

THE DYNKIN DIAGRAM R -GROUP

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ABSTRACT. We define an abelian group from the Dynkin diagram of a split real linear Lie group with abelian Cartan subgroups, G , and show that the $R_{\delta,0}$ -groups defined by Knapp and Stein are subgroups of it. The proof relies on Vogan's approach to the R -groups. The R -group of a Dynkin diagram is easily computed just by looking at the diagram, and so it gives, for instance, quick proofs of the fact that the principal series with zero infinitesimal character of the split groups E_6 , E_8 , G_2 or F_4 are irreducible. The Dynkin diagram subgroup also implicitly describes a small Levi subgroup, which we hope might play a role in computing regular functions on principal nilpotent orbits. We present in the end a conjecture and some evidence in this direction.

1. INTRODUCTION

The reducibility of principal series representations of real reductive Lie groups has been studied extensively. In the approach of Knapp and Stein [1], understanding the reducibility boils down to computing certain small abelian subgroups of the Weyl group, called R -groups. In the case when the group is linear and split, these R -groups can be described quite concretely. We will concentrate on a particular class of principal series for these split groups, those with infinitesimal character equal to zero. They are obtained by parabolic induction as $\text{Ind}_{MAN}^G \delta \otimes 0$, for all representations δ of M , which is a finite abelian group for all the cases which we will consider. There is an action of the Weyl group on the representations of M , and representations in the same Weyl group conjugacy class yield isomorphic principal series. We show that we can reduce the understanding of the decomposition into irreducible components for all principal series with infinitesimal character zero *at once*, to the same problem for a small Levi subgroup, which at the Lie algebra level consists of several copies of $sl(2, \mathbb{R})$. The idea of reducing the understanding of the R_{δ} -groups to $SL(2, \mathbb{R})$ for each δ was used by Knapp and Zuckermann [2] in their approach to the R -groups. We add to that approach the fact that, with a proper choice of a representative in each Weyl group conjugacy class, which we call acceptable, this reduction can be done simultaneously for all principal series with infinitesimal character zero.

As a by-product of this approach, we describe a finite group which we can construct combinatorially by looking at the Dynkin diagram of a simple split group, and show that each $R_{\delta,0}$ group has to be a subgroup of it (Theorem 1). As noted in the abstract, we believe that these results might also be useful in computing regular functions on principal nilpotent orbits. Such computations would be then

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interesting from the point of view of the orbit method, which was our original motivation. In order to make the connection a bit clearer, we mention that the irreducible pieces of principal series with zero infinitesimal character should be regarded as the representations associated to principal nilpotent orbits, in the orbit method picture.

2. SETTING AND BACKGROUND

G will be a real, linear and split group with abelian Cartan subgroups. In order to define the Dynkin diagram subgroup we will assume G to be simple, but we can of course extend the results to semisimple groups in the standard fashion.

The corresponding Lie algebra will be denoted by \mathfrak{g} , and θ will be a Cartan involution yielding a Cartan decomposition of the Lie algebra, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$.

We let K denote the maximal compact subgroup of G ; its Lie algebra is \mathfrak{k} .

\mathfrak{a} is the split Cartan subalgebra of \mathfrak{g} , and let $A = \exp \mathfrak{a}$.

The roots of \mathfrak{g} with respect to the split Cartan subalgebra will be denoted by $\Delta(\mathfrak{g}, \mathfrak{a})$. A choice of positive roots will be denoted by $\Delta^+(\mathfrak{g}, \mathfrak{a})$.

$M = Z_K(A)$ is an abelian group, as G has abelian Cartan subgroups.

M' denotes the normalizer of M in K , and so $W = M'/M$ is the analytic (or real) Weyl group.

δ will denote a representation of M .

We will also fix an $\text{Ad}(G)$ -invariant, θ -invariant, nondegenerate symmetric bilinear form B on $\mathfrak{g} \times \mathfrak{g}$ so that $B_\theta(X, Y) = -B(X, \theta Y)$ is positive definite. From B we construct an inner product $\langle \cdot, \cdot \rangle$ on the dual space of \mathfrak{a} , in the usual way.

Knapp and Stein [1] defined an R -group for each representation δ of M and ν of A , denoted by $R_{\delta, \nu}$. The case when $\nu = 0$ was treated first, as the case of general ν can be reduced to it. Since our interest is in principal series with infinitesimal character zero, we will only talk from here on about the groups $R_{\delta, 0}$, which we will denote by R_δ . As proved by Knapp and Stein, these R -groups are finite abelian groups, with cardinality a power of two, having the property that the dual group \widehat{R}_δ acts transitively on the irreducible components of the principal series $I(\delta \otimes 0)$. Hence $I(\delta \otimes 0)$ splits into $|R_\delta|$ irreducible pieces.

For a more concrete realization of these groups we will follow the construction of R_δ in Vogan's book [8]; for proofs and more details the reader is referred to section 4.3 in [8]. The main idea is to reduce the understanding of R_δ to a small Levi, for each δ .

For each root we can choose an injection $\phi_\alpha : \mathfrak{sl}(2, \mathbb{R}) \longrightarrow \mathfrak{g}$ with the following properties:

$$\begin{aligned} H_\alpha &:= \phi_\alpha \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \in \mathfrak{a}, \\ \phi_\alpha(-X^t) &= \theta \phi_\alpha(X), \\ \phi_\alpha \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right) &\in \alpha \text{ root space}. \end{aligned}$$

Using the above, we then define

$$\begin{aligned} Z_\alpha &= \phi_\alpha \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) \in \mathfrak{k}, \\ \sigma_\alpha &= \exp\left(\frac{\pi}{2} Z_\alpha\right) \in K, \\ m_\alpha &= \sigma_\alpha^2. \end{aligned}$$

Definition 1. A root $\beta \in \Delta^+(\mathfrak{g}, \mathfrak{a})$ is called *bad* if $\delta(m_\alpha) = -1$. A *good* root is a root which is not bad, so $\delta(m_\alpha) = 1$.

