REPRESENTATION THEORY
An Electronic Journal of the American Mathematical Society
Volume 5, Pages 164–190 (June 15, 2001)
S 1088-4165(01)00103-0

TENSOR PRODUCTS OF MINIMAL HOLOMORPHIC REPRESENTATIONS

GENKAI ZHANG

Abstract. Let $D = G/K$ be an irreducible bounded symmetric domain with genus p and $H^\nu(D)$ the weighted Bergman spaces of holomorphic functions for $\nu > p-1$. The spaces $H^\nu(D)$ form unitary (projective) representations of the group G and have analytic continuation in ν; they give also unitary representations when ν in the Wallach set, which consists of a continuous part and a discrete part of r points. The first non-trivial discrete point $\nu = \frac{a}{2}$ gives the minimal highest weight representation of G. We give the irreducible decomposition of tensor product $H^{\frac{a}{2}} \otimes \overline{H^{\frac{a}{2}}}$. As a consequence we discover some new spherical unitary representations of G and find the expansion of the corresponding spherical functions in terms of the K-invariant (Jack symmetric) polynomials, the coefficients being continuous dual Hahn polynomials.

Introduction

Let $D = G/K$ be an irreducible bounded symmetric domain of rank r in a complex vector space V with Lebesgue measure $dm(z)$. The Bergman reproducing kernel of D is of the form $h(z,w)^{-p}$, where p is the genus of D. Let $\nu > p-1$ and consider the weighted Bergman space H^ν with the weighted measure $h(z,z)^{\nu-p}dm(z)$. They give naturally unitary representations of the group G and have analytic continuation in the parameter ν. The set of those ν for which H^ν still form unitary representations is called the Wallach set and has been determined by various methods ([25], [30] and [5]). It is a union of an open interval and a discrete set, the last point in the discrete set is $\nu = 0$ and corresponds to the trivial representation. Suppose that the rank r of D is bigger than 1. The other points in the discrete Wallach set correspond to some singular representations of G; the K-types appearing in the representations form some lower dimensional lattices. The first discrete point $\nu = \frac{a}{2}$ above the trivial point $\nu = 0$ gives the minimal representation and the lattice of K-types is one-dimensional. Minimal and singular representations are of considerable interest since they normally cannot be constructed by standard methods. One may well expect that the representations appearing in the tensor product decomposition are also some minimal (singular) representations, thus it is worthwhile to study. Indeed we discover some new irreducible unitary (minimal) representations that appear in the decomposition. We also find the annihilating

Received by the editors May 23, 2000 and, in revised form, April 10, 2001.
2000 *Mathematics Subject Classification.* Primary 22E46, 47A70, 32M15, 33C52.
Key words and phrases. Bounded symmetric domains, weighted Bergman spaces, unitary highest weight representations, invariant differential operators, tensor product, irreducible decomposition, Clebsch-Gordan coefficients.
Research supported by the Swedish Natural Science Research Council (NFR).

invariant differential operators of the tensor product; we find the expansion of a family of spherical functions in terms of the K-invariant polynomials, which are the Jack symmetric polynomials, the coefficient being the continuous dual Hahn polynomials.

To give a brief background we consider first the case of the unit disk $D = G/K = SU(1,1)/SO(2)$ in the complex plane. The Hilbert spaces H^ν in question are the ones with reproducing kernels $(1 - z\bar{w})^{-\nu}$, $\nu \geq 0$, with $\nu = 0$ giving the trivial (and minimal) representation; so the problem of the tensor product of the minimal representations in this case is trivial; however, the consideration for other parameters of ν will give us ideas for treating higher rank cases. In earlier papers [33] and [22] we studied the explicit spectral decomposition of the tensor products. The main idea there is to study the restriction operator R, $RF(z) = (1 - |z|^2)^\nu F(z,z)$ from the tensor product $H^\nu \otimes \overline{H^\nu}$ to the space $C^\infty(D)$, which was considered earlier also by Repka [23] and [24] (an idea due to Howe, see loc. cit.); the operator R intertwines the tensor product action with the regular action of G. We consider further its polar decomposition, $R = |R|U$. The operator R is bounded and has dense range in the space $L^2(D)$ with the G-invariant measure, for $\nu > 1$. Thus for those ν the operator U is a unitary intertwining operator onto the space $L^2(D)$, whose decomposition is given by the known spherical transform ([7], Introduction). However, for smaller values of ν, the above polar decomposition does not make sense. Let $\phi_\lambda(z)$ be the spherical function on the unit disk. Our idea is simply to consider the power series expansion of the function $(1 - |z|^2)^{-\nu}\phi_\lambda(z)$,

$$(1 - z\bar{z})^{-\nu}\phi_\lambda(z) = \sum_{m=0}^{\infty} p_{\nu,m}(\lambda)(z\bar{z})^m.$$

