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SPHERICAL FUNCTIONS ON MIXED SYMMETRIC SPACES

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Abstract. In this article we compute the spherical functions which are associated to hyperbolically ordered symmetric spaces $H\backslash G$. These spaces are usually not semisimple; one prominent example is given by $H\backslash G = (\mathbb{R}^n \rtimes \mathrm{Gl}(n,\mathbb{R}))\backslash(H_n \rtimes \mathrm{Sp}(n,\mathbb{R}))$ with H_n the $(2n+1)$-dimensional Heisenberg group.

Introduction

In this article we investigate spherical functions on simply connected symmetric spaces $\mathcal{M} := H\backslash G$. Let $(\mathfrak{g}, \tau)$ be the symmetric Lie algebra associated to $\mathcal{M}$ and $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ the corresponding τ-eigenspace decomposition. Write $x_0 \in \mathcal{M}$ for the canonical base point. Our assumptions on $\mathcal{M}$ are purely geometric: We assume that $\mathcal{M}$ is *hyperbolically ordered* which means that there exists an open convex cone C in $\mathfrak{q} \cong T_{x_0}(\mathcal{M})$ which is $\mathrm{Ad}(H)$-invariant and *hyperbolic*, i.e., all operators $\mathrm{ad}\, X$, $X \in C$, are diagonalizable over the real numbers. This includes the non-compact Riemannian symmetric spaces $K\backslash G$ with $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and $C = \mathfrak{p}$, and the non-Riemannian *non-compactly causal* symmetric spaces, for example $\mathrm{SO}(1,1)\backslash \mathrm{Sl}(2,\mathbb{R})$, or, more generally, $\mathrm{SO}(p,q)\backslash \mathrm{Sl}(n,\mathbb{R})$, $\mathrm{Gl}(n,\mathbb{R})_+\backslash \mathrm{Sp}(n,\mathbb{R})$ etc. (cf. [HiÓl96, Th. 3.2.8] for the complete list and Ex. I.13(a) below). Further, there are interesting non-nilpotent solvable examples (cf. Ex. I.13(b)). In general, however, G should be neither reductive nor solvable in order to make $H\backslash G$ a hyperbolically ordered space. We refer to those spaces as *mixed symmetric spaces*.

The guiding mixed example is attached to the *Jacobi group* $G = \mathrm{HSp}(n,\mathbb{R}) := H_n \rtimes \mathrm{Sp}(n,\mathbb{R})$ with H_n the $(2n+1)$-dimensional Heisenberg group (cf. the beginning of Sect. II for a detailed discussion). The symmetric subgroup H is given by $\mathbb{R}^n \rtimes \mathrm{Gl}(n,\mathbb{R})_+$ and we would like to point out that H is not unimodular. The general interest in the Jacobi group stems from the fact that the Schrödinger representation of H_n extends to a unitary highest weight representation of $\mathrm{HSp}(n,\mathbb{R})$; a fact which is going to be very useful for us later on since every hyperbolically symmetric space $H\backslash G$ almost injectively embeds into $(\mathbb{R}^n \rtimes \mathrm{Gl}(n,\mathbb{R})_+)\backslash \mathrm{HSp}(n,\mathbb{R})$ (cf. [KrNe96]).

There are several reasons that make closer study of this class of spaces interesting. One of them is the relation to quantum field theory, in particular, the Osterwalder-Schrader axioms and reflection positivity; see [FOS83], [Jo87], [JoÓl98], [JoÓl00], [Sc86] and the reference therein. In [JoÓl98, JoÓl00] the semisimple case was

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studied, but the general case has yet to be considered. If the group G is semisimple, one can construct a dual simply connected symmetric space $H^c\backslash G^c$ by defining the Lie algebra of G^c to be $\mathfrak{g}^c = \mathfrak{h} + i\mathfrak{q}$. The H-invariant cone $C \subseteq \mathfrak{q}$ then extends to a G-invariant cone $W \subset i\mathfrak{g}^c$, which in turn defines a closed semigroup $\Gamma^c = G^c \exp W$. If π is a suitable generalized complementary series representation of G^c, then there exists a unitary involution $J : \mathcal{H}(\pi) \to \mathcal{H}(\pi)$, where $\mathcal{H}(\pi)$ is the Hilbert space of π, such that $J\pi = \pi \circ \tau J$. Furthermore, there exists a subspace $\mathcal{K} \subseteq \mathcal{H}(\pi)$ such that the bilinear form $(u, v)_J := (u, Jv)$ is non-negative on $\mathcal{K}$. Let $\mathcal{N} = \{u \in \mathcal{K} : (u, u)_J = 0\}$ and let $\mathcal{K}_J$ be the completion of $\mathcal{K}/\mathcal{N}$. Then the representation induced by π on $\mathcal{K}_J$ extends to a holomorphic $*$-representation π_{Γ^c} of Γ^c. Finally, the restriction π_{Γ^c} to G^c is an irreducible unitary highest weight representation of G^c. This allows one to "move" unitary representations from one real form to another by analytic continuation. The spherical functions, that we study in this paper, are—up to a normalizing factor—H-spherical distribution vectors of both of those representations [Kr01, Ól00]. This fact relates the analysis of this duality to the understanding of the *positive definite* spherical functions on hyperbolically ordered symmetric spaces.

Another interesting fact is, that most of the classical Riemannian symmetric spaces H/K_H can be realized as a real form of a bounded symmetric domain G^c/K^c. The fact that G^c/K^c is a bounded symmetric domain implies that $\mathfrak{g}^c$, the Lie algebra of G^c, contains a G^c-invariant pointed generating cone. Hence $H\backslash G$ is hyperbolically ordered. This relation between real forms of bounded symmetric domains, highest weight modules, and generalized complementary series representations has been used to construct *canonical representations* of the group H. It is also closely related to the *generalized equivariant Berezin transform* on symmetric spaces. We refer to [No01, Ól00] and the references therein for further discussion of this topic.

That $(\mathfrak{g}, \tau)$ admits an $\mathrm{Ad}(H)$-invariant hyperbolic convex cone has far reaching consequences for the structure theory of $(\mathfrak{g}, \tau)$ (cf. [HiÓl96], [KrNe96]) and the differential geometry of $\mathcal{M}$ (cf. [La94]). For example the subset $\Gamma := H \exp(C) \subseteq G$ is a semigroup, a so-called *real Ol'shanskiĭ semigroup*. Further, there exists a hyperbolic subspace $\mathfrak{a} \subseteq \mathfrak{q}$ which is maximal abelian in $\mathfrak{q}$. The cone C can be reconstructed from its trace in $\mathfrak{a}$, i.e., $C = \mathrm{Ad}(H).(C \cap \mathfrak{a})$, and we have $\Gamma = H\Gamma_A H$ with $\Gamma_A := \exp(C \cap \mathfrak{a})$.

The set of *positive elements* $\mathcal{M}^+ := x_0.\Gamma$ is the natural domain for spherical functions. If the group H is unimodular and $H^0 := Z_H(\mathfrak{a})$ is compact, then it is natural to define spherical functions as continuous functions φ on $\mathcal{M}^+$ which satisfy the integral equation

$$(\forall s, t \in \Gamma) \quad \varphi(x_0.s)\varphi(x_0.t) = \int_H \varphi(x_0.sht)\, d\mu_H(h)$$

(cf. [FHÓ94]). Note that this implies, in particular, that φ is H-invariant. But in general H is not unimodular, $\mathcal{M}$ carries no invariant measure, and there is no chance to define spherical functions by an integral equation. Write $\mathbb{X}(H/H^0)$ for the group of continuous characters $\chi \colon H \to \mathbb{C}^\times$ of H with $H^0 \subseteq \ker \chi$. For each $\chi \in \mathbb{X}(H/H^0)$ define $\chi^* \in \mathbb{X}(H/H^0)$ by $\chi^*(h) = \overline{\chi(h)}^{-1}$. Then for each $\chi \in \mathbb{X}(H/H^0)$ we have a line bundle $\mathbb{C}_\chi \times_H G \to \mathcal{M}$ and we write $\mathbb{D}(\chi)$ for the corresponding algebra of G-invariant differential operators. A continuous section φ of $\mathbb{C}_\chi \times_H \Gamma \to \mathcal{M}^+$ is called (H, χ)-spherical if φ is right χ^*-semi-invariant and a common eigenfunction of $\mathbb{D}(\chi)$ in the sense of distributions. The reason why there is a character χ involved is that one wants that spherical functions model appropriately the harmonic analysis on $L^2(\mathcal{M}^+)$. Here we need a quasi-invariant measure μ_ρ on $\mathcal{M}^+$ which is constructed by a ρ-function on $\mathcal{M}^+$ which in turn is a C^∞-extension of the character $H \to \mathbb{C}^\times,\ h \mapsto \det \mathrm{Ad}_G(h)^{-1}$.

We write $\mathfrak{n} = \bigoplus_{\alpha \in \Delta^+(\mathfrak{g},\mathfrak{a})} \mathfrak{g}^\alpha$ for the sum of the positive root spaces with respect to a positive system which in some sense is adapted to the cone C. Denote by A and N the analytic subgroups of G corresponding to $\mathfrak{a}$ and $\mathfrak{n}$. Then the map $H \times A \times N \to G,\ (h,a,n) \mapsto han$ is a diffeomorphism onto the open image HAN and we have $\Gamma \subseteq HAN$. Accordingly, every element $s \in \Gamma$ can be written as $s = h_H(s)a_H(s)n_H(s)$ with $h_H(s) \in H$, $a_H(s) \in A$ and $n_H(s) \in n$, everything depending analytically on s. Next we will identify (H,χ)-spherical functions with functions on Γ in the obvious sense. For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $\chi \in \mathbb{X}(H/H^0)$ we define the (H,χ)-*spherical function with parameter* λ by

$$\varphi_\lambda^\chi(s) = \int_{H/H^0} a_H(sh)^{\lambda-\rho}\chi(sh)\overline{\chi(h)}\ d\mu_{H/H^0}(hH^0) \quad \text{ for } \ s \in \Gamma$$

provided the integrals exist. We write $\mathcal{E}^\chi$ for the set of all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ for which all integrals $\varphi_\lambda^\chi(s)$, $s \in \Gamma$, converge. The main result of this paper is the *Factorization Theorem* for φ_λ^χ which asserts that

$$\varphi_\lambda^\chi(a) = \varphi_{\lambda_z}^{\chi_z}(a) \cdot \varphi_{\lambda_l}^{\chi_l \Delta_H^{\frac{1}{2}}}(a)$$

for all $a \in \Gamma_A$ (cf. Theorem III.10). Here the first factor corresponds to the nilradical U of G while $\varphi_{\lambda_l}^{\chi_l \Delta_H^{\frac{1}{2}}}$ corresponds to a reductive complement $L \cong G/U$. Note that there is a shift in the character by the square root of the modular function Δ_H of H which is related to the fact that H is not unimodular. An explicit formula for the function $\varphi_{\lambda_z}^{\chi_z}(a)$ is given as

$$\varphi_{\lambda_z}^{\chi_z}(a) = \frac{2^{\frac{n}{2}}\pi^n a^{\lambda_z}}{\det A_{\lambda_z}} \cdot \frac{\exp\big(-\langle \frac{\mathbf{1}}{a^2-\mathbf{1}}.\overline{w_+}, w_+\rangle_{\lambda_z} - \langle \frac{a^2}{a^2-\mathbf{1}}.w_+, \overline{w_+}\rangle_{\lambda_z} + \langle \frac{a}{a^2-\mathbf{1}}.w_+, w_+\rangle_{\lambda_z}\big)}{\big(\prod_{\alpha\in\Delta_r^+}(\sinh\alpha(\log a))^{m_\alpha}\big)^{\frac{1}{2}}}.$$

We note that the Factorization Theorem gives a concrete formula for φ_λ^χ on solvable symmetric spaces. Our product formula generalizes the one in [HiNe96] for spherical functions on Ol'shanskiĭ spaces $G\backslash G_{\mathbb{C}}$. Spherical functions on reductive symmetric spaces and their asymptotic expansions have been studied extensively by the third author in [Ól97]. See also [ÓlPa00] and [AÓS00].

Spherical functions are closely related to representation theory. In fact, one can think of positive definite spherical functions φ as certain matrix coefficients of a representation $(\pi,\mathcal{H})$ of Γ. More precisely, $\varphi(s) = \langle \pi(s).\nu, \nu\rangle$ where $\nu \in \mathcal{H}^{-\omega}$ is an (H,χ)-spherical vector. Via the Lüscher-Mack correspondence, representations of Γ are related to unitary representations of the simply connected c-dual group G^c (cf. [HiNe97]). It is exactly this correspondence we use to prove our Factorization Theorem. We use the generalized extended metaplectic representation $(\mu_\lambda, \mathcal{H}_\lambda)$ of G^c which is a highest weight representation (cf. [Ne99]). Let $(\mu, \mathcal{H}_\mu)$ denote the extended metaplectic representation of the simply connected Jacobi group $\mathrm{HSp}(n,\mathbb{R}) = H_n \rtimes \mathrm{Sp}(n,\mathbb{R})$ (cf. [Fo89], [Ne99]). We equip the Jacobi group with an involution τ such that $\mathrm{HSp}(n,\mathbb{R})^\tau = \mathbb{R}^n \rtimes \mathrm{GL}(n,\mathbb{R})_+$. Now $(\mu_\lambda, \mathcal{H}_\lambda)$ is obtained by a composition of an appropriate homomorphism $(G^c,\tau) \to (\mathrm{HSp}(n,\mathbb{R}),\tau)$ and the extended metaplectic representation $(\mu, \mathcal{H}_\mu)$ (cf. Section II). The highest weight λ depends on the homomorphism $(G^c,\tau) \to (\mathrm{HSp}(n,\mathbb{R}),\tau)$. In Section II we give a classification of the (H,χ)-spherical distribution and hyper-function vectors of $(\mu_\lambda, \mathcal{H}_\lambda)$ (cf. Theorem II.14). Then we compute an integral over a certain matrix coefficient of $(\mu_\lambda, \mathcal{H}_\lambda)$ (cf. Lemma II.16) which turns out to be crucial for the factorization of φ_λ^χ.

One important point is the asymptotic behaviour of spherical functions at infinity, since this models the growth of certain matrix coefficients (cf. [Wa88, Ch. 4]). The "constant term" of φ_λ^χ at infinity is given by the c-function

$$c_{\mathcal{M}}^\chi(\lambda) = \int_{\overline{N}\cap(HAN)} a_H(\overline{n})^{-\lambda-\rho}\overline{\chi(\overline{n})}\; d\mu_{\overline{N}}(\overline{n}).$$

Using the product formula for the spherical functions and some arguments using analytic continuation, we obtain a product formula for the c-function

$$c_{\mathcal{M}}^\chi(\lambda) = \frac{2^{\frac{n}{2}}\pi^n e^{-\langle w_+,\overline{w_+}\rangle_{\lambda_z}}}{\det A_{\lambda_z}} c_{\mathcal{M}_L}^{\chi_l \Delta_H^{\frac{1}{2}}}(\lambda_l)$$

(cf. Theorem IV.11). We also determine explicitly the domain of convergence $\mathcal{E}^\chi$ for the spherical function φ_λ^χ and the c-function. We conclude this paper with a list of further comments and problems concerning the interplay between spherical functions, representation theory, and harmonic analysis on symmetric spaces. In particular we explain the relation between spherical functions and H-spherical distribution characters of spherical highest weight modules, in particular, holomorphic discrete series representations, and the role of the Factorization Theorem in the theory of Hardy spaces on the c-dual symmetric space $H\backslash G^c$.

I. Hyperbolically causal symmetric spaces

In this section we collect the results on the global and algebraic structure of mixed symmetric spaces that we will need in later sections of this paper.

Symmetric Lie algebras.

Definition I.1. (a) A *symmetric Lie algebra* is a pair $(\mathfrak{g},\tau)$ consisting of a finite dimensional real Lie algebra $\mathfrak{g}$ and an involutive automorphism τ of $\mathfrak{g}$. We put $\mathfrak{h} := \{X \in \mathfrak{g} : \tau.X = X\}$ and $\mathfrak{q} := \{X \in \mathfrak{g} : \tau.X = -X\}$, and note that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$.

(b) An element $X \in \mathfrak{g}$ is called *hyperbolic* if $\operatorname{ad} X$ is diagonalizable over $\mathbb{R}$. An abelian subspace $\mathfrak{a} \subseteq \mathfrak{q}$ is called *abelian maximal hyperbolic* if $\mathfrak{a}$ consists of hyperbolic elements and is maximal w.r.t. this property.

(c) A subspace $\mathfrak{l} \subseteq \mathfrak{q}$ is called a *Lie triple system* if $[\mathfrak{l},[\mathfrak{l},\mathfrak{l}]] \subseteq \mathfrak{l}$. This means that the space $\mathfrak{l}_L := \mathfrak{l} \oplus [\mathfrak{l},\mathfrak{l}]$ is a subalgebra of $\mathfrak{g}$. Recall that all abelian maximal hyperbolic subspaces and all maximal hyperbolic Lie triples systems in $\mathfrak{q}$ are conjugate under $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})$ (cf. [KrNe96, Cor. II.9, Th. III.3]). □

The abelian maximal hyperbolic subspaces generalize the Cartan subspaces occurring in the restricted root decompositions of real semisimple Lie algebras. The maximal hyperbolic Lie triple systems are the infinitesimal version of "maximal non-compact Riemannian subspaces" on the level of the corresponding symmetric spaces.

Definition I.2 (The structure of $\mathfrak{g}$)**.** Let $\mathfrak{r}$ denote the radical of $\mathfrak{g}$ and $\mathfrak{r} = \mathfrak{r}_\mathfrak{h} + \mathfrak{r}_\mathfrak{q}$ its τ-eigenspace decomposition. In the following, subscripts indicate intersections, for example, $\mathfrak{r}_\mathfrak{h} := \mathfrak{r} \cap \mathfrak{h}$, etc. According to [KrNe96, Prop. III.5], there exists a τ-invariant Levi complement $\mathfrak{s} \subseteq \mathfrak{g}$ with the following properties: There exists a maximal hyperbolic Lie triple system $\mathfrak{p} \subseteq \mathfrak{q}$ such that $\mathfrak{p} = \mathfrak{p}_\mathfrak{r} \oplus \mathfrak{p}_\mathfrak{s}$, where $\mathfrak{p}_\mathfrak{s} \subseteq \mathfrak{s}_\mathfrak{q}$ is a maximal hyperbolic Lie triple system in $\mathfrak{s}_\mathfrak{q}$ and $[\mathfrak{p}_\mathfrak{r}, \mathfrak{s}] = \{0\}$. Then each maximal abelian subspace $\mathfrak{a} \subseteq \mathfrak{p}$ is of the form $\mathfrak{a} = \mathfrak{p}_\mathfrak{r} \oplus \mathfrak{a}_\mathfrak{s}$, and $[\mathfrak{p}_L, \mathfrak{s}] \subseteq \mathfrak{s}$. Furthermore, there exists a Cartan involution θ on $\mathfrak{s}$ commuting with $\mathrm{ad}[\mathfrak{p}, \mathfrak{p}]$ and $\tau\,|_\mathfrak{s}$ (cf. [KrNe96, Prop. I.5]). The corresponding Cartan decomposition is denoted by $\mathfrak{s} = \mathfrak{s}_\mathfrak{k} \oplus \mathfrak{s}_\mathfrak{p}$. The largest ideal of $\mathfrak{s}$ contained in $\mathfrak{s}_\mathfrak{h}$, i.e., the kernel of the isotropy representation of corresponding symmetric spaces, is denoted $\mathfrak{s}_{\mathrm{iso}}$. So the semisimple symmetric Lie algebra $(\mathfrak{s}, \tau\,|_\mathfrak{s})$ decomposes as

$$(\mathfrak{s}, \tau\,|_\mathfrak{s}) = (\mathfrak{s}_{\mathrm{iso}}, \tau\,|_{\mathfrak{s}_{\mathrm{iso}}}) \oplus \bigoplus_{i=1}^n (\mathfrak{s}_i, \tau\,|_{\mathfrak{s}_i})$$

with $(\mathfrak{s}_i, \tau\,|_{\mathfrak{s}_i})$ irreducible and effective. □

Definition I.3 (Root decomposition)**.** (a) Let $\mathfrak{a} \subseteq \mathfrak{q}$ be an abelian maximal hyperbolic subspace. For every $\mathrm{ad}\,\mathfrak{a}$-invariant subspace $\mathfrak{b}$ in $\mathfrak{g}$ and for every $\alpha \in \mathfrak{a}^*$ we define

$$\mathfrak{b}^\alpha := \{X \in \mathfrak{b} :\ (\forall Y \in \mathfrak{a})[Y, X] = \alpha(Y)X\}.$$

In particular, we have $\mathfrak{b}^0 := \mathfrak{z}_\mathfrak{b}(\mathfrak{a})$. We write $\Delta := \{\alpha \in \mathfrak{a}^* \backslash \{0\} :\ \mathfrak{g}^\alpha \neq \{0\}\}$ for the set of roots. Then we get the root space decomposition $\mathfrak{g} = \mathfrak{g}^0 \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}^\alpha$. For each $\alpha \in \Delta$ we put $m_\alpha := \dim \mathfrak{g}^\alpha$.

We call a root $\alpha \in \Delta$ *semisimple*, resp. *solvable*, if $\mathfrak{s}^\alpha \neq \{0\}$, resp. $\mathfrak{g}^\alpha \subseteq \mathfrak{r}$. The set of all semisimple, resp. solvable, roots is denoted by Δ_s, resp. Δ_r. Note that $\Delta = \Delta_r \dot\cup \Delta_s$ (cf. [KrNe96, Lemma IV.5(i)]).

A root $\alpha \in \Delta$ is called *compact* if $\mathfrak{p}_L^\alpha \neq \{0\}$ and *non-compact* otherwise. We write Δ_k, Δ_n resp. Δ_p for the set of all compact, non-compact, resp. non-compact semisimple roots. Note that Δ_k is independent of the choice of $\mathfrak{p} \supseteq \mathfrak{a}$ (cf. [KrNe96, Def. V.1]) and that $\Delta = \Delta_k \dot\cup \Delta_n$ holds by definition.

(b) We call an element $X_0 \in \mathfrak{a}$ *regular* if $\alpha(X_0) \neq 0$ for all $\alpha \in \Delta$ and a subset $\Delta^+ \subseteq \Delta$ a *positive system* if there exists a regular element $X_0 \in \mathfrak{a}$ with $\Delta^+ = \{\alpha \in \Delta :\ \alpha(X_0) > 0\}$. To each positive system Δ^+ we associate several subalgebras of $\mathfrak{g}$:

$$\mathfrak{n} = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}^\alpha \qquad \text{and} \qquad \overline{\mathfrak{n}} = \bigoplus_{\alpha \in \Delta^-} \mathfrak{g}^\alpha.$$

Further, we set

$$\mathfrak{n}_n^\pm = \bigoplus_{\alpha \in \Delta_n^\pm} \mathfrak{g}^\alpha, \qquad \mathfrak{n}_r^\pm = \bigoplus_{\alpha \in \Delta_r^\pm} \mathfrak{r}^\alpha, \qquad \mathfrak{n}_p^\pm = \bigoplus_{\alpha \in \Delta_p^\pm} \mathfrak{g}^\alpha \quad \text{and} \quad \mathfrak{n}_k^\pm = \bigoplus_{\alpha \in \Delta_k^\pm} \mathfrak{g}^\alpha.$$

We write ρ, ρ_r, ρ_s, ρ_n and ρ_k, respectively, for $\frac{1}{2}\operatorname{tr}\mathrm{ad}_\mathfrak{n}$, $\frac{1}{2}\operatorname{tr}\mathrm{ad}_{\mathfrak{n}_r}$, $\frac{1}{2}\operatorname{tr}\mathrm{ad}_{\mathfrak{n}_\mathfrak{s}}$, $\frac{1}{2}\operatorname{tr}\mathrm{ad}_{\mathfrak{n}_n}$, and $\frac{1}{2}\operatorname{tr}\mathrm{ad}_{\mathfrak{n}_\mathfrak{k}}$.

(c) The *Weyl group* $\mathcal{W}$ of $(\mathfrak{g}, \tau)$ w.r.t. $\mathfrak{a}$ is defined by

$$\mathcal{W} := N_{\mathrm{Inn}_\mathfrak{g}(\mathfrak{h})}(\mathfrak{a}) / Z_{\mathrm{Inn}_\mathfrak{g}(\mathfrak{h})}(\mathfrak{a}).$$

A positive system is called $\mathfrak{p}$-*adapted* if the set Δ_n^+ of positive non-compact roots is invariant under the Weyl group. □

Definition I.4. The symmetric Lie algebra $(\mathfrak{g}, \tau)$ is called *quasihermitian* if $\mathfrak{z}_{\mathfrak{q}}(\mathfrak{z}(\mathfrak{p})) = \mathfrak{p}$. In this case $\mathfrak{a}$ is maximal abelian in $\mathfrak{q}$ and there exists a $\mathfrak{p}$-adapted positive system Δ^+ (cf. [KrNe96, Prop. V.10]). An irreducible effective quasihermitian symmetric Lie algebra $(\mathfrak{g}, \tau)$ is called *non-compactly Riemannian* (NCR), resp. *non-compactly causal* (NCC), if $\mathfrak{z}(\mathfrak{p}) = \{0\}$, resp. $\mathfrak{z}(\mathfrak{p}) \neq \{0\}$. The property of being quasihermitian is inherited by $\mathfrak{s}$. This means that the irreducible constituents $(\mathfrak{s}_i, \tau|_{\mathfrak{s}_i})$ of $\mathfrak{s}$ are either (NCR) or (NCC) (cf. [KrNe96, Prop. V.9(v)]). □

From now on we will assume that $(\mathfrak{g}, \tau)$ is quasihermitian and that Δ^+ is $\mathfrak{p}$-adapted.

Definition I.5. (a) Let V be a finite dimensional real vector space and V^* its dual. For a subset $E \subseteq V$ the *dual cone* is defined by $E^\star := \{\omega \in V^* : (\forall x \in E)\omega(x) \geq 0\}$ and $\operatorname{cone}(E)$ denotes the smallest closed convex cone containing E. A cone $C \subseteq V$ is called *generating* if $V = C - C$ and *pointed* if $\overline{C} \cap -\overline{C} = \{0\}$.

For a convex subset $C \subseteq V$ we set

$$\lim C := \{v \in V : v + C \subseteq C\} \quad \text{and} \quad B(C) := \{\alpha \in V^* : \inf \alpha|_C > -\infty\}.$$

Note that both $\lim C$ and $B(C)$ are convex cones in V, resp. V^*.

(b) We associate to a positive system of non-compact roots Δ_n^+ the convex cones

$$C_{\min} := \operatorname{cone}\big(\{[X_\alpha, \tau(X_\alpha)] : X_\alpha \in \mathfrak{g}^\alpha, \alpha \in \Delta_n^+\}\big),$$

$$C_{\min,r} := \operatorname{cone}\big(\{[X_\alpha, \tau(X_\alpha)] : X_\alpha \in \mathfrak{g}^\alpha, \alpha \in \Delta_r^+\}\big),$$

$$C_{\min,p} := \operatorname{cone}\big(\{[X_\alpha, \tau(X_\alpha)] : X_\alpha \in \mathfrak{g}^\alpha, \alpha \in \Delta_p^+\}\big),$$

$C_{\max} := (\Delta_n^+)^\star = \{X \in \mathfrak{a} : (\forall \alpha \in \Delta_n^+)\alpha(X) \geq 0\}$ and $C_{\max,p} := (\Delta_p^+)^\star \cap \mathfrak{a}_{\mathfrak{s}}$. □

Definition I.6. In the following, G denotes a simply connected Lie group associated to $\mathfrak{g}$. Then τ integrates to an involution on G also denoted by τ and the fixed point set $H := G^\tau$ is a connected subgroup of G (cf. [Lo69, Th. 3.4]) with Lie algebra $\mathfrak{h}$. Further, we define A, H, N, $\overline{N}$, R and S as the analytic subgroups of G corresponding to $\mathfrak{a}$, $\mathfrak{h}$, $\mathfrak{n}$, $\overline{\mathfrak{n}}$, $\mathfrak{r}$ and $\mathfrak{s}$ (cf. Definition I.3). By subscripts we indicate intersections, for instance $H_R = H \cap R$, etc. □

Proposition I.7 (The HAN-decomposition)**.** *For a simply connected symmetric Lie group (G, τ) associated to $(\mathfrak{g}, \tau)$ the following assertions hold:*

(i) *The groups A, resp. N, are closed, simply connected and diffeomorphic to $\mathfrak{a}$, resp. $\mathfrak{n}$, under the exponential mapping. Moreover, $A \cap N = \{\mathbf{1}\}$.*

(ii) *The map*

$$\varphi\colon H \times A \times N \to G, \quad (h, a, n) \mapsto han$$

is a diffeomorphism onto its open image.