Denote the collection of all good roots for δ by Δ_δ ; it is shown in [8] that they form a root system. Let the corresponding Weyl group be denoted by W_δ^0 . We need one more ingredient before we can define the R -group, namely, the action of the analytic Weyl group on representations of M .

Since M' normalizes M , we can define the following action of $w = [x] \in W$ on δ ($x \in M'$ is any representative for w):

$$w \cdot \delta(y) = \delta(x^{-1}yx).$$

The action is well defined, and W_δ will denote the stabilizer of δ in W with respect to this action. In particular, it turns out that every element in W_δ^0 stabilizes δ , so $W_\delta^0 \subset W_\delta$.

Definition 2. The R -group of δ is defined to be $R_\delta = W_\delta / W_\delta^0$.

We assume fixed an ordering of the positive good roots; then there exists in fact a semidirect product decomposition

$$W_\delta = R_\delta W_\delta^0,$$

where $R_\delta = \{w \in W_\delta \mid w(\Delta_\delta^+) = \Delta_\delta^+\}$. Note, as an aside, that such a decomposition is fairly general, holding also outside of the split group setting, as noted in [1]; what is particularly nice in the split case is the description of good roots, and that of R_δ .

The more concrete description of R_δ follows from the following proposition (see [8], section 4.3):

Proposition 1. Choose an ordering of $\Delta(\mathfrak{g}, \mathfrak{a})$ for which half the sum of the positive good roots, ρ_δ , is dominant. Then the collection of roots $\Delta_S = \{\alpha \mid \langle \alpha, \rho_\delta \rangle = 0\}$ consists of strongly orthogonal simple roots and their negatives. Furthermore, $R_\delta \subset W(\Delta_S)$.

Also we recall Corollary 4.3.20 from [8] which will be used repeatedly:

Proposition 2. If α, β and γ are roots so that $\overset{\vee}{\alpha} + \overset{\vee}{\beta} = \overset{\vee}{\gamma}$, then $m_\gamma = m_\alpha m_\beta$.

We will typically put this proposition to use in the following manner: to show, for instance, that if α and β are bad and $\overset{\vee}{\alpha} + \overset{\vee}{\beta} = \overset{\vee}{\gamma}$, then γ is good, etc.

3. THE DYNKIN DIAGRAM R -GROUP

To the Dynkin diagram of a simple split group, DD , we attach now a finite abelian group denoted by R_{DD} , which can be constructed easily by looking at the diagram. We can of course extend the construction to semisimple groups by taking the product of the Dynkin diagram R -groups of the simple factors. Since the simply laced case is a lot clearer, and also since it includes all the main ideas of the proof, we will first define a preliminary version of the Dynkin diagram R -group in that setting.

Definition 3. As a set, $R_{DD} = \{S \mid S \text{ is a subset of strongly orthogonal simple roots, so that any } x \notin S \text{ is connected to an even number of elements of } S\}$. R_{DD} is made into a group with the operation of symmetric difference of subsets.

Remark 1. It is not clear that R_{DD} is closed under symmetric difference, hence well defined; we will show it later, for the general definition.

As a first example, we consider the Dynkin diagram of E_7 . The only subsets satisfying the condition in the definition are the empty set and the collection of simple roots marked in Figure 3.1. Therefore, the Dynkin diagram R -group of E_7 is \mathbb{Z}_2 .

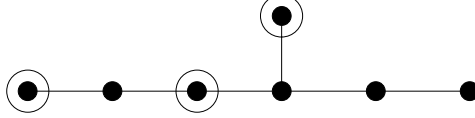


FIGURE 3.1. Nontrivial element in Dynkin diagram R -group of E_7 .

Similarly, one can check that the Dynkin diagram R -group of E_6 or E_8 is trivial.

In the general case, we have to keep track of the different root lengths, for which we will need first a directed, labeled version of the Dynkin diagram.

Definition 4. To each Dynkin diagram DD we associate a labeled directed graph, Γ_{DD} , whose vertices are the vertices of the Dynkin diagram. Two vertices of the Dynkin graph, α and β , are connected by an arrow from α to β labeled by the integer $n_{\alpha,\beta} = \frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}$ whenever α and β are connected by an edge in the Dynkin diagram.

Naturally, for the simply laced groups this graph will only have edges labeled one. Next we define the Dynkin diagram R -group, R_{DD} :

Definition 5. Consider the directed Dynkin graph Γ_{DD} associated to the Dynkin diagram, DD , of a simple split group. As a set, define R_{DD} to be the collection of all subsets of strongly orthogonal roots, S , having the following property: for each vertex $\gamma \notin S$ the sum of the labels on arrows going out of γ and into elements of S is even, namely, $\sum_{\alpha \in S} n_{\gamma,\alpha}$ even. R_{DD} is made into a group with the operation of symmetric difference of subsets.

Remark 2. Proposition 4 will show that R_{DD} is closed under symmetric difference, and hence, well defined.

For example, the directed graph corresponding to the split form of F_4 is given in Figure 3.2, and one can see that there are no subsets S with the property above, hence the Dynkin diagram R -group of F_4 is trivial.

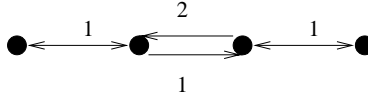


FIGURE 3.2. Directed graph for F_4

A full list of these directed graphs is provided in Figure 7.1, Appendix B. As an easy graph theory diversion, we can state the following propositions, which will show that R_{DD} is closed under symmetric difference.

Proposition 3. *The union of all the elements of R_{DD} is a strongly orthogonal set.*

Proof. We will only give a sketch of the proof. First, all we need to know about the Dynkin diagrams is that they are acyclic, connected graphs with at most one vertex of degree three.

