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$G(F_q)$-INVARIANTS IN IRREDUCIBLE $G(F_{q^2})$-MODULES

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ABSTRACT. We give an explicit formula for the dimension of the space of $G(F_q)$-invariant vectors in an irreducible complex representation of $G(F_{q^2})$, where G is a connected reductive algebraic group defined over a finite field F_q with connected center.

INTRODUCTION

0.1. Let G be a connected reductive group defined over a finite field F_q and let F_{q^2} be a quadratic extension of F_q. The finite group $G(F_{q^2})$ contains $G(F_q)$ as a subgroup. For any irreducible representation ρ of $G(F_{q^2})$ we denote by $\boxed{\rho}$ the dimension of the space of $G(F_q)$-invariant vectors in ρ. The function $\rho \mapsto \boxed{\rho}$ has been studied by Gow [GO], Kawanaka [KA], and Prasad [PR]. Gow assumes G to be GL_n and shows that in this case, $\boxed{\rho} \in \{0,1\}$. Prasad shows that $\boxed{\rho} \in \{0,1\}$ assuming that the character of ρ is constant on the intersection of $G(F_{q^2})$ with any conjugacy class of G (but he does not need G to be reductive); this includes Gow's result as a special case. Kawanaka computes $\boxed{\rho}$ assuming that either G is a classical group with connected center or that ρ is unipotent and the characteristic is good. He gives separately the formula in each case that he discusses but he does not give a closed formula for $\boxed{\rho}$ valid in all cases.

The purpose of this paper is to provide such a closed formula (see Theorem 1.4) valid for any G that has connected center and is simple modulo its center. (The formula makes sense and is expected to hold without assumptions on the center.) In §7 we formulate a variant of this formula for character sheaves. In §8 we extend one of the qualitative features of the formula to general symmetric spaces.

For the benefit of the reader we have included proofs even for results which could be found in [KA]; these proofs are such that they extend to the more general setting of this paper.

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1. Notation and statement of results

1.1. Let k be an algebraic closure of a finite field F_q with $|F_q| = q$. Let F_{q^2} be the subfield of k such that $|F_{q^2}| = q^2$.

Let G be a connected reductive group over k with a fixed F_q-rational structure. Let $F : G \to G$ be the Frobenius map corresponding to the F_q-structure. Let $\mathcal{G}$ be a connected reductive group defined over F_q which is *dual* to G (as in [DL, 5.21]). The Frobenius map of $\mathcal{G}$ is denoted again by F.

We fix a prime number l not dividing q. If Γ is a group, a representation of Γ (or Γ-module) is always assumed to be over $\bar{\mathbf{Q}}_l$ and to factor through a finite quotient of Γ. Let $\mathrm{Irr}\Gamma$ be the set of isomorphism classes of irreducible representations of Γ, or a set of representatives for these isomorphism classes.

Following Frobenius and Schur [FS] define $\psi : \mathrm{Irr}\Gamma \to \{-1, 0, 1\}$ by $\psi(E) = 1$ (resp. $\psi(E) = -1$) if E admits a non-degenerate Γ-invariant symmetric (resp. symplectic) inner product and $\psi(E) = 0$ if E is not self dual.

When Γ is finite we set $\check{\Gamma} = \mathrm{Hom}(\Gamma, \bar{\mathbf{Q}}_l^*)$.

Clearly, G^F is a subgroup of G^{F^2}. For any class function $\phi : G^{F^2} \to \bar{\mathbf{Q}}_l$ we set

$$\boxed{\phi} = |G^F|^{-1} \sum_{g \in G^F} \phi(g) \in \bar{\mathbf{Q}}_l.$$

If ϕ is the character of a G^{F^2}-module E, then $\boxed{\phi}$ is just the dimension of the space of G^F-invariants in E. We will often identify a G^{F^2}-module (or a formal $\bar{\mathbf{Q}}_l$-linear combination of irreducible G^{F^2}-modules) with its character $G^{F^2} \to \bar{\mathbf{Q}}_l$.

For any algebraic group H we denote by H^0 the identity component of H.

If T is a maximal torus of a connected reductive algebraic group H we denote by $N_H(T)$ the normalizer of T in H and we set $W_H(T) = T\backslash N_H(T)$; this acts on T by conjugation.

If Γ' is a group, Γ is a subgroup of Γ' and $g' \in \Gamma'$, we set

$$Z_\Gamma(g') = \{g \in \Gamma | gg' = g'g\}.$$

The center of Γ is denoted by Z_Γ. If f is a map of a set X into itself, we set $X^f = \{x \in X | f(x) = x\}$. The cardinal of a finite set X is denoted by $|X|$ or $\sharp X$.

1.2. Assume that G has connected center. Let s be a semisimple element in $\mathcal{G}^{F^2}$. The centralizer $Z(s)$ of s in $\mathcal{G}$ is connected and F^2-stable. We can find a pair $\mathcal{T}, B$ where B is an F^2-stable Borel subgroup of $Z(s)$ and $\mathcal{T}$ is an F^2-stable maximal torus of B. We write W_s instead of $W_{Z(s)}(\mathcal{T})$. Then W_s is naturally a (finite) Coxeter group with a set of simple reflections S determined in the standard way by B. Now F^2 acts naturally on W_s and $F^2(S) = S$.

Let $\ddot{W}_s$ be the semidirect product of W_s with the infinite cyclic group with generator δ so that in $\ddot{W}_s$ we have the identity $\delta x \delta^{-1} = F^2(x)$ for all $x \in W_s$. Then W_s is naturally a subgroup of $\ddot{W}_s$. Let $\mathrm{Irr}''W_s$ be the set of all $E \in \mathrm{Irr}W_s$ such that there exists a $\ddot{W}_s$-module $\ddot{E}$ whose restriction to W_s is isomorphic to E. By

[L1, 3.2], for $E \in \mathrm{Irr}''W_s$ one can find an $\ddot{E}$ which is defined over $\mathbf{Q}$; moreover, there are exactly two such $\ddot{E}$ (up to isomorphism); one is obtained from the other by replacing the action of δ by that of $-\delta$. In the sequel, $\ddot{E}$ refers to one of these two extensions of E.

Now $\mathrm{Irr}W_s$ is naturally partitioned into subsets called the *families* of W_s (see [L1, 4.2]). The bijection $F^2 : W_s \to W_s$ induces a bijection F^2 of the set of families of W_s with itself. According to [L1, 6.17(i)] we have a natural partition

$$\mathrm{Irr}G^{F^2} = \bigsqcup_{s,\mathcal{F}} \mathrm{Irr}_{s,\mathcal{F}} G^{F^2}$$

where s runs through a set of representatives for the conjugacy classes of semisimple elements in G^{F^2} and $\mathcal{F}$ runs through the set of F^2-stable families of W_s.

Let $s \in \mathcal{G}^{F^2}$ be semisimple and let $\mathcal{F}$ be an F^2-stable family of W_s. Let Γ be the finite group attached to $W_s, \mathcal{F}$ in [L1, §4] (where the notation $\mathcal{G}_{\mathcal{F}}$ is used instead of Γ). Since $F^2(\mathcal{F}) = \mathcal{F}$, we have an induced automorphism $F^2 : \Gamma \to \Gamma$. Let $\ddot{\Gamma}$ be the semidirect product of Γ with the infinite cyclic group with generator δ so that in $\ddot{\Gamma}$ we have the identity $\delta x \delta^{-1} = F^2(x)$ for all $x \in \Gamma$. Then Γ is naturally a subgroup of $\ddot{\Gamma}$.

Consider the set of all pairs (x, ξ) where $x \in \Gamma\delta \subset \ddot{\Gamma}$ and $\xi \in \mathrm{Irr}Z_\Gamma(x)$. On this set we have an action of $\ddot{\Gamma}$ by conjugation; the set of orbits of this action is denoted by $\overline{\mathcal{M}}_{\Gamma,F^2}$. Now [L1, 4.23] provides a bijection

(a) $$\mathrm{Irr}_{s,\mathcal{F}} G^{F^2} \leftrightarrow \overline{\mathcal{M}}_{\Gamma,F^2}.$$

Let $\rho_{x,\xi}$ correspond to (x, ξ) under (a).

1.3. In the special case where $F(s) = s^{-1}$, we note that $Z(s)$ is F-stable and we assume that $\mathcal{T}, B$ are chosen so that B is an F-stable Borel subgroup of $Z(s)$ and $\mathcal{T}$ is an F-stable maximal torus of B. In this case F acts naturally on W_s and $F(S) = S$. Let $\dot{W}_s$ be the semidirect product of W_s with the infinite cyclic group with generator γ so that in $\dot{W}_s$ we have the identity $\gamma x \gamma^{-1} = F(x)$ for all $x \in W_s$. Then $\ddot{W}_s$ is naturally a subgroup of $\dot{W}_s$. (The element $\delta \in \ddot{W}_s$ corresponds to $\gamma^2 \in \dot{W}_s$.) Let $\mathrm{Irr}'W_s$ be the set of all $E \in \mathrm{Irr}W_s$ such that there exists a $\dot{W}_s$-module $\dot{E}$ whose restriction to W_s is isomorphic to E. By [L1, 3.2], for $E \in \mathrm{Irr}'W_s$ one can find an $\dot{E}$ that is defined over $\mathbf{Q}$; moreover, there are exactly two such $\dot{E}$ (up to isomorphism); one is obtained from the other by replacing the action of γ by that of $-\gamma$. In the sequel, $\dot{E}$ refers to one of these two extensions of E. We have clearly $\mathrm{Irr}'W_s \subset \mathrm{Irr}''W_s$. Moreover, if $E \in \mathrm{Irr}'W_s$, then $\ddot{E}$ can be taken to be the restriction of $\dot{E}$ to $\ddot{W}_s$. This $\ddot{E}$ is independent of the choice of $\dot{E}$ (since $(-\gamma)^2 = \gamma^2$) hence in this case we have a canonical $\ddot{E}$.

Let $\mathcal{F}$ be a family of W_s such that $F(\mathcal{F}) = \mathcal{F}$; let Γ be as in 1.2. Since $F(\mathcal{F}) = \mathcal{F}$, we have an induced automorphism $F : \Gamma \to \Gamma$. Let $\dot{\Gamma}$ be the semidirect product of Γ with the infinite cyclic group with generator γ so that in $\dot{\Gamma}$ we have the identity $\gamma x \gamma^{-1} = F(x)$ for all $x \in \Gamma$. We identify $\ddot{\Gamma}$ in 1.2 with a subgroup of $\dot{\Gamma}$ by $x \mapsto x$ for $x \in \Gamma$ and $\delta \mapsto \gamma^2$.

Let (x, ξ) be a pair representing an element of $\overline{\mathcal{M}}_{\Gamma,F^2}$. Let

$$\sqrt{x} = \{x' \in \dot{\Gamma} | x'^2 = x\}.$$

Now $Z_\Gamma(x)$ acts on the set $\sqrt{x}$ by conjugation in $\dot\Gamma$. The corresponding permutation representation of $Z_\Gamma(x)$ is denoted again by $\sqrt{x}$. Let $[\xi : \sqrt{x}]$ be the multiplicity of ξ in this permutation representation.

