathcal{N}$, then $\mathcal{P}_Z \to \mathcal{M}$ is a resolution of the original module $\mathcal{M}$ supported on Z. The complex $\mathcal{P}_Z$ is not a complex of semi-projectives, but it has a resolution that is, namely $\mathcal{P}_{U-Z} \to \mathcal{P} \to \mathcal{P}_Z$. Thus, the total complex of the double complex $\mathcal{P}_{U-Z} \to \mathcal{P}$ is a finite complex of semi-projective sheaves with finitely generated stalks which gives the desired resolution of $\mathcal{M}$. □

With this result in hand, the proof of Proposition 5.7 goes through with sheaves in $\mathcal{F}$ replacing algebraically coherent sheaves (one additional step is necessary; however, one must use the fact that each object in $\mathcal{F}$ has a finite filtration with subquotients each of which is supported on a CR submanifold on which it is locally free of finite rank). Thus, with $D^b_{\mathcal{F}}(\mathcal{D}^t_Y)$ denoting the full subcategory of $D^b(\mathcal{D}^t_Y)$ consisting of objects in $\mathcal{F}_Y$ and similarly for $D^b_{\mathcal{F}}(\mathcal{D})$, we have

Proposition 5.12. *The forget topology functor*

$$FT : D^b_{\mathcal{F}}(\mathcal{D}^t_Y) \to D^b_{\mathcal{F}}(\mathcal{D}_Y)$$

is an equivalence of categories.

We can now state and prove the equivalences that were used in Section 4.

Theorem 5.13. *The forget topology functors*

$$FT : D^b_{K,f}(\mathcal{D}^t_{\lambda,X}) \to D^b_{K,f}(\mathcal{D}_{\lambda,X}); \tag{1}$$

$$FT : D^b_{K,f}(\mathcal{U}^t_0(\mathfrak{g})) \to D^b_{K,f}(\mathcal{U}_0(\mathfrak{g})); \tag{2}$$

and

$$FT : D^b_{G_{\mathbb{R}},f}(\mathcal{D}^t_{\lambda,X}) \to D^b_{G_{\mathbb{R}},f}(\mathcal{D}_{\lambda,X}) \tag{3}$$

are equivalences of categories.

Proof. To simplify the exposition, we omit the twist λ but the proof of this result and the preceding results of this section work equally well in the λ twisted case. The proofs of the three equivalences follow the same model. For (1) we observe that the hearts of $D^b_{K,f}(\mathcal{D}^t_X)$ and $D^b_{K,f}(\mathcal{D}_X)$ both consist of the category of algebraic K equivariant coherent $\mathcal{D}_X$ modules (equipped with the natural DNF topology in the case of $D^b_{K,f}(\mathcal{D}^t_X)$). Each such module is algebraic regular holonomic. Let $\mathcal{M}$ be an object of $D^b_{K,f}(\mathcal{D}^t_X)$. In defining $\mathcal{M}$, we may restrict attention to K free quasi-projective algebraic resolutions $P \to X$ of X. Then, $\mathcal{M}$ assigns to each such resolution $P \to X$, an object $\mathcal{M}(P)$ of $D^b(\mathcal{D}^t_{P_0})$. Now, the cohomology modules of $\mathcal{M}$ belong to the heart of $D^b_{K,f}(\mathcal{D}^t_X)$. Hence, they are algebraic regular holonomic. This means that the image of $\mathcal{M}$ under the forgetful functor $\mathrm{For} : D^b_{K,f}(\mathcal{D}^t_X) \to D^b(\mathcal{D}^t_X)$ has algebraic regular holonomic cohomology. Regular holonomicity is preserved by inverse image and so if $p : P \to X$ is a free resolution of X, then $p^+ \mathrm{For}(\mathcal{M})$ is an object of $D^b(\mathcal{D}^t_P)$ with algebraic regular holonomic cohomology sheaves. However, if $q : P \to P_0$ is the quotient map, then $q^+\mathcal{M}(P)$ is isomorphic in $D^b(\mathcal{D}^t_P)$ to $p^+ \mathrm{For}(\mathcal{M})$ and, hence, it also has algebraic regular holonomic cohomology. It follows that the same is true of $\mathcal{M}(P)$, since it is the K equivariant direct image $q^K_* q^+\mathcal{M}(P)$ of $q^+\mathcal{M}(P)$ under q. In particular, these cohomologies are algebraic coherent. Here we are using the fact that $q^K_* = q_+$ and so q^K_* takes regular holonomic modules to regular holonomic modules (cf. [Bo]

Chapter 7). That is, each of the objects $\mathcal{M}(P)$, defining $\mathcal{M}$, belong to the full subcategory $D^b_{coh}(\mathcal{D}^t_{P_0})$ of $D^b(\mathcal{D}^t_{P_0})$. For the same reasons, the same thing is true of the categories without the superscript "t". Equivalence (1) of the theorem now follows from Proposition 5.7.