Likewise, all we need to know about the corresponding Dynkin graphs defined above is that:

- all edges are labeled ‘1’, possibly with the exception of one edge, which can be labeled ‘2’ or ‘3’ (as a matter of fact just the parity matters, so we could simply take the labels modulo two);
- if the Dynkin diagram has a vertex of degree 3, then all edges are labeled ‘1’.

Now we can assume that the union of elements of R_{DD} is not strongly orthogonal, hence there exists an edge such that the adjacent vertices belong to different elements of R_{DD} . If more exist, then pick the one closest to an endpoint of the graph. There are at this point a few cases, all quickly shown to be impossible given the properties of the Dynkin graphs mentioned above. \square

It is equally easy to show:

Proposition 4. *R_{DD} is closed under symmetric difference of subsets.*

Proof. The symmetric difference of two elements S and P of R_{DD} will again be strongly orthogonal by the previous proposition. Consider any element x adjacent to elements in S and P —since it cannot be in either set S or P , then it must satisfy the parity condition with respect to both S and P , and, as we can see after a quick consideration, also with respect to their symmetric difference. \square

Together these two propositions show that the Dynkin diagram R -group is in fact well defined.

4. ACCEPTABLE REPRESENTATIONS OF M

We want to prove the following result:

Theorem 1. *The R -group R_δ associated to a representation δ of M is a subgroup of the Dynkin diagram R -group, R_{DD} .*

Define first a distinguished (although not unique) member in each W -conjugacy class of representations of M , which we call *acceptable*.

Start by fixing, once and for all, an ordering of all the roots, $\Delta^+(\mathfrak{g}, \mathfrak{a})$. Recall from before that we denoted half the sum of the positive good roots for a certain representation δ by ρ_δ .

Definition 6. A representation of M is called **acceptable** if ρ_δ is dominant.

The following proposition shows that we are not in fact losing any representations by making this choice, but simply choosing one in each Weyl group conjugacy class. In fact, to be more precise, there may be more than one acceptable representation in each conjugacy class. Note that representations of M in the same W -conjugacy class have congruent corresponding R -groups.

Proposition 5. *Any representation δ of M is conjugate to an acceptable one.*

Proof. Conjugate by a Weyl group element $w = [x]$ in order to make ρ_δ dominant. Then let $\delta' = w \cdot \delta$. Since $m_{w \cdot \alpha} = x m_\alpha x^{-1}$, it follows that the good roots for δ' are obtained by applying w to the good roots for δ . In order to conclude that $\rho_{\delta'} = w \cdot \rho_\delta$ which is dominant, and that δ' is acceptable, we still need to prove that w takes good positive roots to good positive roots. In other words, we want to see that $w\Delta_\delta^+ = \Delta_{\delta'}^+$. Assume β is positive, and good for δ . Then

$$\langle w\rho_\delta, w\beta \rangle = \langle \rho_\delta, \beta \rangle > 0.$$

Since $w\rho_\delta$ is dominant, $\langle w\rho_\delta, w\beta \rangle > 0$ only if $w\beta$ is positive, and so we can conclude that $w\Delta_\delta^+ \subset \Delta_{\delta'}^+$. The other inclusion follows similarly. \square

The following is the crucial lemma towards proving Theorem 1:

Lemma 1. *Let δ be an acceptable representation. Let $w = s_{\alpha_1} \dots s_{\alpha_k} \in R_\delta$. Then the collection of simple roots $S(\delta, w) = \{\alpha_1, \dots, \alpha_k\}$ is an element of R_{DD} .*

Proof. Assume the lemma fails—since $\{\alpha_1, \dots, \alpha_k\}$ are certainly strongly orthogonal by Proposition 1, it follows that there must be a simple root α for which $\sum_{\beta \in S(\delta, w)} n_{\alpha, \beta}$ is odd. Denote the set of roots in $S(\delta, w)$ which are also adjacent to α by $\text{Adj}(\alpha)$. We may rewrite the condition above as $\sum_{\beta \in \text{Adj}(\alpha)} n_{\alpha, \beta}$ odd.

First we would like to see that, without loss of generality, we can conjugate δ while leaving it acceptable so that the simple root α is a good root. Assume α bad, and also assume, to fix notation, that $\text{Adj}(\alpha) = \{\beta_1, \dots, \beta_t\}$. Of course, t cannot be, in fact, greater than 3.

Let $\delta' = w \cdot \delta$ —it is still acceptable, since $w\rho_\delta = \rho_{\delta'} = \rho_\delta$ is dominant. The good roots for δ' are obtained from the good roots of δ by applying w , so $\Delta_{\delta'} = w\Delta_\delta$. In particular, we know:

- $w\alpha$ is good for δ , since by definition the elements of R_δ preserve the good roots.
- $w(w\alpha) = \alpha$, as w is a product of orthogonal reflections, and therefore $w^2 = 1$.

It follows that α is good for δ' .

We will need a bit more, namely that after conjugating δ we still have $w \in R_{\delta'}^c$. So show that w preserves the positive good roots for δ' , and also that it stabilizes δ' .

The positive good roots for δ' are given by $w\Delta_\delta^+$, so

$$w\Delta_{\delta'}^+ = ww\Delta_\delta^+ = \Delta_\delta^+ = w\Delta_\delta^+ = \Delta_{\delta'}^+;$$

the positive good roots for δ' are preserved by the action of w . Also,

$$w \cdot \delta'(m_\gamma) = w \cdot \delta(m_{w\gamma}) = \delta(m_{w^2\gamma}) = \delta(m_\gamma) = \delta(m_{w\gamma}) = \delta'(m_\gamma).$$

Therefore, without loss of generality, we can assume that α is good for the acceptable representation δ , and we have not changed the fact that $w \in R_\delta^c$.