Theorem 1.4. *Assume that Z_G is connected and that G/Z_G is simple. Let s be a semisimple element of $\mathcal{G}^{F^2}$ and let $\mathcal{F}$ be an F^2-stable family of W_s. Let $\rho = \rho_{x,\xi} \in \mathrm{Irr}_{s,\mathcal{F}} G^{F^2}$; see 1.2.*

(a) Assume that s is not conjugate under $\mathcal{G}^{F^2}$ to an element s' such that $F(s') = s'^{-1}$. Then $\boxed{\rho} = 0$.

(b) Assume that $F(s) = s^{-1}$ but $F(\mathcal{F}) \neq \mathcal{F}$. Then $\boxed{\rho} = 0$.

(c) Assume that $F(s) = s^{-1}, F(\mathcal{F}) = \mathcal{F}$ and that $|\mathcal{F}| \neq 2$. Then $\boxed{\rho} = [\xi : \sqrt{x}]$.

(d) Assume that $F(s) = s^{-1}, F(\mathcal{F}) = \mathcal{F}$ and that $|\mathcal{F}| = 2$. If $x = \gamma^2$, then $\boxed{\rho} = 1$. If $x \neq \gamma^2$, then $\boxed{\rho} = 0$.

See the introduction for the relation of this theorem to the earlier work of Kawanaka. The strategy of the proof is explained in 2.14. The proof is given in §§2, 3, and 5.

1.5. Let $s \in \mathcal{G}^{F^2}$ be semisimple and let $\mathcal{F}$ be an F^2-stable family of W_s. Let Γ be attached to $s, \mathcal{F}$ as in 1.2. Consider the set of all pairs (y, η) where $y \in \Gamma$ is such that $Z_{\ddot\Gamma}(y)$ meets $\Gamma\gamma^2$ and $\eta \in \mathrm{Irr} Z_{\ddot\Gamma}(y)$ is such that $\eta|_{Z_\Gamma(y)}$ is irreducible. On this set we have an action of $\ddot\Gamma$ by conjugation; the set of orbits of this action is denoted by $\mathcal{M}_{\Gamma,F^2}$. Let M be the group of all roots of 1 in $\bar{\mathbf{Q}}_l^*$. For $\alpha \in M$, let $\epsilon_\alpha \in (\ddot\Gamma)\check{}$ be defined by $\epsilon_\alpha(\gamma^2) = \alpha, \epsilon_\alpha|_\Gamma = 1$. The restriction of ϵ_α to a subgroup of $\ddot\Gamma$ is denoted again by ϵ_α. Then $\alpha : (y, \eta) \mapsto (y, \eta \otimes \epsilon_\alpha)$ defines a free M-action on $\mathcal{M}_{\Gamma,F^2}$.

For any (y, η) as above we set (as in [L1, (4.24.1)])

(a) $$R_{y,\eta} = \sigma(Z(s))\sigma(\mathcal{G}) \sum \{(x,\xi),(y,\eta)\}\Delta(x,\xi)\rho_{x,\xi}$$

sum over all $(x, \xi) \in \overline{\mathcal{M}}_{\Gamma,F^2}$. (An *almost character* of G^{F^2}.) Here $\{(x,\xi),(y,\eta)\} \in \bar{\mathbf{Q}}_l$ is as in [L1, (4.21.13)]; $\Delta(x, \xi)$ is equal to 1 unless $\mathcal{F}$ has exactly two elements in which case it is 1 for $x = \gamma^2$ and is -1 for $x \neq \gamma^2$; $\sigma(H) = (-1)^{F_{q^2}-\mathrm{rank}(H)}$ for a connected algebraic group H defined over F_{q^2}.

Up to a root of 1, $R_{y,\eta}$ depends only on the M-orbit of (y, η) in $\mathcal{M}_{\Gamma,F^2}$. Now

(b) *If $F(s) = s^{-1}$, then in (a) we have $\sigma(Z(s))\sigma(\mathcal{G}) = 1$.*

The proof is given in 1.7.

The following result describes $\boxed{R_{y,\eta}}$.

Corollary 1.6. *Let (y, η) be as above.*

(a) Assume that s is not conjugate under $\mathcal{G}^{F^2}$ to an element s' such that $F(s') = s'^{-1}$. Then $\boxed{R_{y,\eta}} = 0$.

(b) Assume that $F(s) = s^{-1}$ but $F(\mathcal{F}) \neq \mathcal{F}$. Then $\boxed{R_{y,\eta}} = 0$.

(c) Assume that $F(s) = s^{-1}, F(\mathcal{F}) = \mathcal{F}$ and $Z_{\dot\Gamma}(y)$ does not meet $\Gamma\gamma$. Then $\boxed{R_{y,\eta}} = 0$.

(d) Assume that $F(s) = s^{-1}, F(\mathcal{F}) = \mathcal{F}$, that $Z_{\dot\Gamma}(y)$ meets $\Gamma\gamma$ and that for any $\alpha \in M$, $\eta \otimes \epsilon_\alpha$ cannot be extended to a self dual representation of $Z_{\dot\Gamma}(y)$. Then $\boxed{R_{y,\eta}} = 0$.

(e) Assume that $F(s)=s^{-1}, F(\mathcal{F})=\mathcal{F}$, that $|\mathcal{F}|\neq 2$, that $Z_{\dot\Gamma}(y)$ meets $\Gamma\gamma$ and that for some $\alpha\in M$, $\eta\otimes\epsilon_\alpha$ can be extended to a self dual representation $\tilde\eta_\alpha$ of $Z_{\dot\Gamma}(y)$. Then α is unique and $\boxed{R_{y,\eta}}=\psi(\tilde\eta_\alpha)\alpha^{-1}$. ($\psi$ as in 1.1.)

(f) Assume that $F(s)=s^{-1}, F(\mathcal{F})=\mathcal{F}$ and that $|\mathcal{F}|=2$. Then $\boxed{R_{y,\eta}}=1$ if $y=1$ and $\boxed{R_{y,\eta}}=0$ if $y\neq 1$.

Thus, $\boxed{R_{y,\eta}}$ is either 0 or a root of 1. The proof is given in §6.

1.7. Let T be a torus defined over F_q with Frobenius map F. We have

(a) $\sigma(T)=(-1)^{\dim T}$.

Let $\mathcal{L}$ be the lattice of one parameter groups of T. Let $\mathcal{L}_{\mathbf{C}}=\mathbf{C}\otimes\mathcal{L}$. There exists an isomorphism of finite order $f:\mathcal{L}\xrightarrow{\sim}\mathcal{L}$ (inducing an isomorphism $\tilde f: T\xrightarrow{\sim} T$) such that $F(t)=\tilde f(t^q)$ for all $t\in T$. Then $F^2(t)=\tilde f^2(t^{q^2})$ and the F_{q^2}-rank of T is the rank of $\operatorname{Ker}(f^2-1:\mathcal{L}_{\mathbf{C}}\to\mathcal{L}_{\mathbf{C}})$. (We denote $1\otimes f$ again by f.) This rank equals $a+b$ where a (resp. b) is the number of eigenvalues of f equal to 1 (resp. -1.) The eigenvalues other than ±1 of f occur in pairs ζ,ζ^{-1} (since f is given by an integral matrix of finite order). Hence $a+b=\dim T \mod 2$. Thus, (a) follows.

We prove 1.5(b). Let s be as in 1.5(b). Lt $B,\mathcal{T}$ be as in 1.3. Then $\sigma(Z(s))=\sigma(\mathcal{T})$ and, by (a), this equals $(-1)^{\dim\mathcal{T}}$; here, replacing s by 1 we find $\sigma(\mathcal{G})=(-1)^{\dim T}$ where T is a maximal torus of $\mathcal{G}$. Thus

$$\sigma(G)\sigma(Z(s))=(-1)^{\dim\mathcal{T}}(-1)^{\dim T}=1.$$

This proves 1.5(b).

2. Proof of Theorem 1.4(a),(b)

2.1. In this section G is as in 1.1.

Let I be the set of all pairs (T,λ) where T is an F^2-stable maximal torus of G and $\lambda\in(T^{F^2})\check{}$. Let $(T,\lambda)\in I$. Let R_T^λ be the virtual representation of G^{F^2} attached to (T,λ) in [DL] (relative to G,F^2). Let

$$X=\{x\in G|xF(T)x^{-1}=T, xF(x)\in T, \lambda|_{\{t\in T|xF(t)x^{-1}=t\}}=1\}.$$

Then T acts on X by $t:x\mapsto tx$. Clearly, $T\backslash X$ is finite.

Lemma 2.2. *We have $\boxed{R_T^\lambda}=|T\backslash X|$.*

This is an easy consequence of results in [L5]. (In [L5] the characteristic of the ground field was assumed to be odd; but in the special case that is used below, that assumption is unnecessary.)

We consider the F_q-rational structure on the connected reductive algebraic group $G_1=G\times G$ such that the corresponding Frobenius map $F_1:G_1\to G_1$ is $F_1(g,g')=(F(g'),F(g))$. Then $g\mapsto(g,F(g))$ is an isomorphism

(a) $G^{F^2}\xrightarrow{\sim}G_1^{F_1}$.

Now $T_1=T\times F(T)$ is an F_1-stable maximal torus of G_1 and $t\mapsto(t,F(t))$ is an isomorphism $T^{F^2}\xrightarrow{\sim}T_1^{F_1}$. Via this isomorphism, λ becomes $\lambda_1\in(T_1^{F_1})\check{}$. From the definitions it is clear that the G^{F^2}-virtual representation R_T^λ corresponds under (a) to the $G_1^{F_1}$-virtual representation $R_{T_1}^{\lambda_1}$ attached in [DL] to (T_1,λ_1) (relative to G_1,F_1) and

(b) $\boxed{R_T^\lambda}=|G^F|^{-1}\sum_{g\in G^F}\operatorname{tr}((g,g),R_{T_1}^{\lambda_1})$.

The involution $\theta_1 : G_1 \to G_1$ given by $\theta_1(g, g') = (g', g)$ clearly commutes with F_1. The fixed point set of θ_1 may be identified with G.

Applying [L5, 3.3] to $G_1, F_1, \theta_1, T_1, \lambda_1$ we see that the right-hand side of (b) is equal to the number of $T_1^F - G^F$ double cosets in G_1^F represented by elements of the form $(f, F(f))$ where $f \in G^{F^2}$ satisfies $F(f^{-1}Tf) = f^{-1}Tf$ and $\lambda|_{\{t \in T | F(f^{-1}tf) = f^{-1}tf\}} = 1$. (In our case the character ϵ which enters in [L5, 3.3] is 1 and the signs which enter there are also 1.) By the change of variable $y = fF(f^{-1})$ we see that the right-hand side of (b) equals $|T^{F^2} \backslash Y|$ where

$$Y = \{y \in G | F(y) = y^{-1}, yF(T)y^{-1} = T, \lambda|_{\{t \in T | yF(t)y^{-1} = t\}} = 1\}$$

and T^{F^2} acts on Y by $t : y \mapsto tyF(t^{-1})$.