(iii) *The multiplication mapping $\varphi_R\colon H_R \times A_R \times N_R \to R$ is a diffeomorphism.*

(iv) *The set HAN is R-saturated, i.e., left and right R-invariant.*

Proof. This is Proposition II.4 in [KNÓ97]. □

This proposition tells us, in particular, that there exist analytic maps

$$h_H\colon HAN \to H, \quad han \mapsto h, \qquad a_H\colon HAN \to A, \quad han \mapsto a,$$

and

$$n_H \colon HAN \to N, \quad han \mapsto n.$$

Let $\mathfrak{u}$ denote the nilradical of $\mathfrak{g}$. We find a τ-invariant reductive subalgebra $\mathfrak{l} \subseteq \mathfrak{g}$ such that $[\mathfrak{l}, \mathfrak{l}] = \mathfrak{s}$ and $\mathfrak{g} = \mathfrak{u} \rtimes \mathfrak{l}$. Accordingly, we have the decomposition $G = U \rtimes L$. On A this decomposition induces a factorization $A = A_U \times A_L$, where A_U is a central subgroup of G. Let $N_R^\pm$, $N_S^\pm$ and $N_k^\pm$ the analytic subgroups of G corresponding to $\mathfrak{n}_r^\pm$, $\mathfrak{n}_p^\pm$ and $\mathfrak{n}_k^\pm$. In view of Proposition I.7, we have

$$U = H_U A_U N_R \qquad \text{and} \qquad L \supseteq H_L A_L N_L$$

with $N \cong N_R^+ \rtimes N_L$ and $N_L \cong N_S^+ \rtimes N_k^+$. Furthermore, $H_L A_L N_L$ is open in L. Writing each element $a \in A$ as $a = a_u a_l$ with $a_u \in A_U$ and $a_l \in A_L$, we get two analytic maps

$$a_{H,u} \colon HAN \to A_U, \qquad \text{and} \qquad a_{H,l} \colon HAN \to A_L.$$

Lemma I.8. *Let $u \in U$ and $s = s_u s_l \in HAN$. Then*

$$a_H(s_l) = a_{H,l}(s_l) = a_{H,l}(s) = a_{H,l}(su).$$

Proof. Since the homomorphism $G \to L$ preserves the HAN-decomposition, we may write $s_l \in H_L A_L N_L$ as $s_l = h_1 a_1 n_1$. Note that $a_{H,l}(u) = 1$ for all $u \in U$ follows from $U = H_U A_U N_R$. Thus

$$a_{H,l}(s) = a_{H,l}(s_u h_1 a_1 n_1) = a_{H,l}(s_u h_1) a_1 = a_{H,l}(h_1^{-1} s_u h_1) a_1 = a_1.$$

This shows that $a_{H,l}(s) = a_{H,l}(s_l)$, proving the second equality. Now the last equality follows from $(su)_l = s_l$ concluding the proof of the lemma. $\square$

We conclude this subsection with an integral formula which will be useful later on. Since A is simply connected, the exponential mapping $\exp \colon \mathfrak{a} \to A$ is a diffeomorphism and so has an inverse $\log \colon A \to \mathfrak{a}$. In particular, for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ the prescription

$$A \to \mathbb{C}, \quad a \mapsto a^\lambda := e^{\lambda(\log a)}$$

defines an analytic function on A.

Whenever G is a locally compact group, we write μ_G for a left-invariant Haar measure on G.

Lemma I.9. *Let $f \in C_c(HAN)$. Then the Haar measures on G, H, A and N may be normalized in such a way that*

$$\int_G f(g)\, d\mu_G(g) = \int_H \int_A \int_N f(han) a^{2\rho}\, d\mu_N(n)\, d\mu_A(a)\, d\mu_H(h). \tag{1.1}$$

Proof. The open domain $HAN \subseteq G$ is an orbit of the action of $H \times AN$ given by $(h, b).x := hxb^{-1}$. Since, according to [He184, Prop. I.5.1, p. 181], the formula

$$\int_A \int_N f(an) a^{2\rho}\, d\mu_N(n)\, d\mu_A(a)$$

defines a right-invariant Haar measure on the group AN, the right-hand side of (1.1) defines a measure on HAN which is invariant under the action of the group $H \times AN$. On the other hand, the restriction of μ_G to HAN is also invariant under this action because $\det \mathrm{Ad}_G(b) = 1$ holds for each $b \in AN$. Now the uniqueness of invariant measures on homogeneous spaces implies the assertion. $\square$

Symmetric Lie algebras admitting hyperbolic cones.

Definition I.10. We call $(\mathfrak{g},\tau)$ *admissible* if $\mathfrak{q}$ contains an open convex hyperbolic $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})$-invariant subset not containing non-trivial affine subspaces. Then every maximal hyperbolic subspace $\mathfrak{a}\subseteq\mathfrak{q}$ is maximal abelian in $\mathfrak{q}$, $(\mathfrak{g},\tau)$ is quasihermitian and has cone potential (cf. [KrNe96, Th. VI.6, Prop. V.9]). □

From now on we make the assumption that $(\mathfrak{g},\tau)$ is admissible. Further, we assume that $(\mathfrak{g},\tau)$ is *effective*, i.e., $\mathfrak{h}$ does not contain any non-zero ideal of $\mathfrak{g}$. In the following, Δ^+ denotes a $\mathfrak{p}$-adapted positive system.

Lemma I.11. *Let $(\mathfrak{g},\tau)$ be an effective admissible symmetric Lie algebra and let Δ^+ be a $\mathfrak{p}$-adapted positive system. Then:*

(i) $[\mathfrak{u},\mathfrak{u}]\subseteq\mathfrak{z}(\mathfrak{g})$.
(ii) $[\mathfrak{n}_n^+,\mathfrak{n}_n^+]=[\mathfrak{n}_n^-,\mathfrak{n}_n^-]=\{0\}$.
(iii) $[\mathfrak{n}_p^\pm,\mathfrak{n}_r^\mp]\subseteq\mathfrak{n}_r^\pm$.

Proof. (i) In view of [KrNe96, Th. VI.6(iii)], this follows from [KrNe96, Prop. VII.2(iii)(a)].
(ii), (iii) [KrNe96, Th. VII.18(ii)]. □

Remark I.12. Let $V:=\mathfrak{n}_r^+\oplus\mathfrak{n}_r^-$. Then $\mathfrak{u}=V+\mathfrak{z}(\mathfrak{g})$, L acts on V and V carries an L-invariant skew-symmetric bilinear map $\varphi\colon V\times V\to\mathfrak{z}(\mathfrak{g}),(v,w)\mapsto[v,w]$. Note that $\mathfrak{z}(\mathfrak{g})\subseteq\mathfrak{q}$ since $(\mathfrak{g},\tau)$ was assumed to be effective. The bracket on $\mathfrak{g}=V+\mathfrak{z}(\mathfrak{g})+\mathfrak{l}$ is given by

$$[(v,z,X),(v',z',X')]=\big(X.v'-X'.v,\varphi(v,v'),[X,X']\big).$$

Then $G\cong V\times\mathfrak{z}(\mathfrak{g})\times L$ with group multiplication

$$(v,z,l)\cdot(v',z',l')=(v+l.v',z+z'+\tfrac{1}{2}\varphi(v,l.v'),ll').$$

□

Example I.13. We now give some examples of admissible symmetric Lie algebras.
(a) (Semisimple examples) Recall from Definition I.4 and Definition I.10 that a simple admissible symmetric Lie algebra $(\mathfrak{g},\tau)$ is called non-compactly causal. (NCC) symmetric Lie algebras are classified and we refer to [HiÓl96, Th. 3.2.8] for the table. However, we think it might be useful to explain some typical cases.

As a basis of $\mathfrak{sl}(2,\mathbb{R})$ we choose

$$H:=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad T:=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad\text{and}\quad U:=\begin{pmatrix}0&1\\-1&0\end{pmatrix}.$$

Then a symmetric structure is introduced by the choice $\mathfrak{q}:=\mathbb{R}H\oplus\mathbb{R}U$ and $\mathfrak{h}:=\mathbb{R}T$. An abelian maximal hyperbolic subspace of $\mathfrak{q}$ is given by $\mathfrak{a}:=\mathbb{R}H$. Further, we have $\Delta=\Delta_p=\{\pm\alpha\}$ with $\alpha\in\mathfrak{a}^*$ defined by $\alpha(H)=2$. For the choice of $\Delta^+:=\{\alpha\}$ there is only one $e^{\mathrm{ad}\,\mathfrak{h}}$-invariant hyperbolic convex cone in $\mathfrak{q}$ which is given by

$$W=\{hH+uU:\ h\geq 0,\ |u|\leq h\}.$$

This example can be generalized to $(\mathfrak{g},\mathfrak{h}):=(\mathfrak{sl}(n,\mathbb{R}),\mathfrak{so}(p,q))$ with $n=p+q$, $p,q>0$. Here $\mathfrak{a}$ consists of all diagonal matrices in $\mathfrak{g}$ and in the standard notation a $\mathfrak{p}$-adapted positive system is given by $\Delta^+=\{\varepsilon_i-\varepsilon_j:\ 1\leq i<j\leq n\}$ with $\Delta_p^+=\{\varepsilon_i-\varepsilon_j:\ 1\leq i\leq p,\ p+1\leq j\leq n\}$. There is no easy global description of

the maximal cone $W_{\max}$ (which is characterized by $W_{\max} \cap \mathfrak{a} = C_{\max}$). However, the intersection of $W_{\max}$ with $\mathfrak{p}$ is particularly nice:

$$W_{\max} \cap \mathfrak{p} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : A = A^T, B = B^T, \inf \operatorname{Spec}(A) \geq \sup \operatorname{Spec}(B) \right\}.$$

(b) (Solvable example) A very important solvable admissible Lie algebra is the four-dimensional oscillator algebra. This algebra is low-dimensional enough to perform explicit computations and on the other hand, rich enough to feature the solvable situation well. The construction goes as follows. Let $\mathfrak{h}_1 := \mathbb{R}X \oplus \mathbb{R}Y \oplus \mathbb{R}Z$ be the three-dimensional Heisenberg algebra with bracket relation $[X, Y] = Z$. Then the oscillator algebra is the semidirect product $\mathfrak{g} := \mathfrak{h}_1 \rtimes \mathbb{R}A$ with $[A, X] = Y$, $[A, Y] = X$ and $[A, Z] = 0$. A symmetric structure is imposed on $\mathfrak{g}$ by the choice $\mathfrak{h} := \mathbb{R}Y$ and $\mathfrak{q} := \mathbb{R}A \oplus \mathbb{R}X \oplus \mathbb{R}Z$. A possible choice for $\mathfrak{a}$ is $\mathfrak{a} := \mathbb{R}A + \mathbb{R}Z$ and we have $\Delta = \Delta_r = \{\pm\alpha\}$ with $\alpha(aA + zZ) = a$. The maximal cone for $\Delta^+ = \{\alpha\}$ is $W_{\max} := \{aA + xX + zZ : a \geq 0\}$ but there also are pointed $\operatorname{Inn}_{\mathfrak{g}}(\mathfrak{h})$-invariant convex cones, for example,

$$W := \{aA + xX + zZ : a \geq 0, \ z \geq 0, \ x^2 \leq 2az\}.$$

(c) (Mixed examples) The most important example which is neither solvable nor semisimple is the Jacobi algebra which will be discussed in great detail in the beginning of Section II. □

Recall the definition of $\mathfrak{h}^0 = \mathfrak{z}_{\mathfrak{h}}(\mathfrak{a})$ and set $H^0 = Z_H(\mathfrak{a})$. To proceed we first need some structural information about the group H^0. Recall that a subalgebra $\mathfrak{b} \subseteq \mathfrak{g}$ is called *compactly embedded* if $\operatorname{Inn}_{\mathfrak{g}}(\mathfrak{b})$ is relatively compact in $\operatorname{Aut}(\mathfrak{g})$.

Lemma I.14. *For an effective admissible symmetric Lie algebra $(\mathfrak{g}, \tau)$ the following assertions hold:*

(i) *The subgroup H^0 is compactly embedded, i.e., $\overline{\operatorname{Ad}(H^0)}$ is compact.*
(ii) *The group H^0 is contained in L, i.e., $H^0 = H^0_L$. Moreover, $H^0 = H^0_R \times H^0_S$ with $H^0_S Z(G)/Z(G)$ compact and H^0_R simply connected abelian.*

Proof. (i) This follows by replacing the Lie algebra $\mathfrak{h}^0$ in the proof of Proposition VII.12(ii) in [KrNe96] by the group H^0.

(ii) First we prove that $H^0 \subseteq L$. Since $H = H_U \rtimes H_L$, every element $h \in H^0$ can be written in a unique fashion as $h = h_u h_l$ with $h_u \in H_U$ and $h_l \in H_L$. We claim that $h_u, h_l \in H^0$. Let $X \in \mathfrak{a}$ and write $X = X_u + X_l$ according to the decomposition $\mathfrak{a} = \mathfrak{a}_{\mathfrak{u}} \oplus \mathfrak{a}_{\mathfrak{l}}$. Then $\operatorname{Ad}(h).X = X$ implies, in particular, that $\operatorname{Ad}(h_l).X_l = X_l$ for all $X \in \mathfrak{a}$. From $\mathfrak{a}_{\mathfrak{u}} = \mathfrak{z}(\mathfrak{g})$ it follows that $h_l \in H^0$, proving the claim.

Now the fact that H_U is simply connected nilpotent implies that $H^0_U = \exp(\mathfrak{h}^0 \cap \mathfrak{u}) \cong \mathfrak{h}^0 \cap \mathfrak{u}$. According to (i), $\mathfrak{h}^0$ is compactly embedded and therefore $\mathfrak{h}^0 \cap \mathfrak{u}$ is central. Thus $\mathfrak{h}^0 \cap \mathfrak{u} = \{0\}$ by the effectivity of $(\mathfrak{g}, \tau)$. This proves $H^0 \subseteq L$.

Next we show that $H^0 = H^0_R \times H^0_S$. Since L is simply connected, we have $L \cong Z(L)_0 \times S$. Now $\mathfrak{a} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{a}_{\mathfrak{l}}$ and $H^0 \subseteq L$ yield $H^0 = Z_H(A) = Z_H(A_L) = Z_{H_L}(A_L)$. Therefore $H \cap Z(L) \subseteq H^0$ and $H^0_S = Z_{H_S}(A_S) = Z_{H_S}(A_L) = (H^0)_S$, so that $H_L = H_{Z(L)_0} \times H_S$ entails that $H^0 = H_{Z(L)_0} \times H^0_S$. Now (ii) follows from the observation that $H_{Z(L)_0} = H^0_R$ and the group $Z(L)_0$ containing H^0_R is a vector group, so that H^0_R is simply connected and abelian.

Finally, it follows from [FHÓ94, Lemma 5.1] that $H_S^0 Z(G)/Z(G) \cong \mathrm{Ad}(H_S^0)$ is compact, completing the proof of (ii). In view of (i), this follows also from the closedness of $\mathrm{Ad}(H_S^0)$ in $S/(S \cap Z(G))$. □

From now on we fix a maximal hyperbolic Lie triple system $\mathfrak{p} \subseteq \mathfrak{q}$ containing $\mathfrak{a}$. We define

$$\mathfrak{g}(0) := \mathfrak{z}_{\mathfrak{g}}(\mathfrak{z}(\mathfrak{p})).$$

Note that $\mathfrak{g}(0)$ is stable under τ so that $(\mathfrak{g}(0), \tau_0)$ is a symmetric Lie algebra with $\tau_0 := \tau|_{\mathfrak{g}(0)}$. We define

$$G(0) := Z_G(\mathfrak{z}(\mathfrak{p}))$$

and note that $G(0)$ is a τ-stable subgroup of G. We denote the restriction of τ to $G(0)$ also by τ_0.

Lemma I.15. *The following assertions hold:*

(i) *The symmetric Lie algebra* $(\mathfrak{g}(0), \tau_0)$ *is non-compactly Riemannian (NCR) and we have*

$$\mathfrak{g}(0) := \mathfrak{h}^0 + \mathfrak{a} + \bigoplus_{\alpha \in \Delta_k} \mathfrak{g}^\alpha = \mathfrak{h}^0 + [\mathfrak{p}, \mathfrak{p}] + \mathfrak{p}.$$

(ii) $Z(G)H^0 \subseteq G(0)$.

(iii) *The symmetric Lie algebra* $(\mathfrak{g}, \tau)$ *admits a triangular decomposition* $\mathfrak{g} = \mathfrak{n}_n^+ \oplus \mathfrak{g}(0) \oplus \mathfrak{n}_n^-$*, i.e., one has*

$$[\mathfrak{n}_n^+, \mathfrak{n}_n^-] \subseteq \mathfrak{g}(0), \qquad \textit{and} \qquad [\mathfrak{g}(0), \mathfrak{n}_n^\pm] \subseteq \mathfrak{n}_n^\pm.$$

Proof. (i) The second assertion follows from [KrNe96, Prop. V.9(iii), Th. VIII.1(ii), Cor. III.8]. That $(\mathfrak{g}(0), \tau_0)$ is (NCR) follows from Lemma I.14(i) together with [KrNe96, Prop. II.1].

(ii) This is obvious by construction.

(iii) This follows from $(\Delta_k + \Delta_n^\pm) \cap \Delta \subseteq \Delta_n^\pm$ (cf. [KrNe96, Prop. V.4]). □

For the proof of the next proposition the following concept turns out to be crucial.

Definition I.16. Let $(\mathfrak{g}, \tau)$ be a symmetric Lie algebra and $\mathfrak{g}_\mathbb{C}$ the complexification of $\mathfrak{g}$. We extend τ to a complex linear involution τ of $\mathfrak{g}_\mathbb{C}$. The *c-dual* $(\mathfrak{g}^c, \tau^c)$ of $(\mathfrak{g}, \tau)$ is defined by $\mathfrak{g}^c := \mathfrak{h} + i\mathfrak{q}$ and $\tau^c := \tau|_{\mathfrak{g}^c}$. The complex conjugation in $\mathfrak{g}_\mathbb{C}$ w.r.t. the real form $\mathfrak{g}^c$ is denoted by $\widehat{\tau}$. Thus the inclusion map $(\mathfrak{g}, \tau) \hookrightarrow (\mathfrak{g}_\mathbb{C}, \widehat{\tau})$ is an embedding of symmetric Lie algebras. We call $(\mathfrak{g}_\mathbb{C}, \widehat{\tau})$ the *canonical extension* of $(\mathfrak{g}, \tau)$ and write $\widehat{\mathfrak{h}} := \mathfrak{g}^c$ and $\widehat{\mathfrak{q}} := i\mathfrak{g}^c$ for the eigenspaces of $\widehat{\tau}$. □

Example I.17. (a) The c-dual of $(\mathfrak{g}, \mathfrak{h}) := (\mathfrak{sl}(2,\mathbb{R}), \mathfrak{so}(1,1))$ is $(\mathfrak{su}(1,1), \mathfrak{so}(1,1))$. More generally, the c-dual of $(\mathfrak{g}, \mathfrak{h}) := (\mathfrak{sl}(n,\mathbb{R}), \mathfrak{so}(p,q))$ is $(\mathfrak{su}(p,q), \mathfrak{so}(p,q))$. That $\mathfrak{g}^c$ is hermitian in these examples is no coincidence; it is always the case for (NCC) symmetric Lie algebras (cf. [HiÓl96, Ch. 1]).

(b) If $(\mathfrak{g}, \mathfrak{h})$ is as in Example I.13(b), then the c-dual is the Lie algebra $\mathfrak{g}^c = \mathrm{span}\{Y, iX, iZ, iA\}$ which is well known as the Lie algebra of the harmonic oscillator, where A corresponds to the Hamiltonian. For the symmetric subalgebra $(\mathfrak{h}_1, \mathbb{R}Y)$ the mapping

$$\mathfrak{h}_1 \to \mathfrak{h}_1^c, \quad xX + yY + zZ \mapsto ixX + yY + izZ$$

is an isomorphism of Lie algebras. We call $(\mathfrak{h}_1, \mathbb{R}Y)$ *self-dual*. The Lie algebra $\mathfrak{g}$ is not self-dual, as follows directly from the fact that the eigenvalues of $\operatorname{ad} A$ are real, so that the eigenvalues of $\operatorname{ad} iA$ are imaginary. $\square$

Remark I.18. It follows from our assumptions on $(\mathfrak{g}, \tau)$ that its canonical extension $(\mathfrak{g}_{\mathbb{C}}, \widehat{\tau})$ is also admissible. This implies, in particular, that $(\mathfrak{g}_{\mathbb{C}}, \widehat{\tau})$ is quasihermitian so that $\mathfrak{a}$ extends to a maximal abelian hyperbolic subspace $\widehat{\mathfrak{a}} \subseteq \mathfrak{q}$ which is also maximal abelian in $\widehat{\mathfrak{q}}$, i.e., $\mathfrak{t}^c = i\widehat{\mathfrak{a}}$ is a compactly embedded Cartan subalgebra of $\mathfrak{g}^c$. Further, every $\mathfrak{p}$-adapted positive system Δ^+ extends to a $\widehat{\mathfrak{p}}$-adapted positive system $\widehat{\Delta}^+ = \widehat{\Delta}^+(\mathfrak{g}_{\mathbb{C}}, \widehat{\mathfrak{a}})$. In some sense one can say that every structural property of $(\mathfrak{g}, \tau)$ is inherited by its canonical extension $(\mathfrak{g}_{\mathbb{C}}, \widehat{\tau})$ (cf. [KrNe96, Th. VIII.1]). $\square$

Let $G_{\mathbb{C}}$ denote a simply connected Lie group associated to $\mathfrak{g}_{\mathbb{C}}$. Then $\sigma := \widehat{\tau}\tau$ is an involution on $\mathfrak{g}_{\mathbb{C}}$ fixing $\mathfrak{g}$ and integrating to an involution on $G_{\mathbb{C}}$ denoted by the same letter. Let $\mathfrak{p}^{\pm}$ and $\mathfrak{k}_{\mathbb{C}}$ denote the complexifications of $\mathfrak{n}_n^{\pm}$ and $\mathfrak{g}(0)$, respectively and write $P^{\pm}$ and $K_{\mathbb{C}}$ for the corresponding subgroups of $G_{\mathbb{C}}$.

Proposition I.19 (The $N_n^- G(0) N_n^+$-decomposition)**.** *The following assertions hold:*

(i) *The groups $G(0)$ and $N_n^{\pm}$ are closed subgroups of G and $N_n^{\pm}$ is diffeomorphic to $\mathfrak{n}_n^{\pm}$ via the exponential mapping.*

(ii) *The multiplication mapping*

$$N_n^- \times G(0) \times N_n^+ \to G, \quad (n_-, k, n_+) \mapsto n_- k n_+$$

is an analytic diffeomorphism onto its open image.

(iii) *The multiplication mapping $N_R^- \times G(0)_R \times N_R^+ \to R$ is an analytic diffeomorphism.*

(iv) *The set $N_n^- G(0) N_n^+$ is R-saturated.*

Proof. Assume first that $G = G_{\mathbb{C}}^{\sigma}$. In view of the $P^- K_{\mathbb{C}} P^+$-decomposition in $G_{\mathbb{C}}$ (cf. [KNÓ97, Prop. II.5]), all statements in this proposition are true with $N^{\pm}$, $G(0)$, G and R replaced by $P^{\pm}$, $K_{\mathbb{C}}$, $G_{\mathbb{C}}$ and $R_{\mathbb{C}}$. Note that $N^{\pm} = (P^{\pm})^{\sigma}$ and $G_1(0) \subseteq K_{\mathbb{C}}$ so that taking fixed points in the $P^- K_{\mathbb{C}} P^+$-decomposition proves (i)–(iv) provided $G = G_{\mathbb{C}}^{\sigma}$.

The general case is easily deduced from this observation by standard lifting arguments, because $Z(G) \subseteq G(0)$. $\square$

We define mappings

$$\overline{\zeta} : N_n^- G(0) N_n^+ \to \mathfrak{n}_n^-, \quad n_- k n_+ \mapsto \log(n_-)$$

and

$$\zeta \colon N_n^- G(0) N_n^+ \to \mathfrak{n}_n^+, \quad n_- k n_+ \mapsto \log(n_+)\,.$$

By a *hyperbolic cone* $W \subseteq \mathfrak{q}$ we understand a cone for which every element in the non-empty interior is hyperbolic (cf. Definition I.1(b)).

Lemma I.20. *The following assertions hold:*

(i) *We have $HAN = HG(0)_0 N_n^+ \subseteq N_n^- G(0)_0 N_n^+ \subseteq N_n^- G(0) N_n^+$.*

(ii) *The map $\overline{\zeta} \colon N_n^- G(0) N_n^+ \to \mathfrak{n}_n^-$ induces a homeomorphism*

$$HG(0)N_n^+ / G(0)N_n^+ \to \Omega,$$

where $\Omega = \mathfrak{n}_r^- + \Omega_S$ is an open domain in $\mathfrak{n}_n^-$ with $\Omega_S \subseteq \mathfrak{n}_p^-$.

(iii) *The compression semigroup of* $\Gamma_{\rm comp} := \{g \in G : \ g.\Omega \subseteq \Omega\}$ *of* Ω *is given by*

$$\Gamma_{\rm comp} = HR\exp(W_{\max,s}),$$

where $W_{\max,s}$ *is the unique maximal* $\mathrm{Inn}_{\mathfrak g}(\mathfrak h_{\mathfrak s})$*-invariant hyperbolic cone in* $\mathfrak s_{\mathfrak q}$ *with* $W_{\max,s} \cap \mathfrak a = C_{\max,s}$.

(iv) $\overline{N} \cap HAN = \Omega N_k^-$.

Proof. (i) Since $(\mathfrak g(0), \tau_0)$ is (NCR) (cf. Lemma I.15(i)), the group $G(0)_0$ admits an Iwasawa decomposition $G(0)_0 = K(0)_0 A N_k^+$. So the first assertion follows from $N = N_n^+ \rtimes N_k^+$ and $K(0)_0 \subseteq H$.

To prove the second inclusion, it remains to show that $H \subseteq N_n^- G(0) N_n^+$. First we assume that $G = G_{\mathbb C}^\sigma$. Let $G^c := G_{\mathbb C}^{\widehat\tau}$ and note that $H = G \cap G^c$. Then we have $G^c \subseteq P^- K_{\mathbb C} P^+$ (cf. [KNÓ97, Prop. II.5(v)]) and taking σ-fixed points we get $H \subseteq N_n^- G(0) N_n^+$. Using standard covering theory, the general case is easily obtained from this because $Z(G) \subseteq G(0) \cap H$.

(ii) It follows from (i), Proposition I.19(ii) and the fact that HAN is open in G (cf. Proposition I.7(i)) that $HG(0)N_n^+/G(0)N_n^+ \to \Omega$ is a homeomorphism onto an open connected subset $\Omega \subseteq \mathfrak n_n^+$. The R-saturatedness of $HG(0)N_n^+$, which follows from (i) and Proposition I.7(iv), implies that $\Omega + \mathfrak n_r^- = \Omega$ or equivalently $\Omega = \mathfrak n_r^- + \Omega_S$ with $\Omega_S \subseteq \mathfrak n_p^-$ and $\Omega_S \cong H_S G(0) N_S^+ / G(0) N_S^+$.

(iii) Again by the R-bi-invariance of $HG(0)N_n^-$ we obtain $\Gamma_{\rm comp} = R \rtimes \Gamma_{{\rm comp},s}$, where $\Gamma_{{\rm comp},s} = \{g \in S : g.\Omega_S \subseteq \Omega_S\}$. Now $\Gamma_{{\rm comp},s} = H_S \exp(W_{\max,s})$ by [HiÓl96, Th. 5.4.20], proving (iii).

(iv) In view of (i) and Proposition I.19(ii), we have

$$HAN = HG(0)_0 N_n^+ = \Omega G(0)_0 N_n^+ \cong \Omega \times G(0)_0 N_n^+.$$

Since $\overline{N} = N_n^- \rtimes N_k^-$ is adapted to the decomposition of Proposition I.19(ii), we thus obtain

$$HAN \cap \overline{N} = (\Omega \times G(0)_0 N_n^+) \cap (N_n^- \times N_k^-) = \Omega N_k^-.$$

□

Let $W_{\max}$ denote the unique maximal $\mathrm{Inn}_{\mathfrak g}(\mathfrak h)$-invariant hyperbolic cone in $\mathfrak q$ with $W_{\max} \cap \mathfrak a = C_{\max}$,

$$\Gamma := H\exp(\mathrm{int}\, W_{\max}) \subseteq \Gamma_{\rm comp}, \qquad \text{and} \qquad \Gamma_A := \exp(\mathrm{int}\, C_{\max}).$$

According to Lawson's Theorem (cf. [La94]) the *Polar Decomposition*

$$H \times \mathrm{int}\, W_{\max} \to \Gamma, \quad (h, X) \mapsto h\exp(X)$$

is a homeomorphism and Γ is a Lie subsemigroup of G, a so-called *real Ol'shanskiĭ semigroup*. The semigroup Γ defines an ordering on the symmetric space $\mathcal M := H\backslash G$ by

$$Hx < Hy\colon \iff y \in \Gamma x.$$

We call $\mathcal M$ a *hyperbolically causal symmetric space*. Let $x_0 := H.e$ be the base point of $\mathcal M$ and set $\mathcal M^+ := \{x \in \mathcal M\colon x > x_0\}$. Note that $\mathcal M^+ = x_0.\Gamma$.

Recall from Lemma I.14 that H^0 is compactly embedded, so that the quotient space H/H^0 carries a unique H-invariant measure μ_{H/H^0}.