Now we can show that we reach a contradiction, by showing that w must send α to a bad root.

$$\begin{aligned} w \cdot \alpha &= 2\langle \alpha, \alpha \rangle^{-1} s_{\alpha_1} \dots s_{\alpha_k} \alpha \\ &= \frac{2}{\langle \alpha, \alpha \rangle} \left(\alpha - \frac{2\langle \alpha, \beta_1 \rangle}{\langle \beta_1, \beta_1 \rangle} \beta_1 - \dots - \frac{2\langle \alpha, \beta_t \rangle}{\langle \beta_t, \beta_t \rangle} \beta_t \right) \\ &= \overset{\vee}{\alpha} - n_{\alpha, \beta_1} \overset{\vee}{\beta_1} - \dots - n_{\alpha, \beta_t} \overset{\vee}{\beta_t}, \end{aligned}$$

since all roots α_i are strongly orthogonal, and the only ones adjacent to α are in $\text{Adj}(\alpha)$. By our assumption the co-root above is a sum of an odd number of bad

co-roots and a good co-root, hence by Proposition 2 $w \cdot \alpha$ must be bad. We have thus reached a contradiction, since elements in R_δ are supposed to preserve the subset of positive good roots, and so the lemma follows. \square

With the previous lemma we can now prove Theorem 1:

Proof. For any acceptable representation of M , δ , we can construct a map $\phi : R_\delta \rightarrow R_{DD}$ by

$$w = s_{\alpha_1} \dots s_{\alpha_k} \longrightarrow S = \{\alpha_1, \dots, \alpha_k\}.$$

For further use call S the underlying set of w . Lemma 1 shows that the map is well defined; it is clearly an injection. It remains to show that the map also respects the group operations.

For this consider $w_1, w_2 \in R_\delta$. Since the union of the underlying sets of simple roots is still strongly orthogonal by Proposition 3, all the simple reflections commute. Now it is immediate that the underlying set of $w_1 w_2$ is the symmetric difference of the underlying sets of w_1 and w_2 , which shows the map ϕ preserves the group operations. \square

It is now an easy exercise to show that all R_δ are trivial for the split exceptional groups E_6 , E_8 , F_4 and G_2 for instance, or for $SL(2k+1, \mathbb{R})$. We collect all nontrivial Dynkin diagram R -groups in Figure 7.2, Appendix B.

Theorem 1 also provides us with a Levi subgroup: in the usual way a Levi subgroup can be described by a collection of simple roots, namely, all simple roots appearing in the Dynkin diagram subgroup. It has the property that all reducible principal series $I(\delta \otimes 0)$ with δ acceptable already split at the level of this small Levi. We will describe it more carefully and present a conjecture in a later section.

We would like to give more information about the acceptable representations. It would be quite easy to give a full list of all the acceptable representations, at least in the simple classical split groups, but instead we prefer to show how they behave under a certain reduction to a smaller subgroup. Also, we will define a particular acceptable representation, the *maximally bad* one, which defines on its own the Levi subgroup mentioned above, and whose corresponding R -group will be shown to equal R_{DD} .

Definition 7. A representation of M is called *maximally bad* if all simple roots are bad.

Remark 3. If the group G is connected, then M is generated by the m_α for all α simple [5], and hence there is at most one maximally bad representation. Otherwise, there can be more such representations, but the corresponding sets of good roots, and therefore also ρ_δ , are the same for all of them. Unfortunately, it is not always clear when a maximally bad representation exists, due to the restrictions (such as linearity) imposed on the group G . For instance, there is no maximally bad representation for $SO_0(n+1, n)$, but this group happens to have a linear double cover for which such a representation exists.

We will denote a maximally bad representation by δ_0 , and show that it is always acceptable. We can show this in general for the simply laced split groups; however, for the remaining simple split groups ($SO_0(n+1, n)$, $Sp(2n, \mathbb{R})$, the split forms of F_4 and G_2) we only have a case-by-case proof at this point. Since it is not a

particularly enlightening computation, we will relegate it to Appendix A and only do the simply laced case here.

The main tool will be a reduction to a subalgebra containing all roots perpendicular to a certain simple root, so we have to see that acceptable representations remain acceptable under this reduction, and likewise maximally bad representations stay maximally bad. This will be done in the next few lemmas, but first we introduce some notation:

Let α be a simple root, and denote by G^α the centralizer of H_α in G . Its Lie algebra is \mathfrak{g}^α , the centralizer of H_α in \mathfrak{g} , and the corresponding root system consists of all roots γ satisfying $\langle \gamma, \alpha \rangle = 0$. Recall also that we have fixed an ordering and hence a set of positive roots $\Delta^+(\mathfrak{g}, \mathfrak{a})$. Finally, G^α also inherits a notion of good and bad roots from G .

Proposition 6. *Let $\mathfrak{g}' = \mathfrak{g}^\alpha$ as above. Let Π' be the simple roots for \mathfrak{g}' , with order inherited from Δ^+ . Then any α' in Π' is a sum of an odd number of simple roots in \mathfrak{g} , $\alpha' = \sum m_i \alpha_i$, where $\sum m_i = 2k + 1$.*

Remark 4. Combined with Proposition 2, this proposition shows that the restriction of a maximally bad representation to \mathfrak{g}^α is still maximally bad, since any new simple root is a sum of an odd number of old simple roots, and hence also bad. This fact is unfortunately false outside of the simply laced case; as an example take $Sp(4, \mathbb{R})$ and α equal to the long simple root. Then \mathfrak{g}^α has a single simple root, which is good for the restriction of the maximally bad representation to \mathfrak{g}^α .

Proof. The proof consists of adapting some parts of Lemma 5.3.6 in [8]. Show by induction on M , $\sum m_i \leq M$.

Case 1: $M = 1$, clear.