The inclusion $Y \subset X$ induces a map $\kappa : T^{F^2} \backslash Y \to T \backslash X$. To complete the proof, it is enough to show that κ is bijective.

We show that κ is surjective. It is enough to show that, if $x \in X$, then there exists $t \in T$ such that $tx \in Y$. We have $xF(x) = t_1 \in T$ and we must show that there exists $t \in T$ such that $txF(t)x^{-1} = t_1^{-1}$. By Lang's theorem we can write $x = z^{-1}F(z)$ for some $z \in G$. We have $F(zTz^{-1}) = zTz^{-1}$. Thus $F(T') = T'$ where $T' = zTz^{-1}$. Our equation for t is $ztz^{-1}F(z)F(t)F(z^{-1}) = zt_1^{-1}z^{-1}$. Thus, if we set $t' = ztz^{-1}$, $t'_1 = zt_1z^{-1}$, we see that we must solve the equation $t'F(t') = {t'_1}^{-1}$ with $t'_1 \in T'$ and unknown $t' \in T'$. This equation can be solved by Lang's theorem for T' with the Frobenius map $\tau \mapsto F(\tau^{-1})$. Thus, κ is surjective.

We show that κ is injective. It is enough to show that, if $y \in Y, y' \in Y, t_1 \in T$ satisfy $y' = t_1 y$, then $y' = tyF(t^{-1})$ for some $t \in T^{F^2}$. As in the proof of surjectivity of κ we can find $t \in T$ such that $tyF(t^{-1})y^{-1} = t_1$. (We write $y = z^{-1}F(z)$ and we use Lang's theorem for (zTz^{-1}, F).) We have $y' = tyF(t^{-1})$ and it remains to show that $t \in T^{F^2}$. Since $y' = t_1 y \in Y$, we have $F(t_1 y) = y^{-1}t_1^{-1}$. Applying F to $tyF(t^{-1})y^{-1} = t_1$ gives $F(t)y^{-1}F^2(t^{-1})y = F(t_1)$. Substituting here $F(t)y^{-1} = y^{-1}t_1^{-1}t$ gives $y^{-1}t_1^{-1}tF^2(t^{-1})y = F(t_1)$. Hence $F(t_1)y^{-1}tF^2(t^{-1})y = F(t_1)$ and $y^{-1}tF^2(t^{-1})y = 1$. It follows that $tF^2(t^{-1}) = 1$ hence $t \in T^{F^2}$. Thus, κ is injective. The lemma is proved.

2.3. Let $(T, \lambda) \in I$. If $\boxed{R_T^\lambda} \neq 0$, then, by the proof of 2.2, there exists $f \in G^{F^2}$ which conjugates (T, λ) to a pair $(T', \lambda') \in I$ where $F(T') = T'$ and $\lambda'|_{T'^F} = 1$. On the other hand, if $(T, \lambda) \in I$ is such that $F(T) = T$ and $\lambda|_{T^F} = 1$, then, by 2.2,

$$\boxed{R_T^\lambda} = \sharp(w \in W_G(T); F(w) = w^{-1}, \lambda|_{\{t \in T | w(F(t)) = t\}} = 1).$$

2.4. Let T be a torus defined over k. Let p be the characteristic of k. For any integer $n \geq 1$, let $T_n = \{t \in T | t^{p^n - 1} = 1\}$. If $n' \geq 1$ is an integer divisible by n we have a (surjective) norm map $N_{n',n} : T_{n'} \to T_n$ given by $t \mapsto t^{(p^{n'}-1)/(p^n-1)}$. Thus we obtain a projective system of finite groups $(T_n, N_{n',n})$. By passage to transposes we obtain an inductive system of finite groups $(\check{T_n}, {}^{tr}N_{n',n})$ whose transition maps are injective. The union of this inductive system is denoted by $\hat{T}$. Now $T \mapsto \hat{T}$ is a contravariant functor from the category of tori defined over k (and homomorphisms of algebraic groups between such tori) and the category of torsion abelian groups. If $f : T_1 \to T_2$ is a morphism of tori, we denote by $f^* : \hat{T}_2 \to \hat{T}_1$ the corresponding homomorphism of torsion abelian groups.

Assume now that $f : T \to T$ is a morphism such that for some integers $m, e \geq 1$ we have $f^m(t) = t^{em}$ for all $t \in T$. We define a surjective homomorphism $T_{em} \to T^f$ by $t \mapsto tf(t)f^2(t) \dots f^{m-1}(t)$. Taking transposes we obtain an injective homomorphism $(T^f)\check{} \to T_{em}\check{}$. Composing with the canonical imbedding $T_{em}\check{} \to \hat{T}$ we obtain an injective homomorphism $a : (T^f)\check{} \to \hat{T}$ which is in fact independent of the choice of m. Consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & (T^{f^2})\check{} \arrow[r, "a'"] \arrow[d, "c"'] & \hat{T} \arrow[r, "b'"] \arrow[d, "d"'] & \hat{T} \arrow[r] \arrow[d, "1"'] & 0 \\ 0 \arrow[r] & (T^f)\check{} \arrow[r, "a"] & \hat{T} \arrow[r, "b"] & \hat{T} \arrow[r] & 0 \end{tikzcd}$$

where a is above, a' is the map analogous to a defined in terms of f^2 instead of f, $b(x) = f^*(x)x^{-1}$, $b'(x) = (f^2)^*(x)x^{-1}$, c is the transpose of the obvious inclusion $T^f \subset T^{f^2}$ and $d(x) = f^*(x)x$.

(a) *This diagram is commutative, with exact rows.*

(The proof is left to the reader.)

Let $(T, \lambda) \in I$. Let $\Lambda \in \hat{T}$ be the image of $\lambda \in (T^{F^2})\check{}$ under a (with $f = F^2 : T \to T$).

Lemma 2.5. *Let $X' = \{x \in G | xF(T)x^{-1} = T, xF(x) \in T, f_x^*(\Lambda) = \Lambda^{-1}\}$, where $f_x : T \to T$ is defined by $f_x(t) = xF(t)x^{-1}$. Then X in 2.1 coincides with X'.*

If $x \in G$ satisfies $xF(T)x^{-1} = T$, $xF(x) \in T$, then $f_x : T \to T$ is well defined and satisfies $f_x^2 = F^2 : T \to T$. Hence 2.4(a) is applicable to $f = f_x$ and shows that the condition that $\lambda|_{\{t \in T | xF(t)x^{-1} = t\}} = 1$, that is, the condition that $c(\lambda) = 1$, is equivalent to the condition that $da'(\lambda) = 1$, that is, the condition that $f_x^*(\Lambda) = \Lambda^{-1}$. Thus $X' = X$. The lemma is proved.

2.6. Let $\mathcal{I}$ be the set of all pairs $(\mathcal{T}, s)$ where $\mathcal{T}$ is an F^2-stable maximal torus of $\mathcal{G}$ and $s \in \mathcal{T}^{F^2}$. Now G^{F^2} acts by conjugation on I and $\mathcal{G}^{F^2}$ acts by conjugation on $\mathcal{I}$. Applying [DL, (5.21.5)] for F^2 instead of F we obtain a canonical bijection

(a) $G^{F^2} \backslash I \leftrightarrow \mathcal{G}^{F^2} \backslash \mathcal{I}$.

Using the definition and 2.5, we see that this bijection has the following property:

(b) *a G^{F^2}-orbit in I contains a pair (T, λ) such that $FT = T, \lambda|_{T^F} = 1$ if and only if the corresponding $\mathcal{G}^{F^2}$-orbit in $\mathcal{I}$ contains a pair $(\mathcal{T}, s)$ such that $F(\mathcal{T}) = \mathcal{T}$ and $F(s) = s^{-1}$; for such $(T, \lambda), (\mathcal{T}, s)$ the number of $w \in W(T)$ such that $F(w) = w^{-1}$ and $\lambda|_{\{t \in T | w(F(t)) = t\}} = 1$ is equal to the number of $w \in W_{\mathcal{G}}(\mathcal{T})$ such that $F(w) = w^{-1}$ and $w(F(s)) = s^{-1}$ (that is $w(s) = s$).*

For any $(\mathcal{T}, s) \in \mathcal{I}$ we define $R_{\mathcal{T}}^s$ to be R_T^λ where the G^{F^2}-orbit of (T, λ) corresponds to the $\mathcal{G}^{F^2}$-orbit of $(\mathcal{T}, s)$ under (a). Now $R_{\mathcal{T}}^s$ is well defined since R_T^λ depends only on the G^{F^2}-orbit of (T, λ).

2.7. *In the remainder of this section we assume that Z_G is connected.*

Using 2.6(b) we can reformulate the results in 2.3 as follows. Let $(\mathcal{T}, s) \in \mathcal{I}$.

(a) *If the $\mathcal{G}^{F^2}$-orbit of $(\mathcal{T}, s)$ does not contain a pair $(\mathcal{T}', s')$ such that $F(\mathcal{T}') = \mathcal{T}'$ and $F(s') = s'^{-1}$, then* $\boxed{R_{\mathcal{T}}^s} = 0$.

(b) *If $F(\mathcal{T}) = \mathcal{T}$ and $F(s) = s^{-1}$, then*

$$\begin{aligned}\boxed{R^s_{\mathcal{T}}} &= \sharp(w \in W_{\mathcal{G}}(\mathcal{T}); F(w) = w^{-1}, w(s) = s)\\ &= \sharp(w \in W_{Z(s)}(\mathcal{T}); F(w) = w^{-1}).\end{aligned}$$

By Lang's theorem for $(Z(s), F^2)$, for any $w \in W_s$ we can find an element $g_w \in Z(s)$ such that $g_w^{-1}F^2(g_w)$ is in $N_{Z(s)}(\mathcal{T})$ and represents w. Then $\mathcal{T}_w = g_w\mathcal{T}g_w^{-1}$ is an F^2-stable maximal torus of $Z(s)$ whose $Z(s)^{F^2}$-conjugacy class is independent of the choice of g_w. Hence we may set $R^s_w = R^s_{\mathcal{T}_w}$; this is well defined by s, w.

Lemma 2.8. *Let $s \in \mathcal{G}^{F^2}$ be semisimple and let $w \in W_s$.*

(a) If s is not conjugate under $\mathcal{G}^{F^2}$ to an element s' such that $F(s') = s'^{-1}$ (or equivalently, $F(s)$ is not conjugate to s^{-1} in $\mathcal{G}$), then $\boxed{R^s_w} = 0$.

(b) If $F(s) = s^{-1}$, then $\boxed{R^s_w} = \sharp(y \in W_s; w = yF(y))$.