Conceptually the l.h.s. is the restriction to diagonal (z,z) of the eigenfunction $R^{-1}\phi_\lambda$ of the Casimir element on the tensor product, and the formula is its expansion in terms of the $K = SO(2)$-invariant elements $(z\bar{w})^m$ in the tensor product. The action of the Casimir element is equivalent to a multiplication by $(\lambda^2 + \frac{1}{4})$ on the coefficients $p_{\nu,m}(\lambda)$. It turns out that $p_{\nu,m}(\lambda)$ are the continuous dual Hahn polynomials, whose orthogonality relation has been proved by Wilson (see [31], [1] and [14]). So by using the orthogonality relation, we found in [20] the irreducible decomposition of the tensor product $H^\nu \otimes \overline{H^\nu}$ for all $\nu > 0$.

Consider a general irreducible bounded symmetric domain $D = G/K$ of rank $r \geq 2$. By a general consideration we know that the representations appearing in the decomposition of the tensor product $H^\nu \otimes \overline{H^\nu}$ are spherical, for all ν in the Wallach set. Our interests will be the tensor product when ν is in the discrete Wallach set. So let $\nu = \frac{a}{2}(j-1)$ be such a point. We consider the expansion of the functions $R^{-1}\phi_{\underline{\boldsymbol{\lambda}}}$ in terms of the K-invariant polynomials; more precisely

$$h^{-\nu}(z,z)\phi_{\underline{\boldsymbol{\lambda}}}(z) = \sum_{\underline{\mathbf{m}}} p_{\underline{\mathbf{m}}}(\underline{\boldsymbol{\lambda}}) K_{\underline{\mathbf{m}}}(z,z), \tag{0.1}$$

in a neighborhood of $z = 0$. (See Section 1 for the definition of $K_{\underline{\mathbf{m}}}$.) This formula makes sense for all ν. For any invariant differential operator $\mathcal{M} \in \mathcal{D}^G(D)$ with eigenvalue $\mathcal{M}(\underline{\boldsymbol{\lambda}})$ on the spherical function $\phi_{\underline{\boldsymbol{\lambda}}}$, the invariant differential operator $R^{-1}\mathcal{M}R$ on the basis vectors $\{K_{\underline{\mathbf{m}}}(z,w)\}$ is then unitarily equivalent to the multiplication operator by $\mathcal{M}(\underline{\boldsymbol{\lambda}})$ on the coefficients $\{p_{\underline{\mathbf{m}}}(\underline{\boldsymbol{\lambda}})\}$, see Proposition 2.5. Our problem will be partly to identify those polynomials.

The algebra $\mathcal{D}^G(D)$ is commutative with r generators. In [27] Shimura constructs an r-tuple of generators $(\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_r)$. Later the author [36] finds the eigenvalues of the generators on the spherical functions. When $\nu = \frac{a}{2}$ we prove that the image $R(H^{\frac{a}{2}} \otimes \overline{H^{\frac{a}{2}}})$ of the tensor product under R is annihilated by the operators $\mathcal{L}_2, \dots, \mathcal{L}_r$; see Proposition 4.1. By our early results [36] on the eigenvalues of the Shimura operators we know that a spherical function $\phi_{\underline{\boldsymbol{\lambda}}}$ is annihilated by those Shimura operators if and only $\underline{\boldsymbol{\lambda}}$ is in a certain one-dimensional hyperplane. For those $\phi_{\underline{\boldsymbol{\lambda}}}$ we find the above expansion, where only those $\underline{\mathbf{m}} = (m, 0, \dots, 0)$ appear. The coefficients turn out also to be the continuous dual Hahn polynomials, their orthogonality relation then gives the spectrum of the multiplication operator by $\mathcal{L}_1(\underline{\boldsymbol{\lambda}})$ and thus the operator of $R^{-1}\mathcal{L}_1 R$; see Theorem 6.1. For type I domain $SU(2,2)/(S(U(2)\times U(2))$, this has been done in [21] by using the explicit (Berezin's) formula for spherical functions [9].

It turns out that when (and only when) D is a non-tube domain of type one $SU(r, r+b)/S(U(r) \times U(r+b))$ with $b \geq 2$ there are discrete parts, to be called complementary series, appearing; and they naturally deserve further study. Viewing the tensor product as the space of Hilbert-Schmidt operators we thus get a *quantization* of the complementary series; see Theorem 7.1. In fact, we find that a larger family of spherical representations can be quantized as operators on $H^{\frac{a}{2}}$ of the Schatten-von Neumann class $\mathfrak{S}_q$, so that we get some invariant Banach spaces (instead of Hilbert spaces) generated by the spherical functions. See also [4] for the rank one case.

We mention that the tensor product decomposition has been an important method in producing new representations, and has been studied extensively in the literature, in particular, in its relation to dual pairs; see [11]. In [3] Sahi and Dvorsky study the tensor products $H_1 \otimes \cdots \otimes H_l$ of several more general singular representations and construct dual pairs. See also [8], [17], [18] and the references therein. Also, there is some renewed interest in complementary series in connection with some other analytical problems; see e.g. [12].