Case 2: $M = 2$. Assume we have a new simple root $\alpha' = \alpha_1 + \alpha_2$. Then $\langle \alpha_1 + \alpha_2, \alpha \rangle = 0$, but we can easily see that the only way we could get this would be if $\langle \alpha_i, \alpha \rangle = 0$ for $i = 1, 2$, contradicting the assumption that α' was simple.

Induction step: reduce from M to $M - 2$. If $\sum m_i > 2$, then there exists a simple root γ so that $\alpha' - \gamma$ is a root. If $\langle \gamma, \alpha \rangle = 0$, then $\alpha' = \gamma + (\alpha' - \gamma)$, not simple in \mathfrak{g}' , contradiction. So we must have $\langle \gamma, \alpha \rangle = -1$, hence $\alpha' - \gamma - \alpha$ is a root. Note it is also perpendicular to γ :

$$\langle \alpha' - \gamma - \alpha, \gamma \rangle = 1 - 2 + 1 = 0.$$

Now show that, in fact, $\beta = \alpha' - \gamma - \alpha, \gamma$ is simple in \mathfrak{g}^β ; if not, we have $\beta = \eta_1 + \eta_2$, $\langle \eta_i, \gamma \rangle = 0$. This further gives

$$\alpha' = \gamma + \alpha + \eta_1 + \eta_2.$$

Note that $\langle \eta_1 + \eta_2, \alpha \rangle = \langle \alpha' - \gamma - \alpha, \alpha \rangle = 1 - 2 = -1$, so without loss of generality, say $\langle \eta_1, \alpha \rangle = 0$ and $\langle \eta_2, \alpha \rangle = -1$. Then

$$\alpha' = \eta_1 + (\gamma + \alpha + \eta_2)$$

and we only need to show that $\gamma + \alpha + \eta_2$ is a root to get to a contradiction with the fact that α' is simple. But this is clear, since $\langle \eta_2 + \alpha, \gamma \rangle = -1$. So we can conclude that β must, in fact, be simple. Since $\beta = \sum m_i \alpha_i$ with $\sum \alpha_i \leq M - 2$, by induction it is a sum of an odd number of simple roots, and hence so is α' . \square

In order to see that acceptable roots stay acceptable under a restriction of the kind mentioned above, we need to understand how the half sum of positive good

roots ρ_δ behaves under restriction to \mathfrak{g}^α . It turns out to be easier to understand how $2\rho_\delta - \rho = \rho_\delta - \rho_{\text{bad}}$ behaves, in the same way that it is easy to understand how the difference between the sum of positive compact roots and positive noncompact roots behaves, as in [6] or [8]. The reason for the similarity is the following: consider α to be a simple bad root, and write all the other roots as belonging to α -strings. Since we have limited ourselves to the simply laced case, the strings will have length 1 or 2. Each string of length two will contain a good and a bad root, by Proposition 2. Similarly, if α had been a simple noncompact root, then the α -strings through various roots would have contained alternating compact and noncompact roots. Then the same proofs as in [6] or [8] can be used here essentially with no modification, as we will show.

Proposition 7. *Let α be a simple root, $\mathfrak{g}' := \mathfrak{g}^\alpha$ as before, L^α the corresponding Levi. We can restrict our representation δ to $L^\alpha \cap M$, and there we have the same notions of good, bad roots and so on, with \mathfrak{g}' in place of \mathfrak{g} . Then the following relation holds:*

$$2\rho'_\delta - \rho' = (2\rho_\delta - \rho)|_{\mathfrak{g}'}.$$

Proof. The proof is the same as for the equivalent result for compact vs. noncompact roots (Lemma 5.3.5 in [8]); we write all roots as α strings, which will have length at most two, since we are in the simply laced case. Strings of length two will contribute nothing in \mathfrak{g}' since they will contain a good and a bad root, and therefore their difference will be a multiple of α , which is perpendicular to all elements of \mathfrak{g}' . \square

Proposition 8. *In the setting above, if ρ_δ is dominant, then $\rho_{\delta'}$ is dominant in \mathfrak{g}' , hence the restriction of δ is also acceptable.*

Keeping in mind the analogy with compact-noncompact roots, this is in fact a particular case of Lemma 5.3.6 in [8]; we will however recall in this setting the relevant parts of the proof.

Proof. It suffices to show that, for any simple root of \mathfrak{g}' , α' , which is not also a simple root of \mathfrak{g} , we have $\langle 2\rho_{\delta'} - \rho', \alpha' \rangle \geq -1$. For simple roots of \mathfrak{g}' which are also simple for \mathfrak{g} this relation is satisfied because of Proposition 7. For $\alpha' = \sum m_i \alpha_i$ we can proceed by induction on $\sum m_i$.

Base case: $\sum m_i = 3$ – recall from Proposition 6 that this is the smallest case we need to consider.

Then $\alpha' = \alpha + \beta + \gamma$, and α is connected to both β and γ in the Dynkin diagram, from the proof of Proposition 6. There are two possibilities:

1. β is good. Then $\langle 2\rho_\delta - \rho, \beta \rangle = 2 - 1 = 1$, so $\langle 2\rho_\delta - \rho, \gamma + \alpha + \beta \rangle \geq -1 - 1 + 1 = -1$, done.
2. β is bad, hence $\beta + \alpha$ is good. Then $\langle 2\rho_\delta - \rho, \alpha + \gamma + \beta \rangle \geq -1 + 2 - 2 = -1$, done.