(a) follows immediately from 2.7(a). We prove (b). We first show that

(c) *if $\boxed{R^s_w} \neq 0$, then $w = yF(y)$ for some $y \in W_s$.*

Using 2.7(a) we see that there exists $g \in \mathcal{G}^{F^2}$ such that $F(g\mathcal{T}_w g^{-1}) = g\mathcal{T}_w g^{-1}$, $F(gsg^{-1}) = gs^{-1}g^{-1}$. Then $F(g)sF(g^{-1}) = gsg^{-1}$ so that $g^{-1}F(g) \in Z(s)$. By Lang's theorem for $Z(s)$ we can find $z \in Z(s)$ so that $g^{-1}F(g) = z^{-1}F(z)$. Then $gz^{-1} \in \mathcal{G}^F$ and $z \in Z(s)^{F^2}$. We have $F(z\mathcal{T}_w z^{-1}) = z\mathcal{T}_w z^{-1}$. Since $\boxed{R^s_{z\mathcal{T}_w z^{-1}}} = \boxed{R^s_w} \neq 0$, we see from 2.7(b) that there exists

$$u \in W_{Z(s)}(z\mathcal{T}_w z^{-1}) = W_{Z(s)}(zg_w\mathcal{T}g_w^{-1}z^{-1})$$

such that $F(u) = u^{-1}$. We have $F(zg_w\mathcal{T}g_w^{-1}z^{-1}) = zg_w\mathcal{T}g_w^{-1}z^{-1}$. Since $F^2(z) = z$, the element zg_w has the same properties as g_w. Hence we may replace zg_w by g_w. Thus we may assume that $F(g_w\mathcal{T}g_w^{-1}) = g_w\mathcal{T}g_w^{-1}$ and there exists $u \in W_{Z(s)}(g_w\mathcal{T}g_w^{-1})$ such that $F(u) = u^{-1}$. Now $x \mapsto g_w x g_w^{-1}$ induces a bijection $\iota : W = W_{Z(s)}(\mathcal{T}) \to W_{Z(s)}(\mathcal{T}_w)$. If $x \in W_{Z(s)}(\mathcal{T})$, then the condition $F(\iota(x)) = \iota(x)^{-1}$ is equivalent to $w_1F(x)w_1^{-1} = x^{-1}$ where w_1 is the element of W_s represented by $g_w^{-1}F(g_w)$. Thus if $x = \iota^{-1}(u)$, then $w_1F(x)w_1^{-1} = x^{-1}$. Hence $y = x^{-1}w_1$ satisfies $yF(y) = w$. This proves (c).

Next we assume that $w = w_1F(w_1)$ for some $w_1 \in W_s$. By Lang's theorem for $(Z(s), F)$ we can find an element $h_w \in Z(s)$ such that $h_w^{-1}F(h_w)$ is in $N_{Z(s)}(\mathcal{T})$ and represents w_1. Then $h_w^{-1}F^2(h_w) = h_w^{-1}F(h_w)F(h_w^{-1}F(h_w)) \in N_{Z(s)}(\mathcal{T})$ represents $w_1F(w_1) = w$. Hence we may assume that $\mathcal{T}_w = h_w\mathcal{T}h_w^{-1}$. By the definition of h_w we have $F(\mathcal{T}_w) = \mathcal{T}_w$. By 2.7(b) we have

$$\boxed{R^s_w} = \sharp(w' \in W_{Z(s)}(\mathcal{T}_w); F(w') = w'^{-1}).$$

Now $x \mapsto h_w x h_w^{-1}$ induces a bijection $N_{Z(s)}(\mathcal{T}) \to N_{Z(s)}(\mathcal{T}_w)$ and a bijection $\iota : W_s = W_{Z(s)}(\mathcal{T}) \to W_{Z(s)}(\mathcal{T}_w)$. If $x \in W_{Z(s)}(\mathcal{T})$, then the condition $F(\iota(x)) = \iota(x)^{-1}$ is equivalent to $w_1F(x)w_1^{-1} = x^{-1}$. Thus

$$\boxed{R^s_w} = \sharp(x \in W_s; w_1F(x)w_1^{-1} = x^{-1}).$$

The change of variable $x \mapsto y$ where $y^{-1} = w_1^{-1}x$ transforms the equation $w_1F(x)w_1^{-1} = x^{-1}$ into the equation $w_1F(w_1)F(y^{-1}) = y$, that is, $yF(y) = w$. Hence $\boxed{R^s_w} = \sharp(y \in W_s; w = yF(y))$. This together with (c) completes the proof.

2.9. Let Γ be a finite group. According to [FS], if $x \in \Gamma$ and ψ is as in 1.1, we have

(a) $$\sum_{E \in \mathrm{Irr}\Gamma} \mathrm{tr}(x, E)\psi(E) = \sharp(x' \in \Gamma; x'^2 = x).$$

2.10. Assume that we are given an automorphism F of the finite group Γ. Let $\dot{\Gamma}$ be the semidirect product of Γ with the infinite cyclic group with generator γ so that in $\dot{\Gamma}$ we have the identity $\gamma x \gamma^{-1} = F(x)$ for all $x \in \Gamma$. Then Γ is naturally a subgroup of $\dot{\Gamma}$. Let $\mathrm{Irr}'\Gamma$ be the set of all $E \in \mathrm{Irr}\Gamma$ such that there exists a $\dot{\Gamma}$-module $\dot{E}$ and such that the restriction of $\dot{E}$ to Γ is isomorphic to E.

For $E \in \mathrm{Irr}'\Gamma$ we define $\psi'(E) \in \{-1, 0, 1\}$ as follows. If there exists $\dot{E}$ as above such that $\dot{E}$ is self dual, we set $\psi'(E) = \psi(\dot{E}) \in \{-1, 1\}$. (In this case there are exactly two choices for a self dual $\dot{E}$ and both have the same ψ-value.) If there is no $\dot{E}$ as above such that $\dot{E}$ is self dual, we set $\psi'(E) = 0$.

In this setup we have the following result which is a generalization of 2.9(a).

Lemma 2.11. *For any $x \in \Gamma$ we have*

$$\sum_{\substack{E \in \mathrm{Irr}'\Gamma \\ \psi'(E) = \pm 1}} \mathrm{tr}(x\gamma^2, \dot{E})\psi'(E) = \sharp(y \in \Gamma; yF(y) = x).$$

(In the last formula $\dot{E}$ is assumed to be one of the two self dual extensions of E; they are obtained from each other by replacing the action of γ by that of $-\gamma$. Hence $\mathrm{tr}(x\gamma^2, \dot{E})$ is independent of the choice of $\dot{E}$.)

Let n be an integer ≥ 2 such that $F^n = 1$ on Γ. Let Γ_1 be the semidirect product of Γ with the cyclic group of order $2n$ with generator γ_1 so that in Γ_1 we have the identity $\gamma_1 x' \gamma_1^{-1} = F(x')$ for all $x' \in \Gamma$. The solutions $y' \in \Gamma_1$ of the equation $y'^2 = x\gamma_1^2$ are of two kinds: $y' = y\gamma_1$ with $y \in \Gamma$ such that $yF(y) = x$ and $y' = y\gamma_1^{1+n}$ with $y \in \Gamma$ such that $yF(y) = x$. Hence

$$\sharp(y \in \Gamma; yF(y) = x) = (1/2)\sharp(y' \in \Gamma_1; y'^2 = x\gamma_1^2).$$

By 2.9(a) (for Γ_1), the last expression is equal to

(a) $$(1/2) \sum_{E_1 \in \mathrm{Irr}\Gamma_1} \mathrm{tr}(x\gamma_1^2, E_1)\psi(E_1).$$

Let $E_1 \in \mathrm{Irr}\Gamma_1$. The restriction $E_1|_\Gamma$ is multiplicity free hence it decomposes canonically into a direct sum of irreducible Γ-modules which are cyclically permuted by $\gamma_1 : E_1 \to E_1$. If the number of summands is 3 or more, then clearly $\mathrm{tr}(x\gamma_1^2, E_1) = 0$. If the number of summands is two, then n is even and the number of distinct $E_1' \in \mathrm{Irr}\Gamma_1$ such that $E_1'|_\Gamma \cong E_1|_\Gamma$ is n. These E_1' can be arranged into $n/2$ pairs so that the two representations in the same pair have the same value of ψ and opposite values of $\mathrm{tr}(x\gamma_1^2,)$. Hence the sum (a) may be restricted to those E_1 whose restriction to Γ is irreducible. From this the lemma follows easily.

2.12. Let $s \in \mathcal{G}^{F^2}$ be semisimple. Following [L1, 3.7], for $E \in \mathrm{Irr}''W_s$ we define

(a) $$R^s_{\ddot{E}} = |W_s|^{-1} \sum_{w \in W_s} \mathrm{tr}(w\delta, \ddot{E})R^s_w$$

(an element of the Grothendieck group of representations of G^{F^2} tensored with $\mathbf{Q}$).

Proposition 2.13. *(a) If s is not conjugate under $\mathcal{G}^{F^2}$ to an element s' such that $F(s') = s'^{-1}$, then $\boxed{R^s_{\ddot{E}}} = 0$ for any $E \in \mathrm{Irr}''W_s$ (for the two choices of $\ddot{E}$).*

(b) If $F(s) = s^{-1}$ and $E \in \mathrm{Irr}''W_s - \mathrm{Irr}'W_s$, then $\boxed{R^s_{\ddot{E}}} = 0$ (for the two choices of $\ddot{E}$).

(c) If $F(s) = s^{-1}$ and $E \in \mathrm{Irr}'W_s$, then $\boxed{R^s_{\ddot{E}}} = 1$ (for the canonical $\ddot{E}$).

(a) follows immediately from 2.8(a). Assume now that $F(s) = s^{-1}$. By 2.11, for any $w \in W_s$ we have

$$\sum_{E' \in \mathrm{Irr}'W_s} \mathrm{tr}(w\gamma^2, \dot{E}') = \sharp(y \in W_s; yF(y) = w),$$

or equivalently

$$\sum_{E' \in \mathrm{Irr}'W_s} \mathrm{tr}(w\delta, \ddot{E}') = \sharp(y \in W_s; yF(y) = w),$$

where $\ddot{E}'$ is the canonical one. (In the present case we have $\psi'(E') = 1$ for any $E' \in \mathrm{Irr}'W_s$; see [L1, 3.2].) Combining this with 2.8(b) gives

$$\boxed{R^s_w} = \sum_{E' \in \mathrm{Irr}'W_s} \mathrm{tr}(w\delta, \ddot{E}')$$

for any $w \in W_s$. Now let $E \in \mathrm{Irr}''W_s$ and consider one of the two choices of $\ddot{E}$. We have

$$\boxed{R^s_{\ddot{E}}} = |W_s|^{-1} \sum_{w \in W_s} \mathrm{tr}(w\delta, \ddot{E}) \boxed{R^s_w} = |W_s|^{-1} \sum_{w \in W_s} \mathrm{tr}(w\delta, \ddot{E}) \sum_{E' \in \mathrm{Irr}'W} \mathrm{tr}(w\delta, \ddot{E}')$$
$$= |W_s|^{-1} \sum_{E' \in \mathrm{Irr}'W_s} \sum_{w \in W_s} \mathrm{tr}(w\delta, \ddot{E}) \mathrm{tr}(w\delta, \ddot{E}').$$

Now $|W_s|^{-1} \sum_{w \in W_s} \mathrm{tr}(w\delta, \ddot{E}) \mathrm{tr}(w\delta, \ddot{E}')$ is 0 if $E \neq E'$ and is 1 if $\ddot{E} = \ddot{E}'$. (See [L1, p. 75].) Hence (b) and (c) follow. The proposition is proved.