The paper is organized as follows. In Section 1 we recall some well-known results on holomorphic spaces on bounded symmetric domains, and we prove a decomposition result on the point-wise product of two irreducible polynomial spaces. In Section 2 we incorporate the known results on the tensor product of $H^\nu \otimes \overline{H^\nu}$ for regular parameter ν. Sections 3 and 4 are devoted to Shimura invariant differential operators and their annihilating property. In Sections 5 and 6 we find the irreducible decomposition of the tensor product. In Section 7 we study, in particular, the complementary series appearing in the decomposition. Section 8 is devoted to the proof of positive definiteness of the spherical functions, which, with the help of our explicit formula for the Clebsch-Gordan coefficients and their orthogonality relation, is straightforward and of an abstract nature. For the convenience of the reader we list some known properties of the continuous dual Hahn polynomials in the last section.

Acknowledgement. I would like to thank Miroslav Engliš for his careful reading of an earlier version of the manuscript and for some helpful remarks, and Bent Ørsted for his encouragement. It is also my pleasure to thank the anonymous referee for his or her clarifying remarks, in particular, for bringing the reference [16] to my attention.

1. Bounded symmetric domains and the polynomial spaces

In this section we fix notations and recall some necessary results on bounded symmetric domains. The notation and setup will be the same as in [36] and [35], so that we will be very brief and the unexplained notation can be found there.

Let G/K be an irreducible Hermitian symmetric space. It can be realized as a bounded convex domain D in a complex n-dimensional space V with G realized as the identity component of the group of biholomorphic mappings and K the isotropic subgroup of $0 \in D$. Let $\mathfrak{g}$ be the Lie algebra of G, and let $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ be its Cartan decomposition. The Lie algebra $\mathfrak{k}$ has one-dimensional center. Let $\mathfrak{g}^{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p}^-$ be the corresponding eigenspace decomposition of $\mathfrak{g}^{\mathbb{C}}$, the complexification of $\mathfrak{g}$. The vector space $V = \mathfrak{p}^+$ can be identified with the holomorphic tangent space.

The vector space V has a structure of a Jordan triple system so that $\mathfrak{p} = \{\xi_v(z) = v - Q(z)\bar{v}; v \in V\}$ where $Q(z) \in Aut(\bar{V}, V)$ is a quadratic operator. We normalize the K-invariant inner product $\langle z, w\rangle$ on V as in [35], so that a minimal tripotent has norm 1.

We fix $\{e_1, \dots, e_r\}$ as a frame of V and $\mathfrak{a} = \mathbb{R}\xi_{e_1} + \cdots + \mathbb{R}\xi_{e_r}$. Then $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$ with basis vectors $\xi_{e_1}, \xi_{e_2}, \cdots, \xi_{e_r}$. Let $\{\beta_j\}_{j=1}^r \subset \mathfrak{a}^*$ be the basis of $\mathfrak{a}^*$ determined by

$$\beta_j(\xi_{e_k}) = 2\delta_{j,k}, \; 1 \le j, k \le r,$$

and define an ordering on $\mathfrak{a}^*$ via

$$\beta_r > \beta_{r-1} > \cdots > \beta_1 > 0.$$

We will write an element $\underline{\boldsymbol{\lambda}} \in (\mathfrak{a}^*)^{\mathbb{C}}$ as

$$\underline{\boldsymbol{\lambda}} = \sum_{j=1}^r \lambda_j \beta_j,$$

and identify $\underline{\boldsymbol{\lambda}}$ with $(\lambda_1, \lambda_2, \cdots, \lambda_r)$. The half sum of the positive roots is given by

$$\underline{\boldsymbol{\rho}} = \sum_{j=1}^r \rho_j \beta_j = \sum_{j=1}^r \frac{b + 1 + a(j-1)}{2} \beta_j, \tag{1.1}$$

where a is the root multiplicity of $\frac{\beta_j \pm \beta_k}{2}$ and $2b$ the root multiplicity of $\frac{\beta_j}{2}$.