Induction step: As in the proof of Proposition 6, we can write $\alpha' = \beta + \gamma + \alpha$, where β is simple, and γ is simple in the subalgebra \mathfrak{g}^β . It follows by induction that $\langle 2\rho_\delta - \rho, \gamma \rangle \geq -1$. We have the same two cases as above, depending on whether γ is good or bad, and the proof works the same way. \square

Propositions 6 and 7 can now be used to prove:

Proposition 9. *In a simply laced split group, a maximally bad representation is always acceptable.*

Proof. We will do an induction on the rank of the split group; for the induction step to work properly, we need to assume that $\text{rk } \mathfrak{g} > 4$. However the cases when the rank is at most 4 can be checked quickly by hand (there are only 5 such cases, as we only need to look at connected groups), so we assume $\text{rk } \mathfrak{g} > 4$. Assume the conclusion fails; then ρ_{δ_0} is not dominant, hence there exists a simple root α for which $\langle \rho_{\delta_0}, \alpha \rangle < 0$. Since the rank of the algebra \mathfrak{g} is at least 5, there exists a simple root β so that $\alpha \in \mathfrak{g}^\beta$. Then \mathfrak{g}^β is a split simply laced algebra of rank less than that of \mathfrak{g} , and denote the restriction of ρ and ρ_{δ_0} to \mathfrak{g}^β by ρ' and ρ'_{δ_0} respectively. By Proposition 7 we have $\langle 2\rho_{\delta_0} - \rho, \alpha \rangle = \langle 2\rho'_{\delta_0} - \rho', \alpha \rangle$. But the restriction of δ_0 to \mathfrak{g}^β is still maximally bad, hence acceptable by the induction hypothesis. Hence $\langle 2\rho_{\delta_0} - \rho, \alpha \rangle \geq -1$, which contradicts the assumption that $\langle \rho_{\delta_0}, \alpha \rangle < 0$. \square

Using the case-by-case computations in Appendix A, we can remove the simply laced requirement from the statement of this proposition, and so from now on we will use the fact that all maximally bad representations are acceptable in all split groups.

Finally, we want to show that the R_{DD} group defined is, in a sense, no larger than necessary; the principal series induced from a maximally bad representation splits in as many pieces as the cardinality of the Dynkin diagram R -group.

Proposition 10. *Assume G is connected, and assume it has a maximally bad representation of M , δ_0 . Then $R_{DD} \cong R_{\delta_0}$.*

Proof. To any element on R_{DD} , $S = \{\alpha_1, \dots, \alpha_k\}$, we can associate an element of the Weyl group $W(G, A) = W(\mathfrak{g}, \mathfrak{a})$ as follows:

$$S \longrightarrow w_S = s_{\alpha_1} \dots s_{\alpha_k}.$$

We want to show that $w_S \in R_{\delta_0}$, namely that $w_S(\Delta_{\delta_0}^+) \subset \Delta_{\delta_0}^+$, and that $w_S \in W^{\delta_0}$. First we show that w_S sends positive good roots to positive roots. We know that w_S has length k , and sends all the simple roots in S to their negative (recall that S is a subset of strongly orthogonal roots). Since the length of a Weyl group element also equals the number of positive roots that it sends to negative roots, it follows necessarily that w_S sends all other positive roots to positive roots. Next, we show that w_S sends all simple roots, which are all bad by the definition of the maximally bad representation, to bad roots. Say β is some other simple root. Denote the simple roots in S adjacent to β in the Dynkin diagram by $\text{Adj}(\beta) := \{\alpha_{i_1}, \dots, \alpha_{i_t}\}$.

Then $(w_S\beta)^\vee = \beta^\vee + n_{\beta, \alpha_{i_1}} \alpha_{i_1}^\vee + \dots + n_{\beta, \alpha_{i_t}} \alpha_{i_t}^\vee$, and also

$$\begin{aligned} \delta_0(m_{w_S\beta}) &= \delta_0(m_\beta) \delta_0^{n_{\beta, \alpha_{i_1}}}(m_{\alpha_{i_1}}) \dots \delta_0^{n_{\beta, \alpha_{i_t}}}(m_{\alpha_{i_t}}) \\ &= \delta_0(m_\beta) (-1)^{\sum n_{\beta, \alpha_{i_j}}} = \delta_0(m_\beta), \end{aligned}$$

as $\sum n_{\beta, \alpha_{i_j}}$ is even. Therefore, all simple roots are sent to bad roots. But good co-roots are sums of an even number of bad simple co-roots, and hence it follows from the above that all good roots are sent to good roots under the action of w_S . To complete the proof that $w_S \in R_{\delta_0}$ we only need to show that $w_S \in W^{\delta_0}$, or in other words, that the action of w_S stabilizes the maximally bad representation of M , δ_0 . This is straightforward: choose a representative in M' for w_S , $[x] = w_S$.

Then $xm_\alpha x^{-1} = m_{w_S \alpha}$, so $w_S \cdot \delta(m_\alpha) = \delta(m_{w_S \alpha}) = \delta(m_\alpha)$, since the action of w_S preserves the good and respectively bad roots. However, G is connected and so M is generated by the collection of m_α , for α simple, and hence we can conclude that $w_S \in W^{\delta_0}$, and therefore $w_S \in R_{\delta_0}$. By Theorem 1 we have, in fact, equality, and so $R_{DD} = R_{\delta_0}$. \square

5. A CANONICAL LEVI SUBGROUP

As explained in the introduction, our original motivation was computing regular functions on principal nilpotent orbits; in order to explain the problem we need to fix some more notation. We let $K^\mathbb{C}$ denote the complexification of the maximal compact subgroup K .

The Cartan decomposition fixed in Section 2 gives by complexification a Cartan decomposition of the complex Lie algebra $\mathfrak{g}^\mathbb{C}$, namely, $\mathfrak{g}^\mathbb{C} = \mathfrak{p}^\mathbb{C} + \mathfrak{k}^\mathbb{C}$. $K^\mathbb{C}$ acts with finitely many orbits on the nilpotent elements in $\mathfrak{p}^\mathbb{C}$, and these orbits correspond via the Kostant-Sekiguchi [7] bijection to the nilpotent adjoint orbits in \mathfrak{g} . We will denote the nilpotent cone in $\mathfrak{p}^\mathbb{C}$ by \mathcal{N}^θ .