2.14. Our strategy to prove Theorem 1.4 is as follows. Let $s, \mathcal{F}$ be as in 1.4. In [L1, 6.17(ii)], certain (not necessarily) irreducible representations $\pm R_{\gamma^2 x}$ of G^{F^2} are described, where x is an element in the two-sided cell of W_s corresponding to $\mathcal{F}$. (In *loc. cit.* these representations are denoted by $\pm R_{\gamma x}$.) These representations are on the one hand $\mathbf{Z}$-linear combinations of various $R^s_{\ddot{E}}$ with $E \in \mathrm{Irr}''W_s$ with coefficients explicitly known from [L4]. Hence $\boxed{\pm R_{\gamma^2 x}}$ are explicitly known from 2.13. On the other hand, these representations are linear combinations of various $\rho \in \mathrm{Irr}_{s,\mathcal{F}} G^{F^2}$ whose coefficients (in $\mathbf{N}$) are explicitly known from [L1, 4.23]. This gives rise to a system of linear equations for the unknowns $\boxed{\rho}$, $\rho \in \mathrm{Irr}_{s,\mathcal{F}} G^{F^2}$, where the coefficients and the constant terms are in $\mathbf{N}$ and are rather small. Also by [L1, (6.18.2)], any $\rho \in \mathrm{Irr}_{s,\mathcal{F}} G^{F^2}$ appears with > 0 coefficient in at least one of these equations. Although this system has in general more unknowns than equations, the fact that the unknowns must be in $\mathbf{N}$ provides very strong constraints. In particular, the unknowns are bounded above. Also, if one of the equations has constant term 0, then all unknowns which enter into that equation must be 0. In the case of classical groups this is already sufficient to determine all unknown $\boxed{\rho}$ as we will see in §3. For the exceptional groups the method above determines the $\boxed{\rho}$

up to a small indeterminacy. To remove that indeterminacy we will need additional information which comes from the theory of character sheaves.

2.15. Proof of Theorem 1.4(a),(b). We carry out the strategy in 2.14 in the setup of 1.4(a) or (b). Let $s, \mathcal{F}, \rho$ be as in 1.4(a) or (b). As in 2.14, there exists a representation ρ' of G^{F^2} such that ρ is a direct summand of ρ' and such that the character of ρ' is a $\mathbf{Z}$-linear combination of characters of $R^s_{\ddot{E}}$ with $E \in \mathrm{Irr}''W_s \cap \mathcal{F}$. Since $0 \le \boxed{\rho} \le \boxed{\rho'}$, it is enough to show that $\boxed{\rho'} = 0$. Hence it is enough to show that $\boxed{R^s_{\ddot{E}}} = 0$ for any $E \in \mathrm{Irr}''W_s \cap \mathcal{F}$.

Under the assumption of 1.4(a), this follows from 2.13(a). Under the assumption of 1.4(b), this follows from 2.13(b) since in this case any $E \in \mathrm{Irr}''W_s \cap \mathcal{F}$ is outside $\mathrm{Irr}'W_s$. This completes the proof of 1.4(a),(b).

3. Proof of Theorem 1.4(c) for classical groups

3.1. In this section we assume that G is as in 1.4, of classical type and that $F(s) = s^{-1}$ and $F(\mathcal{F}) = \mathcal{F}$.

3.2. We prove 1.4(c) in the case where $\Gamma = \{1\}$. In this case, $\mathcal{F}$ consists of a single representation E (necessarily in $\mathrm{Irr}'W_s$) and $\overline{\mathcal{M}}_{\Gamma, F^2}$ consists of a single element $(\gamma^2, 1)$. By [L1] and 1.5(b) we have $\rho_{\gamma^2,1} = R^s_{\ddot{E}}$. Hence by 2.13 we have $\boxed{\rho_{\gamma^2,1}} = \boxed{R^s_{\ddot{E}}} = 1$. On the other hand, $\sqrt{\gamma^2} = \{\gamma\}$ is a point, hence $[\xi : \sqrt{x}] = 1$. Thus 1.4(c) is proved in the present case.

3.3. In general, we can write canonically $W_s = W_1 \times W_2$ where W_1 is a product of Weyl groups of type A and W_2 is a Weyl group without factors of type A. Both W_1 and W_2 are F-stable. If $W_s = W_1$, then 1.4(c) holds by 3.2. Similarly, the proof of 1.4(c) for general W_s is exactly the same as the proof in the case where $W_s = W_2$. Therefore we may assume that

(a) W_s has no factors of type A.

3.4. Assume that W_s is as in 3.3(a), that F maps each irreducible factor of W_s into itself and that F^2 acts trivially on W_s. In this case we have $\mathcal{F} \cap \mathrm{Irr}'W_s = \mathcal{F} \cap \mathrm{Irr}''W_s$. We shall use arguments from [L1, §6]. As in *loc.cit.* to $\mathcal{F}$ we can associate an F_2-vector space Y with a basis $e_1, e_2, \dots, e_n$ and a symplectic form $(,) : Y \times Y \to F_2$ such that $(e_i, e_j) = 0$ if $|i - j| \ne 1$. Let $\mathcal{R}$ be the radical of $(,)$. We consider the family $\mathcal{T}(Y)$ of subspaces of Y defined as in [L1, p. 270]. Each subspace $C \in \mathcal{T}(Y)$ contains $\mathcal{R}$ and we have $C/\mathcal{R} \cong \Gamma$.

The union $\tilde{Y}$ of all C in $\mathcal{T}(Y)$ is therefore a union of $\mathcal{R}$-cosets in Y and there is a natural bijection between the set $\mathcal{F}$ and the set of $\mathcal{R}$-cosets in $\tilde{Y}$. Let $E_y \in \mathcal{F}$ correspond to the coset of $y \in \tilde{Y}$.

Let X be the set of linear forms $\eta : Y \to F_2$ such that $\eta|_{\mathcal{R}} = 0$. There is a natural bijection between X and $\mathrm{Irr}_{s,\mathcal{F}} G^{F^2}$. Let $\rho_\eta \in \mathrm{Irr}_{s,\mathcal{F}} G^{F^2}$ correspond to $\eta \in X$. Then for any $\eta \in X, y \in \tilde{Y}$ we have $(\rho_\eta : R^s_{\ddot{E}_y}) = (-1)^{\eta(y)}$. Moreover, for any $C \in \mathcal{T}(Y)$ and any linear form $\xi : C \to F_2$ such that $\xi|_{\mathcal{R}} = 0$ we have

$$\text{(a)} \qquad \sum_{y \in C/\mathcal{R}} (-1)^{\xi(y)} R^s_{\ddot{E}_y} = \sum_{\eta \in X; \eta|_C = \xi} \rho_\eta.$$

In our case, 1.4(c) can be reformulated as follows:

(b) $\boxed{\rho_0} = |\Gamma|$,
(c) $\boxed{\rho_\eta} = 0$ if $\eta \neq 0$.

Using 2.13 and (a) we see that for any $C \in \mathcal{T}(Y)$ and any $\xi \in \mathrm{Hom}(C, F_2)$ such that $\xi|_{\mathcal{R}} = 0$ we have

$$\sum_{\eta \in X; \eta|_C = \xi} \boxed{\rho_\eta} = \sum_{y \in C/\mathcal{R}} (-1)^{\xi(y)} \boxed{R^s_{\ddot{E}_y}} = \sum_{y \in C/\mathcal{R}} (-1)^{\xi(y)}.$$

Hence

(d) $$\sum_{\eta \in X; \eta|_C = \xi} \boxed{\rho_\eta} = 0 \text{ if } \xi \neq 0,$$

(e) $$\sum_{\eta \in X; \eta|_C = 0} \boxed{\rho_\eta} = |C/\mathcal{R}| = |\Gamma|.$$

Since all terms in the sum (d) are ≥ 0, it follows that $\boxed{\rho_\eta} = 0$ for any $\eta \in X$ such that $\eta|_C \neq 0$ for some C.

Assume that $\eta \in X - \{0\}$. Then $\eta(e_i) \neq 0$ for some i. By the definition of $\mathcal{T}(Y)$ we can find $C \in \mathcal{T}(Y)$ such that $e_i \in C$. Since $\eta(e_i) \neq 0$, we have $\eta|_C \neq 0$. Hence $\boxed{\rho_\eta} = 0$. Thus (c) is proved. Now let $C \in \mathcal{T}(Y)$. We rewrite (e) using (c); (b) follows. Thus 1.4(c) is proved in our case.

3.5. Assume that W_s is as in 3.3(a) and that F does not preserve each irreducible factor of W. Since G/Z_G is assumed to be simple, W is a product $W_1 \times W_1$ where W_1 is a Weyl group of type B_n, C_n or D_n and F interchanges the two factors. We shall use arguments from [L1, §6]. To $\mathcal{F}$ we can associate $Y, e_1, e_2, \dots, e_n, (,), \mathcal{R}, \mathcal{T}(Y), \tilde{Y}$ as in 3.4 so that the set $\mathcal{F}$ is naturally in bijection with the set of ordered pairs of $\mathcal{R}$-cosets in $\tilde{Y}$. Let $E_{y,y'} \in \mathcal{F}$ correspond to the pair of cosets of y and y' in $\tilde{Y}$. Let X be the set of linear forms $\eta : Y \to F_2$ such that $\eta|_{\mathcal{R}}$ is 0 if $F^2 = 1$ on W and is an isomorphism $\mathcal{R} \xrightarrow{\sim} F_2$ if $F^2 \neq 1$ on W. There is a natural bijection between $X \times X$ and $\mathrm{Irr}_{s,\mathcal{F}} G^{F^2}$. Let $\rho_{\eta,\eta'} \in \mathrm{Irr}_{s,\mathcal{F}} G^{F^2}$ correspond to $(\eta, \eta') \in X \times X$.

For simplicity we assume that $F^2 = 1$ on W. (The case where $F^2 \neq 1$ is similar.)