Proposition I.21. *The following assertions hold:*

(i) *The set* $\overline{N}H^0AN$ *is open in* G *and for* $f \in L^1(G)$ *with* $\mathrm{supp}(f) \subseteq \overline{N}H^0AN$ *we have*

$$\int_G f(g)\ d\mu_G(g) = \int_{\overline{N}} \int_{H^0} \int_A \int_N f(\overline{n}han)a^{2\rho}\ d\mu_N(n)\ d\mu_A(a)\ d\mu_{H^0}(h)\ d\mu_{\overline{N}}(\overline{n}).$$

(ii) *For all* $f \in L^1(H/H^0)$ *we have*

$$\int_{H/H^0} f(hH^0)\ d\mu_{H/H^0}(hH^0) = \int_{\overline{N}\cap HAN} f(h_H(\overline{n}))a_H(\overline{n})^{-2\rho}\ d\mu_{\overline{N}}(\overline{n}).$$

Proof (cf. [Ól87]). In view of Proposition I.19(iv), we have that $\overline{N}H^0AN = R\overline{N}_SH_S^0A_SN_S$. Further $\mathfrak{a}_\mathfrak{s}$ is also maximal abelian in $\mathfrak{p}_\mathfrak{s}$, therefore Bruhat's Theorem shows that $\overline{N}_SZ_{K_S}(A_S)A_SN_S$ is open in S with a complement of Haar measure zero. Note that $H_S \subseteq Z_{K_S}(A_S)$ is an open subgroup which in general is proper. So the first assertion follows from $m_*(\mu_R \otimes \mu_S) = \mu_G$, where $m\colon R \times S \to G$ is the multiplication mapping. The integration formula now follows by a similar argument as in the proof of Lemma I.9, here applied to the action of the group $(\overline{N}H^0) \times AN$ on G (see also [Ól87, Th. 7.1]).

(ii) (cf. [Ól87, Lemma 1.3]) Choose a function $\psi \in C_c(AN)$ such that

$$\int_A \int_N \psi(an)a^{2\rho}\ d\mu_N(n)\ d\mu_A(a) = 1$$

and set

$$f_1(xH^0) = \begin{cases} f(h_H(x)H^0)\psi(a_H(x)n_H(x)) & \text{for } x \in HAN, \\ 0 & \text{otherwise.} \end{cases}$$

Then $f_1 \in L^1(G/H^0)$, and, according to Lemma I.9, we have

$$\int_{G/H^0} f_1(x)\ d\mu_{G/H^0}(x) = \int_{H/H^0} f(hH^0)\ d\mu_{H/H^0}(hH^0).$$

Thus it follows from the integration formula in (i) that

$$\begin{aligned}
&\int_{H/H^0} f(hH^0)\ d\mu_{H/H^0}(hH^0) \\
&\quad= \int_{HAN} f_1(\overline{n}an)a^{2\rho}\ d\mu_N(n)\ d\mu_A(a)\ d\mu_{\overline{N}}(\overline{n}) \\
&\quad= \int_{\overline{N}AN} f_1(h_H(\overline{n})an)a_H(\overline{n})^{-2\rho}a^{2\rho}\ d\mu_N(n)\ d\mu_A(a)\ d\mu_{\overline{N}}(\overline{n}) \\
&\quad= \int_{\overline{N}\cap HAN} f(h_H(\overline{n}))a_H(\overline{n})^{-2\rho}\ d\mu_{\overline{N}}(\overline{n}),
\end{aligned}$$

as was to be shown. □

II. The extended metaplectic representation

In this section we discuss in detail the generalized extended metaplectic representation $(\mu_\lambda, \mathcal{H}_\lambda)$ of G. This representation is obtained by a composition of an appropriate homomorphism of G^c into the Jacobi group $\mathrm{HSp}(n,\mathbb{R}) := H_n \rtimes \mathrm{Sp}(n,\mathbb{R})$ and the extended metaplectic representation $(\mu, \mathcal{H}_\mu)$ of $\mathrm{HSp}(n,\mathbb{R})$. We classify the (H,χ)-spherical distribution and hyper-function vectors of $(\mu_\lambda, \mathcal{H}_\lambda)$ (cf. Theorem II.14). We further compute an integral over a certain matrix coefficient of $(\mu_\lambda, \mathcal{H}_\lambda)$

(cf. Lemma II.16) that will be a key ingredient in the factorization of spherical functions in Section III.

The symmetric Jacobi group. We recall from [KrNe96, Ex. VII.17(b)] the construction of the symmetric Jacobi algebra. Denote by $\mathfrak{h}_n = \mathbb{R}^{2n} \oplus \mathbb{R}$ the $2n+1$-dimensional Heisenberg algebra with the Lie bracket

$$[(v,t),(v',t')] = (0, \Omega(v,v')) \quad \text{for} \quad v, v' \in \mathbb{R}^{2n}, t, t' \in \mathbb{R},$$

where Ω is the skew symmetric bilinear form represented by

$$\Omega = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

The *Jacobi algebra* is defined by $\mathfrak{hsp}(n,\mathbb{R}) := \mathfrak{h}_n \rtimes \mathfrak{sp}(n,\mathbb{R})$. Set $I_{n,n} := \operatorname{diag}(I_n, -I_n) \in \mathfrak{gl}(2n,\mathbb{R})$. The prescription

$$\tau(v,t,X) := (I_{n,n}.v, -t, I_{n,n} X I_{n,n})$$

defines a symmetric structure on $\mathfrak{hsp}(n,\mathbb{R})$ (cf. [Kn96, Ex. VII.17(b)]). We call $(\mathfrak{hsp}(n,\mathbb{R}),\tau)$ the *symmetric Jacobi algebra*.

Denote by H_n the simply connected Heisenberg group corresponding to $\mathfrak{h}_n$. Then

$$(\mathrm{HSp}(n,\mathbb{R}),\tau) := (H_n \rtimes \widetilde{\mathrm{Sp}}(n,\mathbb{R}),\tau)$$

is a simply connected Lie group with $\mathbf{L}(\mathrm{HSp}(n,\mathbb{R})) = \mathfrak{hsp}(n,\mathbb{R})$, called the *symmetric Jacobi group*. We put $(G^c,\tau^c) := (\mathrm{HSp}(n,\mathbb{R}),\tau)$ and accordingly $(\mathfrak{g}^c,\tau^c) := (\mathfrak{hsp}(n,\mathbb{R}),\tau)$ (the terminology will be justified later on). Then we have

$$\mathfrak{h} = \{0\} \oplus (\mathbb{R}^n \oplus \{0\}) \rtimes \left\{ \begin{pmatrix} X & 0 \\ 0 & -X^t \end{pmatrix} : X \in \mathfrak{gl}(n,\mathbb{R}) \right\}.$$

Thus we can identify $\mathfrak{h}$ with $\mathbb{R}^n \rtimes \mathfrak{gl}(n,\mathbb{R})$, where $\mathfrak{gl}(n,\mathbb{R})$ acts on $\mathbb{R}^n$ by the identical representation.

Lemma II.1. *The following assertions hold:*

(i) *The fixed point group H is isomorphic to $\mathbb{R}^n \rtimes \mathrm{GL}(n,\mathbb{R})_+$.*
(ii) *The group H is not unimodular and the modular function is given by*

$$\Delta_H(p,g) = (\det g)^{-1} \quad \textit{for} \quad (p,g) \in H.$$

Proof. (i) We only have to show that $\widetilde{\mathrm{Sp}}(n,\mathbb{R})^\tau \cong \mathrm{GL}(n,\mathbb{R})_+$. Denote by K^c the maximal compactly embedded subgroup of $\widetilde{\mathrm{Sp}}(n,\mathbb{R})$. As $\widetilde{\mathrm{Sp}}(n,\mathbb{R})^\tau$ admits a polar decomposition, we only have to show that $\widetilde{\mathrm{Sp}}(n,\mathbb{R})^\tau \cap K^c \cong \mathrm{SO}(n,\mathbb{R})$. But this follows from $K^c \cong \mathrm{SU}(n) \times \mathbb{R}$.

(ii) Since $\mathrm{Ad}_H(p)$ is unipotent, this follows from

$$\Delta_H(p,g) = |\det \mathrm{Ad}_H(p,g)^{-1}| = \det \mathrm{Ad}_H(g^{-1}) = (\det g)^{-1}$$

for $(p,g) \in H$. $\square$

If $T = \operatorname{diag}(t_1, \ldots, t_n)$ is the diagonal matrix with entries $t_1, \ldots, t_n$, then we set

$$X(t_1,\ldots,t_n) := \begin{pmatrix} 0 & T \\ -T & 0 \end{pmatrix}.$$

The symplectic Lie algebra $\mathfrak{sp}(n,\mathbb{R})$ admits a τ-stable Cartan subalgebra $\mathfrak{t}_{\mathfrak{l}}^c$ given by

$$\mathfrak{t}_{\mathfrak{l}}^c := \{X(t_1,\dots,t_n):\ t_1,\dots,t_n \in \mathbb{R}\}.$$

Note that $\mathfrak{t}^c = \mathfrak{z}(\mathfrak{g}^c) \oplus \mathfrak{t}_{\mathfrak{l}}^c$ is a compactly embedded Cartan subalgebra of $\mathfrak{g}^c$ and write $T^c = Z(G^c) \times T_L^c$ for the corresponding subgroup of G^c.

Remark II.2. (a) The symmetric Jacobi algebra is self-dual, since both $(\mathfrak{h}_n, \tau\,|_{\mathfrak{h}_n})$ and $(\mathfrak{sp}(n,\mathbb{R}), \tau\,|_{\mathfrak{sp}(n,\mathbb{R})})$ are.

(b) The c-dual $(\mathfrak{g},\tau)$ of $(\mathfrak{g}^c,\tau)$ is an admissible effective symmetric Lie algebra which, according to (i), is isomorphic to $(\mathfrak{g}^c,\tau)$.

The subspace $\mathfrak{a} := i\mathfrak{t}^c$ is a maximal abelian hyperbolic subspace of $(\mathfrak{g},\tau)$ and so $\widehat{\Delta}(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}^c) = \Delta(\mathfrak{g},\mathfrak{a})$. For $1 \le j \le n$ we define $\varepsilon_j \in \mathfrak{a}^*$ by $\varepsilon_j(iX(t_1,\dots,t_n)) = t_j$. Then an easy calculation shows that

$$\Delta_r = \{\pm\varepsilon_j :\ 1 \le j \le n\}, \qquad \Delta_p = \{\pm(\varepsilon_j + \varepsilon_k) :\ 1 \le j,k \le n\},$$

and

$$\Delta_k = \{\pm(\varepsilon_j - \varepsilon_k) :\ 1 \le j < k \le n\}$$

(cf. [Ne99, Ex. VII.2.30]). Moreover, a positive $\mathfrak{p}$-adapted system of roots is given by

$$\Delta^+ = \{\varepsilon_j :\ 1 \le j \le n\} \cup \{\varepsilon_j + \varepsilon_k :\ 1 \le j,k \le n\} \cup \{\varepsilon_j - \varepsilon_k :\ 1 \le j < k \le n\}$$

and related to Δ^+ we have

$$C_{\max,s} = \{iX(t_1,\dots,t_n) :\ (\forall 1 \le j \le n)\ t_j \ge 0\}.$$

□

The extended metaplectic representation. Recall the definition of the *Schrödinger representation* $(\sigma, \mathcal{H}_\sigma)$ of the Heisenberg group H_n modelled on $\mathcal{H}_\sigma = L^2(\mathbb{R}^n)$. The explicit formula for σ is given by

$$\big(\sigma\big((p,q),t\big).f\big)(x) = e^{2\pi i t + 2\pi i\langle q,x\rangle + \pi i\langle p,q\rangle} f(x+p)$$

for all $f \in \mathcal{H}_\sigma$ (cf. [Fo89, Ch. 1]). It is well known that $(\sigma,\mathcal{H})$ extends to a unitary representation of the Jacobi group G^c, called the *extended metaplectic representation* and which we denote by $(\mu, \mathcal{H}_\mu)$ (cf. [Fo89, Ch. 4]).

In the following μ_n denotes Lebesgue measure on $\mathbb{R}^n$.

Lemma II.3. *The following assertions hold:*

(i) *For all* $h = (p,g) \in H = \mathbb{R}^n \rtimes \mathrm{GL}(n,\mathbb{R})_+$, $f \in L^2(\mathbb{R}^n)$ *and* $x \in \mathbb{R}^n$ *we have*

$$(\mu(h).f)(x) := \Delta_H(h)^{\frac{1}{2}} f\big(g^{-1}.(x+p)\big) = \det(g)^{-\frac{1}{2}} f\big(g^{-1}.(x+p)\big).$$

(ii) *For* $X = X(t_1,\dots,t_n) \in -iC_{\max,s}^0$ *with* $\det(\cos T) \neq 0$ *we have*

$$(\mu(\exp X).f)(x) = \det(\cos T)^{-\frac{1}{2}} \cdot \int_{\mathbb{R}^n} e^{\pi i\langle \tan(T).x,x\rangle + \pi i\langle \tan(T).y,y\rangle + 2\pi i\langle \cos(T)^{-1}.x,y\rangle} \widehat{f}(y)\ d\mu_n(y)$$

for $f \in \mathcal{H}_\mu$ *and* $x \in \mathbb{R}^n$, *where for* $f \in \mathcal{H}_\mu \cap L^1(\mathbb{R}^n)$ *the Fourier transform is given by* $\widehat{f}(y) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i\langle x,y\rangle} d\mu_n(x)$.

Proof. (i) This is a special case of [Fo89, Th. 4.51]. (ii) [Fo89, Cor. 4.55]. □

For any topological group G we denote by $\mathbb{X}(G)$ the group of all continuous characters $\chi\colon G \to \mathbb{C}^\times$.

Definition II.4. Let G be a Lie group and $\mathcal{H}$ a Hilbert space.

(a) For a unitary representation $(\pi, \mathcal{H})$ of G we denote by $\mathcal{H}^\infty$ and $\mathcal{H}^\omega$ the space of all smooth, resp. analytic vectors of $(\pi, \mathcal{H})$. The corresponding strong antiduals are denoted by $\mathcal{H}^{-\infty}$ and $\mathcal{H}^{-\omega}$ and their elements are called *distribution*, resp. *hyperfunction vectors* (see [KNÓ97, Appendix] for the definition of the topology of $\mathcal{H}^\omega$). Note that there is a natural chain of continuous inclusions

$$\mathcal{H}^\omega \hookrightarrow \mathcal{H}^\infty \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{H}^{-\infty} \hookrightarrow \mathcal{H}^{-\omega}.$$

The natural extension of $(\pi, \mathcal{H})$ to a representation on the space of hyperfunction vectors is denoted by $(\pi^{-\omega}, \mathcal{H}^{-\omega})$ and given explicitly by

$$\langle \pi^{-\omega}(g).\nu, v\rangle := \langle \nu, \pi^\omega(g^{-1}).v\rangle.$$

(b) Let $H \subseteq G$ be a closed subgroup and $\chi \in \mathbb{X}(H)$ a continuous character of H. For a unitary representation $(\pi, \mathcal{H})$ of G we write $(\mathcal{H}^{-\omega})^{(H,\chi)}$ for the set of all those elements $\nu \in \mathcal{H}^{-\omega}$ satisfying $\pi^{-\omega}(h).\nu = \chi(h).\nu$ for all $h \in H$. The unitary representation $(\pi, \mathcal{H})$ is called (H, χ)-*spherical* if there exists a cyclic vector $\nu \in (\mathcal{H}^{-\omega})^{(H,\chi)}$. $\square$

Remark II.5. (a) Even though in general the topology on the space of analytic vectors is hard to deal with, one has quite an explicit picture for unitary highest weight representations. If G is a connected Lie group and $(\pi, \mathcal{H})$ a unitary highest weight representation of G with discrete kernel, then $(\pi, \mathcal{H})$ naturally extends to a holomorphic representation of a bigger complex Ol'shanskiĭ semigroup $\Gamma = G\operatorname{Exp}(W)$, where W denotes a closed convex $\operatorname{Ad}(G)$-invariant cone in $i\mathfrak{g}$ with non-empty interior W^0. For each $X \in W^0$ we then have

$$\mathcal{H}^\omega = \bigcup_{t>0} \pi(\operatorname{Exp}(tX)).\mathcal{H},$$

and the topology on $\mathcal{H}^\omega$ is the finest locally convex topology on $\mathcal{H}^\omega$ making for all $t > 0$ the maps $\mathcal{H} \to \mathcal{H}^\omega$, $\ v \mapsto \pi(\operatorname{Exp}(tX)).v$ continuous (cf. [KNÓ97, Appendix]). The action of Γ on $\mathcal{H}^{-\omega}$ is given by

$$\langle \pi^{-\omega}(s).\nu, v\rangle := \langle \nu, \pi^\omega(s^*).v\rangle.$$

(b) Let $G \cong G^c$ be the Jacobi group, and Δ^+ as in Remark II.2(b). Let $\varepsilon_0 \in \mathfrak{z}(\mathfrak{g})^* \cong \mathfrak{a}_{\mathfrak{l}}^\perp$ be defined by $\varepsilon_0(i) = -1$. Then one knows that the extended metaplectic representation $(\mu, \mathcal{H}_\mu)$ is a unitary highest weight representation of the Jacobi group G^c w.r.t. $\widehat{\Delta}^+$ and highest weight $\lambda = 2\pi\varepsilon_0 - \rho_r$. Moreover, $(\mu, \mathcal{H}_\mu)$ extends to a highest weight representation of the complex Ol'shanskiĭ semigroup $\Gamma^c = G^c\operatorname{Exp}(\widehat{W}_{\max,s})$ (cf. [Ne99, Ch. X]). $\square$

Lemma II.6. *Let $\mathcal{S}(\mathbb{R}^n)$ denote the Schwartz space of $\mathbb{R}^n$. Then the smooth vectors of the Schrödinger representation $(\sigma, \mathcal{H}_\sigma)$ and the extended metaplectic representation $(\mu, \mathcal{H}_\mu)$ coincide and we have*

$$\mathcal{H}_\sigma^\infty = \mathcal{H}_\mu^\infty = \mathcal{S}(\mathbb{R}^n).$$

Proof. First we claim that $\mathcal{H}_\sigma^\infty = \mathcal{S}(\mathbb{R}^n)$. From the explicit formula for the derived representation of the Schrödinger representation (cf. [Fo89, Ch. 1]) we deduce that

$$\mathcal{H}_\sigma^\infty = \Big\{ f \in L^2(\mathbb{R}^n) : \ (\forall P, Q \in \mathbb{C}[X_1, \ldots, X_n]) \\ P(x_1, \ldots, x_n) Q\Big(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\Big).f \in L^2(\mathbb{R}^n) \Big\}.$$

Thus $\mathcal{H}_\sigma^\infty = \mathcal{S}(\mathbb{R}^n)$ follows from the Sobolev Lemma.

In view of $\mathcal{H}_\mu^\infty \subseteq \mathcal{H}_\sigma^\infty$, it only remains to show that $\mathcal{S}(\mathbb{R}^n) \subseteq \mathcal{H}_\mu^\infty$. But this follows from the fact that all operators in $d\mu(\mathfrak{g})$ are contained in the associative algebra generated by $d\sigma(\mathfrak{h}_n)$ (cf. [Fo89, Th. 4.40]). □

Remark II.7 (cf. [Fo89, Ch. 1.6]). We denote by $\langle \cdot, \cdot \rangle$ the standard hermitian inner product on $\mathbb{C}^n$ and define the Fock space

$$\mathcal{F}(\mathbb{C}^n) := \{ f \in \mathrm{Hol}(\mathbb{C}^n) : \ \|f\|^2 := \int_{\mathbb{C}^n} |f(z)|^2 e^{-\pi\|z\|^2} \, d\mu_{\mathbb{C}^n}(z) < \infty \}.$$

This is a Hilbert space with the reproducing kernel $K(z, w) = e^{\pi\langle z, w\rangle}$, i.e., the holomorphic functions $K_w \colon z \mapsto K(z, w)$ are contained in $\mathcal{F}(\mathbb{C}^n)$ and satisfy $\langle f, K_z\rangle = f(z)$ for all $f \in \mathcal{F}(\mathbb{C}^n)$. The Fock space is related to $L^2(\mathbb{R}^n)$ by the Bargmann transform

$$B \colon L^2(\mathbb{R}^n) \to \mathcal{F}(\mathbb{C}^n), \quad (Bf)(z) = 2^{\frac{n}{4}} e^{-\frac{\pi}{2}\langle z, \overline{z}\rangle} \int_{\mathbb{R}^n} f(x) e^{2\pi\langle z, x\rangle} e^{-\pi\|x\|^2} \, d\mu_n(x)$$

which is an isometric isomorphism. Writing $w = p + iq$ for $p, q \in \mathbb{R}^n$, the Schrödinger representation in the Fock model is given by

$$\big(\sigma\big(w, t).f\big)(z) = e^{-\frac{\pi}{2}\langle w, w\rangle - \pi\langle z, w\rangle + 2\pi i t} f(z + w). \tag{2.1}$$

Note that μ and σ naturally extend to representations on all holomorphic functions on $\mathbb{C}^n$.

The advantage of the Fock model of the Schrödinger representation is that one also gets a good picture of the hyperfunction vectors. If we realize the representation $(\mu_1, \mathcal{H}_{\mu_1})$ in $\mathcal{F}(\mathbb{C}^n)$, then the mapping

$$\mathcal{H}_{\mu_1}^{-\omega} \to \mathrm{Hol}(\mathbb{C}^n), \quad \nu \mapsto \big(z \mapsto \nu(K_z)\big)$$

is a G_1^c-equivariant realization of $\mathcal{H}_{\mu_1}^{-\omega}$ (cf. [KNÓ97, Sect. VI]). Thus we obtain a realization of the hyperfunction vectors of $(\mu_1, \mathcal{H}_{\mu_1})$ by holomorphic functions on $\mathbb{C}^n$ with H_n-action given by (2.1).

According to [Fo89, (1.80)], the inverse of the Bargmann transform is given for $f \in \mathcal{F}(\mathbb{C}^n)$ by

$$(B^{-1}.f)(x) = 2^{\frac{n}{4}} e^{-\pi x^2} \int_{\mathbb{C}^n} e^{2\pi\langle x, z\rangle} e^{-\frac{\pi}{2}\langle \overline{z}, z\rangle} f(z) e^{-\pi\|z\|^2} \, d\mu_{\mathbb{C}^n}(z),$$

where the right-hand side has to be interpreted as the limit of the corresponding expressions for f_n, where $f_n \to f$ holds in $\mathcal{F}(\mathbb{C}^n)$, and the functions f_n are polynomials. For $f = 1$ we obtain, in particular,

$$(B^{-1}.1)(x) = 2^{\frac{n}{4}} e^{-\pi\|x\|^2}$$

(cf. [Fo89, App. A]).

For later reference we record the following formula for Gaussian integrals (cf. [Fo89, p. 256]):

$$\int_{\mathbb{R}^n} e^{2\pi i\langle x,w\rangle} e^{2\pi\langle z,x\rangle} e^{-\pi\|x\|^2}\, d\mu_n(x) = e^{-\pi\langle iz-\overline{w},-i\overline{z}-w\rangle}.$$

For $w=0$ we obtain, in particular,

$$\int_{\mathbb{R}^n} e^{2\pi\langle z,x\rangle} e^{-\pi\|x\|^2}\, d\mu_n(x) = e^{\pi\langle z,\overline{z}\rangle}. \tag{2.2}$$

Since the actions of some groups are more transparent in the realization of $\mathcal{H}_{\mu_1}$ as $L^2(\mathbb{R}^n)$, it is instructive to see how the hyperfunction vectors can be Bargmann transformed. We consider the function $f_w \in \mathrm{Hol}(\mathbb{C}^n)$ given by

$$f_w(z) = e^{2\pi i\langle z,w\rangle} e^{\frac{\pi}{2}\langle z,\overline{z}\rangle}.$$

We will see below in Theorem II.9 that $f_w \in \mathcal{H}_{\mu_1}^{-\omega}$. For $h \in \mathcal{H}_{\mu_1}^{\omega} \subseteq L^2(\mathbb{R}^n)$ we obtain with

$$\widehat{h}(z) := e^{\frac{\pi}{2}\langle z,\overline{z}\rangle}(Bh)(z) = 2^{\frac{n}{4}} \int_{\mathbb{R}^n} h(x) e^{2\pi\langle z,x\rangle} e^{-\pi\|x\|^2}\, d\mu_n(x)$$

the relation

$$\begin{aligned}\langle f_w, Bh\rangle &= \int_{\mathbb{C}^n} e^{2\pi i\langle z,w\rangle} e^{\frac{\pi}{2}\langle z,\overline{z}\rangle} e^{-\frac{\pi}{2}\langle \overline{z},z\rangle} \overline{\widehat{h}(z)} e^{-\pi\|z\|^2}\, d\mu_{\mathbb{C}^n}(z)\\ &= \int_{\mathbb{C}^n} e^{2\pi i\langle z,w\rangle} e^{\pi i\,\mathrm{Im}(\langle z,\overline{z}\rangle)} \overline{\widehat{h}(z)} e^{-\pi\|z\|^2}\, d\mu_{\mathbb{C}^n}(z).\end{aligned}$$

Since the function $z \mapsto e^{\pi i\,\mathrm{Im}(\langle z,\overline{z}\rangle)}$ is bounded, Fubini's Theorem leads to

$$\begin{aligned}\langle f_w, Bh\rangle = 2^{\frac{n}{4}} &\int_{\mathbb{R}^n} \overline{h(x)} e^{-\pi\|x\|^2}\\ &\cdot \int_{\mathbb{C}^n} e^{2\pi(\langle x,z\rangle + i\langle z,w\rangle)} e^{\pi i\,\mathrm{Im}\langle z,\overline{z}\rangle} e^{-\pi\|z\|^2}\, d\mu_{\mathbb{C}^n}(z)\, d\mu_n(x)\end{aligned}$$

which means that $B^*(f_w) = B^{-1}(f_w)$ is represented by the function

$$B^*(f_w)(x) = 2^{\frac{n}{4}} e^{-\pi\|x\|^2} \int_{\mathbb{C}^n} e^{2\pi(\langle x,z\rangle + i\langle z,w\rangle)} e^{\pi i\,\mathrm{Im}\langle z,\overline{z}\rangle} e^{-\pi\|z\|^2}\, d\mu_{\mathbb{C}^n}(z).$$

In view of Lebesgue's Dominated Convergence Theorem, we have

$$B^*(f_w)(x) = 2^{\frac{n}{4}} e^{-\pi\|x\|^2} \lim_{\substack{s\to\pi\\ s<\pi}} \int_{\mathbb{C}^n} e^{2\pi(\langle x,z\rangle + i\langle z,w\rangle)} e^{\frac{s}{2}\langle z,\overline{z}\rangle} e^{-\frac{s}{2}\langle \overline{z},z\rangle} e^{-\pi\|z\|^2}\, d\mu_{\mathbb{C}^n}(z).$$

To evaluate the integrals on the right-hand side we use the general formula [Fo89, p. 258] with $v = 2x$, $u = 2i\overline{w}$, $A = \frac{s}{\pi}\mathbf{1}$ and $D = -A$. This leads to

$$\begin{aligned}B^*(f_w)(x) &= 2^{\frac{n}{4}} e^{-\pi\|x\|^2} 2^{-\frac{n}{2}} e^{\pi(\|x\|^2 + 2i\langle \overline{w},x\rangle + \langle i\overline{w},-iw\rangle)}\\ &= 2^{-\frac{n}{4}} e^{-\pi\langle \overline{w},w\rangle} e^{2\pi i\langle x,w\rangle}.\end{aligned} \tag{2.3}$$

□

Lemma II.8. *Let $H_1 = H_U \rtimes H_{1,L}$ be a semidirect product group, where H_U is a real vector space which is a semisimple $H_{1,L}$-module. If $H_{U,\mathrm{fix}}$ is the subspace of $H_{1,L}$-fixed vectors, then the mapping*

$$\mathbb{X}(H_1) \to \mathbb{X}(H_{1,L}) \times \mathbb{X}(H_{U,\mathrm{fix}}), \quad \chi \mapsto (\chi|_{H_{1,L}}, \chi|_{H_{U,\mathrm{fix}}})$$

is an isomorphism of groups.