We next prove (3) analogously, using Proposition 5.12. In this case the hearts of the two categories are both equal to the $G_{\mathbb{R}}$ equivariant sheaves in $\mathcal{F}_X$, where recall that, for an analytic CR manifold Y, $\mathcal{F}_Y$ is the space of $\mathcal{D}^t$ modules which are locally free of finite rank as $\mathcal{O}_Y$ modules on each set in some finite stratification of Y by locally closed CR submanifolds. To proceed as in the previous paragraph, we need to observe two things: (a) If $p : P \to X$ is a resolution of X by a free $G_{\mathbb{R}}$ space, then the inverse image under p of a $G_{\mathbb{R}}$ equivariant sheaf in $\mathcal{F}_X$ is a $G_{\mathbb{R}}$ equivariant sheaf in $\mathcal{F}_P$. (b) The $G_{\mathbb{R}}$ equivariant direct image of such a sheaf under $q : P \to P_0$ belongs to $\mathcal{F}_{P_0}$. Then it follows as in the previous paragraph that an object $\mathcal{M}$ in $D^b_{G_{\mathbb{R}},f}(\mathcal{D}^t_X)$ has the property that each $\mathcal{M}(P)$ belongs to $D^b_{\mathcal{F}}(\mathcal{D}^t_{P_0})$. The analogous statement holds for objects of $D^b_{G_{\mathbb{R}},f}(\mathcal{D}_X)$ and so the second statement of the theorem follows from Proposition 5.12.

(2) The heart of the two categories in this case is the category of finitely generated K equivariant $\tilde{\mathcal{D}}_{pt} = \mathcal{U}_0(\mathfrak{g})$ modules. Note that $\mathcal{U}_0(\mathfrak{g})$ is Noetherian and of finite homological dimension and so such modules have finite rank, free resolutions of finite length. For any resolution $P \to pt$ of a point, the sheaves of algebras $\tilde{\mathcal{D}}_P$ and $\tilde{\mathcal{D}}_{P_0}$ are locally just tensor products of $\mathcal{U}_0(\mathfrak{g})$ with $\mathcal{D}_P$ (resp. $\mathcal{D}_{P_0}$). These are naturally DNF sheaves, since as $\mathcal{O}$ modules they are free of countable dimension. For each CR manifold Y let $\mathcal{G}_Y$ denote the category of $\tilde{\mathcal{D}}_Y$ modules which are locally free of countable dimension as $\mathcal{O}$ modules. If $p : P \to pt$ is a K free resolution of a point, and M is a K equivariant $\tilde{\mathcal{D}}_{pt} = \mathcal{U}_0(\mathfrak{g})$ module, then p^+M is just $\mathcal{O}_P \otimes M$ and, hence, is a K equivariant object in $\mathcal{G}_P$. If $q : P \to P_0$ is the quotient map, then q^K_* maps K equivariant modules in $\mathcal{G}_P$ to objects in $\mathcal{G}_{P_0}$. It follows that an object $\mathcal{M}$ of $D^b_{K,f}(\tilde{\mathcal{D}}_X)$ has the property that for each resolution P, the object $\mathcal{M}(P)$ lies in the full subcategory $D^b_{\mathcal{G}}(\tilde{\mathcal{D}}_{P_0})$ of $D^b_{K,f}(\tilde{\mathcal{D}}_X)$ consisting of objects with cohomology in $\mathcal{G}_{P_0}$. As in the proof of Proposition 5.9, objects in $\mathcal{G}$ have finite free $\tilde{\mathcal{D}}_\lambda$ module resolutions. Hence, the proof of Proposition 5.12 goes through with $\mathcal{D}$ replaced by $\tilde{\mathcal{D}}_\lambda$ and with $\mathcal{F}$ replaced by $\mathcal{G}$. Thus, on each CR manifold Y, the forget topology functor yields an equivalence

$$FT : D^b_{\mathcal{G}}(\mathcal{D}^t) \to D^b_{\mathcal{G}}(\mathcal{D}).$$

Equivalence (2) follows from this and the preceding discussion.

Finally, we mention that the same proof given for (2) above establishes that

$$FT : D^b_{G_{\mathbb{R}},fg}(\mathcal{U}^t_0(\mathfrak{g})) \to D^b_{G_{\mathbb{R}},fg}(\mathcal{U}_0(\mathfrak{g}))$$

is an equivalence but the full subcategory consisting of objects with finitely generated cohomology modules is not large enough to be of interest here. In particular, minimal globalizations of Harish-Chandra modules are not generally finitely generated as $\mathcal{U}_0(\mathfrak{g})$ modules. □

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