For any $y, y' \in \tilde{Y}$ we have $E_{y,y'} \in \mathrm{Irr}''W$; we have $E_{y,y'} \in \mathrm{Irr}'W$ if and only if $y = y' \mod \mathcal{R}$. For any $y, y' \in \tilde{Y}$ let $\ddot{E}_{y,y'}$ be the extension of $E_{y,y'}$ to $\ddot{W}$ on which γ^2 acts trivially. Then for any $\eta, \eta' \in X$ and $y, y' \in \tilde{Y}$ we have $(\rho_\eta : R^s_{\ddot{E}_{y,y'}}) = (-1)^{\eta(y)+\eta'(y')}$. Moreover, for any $C, C' \in \mathcal{T}(Y)$ and any $\xi \in \mathrm{Hom}(C, F_2), \xi' \in \mathrm{Hom}(C', F_2)$ such that $\xi|_{\mathcal{R}} = \xi'|_{\mathcal{R}} = 0$ we have

(a) $$\sum_{y \in C/\mathcal{R}, y' \in C'/\mathcal{R}} (-1)^{\xi(y)+\xi'(y')} R^s_{\ddot{E}_{y,y'}} = \sum_{\eta,\eta' \in X; \eta|_C = \xi, \eta'|_{C'} = \xi'} \rho_{\eta,\eta'}.$$

In our case, 1.4(c) can be reformulated as follows:
(b) $\boxed{\rho_{\eta,\eta'}} = 1$ if $\eta = \eta'$,
(c) $\boxed{\rho_{\eta,\eta'}} = 0$ if $\eta \neq \eta'$.

Using 2.13 and (a) we see that for any $C, C' \in \mathcal{T}(Y)$ and any $\xi \in \mathrm{Hom}(C, F_2), \xi' \in \mathrm{Hom}(C', F_2)$ such that $\xi|_{\mathcal{R}} = \xi'|_{\mathcal{R}} = 0$, we have

$$\sum_{\substack{\eta,\eta' \in X \\ \eta|_C=\xi \\ \eta'|_{C'}=\xi'}} \boxed{\rho_{\eta,\eta'}} = \sum_{\substack{y \in C/\mathcal{R} \\ y' \in C'/\mathcal{R}}} (-1)^{\xi(y)+\xi'(y')} \boxed{R^s_{\ddot{E}_{y,y'}}} = \sum_{y \in (C\cap C')/\mathcal{R}} (-1)^{\xi(y)+\xi'(y)}.$$

Hence

(d)
$$\sum_{\eta,\eta' \in X; \eta|_C=\xi, \eta'|_{C'}=\xi'} \boxed{\rho_{\eta,\eta'}} = 0$$

if $\xi \neq \xi'$,

(e)
$$\sum_{\eta,\eta' \in X; \eta|_C=\xi, \eta'|_{C'}=\xi'} \boxed{\rho_{\eta,\eta'}} = |(C \cap C')/\mathcal{R}|$$

if $\xi = \xi'$. Since each term in the sum (d) is ≥ 0, we see that $\boxed{\rho_{\eta,\eta'}} = 0$ for any $\eta, \eta' \in X$ such that $\eta|_C \neq \eta'|_C$ for some $C \in \mathcal{T}(Y)$. If $\eta \neq \eta'$, then, by an argument in 3.4, we see that there exists $C \in \mathcal{T}(Y)$ such that $(\eta - \eta')|_C \neq 0$. Hence $\boxed{\rho_{\eta,\eta'}} = 0$ and (c) is proved. We can find C, C' in $\mathcal{T}(Y)$ such that $C \cap C' = \mathcal{R}$ and $C + C' = Y$. Let $\eta_0 \in X$. Let $\xi = \eta_0|_C, \xi' = \eta_0|_{C'}$. Then in the sum (e) for C, C', ξ, ξ' as above we may restrict ourselves to $\eta = \eta'$ (by (c)) such that $\eta|_C = \xi, \eta|_{C'} = \xi'$, that is, such that $\eta = \eta_0$; we see that $\boxed{\rho_{\eta_0,\eta_0}} = 1$. Thus 1.4(c) holds in our case.

3.6. Assume that W_s is as in 3.3(a), that F maps each irreducible factor of W_s into itself and that F^2 acts non-trivially on W_s. In this case W_s is of type D_4 and $F : W_s \to W_s$ has order 3. This case is similar to (but simpler than) that in 3.4. We omit the proof.

4. Some results on character sheaves

4.1. In this section G is as in 1.1. We assume that the characteristic of F_q is restricted as in [L2, (23.0.1)] so that the results of [L2] on character sheaves are valid.

Let A be a character sheaf on G such that there exist an isomorphism $\phi : (F^2)^*(A) \xrightarrow{\sim} A$. Let $\chi_{A,\phi} : G^{F^2} \to \bar{\mathbf{Q}}_l$ be the class function defined by

$$\chi_{A,\phi}(g) = \sum_i (-1)^i \mathrm{tr}(\phi_g, \mathcal{H}^i_g(A))$$

where $\mathcal{H}^i(A)$ denotes the i-th cohomology sheaf of A and $\chi_g(A)$ is its stalk at g. Let DA be the Verdier dual of A.

Lemma 4.2. *Let A, ϕ be as above. Assume that $H^i_c(G, F^*(A) \otimes A) = 0$ for all i. Then $\boxed{\chi_{A,\phi}} = 0$.*

We define an isomorphism $\tilde{\phi} : F^*(F^*(A) \otimes A) \to F^*(A) \otimes A$ as the composition

$$(F^2)^*(A) \otimes F^*(A) \xrightarrow{\phi \otimes 1} A \otimes F^*(A) \to F^*(A) \otimes A$$

where the last isomorphism is switching the two factors. If $g \in G^F$, the map induced by $\tilde{\phi}$ on the stalk

$$\mathcal{H}^i_g(A \otimes A) = \bigoplus_{i'+i''=i} \mathcal{H}^{i'}_g(A) \otimes \mathcal{H}^{i''}_g(A)$$

is $\bigoplus_{i'+i''=i}\phi_g^{i'}\otimes 1$ composed with the isomorphism that switches the factors. It follows that $\operatorname{tr}(\tilde\phi_g,\mathcal{H}_g^i(A\otimes A))$ is $\operatorname{tr}(\phi_g,\mathcal{H}_g^{i/2}(A))$ if i is even and is 0 if i is odd. Hence

$$\chi_{F^*(A)\otimes A,\tilde\phi}=\sum_{i'}\operatorname{tr}(\phi_g,\mathcal{H}_g^{i'}(A))=\pm\chi_{A,\phi}.$$

The last equality holds since $\mathcal{H}^{i'}(A)\neq 0$ implies $i'=\dim\operatorname{supp}A \mod 2$ (see [L2, 24.11]). Hence

$$\pm\boxed{\chi_{A,\phi}}=|G^F|^{-1}\sum_{g\in G^F}\chi_{F^*(A)\otimes A,\tilde\phi}(g).$$

By the trace formula for $F:G\to G$, the last sum is equal to

$$\sum_i(-1)^i\operatorname{tr}(F_1^*,H_c^i(G,F^*(A)\otimes A))$$

where F_1^* is induced by $F,\tilde\phi$. Since $H_c^i(G,F^*(A)\otimes A)=0$, each term in the last sum is zero. The lemma is proved.

4.3. Let P be a parabolic subgroup of G and let L be a Levi subgroup of P. Assume that $FL=L$. Let K_0 be a cuspidal character sheaf on L. Let K be the perverse sheaf on G obtained by inducing K_0 from P to G (see [L2, 4.1]). Then $F^*(K)$ is the perverse sheaf on G obtained by inducing $F^*(K_0)$ from $F^{-1}(P)$ to G. Assume that for any isomorphism $f:L\xrightarrow{\sim}L$ induced by conjugation by an element in G we have $F^*(K_0)\not\cong f^*(DK_0)$. (Here D is a Verdier duality on L.)

Lemma 4.4. *Let A,ϕ be as in 4.1. Assume that A is a summand of K as in 4.3. Then* $\boxed{\chi_{A,\phi}}=0$.

By [L2, 7.8] we have $H_c^i(L,F^*(K_0)\otimes f^*(K_0))=0$ for all i and all f as in 4.3. Hence by [L2, 7.2] applied to $F^*(K)$ and K we have $H_c^i(G,F^*(A)\otimes A)=0$ for all i. Using 4.2 we deduce that $\boxed{\chi_{A,\phi}}=0$.

4.5. The hypothesis of 4.4 is verified in the following cases (here we assume that Z_G is connected):

(a) G/Z_G is of type E_6,E_7 or E_8, L/Z_L is of type E_6, K_0 is any cuspidal character sheaf on L, A is any simple summand of K.

(b) G/Z_G is of type E_7 or E_8, L/Z_L is of type E_7, K_0 is any cuspidal character sheaf on L, A is any simple summand of K.

(c) $G=G/Z_G$ is of type E_8,F_4 or G_2, $L=G$ and $K_0=K=A$ is a cuspidal character sheaf on G which is not uniquely determined by its support.

For these cases the restrictions in 4.1 can be competely removed. (See Shoji [S].)

5. Proof of Theorem 1.4(c),(d) for exceptional groups

5.1. In this section we assume that G is as in 1.4, of exceptional type and that $F(s)=s^{-1}$ and $F(\mathcal{F})=\mathcal{F}$.

5.2. The proof of 1.4(c) in the case where $\Gamma=\{1\}$ is exactly the same as in 3.2.

5.3. We prove 1.4(c) in the case where $\Gamma \cong \mathbf{Z}/2\mathbf{Z}$. In this case $\mathrm{Irr}_{s,\mathcal{F}} G^{F^2}$ consists of four elements $\rho_1, \rho_2, \rho_3, \rho_4$ and $\mathrm{Irr}' W_s \cap \mathcal{F}$ consists of three representations E_1, E_2, E_3. By [L1] we have

$$\rho_1 + \rho_2 = R_1 + R_2, \rho_1 + \rho_3 = R_1 + R_3, \rho_3 + \rho_4 = R_1 - R_2, \rho_2 + \rho_4 = R_1 - R_3$$

where $R_i = R^s_{\ddot{E}_i}$. Let $a_i = \boxed{\rho_i}$. Using 2.13(c) we obtain

$$a_1 + a_2 = 2, a_1 + a_3 = 2, a_3 + a_4 = 0, a_2 + a_4 = 0.$$

Since $a_i \geq 0$, the last two equations imply $a_2 = a_3 = a_4 = 0$. It follows that $a_1 = 2$. From this 1.4(c) follows easily in our case.