Proof. Since H_U is assumed to be semisimple as an $H_{1,L}$-module, we have $H_U = H_{U,\mathrm{fix}} \oplus H_{u,\mathrm{eff}}$, where $H_{U,\mathrm{eff}} = \mathrm{span}\{h.w - w : w \in H_U, h \in H_{1,L}\}$. If $\chi \in \mathbb{X}(H_1)$ is a character, then $\chi|_{H_U}$ vanishes on $H_{U,\mathrm{eff}}$ which is contained in the commutator group of H_1. Now the assertion of the lemma follows easily. □

Theorem II.9. *Let $(\mathfrak{g}_1^c, \tau_1^c)$ be a symmetric subalgebra of $(\mathfrak{g}^c, \tau^c)$ such that $(\mathfrak{g}_1, \tau_1)$ is effective and admissible, $\widehat{W}_{\max}^0 \cap i\mathfrak{g}_1^c \neq \emptyset$, $\mathfrak{g}_1^c = \mathfrak{h}_n \rtimes \mathfrak{l}_1^c$ with $\mathfrak{l}_1^c$ reductive, and that $\mathfrak{t}_1^c := \mathfrak{g}_1^c \cap \mathfrak{t}^c$ is a compactly embedded Cartan subalgebra of $\mathfrak{g}_1^c$. Denote by G_1^c the corresponding analytic subgroup of G^c and by $(\mu_1, \mathcal{H}_{\mu_1})$ the restriction of the extended metaplectic representation to G_1^c. Then $H_1 = H_U \rtimes H_{1,L}$ and for $\chi \in \mathbb{X}(H_1)$ the following assertions hold:*

(i) *The representation $(\mu_1, \mathcal{H}_{\mu_1})$ is (H_1, χ)-spherical if and only if $\chi_l = \Delta_{H_1}^{\frac{1}{2}}|_{H_{1,L}}$ where $\chi_l = \chi|_{H_{1,L}}$ and in this case we have*

$$(\mathcal{H}_{\mu_1}^{-\omega})^{(H_1,\chi)} = \mathbb{C}\chi_u,$$

where $\chi_u := \chi|_{H_U}$ is viewed as an antilinear functional on $\mathcal{H}_{\mu_1}^{\omega}$ by

$$\langle \chi_u, \varphi \rangle = \int_{\mathbb{R}^n} \chi_u(x)\overline{\varphi(x)}\, d\mu_n(x).$$

(ii) *We have $(\mathcal{H}_{\mu_1}^{-\omega})^{(H_1,\chi)} = (\mathcal{H}_{\mu_1}^{-\infty})^{(H_1,\chi)}$ if and only if χ_u is unitary. In particular,*

$$(\mathcal{H}_{\mu_1}^{-\infty})^{(H_1,\chi)} = \{0\}$$

whenever χ_u is not unitary.

Proof. (i) We recall the notation from Remark II.7. Let $f \in (\mathcal{H}_{\mu_1}^{-\omega})^{(H_1,\chi)}$ and $w \in \mathbb{C}^n$ such that $\chi_u(x) = e^{2\pi i\langle x,w\rangle}$. Note that the condition that χ_u extends to a character of $H_1 = H_U \rtimes H_{1,L}$ means that $w \in \mathbb{C}^n$ is fixed under the group $H_{1,L}$ (cf. Lemma II.8). In view of (2.1) in Remark II.7, we have for all $p \in \mathbb{R}^n \subseteq H_1$ and $z \in \mathbb{C}^n$

$$e^{2\pi i\langle p,w\rangle} f(z) = (\pi_{\mu_1}^{-\omega}(p).f)(z) = e^{-\frac{\pi}{2}\langle p,p\rangle - \pi\langle z,p\rangle} f(z+p).$$

Putting $z = 0$ we derive

$$f(x) = e^{2\pi i\langle x,w\rangle + \frac{\pi}{2}\langle x,x\rangle} f(0)$$

for all $x \in \mathbb{R}^n$. Since f is holomorphic, it is uniquely determined by its restriction to $\mathbb{R}^n$. Therefore $(\mathcal{H}_{\mu_1}^{-\omega})^{(H_1,\chi)} \subseteq \mathbb{C}f_\chi$ with $f_\chi(z) = e^{2\pi i\langle z,w\rangle + \frac{\pi}{2}\langle z,\overline{z}\rangle}$.

We claim that $f_\chi \in \mathcal{H}_{\mu_1}^{-\omega}$. Let $\Gamma_1^c = G_1^c \operatorname{Exp}(\widehat{W}_1)$ be the complex Ol'shanskiĭ semigroup with $\widehat{W}_1 = \widehat{W}_{\max} \cap i\mathfrak{g}_1^c$ (this cone is non-trivial by assumption). It follows from Remark II.5 that $(\mu_1, \mathcal{H}_{\mu_1})$ extends to a holomorphic representation of Γ_1^c. By our assumptions on $(\mathfrak{g}_1^c, \tau_1^c)$, we find an element $X = iX(t_1, \ldots, t_n) \in \widehat{W}_1^0$. Note that $t_j > 0$ for all $1 \leq j \leq n$. In view of Remark II.4(a), it suffices to show that

$$(\forall s > 0) \;\; \mu_1^{-\omega}(\operatorname{Exp}(sX)).f_\chi \in \mathcal{F}(\mathbb{C}^n). \tag{2.4}$$

According to [Fo89, Prop. 4.39], we have

$$\big(\mu_1^{-\omega}(\operatorname{Exp}(sX)).f_\chi\big)(z) = \Big(\prod_{j=1}^n e^{-\frac{st_j}{2}}\Big) f_\chi(e^{-st_1}z_1, \ldots, e^{-st_n}z_n),$$

so that the explicit formula for f_χ shows that (2.4) is satisfied, proving our claim. Hence

$$(\mathcal{H}_{\mu_1}^{-\omega})^{(H_1,\chi)} \subseteq \mathbb{C} f_\chi \subseteq \mathcal{H}_{\mu_1}^{-\omega}. \tag{2.5}$$

Now we turn to the $L^2(\mathbb{R}^n)$-model of the extended metaplectic representation, where the H-action is much simpler than in the Fock model. Using the Bargmann transform, we obtain

$$(\mathcal{H}_{\mu_1}^{-\omega})^{(H_1,\chi)} \subseteq \mathbb{C}\chi_u$$

because we have seen in (2.3) in Remark II.7 that $B^*(f_w) = c\chi_u$ holds for some $c \in \mathbb{R}$.

For $g \in H$ we have

$$\big(\mu(g).f\big)(x) = (\det g)^{-\frac{1}{2}} f(g^{-1}.x)$$

and, in particular,

$$\big(\mu(g).\chi_u\big)(x) = (\det g)^{-\frac{1}{2}} e^{2\pi i \langle g^{-1}.x, w\rangle}.$$

The right-hand side equals $\chi(g)\chi_u$ for all $g \in H_1$ if and only if $\chi_l = \Delta_{H_1}^{\frac{1}{2}}$ because w is fixed by the group $H_{1,L}$ (see Lemma II.8). According to Lemma II.3(i), this condition is equivalent to $(\mathcal{H}_{\mu_1}^{-\omega})^{(H_1,\chi)} \neq \{0\}$. In view of (2.5), this proves (i).

(ii) Note that the function χ_u is a tempered distribution if and only if χ_u is unitary. Hence the assertion follows from $\mathcal{S}(\mathbb{R}^n) = \mathcal{H}_{\mu_1}^\infty$ (cf. Lemma II.6). □

Corollary II.10. *The extended metaplectic representation is (H,χ)-spherical if and only if $\chi = \Delta_H^{\frac{1}{2}}$.* □

In the following we write $f^*(x) := \overline{f(x^{-1})}$ for a function f on a group G.

Lemma II.11. *Let $\chi = \chi_u \Delta_H^{\frac{1}{2}} \in \mathbb{X}(H_1)$ and $w \in \mathbb{C}^n$ with $\chi_u(x) = e^{2\pi i\langle x, w\rangle}$. Further, let $v_\lambda(x) = 2^{\frac{n}{4}} e^{-\pi\|x\|^2}$ be the normalized highest weight vector of $(\mu_1, \mathcal{H}_{\mu_1})$. Then we have for all $a = \operatorname{Exp}(is, iX(t_1, \ldots, t_n)) \in \Gamma_A^c$ and $h \in H_{1,L}$ the formula*

$$\begin{aligned}
&\int_{H_U} \langle \chi_u, \mu_1(ahp).v_\lambda\rangle \chi_u^*(p)\, d\mu_{H_U}(p) \\
&\quad = 2^{\frac{n}{4}} \frac{\Delta_H(h)^{\frac{1}{2}} e^{-2\pi s}}{\det(\sinh T)^{\frac{1}{2}}} \exp\big(-\pi(\langle(\coth T).\overline{w}, w\rangle \\
&\qquad\qquad + \langle e^T(\sinh T)^{-1}.w, \overline{w}\rangle - 2\langle(\sinh T)^{-1}.w, w\rangle)\big).
\end{aligned}$$

Proof. According to Remark II.5 and $a^* = a$, we have

$$\langle \chi_u, \mu_1(ahp).v_\lambda\rangle = \langle \mu_1^{-\omega}(a).\chi_u, \mu_1(hp).v_\lambda\rangle.$$

In view of Lemma II.3(i), we have for all $x \in \mathbb{R}^n$

$$(\mu_1(hp).v_\lambda)(x) = \Delta_H(h)^{\frac{1}{2}} e^{-\pi\langle h^{-1}.x+p, h^{-1}.x+p\rangle}.$$

If w is real, then χ_u is a tempered distribution with $\widehat{\chi_u} = \delta_w$ (the evaluation in $w \in \mathbb{R}^n$) and $\widehat{\delta_w} = \chi_{-u}$, so that Lemma II.3(ii) yields

$$\begin{aligned}
&(\mu_1^{-\omega}(a).\chi_u)(x) \\
&\quad = e^{-2\pi s} \det(\cosh T)^{-\frac{1}{2}} e^{-\pi(\langle\tanh T.x, x\rangle + \langle\tanh T.\overline{w}, w\rangle)} e^{2\pi i\langle(\cosh T)^{-1}.x, w\rangle}.
\end{aligned} \tag{2.6}$$

If w denotes an element of $\mathbb{C}^n$ and not only of $\mathbb{R}^n$, then the right-hand side is a Schwartz function depending antiholomorphically on w. We claim that (2.6) remains true for every $w \in \mathbb{C}^n$ which is not a priori clear since χ_u need not be bounded, i.e., a distribution vector for the representation on $L^2(\mathbb{R}^n)$ (cf. Lemma II.6).

In view of Lebesgue's Dominated Convergence Theorem, the antiholomorphic dependence of the right-hand side of w implies that for each $f \in L^2(\mathbb{R}^n)^\omega$ the mapping $w \mapsto \langle \chi_u, f \rangle$ is antiholomorphic. Let $s \in \operatorname{int}(\Gamma^c)$. Then the mapping

$$\mu_1^{-\omega}(s)\colon L^2(\mathbb{R}^n)^{-\omega} \to L^2(\mathbb{R}^n)^\infty \cong \mathcal{S}(\mathbb{R}^n)$$

is continuous, and therefore the mapping $\mathbb{C}^n \to \mathcal{S}(\mathbb{R}^n), w \mapsto \mu_1^{-\omega}(a).\chi_u$ is weakly antiholomorphic. Since point evaluations are continuous linear functionals on $\mathcal{S}(\mathbb{R}^n)$, it follows that for each $x \in \mathbb{R}^n$ the function

$$\mathbb{C}^n \to \mathbb{R}, \quad w \mapsto \big(\mu_1^{-\omega}(a).\chi_u\big)(x)$$

is antiholomorphic, and therefore that (2.6) also holds for all $w \in \mathbb{C}^n$.

We recall that $H_U \cong \mathbb{R}^n$ and that in this sense μ_{H_U} corresponds to μ_n. Using $h.w = w$ for all $h \in H_{1,L}$ (Lemma II.8), we calculate

$$\begin{aligned}
&\int_{H_U} \langle \chi_u, \mu_1(ahp).v_\lambda \rangle \chi_u^*(p)\; d\mu_{H_U}(p) \\
&\quad = 2^{\frac{n}{4}} e^{-2\pi s} \Delta_H(h)^{\frac{1}{2}} \frac{e^{-\pi\langle \tanh T.\overline{w}, w\rangle}}{\det(\cosh T)^{\frac{1}{2}}} \\
&\quad\quad \cdot \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-\pi\langle h^{-1}.x+p, h^{-1}.x+p\rangle} \\
&\quad\quad e^{-\pi\langle \tanh T.x, x\rangle + 2\pi i\langle (\cosh T)^{-1}.x, w\rangle} \cdot e^{2\pi i\langle p, \overline{w}\rangle}\; d\mu_n(x)\; d\mu_n(p) \\
&\quad = 2^{\frac{n}{4}} e^{-2\pi s} \Delta_H(h)^{\frac{1}{2}} \frac{e^{-\pi\langle \tanh T.\overline{w}, w\rangle}}{\det(\cosh T)^{\frac{1}{2}}} \cdot \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-\pi\langle h^{-1}.x+p, h^{-1}.x+p\rangle} e^{2\pi i\langle p, \overline{w}\rangle} \\
&\quad\quad e^{-\pi\langle \tanh T.x, x\rangle + 2\pi i\langle (\cosh T)^{-1}.x, w\rangle}\; d\mu_n(p)\; d\mu_n(x) \quad \text{(Fubini)} \\
&\quad = 2^{\frac{n}{4}} e^{-2\pi s} \Delta_H(h)^{\frac{1}{2}} \frac{e^{-\pi\langle \tanh T.\overline{w}, w\rangle}}{\det(\cosh T)^{\frac{1}{2}}} \cdot \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-\pi\langle p, p\rangle} e^{2\pi i\langle p - h^{-1}.x, \overline{w}\rangle} \\
&\quad\quad e^{-\pi\langle \tanh T.x, x\rangle + 2\pi i\langle (\cosh T)^{-1}.x, w\rangle}\; d\mu_n(p)\; d\mu_n(x) \quad \text{(translation inv.)} \\
&\quad = 2^{\frac{n}{4}} e^{-2\pi s} \Delta_H(h)^{\frac{1}{2}} \frac{e^{-\pi\langle \tanh T.\overline{w}, w\rangle}}{\det(\cosh T)^{\frac{1}{2}}} \cdot \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} e^{-\pi\langle p, p\rangle} e^{2\pi i\langle p, \overline{w}\rangle}\; d\mu_n(p) \Big) \\
&\quad\quad e^{-\pi\langle \tanh T.x, x\rangle + 2\pi i\langle (\cosh T)^{-1}.x, w\rangle} e^{-2\pi i\langle x, \overline{w}\rangle}\; d\mu_n(x) \quad (H_{1,L}\text{-inv. of w}) \\
&\quad = 2^{\frac{n}{4}} e^{-2\pi s} \Delta_H(h)^{\frac{1}{2}} e^{-\pi\langle w, \overline{w}\rangle} \frac{e^{-\pi\langle \tanh T.\overline{w}, w\rangle}}{\det(\cosh T)^{\frac{1}{2}}} \\
&\quad\quad \cdot \int_{\mathbb{R}^n} e^{-\pi\langle \tanh T.x, x\rangle + 2\pi i\langle (\cosh T)^{-1}.x, w\rangle} e^{-2\pi i\langle x, \overline{w}\rangle}\; d\mu_n(x)
\end{aligned}$$

$$
\begin{aligned}
&= 2^{\frac{n}{4}} e^{-2\pi s}\Delta_H(h)^{\frac{1}{2}} e^{-\pi\langle w,\overline{w}\rangle} \frac{e^{-\pi\langle \tanh T.\overline{w},w\rangle}}{\det(\sinh T)^{\frac{1}{2}}} \\
&\quad \cdot \int_{\mathbb{R}^n} e^{2\pi i\langle(\sinh T\cosh T)^{-\frac{1}{2}}.x,w\rangle} e^{-2\pi i\langle(\coth T)^{\frac{1}{2}}.x,\overline{w}\rangle} e^{-\pi\|x\|^2}\ d\mu_n(x) \\
&= 2^{\frac{n}{4}} e^{-2\pi s}\Delta_H(h)^{\frac{1}{2}} e^{-\pi\langle w,\overline{w}\rangle} \frac{e^{-\pi\langle \tanh T.\overline{w},w\rangle}}{\det(\sinh T)^{\frac{1}{2}}} \\
&\quad e^{-\pi\langle(\sinh T\cosh T)^{-\frac{1}{2}}.\overline{w}-(\coth T)^{\frac{1}{2}}.w,(\sinh T\cosh T)^{-\frac{1}{2}}.w-(\coth T)^{\frac{1}{2}}.\overline{w}\rangle} \\
&= 2^{\frac{n}{4}} \frac{e^{-2\pi s}\Delta_H(h)^{\frac{1}{2}}}{\det(\sinh T)^{\frac{1}{2}}} e^{-\pi\langle(\tanh T+(\sinh T\cosh T)^{-1}).\overline{w},w\rangle} \\
&\quad e^{-\pi\langle(\mathbf{1}+\coth T).w,\overline{w}\rangle} e^{2\pi\langle(\sinh T)^{-1}.w,w\rangle} \\
&= 2^{\frac{n}{4}} \frac{e^{-2\pi s}\Delta_H(h)^{\frac{1}{2}}}{\det(\sinh T)^{\frac{1}{2}}} e^{-\pi\langle(\coth T).\overline{w},w\rangle} e^{-\pi\langle e^T(\sinh T)^{-1}.w,\overline{w}\rangle} e^{2\pi\langle(\sinh T)^{-1}.w,w\rangle}.
\end{aligned}
$$

□

Embedding into the symmetric Jacobi algebra. In this subsection we construct a homomorphism of an effective admissible symmetric Lie algebra into the symmetric Jacobi algebra. With the aid of this homomorphism we can also use the explicit computations of the preceding subsection in an appropriately general framework.

Let $(\mathfrak{g},\tau)$ be an effective admissible symmetric Lie algebra and $\mathfrak{g}'' := \mathfrak{z}_{\mathfrak{l}}(\mathfrak{u})$ be the largest ideal of $\mathfrak{g}$ contained in $\mathfrak{l}$. We write $\mathfrak{l} = \mathfrak{l}' \oplus \mathfrak{g}''$, where $\mathfrak{l}'$ is an ideal of $\mathfrak{l}$ complementing $\mathfrak{g}''$. Then $\mathfrak{g} \cong \mathfrak{g}' \oplus \mathfrak{g}''$ with $\mathfrak{g}' = \mathfrak{u} \rtimes \mathfrak{l}'$. Since all subspaces in this decomposition are τ-invariant, we obtain the decomposition

$$(\mathfrak{g},\tau) = (\mathfrak{g}',\tau') \oplus (\mathfrak{g}'',\tau'').$$

By our assumptions on $(\mathfrak{g},\tau)$, the canonical extension $(\mathfrak{g}_{\mathbb{C}},\widehat{\tau})$ has strong cone potential, i.e., for all $\omega \in -\operatorname{int} C^{\star}_{\min,r}$ the hermitian form

$$\langle\cdot,\cdot\rangle_\omega \colon \mathfrak{p}_r^+ \times \mathfrak{p}_r^+ \to \mathbb{C}, \quad (X,Y) \mapsto -\omega([X,\widehat{\tau}(Y)])$$

on $\mathfrak{p}_r^+$ is positive definite (cf. [KrNe96, Th. VIII.1(vii)]). Note that for all $\alpha \neq \beta \in \Delta_r^+$ the corresponding root spaces are orthogonal with respect to $\langle\cdot,\cdot\rangle_\omega$.

Since the form $\Omega(v,w) := -i\omega([v,w])$ is a non-degenerate skew-symmetric bilinear form on $V_{\mathbb{C}} := \mathfrak{p}_r^+ \oplus \mathfrak{p}_r^-$, we obtain for $n := \dim_{\mathbb{C}} \mathfrak{p}_r^+$ a homomorphism

$$\varphi_\omega := \operatorname{ad}_{V_{\mathbb{C}}} \colon \mathfrak{l} \to \mathfrak{sp}(V_{\mathbb{C}},\Omega) \cong \mathfrak{sp}(n,\mathbb{C})$$

with $\ker\varphi_\omega = \mathfrak{g}''_{\mathbb{C}}$. According to the vector space decomposition $\mathfrak{g}_{\mathbb{C}} = V_{\mathbb{C}} \oplus \mathfrak{z}(\mathfrak{g})_{\mathbb{C}} \oplus \mathfrak{l}_{\mathbb{C}}$, we write the elements of $\mathfrak{g}_{\mathbb{C}}$ as triples (v,z,X). Our considerations lead to the following result.

Proposition II.12. *Let $(\mathfrak{g},\tau)$ be an effective admissible symmetric Lie algebra and $n := \dim_{\mathbb{C}} \mathfrak{p}_r^+$. Then for each $\omega \in -\operatorname{int} C^{\star}_{\min,r}$ the mapping*

$$j_\omega \colon (\mathfrak{g}_{\mathbb{C}},\widehat{\tau}) \to (\mathfrak{hsp}(n,\mathbb{C}),\tau), \quad (v,z,X) \mapsto \big(v,-i\omega(z),\varphi_\omega(X)\big)$$

is a morphism of symmetric Lie algebras with $j_\omega(\mathfrak{g}^c) \subseteq \mathfrak{hsp}(n,\mathbb{R})$ and $\ker j_\omega \subseteq \mathfrak{z}(\mathfrak{g})_{\mathbb{C}} \oplus \mathfrak{g}''_{\mathbb{C}}$. Moreover, it maps the sum $\mathfrak{n}_n^+$ of the positive non-compact root spaces into the corresponding subspace of $\mathfrak{hsp}(n,\mathbb{R})$. □

Definition II.13 (cf. [Ne96]). Let G^c be a simply connected Lie group with Lie algebra $\mathfrak{g}^c$, $\omega \in -\operatorname{int} C^\star_{\min,r}$ and $j_\omega \colon G^c \to \mathrm{HSp}(n,\mathbb{R})$ the homomorphism induced from $\mathfrak{g}^c \to \mathfrak{hsp}(n,\mathbb{R})$. If $(\mu, \mathcal{H}_\mu)$ is the extended metaplectic representation of $\mathrm{HSp}(n,\mathbb{R})$, then the prescription $g \mapsto \mu(j_\omega(g))$ gives rise to a unitary representation of G^c which we call the *extended metaplectic representation with parameter* $\lambda = 2\pi\omega - \rho_r$ and which we denote by $(\mu_\lambda, \mathcal{H}_\lambda)$. We note that $(\mu_\lambda, \mathcal{H}_\lambda)$ is a unitary highest weight representation of G^c w.r.t. $\widehat{\Delta}^+$ and highest weight λ. □

Now we can adapt Theorem II.9 and Lemma II.11 to the more general setting. Let H denote the τ-fixed point group of G^c. Then $H = H_U \rtimes H_L$, where $H_U \cong \mathbb{R}^n$ is a vector group and it is H_L reductive. In this sense we may identify $L^2(\mathbb{R}^n)$ with $L^2(H_U)$.

Theorem II.14. *Let $\omega \in -\operatorname{int} C^\star_{\min,r}$ and $\lambda = 2\pi\omega - \rho_r$.*

(i) *The extended metaplectic representation $\big(\mu_\lambda, L^2(\mathbb{R}^n)\big)$ is (H,χ)-spherical if and only if $\chi_l = \Delta_H^{\frac{1}{2}}$ and in this case we have*
$$(L^2(\mathbb{R}^n)^{-\omega})^{(H,\chi)} = \mathbb{C}\chi_u.$$

(ii) *We have*
$$(L^2(\mathbb{R}^n)^{-\omega})^{(H,\chi)} = \big(L^2(\mathbb{R}^n)^{-\infty}\big)^{(H,\chi)}$$
if and only if χ_u is unitary, and $\big(L^2(\mathbb{R}^n)^{-\infty}\big)^{(H,\chi)} = \{0\}$ otherwise.

Proof. These are immediate consequences of Theorem II.9 and Proposition II.12. □

Pick $\omega_0 \in -\operatorname{int} C^\star_{\min,r}$ and consider the corresponding positive definite real symmetric form $\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_{\omega_0}$ on $\mathfrak{n}_r^+ = \mathfrak{p}_r^+ \cap \mathfrak{g}$. Since for each $\omega \in -\operatorname{int} C^\star_{\min,r}$ the form $\langle\cdot,\cdot\rangle_\omega$ is positive definite on $\mathfrak{p}_r^+$, there exists a unique operator $A_\omega \in \operatorname{End}(\mathfrak{n}_r^+) \subseteq \operatorname{End}(\mathfrak{p}_r^+)$, positive definite w.r.t. $\langle\cdot,\cdot\rangle$, such that
$$\langle v, w\rangle_\omega = \langle A_\omega.v, A_\omega.w\rangle \tag{2.7}$$
holds for all $v, w \in \mathfrak{p}_r^+$.

We note that the mapping
$$\mathfrak{h}_\mathfrak{u} \to \mathfrak{n}_r^+, \quad v \mapsto v_+$$
which is uniquely determined by $v = v_+ + \tau.v_+$ is an isomorphism of vector spaces and extends to a complex linear isomorphism
$$(\mathfrak{h}_\mathfrak{u})_\mathbb{C} \to \mathfrak{p}_r^+, \quad v \mapsto v_+.$$
Therefore we obtain a hermitian scalar product on $(\mathfrak{h}_\mathfrak{u})_\mathbb{C}$ by $\langle v, w\rangle_\omega := \langle v_+, w_+\rangle_\omega$ for $v, w \in (\mathfrak{h}_\mathfrak{u})_\mathbb{C}$. We normalize the left Haar measure μ_{H_U} on $H_U \cong \mathfrak{h}_\mathfrak{u}$ by the requirement that
$$\int_{H_U} \varphi(h)\, d\mu_{H_U}(h) = \int_{\mathfrak{h}_\mathfrak{u}} \varphi(\exp X)\, d\mu^{\omega_0}_{\mathfrak{h}_\mathfrak{u}}(X),$$
where $\mu^{\omega_0}_{\mathfrak{h}_\mathfrak{u}}$ is the Lebesgue measure corresponding to the scalar product $\langle\cdot,\cdot\rangle_\omega$. Likewise, we normalize a Haar measure $\mu_{\mathfrak{n}_r^+}$ on $\mathfrak{n}_r^+$.

Lemma II.15. *For each $\omega \in -\operatorname{int} C^\star_{\min,r}$ we have $\mu^{\omega_0}_{\mathfrak{h}_\mathfrak{u}} = (\det A_\omega)^{-1}\mu^{\omega}_{\mathfrak{h}_\mathfrak{u}}$.*

Proof. Since the mapping $\mathfrak{h}_{\mathfrak{u}} \to \mathfrak{n}_r^+, v \mapsto v_+$ is an isometry, it suffices to prove the corresponding assertion for the measures $\mu^{\omega}_{\mathfrak{n}_r^+}$.

In view of (2.7), A_ω maps an orthonormal basis with respect to $\langle \cdot, \cdot \rangle_\omega$ into an orthonormal basis with respect to $\langle \cdot, \cdot \rangle_{\omega_0}$. Comparing the volume of the corresponding cube W, we see that

$$1 = \mu^{\omega_0}_{\mathfrak{n}_r^+}(A_\omega.W) \quad \text{and} \quad \mu^{\omega}_{\mathfrak{n}_r^+}(A_\omega.W) = (\det A_\omega)\mu^{\omega}_{\mathfrak{n}_r^+}(W) = \det A_\omega.$$

$\square$

Lemma II.16. *For $w \in (\mathfrak{h}_{\mathfrak{u}})_{\mathbb{C}}$ we define $\chi_u \colon \mathbb{R}^n \to \mathbb{C}^\times$ by $\chi_u(x) = e^{2\pi i \langle x, w\rangle}$. Let $\chi = \chi_u \Delta_H^{\frac{1}{2}} \in \mathbb{X}(H)$, $\omega \in -\operatorname{int} C^\star_{\min,r}$ and $\lambda = 2\pi\omega - \rho_r$. Further, let $v_\chi := \chi_u \in (\mathcal{H}_\lambda^{-\omega})^{(H,\chi)}$.*

(i) *For $a \in \Gamma_A^c$ and $h \in H_L$ we have*

$$\int_{H_U} \langle v_\chi, \mu_\lambda(ahp).v_\lambda\rangle \chi_u^*(p)\ d\mu_{H_U}(p) = \frac{\Delta(h)^{\frac{1}{2}} a^{2\pi\omega}}{\det A_\omega}$$
$$\cdot \frac{\exp\Big(-2\pi\big(\langle \frac{a^2+\mathbf{1}}{2(a^2-\mathbf{1})}.\overline{w_+}, w_+\rangle_\omega + \langle \frac{a^2}{a^2-\mathbf{1}}.w_+, \overline{w_+}\rangle_\omega - \langle \frac{a}{a^2-\mathbf{1}}.w_+, w_+\rangle_\omega\big)\Big)}{\big(\prod_{\alpha\in\Delta_r^+} \sinh\alpha(\log(a))^{m_\alpha}\big)^{\frac{1}{2}}}.$$

(ii) *For all $s \in \Gamma$ we have*

$$\langle v_\chi, \mu_\lambda(s).v_\lambda\rangle = 2^{\frac{n}{4}} \chi\big(h_H(s)\big)^{-1} a_H(s)^\lambda e^{-\pi\langle \overline{w}, w\rangle_\omega}.$$

Proof. (i) To make the formula from Lemma II.11 available, we have to write for an element $iX \in \mathfrak{a}$ the operator $\operatorname{ad}_{V_{\mathbb{C}}}(X)$ in the appropriate block form. We have $\operatorname{ad} X(\mathfrak{h}_{\mathfrak{u}}) \subseteq i\mathfrak{q}_{\mathfrak{u}}$ and $\operatorname{ad} X(i\mathfrak{q}_{\mathfrak{u}}) \subseteq \mathfrak{h}_{\mathfrak{u}}$.