Let $\mathcal{P}(V)$ be the space of all holomorphic polynomials on V. The group K acts naturally on $\mathcal{P}(V)$ induced from its regular action on V. Its irreducible decomposition is now well known; see [10], [26] and [5]. To state the result we let $\mathfrak{h}^{\mathbb{C}}$ be a Cartan subalgebra of $\mathfrak{k}^{\mathbb{C}}$ that contains the elements $D(e_j, e_j)$, $j = 1, 2, \dots, r$. Let $\gamma_1 > \gamma_2 \cdots > \gamma_r$ be the corresponding Harish-Chandra strongly orthogonal roots. Thus $\gamma_k(D(e_j, e_j)) = 2\delta_{jk}$. The space $V = \mathfrak{p}^+$ is now of highest weight γ_1 with highest weight vector e_1; and dual space $V' = \mathfrak{p}^-$ is of lowest weight $-\gamma_1$. The subspace $\mathcal{P}_m$ of homogeneous polynomial of degree m is decomposed into irreducible subspaces with multiplicity one as

$$\mathcal{P}_m = \sum_{\underline{\mathbf{m}}} \mathcal{P}_{\underline{\mathbf{m}}}$$

where each $\mathcal{P}_{\underline{\mathbf{m}}}$ is of lowest weight $-\underline{\mathbf{m}} = -(m_1\gamma_1 + \cdots + m_r\gamma_r)$ with $m_1 \ge m_2 \ge \cdots \ge m_r \ge 0$ being integers, and the summation is over all $\underline{\mathbf{m}}$ with $|\underline{\mathbf{m}}| = m_1 + m_2 + \cdots + m_r = m$.

We define a K-invariant function $h(z)$ on D so that

$$h(c_1e_1 + c_2e_2 + \cdots + c_re_r) = (1 - |c_1|^2)(1 - |c_2|^2)\dots(1 - |c_r|^2)$$

and let $h(z, w)$ be its polarization, holomorphic in z and antiholomorphic in w so that $h(z, z) = h(z)$. Consider the weighted probability measure

$$d\mu_\nu(z) = c_\nu h(z)^{\nu - p} dm(z) \tag{1.2}$$

with $\nu > p - 1$ and c_ν a normalization constant. We denote H^ν the corresponding weighted Bergman space; it has reproducing kernel $h(z, w)^{-\nu}$.

The group G acts unitarily on H^ν via

$$\pi(\nu)(g)f(z) = (J_{g^{-1}}(z))^{\frac{\nu}{p}} f(g^{-1}z), \quad g \in G, \tag{1.3}$$

and it gives irreducible unitary (projective) representation of G. One may also consider more generally the actions of G on vector-valued C^∞-functions on D; see (3.1) below.

We recall now the Faraut-Koranyi expansion of the reproducing kernel $h(z, w)^{-\nu}$. Let $K_{\underline{\mathbf{m}}}$ be as in [5] the reproducing kernel of the subspace $\mathcal{P}_{\underline{\mathbf{m}}}$ with the Fock norm.

Theorem 1.1 (Faraut and Koranyi [5], Theorem 3.8)**.** *The function $h(z, w)^{-\nu}$ has the expansion*

$$h^{-\nu}(z, w) = \sum_{\underline{\mathbf{m}}} (\nu)_{\underline{\mathbf{m}}} K_{\underline{\mathbf{m}}}(z, w) \tag{1.4}$$

for all $\nu \in \mathbb{C}$, and the convergence is uniform on compact subsets of $D \times D$. Here

$$(\nu)_{\underline{\mathbf{m}}} = \prod_{j=1}^{r} (\nu - \frac{a}{2}(j-1))_{m_j} = \prod_{j=1}^{r} \prod_{k=1}^{m_j} (\nu - \frac{a}{2}(j-1) + k - 1).$$

It follows from this expansion that the kernel $h^{-\nu}(z, w)$ is positive definite and defines a Hilbert space if and only if ν is in

$$W(D) = \{0, \frac{a}{2}, \dots, \frac{a}{2}(r-1)\} \cup (\frac{a}{2}(r-1), \infty), \tag{1.5}$$

also called the Wallach set.

If $\nu = \frac{a}{2}(j-1)$ in the discrete Wallach set, only certain subspaces $\mathcal{P}_{\underline{\mathbf{m}}}$ are in the Hilbert space $H^\nu = H^{\frac{a}{2}(j-1)}$; more precisely,

$$H^{\frac{a}{2}(j-1)} = \sum_{\underline{\mathbf{m}}: m_j = 0} \mathcal{P}_{\underline{\mathbf{m}}}. \tag{1.6}$$

Moreover, it forms an irreducible representation of G with the action $\pi(\nu)$. In particular, the algebraic sum of all $\mathcal{P}_{\underline{\mathbf{m}}}$ with $m_j = 0$ forms an irreducible representation of the Lie algebra $\mathfrak{g}^{\mathbb{C}}$.

The next lemma will be used in the proof of Theorem 5.2.