5.4. Assume that G is adjoint of type E_6 and that $\Gamma \cong \mathfrak{S}_3$ (symmetric group in 3 letters). In this case F acts trivially on Γ. The eight elements of $\overline{\mathcal{M}}_{\Gamma, F^2}$ can be represented in the form $(1,1), (g_2,1), (g_3,1), (1,r), (1,\epsilon), (g_2,\epsilon), (g_3,\theta), (g_3,\theta^2)$ (notation of [L1, 4.3]; we write (g,ξ) instead of $(g\gamma^2,\xi)$). The corresponding elements of $\mathrm{Irr}_{s,\mathcal{F}} G^{F^2}$ are denoted by $\rho_1, \rho_2, \dots, \rho_8$. The elements of $\mathrm{Irr}' W_s \cap \mathcal{F}$ are denoted by E_1, E_2, E_3, E_4, E_5. By [L1] we have

(a)
$$\begin{aligned}
&\rho_1 + \rho_2 + \rho_3 = R_1 + R_2 + R_3, \rho_1 + \rho_2 + \rho_4 = R_1 + R_2 + R_4,\\
&\rho_2 + \rho_4 + \rho_6 + \rho_7 + \rho_8 = 2R_1 - R_3, \rho_2 + \rho_3 + \rho_6 + \rho_7 + \rho_8 = 2R_1 - R_4,\\
&\rho_3 + \rho_5 + \rho_6 = R_1 - R_2 + R_3, \rho_4 + \rho_5 + \rho_6 = R_1 - R_2 + R_4, \rho_2 + \rho_6 = R_1 - R_5,
\end{aligned}$$

where $R_i = R^s_{\ddot{E}_i}$. Let $a_i = \boxed{\rho_i}$. Using 2.13(c) we obtain

$$\text{(b)} \quad \begin{aligned}
&a_1 + a_2 + a_3 = 3, a_1 + a_2 + a_4 = 3, a_2 + a_4 + a_6 + a_7 + a_8 = 1,\\
&a_2 + a_3 + a_6 + a_7 + a_8 = 1, a_3 + a_5 + a_6 = 1, a_4 + a_5 + a_6 = 1, a_2 + a_6 = 0.
\end{aligned}$$

Since $a_i \geq 0$ we deduce that $a_2 = a_6 = 0$. Since ρ_7, ρ_8 are algebraically conjugate we see that a_7, a_8 are algebraically conjugate. Since they are integers we have $a_7 = a_8$. The third equation in (b) becomes $a_4 + 2a_7 = 1$. Since $a_i \in \mathbf{N}$, it follows that $a_4 = 1, a_7 = 0$. Now (b) yields $(a_1, a_2, \dots, a_8) = (2,0,1,1,0,0,0,0)$. From this 1.4(c) follows easily in our case.

5.5. Assume that G/Z_G is of type E_6, E_7 or E_8 and that $\Gamma \cong \mathfrak{S}_3$. In this case the elements of $\mathrm{Irr}_{s,\mathcal{F}} G^{F^2}$ and of $\mathrm{Irr}' W_s \cap \mathcal{F}$ can be labelled as in 5.4. The equations 5.4(a) and their consequence 5.4(b) still hold in our case. But in this case we do not know a priori that $a_7 = a_8$. Thus we need an additional equation. As stated in [L2] and proved in [L3],[L6],[S], the almost characters

$$\rho_1 - \rho_3 - \rho_4 + \rho_5 + 2\rho_7 - \rho_8, \quad \rho_1 - \rho_3 - \rho_4 + \rho_5 - \rho_7 + 2\rho_8$$

are of the form $\chi_{A,\phi}$ where A is as in 4.5(a) for a suitable ϕ. Using 4.4 we deduce that

$$a_1 - a_3 - a_4 + a_5 + 2a_7 - a_8 = 0, \quad a_1 - a_3 - a_4 + a_5 - a_7 + 2a_8 = 0.$$

Substracting, we get $a_7 = a_8$. The argument continues as in 5.4 and the a_i are determined as there. We see that 1.4(c) holds in our case.

5.6. Assume that G/Z_G is of type G_2 and that $\Gamma \cong \mathfrak{S}_3$. Since in this case $G = Z_G \times (G/Z_G)$, we may assume that $Z_G = \{1\}$. The elements of $\mathrm{Irr}_{s,\mathcal{F}} G^{F^2}$ can be labelled as in 5.4 but $\mathrm{Irr}' W_s \cap \mathcal{F}$ consists now of E_1, E_2, E_3, E_4 so that the equations 5.4(a) (except the last one) hold. As in 5.4 from these equations we deduce

(a)
$$a_1 + a_2 + a_3 = 3, a_1 + a_2 + a_4 = 3, a_2 + a_4 + a_6 + a_7 + a_8 = 1,$$
$$a_2 + a_3 + a_6 + a_7 + a_8 = 1, a_3 + a_5 + a_6 = 1, a_4 + a_5 + a_6 = 1.$$

Since there is one less equation than in 5.4(b), we need additional information. As in 5.5, the almost characters

$$\rho_1 - \rho_3 - \rho_4 + \rho_5 + 2\rho_7 - \rho_8, \rho_1 - \rho_3 - \rho_4 + \rho_5 - \rho_7 + 2\rho_8$$

are of the form $\chi_{A,\phi}$ where A is as in 4.5(c) for a suitable ϕ. Using 4.4 we deduce

(b) $\quad a_1 - a_3 - a_4 + a_5 + 2a_7 - a_8 = 0, \quad a_1 - a_3 - a_4 + a_5 - a_7 + 2a_8 = 0.$

It is easy to see that the equations (a) and (b) have a unique solution with $a_i \in \mathbf{N}$ namely $(a_1, a_2, \ldots, a_8) = (2, 0, 1, 1, 0, 0, 0, 0)$. From this 1.4(c) follows easily in our case.

5.7. Assume that $\mathcal{F}$ consists of two irreducible representations E_1, E_2. They are in $\mathrm{Irr}' W_s$. In this case, G/Z_G is of type E_7 or E_8 and $\Gamma \cong \mathbf{Z}/2\mathbf{Z}$. The four elements of $\mathrm{Irr}_{s,\mathcal{F}} G^{F^2}$ are denoted by $\rho_1, \rho_2, \rho_3, \rho_4$ so that

$$\rho_1 + \rho_2 = R_1 + R_2, \rho_3 + \rho_4 = R_1 - R_2,$$

where $R_i = R^s_{\ddot{E}_i}$. Let $a_i = \boxed{\rho_i}$. Using 2.13(c) we obtain

$$a_1 + a_2 = 2, a_3 + a_4 = 0.$$

Since $a_i \in \mathbf{N}$, we have $a_3 = a_4 = 0$. As in 5.5, the almost character

$$\rho_1 - \rho_2 + \rho_3 - \rho_4$$

is of the form $\chi_{A,\phi}$ where A is as in 4.5(b) for a suitable ϕ. Using 4.4 we deduce that $a_1 - a_2 + a_3 - a_4 = 0$ hence $a_1 = a_2$. It follows that $(a_1, a_2, a_3, a_4) = (1, 1, 0, 0)$. Hence 1.4(d) holds.

5.8. Assume that G/Z_G is of type F_4 and that $\Gamma \cong \mathfrak{S}_4$. We may assume that $Z_G = \{1\}$. We have $s = 1$. Here $\mathrm{Irr}_{1,\mathcal{F}} G^{F^2}$ consists of 21 irreducible representations ρ_i and $\mathcal{F}$ consists of 11 representations E_i. We consider the analogues of the equations 5.4)(a); as explained in 2.14, they are obtained from the elements

$$\alpha_x = \sum_i c_{x,E_i} E_i$$

(see [L1, (5.10.3)]) by expressing $\sum_i c_{x,E_i} R^s_{\ddot{E}_i}$ as linear combinations with coefficients in $\mathbf{N}$ of ρ_j (using [L1, 4.23]). Now the various α_x are computed in [L4]. As in 5.4 from these equations we get a system of equations with coefficients in $\mathbf{N}$ for the unknown $a_i = \boxed{\rho_i}$. As in 5.5 we get four additional equations for the a_i attached to four almost characters which are of the form $\chi_{A,\phi}$ with A as in 4.5(c). These equations determine uniquely the a_i subject to $a_i \in \mathbf{N}$. From this 1.4(c) follows easily in our case.

The same argument applies in the case where G/Z_G is of type E_8 and $\Gamma \cong \mathfrak{S}_5$. (Here the situation is simpler as the additional equations coming from 4.4 are not needed. Instead we use, as in 5.4, the equations $a_i = a_j$ whenever ρ_i, ρ_j are algebraically conjugate.)

This completes the proof of Theorem 1.4.

6. Proof of Corollary 1.6

6.1. 1.6(a),(b),(f) follow immediately from 1.4(a),(b),(d). In the remainder of the proof we assume that $F(s) = s^{-1}, F(\mathcal{F}) = \mathcal{F}$ and that $|\mathcal{F}| \neq 2$. Since the transition from $\rho_{x,\xi}$ to $R_{y,\eta}$ is given by an invertible matrix [L1, (4.21.10)], it is enough to show that, if we assume that 1.6(c),(d),(e) hold, then 1.4(c) holds. Using 1.5(b) and [L1, (4.21.10)], we have

$$\rho_{x,\xi} = |Z_\Gamma(x)|^{-1}|Z_\Gamma(y)^{-1}| \sum_{y,\eta} \sum_{\substack{g\in\Gamma \\ xgyg^{-1}=gyg^{-1}x}} \mathrm{tr}(g^{-1}x^{-1}g,\eta)\mathrm{tr}(gyg^{-1},\xi)R_{y,\eta}$$

where (y,η) runs over a set of representatives for the M-orbits in $\mathcal{M}_{\Gamma,F^2}$. Hence, by our assumption, we have

$$\boxed{\rho_{x,\xi}} = |\Gamma|^{-1}|Z_\Gamma(x)|^{-1} \sum_{y}\sum_{\eta} \sum_{\substack{g\in\Gamma \\ xgyg^{-1}=gyg^{-1}x}} \mathrm{tr}(g^{-1}x^{-1}g,\eta)\mathrm{tr}(gyg^{-1},\xi)\psi(\tilde\eta)$$

where y runs over the elements of Γ and is subject to the condition that $Z_{\dot\Gamma}(y)$ meets $\Gamma\gamma$ and η runs over the irreducible representations of $Z_{\dot\Gamma}(y)$ such that η extends to a self dual representation $\tilde\eta$ of $Z_{\dot\Gamma}(y)$. By 2.11, for any $y, g\in\Gamma$ such that $xgyg^{-1} = gyg^{-1}x$, we have

$$\sum_\eta \mathrm{tr}(g^{-1}xg,\eta)\psi(\tilde\eta) = \sharp(z\in Z_{\dot\Gamma}(y); z^2 = g^{-1}xg)$$

hence

$$\begin{aligned}\boxed{\rho_{x,\xi}} &= |\Gamma|^{-1}|Z_\Gamma(x)|^{-1}\sum_y \sum_{g\in\Gamma; xgyg^{-1}=gyg^{-1}x} \mathrm{tr}(gyg^{-1},\xi)\sharp(z\in Z_{\dot\Gamma}(y); z^2=g^{-1}xg)\\ &= |\Gamma|^{-1}|Z_\Gamma(x)|^{-1} \sum_{y,g\in\Gamma, z\in\Gamma\gamma; zy=yz, z^2=g^{-1}xg} \mathrm{tr}(gyg^{-1},\xi).\end{aligned}$$

Setting $y' = gyg^{-1}, z' = gzg^{-1}$ we obtain

$$\begin{aligned}\boxed{\rho_{x,\xi}} &= |Z_\Gamma(x)|^{-1} \sum_{\substack{y'\in\Gamma \\ z'\in\Gamma\gamma \\ z'y'=y'z' \\ z'^2=x}} \mathrm{tr}(y',\xi)\\ &= |Z_\Gamma(x)|^{-1}\sum_{y'\in Z_\Gamma(x)} \mathrm{tr}(y',\sqrt{x})\mathrm{tr}(y',\xi) = [\xi:\sqrt{x}],\end{aligned}$$

as required.