Let $v_1, \ldots, v_n$ be a basis of root vectors in $\mathfrak{n}_r^+$ and $\alpha_1, \ldots, \alpha_n$ the corresponding roots. Then $(v_j + \tau.v_j)_{j=1,\ldots,n}$ is a basis of $\mathfrak{h}_{\mathfrak{u}}$ and $(iv_j - i\tau.v_j)_{j=1,\ldots,n}$ is a basis of $[\mathfrak{a}, i\mathfrak{q}_{\mathfrak{u}}] = i\mathfrak{h}_{\mathfrak{u}}$. Moreover,

$$[X, v_j + \tau.v_j] = -\alpha_j(iX)i(v_j - \tau.v_j) \quad \text{and} \quad [X, i(v_j - \tau.v_j)] = \alpha_j(iX)(v_j + \tau.v_j).$$

Therefore $\operatorname{ad}_{V_{\mathbb{C}}} X$ is represented by a block matrix $\begin{pmatrix} 0 & T \\ -T & 0 \end{pmatrix}$ with

$$T = \operatorname{diag}\big(\alpha_1(iX), \ldots, \alpha_n(iX)\big),$$

considered as an operator on $(\mathfrak{h}_{\mathfrak{u}})_{\mathbb{C}}$. The calculation above shows that

$$(T.v)_+ = \operatorname{ad}(iX).v_+$$

because

$$\operatorname{ad}(iX).v_j = \alpha_j(iX).v_j = \alpha_j(iX).(v_j + \tau.v_j)_+ = \big(T.(v_j + \tau.v_j)\big)_+$$

for all j. Hence we have in the terminology of Lemma II.11 for $a = \operatorname{Exp}(iX)$:

$$\Big((\coth T).w\Big)_+ = \frac{a + a^{-1}}{a - a^{-1}}.w_+ = \frac{a^2 + \mathbf{1}}{a^2 - \mathbf{1}}.w_+,$$

$$\Big(e^T(\sinh T)^{-1}.w\Big)_+ = \frac{2a}{a - a^{-1}}.w_+ = \frac{2a^2}{a^2 - \mathbf{1}}.w_+,$$

and therefore

$$\langle(\coth(\operatorname{ad} iX)).\overline{w}, w\rangle_\omega + \langle e^{\operatorname{ad} iX}(\sinh \operatorname{ad} iX)^{-1}.w, \overline{w}\rangle_\omega - 2\langle(\sinh \operatorname{ad} iX)^{-1}.w, w\rangle_\omega$$
$$= \langle\frac{a^2+\mathbf{1}}{a^2-\mathbf{1}}.\overline{w_+}, w_+\rangle_\omega + \langle\frac{2a^2}{a^2-\mathbf{1}}.w_+, \overline{w_+}\rangle_\omega - \langle\frac{2a}{a^2-\mathbf{1}}.w_+, w_+\rangle_\omega.$$

In view of Lemma II.11, Proposition II.12, and Lemma II.15, this proves (i).

(ii) According to the HAN-decomposition, we have $s = h_H(s)a_H(s)n_H(s)$. Then the same calculation as in the proof of [Ne94, Prop. III.9] yields

$$\langle v_\chi, \mu_\lambda(s).v_\lambda\rangle = \chi\big(h_H(s)\big)^{-1} a_H(s)^\lambda \langle v_\chi, v_\lambda\rangle$$

and formula (2.2) in Remark II.7 yields

$$\langle v_\chi, v_\lambda\rangle = 2^{\frac{n}{4}} \int_{\mathfrak{h}_\mathfrak{u}} e^{2\pi i\langle x,w\rangle_\omega} e^{-\pi\|x\|^2_\omega}\, d\mu^\omega_{\mathfrak{h}_\mathfrak{u}}(x) = 2^{\frac{n}{4}} e^{-\pi\langle\overline{w},w\rangle_\omega},$$

proving (ii). □

III. Spherical functions

In this section we define the (H,χ)-spherical functions φ^χ_λ with parameter λ and study their domain of convergence $\mathcal{E}^\chi$ and analytic dependence on λ. Using the results developed in Section II, we prove the Factorization Theorem for φ^χ_λ into a unipotent and a reductive part (cf. Theorem III.10). We conclude this section with a discussion of the structure of the algebra $\mathbb{D}(\chi)$ of G-invariant differential operators on the line bundle $\mathbb{C}_\chi \times_H G \to H\backslash G$.

Definitions and basic properties. For each $\chi \in \mathbb{X}(H)$ we write $(\chi, \mathbb{C}_\chi)$ for the corresponding one-dimensional representation of H. We let H act on $\mathbb{C}_\chi \times G$ by $h.(z,g) = (\chi(h).z, hg)$ and write $\mathbb{C}_\chi \times_H G$ for the corresponding quotient space and $\mathbb{C}_\chi \times_H E$ for a subset $E \subseteq G$ with $HE = E$. Then $\mathbb{C}_\chi \times_H \Gamma$ is a smooth vector bundle over $\mathcal{M}^+ := H\backslash\Gamma$. We identify the smooth sections of this bundle with the space

$$C^\infty(H\backslash\Gamma, \chi) := \{f \in C^\infty(\Gamma) : (\forall s \in \Gamma)(\forall h \in H)\ f(hs) = \chi(h)f(s)\}.$$

We collect some facts from [Hel78, Ch. II] and [Sh90] on invariant differential operators. We write $\mathbb{D}(\chi)$ for the algebra of all G-right invariant differential operators on $C^\infty(H\backslash G, \chi)$ and $\mathbb{D}(G)$ for the algebra of G-right invariant differential operators on G. The derivation of the left regular representation of G on $C^\infty(G)$ yields a homomorphism

$$\mathfrak{g} \to \mathbb{D}(G), \quad X \mapsto L_X;\ (L_X.f)(g) = \frac{d}{dt}\Big|_{t=0} f(\exp(-tX).g)$$

extending to an isomorphism $\mathcal{U}(\mathfrak{g}_\mathbb{C}) \to \mathbb{D}(G)$. The operator L_U, $u \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$, defines an operator in $\mathbb{D}(\chi)$ if and only if $u \in \mathcal{U}(\mathfrak{g}_\mathbb{C})^H := \{v \in \mathcal{U}(\mathfrak{g}_\mathbb{C}) : (\forall h \in H)\operatorname{Ad}(h).v = v\}$. Let I_χ be the left ideal in $\mathcal{U}(\mathfrak{g}_\mathbb{C})$ generated by $X + d\chi(X).\mathbf{1}$, $X \in \mathfrak{h}$. Then the isomorphism from above induces an isomorphism

$$\mathbb{D}(\chi) \cong \mathcal{U}(\mathfrak{g}_\mathbb{C})^H / \big(\mathcal{U}(\mathfrak{g}_\mathbb{C})^H \cap I_\chi\big). \tag{3.1}$$

Let $\langle\cdot,\cdot\rangle$ denote an inner product of $L^2(G)$ with respect to a left Haar measure on G. For each $D \in \mathbb{D}(G)$ we define its transpose $D^\top$ by $\langle D^\top.f, \overline{g}\rangle : = \langle f, \overline{D.g}\rangle$ for all $f \in C^\infty(G)$ and $g \in C^\infty_c(G)$. In the sequel we consider elements $\varphi \in L^\infty(G)$ also as distributions via $C^\infty_c(G) \ni f \mapsto \langle f, \overline{\varphi}\rangle$.

Definition III.1. A continuous section φ of the bundle $\mathbb{C}_\chi \times_H \Gamma \to \mathcal{M}^+$ is called (H, χ)-spherical if the following conditions are satisfied:

(S1) The section φ is right χ^*-semi-invariant, i.e., $\varphi(sh) = \chi^*(h)\varphi(s)$ holds for all $s \in \Gamma$ and $h \in H$.
(S2) For every $D \in \mathbb{D}(\chi)$ there exists a complex number λ_D such that $D.\varphi = \lambda_D \varphi$ holds in the sense of distributions on Γ. □

Remark III.2. (a) If χ is trivial and G is semisimple, then Definition II.1 coincides with the definition of spherical functions on a non-compactly causal symmetric space (cf. [Ól97]).

(b) In the case, where $(\mathfrak{g}, \tau)$ is a non-compactly Riemannian symmetric Lie algebra of hermitian type, the (H, χ)-spherical functions have been studied by Heckman and Opdam (cf. [HS94, Part I]). □

We write $\mathbb{X}(H/H^0)$ for all those elements of $\mathbb{X}(H)$ which are trivial on H^0. For a fixed $\chi \in \mathbb{X}(H/H^0)$ there exists an interesting family of spherical functions $(\varphi_\lambda^\chi)_\lambda$, where λ runs over a certain subset of $\mathfrak{a}_\mathbb{C}^*$, whose construction we describe below. The construction is motivated by the special cases considered in [FHÓ94], [HiNe96] and [Ól97].

Using the HAN-decomposition, we extend each $\chi \in \mathbb{X}(H/H^0)$ to an analytic function on HAN by $\chi(s) := \chi(h_H(s))$ for all $s \in HAN$.

Definition III.3. For $\lambda \in \mathfrak{a}_\mathbb{C}^*$ and $\chi \in \mathbb{X}(H/H^0)$ define

$$\varphi_\lambda^\chi(s) = \int_{H/H^0} a_H(sh)^{\lambda-\rho} \chi(sh)\overline{\chi(h)}\; d\mu_{H/H^0}(hH^0)$$

for $s \in \Gamma$ provided the integral exists. If $\chi = \mathbf{1}$ is trivial, then we write φ_λ instead of φ_λ^χ. We write $\mathcal{E}^\chi$ for the set of all $\lambda \in \mathfrak{a}_\mathbb{C}^*$ for which all integrals $\varphi_\lambda^\chi(s)$, $s \in \Gamma$, converge. We will see below (Proposition III.17) that φ_λ^χ is spherical. We call φ_λ^χ the (H, χ)-*spherical function with parameter* λ. □

Remark III.4. If the group H is unimodular, $Z(H)$ is compact, and χ is trivial, then there exists another definition of spherical functions in the literature, namely by integral equations: an H-bi-invariant continuous function on Γ is called spherical if

$$\varphi(x)\varphi(y) = \int_H \varphi(xhy)\; d\mu_H(h)$$

holds for all $x, y \in \Gamma$, cf. [FHÓ94]. It is not clear which of the functions φ_λ obtained by analytic continuation in the parameter λ satisfy this integral equation because the integral does not have to exist in general. This motivates the definition of spherical functions as certain eigenfunctions of invariant differential operators, which in some respects is easier to deal with. □

Lemma III.5. *For $\chi \in \mathbb{X}(H/H^0)$ and $\lambda \in \mathcal{E}^\chi$ the function φ_λ^χ is left χ- and right χ^*-semi-invariant.*

Proof. For $h_1, h_2 \in H$ and $s \in \Gamma$ we have

$$\begin{aligned}
\varphi_\lambda^\chi(h_1 s h_2) &= \int_{H/H^0} a_H(h_1 s h_2 h)^{\lambda-\rho} \chi(h_1 s h_2 h) \overline{\chi(h)}\, d\mu_{H/H^0}(hH^0) \\
&= \chi(h_1) \int_{H/H^0} a_H(s h_2 h)^{\lambda-\rho} \chi(s h_2 h) \overline{\chi(h)}\, d\mu_{H/H^0}(hH^0) \\
&= \chi(h_1) \int_{H/H^0} a_H(sh)^{\lambda-\rho} \chi(sh) \overline{\chi(h_2^{-1} h)}\, d\mu_{H/H^0}(hH^0) \\
&= \chi(h_1)\chi^*(h_2)\varphi_\lambda^\chi(s),
\end{aligned}$$

proving the lemma. □

Definition III.6. Let μ be a positive measure on the σ-algebra of all Borel subsets of the finite dimensional real vector space V. The *Laplace transform* of μ is defined by

$$\mathcal{L}(\mu)\colon D_\mu \to \mathbb{R}, \quad \alpha \mapsto \int_V e^{-\alpha(x)}\, d\mu(x),$$

where $D_\mu = \{\alpha \in V^* : \mathcal{L}(\mu)(\alpha) < \infty\}$. □

We recall from Definition I.5 the notation used in the following lemma.

Lemma III.7. *For a positive measure μ on the finite dimensional vector space V the following assertions hold:*

(i) *The domain D_μ is convex and $\mathcal{L}(\mu)$ is a convex function on D_μ.*
(ii) *If $C_\mu = \overline{\operatorname{conv}(\operatorname{supp}\mu)}$, then $D_\mu + B(C_\mu) \subseteq D_\mu$, i.e., $B(C_\mu) \subseteq \lim D_\mu$.*
(iii) *The Laplace transform of μ extends uniquely to a holomorphic function, also denoted $\mathcal{L}(\mu)$, on $\operatorname{int} D_\mu + iV^* \subseteq V_{\mathbb{C}}^*$ which is given by the integral formula*

$$\mathcal{L}(\mu)\colon \operatorname{int} D_\mu + iV^* \to \mathbb{C}, \quad \alpha \mapsto \int_V e^{-\alpha(x)}\, d\mu(x).$$

(iv) *The function $\mathcal{L}(\mu)$ admits no analytic continuation to an open domain strictly larger than $\operatorname{int} D_\mu + iV^*$.*

Proof. (i),(ii) [Ne99, Prop. V.4.3].
(iii) [Ne99, Prop. V.4.6].
(iv) This is a special case of [Ne98b, Th. III.4]. □

Lemma III.8. *Let $\chi \in \mathbb{X}(H/H^0)$.*

(i) *The domain of definition $\mathcal{E}^\chi$ for the (H,χ)-spherical function φ_λ^χ is of the shape*

$$\mathcal{E}^\chi = i\mathfrak{a}^* + \mathcal{E}_{\mathbb{R}}^\chi,$$

where $\mathcal{E}_{\mathbb{R}}^\chi$ is a convex set with $-C_{\min}^\star \subseteq \lim \mathcal{E}_{\mathbb{R}}^\chi$.
(ii) *For each fixed $s \in \Gamma$, the function*

$$\operatorname{int} \mathcal{E}^\chi \to \mathbb{C}, \quad \lambda \to \varphi_\lambda^\chi(s)$$

is holomorphic. If, in addition, χ is positive, then it is a convex function on the domain where it is defined.

Proof. We may w.l.o.g. assume that χ is positive. For every $a \in \Gamma_A = \exp(\operatorname{int} C_{\max})$ (cf. Lemma I.20) we consider the mapping

$$\psi_a\colon H/H^0 \to \mathfrak{a}, \quad hH^0 \mapsto \log a_H(ah).$$

Let ν_a be the measure on $\mathfrak{a}$ defined by the push forward of $\chi(ah)\chi(h)d\mu_{H/H^0}(hH^0)$ under the map ψ_a. Then we have

$$\mathcal{L}(\nu_a)(\lambda) = \int_{\mathfrak{a}} e^{-\lambda(x)}\ d\nu_a(x) = \int_{H/H^0} a_H(ah)^{-\lambda}\chi(ah)\chi(h)\ d\mu_{H/H^0}(hH^0),$$

and so

$$\mathcal{L}(\nu_a)(\rho - \lambda) = \varphi_\lambda^\chi(a). \tag{3.2}$$

As φ_λ^χ is left χ- and right χ^*-semi-invariant (cf. Lemma III.5), the function φ_λ^χ is uniquely determined by its restriction to Γ_A. Thus (3.2) implies that

$$\mathcal{E}_{\mathbb{R}}^\chi = \bigcap_{a\in\Gamma_A} (\rho - D_{\nu_a}), \tag{3.3}$$

and so it follows from Lemma III.7(i), (iii) that $\mathcal{E}^\chi$ is convex and $i\mathfrak{a}^* \subseteq \lim \mathcal{E}^\chi$. Further,

$$\operatorname{im} \psi_a \subseteq \operatorname{conv}(\mathcal{W}.\log a) + C_{\min}$$

by [Ne97a, Th. II.8], so that $C_{\nu_a} \subseteq \operatorname{conv}(\mathcal{W}.\log a) + C_{\min}$. We conclude that

$$B(\rho - C_{\nu_a}) = B(-C_{\nu_a}) \supseteq B(-\operatorname{conv}(\mathcal{W}.\log a) - C_{\min}) = -C_{\min}^\star.$$

In view of (3.2) and (3.3), this proves (i). Finally, (3.2) and Lemma III.7(i),(iii) imply (ii). $\square$

Factorization of spherical functions. In this subsection we take a closer look at the (H,χ)-spherical functions φ_λ^χ with parameter λ. Using the results of Section II, we prove the Factorization Theorem for φ_λ^χ in unipotent and reductive part (cf. Theorem III.10).

Lemma III.9. *The following assertions hold:*

(i) *The multiplication mapping*

$$m\colon H_U \times (H_L/H^0) \to H/H^0, \quad (u,l) \mapsto ulH^0$$

is a diffeomorphism of homogeneous spaces.

(ii) *Denote by Δ_H the modular function of H. Then $H_U, H^0 \subseteq \ker \Delta_H$.*

(iii) *Consider Δ_H as a function on H_L/H^0. Then the pull back of μ_{H/H^0} under m is given by $\mu_{H_U} \otimes \Delta_H \mu_{H_L/H^0}$, i.e., for $f \in L^1(H/H^0)$ we have*

$$\int_{H/H^0} f(hH^0)\ d\mu_{H/H^0}(hH^0) = \int_{H_U}\int_{H_L/H^0} f(h_u h_l H^0)\Delta_H(h_l)\ d\mu_{H_L/H^0}(h_l H^0)\ d\mu_{H_U}(h_u).$$

Moreover, we have

$$\int_{H/H^0} f(hH^0)\ d\mu_{H/H^0}(hH^0) = \int_{H_L/H^0}\int_{H_U} f(h_l h_u H^0)\ \ d\mu_{H_L/H^0}(h_l H^0)\ d\mu_{H_U}(h_u).$$

Proof. (i) This follows from $H = H_U \rtimes H_L$ and $H^0 \subseteq H_L$ (cf. Lemma I.14(ii)).

(ii) Note that $\Delta_H(h) = |\det \mathrm{Ad}(h)^{-1}|$. Since $\mathfrak{h}$ is a nilpotent $\mathfrak{h}_\mathfrak{u}$-module, the subgroup H_U is contained in the kernel of the modular function Δ_H. As H^0 is compactly embedded (cf. Lemma I.14(i)), we also have $H^0 \subseteq \ker \Delta_H$.

(iii) First we note that the measure $\nu := \mu_{H_U} \otimes (\Delta_H \mu_{H_L})$ is an H_U-left invariant measure on $H = H_U \rtimes H_L$ which we consider as the product of the two locally compact spaces H_U and H_L. For $g \in G$ we define $I_g \colon G \to G$ by $I_g(x) = gxg^{-1}$. For $h \in H_L$ we then have

$$\begin{aligned}(\lambda_h)^*\big(\mu_{H_U} \otimes (\Delta_H \mu_{H_L})\big) &= \big(I_h^* \mu_{H_U} \otimes \lambda_h^*(\Delta_H \mu_{H_L})\big) \\ &= \big(\Delta_H(h)\mu_{H_U} \otimes (\Delta_H \circ \lambda_h^{-1})\mu_{H_L}\big) \\ &= \mu_{H_U} \otimes \Delta_H \mu_{H_L}.\end{aligned}$$

Hence ν is a left Haar measure on H. From this, it directly follows that ν induces on the quotient space H/H^0 the left invariant measure

$$\mu_{H_U} \otimes \Delta_H \mu_{H_L/H^0}.$$

Here we use (ii) to see that Δ_H factors to a function on H/H^0.

To obtain the formula for integration in the reversed order, we first observe that

$$\begin{aligned}\int_H f(h)\, d\nu(h) &= \int_{H_U}\int_{H_L} f(h_u h_l)\Delta_H(h_l)\, d\mu_{H_L}(h_l)\, d\mu_{H_U}(h_u) \\ &= \int_{H_L}\int_{H_U} f(h_u h_l)\Delta_H(h_l)\, d\mu_{H_U}(h_u) d\mu_{H_L}(h_l) \\ &= \int_{H_L}\int_{H_U} f(h_l h_l^{-1} h_u h_l)\Delta_H(h_l)\, d\mu_{H_U}(h_u) d\mu_{H_L}(h_l) \\ &= \int_{H_L}\int_{H_U} f(h_l h_u)\, d\mu_{H_U}(h_u) d\mu_{H_L}(h_l).\end{aligned}$$

For $h_0 \in H^0$ we now obtain from $\Delta_H(h_0) = 1$:

$$\begin{aligned}\int_H f(h_u h_l h_0)\, d\nu(h) &= \int_{H_L}\int_{H_U} f(h_l h_u h_0)\, d\mu_{H_U}(h_u) d\mu_{H_L}(h_l) \\ &= \int_{H_L}\int_{H_U} f(h_l h_0 h_0^{-1} h_u h_0)\, d\mu_{H_U}(h_u) d\mu_{H_L}(h_l) \\ &= \int_{H_L} \Delta_H(h_0)^{-1} \int_{H_U} f(h_l h_0 h_u)\, d\mu_{H_U}(h_u) d\mu_{H_L}(h_l) \\ &= \int_{H_L}\int_{H_U} f(h_l h_0 h_u)\, d\mu_{H_U}(h_u) d\mu_{H_L}(h_l).\end{aligned}$$

Thus the invariant measure on H/H^0 can be written as in the second formula above. □

According to the decomposition $\mathfrak{a} = \mathfrak{a}_u \oplus \mathfrak{a}_\mathfrak{l}$, we write each element $\lambda \in \mathfrak{a}_\mathbb{C}^*$ as $\lambda = \lambda_z + \lambda_l$ with $\lambda_z \in (\mathfrak{a}_\mathfrak{u})_\mathbb{C}^*$ and $\lambda_l \in (\mathfrak{a}_\mathfrak{l})_\mathbb{C}^*$.

Theorem III.10 (The Factorization Theorem)**.** *Let* $\chi \in \mathbb{X}(H/H^0)$ *with* $\chi_u(\exp x) = e^{2\pi i\langle x,w\rangle}$ *for* $x \in \mathfrak{h}_\mathfrak{u}$ *and* $n := \dim \mathfrak{n}_r^+$.

(i) *If* $\lambda \in \mathcal{E}^\chi$*, then* $\operatorname{Re}\lambda_z \in -\operatorname{int} C^\star_{\min,r}$ *and we have that for all* $a \in \Gamma_A = \exp(\operatorname{int} C_{\max})$ *the product formula*

$$\varphi^\chi_\lambda(a) = \frac{2^{\frac{n}{2}}\pi^n a^{\lambda_z}}{\det A_{\lambda_z}} \cdot \frac{\exp\big(-\big(\langle \frac{\mathbf{1}}{a^2-\mathbf{1}}.\overline{w_+}, w_+\rangle_{\lambda_z} + \langle \frac{a^2}{a^2-\mathbf{1}}.w_+, \overline{w_+}\rangle_{\lambda_z} - \langle \frac{a}{a^2-\mathbf{1}}.w_+, w_+\rangle_{\lambda_z}\big)\big)}{\big(\prod_{\alpha\in\Delta_r^+}(\sinh\alpha(\log a))^{m_\alpha}\big)^{\frac{1}{2}}} \cdot \varphi^{\chi_l\Delta_H^{\frac{1}{2}}}_{\lambda_l}(a),$$

where

$$\varphi^{\chi_l\Delta_H^{\frac{1}{2}}}_{\lambda_l}(a) = \int_{H_L/H^0} a_{H,l}(ah)^{\lambda_l-\rho_s}(\chi_l\Delta_H^{\frac{1}{2}})(ah)\overline{(\chi_l\Delta_H^{\frac{1}{2}})(h)}\, d\mu_{H_L/H^0}(hH^0).$$

(ii) *If* $\mathcal{E}_L^{\chi_l\Delta_H^{\frac{1}{2}}} \subseteq (\mathfrak{a}_l)^*_{\mathbb{C}} \subseteq \mathfrak{a}^*_{\mathbb{C}}$ *denotes the domain of convergence of* $\varphi^{\chi_l\Delta_H^{\frac{1}{2}}}_{\lambda_l}$*, then*

$$\mathcal{E}^\chi = (i\mathfrak{a}^* - \operatorname{int} C^\star_{\min,r}) \cap \mathcal{E}_L^{\chi_l\Delta_H^{\frac{1}{2}}}.$$

In particular, $\mathcal{E}^\chi$ *is independent of* χ_u.

Proof. (i), (ii) Suppose first that $\operatorname{Re}\lambda_z \in -\operatorname{int} C^\star_{\min,r} \cap \mathcal{E}^\chi$. Since both sides of the formula asserted in (i) depend holomorphically on λ (cf. Lemma III.8(ii)), it suffices to assume that $\lambda_z \in -\operatorname{int} C^\star_{\min,r}$. In view of Lemma III.9, we have

(3.4)

$$\begin{aligned}
\varphi^\chi_\lambda(a) &= \int_{H/H^0} a_H(ah)^{\lambda-\rho}\chi(ah)\overline{\chi(h)}\, d\mu_{H/H^0}(hH^0)\\
&= \int_{H_L/H^0}\int_{H_U} a_H(ah_lh_u)^{\lambda-\rho}\chi(ah_lh_u)\overline{\chi(h_lh_u)}\, d\mu_{H_U}(h_u)\, d\mu_{H_L/H^0}(h_lH^0)\\
&= \int_{H_L/H^0} a_{H,l}(ah_l)^{\lambda_l-\rho}\chi_l(ah_l)\overline{\chi_l(h_l)}\\
&\qquad \cdot\Big(\int_{H_U} a_{H,u}(ah_lh_u)^{\lambda_z}\chi_u(ah_lh_u)\overline{\chi_u(h_u)}\, d\mu_{H_U}(h_u)\Big)\, d\mu_{H_L/H^0}(h_lH^0),
\end{aligned}$$

where we have used Lemma I.8 and its analogue for the H-projection.

To obtain the product formula, we first have to evaluate the integral

$$I(a,h_l) = \int_{H_U} a_{H,u}(ah_lh_u)^{\lambda_z}\chi_u(ah_lh_u)\overline{\chi_u(h_u)}\, d\mu_{H_U}(h_u).$$

Let $\sigma := \lambda_z - \rho_r \in \mathfrak{a}^*$ and $(\mu_\sigma, \mathcal{H}_\sigma)$ denote the extended metaplectic representation of G^c with parameter σ. Put $\psi = \chi_u^{-1}\Delta_H^{\frac{1}{2}}$. In view of Lemma II.16(ii), we have for $h_l \in H_L$, $h_u \in H_U$ and $n = \dim\mathfrak{n}_r^+$:

$$\begin{aligned}
\langle \nu_\psi, \mu_\sigma(ah_lh_u).v_\sigma\rangle &= 2^{\frac{n}{4}}\psi^{-1}(ah_lh_u)a_H(ah_lh_u)^\sigma e^{-\frac{1}{2}\langle\overline{w},w\rangle_{\lambda_z}}\\
&= 2^{\frac{n}{4}}\Delta_H(ah_l)^{-\frac{1}{2}}\chi_u(ah_lh_u)e^{-\frac{1}{2}\langle\overline{w},w\rangle_{\lambda_z}}a_{H,u}(ah_lh_u)^{\lambda_z}a_{H,l}(ah_l)^{-\rho_r},
\end{aligned}$$

so that we obtain

$$\begin{aligned}
&a_{H,u}(ah_lh_u)^{\lambda_z}\chi_u(ah_lh_u)\overline{\chi_u(h_u)}\\
&\quad = 2^{-\frac{n}{4}}\Delta_H(ah_l)^{\frac{1}{2}}e^{\frac{1}{2}\langle\overline{w},w\rangle_{\lambda_z}}a_{H,l}(ah_l)^{\rho_r}\langle \nu_\psi, \mu_\sigma(ah_lh_u).v_\sigma\rangle\overline{\chi_u(h_u)}
\end{aligned}$$

(note the different signs of w in ψ and the character χ used in Lemma II.16). Integrating over H_U and using Lemma II.16(i) now gives

$$I(a,h_l) = a_{H,l}(ah_l)^{\rho_r} 2^{-\frac{n}{2}} \frac{\Delta_H(ah_l)^{\frac{1}{2}}\Delta_H(h_l)^{\frac{1}{2}}}{\det A_{\frac{\lambda_z}{2\pi}}} \cdot \frac{e^{-\left(\langle \frac{1}{a^2-1}.\overline{w_+},w_+\rangle_{\lambda_z} + \langle \frac{a^2}{a^2-1}.w_+,\overline{w_+}\rangle_{\lambda_z} - \langle \frac{a}{a^2-1}.w_+,w_+\rangle_{\lambda_z}\right)}}{\left(\prod_{\alpha\in\Delta_r^+}(\sinh\alpha(\log a))^{m_\alpha}\right)^{\frac{1}{2}}}.$$

Putting this expression in (3.4), we finally obtain

$$\begin{aligned}
\varphi_\lambda^\chi(a) &= \frac{2^{\frac{n}{2}}\pi^n a^{\lambda_z}}{\det A_{\lambda_z}} \cdot \frac{e^{-\left(\langle \frac{1}{a^2-1}.\overline{w_+},w_+\rangle_{\lambda_z} + \langle \frac{a^2}{a^2-1}.w_+,\overline{w_+}\rangle_{\lambda_z} - \langle \frac{a}{a^2-1}.w_+,w_+\rangle_{\lambda_z}\right)}}{\left(\prod_{\alpha\in\Delta_r^+}(\sinh\alpha(\log a))^{m_\alpha}\right)^{\frac{1}{2}}} \\
&\quad \cdot \int_{H_L/H^0} a_{H,l}(ah)^{\lambda_l-\rho_s}\chi_l(ah_l)\Delta_H(ah_l)^{\frac{1}{2}}\overline{\chi_l(h_l)}\Delta_H(h_l)^{\frac{1}{2}}\; d\mu_{H_L/H^0}(h_lH^0) \\
&= \frac{2^{\frac{n}{2}}\pi^n a^{\lambda_z}}{\det A_{\lambda_z}} \cdot \frac{e^{-\left(\langle \frac{1}{a^2-1}.\overline{w_+},w_+\rangle_{\lambda_z} + \langle \frac{a^2}{a^2-1}.w_+,\overline{w_+}\rangle_{\lambda_z} - \langle \frac{a}{a^2-1}.w_+,w_+\rangle_{\lambda_z}\right)}}{\left(\prod_{\alpha\in\Delta_r^+}(\sinh\alpha(\log a))^{m_\alpha}\right)^{\frac{1}{2}}} \cdot \varphi_{\lambda_l}^{\chi_l\Delta_H^{\frac{1}{2}}}(a).
\end{aligned}$$

To complete the proof of the theorem, we have to show that $\operatorname{Re}\lambda_z \in -\operatorname{int} C^\star_{\min,r}$, whenever φ_λ^χ is defined. If not, then the convexity of $\mathcal{E}_\mathbb{R}^\chi$ and $-C^\star_{\min} \subseteq \lim \mathcal{E}_\mathbb{R}^\chi$ (Lemma III.8) implies the existence of $\lambda \in \operatorname{int}\mathcal{E}_\mathbb{R}^\chi$ with $\lambda_z \in -\partial C^\star_{\min}$. Choose a sequence $(\lambda^n)_{n\in\mathbb{N}}$ in $\operatorname{int}\mathcal{E}_\mathbb{R}^\chi$ converging to λ. Then

$$\lim_{n\to\infty} \det A_{\lambda_z^n} = \det A_{\lambda_z} = 0,$$

so that

$$\lim_{n\to\infty} \varphi_{\lambda^n}^{|\chi|}(a) = \infty$$

for all $a \in \Gamma_A$. In view of the convexity of $\lambda \mapsto \varphi_\lambda^{|\chi|}(a)$ (cf. Lemma III.8(ii)), this shows that $\operatorname{Re}\lambda_z \in -\operatorname{int}\ C^\star_{\min,r}$ whenever $\lambda \in \operatorname{int}\mathcal{E}^\chi$. □

Remark III.11. (a) Our product formula generalizes the product formula for spherical functions on Ol'shanskiĭ spaces $G_\mathbb{C}/G_\mathbb{R}$ which has been derived by J. Hilgert and the second author in [HiNe96, Prop. II.5].