The closure of a nilpotent $K^\mathbb{C}$ -orbit is an affine subvariety of $\mathfrak{p}^\mathbb{C}$, and as such it has a ring of regular functions. Such a ring of regular functions carries an action of $K^\mathbb{C}$, and thus is an interesting representation that comes naturally with an orbit. There are no formulas for computing the regular functions on the closure of a $K^\mathbb{C}$ orbit in general.

We hope to be able to reduce computing regular functions on principal (generic) $K^\mathbb{C}$ -orbits in $\mathfrak{p}^\mathbb{C}$ to the same problem in a smaller subgroup, namely, the Levi subgroup provided by the Dynkin diagram R -group. We will first describe the Levi subgroup, then state the more precise conjecture.

As in [8], the R -group computations in the previous section provide implicitly a Levi at which the reducibility of the principal series can be seen. More precisely, given any subset of the simple roots, S , we can define M_S^0 to be the subgroup generated by all the root spaces M^α , $\alpha \in S$. Further, we consider the subspace of the split Cartan subalgebra

$$\mathfrak{a}_S = \{X \in \mathfrak{a} \mid \alpha(X) = 0, \forall \alpha \in S\}$$

which by exponentiation yields A_S . Finally, complete the construction of the desired Levi by setting

$$\begin{aligned} M_S &= Z_K(\mathfrak{a}_S)M_S^0, \\ L &= M_S A_S. \end{aligned}$$

The Dynkin diagram R -group provides such a Levi, which we may denote by $M_{DD}A_{DD}$, corresponding to the subset of the simple roots given by the union of all elements of R_{DD} —they are all the simple roots circled in Figure 7.2 in Appendix B, for instance. By Proposition 3, this union is a subset of strongly orthogonal roots. For example if R_{DD} is trivial, then the subset of the set of simple roots S is the empty set, and $\mathfrak{a}_S = \mathfrak{a}$, the maximally split Cartan subalgebra. Thus for the corresponding Levi we obtain $L = MA$, the maximally split Cartan subgroup.

The upshot is that the R_δ -groups for M_{DD} are the same as the R_δ groups of G , for δ acceptable [8]. The Levi subgroup obtained in this fashion has automatically two nice properties: it is locally a product of $SL(2, \mathbb{R})$'s, and the principal series with infinitesimal character zero already split in the maximum number of pieces at this level. It seems reasonable to believe that it also has the property that orbit

induction takes single principal orbits to single principal orbits. We list all such Levi subgroups in Appendix B, Figure 7.2. We should add that, although we only worked with split groups so far, we could, in fact, consider quasisplit groups; we append at the end of the list in Appendix A what the corresponding Levi subgroup should be for $SU(n, n)$.

We will recall the definitions of real orbit induction and explain in what way we hope that the geometry of this Levi subgroup reflects something about the geometry of the group G .

The notion of real (coadjoint) orbit induction is similar to the corresponding one for complex orbits introduced by Lusztig and Spaltenstein [3]. Consider a parabolic subgroup P with Levi decomposition $P = LU$, and \mathcal{O}^L a nilpotent orbit in L . Since we have a natural inclusion of \mathfrak{p} into \mathfrak{g} , and a projection from \mathfrak{p} onto \mathfrak{l} , we get the corresponding maps for the dual algebras

$$\mathfrak{l}^* \xrightarrow{P} \mathfrak{p}^* \xleftarrow{i} \mathfrak{g}^*.$$

Then the G -saturation $G \cdot (i^{-1}(p(\mathcal{O}^L)))$ is a subset of the nilpotent elements of \mathfrak{g}^* which may be reducible. Any nilpotent orbit \mathcal{O}^G included in this subset and additionally having $\dim(\mathcal{O}^G) = \dim(\mathcal{O}^L) + \dim(G/L)$ will be said to be induced from the orbit \mathcal{O}^L . The collection of all such orbits will be denoted by $\text{Ind}_L^G(\mathcal{O}^L)$. There are a few important properties of this construction:

1. The orbit induction construction preserves codimension.
2. Real orbit induction may take a single orbit to multiple orbits, as suggested by the definition; this fact makes it more complicated than the analogous situation for the complex case.

Briefly, we hope that, for the principal (generic) orbits, regular functions might commute with induction, in the following sense:

$$\text{Ind}_{(L \cap K)^c}^{K^c} \mathcal{R}(\overline{\mathcal{O}}) = \overline{\mathcal{R}(\text{Ind}_{LU}^G(\mathcal{O}))},$$

for a principal nilpotent orbit \mathcal{O} in \mathfrak{l} . Note that we move freely between a G -nilpotent orbit, and the K^c -orbit corresponding to it via the Kostant-Sekiguchi correspondence. When talking about regular functions on an orbit we mean the K^c -orbit. However, when we consider orbit induction from a real parabolic, the orbit being induced is a real coadjoint orbit.

As mentioned before, when the R -groups are trivial we have automatically $L = MA$, the maximally split torus. Also, we necessarily have a single principal orbit, and its closure is the whole nilpotent cone. In this setting, the formula above becomes the Kostant-Rallis formula for multiplicities of regular functions on the nilpotent cone

$$\mathcal{R}(\mathcal{N}) = \text{Ind}_M^K 0.$$

We formulate the hope described above as a conjecture; the evidence so far consists only of the families of groups $Sp(2n, \mathbb{R})$ and $SU(n, n)$, for which we proved this result in [4].

Conjecture. *Let G be a simple quasisplit group satisfying the usual conditions in this paper. Let L be the Levi subgroup defined by the Dynkin diagram R -group as*

above. Then regular functions on the closure of a principal nilpotent $K^\mathbb{C}$ -orbit \mathcal{O}_G in $\mathfrak{p}^\mathbb{C}$ can be computed as

$$\mathcal{R}(\overline{\mathcal{O}_G}) = \text{Ind}_{(L \cap K)^\mathbb{C}}^{K^\mathbb{C}} \mathcal{R}(\overline{\mathcal{O}_L}),$$

where \mathcal{O}_L is a principal nilpotent $(L \cap K)^\mathbb{C}$ -orbit in $(l \cap p)^\mathbb{C}$.