7. More on character sheaves

7.1. Let G be as in 1.1. In this section we assume that

(a) the connection stated in [L2] between the almost characters of G^{F^2} and the character sheaves of G holds;

(b) q is sufficiently large.

Under these assumptions and with the notation of 4.1 we have the following result suggested by Corollary 1.6. (It is likely that the same holds without any assumptions.)

Proposition 7.2. *Let A be a character sheaf of G such that there exists an isomorphism $\phi : (F^2)^*(A) \xrightarrow{\sim} A$.*

(a) If $F^*(A) \ncong DA$, then $\boxed{\chi_{A,\phi}} = 0$ for any ϕ as above.

(b) If $F^*(A) \cong DA$, then there is a canonical choice for ϕ. For this ϕ we have $\boxed{\chi_{A,\phi}} = (-1)^{\dim \operatorname{supp}(A)}$.

In case (b), ϕ is defined as follows: we choose an isomorphism $\phi' : F^*(A) \xrightarrow{\sim} DA$ and we define ϕ to be the composition

$$(F^2)^*A \xrightarrow{F^*(\phi')} F^*(DA) = D(F^*(A)) \xrightarrow{D(\phi')^{-1}} A.$$

This is clearly independent of the choice of ϕ'.

7.3. We sketch a proof of 7.2. It is known that there exists a canonical finite partition $G = \bigcup_{j \in J} X_j$ of G into locally closed smooth irreducible subvarieties X_j with $\dim X_j = d_j$ such that $\bar{X}_j$ is a union of subvarieties $X_{j'}$ and the following holds:

(a) If A is any character sheaf of G, then for a unique $j \in J$ we have
$\operatorname{supp}(A) \subset \bar{X}_j$;
$\mathcal{H}^{-d_j}(A)|_{X_j}$ is an irreducible local system;
$\mathcal{H}^i(A)|_{X_j} = 0$ if $i \neq -d_j$; if $j' \neq j$, then $\mathcal{H}^i(A)|_{X_{j'}} \neq 0 \implies d_{j'} + i < 0$.

Moreover, there exists an integer $N \geq 1$ (depending only on G) such that for any character sheaf A of G we have

(b) $$\sum_i \dim \mathcal{H}^i_g(A) \leq N \text{ for all } g \in G.$$

Assume now that A is a character sheaf of G such that $(F^2)^*(A) \cong A$. It is known that we can choose $\phi : (F^2)^*(A) \xrightarrow{\sim} A$ such that, if j is as in (a), then for any $n \geq 1$ and any $g \in G^{F^{2n}} \cap X_j$, the map $\phi^n_g : \mathcal{H}^{-d_j}_g(A) \to \mathcal{H}^{-d_j}_g(A)$ it is equal to $(q^2)^{(a-d_j)/2}$ times a linear transformation of finite order. Here $a = \dim G$. For such ϕ, the following holds, by Gabber's purity theorem:

(c) *for any i and any $g \in G^{F^2}$, any eigenvalues μ of $\phi_g : \mathcal{H}^i_g(A) \to \mathcal{H}^i_g(A)$ satisfies $|\mu| \leq (q^2)^{(a+i)/2}$.*

Here $||$ is complex absolute value. (This notation diverges from that in 1.1.) We have

(d) $$\boxed{\chi_{A,\phi}} = \sum_{j' \in J; F(X_{j'}) = X_{j'}} \alpha_{j'}$$

where

$$\alpha_{j'} = |G^F|^{-1} \sum_{g \in X^F_{j'}} \sum_i (-1)^i \operatorname{tr}(\phi_g, \mathcal{H}^i_g(A)).$$

If $j' \neq j$, then by (a) we have

$$\alpha_{j'} = |G^F|^{-1} \sum_{g \in X^F_{j'}} \sum_{i; d_{j'}+i<0} (-1)^i \operatorname{tr}(\phi_g, \mathcal{H}^i_g(A)).$$

By (b),(c) any eigenvalue μ of the last ϕ_g in the sum satisfies $|\mu| \leq q^{a+i} \leq q^{a-d_{j'}-1}$ hence

$$|\alpha_{j'}| \leq |G^F|^{-1} \sharp(X^F_{j'}) N q^{a-d_{j'}-1}.$$

Now $|\sharp(X_{j'}^F) - q^{d_{j'}}| \le Cq^{d_{j'}-1/2}$ where C is a constant. Hence

$$|\alpha_{j'}| \le |G^F|^{-1}N(q^{a-1} + Cq^{a-3/2}),$$

or

(e) $$|\alpha_{j'}| \le Nq^{-1} + C'q^{a-3/2}$$

where C' is a constant. Let $\mathcal{L}$ be the irreducible local system $\mathcal{H}^{-d_j}(A)|_{X_j}$. By (a) we have $\alpha_j = |G^F|^{-1}(-1)^{d_j}\alpha'$ where

$$\alpha' = \sum_{g \in X_j^F} \mathrm{tr}(\phi_g, \mathcal{L}_g).$$

As in the proof of 4.2, we see that α' is equal to the alternating sum of traces of Frobenius on $H_c^i(X_j, F^*(\mathcal{L}) \otimes \mathcal{L})$ with respect to an isomorphism $F^*(F^*(\mathcal{L}) \otimes \mathcal{L}) \xrightarrow{\tilde{\phi}} F^*(\mathcal{L}) \otimes \mathcal{L}$ induced by ϕ. If $F^*(A) \not\cong DA$, then $F^*(\mathcal{L}) \otimes \mathcal{L}$ contains no summand $\bar{\mathbf{Q}}_l$, hence $H_c^{2d_j}(X_j, F^*(\mathcal{L}) \otimes \mathcal{L}) = 0$. If $F^*(A) \cong DA$, then $F^*(\mathcal{L}) \otimes \mathcal{L}$ contains exactly one summand $\bar{\mathbf{Q}}_l$ hence $H_c^{2d_j}(X_j, F^*(\mathcal{L}) \otimes \mathcal{L}) = \bar{\mathbf{Q}}_l$ with Frobenius acting as q^a (by our choice of ϕ). In both cases, for $i < 2d_c$ we have

$$|\mathrm{tr}(F^*, H_c^i(X_j, F^*(\mathcal{L}) \otimes \mathcal{L}))| \le N'q^{a-1/2}$$

where N' is a constant. It follows that in case 7.2(a) we have

(f) $$|\alpha_j| \le N_1 q^{-1/2}$$

and in case 7.2(b) we have $|\alpha_j - (-1)^{d_j}|G^F|^{-1}q^a| \le N_1 q^{a-1/2}$, or

(g) $$|\alpha_j - (-1)^{d_j}| \le N_2 q^{-1/2}$$

where N_1, N_2 are constants. From (d),(e),(f),(g) we deduce

(h) $|\boxed{\chi_{A,\phi}}| \le N_1' q^{-1/2}$ in case 7.2(a),

(i) $|\boxed{\chi_{A,\phi}} - (-1)^{d_j}| \le N_1' q^{-1/2}$ in case 7.2(b).

where N_1' is a constant. From (h),(i) we see using 7.1(b) that

(j) $\boxed{\chi_{A,\phi}}$ *is very close to* 0 *in case 7.2(a) and very close to* $(-1)^{d_j}$ *in case 7.2(b).*

From 7.1(a) we see that for some integer $n \ge 1$ (depending only on G), $n\chi_{A,\phi}$ is a linear combination of irreducible characters of G^{F^2} with coefficients cyclotomic integers. It follows that $n\boxed{\chi_{A,\phi}}$ is an integer in a (fixed) cyclotomic field. Using this and (j) (which holds with respect to any complex absolute value) we deduce that $\boxed{\chi_{A,\phi}}$ equals 0 in case 7.2(a) and equals $(-1)^{d_j}$ in case 7.2(b), as required.

8. A generalization to symmetric spaces

8.1. Theorem 1.4 implies that, when ρ varies through $\mathrm{Irr}G^{F^2}$, $\boxed{\rho}$ is bounded above by a bound which depends only on G and not on the F_q-rational structure. This fact can be generalized as follows.

Let G be as in 1.1. Assume that q is odd. Let $\theta : G \to G$ be an involutive automorphism which commutes with F. Let K be an F-stable closed subgroup of the fixed point set G^θ which contains $(G^\theta)^0$.

Theorem 8.2. *There exists a constant* $C > 0$ *depending only on* G *(and not on the* F_q*-rational structure or on* K*) such that the following holds: for any* $\rho \in \mathrm{Irr}G^F$*, the dimension of the space* ρ^{K^F} *of* K^F*-invariant vectors in* ρ *has dimension* $\le C$*.*

For simplicity we assume that G has connected center. We use the same strategy as in 2.14. Let T be an F-stable maximal torus of G, let $\lambda \in (T^F)\check{}$ and let R_T^λ be the virtual representation of G^F attached to (T, λ) in [DL] (relative to G, F). (This notation disagrees with that in 2.1.) Now $\rho \mapsto \dim \rho^{K^F}$ extends by linearity to a function $f \mapsto a(f)$ from formal $\mathbf{Q}$-linear combinations of irreducible representations of G^F to $\mathbf{Q}$. In particular, $a(R_T^\lambda) \in \mathbf{Z}$ is well defined. From the explicit formula for $a(R_T^\lambda)$ given in [L5, 3.3] we see that $|a(R_T^\lambda)| \le C_1$ where C_1 is an integer ≥ 1 which depends only on G. Let $\rho \in \mathrm{Irr} G^F$. As in 2.14 we see, using [L1, 6.17(ii), (6.18.2)], that there exists a representation ρ' of G^F such that ρ is a direct summand of ρ' and such that the character of ρ' is a $\mathbf{Z}$-linear combination of characters of $R_{\tilde{E}}$ (as in [L1, 6.17(i)]). Moreover, the sum of absolute values of coefficients in this linear combination is bounded above by a constant $C_2 \ge 1$. By the definition of $R_{\tilde{E}}$ we see that $R_{\tilde{E}}$ is a $\mathbf{Q}$-linear combination of various R_T^λ and the sum of absolute values of coefficients in this linear combination is bounded above by a constant $C_3 \ge 1$. It follows that $a(R_{\tilde{E}}) \le C_1 C_3$ and $a(\rho') \le C_1 C_2 C_3$. Clearly $0 \le a(\rho) \le a(\rho')$. Hence $a(\rho) \le C_1 C_2 C_3$. The theorem is proved.

8.3. It would be very interesting to carry out completely the strategy of 2.14 in the present case and compute explicitly $a(\rho)$ for any ρ.

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