(b) There exists a close connection between spherical functions and distribution characters of spherical highest weight representations of the group G^c. These distribution characters occur naturally in the Plancherel formula of G^c-invariant Hilbert spaces of holomorphic functions on certain G^c-invariant complex domains in $G_\mathbb{C}/H_\mathbb{C}$ or coverings thereof. But in this application of spherical functions to representation theory only the character $\chi = \Delta_H^{-\frac{1}{2}}$ occurs. In this case our product formula for $\varphi_\lambda^{\Delta_H^{-\frac{1}{2}}}$ boils down to

$$\varphi_\lambda^{\Delta_H^{-\frac{1}{2}}}(a) = \frac{2^{\frac{n}{2}}\pi^n a^{\lambda_z}}{\det A_{\lambda_z}} \cdot \frac{1}{\left(\prod_{\alpha\in\Delta_r^+}(\sinh\alpha(\log a))^{m_\alpha}\right)^{\frac{1}{2}}}\varphi_{\lambda_l}(a).$$

□

The algebra $\mathbb{D}(\chi)$. We now discuss the algebra $\mathbb{D}(\chi)$ in detail. We prove a version of the Harish-Chandra homomorphism which relates $\mathbb{D}(\chi)$ with the Weyl group invariants in the symmetric algebra $\mathcal{S}(\mathfrak{a}_{\mathbb{C}})$ of $\mathfrak{a}_{\mathbb{C}}$ (cf. Theorem III.13). After that we give the still missing proof of the fact that the functions φ_λ^χ are (H,χ)-spherical.

Define the *big Weyl group of* $\mathfrak{a}$ by $\widetilde{\mathcal{W}} := N_{\operatorname{Inn}\mathfrak{g}}(\mathfrak{a})/Z_{\operatorname{Inn}\mathfrak{g}}(\mathfrak{a})$ and note that $\mathcal{W} \subseteq \widetilde{\mathcal{W}}$.

Lemma III.12. *We have*

$$\widetilde{\mathcal{W}} \cong N_{\operatorname{Inn}_{\mathfrak{g}}(\mathfrak{l})}(\mathfrak{a})/Z_{\operatorname{Inn}_{\mathfrak{g}}(\mathfrak{l})}(\mathfrak{a}) \cong N_{\operatorname{Inn}_{\mathfrak{g}}(\mathfrak{s})}(\mathfrak{a})/Z_{\operatorname{Inn}_{\mathfrak{g}}(\mathfrak{s})}(\mathfrak{a})$$

and $\widetilde{\mathcal{W}}$ is generated by the reflections $s_\alpha\colon \mathfrak{a} \to \mathfrak{a}, X \mapsto X - \alpha(X)\check{\alpha}$ corresponding to the semisimple roots $\alpha \in \Delta_s$.

Proof (cf. [KrNe96, Lemma III.6]). First we show that

$$N_{\operatorname{Inn}\mathfrak{g}}(\mathfrak{a}) = N_{\operatorname{Inn}_{\mathfrak{g}}(\mathfrak{l})}(\mathfrak{a}) Z_{\operatorname{Inn}_{\mathfrak{g}}(\mathfrak{u})}(\mathfrak{a}). \tag{3.5}$$

The inclusion "$\supseteq$" is obvious. For the converse let $g \in N_{\operatorname{Inn}\mathfrak{g}}(\mathfrak{a})$ and write $g = lu$ with $l \in \operatorname{Inn}_{\mathfrak{g}}(\mathfrak{l})$ and $u \in \operatorname{Inn}_{\mathfrak{g}}(\mathfrak{u})$. As $\mathfrak{a} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{a}_{\mathfrak{l}}$ and $\mathfrak{u}$ is an ideal in $\mathfrak{g}$ with $\mathfrak{l} \cong \mathfrak{g}/\mathfrak{u}$, it follows that $l \in N_{\operatorname{Inn}_{\mathfrak{g}}(\mathfrak{l})}(\mathfrak{a})$. Thus $u \in N_{\operatorname{Inn}_{\mathfrak{g}}(\mathfrak{u})}(\mathfrak{a})$ and it remains to show that

$$N_{\operatorname{Inn}_{\mathfrak{g}}(\mathfrak{u})}(\mathfrak{a}) = Z_{\operatorname{Inn}_{\mathfrak{g}}(\mathfrak{u})}(\mathfrak{a}). \tag{3.6}$$

Let $u \in N_{\operatorname{Inn}_{\mathfrak{g}}(\mathfrak{u})}(\mathfrak{a})$. As $\mathfrak{g}$ is a nilpotent $\mathfrak{u}$-module, we find $X \in \mathfrak{u}$ such that $u = e^{\operatorname{ad} X}$. For the same reason we have $\operatorname{ad} X = \log e^{\operatorname{ad} X}$ and so X normalizes $\mathfrak{a}$, i.e., $X \in \mathfrak{n}_{\mathfrak{u}}(\mathfrak{a})$. Now $\mathfrak{a}$ being a maximal hyperbolic subspace, the ideal $\mathfrak{u}$ is a semisimple $\mathfrak{a}$-module. Hence $\mathfrak{u} = \mathfrak{z}_{\mathfrak{u}}(\mathfrak{a}) \oplus [\mathfrak{u},\mathfrak{a}]$ and so $\mathfrak{z}_{\mathfrak{u}}(\mathfrak{a}) = \mathfrak{n}_{\mathfrak{u}}(\mathfrak{a})$. This proves (3.6) and hence (3.5).

It follows from (3.5) that

$$Z_{\operatorname{Inn}\mathfrak{g}}(\mathfrak{a}) = Z_{\operatorname{Inn}_{\mathfrak{g}}(\mathfrak{l})}(\mathfrak{a}) Z_{\operatorname{Inn}_{\mathfrak{g}}(\mathfrak{u})}(\mathfrak{a}).$$

This proves the first assertion. The second assertion is immediate by the standard structure theory of semisimple symmetric Lie algebras (cf. [Kn96, Ch. VII, Lemma 7.22, Prop. 7.32]). □

Theorem III.13. *The following assertions hold:*

(i) *With $I_\chi = \sum_{H\in\mathfrak{h}} \mathcal{U}(\mathfrak{g}_{\mathbb{C}})(H + d\chi(H).\mathbf{1})$ we have*

$$\mathcal{U}(\mathfrak{g}_{\mathbb{C}}) = \mathcal{U}(\mathfrak{a}_{\mathbb{C}}) \oplus (I_\chi + \mathfrak{n}\mathcal{U}(\mathfrak{g}_{\mathbb{C}})).$$

Further, if χ_u is trivial, then we also have

$$\mathcal{U}(\mathfrak{g}_{\mathbb{C}}) = \mathcal{U}(\mathfrak{a}_{\mathbb{C}}) \oplus (I_\chi + (\mathfrak{n} + [\mathfrak{a},\mathfrak{u}])\mathcal{U}(\mathfrak{g}_{\mathbb{C}})).$$

(ii) *For $u \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ let u_0 denote the first component of u in the decomposition of* (i)*. If $\mathfrak{g}$ is solvable or χ_u is trivial, then the mapping*

$$\gamma_\chi\colon \mathcal{U}(\mathfrak{g}_{\mathbb{C}})^H \to \mathcal{S}(\mathfrak{a}_{\mathbb{C}})^{\widetilde{\mathcal{W}}}, \quad u \mapsto u_0 - \rho(u_0)$$

is an algebra homomorphism, inducing a homomorphism $\gamma_\chi\colon \mathbb{D}(\chi) \to \mathcal{S}(\mathfrak{a}_{\mathbb{C}})^{\widetilde{\mathcal{W}}}$.

Proof. (i) (cf. [Sh90, Prop. 2.1].) From $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{a} \oplus \mathfrak{n}$ and the Poincaré-Birkhoff-Witt Theorem we deduce that $\mathcal{U}(\mathfrak{g}_{\mathbb{C}}) = \mathcal{U}(\mathfrak{h}_{\mathbb{C}}) \otimes \mathcal{U}(\mathfrak{a}_{\mathbb{C}}) \otimes \mathcal{U}(\mathfrak{n}_{\mathbb{C}})$. For each $\chi \in \mathbb{X}(H)$, the mapping

$$i_\chi\colon \mathfrak{h}_{\mathbb{C}} \to \mathcal{U}(\mathfrak{h}_{\mathbb{C}}), \quad H \mapsto H - d\chi(H)\mathbf{1}$$

is a homomorphism of Lie algebras, hence extends to an algebra homomorphism $i_\chi\colon \mathcal{U}(\mathfrak{h}_\mathbb{C}) \to \mathcal{U}(\mathfrak{h}_\mathbb{C})$. Since $i_\chi \circ i_{\chi^{-1}} = i_{\chi^{-1}} \circ i_\chi = \mathrm{id}$, the map i_χ is an isomorphism. Hence $\mathcal{U}(\mathfrak{g}_\mathbb{C}) = \mathcal{U}(i_\chi(\mathfrak{h}_\mathbb{C})) \otimes \mathcal{U}(\mathfrak{a}_\mathbb{C}) \otimes \mathcal{U}(\mathfrak{n}_\mathbb{C})$. This proves the first assertion.

If $\chi_u = \mathbf{1}$, then $I_\chi = \mathcal{U}(\mathfrak{g}_\mathbb{C})\mathfrak{h}_\mathfrak{u} \oplus \sum_{H\in\mathfrak{h}_\mathfrak{l}} \mathcal{U}(\mathfrak{g}_\mathbb{C})(H + d\chi(H).\mathbf{1})$. Further, we have $\mathfrak{g} = \mathfrak{h}_\mathfrak{l} \oplus \mathfrak{a} \oplus ([\mathfrak{a},\mathfrak{u}] + \mathfrak{n})$. Thus if $\mathcal{U}((\mathfrak{n} + [\mathfrak{a},\mathfrak{u}])_\mathbb{C})$ denotes the subspace of $\mathcal{U}(\mathfrak{g}_\mathbb{C})$ spanned by all ordered products of elements of an ordered basis of $(\mathfrak{n} + [\mathfrak{a},\mathfrak{u}])_\mathbb{C}$, then $\mathcal{U}(\mathfrak{g}_\mathbb{C}) = \mathcal{U}\big(i_\chi((\mathfrak{h}_\mathfrak{l})_\mathbb{C})\big) \otimes \mathcal{U}(\mathfrak{a}_\mathbb{C}) \otimes \mathcal{U}((\mathfrak{n} + [\mathfrak{a},\mathfrak{u}])_\mathbb{C})$, proving the second assertion.

(ii) First we show that γ_χ is a homomorphism. For that it suffices to prove that $u \mapsto u_0$ is homomorphic. Let $u, v \in \mathcal{U}(\mathfrak{g}_\mathbb{C})^H$. Write $u = u_0 + u_1 + u_2$ with $u_0 \in \mathcal{U}(\mathfrak{a}_\mathbb{C})$, $u_1 = \sum_{X\in\mathfrak{h}} u_X(X + d\chi(X).\mathbf{1}) \in I_\chi$ and $u_2 \in \mathfrak{n}\mathcal{U}(\mathfrak{g}_\mathbb{C})$. Similarly, for v. Then

$$\begin{aligned}
uv &= (u_0 + u_1 + u_2)v \\
&\equiv u_0 v + \sum_{X\in\mathfrak{h}} u_X(X + d\chi(X).\mathbf{1})v \qquad \mod I_\chi + \mathfrak{n}\mathcal{U}(\mathfrak{g}_\mathbb{C}) \\
&\equiv u_0 v + \sum_{X\in\mathfrak{h}} u_X v(X + d\chi(X).\mathbf{1}) \qquad \mod I_\chi + \mathfrak{n}\mathcal{U}(\mathfrak{g}_\mathbb{C}) \\
&\equiv u_0 v \qquad \mod I_\chi + \mathfrak{n}\mathcal{U}(\mathfrak{g}_\mathbb{C}) \\
&\equiv u_0 v_0 + u_0 v_2 \qquad \mod I_\chi + \mathfrak{n}\mathcal{U}(\mathfrak{g}_\mathbb{C}) \\
&\equiv u_0 v_0 \qquad \mod I_\chi + \mathfrak{n}\mathcal{U}(\mathfrak{g}_\mathbb{C}),
\end{aligned}$$

where the last equality follows from the fact that $\mathfrak{a}$ normalizes $\mathfrak{n}$.

Now we show that $\operatorname{im}\gamma_\chi \subseteq \mathcal{S}(\mathfrak{a}_\mathbb{C})^{\widetilde{\mathcal{W}}}$ which is the hard part in the proof of this theorem.

If $\mathfrak{g}$ is solvable, then $\widetilde{\mathcal{W}}$ is trivial by Lemma III.12 and the assertion is clear. Thus we may assume that χ_u is trivial. According to Lemma III.12, the big Weyl group $\widetilde{\mathcal{W}}$ is generated by the reflections s_α corresponding to base roots α of Δ_s^+. Thus we only have to show that $\operatorname{im}\gamma_\chi \subseteq \mathcal{S}(\mathfrak{a}_\mathbb{C})^{\{\mathbf{1},s_\alpha\}}$ holds for all base roots α of Δ_s^+.

We adapt the method in the proof of [Wa88, Th. 3.2.3] to our situation. For a base root $\alpha \in \Delta_s^+$ let $\mathfrak{m}_\alpha$ be the Lie algebra generated by $\mathfrak{a}$ and $\bigoplus_{\beta\in\mathbb{Z}\alpha} \mathfrak{g}^\beta$, i.e.,

$$m_\alpha = \langle \mathfrak{a} \oplus \bigoplus_{\beta\in\mathbb{Z}\alpha} \mathfrak{g}^\beta \rangle \subseteq \mathfrak{a} \oplus \mathfrak{h}^0 \oplus \bigoplus_{\beta\in\mathbb{Z}\alpha} \mathfrak{g}^\beta.$$

Since $(\mathfrak{g},\tau)$ is assumed to be admissible (cf. Definition I.10), hence has cone potential (cf. [KrNe96, Th. VI.6]), it follows that $\mathfrak{g}^\gamma \subseteq \mathfrak{s}$ for all $\gamma \in \Delta_s$ (cf. [KrNe96, Prop. VII.8]). In particular, $\mathfrak{m}_\alpha$ is a τ-invariant reductive subalgebra of $\mathfrak{l} + \mathfrak{a}$ and its commutator algebra $\mathfrak{m}_\alpha'$ is semisimple of rank one. Set $\mathfrak{h}_\alpha = \mathfrak{m}_\alpha \cap \mathfrak{h}$. Further, we define

$$\mathfrak{n}(\alpha) := \bigoplus_{\beta\in\Delta_s^+\setminus\mathbb{N}\alpha} \mathfrak{g}^\beta = \bigoplus_{\beta\in\Delta_s^+\setminus\mathbb{N}\alpha} \mathfrak{s}^\beta \quad \text{and} \quad \mathfrak{h}(\alpha) = \{X + \tau(X) :\ X \in \mathfrak{n}(\alpha) + \mathfrak{u}\}.$$

We claim that

(3.7) $[\mathfrak{m}_\alpha, \mathfrak{n}(\alpha)] \subseteq \mathfrak{n}(\alpha)$.
(3.8) $[\mathfrak{h}_\alpha, \mathfrak{h}(\alpha)] \subseteq \mathfrak{h}(\alpha)$.

The first statement follows from the fact that α is a base root. For (3.8) we observe that $\mathfrak{h}(\alpha) = \mathfrak{h}_\mathfrak{u} \oplus \mathfrak{h}(\alpha)_\mathfrak{l}$ and so we only have to show that $[\mathfrak{h}_\alpha, \mathfrak{h}(\alpha)_\mathfrak{l}] \subseteq \mathfrak{h}(\alpha)$.

Let $X \in \mathfrak{h}_\alpha$ and $Y \in \mathfrak{h}(\alpha)_\mathfrak{l}$. Then $Y = Z + \tau(Z)$ for some $Z \in \mathfrak{n}(\alpha)$ and we have

$$[X, Y] = [X, Z] + [X, \tau(Z)] = [X, Z] + \tau([X, Z]).$$

Since $[X, Z] \in \mathfrak{n}(\alpha)$ by (3.7), it follows that $[X, Y] \in \mathfrak{h}(\alpha)$, proving (3.8).

Let $I_\chi^\alpha := \sum_{H \in \mathfrak{h}(\alpha)} \mathcal{U}(\mathfrak{g}_\mathbb{C})(H + d\chi(H).\mathbf{1})$ and note that

$$I_\chi = \mathcal{U}(\mathfrak{g}_\mathbb{C})\mathfrak{h}_\mathfrak{u} \oplus \sum_{H \in \mathfrak{h}(\alpha)_\mathfrak{l}} \mathcal{U}(\mathfrak{g}_\mathbb{C})(H + d\chi(H).\mathbf{1}),$$

since χ_u is trivial. As $\mathfrak{g} = \mathfrak{m}_\alpha \oplus \mathfrak{h}(\alpha)_\mathfrak{l} \oplus \mathfrak{n}(\alpha) \oplus [\mathfrak{a}, \mathfrak{u}]$, a similar argument as in (i) shows that

$$\mathcal{U}(\mathfrak{g}_\mathbb{C}) = \mathcal{U}((\mathfrak{m}_\alpha)_\mathbb{C}) \oplus \big(I_\chi^\alpha + (\mathfrak{n}(\alpha) + [\mathfrak{a}, \mathfrak{u}])\mathcal{U}(\mathfrak{g}_\mathbb{C})\big). \tag{3.9}$$

It follows from $[\mathfrak{l}, [\mathfrak{a}, \mathfrak{u}]] \subseteq [\mathfrak{a}, \mathfrak{u}]$ together with (3.7) and (3.8) that both summands are $\operatorname{ad}\mathfrak{h}_\alpha$-invariant. Thus we get a projection $q\colon \mathcal{U}(\mathfrak{g}_\mathbb{C})^H \to \mathcal{U}((\mathfrak{m}_\alpha)_\mathbb{C})^{H_\alpha}$, where $H_\alpha := \operatorname{Inn}_{\mathfrak{m}_\alpha}(\mathfrak{h}_\alpha)$. Finally, let $\sigma \in \operatorname{Aut}\big(\mathcal{U}((\mathfrak{m}_\alpha)_\mathbb{C})\big)$ be given by

$$\sigma(X) = X - \frac{1}{2}(\operatorname{tr}\operatorname{ad} X\,|_{\mathfrak{n}(\alpha)+\mathfrak{n}_r^+}), \quad X \in \mathfrak{m}_\alpha,$$

and set $q^\alpha = \sigma \circ q$. Let γ_χ^α denote the Harish-Chandra homomorphism associated to $\mathfrak{m}_\alpha$. We claim that $\gamma_\chi = \gamma_\chi^\alpha \circ q^\alpha$ holds.

In view of the Poincaré-Birkhoff-Witt Theorem, we have

$$\mathcal{U}((\mathfrak{m}_\alpha)_\mathbb{C}) = \mathcal{U}(\mathfrak{a}_\mathbb{C}) \oplus \Big(\sum_{H \in \mathfrak{h}_\alpha} \mathcal{U}((\mathfrak{m}_\alpha)_\mathbb{C})(H + d\chi(H).\mathbf{1}) + \mathfrak{n}_\alpha \mathcal{U}((\mathfrak{m}_\alpha)_\mathbb{C})\Big), \tag{3.10}$$

and

$$\begin{aligned} I_\chi + (\mathfrak{n} + [\mathfrak{a}, \mathfrak{u}])\mathcal{U}(\mathfrak{g}_\mathbb{C}) &\supseteq \big(I_\chi^\alpha + (\mathfrak{n}(\alpha) + [\mathfrak{a}, \mathfrak{u}])\mathcal{U}(\mathfrak{g}_\mathbb{C})\big) \\ &\quad \oplus \Big(\sum_{H \in \mathfrak{h}_\alpha} \mathcal{U}((\mathfrak{m}_\alpha)_\mathbb{C})(H + d\chi(H).\mathbf{1}) + \mathfrak{n}_\alpha \mathcal{U}((\mathfrak{m}_\alpha)_\mathbb{C})\Big). \end{aligned} \tag{3.11}$$

Thus it follows from (3.9), (3.10) and (i) that in (3.11) equality holds. Now our claim is immediate from (3.9) and (i).

It remains to show that $\operatorname{im}\gamma_\chi^\alpha \subseteq \mathcal{S}(\mathfrak{a}_\mathbb{C})^{\{\mathbf{1}, s_\alpha\}}$. We may assume that $\mathfrak{m}_\alpha$ is semisimple. We distinguish two cases:

Case 1: The root α is compact. In this case all roots of $\mathfrak{m}_\alpha$ are compact and $\mathfrak{m}_\alpha$ is (NCR). If $\mathfrak{h}_\alpha$ is semisimple, then $d\chi\,|_{\mathfrak{h}_\alpha} = 0$ and the assertion follows from [He184, Ch. II, Th. 5.18]. If $\mathfrak{h}_\alpha$ is not semisimple, then $\mathfrak{m}_\alpha$ is hermitian, and the assertion follows from [HS94, Part I, Th. 5.1.10]. □

Case 2: The root α is non-compact. In this case $\mathfrak{m}_\alpha$ is (NCC). If $\mathfrak{h}_\alpha$ is semisimple, then $d\chi\,|_{\mathfrak{h}_\alpha} = 0$ and the assertion follows from [HS94, Part II, Th. 4.3]. If $\mathfrak{h}_\alpha$ is not semisimple, then $\mathfrak{m}_\alpha$ is (CT) and therefore its Riemannian dual $\mathfrak{m}_\alpha^r$ is hermitian (cf. [HiÓl96, Th. 1.3.11]). Since the complexifications of $\mathfrak{m}_\alpha$ and $\mathfrak{m}_\alpha^r$ coincide, we may assume that $\mathfrak{m}_\alpha = \mathfrak{m}_\alpha^r$ (cf. the method in the proof of [HS94, Part II, Th. 4.3]). Now the assertion follows from [HS94, Part I, Th. 5.1.10]. □

Remark III.14. In general, it is not true that γ_χ is injective, since this becomes false for non-abelian solvable admissible symmetric Lie algebras ($\mathfrak{h}$ is abelian in this case). But still an open problem is whether γ_χ is onto or not. □

Lemma III.15. *Let $(\mathfrak{g}, \tau)$ be a non-compactly causal symmetric Lie algebra, G a simply connected Lie group with Lie algebra $\mathfrak{g}$ and $K \subseteq G$ a τ-stable maximal compactly embedded subgroup of G. Then the following assertions hold:*

(i) *The subgroup* $H^0 = Z_H(\mathfrak{a})$ *is compact.*
(ii) *For each compact subset* $Q \subseteq \Gamma_A$*, the set* $\overline{Q.(K \cap HAN)}$ *is a compact subset of* HAN*.*

Proof. (i) Let $G_{\mathbb{C}}$ denote a simply connected Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$ and set $H_1 := \langle \exp_{G_{\mathbb{C}}}(\mathfrak{h})\rangle$. Since G is simply connected, the covering $q_G\colon G \to G_1 \subseteq G_{\mathbb{C}}$ restricts to a covering $q\colon H \to H_1$. By [FHÓ94, Lemma 5.1] the group H_1^0 is compact. Thus we only have to show that the fibers of q are compact, i.e., $q^{-1}(\mathbf{1})$ is compact. Let K denote a τ-stable maximal compactly embedded subgroup of G and $\mathfrak{k}$ the corresponding Lie algebra. Note that $Z(G) \subseteq Z(K)$. Since G and hence K is simply connected, we have

$$Z(K) \cong \exp(\mathfrak{z}(\mathfrak{k})) \times F \cong \mathfrak{z}(\mathfrak{k}) \times F,$$

where F, the set of elements of finite order, is a finite group. Now

$$q^{-1}(\mathbf{1}) \subseteq H \cap Z(G) \subseteq H \cap Z(K).$$

Since $\exp(\mathfrak{z}(\mathfrak{k})) \cong \mathfrak{z}(\mathfrak{k})$ and F is a finite group, it follows that $\tau(F) \subseteq F$. Using the table [HiÓl96, Th. 3.2.8], we deduce that $\mathfrak{z}(\mathfrak{k}) \subseteq \mathfrak{q}$. Thus $H \cap Z(K) = H \cap (\exp(\mathfrak{z}(\mathfrak{k})) \times F) = \exp(\mathfrak{z}(\mathfrak{k}))^\tau \times F^\tau$. Now $\exp(\mathfrak{z}(\mathfrak{k}))^\tau$ is a 2-subgroup of $\exp(\mathfrak{z}(\mathfrak{k})) \cong \mathbb{R}$, hence is trivial. Thus $H \cap Z(K) = F^\tau$ is finite, proving (i).

(ii) As Ω is bounded (cf. [HiÓl96, Th. 5.1.8]) and Q is compact, it follows that $\Omega_Q := Q.\overline{\Omega} \subseteq \Omega$ is compact (cf. Lemma I.20(iii)). In view of Lemma I.20(i), we have

$$QHAN = Q.\Omega H^0 AN \subseteq \Omega_Q H^0 AN \subseteq HAN.$$

Since both Ω_Q and H^0 are compact, $\Omega_Q H^0$ is compact and so $\Omega_Q H^0 AN = C_K AN$ for some compact subset $C_K \subseteq K$ by the Iwasawa decomposition of G. On the other hand, the compactness of K/Z implies that QK/Z is compact in G/Z. Thus by the Iwasawa-decomposition of G/Z we obtain $QK \subseteq KC_AC_N$ with compact subsets $C_A \subseteq A$ and $C_N \subseteq N$. Therefore we have

$$Q(K \cap (HAN)) \subseteq KC_AC_N \cap CAN = C_KC_AC_N.$$

Since $C_KC_AC_N$ is compact and the set HAN is H^0-right invariant (Lemma I.20(i)), the set C_K and therefore $C_KC_AC_N$ is contained in HAN, proving (ii). ☐

Lemma III.16. *The mapping*

$$H \times \Gamma_A \times H \to \Gamma, \quad (h_1, a, h_2) \mapsto h_1 a h_2$$

is an identification map.

Proof. Recall from Section I that the polar decomposition

$$H \times W \to \Gamma, \quad (h, W) \mapsto h\exp(X)$$

is a homeomorphism. In view of

$$W \cong \exp(W) \quad \text{and} \quad h_1ah_2 = h_1h_2\exp(\mathrm{Ad}(h_2)^{-1}.\log(a)),$$

we only have to show that the mapping

$$H \times (W \cap \mathfrak{a}) \to W, \quad (h, X) \mapsto \mathrm{Ad}(h).X$$

is an identification mapping. But this follows from [Ne98a, Lemma I.3], proving the lemma. ☐

Proposition III.17. *Let $\chi \in \mathbb{X}(H/H^0)$ and $\lambda \in \mathcal{E}^\chi$. Then the function φ_λ^χ is (H,χ)-spherical and we have for all $D \in \mathbb{D}(\chi)$*

$$D.\varphi_\lambda^\chi = \gamma_\chi(D)\varphi_\lambda^\chi.$$

Proof. It follows from Lemma III.5 that φ_λ^χ is left χ- and right χ^*-semiinvariant. Thus it remains to show that φ_λ^χ is a continuous common eigendistribution for $\mathbb{D}(\chi)$ to the character γ_χ.

Step 1: The function φ_λ^χ is continuous.