6. APPENDIX A—COMPUTATIONS

We give here the full computations that show that the maximally bad representation is also acceptable in all simple split groups. Since the simply laced case was done in Section 4, we only need to consider, case-by-case, the following groups: $Sp(2n, \mathbb{R})$, $SO_0(n+1, n)$, and the split exceptional groups F_4 and G_2 .

In all of the following cases, the simple roots will be bad (as we are dealing with the maximally bad representation). Therefore, the good roots are those whose corresponding co-root can be written as a sum of an even number of simple co-roots.

The split F_4 . Consider the realization of the Lie algebra f_4 in which the simple roots are $\{e_2 - e_3, e_3 - e_4, e_4, \frac{1}{2}(e_1 - e_2 - e_3 - e_4)\}$. We simply list the good co-roots (the number in the second column indicates that the corresponding co-roots can be written as a sum of n simple co-roots).

$e_2 - e_4, e_3 + e_4, e_1 - e_2 - e_3 + e_4$	2
$e_1 - e_2 + e_3 - e_4, e_1 - e_3, e_2 + e_3$	4
$e_1 + e_2 - e_3 - e_4, e_1 + e_4, 2e_1$	6
$e_1 + e_2$	8
$e_1 + e_2 + e_3 + e_4$	10

Hence, the good roots are

$e_2 - e_4, e_3 + e_4, \frac{1}{2}(e_1 - e_2 - e_3 + e_4)$	2
$\frac{1}{2}(e_1 - e_2 + e_3 - e_4), e_1 - e_3, e_2 + e_3$	4
$\frac{1}{2}(e_1 + e_2 - e_3 - e_4), e_1 + e_4, 2e_1$	6
$e_1 + e_2$	8
$\frac{1}{2}(e_1 + e_2 + e_3 + e_4)$	10

$$\rho_{\delta_0} = \frac{1}{2}(6e_1 + 3e_2 + e_3 + e_4)$$

Thus ρ_δ is dominant, and the maximally bad representation is acceptable.

The split G_2 . Consider the realization of the algebra g_2 which has as good simple roots $\{e_2 - e_3, e_1 + e_3 - 2e_2\}$. Once again, we list the good co-roots first:

$\frac{1}{2}(e_1 + e_2 - 2e_3)$	2
$e_1 - e_2$	4

Hence the sum of the corresponding simple roots is $\rho_{\delta_0} = e_1 - e_3$, again dominant.

The symplectic group, $Sp(2n\mathbb{R})$. We consider the simple roots to be $\{e_1 - e_2, \dots, 2e_n\}$. We list directly the simple roots. Recall that M is isomorphic to \mathbb{Z}_2^n , embedded diagonally inside of the maximal torus of the maximal compact subgroup, $U(n)$. The maximally bad representation can be described, as a weight of the torus, as $(0 \ 1 \ \dots \ 0 \ 1)$, if n is even, and as $(1 \ 0 \ 1 \ \dots \ 0 \ 1)$. Consider the two cases separately.

1. If $n = 2k$, then the good roots are $e_i \pm e_j$, for all i, j of the same parity, together with $2e_i$, for i odd. A quick computation shows that $\rho_{\delta_0} = ke_1 + (k-1)e_2 + (k-1)e_3 + (k-2)e_4 + \dots + e_{n-1}$, which is dominant.

2. If $n = 2k + 1$, then similarly the good roots are $e_i \pm e_j$, for all i, j of the same parity, together with $2e_i$, for i even. We obtain $\rho_{\delta_0} = ke_1 + ke_2 + (k-1)e_3 + (k-1)e_4 + \dots + e_{n-1}$, again dominant.

The orthogonal groups, $SO_0(n, n+1)$. The groups themselves have no maximally bad representation, but their linear double covers do, so we will assume we are in that setting. Take the usual simple roots, $\{e_1 - e_2, \dots, e_n\}$ —they are all bad for the maximally bad representation. Since we can write, for instance, $e_{n-1}^\vee = 2(e_{n-1} - e_n)^\vee + e_n^\vee$, we can see that e_{n-1} is also bad. Likewise, all roots of the form e_i will be bad. Similar computations show that $e_i \pm e_j$ will be good if and only if i and j have the same parity. Then the half sum of positive roots will be of the same form as in the case of $Sp(2n, \mathbb{R})$, hence also dominant.

7. APPENDIX B—TABLES

Split group	Directed Dynkin Diagram
$Sp(2n, \mathbb{R})$	
$SO(2n+1, 2n)$	
G_2	
F_4	

FIGURE 7.1. Directed Dynkin graphs for the nonsimply laced groups. For the simply laced groups they are simply the Dynkin diagrams, with all edges labeled one.

Split group	Levi subgroup in the notation of Section 5	Dynkin Diagram R -group
$SL(2n, \mathbb{R})$		\mathbb{Z}_2
$SO_0(n, n)$, $n = 2k$		$\mathbb{Z}_2 \times \mathbb{Z}_2$
$SO_0(n, n)$, $n = 2k + 1$		\mathbb{Z}_2
$SO_0(n + 1, n)$, $n = 2k$		\mathbb{Z}_2
$SO_0(n + 1, n)$, $n = 2k + 1$		\mathbb{Z}_2
$Sp(2n, \mathbb{R})$		\mathbb{Z}_2
E_7		\mathbb{Z}_2
$SU(n, n)$ (quasisplit)		\mathbb{Z}_2

FIGURE 7.2. Dynkin diagram R -groups and corresponding Levi subgroups. The simple split groups not included have trivial Dynkin diagram R -group.

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