Lemma 1.2. *Let $1 \le j \le r - 1$. Consider the product $\mathcal{P}_{\underline{\mathbf{m}}} \cdot \mathcal{P}_{\underline{\mathbf{m}}'}$ (consisting of the sum of point-wise products of two polynomials in the respective spaces) of a subspace $\mathcal{P}_{\underline{\mathbf{m}}}$ with signature $\underline{\mathbf{m}} = (m_1, 0, \dots, 0)$ and $\mathcal{P}_{\underline{\mathbf{m}}'}$ with $\underline{\mathbf{m}}' = (m'_1, \dots, m'_{j-1}, 0, \dots, 0)$. Let*

$$\mathcal{P}_{\underline{\mathbf{m}}} \cdot \mathcal{P}_{\underline{\mathbf{m}}'} = \sum_{\underline{\mathbf{n}}} \mathcal{P}_{\underline{\mathbf{n}}}$$

be its irreducible decomposition under K (which is multiplicity free by Theorem 1.1). Then the signatures $\underline{\mathbf{n}} = (n_1, \dots, n_r)$ that appear in the decomposition satisfy $n_{j+1} = 0$.

To prove the lemma we give a more general result on tensor products of Hilbert spaces with reproducing kernels; it might have been proved previously, but we include here a simple proof. (Similar arguments have also been used in studying the tensor product of a holomorphic Hilbert space with its conjugate; see e.g. [4].)

Let temporarily $\mathcal{H}(K_1)$ and $\mathcal{H}(K_2)$ be two Hilbert spaces of holomorphic functions on a bounded (not necessarily symmetric) domain D in $\mathbb{C}^n$ with reproducing kernels $K_1(z, \xi)$ and $K_2(z, \xi)$, so that the point evaluation is a continuous linear functional. Thus the reproducing kernels $K_1(z, \xi)$ and $K_2(z, \xi)$ are positive definite. The point-wise (Schur) product $K_1K_2(z, \xi) = K_1(z, \xi)K_2(z, \xi)$ is again semi-positive definite. Let $\mathcal{H}(K_1K_2)$ denote the corresponding Hilbert space determined by the reproducing kernel and $H(D)$ the space of all holomorphic functions on D. The tensor product $\mathcal{H}(K_1)\otimes\mathcal{H}(K_2)$ can be realized as a space of holomorphic functions $F(z, w)$ in two variables.

Lemma 1.3. *Consider the operator $R : \mathcal{H}(K_1)\otimes\mathcal{H}(K_2) \to H(D)$ by the restriction to the diagonal $Rf(z) = f(z, z)$. Then R extends to an isometry from $(\operatorname{Ker} R)^\perp$ onto $\mathcal{H}(K_1K_2)$.*

Proof. Clearly $\operatorname{Ker} R$ is a closed subspace of $\mathcal{H}(K_1) \otimes \mathcal{H}(K_2)$ by the continuity of the point evaluation. Thus R defines a one-to-one map from $(\operatorname{Ker} R)^\perp$ into $H(D)$. Consider the elements in $\mathcal{H}(K_1) \otimes \mathcal{H}(K_2)$ of the form $g = \sum c_j K_1(z, \xi_j)K_2(w, \xi_j)$. First, these elements are in $(\operatorname{Ker} R)^\perp$; indeed for any $f \in \operatorname{Ker} R$, by the reproducing property,

$$\langle f, g\rangle = \sum \bar{c_j} f(\xi_j, \xi_j) = 0.$$

Second, they are dense in $(\operatorname{Ker} R)^\perp$, since if $g_0 \in (\operatorname{Ker} R)^\perp$ is orthogonal to all g, it is, in particular, orthogonal to $g = K_1(z, \xi)K_2(w, \xi)$ for any fixed ξ, and

$$Rg_0(\xi) = g_0(\xi, \xi) = \langle g_0, g\rangle = 0,$$

namely $g_0 \in \operatorname{Ker} R \cap (\operatorname{Ker} R)^\perp = \{0\}$.

The images of g are

$$Rg(z) = \sum c_j K_1(z, \xi_j)K_2(z, \xi_j),$$

and they form a dense subspace of $\mathcal{H}(K_1K_2)$. Moreover, R is an isometry, again by the reproducing property. Thus R extends to an isometry from the closure of those g, which is $(\operatorname{Ker} R)^\perp$, onto $\mathcal{H}(K_1K_2)$. $\square$

Specializing the above result with $\mathcal{H}(K_1)$ and $\mathcal{H}(K_2)$ replaced by $H^{-\frac{a}{2}}$ respectively $H^{-\frac{a}{2}(j-1)}$ we see that the products $\mathcal{P}_{\underline{\mathbf{m}}} \cdot \mathcal{P}_{\underline{\mathbf{m}}'}$ of two irreducible subspaces in $H^{-\frac{a}{2}}$ and $H^{-\frac{a}{2}(j-1)}$ is a subspace in $H^{-\frac{a}{2}j}$, whose decomposition under K is given by the Faraut-Koranyi expansion in Theorem 1.1.