In view of Lemma III.16 and the H-semi-invariance properties of φ_λ^χ, it suffices to check that $\varphi_\lambda^\chi\,|_{\Gamma_A}$ is continuous. Now our product formula for φ_λ^χ (cf. Theorem III.10) shows that we may w.l.o.g. assume that $(\mathfrak{g},\tau)$ is reductive. Note that $\varphi_\lambda^\chi(aa_0) = a_0^{\lambda_z}\varphi_\lambda^\chi(a)$ holds for all $a \in \Gamma_A$ and $a_0 \in Z(G)\cap A$. Therefore we may assume that $(\mathfrak{g},\tau)$ is semisimple. Splitting $(\mathfrak{g},\tau)$ in irreducibles, we even may assume that $(\mathfrak{g},\tau)$ is irreducible. We have to distinguish two cases (cf. Definition I.4).

Case 1: $(\mathfrak{g},\tau)$ is non-compactly Riemannian (NCR).

In this case H/H^0 is compact and $G = HAN$ is the Iwasawa decomposition, and therefore φ_λ^χ is continuous for all $\lambda \in \mathfrak{a}_\mathbb{C}^*$ and $\chi \in \mathbb{X}(H/H^0)$.

Case 2: $(\mathfrak{g},\tau)$ is non-compactly causal (NCC) (cf. [FHÓ94, Prop. 5.3]).

Since H^0 is compact (cf. Lemma III.15), we can apply [Ól87, Lemma 1.3] and get

$$\int_{H/H^0} f(hH^0)\; d\mu_{H/H^0)}(hH^0) = \int_{K\cap(HAN)} f(h(k))a_H(k)^{-2\rho}\; d\mu_K(k)$$

for all $f \in L^1(H/H^0)$. In particular, we obtain

$$\begin{aligned}\varphi_\lambda^\chi(a) &= \int_{K\cap(HAN)} a_H(ah(k))^{\lambda-\rho}\chi(ah(k))\overline{\chi(h(k))}a_H(k)^{-2\rho}\; d\mu_K(k)\\ &= \int_{K\cap(HAN)} a_H(ak)^{\lambda-\rho}\chi(ak)a_H(k)^{-\lambda-\rho}\overline{\chi(k)}\; d\mu_K(k).\end{aligned} \tag{3.12}$$

Let $Q \subseteq \Gamma_A$ be a compact subset. Then Lemma III.15(ii) implies $Q(\overline{K\cap(HAN)})$ is a compact subset of HAN. We conclude, in particular, that

$$Q\times(K\cap(HAN)) \to \mathbb{C}, \quad (a,k)\mapsto |\chi(ak)a_H(ak)^{\lambda-\rho}|$$

is a continuous function, bounded from above and below by positive constants. Thus it follows from (3.12) that φ_λ^χ is continuous, concluding the proof of Step 1.

Step 2: The function φ_λ^χ is a common eigendistribution for $\mathbb{D}(\chi)$ w.r.t. the character γ_χ.

We know already that φ_λ^χ is continuous and therefore defines a distribution on Γ. We define a function

$$k_\lambda^\chi\colon \Gamma \to \mathbb{C}, \quad s\mapsto a_H(s)^{\lambda-\rho}\chi(s).$$

Now we have for all $f \in C_c^\infty(\Gamma)$ and $D \in \mathbb{D}(G)$,

$$\begin{aligned}(D.\varphi_\lambda^\chi)(f) &= \int_\Gamma (D^\top.f)(g)\varphi_\lambda^\chi(g)\ d\mu_G(g)\\ &= \int_\Gamma \int_{H/H^0} (D^\top.f)(g)k_\lambda^\chi(gh)\overline{\chi(h)}\ d\mu_{H/H^0}(hH^0)\ d\mu_G(g)\\ &= \int_{H/H^0} \int_\Gamma (D^\top.f)(g)k_\lambda^\chi(gh)\overline{\chi(h)}\ d\mu_G(g)\ d\mu_{H/H^0}(hH^0)\\ &= \int_{H/H^0} \int_\Gamma f(g)(D.k_\lambda^\chi)(gh)\overline{\chi(h)}\ d\mu_G(g)\ d\mu_{H/H^0}(hH^0).\end{aligned}$$

To prove Step 2, it therefore suffices to check that

$$(\forall D \in \mathbb{D}(\chi))\ \ (D.k_\lambda^\chi) = \gamma_\chi(D)k_\lambda^\chi. \tag{3.13}$$

This is now seen as follows. Recall from (3.1) that $\mathbb{D}(\chi)$ is a certain quotient of $\mathcal{U}(\mathfrak{g}_\mathbb{C})^H$. Let $u \in \mathcal{U}(\mathfrak{g}_\mathbb{C})^H$. Then the H-invariance of u implies

$$(u.k_\lambda^\chi)(s) = \chi(s)(u.k_\lambda^\chi)(a_H(s))$$

for all $s \in \Gamma$. Now write $u = u_0 + u_1 + u_2$ with $u_0 \in \mathcal{U}(\mathfrak{a}_\mathbb{C})$, $u_1 \in \sum_{H\in\mathfrak{h}} \mathcal{U}(\mathfrak{g}_\mathbb{C})(H + d\chi(H).\mathbf{1})$ and $u_2 \in \mathfrak{n}\mathcal{U}(\mathfrak{g}_\mathbb{C})$. Then $(u_2.k_\lambda^\chi)(a_H(s)) = 0$, because A normalizes $\mathfrak{n}$ and χ is constant on right N-cosets. Similarly, we get $(u_1.k_\lambda^\chi)(a_H(s)) = 0$. Thus

$$(u.k_\lambda^\chi)(s) = \chi(s)(u_0.k_\lambda^\chi)(a_H(s)) = \chi(s)(\lambda(u_0) - \rho(u_0))a_H(s)^{\lambda-\rho} = \gamma_\chi(u)k_\lambda^\chi(s),$$

proving (3.13) and thus Step 2. □

Problems III.18. (a) Is there any defining integral equation for (H, χ)-spherical functions? In view of the product formula for the φ_λ^χ (cf. Theorem III.10), this should be the case at least for the character $\chi = \Delta_H^{-\frac{1}{2}}$ (cf. Remark III.11).

(b) Spherical functions can be defined for all $\chi \in \mathbb{X}(H)$. How natural is the assumption $\chi \in \mathbb{X}(H/H^0)$? □

IV. c-functions and asymptotic behaviour of spherical functions

In this last section we study the asymptotic behaviour of the (H, χ)-spherical functions φ_λ^χ. We discuss the "constant term $c_\mathcal{M}^\chi(\lambda)$ at infinity," the so-called c-function. Using our product formula from Theorem III.10 for the spherical functions and an analytic continuation technique, we obtain a product formula the $c_\mathcal{M}^\chi(\lambda)$ in unipotent and reductive part (cf. Theorem IV.11). We conclude the section with a list of problems relating the theory of spherical functions to harmonic analysis on complexified symmetric spaces.

Lemma IV.1. *Let $\chi \in \mathbb{X}(H/H^0)$ and $\lambda \in \mathcal{E}^\chi$. Then we have for all $s \in \Gamma$*

$$\varphi_\lambda^\chi(s) = \int_{\overline{N}\cap(HAN)} a_H(s\overline{n})^{\lambda-\rho}a_H(\overline{n})^{-\lambda-\rho}\chi(s\overline{n})\overline{\chi(\overline{n})}\ d\mu_{\overline{N}}(\overline{n}).$$

In particular, if $s = a \in \Gamma_A$, then

$$\varphi_\lambda^\chi(a) = a^{\lambda-\rho}\int_{\overline{N}\cap(HAN)} a_H(a\overline{n}a^{-1})^{\lambda-\rho}a_H(\overline{n})^{-\lambda-\rho}\chi(a\overline{n}a^{-1})\overline{\chi(\overline{n})}\ d\mu_{\overline{N}}(\overline{n}).$$

Proof. In view of Proposition I.21(ii), we have for all $s \in \Gamma$

$$\begin{aligned}\varphi_\lambda^\chi(s) &= \int_{H/H^0} a_H(sh)^{\lambda-\rho}\chi(sh)\overline{\chi}(h)\ d\mu_{H/H^0}(hH^0)\\ &= \int_{\overline{N}\cap(HAN)} a_H(sh_H(\overline{n}))^{\lambda-\rho}\chi(sh_H(\overline{n}))\overline{\chi(h_H(\overline{n}))}a_H(\overline{n})^{-2\rho}\ d\mu_{\overline{N}}(\overline{n}).\end{aligned}$$

Writing $\overline{n} = h_H(\overline{n})a_H(\overline{n})n_H(\overline{n})$ we obtain $h_H(\overline{n}) = \overline{n}a_H(\overline{n})^{-1}n$ for some $n \in N$, and so

$$\begin{aligned}\varphi_\lambda^\chi(s) &= \int_{\overline{N}\cap(HAN)} a_H(s\overline{n})^{\lambda-\rho}a_H(\overline{n})^{-\lambda+\rho}\chi(s\overline{n})\overline{\chi(\overline{n})}a_H(\overline{n})^{-2\rho}\ d\mu_{\overline{N}}(\overline{n})\\ &= \int_{\overline{N}\cap(HAN)} a_H(s\overline{n})^{\lambda-\rho}a_H(\overline{n})^{-\lambda-\rho}\chi(s\overline{n})\overline{\chi(\overline{n})}\ d\mu_{\overline{N}}(\overline{n}).\end{aligned}$$

This proves the first assertion.

For $s = a \in \Gamma_A$ we further get

$$\begin{aligned}\varphi_\lambda^\chi(a) &= \int_{\overline{N}\cap(HAN)} a_H(a\overline{n})^{\lambda-\rho}a_H(\overline{n})^{-\lambda-\rho}\chi(a\overline{n})\overline{\chi(\overline{n})}\ d\mu_{\overline{N}}(\overline{n})\\ &= a^{\lambda-\rho}\int_{\overline{N}\cap(HAN)} a_H(a\overline{n}a^{-1})^{\lambda-\rho}a_H(\overline{n})^{-\lambda-\rho}\chi(a\overline{n}a^{-1})\overline{\chi(\overline{n})}\ d\mu_{\overline{N}}(\overline{n}),\end{aligned}$$

concluding the proof of the lemma. ☐

Definition IV.2. Fix $\chi \in \mathbb{X}(H)$. For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ we set

$$\begin{aligned}c_{\mathcal{M}}^\chi(\lambda) &= \int_{\overline{N}\cap(HAN)} a_H(\overline{n})^{-\lambda-\rho}\overline{\chi(\overline{n})}\ d\mu_{\overline{N}}(\overline{n}),\\ c_\Omega^\chi(\lambda) &= \int_\Omega a_H(\overline{n})^{-\lambda-\rho}\overline{\chi(\overline{n})}\ d\mu_{N_n^-}(\overline{n}),\end{aligned}$$

and

$$c_0^\chi(\lambda) = \int_{N_k^-} a_H(\overline{n_k})^{-\lambda-\rho_k}\overline{\chi(\overline{n_k})}\ d\mu_{N_k^-}(\overline{n_k})$$

provided the integrals exist. We write $\mathcal{E}_{\mathcal{M}}^\chi$ for the set of all λ for which $c_{\mathcal{M}}^\chi$ is defined. Accordingly, we define $\mathcal{E}_\Omega^\chi$ and $\mathcal{E}_0^\chi$. ☐

Before we turn to the properties of the c-functions, we need some additional geometric information on the image of N_n^- under the projection $L := \log a_H \colon HAN \to \mathfrak{a}, han \mapsto \log a$.

Proposition IV.3. *Let $X \in \Delta_k^\perp \cap \operatorname{int}(\Delta_n^+)^\star$, $a = \exp X$, and $\overline{n} \in N_n^- \cap HAN$. Then $a\overline{n}a^{-1} \in HAN$ and*

$$a_H(a\overline{n}a^{-1}) - a_H(\overline{n}) \in C_{\min}.$$

Proof. First we note that $L(xy) = L\big(xh_H(y)\big) + L(y)$, showing that

$$L(a\overline{n}a^{-1}) = L\big(ah_H(\overline{n})\big) + L(\overline{n}) - \log a.$$

Now the Non-Linear Convexity Theorem ([Ne97a, Th. II.8]) shows that

$$L\big(ah_H(\overline{n})\big) \in L(aH) \subseteq \operatorname{conv}(\mathcal{W}.\log a) + C_{\min} = \log a + C_{\min},$$

and the assertion follows. ☐

Corollary IV.4. $L(N_n^- \cap HAN) = L(\Omega) \subseteq -C_{\min}$.

Proof. Let X be as in Proposition IV.3 and $\overline{n} \in N_n^- \cap HAN = \Omega$. Then

$$\lim_{t\to\infty} \exp(tX)\overline{n}\exp(-tX) = \mathbf{1}$$

and therefore Proposition IV.3 implies that

$$\lim_{t\to\infty} L\big(\exp(tX)\overline{n}\exp(-tX)\big) - L(\overline{n}) = -L(\overline{n}) \in C_{\min}.$$

□

Lemma IV.5. *For the domains of convergence $\mathcal{E}_\Omega^\chi$ resp. $\mathcal{E}_\mathcal{M}^\chi$, of c_Ω^χ, resp. $c_\mathcal{M}^\chi$, the following assertions hold:*

(i) *We have $\mathcal{E}_\Omega^\chi = i\mathfrak{a}^* + \mathcal{E}_{\Omega,\mathbb{R}}^\chi$, where $\mathcal{E}_{\Omega,\mathbb{R}}^\chi \subseteq \mathfrak{a}^*$ is a convex subset with $-C_{\min}^\star \subseteq \lim \mathcal{E}_{\Omega,\mathbb{R}}^\chi$. The restriction of the function c_Ω^χ to $\operatorname{int}\mathcal{E}_\Omega^\chi$ is holomorphic.*

(ii) *We have $\mathcal{E}_\mathcal{M}^\chi = i\mathfrak{a}^* + \mathcal{E}_{\mathcal{M},\mathbb{R}}^\chi$ with $\mathcal{E}_{\mathcal{M},\mathbb{R}}^\chi \subseteq \mathfrak{a}^*$ a convex subset. The restriction of $c_\mathcal{M}^\chi$ to $\operatorname{int}\mathcal{E}_\mathcal{M}^\chi$ is holomorphic.*

Proof. (i) We may assume that $\chi = |\chi|$. We define a measure μ_χ on Ω by $\mu_\chi = (\chi\mu_{N_n^-})|_\Omega$ Consider the mapping

$$\psi\colon \Omega \to \mathfrak{a}, \quad \overline{n} \mapsto \log a_H(\overline{n})$$

and let ν_χ be the push forward of μ_χ under this map. Then we have

$$\begin{aligned}\mathcal{L}(\nu_\chi)(\lambda) = \int_\mathfrak{a} e^{-\lambda(X)}\, d\nu_\chi(X) &= \int_\Omega a_H(\overline{n})^{-\lambda}\overline{\chi(\overline{n})}\, d\mu_{N_n^-}(\overline{n})\\ &= \int_\Omega a_H(\overline{n})^{-\lambda}\chi(\overline{n})\, d\mu_{N_n^-}(\overline{n}),\end{aligned}$$

and so

$$\mathcal{L}(\nu_\chi)(\rho+\lambda) = c_\Omega^\chi(\lambda). \tag{4.1}$$

Thus Lemma III.7(i) implies that $\mathcal{E}_\Omega^\chi$ is convex and $i\mathfrak{a}^* \subseteq \mathcal{E}_\Omega^\chi$. Further, Lemma III.7(iii) implies that c_Ω^χ is holomorphic on $\operatorname{int}\mathcal{E}_\Omega^\chi$.

In view of Corollary IV.4, $\operatorname{im}\psi \subseteq C_{\min}$, hence $C_{\nu_\chi} \subseteq -C_{\min}$ and therefore $B(C_{\nu_\chi}) \supseteq C_{\min}^\star$. In view of (4.1) and Lemma III.7(ii), this shows that $-C_{\min}^\star \subseteq \lim \mathcal{E}_\Omega^\chi$, concluding the proof of (i).

(ii) This is analogous to the first part of the proof of (i). □

Lemma IV.6. *The following assertions hold:*

(i) *If $s \in HAN$ let $b_H(s) := a_H(s)n_H(s)$. Then we have for all $s \in HAN$ and $u \in U$*

$$h_H(su) = h_H(s)h_{H_U}(b_H(s)ub_H(s)^{-1}).$$

In particular, we have for all $\chi \in \mathbb{X}(H)$ that $\chi(su) = \chi(s)\chi_u(b_H(s)ub_H(s)^{-1})$.

(ii) *For all $u \in U$ we have*

$$h_H(u^{-1}) = h_H(u)^{-1}.$$

In particular, we have $\chi(u^{-1}) = \chi(u)^{-1}$ for all $\chi \in \mathbb{X}(H_U)$.

(iii) *For all $\overline{n_p} \in N_p^- \cap HAN$ and $\overline{n_r} \in N_r^-$ we have*

$$a_H(\overline{n_p n_r}) = a_H(\overline{n_p}(\overline{n_r})^{-1}).$$

Proof. (i) This follows from

$$h_H(su) = h_H(h_H(s)b_H(s)u) = h_H(s)h_H(b_H(s)u) = h_H(s)h_{H_U}(b_H(s)ub_H(s)^{-1}).$$

(ii) In view of Proposition I.7(iii), we have $u = h_H(u)a_H(u)n_H(u)$. As $[\mathfrak{u},\mathfrak{u}] \subseteq \mathfrak{z}(\mathfrak{g}) = \mathfrak{a}_\mathfrak{u}$, we therefore get

$$u^{-1} = n_H(u)^{-1}a_H(u)^{-1}h_H(u)^{-1} \in h_H(u)^{-1}A_U N_R,$$

as was to be shown.

(iii) This follows from [HiNe96, Lemma I.2.8] by embedding into the canonical extension (cf. Definition I.16, Remark I.18). □

Lemma IV.7. *For all $\lambda \in \mathcal{E}^\chi_\mathcal{M}$ we have*

$$c^\chi_\mathcal{M}(\lambda) = c^\chi_0(\lambda)c^\chi_\Omega(\lambda).$$

In particular, we have

$$\mathcal{E}^\chi_\mathcal{M} = \mathcal{E}^\chi_\Omega \cap \mathcal{E}^\chi_0.$$

If, in addition, $\chi \in \mathbb{X}(H/H^0)$, then $\mathcal{E}_0 := \mathcal{E}^\chi_0 = \{\lambda \in \mathfrak{a}^*_\mathbb{C} :\ (\forall \alpha \in \Delta^+_k)\operatorname{Re}\lambda(\check\alpha) > 0\}$ is independent of χ.

Proof (cf. [FHÓ94, Prop. 6.5], [HiNe96, Prop. II.14]). According to $\overline{N} = N^-_n \rtimes N^-_k$, every $\overline{n} \in \overline{N}$ can be written as $\overline{n} = \overline{n_n n_k}$ with $\overline{n_n} \in N^-_n$ and $\overline{n_k} \in N^-_k$. We have for $\overline{n} \in \overline{N} \cap HAN$:

$$a_H(\overline{n_n n_k}) = a_H(\overline{n_n}h_H(\overline{n_k}))a_H(\overline{n_k})$$

with $h_H(\overline{n_k}) \in K(0)_0 \subseteq G(0)_0$ (cf. Lemma I.15). Note that $\mu_{N^-_n}$ is $K(0)_0$-invariant. Thus, in view of Lemma I.20(iv), we get

$$\begin{aligned}
c^\chi_\mathcal{M}(\lambda) &= \int_{N^-_k}\int_\Omega a_H(\overline{n_n n_k})^{-\lambda-\rho}\overline{\chi}(\overline{n_n n_k})\ d\mu_{N^-_n}(\overline{n_n})\ d\mu_{N^-_k}(\overline{n_k})\\
&= \int_{N^-_k} a_H(\overline{n_k})^{-\lambda-\rho}\\
&\quad\cdot\int_\Omega a_H(h_H(\overline{n_k})^{-1}\overline{n_n}h_H(\overline{n_k}))^{-\lambda-\rho}\overline{\chi}(\overline{n_n n_k})\ d\mu_{N^-_n}(\overline{n_n})\ d\mu_{N^-_k}(\overline{n_k})\\
&= \int_{N^-_k} a_H(\overline{n_k})^{-\lambda-\rho}\\
&\quad\cdot\int_\Omega a_H(\overline{n_n})^{-\lambda-\rho}\overline{\chi}(h_H(\overline{n_k})\overline{n_n}h_H(\overline{n_k}^{-1})\overline{n_k})\ d\mu_{N^-_n}(\overline{n_n})\ d\mu_{N^-_k}(\overline{n_k})\\
&= \int_{N^-_k} a_H(\overline{n_k})^{-\lambda-\rho}\int_\Omega a_H(\overline{n_n})^{-\lambda-\rho}\overline{\chi}(h_H(\overline{n_k})\overline{n_n})\ d\mu_{N^-_n}(\overline{n_n})\ d\mu_{N^-_k}(\overline{n_k})\\
&= \int_{N^-_k} a_H(\overline{n_k})^{-\lambda-\rho}\overline{\chi}(\overline{n_k})\ d\mu_{N^-_k}(\overline{n_k})\int_\Omega a_H(\overline{n_n})^{-\lambda-\rho}\overline{\chi}(\overline{n_n})\ d\mu_{N^-_n}(\overline{n_n})\\
&= c^\chi_0(\lambda)c^\chi_\Omega(\lambda),
\end{aligned}$$

where the last equality follows from $a_H(\overline{n_k})^{\rho_n} = 1$ (since ρ_n is $\mathcal{W}$-fixed).

It follows, in particular, that $\mathcal{E}^\chi_\mathcal{M} = \mathcal{E}^\chi_0 \cap \mathcal{E}^\chi_\Omega$. If, in addition, $\chi \in \mathbb{X}(H/H^0)$, then $\chi|_{K(0)_0}$ is unitary. Thus $\mathcal{E}_0 := \mathcal{E}^\chi_0$ is independent of χ. Finally, [He184, Ch. IV, Th. 6.13] implies that

$$\mathcal{E}_0 = \{\lambda \in \mathfrak{a}^*_\mathbb{C} :\ (\forall \alpha \in \Delta^+_k)\operatorname{Re}\lambda(\check\alpha) > 0\},$$

concluding the proof of the lemma. □

Next we consider the analytic behaviour of the c-functions and its dependence of χ_u, resp. w. Recall that there is an isomorphism $\mathbb{X}(H/H^0) \cong \mathbb{X}(H_L/H^0) \times \mathbb{X}(H_{U,\mathrm{fix}})$ (cf. Lemma I.14, Lemma II.8). Thus we find a complex subspace $W \subseteq \mathbb{C}^n \cong (\mathfrak{h}_\mathfrak{u})^*_\mathbb{C}$ such that

$$W \to \mathbb{X}(H_{U,\mathrm{fix}}), \quad w \mapsto \chi_w; \ \chi_w(\exp(x)) := e^{2\pi i \langle x, w\rangle}$$

is a surjective homomorphism. For fixed $\chi \in \mathbb{X}(H)$ and $\lambda \in \mathcal{E}^\chi_\mathcal{M}$ we set

$$\mathcal{E}^{\chi,\lambda}_\Omega := \{w \in W : c^{\chi_s \chi_w}_\Omega(\lambda) \ \text{ is finite}\}.$$

Note that $\mathcal{E}^{\chi,\lambda}_\Omega \neq \emptyset$, because $c^\chi_\Omega(\lambda)$ is finite. Similarly, we define $\mathcal{E}^{\chi,\lambda}_\mathcal{M}$. Finally, we set $W_\mathbb{R} = W \cap \mathbb{R}^n$ and note that $W = W_\mathbb{R} \oplus iW_\mathbb{R}$.

Lemma IV.8. *Let $\chi \in \mathbb{X}(H)$ and $\lambda \in \mathcal{E}^\chi$. Then the following assertions hold:*

(i) *The domain $\mathcal{E}^{\chi,\lambda}_\Omega$ has the form*

$$\mathcal{E}^{\chi,\lambda}_\Omega = W_\mathbb{R} + \mathcal{E}^{\chi,\lambda}_{\Omega,\mathbb{R}},$$

where $\mathcal{E}^{\chi,\lambda}_{\Omega,\mathbb{R}} \subseteq iW_\mathbb{R}$ is a non-empty convex subset with $\mathcal{E}^{\chi,\lambda}_{\Omega,\mathbb{R}} = -\mathcal{E}^{\chi,\lambda}_{\Omega,\mathbb{R}}$; in particular, $0 \in \mathcal{E}^{\chi,\lambda}_{\Omega,\mathbb{R}}$. Further, the mapping

$$\operatorname{int} \mathcal{E}^{\chi,\lambda}_\Omega \to \mathbb{C}, \quad w \mapsto c^{\chi_s\chi_w}_\Omega(\lambda)$$

is holomorphic.

(ii) *If $\chi \in \mathbb{X}(H/H^0)$, then an analogous statement holds for $c^\chi_\mathcal{M}$ and $\mathcal{E}^{\chi,\lambda}_\mathcal{M}$.*

Proof. (i) In the following we write the elements of $\overline{n}$ of $N_n^- = N_p^- \times N_R$ as products $\overline{n} = \overline{n_p n_r}$ with $\overline{n_p} \in N_p^-$ and $\overline{n_r} \in N_R$.

Consider the mapping

$$\psi\colon \Omega \to i\mathfrak{h}_\mathfrak{u}, \quad \overline{n} \mapsto 2\pi i \log h_H(b_H(\overline{n_p})\overline{n_u} b_H(\overline{n_p})^{-1})$$

and let ν be the push forward of the measure $a_H(\overline{n})^{-\lambda-\rho}\overline{\chi_s(\overline{n_p})} d\mu_{N_n^-}(\overline{n})$ under ψ. Then Lemma IV.6(i) entails that $\mathcal{L}(\nu)(w) = c^{\chi_s\chi_w}_\Omega$ for all $w \in W$, and all the assertions except for $\mathcal{E}^{\chi,\lambda}_{\Omega,\mathbb{R}} = -\mathcal{E}^{\chi,\lambda}_{\Omega,\mathbb{R}}$ follow from Lemma III.7. Let $w \in W$. Then we compute with Lemma IV.6(ii),(iii)

$$\begin{aligned}
&\mathcal{L}(\nu)(w) = \int_{i\mathfrak{h}_\mathfrak{u}} e^{-\langle w, x\rangle} \, d\nu(x) \\
&= \int_\Omega a_H(\overline{n})^{-\lambda-\rho} \overline{\chi_s(\overline{n_p})\chi_w(b_H(\overline{n_p})\overline{n_r} b_H(\overline{n_p})^{-1})} \, d\mu_{N_n^-}(\overline{n}) \\
&= \int_{\Omega_S}\int_{N_R} a_H(\overline{n_p n_r})^{-\lambda-\rho} \overline{\chi_s(\overline{n_p})\chi_w(b_H(\overline{n_p})\overline{n_r} b_H(\overline{n_p})^{-1})} \, d\mu_{N_p^-}(\overline{n_p}) \, d\mu_{N_R}(\overline{n_r}) \\
&= \int_{\Omega_S}\int_{N_R} a_H(\overline{n_p n_r}^{-1})^{-\lambda-\rho} \overline{\chi_s(\overline{n_p})}\overline{\chi_w(b_H(\overline{n_p})\overline{n_r}^{-1} b_H(\overline{n_p})^{-1})} \, d\mu_{N_p^-}(\overline{n_p}) \, d\mu_{N_R}(\overline{n_r}) \\
&= \int_{\Omega_S}\int_{N_R} a_H(\overline{n_p n_r})^{-\lambda-\rho} \overline{\chi_s(\overline{n_p})}\chi_w(b_H(\overline{n_p})\overline{n_r} b_H(\overline{n_p})^{-1})^{-1} \, d\mu_{N_p^-}(\overline{n_p}) \, d\mu_{N_R}(\overline{n_r}) \\
&= \mathcal{L}(\nu)(-w),
\end{aligned}$$

concluding the proof of (i).

(ii) Our technique of using Laplace transforms shows that the domain of convergence is a tube domain as well as holomorphic dependence in the interior (Lemma

III.7). If, in addition, $\chi \in \mathbb{X}(H/H^0)$, then the symmetry of $\mathcal{E}^\chi_{\mathcal{M},\mathbb{R}}$ follows from (i) and Lemma IV.5 since $\mathcal{E}^\chi_0 = \mathcal{E}_0$ does not depend on χ. □

Lemma IV.9. *If G is reductive and $\chi \in \mathbb{X}(H/H^0)$, then the following assertions hold:*

(i) *The function φ^χ_λ is defined if and only if $c^\chi_\Omega(\lambda)$ is finite, i.e., $\mathcal{E}^\chi = \mathcal{E}^\chi_\Omega$. Moreover, $\mathcal{E}^\chi$ is open.*

(ii) *If $\lambda \in \mathcal{E}^\chi_\mathcal{M}$ and $X \in \operatorname{int}(\Delta^+)^\star$, then*

$$\lim_{t\to\infty} e^{(\rho-\lambda)(tX)}\varphi^\chi_\lambda(\exp(tX)) = c^\chi_\mathcal{M}(\lambda).$$

Proof. (i) If $\chi = \mathbf{1}$, then, in view of Definition I.4, the first assertion follows from [FHÓ94, Th. 6.3]. The general case is obtained by a slight modification of their arguments (cf. the proof of Case 2 in the proof of Proposition III.17). The openness of $\mathcal{E}^\chi$ follows from the same type of arguments as in [Ól97, Th. 3.10].

(ii) This follows from [FHÓ94, Th. 6.8] for χ being trivial. The general case is obtained exactly along the same lines. □

Remark IV.10. Recall the content of Lemma IV.7(iv). In view of our characterization of the various c-functions as Laplace transforms (cf. Lemma IV.5), we conclude, in particular, that the c-functions do not admit a holomorphic continuation to an open domain bigger than the interior of the domain of convergence of the Laplace transform. In the following we will use this fact frequently without mentioning it explicitly. □

Now we have all the technical tools to prove the two main results of this section.

Theorem IV.11 (Asymptotic behaviour of spherical functions)**.** *If $\chi\in\mathbb{X}(H/H^0)$, then we have for all $\lambda \in \mathcal{E}^\chi_\mathcal{M}$ and $X \in \operatorname{int}(\Delta^+)^\star$*

$$\lim_{t\to\infty} e^{(\rho-\lambda)(tX)}\varphi^\chi_\lambda(\exp(tX)) = c^\chi_\mathcal{M}(\lambda).$$

Further, if $\chi_u(\exp(x)) = e^{2\pi i\langle x,w\rangle}$ for $x \in \mathfrak{h}_\mathfrak{u}$ and $n := \dim \mathfrak{n}_r^+$, then the following product formula

$$c^\chi_\mathcal{M}(\lambda) = \frac{2^{\frac{n}{2}}\pi^n e^{-\langle w_+,\overline{w_+}\rangle_{\lambda_z}}}{\det A_{\lambda_z}} \cdot c^{\chi_l\Delta_H^{\frac12}}_{\mathcal{M}_L}(\lambda_l)$$

holds. The domain $\mathcal{E}^\chi_\mathcal{M}$ of convergence is independent of χ_u and given by

$$\mathcal{E}^\chi_\mathcal{M} = (i\mathfrak{a}^* - \operatorname{int} C^\star_{\min,r}) \cap \mathcal{E}^{\chi_l\Delta_H^{\frac12}}_{\mathcal{M}_L}.$$

Proof. Fix $\chi_l \in \mathbb{X}(H_L/H^0) \subseteq \mathbb{X}(H/H^0)$ and $X \in \operatorname{int}(\Delta^+)^\star$.

We define

$$f(\lambda, w) := \lim_{t\to\infty} e^{(\rho-\lambda)(tX)}\varphi^{\chi_l\chi_w}_\lambda(\exp(tX)),$$

whenever the right-hand side exists and set

$$\mathcal{D}(\chi_l) := \{(\lambda, w) \in \mathcal{E}^{\chi_l\chi_w} \times W : \ f(\lambda, w) \text{ exists}\}.$$

Then Theorem III.10(i) implies for all $(\lambda, w) \in \mathcal{D}(\chi_l)$ that

$$f(\lambda, w) = \frac{2^{\frac{n}{2}}\pi^n e^{-\langle w_+,\overline{w_+}\rangle_{\lambda_z}}}{\det A_{\lambda_z}} \cdot \lim_{t\to\infty} e^{(\rho_l-\lambda_l)(tX)}\varphi^{\chi_l\Delta_H^{\frac12}}_{\lambda_l}(\exp(tX)), \tag{4.2}$$

where $n = \dim \mathfrak{n}_r^+$ and $\rho_l := \rho - \rho_r$ denotes the half sum of positive roots corresponding to $\mathfrak{l}$ counted with their multiplicities. In view of Lemma IV.9, it thus follows from (4.2) that

$$(4.3) \qquad (\forall (\lambda, w) \in \mathcal{D}(\chi_l)) \quad f(\lambda, w) = \frac{2^{\frac{n}{2}} \pi^n e^{-\langle w_+, \overline{w_+} \rangle_{\lambda_z}}}{\det A_{\lambda_z}} \cdot c_{\mathcal{M}_L}^{\chi_l \Delta_H^{\frac{1}{2}}}(\lambda_l).$$

We conclude from (4.3) and Theorem III.10(ii) that

$$\mathcal{D}(\chi_l) = \Big((i\mathfrak{a}^* - \operatorname{int} C_{\min,r}^\star) \cap \mathcal{E}_{\mathcal{M}_L}^{\chi_l} \Big) \times W.$$

In particular, $\mathcal{D}(\chi_l)$ is open by Lemma IV.9 and Lemma IV.7. Further, (4.3) together with Lemma IV.5(ii) imply that f is holomorphic on $\mathcal{D}(\chi_l)$.

Now we compute the limit in a second way. Let $\chi \in \mathbb{X}(H/H^0)$ such that $\chi|_{H_L} = \chi_l$. As $\lim \mathcal{E}_{\mathbb{R}}^\chi \supseteq -C_{\min}^\star$ (cf. Lemma III.8), we deduce that

$$\mathcal{E}_\rho^\chi := \{\lambda \in \mathcal{E}^\chi : \ \mathrm{Re}(\lambda - \rho) \in -C_{\min}^\star \cap (\check{\Delta}_k^+)^\star\}$$

is a convex subset of $\mathcal{E}^\chi$ with non-empty interior. Let $\lambda \in \mathcal{E}_\rho^\chi$ and assume first that χ_u is unitary, i.e., $\chi_u = \chi_w$ for some $w \in W_{\mathbb{R}}$. In view of Lemma IV.1, we have

$$\begin{aligned} e^{(\rho-\lambda)(tX)} \varphi_\lambda^\chi(\exp(tX)) = \int_{\overline{N} \cap HAN} & a_H(\exp(tX)\overline{n}\exp(-tX))^{\lambda-\rho} a_H(\overline{n})^{-\lambda-\rho} \\ & \cdot \chi(\exp(tX)\overline{n}\exp(-tX))\overline{\chi(\overline{n})}\, d\mu_{\overline{N}}(\overline{n}). \end{aligned}$$

Thus we only have to justify taking the limit under the integral sign. For that we may assume that $\chi = |\chi|$ and since χ_u was supposed to be unitary, we even may assume that $\chi = |\chi_l|$. Recall that $\overline{N} \cap HAN = \overline{N}_U \cdot \Omega_L$ and that Ω_L is bounded. As $\exp(tX).\overline{\Omega_L} \subseteq \exp(sX).\overline{\Omega_L} \subseteq \Omega_L$ for $0 < s \leq t$, we deduce that $\exp X.\overline{\Omega_L} = \bigcup_{t \geq 1} \exp(tX).\overline{\Omega_L}$ is a compact subset of Ω_L. In particular, the function

$$(\overline{N} \cap HAN) \times [1, \infty[\to \mathbb{R}^+, \quad t \mapsto |\chi(\exp(tX)\overline{n}\exp(-tX))|$$

is uniformly bounded from above and below by positive constants. Moreover, $\lambda \in \mathcal{E}_\rho^\chi$ implies that $t \mapsto a_H(\exp(tX)\overline{n}\exp(-tX))^{\lambda-\rho}$ is monotonically decreasing for all $\overline{n} \in \overline{N} \cap HAN$ (cf. [Ne97a, Prop. II.10]). Hence pushing through the limit is justified, $c_{\mathcal{M}}^\chi(\lambda)$ is finite and we have

$$(4.4) \qquad (\forall w \in W_{\mathbb{R}})(\forall \lambda \in \mathcal{E}_\rho^{\chi_l \chi_w}) \quad f(\lambda, w) = c_{\mathcal{M}}^{\chi_l \chi_w}(\lambda).$$

On the other hand, $c_{\mathcal{M}}^{\chi_l \chi_w}(\lambda)$ depends holomorphically on w and λ (cf. Lemma IV.7(ii), Lemma IV.8(ii)). Further, Lemma IV.8(ii) shows that $\mathcal{E}_{\mathcal{M}}^{\chi_l \chi_w, \lambda} \neq \emptyset$ implies that $\mathcal{E}_{\mathcal{M}}^{\chi_l, \lambda} \neq 0$ for all $w \in W$. Thus analytic continuation in (4.4) in both variables w and λ gives

$$(4.5) \quad (\forall (\lambda, w) \in \mathcal{D}(\chi_l)) \quad c_{\mathcal{M}}^{\chi_l \chi_w}(\lambda) = \frac{2^{\frac{n}{2}} \pi^n e^{-\langle w_+, \overline{w_+} \rangle_{\lambda_z}}}{\det A_{\lambda_z}} \cdot c_{\mathcal{M}_L}^{\chi_l \Delta_H^{\frac{1}{2}}}(\lambda_l) = f(\lambda, w).$$

In view of Remark IV.10, the proof of the theorem is now complete. ☐

Corollary IV.12 (Product decomposition for the c_Ω-functions)**.** *If $\chi \in \mathbb{X}(H/H^0)$ and $\lambda \in \mathcal{E}_\Omega^\chi$, then we have*

$$c_\Omega^\chi(\lambda) = \frac{2^{\frac{n}{2}} \pi^n e^{-\langle w_+, \overline{w_+} \rangle_{\lambda_z}}}{\det A_{\lambda_z}} \cdot c_{\Omega_L}^{\chi_l \Delta_H^{\frac{1}{2}}}(\lambda_l).$$

The domain of convergence $\mathcal{E}^\chi_\Omega$ *of* c^χ_Ω *coincides with the domain of convergence* $\mathcal{E}^\chi$ *of* φ^χ_λ *and we have*

$$\mathcal{E}^\chi = \mathcal{E}^\chi_\Omega = (i\mathfrak{a}^* - \operatorname{int} C^\star_{\min,r}) \cap \mathcal{E}^{\chi_l \Delta_H^{\frac{1}{2}}}_{\Omega_L}.$$

Proof. We deduce from Theorem IV.11 and Lemma IV.7 that

$$(\forall \lambda \in \mathcal{E}^\chi_{\mathcal{M}}) \qquad c^\chi_\Omega(\lambda) = \frac{2^{\frac{n}{2}} \pi^n e^{-\langle w_+, \overline{w_+}\rangle_{\lambda_z}}}{\det A_{\lambda_z}} \cdot c^{\chi_l \Delta_H^{\frac{1}{2}}}_{\Omega_L}(\lambda_l).$$

As both sides are holomorphic functions on the interior of their domain of definition (cf. Lemma IV.5(i)), we see that equality holds for all $\lambda \in \operatorname{int} \mathcal{E}^\chi_\Omega$. Then Theorem III.10(ii) together with Lemma IV.9(ii) imply

$$\mathcal{E}^\chi = (i\mathfrak{a}^* - \operatorname{int} C^\star_{\min,r}) \cap \mathcal{E}^{\chi_l \Delta_H^{\frac{1}{2}}}_{\Omega_L}.$$

In view of Remark IV.10, the proof of the corollary is now complete. □

Remark IV.13. If G is reductive and $\chi = \mathbf{1}$, then $c_{\mathcal{M}}$ is known. In fact, it is a product of c-functions corresponding to (NCR) and (NCC) spaces. For (NCR) spaces the c-function was computed by Gindikin and Karpelevic (cf. [He184, Ch. IV, Th. 6.13]) while for (NCC) spaces the exact formula for $c_{\mathcal{M}}$ was recently obtained in [KrÓl99]. Hence for general G and the canonical character $\chi = \Delta_H^{-\frac{1}{2}}$, Theorem IV.11 gives us a precise expression for the c-function $c^\chi_{\mathcal{M}}$. □

V. Further remarks and problems

We conclude our paper with some remarks related to earlier work and we give a list of problems concerning the interplay of the theory of spherical functions and harmonic analysis. See also [Ól98] and [Ól00].

Spherical functions on reductive symmetric spaces. In view of our decomposition results (Theorem III.10, Theorem IV.11), the study of spherical functions on hyperbolically ordered symmetric spaces is reduced to reductive symmetric spaces consisting of (NCR) and (NCC) components. For the theory of spherical functions on non-compactly Riemannian symmetric spaces we refer to [He184] and [GaVa88] for the case $\chi = \mathbf{1}$ and to [HS94, Part I] for the theory on hermitian symmetric spaces with χ arbitrary. For $\chi = \mathbf{1}$ and $(\mathfrak{g}, \tau)$ non-compactly causal, spherical functions, their asymptotic expansions, meromorphic continuations and domains of convergence have been discussed by the third author in [Ól97]. More detailed results on the analytic continuation and the singularities of the spherical functions have been obtained by the third author and A. Pasquale in [ÓlPa00] by using the Bernstein polynomial to obtain the analytic continuation of the spherical function. The analytic continuation of the spherical functions and estimates that can be derived from the integral formula defining the spherical function and its analytic continuation (cf. [Ól97] and [ÓlPa00]) are the fundamental tools in harmonic analysis on $H \backslash \operatorname{int} \Gamma$. Those results are used to prove analytic continuation of the spherical Laplace transform on $H\backslash G$ in case G is simple and to characterize the space of the image of the space of function on $H \backslash \operatorname{int} \Gamma$ with compact support modulo H. It has been shown in [AÓS00] that the image is contained in a Paley-Wiener space of holomorphic function with exponential growth similar to the Riemannian case. But the surjectivity of the Laplace transform is still an open problem.

Analytic continuation, estimates and Paley-Wiener Theorems. Fix $\chi \in \mathbb{X}(H/H^0)$ and $a \in \Gamma_A$. Prove that the map

$$\mathcal{E}^\chi \mapsto \mathbb{C}, \quad \lambda \mapsto \frac{1}{c^\chi_{\mathcal{M}}(\lambda)} \varphi^\chi_\lambda(a)$$

admits a meromorphic continuation on $\mathfrak{a}_\mathbb{C}$. If $(\mathfrak{g}, \tau)$ is solvable, then this follows from Theorem III.10 and Theorem IV.11. If $\chi = \mathbf{1}$ and $(\mathfrak{g}, \tau)$ is (NCR), then this is a consequence of the fact that the $c^\chi_{\mathcal{M}} = c_0$ is meromorphic (cf. [He184, Ch. IV, Th. 6.13]), and for $\chi = \mathbf{1}$ and $(\mathfrak{g}, \tau)$ being (NCC) this has been proved by the third author in [Ól97]. More detailed information of the location of the poles was obtained in [ÓlPa00]. Thus in view of our product formulas in Theorem III.10 and Theorem IV.11, the situation is reduced to the (NCC)-case for characters $\chi \neq \mathbf{1}$. It also remains an open problem to prove that the meromorphic continuations of the *normalized spherical functions* $\frac{1}{c^\chi_{\mathcal{M}}(\lambda)} \varphi^\chi_\lambda$ exhaust, up to scalar multiples, all (H, χ)-spherical functions (cf. Definition III.1). It is also natural to try to generalize the Laplace transform and prove Paley-Wiener type theorems in the general case considered in this article.

Spherical functions and character theory. Let $(\pi_\lambda, \mathcal{H}_\lambda)$ be an (H, χ)-spherical unitary highest weight representation of G^c which we also consider as a holomorphic representation of Γ^c (cf. Remark II.5). Set $\operatorname{int} \Gamma^c := G^c \operatorname{Exp}(\widehat{W}^0_{\max})$. If $0 \neq \nu \in (\mathcal{H}^{-\omega}_\lambda)^{(H,\chi)}$ and v_λ is a highest weight vector, then we define the *spherical character* Θ^χ_λ of $(\pi_\lambda, \mathcal{H}_\lambda)$ by

$$\Theta^\chi_\lambda \colon \operatorname{int} \Gamma^c \to \mathbb{C}, \quad s \mapsto \frac{\langle v_\lambda, v_\lambda \rangle}{|\langle \nu, v_\lambda \rangle|^2} \langle \pi_\lambda(s).\nu, \nu \rangle. \tag{5.1}$$

Note that Θ^χ_λ is a holomorphic function on $\operatorname{int} \Gamma^c$ (cf. [KNÓ97, Lemma V.6]). Prove for all parameters $\lambda \in \mathfrak{a}^*$ which correspond to non-singular unitary spherical highest weight representations the relation

$$(\forall s \in \operatorname{int} \Gamma^c \cap HAN) \quad \Theta^\chi_\lambda(s) = \frac{1}{c^\chi_{\mathcal{M}}(\lambda + \rho)} \varphi^\chi_{\lambda+\rho}(s).$$

If $(\mathfrak{g}, \tau)$ is (NCC) and $\chi = \mathbf{1}$, then this has been proved in [Kr01, Th. V.3] and [HiKr99a, Th. 4.1.2]. This relation has far reaching consequences for the harmonic analysis on G^c-invariant subdomains D of the Stein manifold $\Xi^0_{\max} := G^c \times_H W^0_{\max}$ (cf. [KNÓ97], [Ne98a]). For instance, it implies the *Plancherel Theorem* for G^c-invariant Hilbert spaces of holomorphic functions on the domains D (cf. [HiKr99a], [Kr98, Kr99, Kr01]). This is also used to decompose the Cauchy-Szegö kernel of the Hardy space on Ξ^0 into H-spherical distribution characters of holomorphic discrete series representation and derive a formula for those characters using the Harish-Chandra type expansion in [Ól97] for the spherical function, [Ól98], This gives a Harish-Chandra type formula for the H-spherical distribution character of holomorphic discrete series representations generalizing the results of Harish-Chandra for the holomorphic discrete series representation of G^c.

It was proved in [FHÓ94] in case χ is trivial and G is semisimple that the spherical functions correspond to H-spherical characters of a prinicpal series representation of the group G. This shows that on the level of characters, principal series representations of G correspond to H-spherical representations of the group G. It is still an open problem to generalize those results to our setting.

Basic inequality and boundary behaviour. Let $\mathfrak{a}^+ := \mathrm{int}(\Delta^+)^\star$ and $A := \exp(\mathfrak{a}^+)$. We define a function Δ on A^+ by

$$\Delta(a) = \prod_{\alpha \in \Delta_n^+} (1 - e^{2\alpha(\log a)})^{m_\alpha}$$

and let $\|\cdot\|$ denote an arbitrary norm on $\mathfrak{a}$. Then for the generic character $\chi = \Delta_H^{-\frac{1}{2}}$, it follows from Theorem III.10 and [HiKr99b, Th. 4.2.1] that there exist constants $\kappa > 0$ and $d \in \mathbb{N}$ such that for all integral parameters $\lambda \in \mathcal{E}^\chi$ one has

$$(\forall a \in A^+) \quad \varphi_\lambda^\chi(a)\Delta(a) \leq \kappa \frac{c_{\mathcal{M}}^\chi(\lambda)}{\det A_{\lambda_z}} a^{\lambda - \rho}(1 + \|\log a\|)^d.$$

Further, using [HiKr99b, Th. 4.3.2] one can deduce that the inequality holds for all parameters lying in an affine cone in $\mathcal{E}^\chi$. One should consider this estimate as a generalization of Harish-Chandra's estimates for spherical functions on non-compact Riemannian symmetric spaces (cf. [Wa88, Prop. 4.6.3]). It implies, in particular, that the function $a \mapsto \Delta(a)\varphi_\lambda^\chi(a)$ admits a continuous continuation in the origin. This in turn has far reaching consequences for the determination of the spectrum of various G^c-invariant Hilbert spaces of holomorphic functions, for instance, for Bergman spaces on the domains D (cf. [HiKr99c], [Kr98]).

Reflection positivity. The spherical functions relate generalized principal series representations of G to highest weight modules for G^c. This idea was further developed in [JoÓl98, JoÓl00], where it was shown, using the Lüscher-Mack Theorem, that this correspondence can be explained using the representations themselves. The main idea is to take the part of the principal series representation supported on an open H-orbit in a flag manifold and invariant under the semigroup $\Gamma \cap G$. Using a new inner product related to the construction of the complementary series representation this gives rise to a $*$-representation of Γ, which by the Lüscher-Mack correspondence corresponds to an irreducible highest weight representation of G^c. It is an open problem how much of this construction can be generalized to the more general framework considered in this paper or even infinitely dimensional Lie groups. We refer to [JoÓl00] for a discussion on how this relates to the Osterwalder-Schrader duality in quantum field theory.

Hardy spaces. We now explain how the results of this paper can be used for the L^2-harmonic analysis on the c-dual symmetric space $H\backslash G^c$, in particular, for the theory of Hardy spaces. However, we will omit the proofs which are more or less straightforward generalizations of arguments existing in the literature, in particular, [HÓØ91].

Let G, G^c be connected Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{g}^c$. To facilitate the exposition, we will assume in the following that both G and G^c are contained in a complex Lie group $G_{\mathbb{C}}$ associated to $\mathfrak{g}_{\mathbb{C}}$ (one can overcome this difficulty by using the theory developed in [KNÓ97, Sect. IV]). We also assume that τ integrates to an involution of $G_{\mathbb{C}}$ and we set $H := (G^\tau)_0$ and $H_{\mathbb{C}} := (G_{\mathbb{C}}^\tau)_0$.

Let $W \subseteq W_{\max}$ be a closed H-invariant convex cone which is pointed (i.e., contains no affine lines) and has non-empty interior. Then W extends to a $\{G^c, -\tau\}$-invariant convex cone $\widehat{W} \subseteq i\mathfrak{g}^c$ with $\widehat{W} \cap \mathfrak{q} = W$ (cf. [KrNe96, Th. X.7]). In particular, $\Gamma = H\exp(W)$ extends to a complex Ol'shanskiĭ semigroup $\widehat{\Gamma} = G^c \exp(\widehat{W})$.

Now we collect some facts from [KNÓ97, Sect. IV] about the domains $\Xi := W \times_H G^c$. By the embedding

$$\Xi \to H_{\mathbb{C}}\backslash G_{\mathbb{C}}, \quad [X, g] \mapsto H_{\mathbb{C}} \exp(X) g$$

a complex structure is induced on $\Xi^0 := (\operatorname{int} W) \times_H G^c$. Further, $\widehat{\Gamma}$ acts from the right on Ξ and we have $\Xi = [0, \mathbf{1}].\widehat{\Gamma}$ as well as $\Xi^0 = [0, \mathbf{1}]. \operatorname{int} \widehat{\Gamma}$. Set $K^c := \exp(\mathfrak{g}(0)^c)$, $P^+ := (N_n^+)_{\mathbb{C}}$ and recall from [KNÓ97, Lemma III.7] that $\widehat{\Gamma} \subseteq H_{\mathbb{C}} K_{\mathbb{C}}^c P^+$. Further, we have $H_{\mathbb{C}} K_{\mathbb{C}}^c P^+ \cong H_{\mathbb{C}} \times_{K_{\mathbb{C}}^c \cap H_{\mathbb{C}}} K_{\mathbb{C}}^c P^+$ (cf. [KNÓ97, Prop. II.6]).

Write $\Delta_H \colon H \to \mathbb{R}^+$ for the modular function of H. Then Δ_H extends to a holomorphic character $\Delta_{H_{\mathbb{C}}} \colon H_{\mathbb{C}} \to \mathbb{C}^\times$. Finally, a simple argument using covering semigroups leads to a continuous map

$$\Delta_0 \colon \widehat{\Gamma} \to \mathbb{C}^\times, \quad \Delta_0(hkp^+) := \Delta_{H_{\mathbb{C}}}(h) \qquad (h \in H_{\mathbb{C}}, k \in K_{\mathbb{C}}^c, p^+ \in P^+)$$

which is holomorphic on $\operatorname{int} \widehat{\Gamma}$. That Δ_0 is well defined can be deduced from $\Delta|_{H \cap K^c} \equiv 1$ (cf. Lemma III.9) together with some covering arguments.

Recall that G^c has to be unimodular by the existence of the open elliptic cone $i \operatorname{int} W \subseteq \mathfrak{g}^c$. Set $\rho := |\Delta_0|$ and note that ρ satisfies

$$(\forall h \in H)(\forall s \in \widehat{\Gamma}) \qquad \rho(hs) = \frac{\Delta_H(h)}{\Delta_{G^c}(h)} \rho(s). \tag{5.2}$$

Hence ρ is a *rho-function* in the sense of [War72, App. 1]. In particular, $\rho|_{G^c}$ induces a quasi-invariant measure μ_ρ on $H\backslash G^c$.

In the sequel we identify Ξ with the corresponding subset of $H_{\mathbb{C}}\backslash G_{\mathbb{C}}$. Using again an argument involving covering manifolds, we obtain a continuous cocycle

$$J \colon \Xi \times \widehat{\Gamma} \to \mathbb{C}^\times; \; J(H_{\mathbb{C}} x, \gamma) := \Big(\frac{\Delta_0(x\gamma)}{\Delta_0(x)}\Big)^{\frac{1}{2}}$$

with $J(H_{\mathbb{C}}, \mathbf{1}) = 1$. We note that J is holomorphic when restricted to the interior of its domain of definition.

By definition of the measure μ_ρ we have for all $f \in L^1(H\backslash G^c, \mu_\rho)$ and $g \in G^c$ that

$$\int_{H\backslash G^c} f(Hxg) |J(Hx, g)|^2 \, d\mu_\rho(Hg) = \int_{H\backslash G^c} f(Hx) \, d\mu_\rho(Hx). \tag{5.3}$$

A holomorphic representation of $\widehat{\Gamma}$ on $\operatorname{Hol}(\Xi^0)$ given by

$$(\pi(\gamma).f)(m) := J(m, \gamma) f(m\gamma) \qquad (f \in \operatorname{Hol}(\Xi^0), \gamma \in \widehat{\Gamma}, m \in \Xi^0).$$

Now we can define the *Hardy space* associated to Ξ^0 by

$$\mathcal{H}^2(\Xi^0) := \Big\{ f \in \operatorname{Hol}(\Xi^0) : \sup_{\gamma \in \widehat{\Gamma}} \int_{H\backslash G^c} |(\pi(\gamma).f)(m)|^2 \, d\mu_\rho(m) < \infty \Big\}.$$

Using standard techniques, for example, as presented in [Ne99, Ch. XIV], one can show that $\mathcal{H}^2(\Xi^0)$ is a Hilbert space on which G^c acts unitarily (by (5.3)) and $\widehat{\Gamma}$ by contractions. Further, we have an isometric boundary value map

$$b \colon \mathcal{H}^2(\Xi^0) \to L^2(H\backslash G^c, \mu_\rho), \quad f \mapsto b(f) := \lim_{\substack{\gamma \to \mathbf{1} \\ \gamma \in \operatorname{int} \widehat{\Gamma}}} (\pi(\gamma).f)|_{H\backslash G^c}.$$

The Hardy space admits a reproducing kernel $K \colon \Xi^0 \times \Xi^0 \to \mathbb{C}$, called the *Cauchy-Szegö kernel*. The function $\Xi^0 \times \Xi^0 \ni (z, w) \mapsto K(z, w)$ extends in the second variable to a smooth function on Ξ and hence $\Theta(m) := K(m, x_0)$ is a well defined positive definite function on Ξ^0

The case where G^c is semisimple and χ is trivial was discussed in detail in [Ól98]. In particular, it was shown that if $f \in \mathcal{H}^2(\Xi^0)^\infty$, then $b(f)$ is a pointwise limit along the center of $\mathfrak{k}$. By [HÓØ91] it is known that $\mathcal{H}^2(\Xi^0) = \bigoplus \mathcal{H}_\lambda$, where $\mathcal{H}_\lambda$ is the holomorphic discrete series representation constructed in [ÓØ91] and λ stands for the highest weight of the minimal K-type. Define the H-spherical distribution Θ_λ on $H\backslash G$ by

$$C_c^\infty(H\backslash G) \ni f \mapsto p_\lambda(f)(x_0) \in \mathbb{C}$$

where p_λ stands for the orthogonal projection onto $\mathcal{H}_\lambda$. Note that Θ_λ is the boundary value of the spherical character defined in (5.1) for $\chi = \mathbf{1}$. It is proved in [Ól98] using the results in [Kr01] that this function is just $d(\lambda)\varphi_{\lambda+\rho}$. Here $d(\lambda)$ is the analytic continuation of the formal dimension of $\mathcal{H}_\lambda$ when realized in $L^2(G^c)$. The Harish-Chandra type expansion of φ_λ in [Ól98]

$$\varphi_\lambda(a) = \sum_{w \in \mathcal{W}} c(w.\lambda)\varphi_{w.\lambda}(a), \tag{5.4}$$

therefore gives a formula for the H-spherical distribution character of $\mathcal{H}_\lambda$ generalizing the corresponding results of Harish-Chandra for the group case. It follows also that Θ decomposes as $\Theta = \sum d(\lambda)\varphi_{\lambda+\rho}$. It is therefore natural to conjecture that in the general case Θ has a similar decomposition:

$$\Theta(m) = \sum_{\substack{\lambda \text{ unitary highest weight} \\ \lambda+\rho \in -\operatorname{int} C_{\min}^\star, \\ \lambda \in -(W \cap \mathfrak{a})^\star}} d(\lambda)\varphi_{\lambda+\rho}^{\Delta_H^{-\frac{1}{2}}}(m), \tag{5.5}$$

where again $d(\lambda)$ is the formal dimension of $(\pi_\lambda, \mathcal{H}_\lambda)$ when realized as a subrepresentation of $L^2(G^c)$ (cf. [Ne97b] for the formula for $d(\lambda)$). Note that for small parameters the right-hand side in (5.5) is actually an analytic continuation as explained in [Kr99] (the poles of $\varphi_{\lambda+\rho}$ get canceled by the zeros of $d(\lambda)$). In particular, we can now see how our Factorization Theorem III.10 (see also Remark III.11) yields important information on the Cauchy-Szegö function of the Hardy space. It is also an open problem to show that the character formula (5.4) holds for other discrete series representations of $H\backslash G^c$.

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