2. Tensor product $\pi(\nu) \otimes \overline{\pi(\nu)}$ and the Berezin transform: Some general results

The tensor product of $\pi(\nu) \otimes \overline{\pi(\nu)}$ for the large parameter $\nu > p - 1$ has been studied in several contexts; see [19], [37] and the reference therein. We recall some of the results in the literature. Consider the tensor product $H^\nu \otimes \overline{H^\nu}$, realized as the space of Hilbert-Schmidt operators F on H^ν with kernel $F(z,w)$ holomorphic in z and anti-holomorphic in w. The group G acts on the tensor product via $\pi(\nu)\otimes\overline{\pi(\nu)}$, and it gives a (genuine) representation. To study the irreducible decomposition we consider the map $R : H^\nu \otimes \overline{H^\nu} \to C^\infty(D)$, defined by

$$RF(z) = F(z,z)h(z,z)^\nu. \tag{2.1}$$

Then R intertwines the action $\pi(\nu)\otimes\overline{\pi(\nu)}$ with the regular action $\pi(0)$ on $C^\infty(D)$. Clearly R is one-to-one. Note that the inverse operator R^{-1} is defined on the space of all real analytic functions on D and is essentially the so-called polarization. (However, generally, $R^{-1}f(z,w)$ for a real analytic function f is defined only on a small set near the diagonal, so it is not a function on $D\times D$; we thank Bo Berndtsson and Miroslav Engliš for some kind correspondence. In our case we shall only consider the operator R^{-1} as defined on $R(H^\nu \otimes \overline{H^\nu})$.) When considering functions $F(z,w)$ holomorphic in z and anti-holomorphic in w we will frequently identify F with its restriction $F(z,z)$ to the diagonal and simply write $F(z)$.

Let $L^2(D)$ be the L^2-space on D with respect to the G-invariant measure $\frac{dm(z)}{h(z,z)^p}$. When $\nu > p-1$ the operator R is an injective bounded operator into the space $L^2(D)$ with dense image; the decomposition of the later space is well known by the theory of spherical transform. Moreover, the operator RR^* is actually the Berezin transform on $L^2(D)$. However, for smaller ν the operator R is no longer bounded and the above method does not work. Our approach is to find directly the decomposition of the tensor product without referring to the bounded property of R. For that purpose we first establish some general results about the nature of the eventual irreducible decomposition.

Lemma 2.1. *Let ν be in the Wallach set* (1.5). *Suppose*

$$\pi(\nu)\otimes\overline{\pi(\nu)} \equiv \int_\Sigma \pi(\tau)d\mu(\tau)$$

is the irreducible decomposition of $\pi(\nu)\otimes\overline{\pi(\nu)}$. Then for almost all τ the representations $\pi(\tau)$ are spherical.

Proof. Take a subset $\Delta\subset\Sigma$, with $\mu(\Delta)\neq 0$ and let P_Δ be the orthogonal projection onto $\int_\Delta \pi(\tau)d\mu(\tau)$. The space $H^\nu\otimes\overline{H^\nu}$ has $1\otimes 1$ as a cyclic vector under the action $\pi(\nu)\otimes\overline{\pi(\nu)}$ as easily seen by using the reproducing property (see e.g. [4], see also [21] in terms of the universal enveloping algebra). Thus $P_\Delta(1\otimes 1)$ is a K-invariant element of $\int_\Delta \pi(\tau)d\mu(\tau)$ and is cyclic; moreover, $P_\Delta(1\otimes 1)\neq 0$ since $\int_\Delta \pi(\tau)d\mu(\tau)\neq 0$. This proves our claim. □

Denote $(H^\nu\otimes\overline{H^\nu})_0$ the subspace of K-invariant elements. In that subspace there is an orthogonal basis given by $K_{\underline{\mathbf{m}}}(z,w)$, the spherical transforms of their images $R(K_{\underline{\mathbf{m}}})$ under R provide then the Clebsch-Gordan coefficients; moreover, they give also the coefficients in the expansion of the spherical function in terms of $K_{\underline{\mathbf{m}}}$; see below.

We let

$$E_{\underline{\mathbf{m}}}(z,z) = E_{\underline{\mathbf{m}},\nu}(z,z) = (\nu)_{\underline{\mathbf{m}}}K_{\underline{\mathbf{m}}}(z,z), \tag{2.2}$$

$$e_{\underline{\mathbf{m}}}(z) = e_{\underline{\mathbf{m}},\nu}(z,z) = \frac{E_{\underline{\mathbf{m}},\nu}}{d_{\underline{\mathbf{m}}}^{\frac{1}{2}}} = \frac{(\nu)_{\underline{\mathbf{m}}}K_{\underline{\mathbf{m}}}(z,z)}{d_{\underline{\mathbf{m}}}^{\frac{1}{2}}}, \tag{2.3}$$

where $d_{\underline{\mathbf{m}}} = \dim \mathcal{P}_{\underline{\mathbf{m}}}$. The following result is then a direct consequence of (and in fact is equivalent to) the expansion (1.4).

Lemma 2.2. *If $\nu > \frac{a}{2}(r-1)$, then the functions $e_{\underline{\mathbf{m}},\nu}$ form an orthonormal basis of $(H_\nu \otimes \overline{H_\nu})_0$; if $\nu = \frac{a}{2}(j-1)$ for some $j = 1, \dots, r$, then the functions $e_{\underline{\mathbf{m}},\nu}$ for $\underline{\mathbf{m}} = (m_1, \dots, m_{j-1}, 0, \dots, 0)$ form an orthonormal basis of $(H_\nu \otimes \overline{H_\nu})_0$.*

The next result is proved in [35] (see (5.5) there). Let $\phi_{\underline{\boldsymbol{\lambda}}}(z)$ be the spherical function D. We denote $\widehat{f}(\underline{\boldsymbol{\lambda}})$ the spherical transform of a K-invariant function f on D,

$$\widehat{f}(\underline{\boldsymbol{\lambda}}) = \int_D f(z)\phi_{\underline{\boldsymbol{\lambda}}}(z)\frac{dm(z)}{h(z,z)^p}.$$

The constant c_ν below is that in (1.2). Note first that the function $h^{-\nu}(z)\phi_{\underline{\boldsymbol{\lambda}}}(z)$ is formally the restriction to the diagonal of the polarization $R^{-1}\phi_{\underline{\boldsymbol{\lambda}}}$ of $\phi_{\underline{\boldsymbol{\lambda}}}$.

Lemma 2.3. *Consider the power series expansion of $h^{-\nu}(z)\phi_{\underline{\boldsymbol{\lambda}}}(z)$*

$$(R^{-1}\phi_{\underline{\boldsymbol{\lambda}}})(z) = h^{-\nu}(z)\phi_{\underline{\boldsymbol{\lambda}}}(z) = \sum_{\underline{\mathbf{m}}} \mathcal{E}_{\underline{\mathbf{m}},\nu}(\underline{\boldsymbol{\lambda}})E_{\underline{\mathbf{m}}}(z,z) \tag{2.4}$$

in terms of the K-invariant polynomials $E_{\underline{\mathbf{m}}}(z,z)$. Let

$$\varepsilon_{\underline{\mathbf{m}}}(\underline{\boldsymbol{\lambda}}) = \varepsilon_{\underline{\mathbf{m}},\nu}(\underline{\boldsymbol{\lambda}}) = d_{\underline{\mathbf{m}}}^{\frac{1}{2}}\mathcal{E}_{\underline{\mathbf{m}},\nu}(\underline{\boldsymbol{\lambda}}). \tag{2.5}$$

Suppose $\nu > p-1$. Then the coefficients $\mathcal{E}_{\underline{\mathbf{m}},\nu}(\underline{\boldsymbol{\lambda}})$ can be obtained by the spherical transform,

$$b_\nu(\underline{\boldsymbol{\lambda}})d_{\underline{\mathbf{m}}}\mathcal{E}_{\underline{\mathbf{m}},\nu}(\underline{\boldsymbol{\lambda}}) = (\widehat{c_\nu h^\nu K_{\underline{\mathbf{m}}}})(\underline{\boldsymbol{\lambda}}) \tag{2.6}$$

and

$$b_\nu(\underline{\boldsymbol{\lambda}})\varepsilon_{\underline{\mathbf{m}},\nu}(\underline{\boldsymbol{\lambda}}) = (\widehat{c_\nu h^\nu e_{\underline{\mathbf{m}}}})(\underline{\boldsymbol{\lambda}}). \tag{2.7}$$

Remark 2.4. The above expansion (2.4) is valid a priori only in a neighborhood of $z = 0$. There arises naturally an interesting question as whether it holds for all $z \in D$.

Proposition 2.5. *Suppose ν is in the Wallach set* (1.5)*. Let $\mathcal{M}$ be an invariant differential operator on $C^\infty(D)$ with eigenvalue $\mathcal{M}(\underline{\boldsymbol{\lambda}})$ on the spherical function $\phi_{\underline{\boldsymbol{\lambda}}}$. Then the invariant differential operator $R^{-1}\mathcal{M}R$ on the orthonormal basis $\{e_{\underline{\mathbf{m}}}(z)\}$ of $(H^\nu \otimes \overline{H^\nu})_0$ and the multiplication operator $\mathcal{M}(\underline{\boldsymbol{\lambda}})$ on the system of polynomials $\varepsilon_{\underline{\mathbf{m}}}(\underline{\boldsymbol{\lambda}})$ have the same matrix form. Namely, if*

$$(R^{-1}\mathcal{M}R)e_{\underline{\mathbf{m}}}(z) = \sum_{\underline{\mathbf{m}}'} a_{\underline{\mathbf{m}}}(\underline{\mathbf{m}}')e_{\underline{\mathbf{m}}'}(z), \tag{2.8}$$

then

$$\mathcal{M}(\underline{\boldsymbol{\lambda}})\varepsilon_{\underline{\mathbf{m}},\nu}(\underline{\boldsymbol{\lambda}}) = \sum_{\underline{\mathbf{m}}'} a_{\underline{\mathbf{m}}}(\underline{\mathbf{m}}')\varepsilon_{\underline{\mathbf{m}}',\nu}(\underline{\boldsymbol{\lambda}}). \tag{2.9}$$

Proof. Suppose $\nu > p-1$. The matrix form (2.8) can also be written as, after multiplying by the constant c_ν,

$$\mathcal{M}(c_\nu h^\nu e_{\underline{\mathbf{m}}})(z) = \sum_{\underline{\mathbf{m}}'} a_{\underline{\mathbf{m}}}(\underline{\mathbf{m}}')c_\nu h^\nu(z)e_{\underline{\mathbf{m}}'}(z).$$

We perform the spherical transform on the equation and use (2.6). The l.h.s. then becomes $\mathcal{M}(\underline{\boldsymbol{\lambda}})b_\nu(\underline{\boldsymbol{\lambda}})\varepsilon_{\underline{\mathbf{m}},\nu}(\underline{\boldsymbol{\lambda}})$ and the r.h.s. is

$$\sum_{\underline{\mathbf{m}}'} a_{\underline{\mathbf{m}}}(\underline{\mathbf{m}}')b_\nu(\underline{\boldsymbol{\lambda}})\varepsilon_{\underline{\mathbf{m}}',\nu}(\underline{\boldsymbol{\lambda}});$$

dividing both sides by $b_\nu(\underline{\boldsymbol{\lambda}})$ proves the claim for $\nu > p-1$. It is easily seen that the matrix forms (2.8) and (2.9) depend polynomially on the parameter ν. Thus it holds for all ν and $\underline{\mathbf{m}}$ whenever $e_{\underline{\mathbf{m}},\nu}$ is well-defined. $\square$

Remark 2.6. The above Proposition is a justification of a simple formal calculation. Let L be a self-adjoint operator on a Hilbert space, and $\{e_m\}$ an orthogonal basis so that each vector is in the domain of L. Suppose

$$Le_m = a_m e_{m-1} + b_m e_m + c_m e_{m+1}$$

and that $\psi = \sum_{m=0}^{\infty} f_m e_m$ is an eigenvector of L with eigenvalue λ (in some proper sense). Then the coefficients f_n satisfy the following recurrence relation

$$\lambda f_m = a_{m+1} f_{m+1} + b_m f_m + c_{m-1} f_{m-1}$$

which we may refer to as the dual relation to the former. On the other hand, if the two relations hold, then the series $\psi = \sum_{m=0}^{\infty} f_m e_m$ is an eigenvector of L whenever it makes sense. If, moreover, $\{e_m\}$ is an orthonormal basis, then $a_{m+1} = c_m$ since L is self-adjoint, and the two relations are exactly the same. (See Lemma 6.1 and Remark 6.2 concerning the self-adjointness of the conjugates of the Shimura operators by R.)

3. Invariant differential operators

We introduce the Shimura system of generators of all invariant differential operators on D. Let (X, τ) be a holomorphic representation of $K^{\mathbb{C}}$ on a finite dimensional vector space X. This then induces a homogeneous bundle W on D. A smooth section of the bundle will be identified with an element in $C^\infty(D, X)$, the space of X-valued C^∞-functions on $D = G/K$. The induced action of G on $C^\infty(D, X)$ is

$$\pi_X(g)f(z) = \tau(dg^{-1}(z))^{-1} f(g^{-1}z). \tag{3.1}$$

In particular, if $X = \mathbb{C}$ is one-dimensional with τ on $K^{\mathbb{C}}$ being $(\det(k))^{-\frac{\nu}{p}}$, we get the action (1.3) on $C^\infty(D)$.

We let $\mathcal{D} = \mathcal{D}_X$ be as in [36] the holomorphic covariant differentiation operator on $C^\infty(D, X)$. It maps $C^\infty(D, X)$ into $C^\infty(D, V' \otimes X)$, where V' is the dual space of V, viewed as the holomorphic cotangent space. More importantly, it intertwines the corresponding actions of G,

$$\mathcal{D}_X(\pi_X(g)f) = \pi_{V'\otimes X}(g)(\mathcal{D}_X f).$$

Let $\mathcal{D}_X^m = \mathcal{D}_{\bigotimes^{m-1} V'\otimes X} \dots D_{V'\otimes X} D_X$ be the iterate of $\mathcal{D}$. It has been proved in [36] that $\mathcal{D}^m$ actually maps $C^\infty(D, X)$ into $C^\infty(D, S_m(V') \otimes X)$, where $S_m(V')$ stands for the subspace of symmetric tensors of $\bigotimes^m V'$. The space $S_m(V')$, as a K-module, is equivalent to the space of all homogeneous polynomials of degree m, thus can be decomposed under K as irreducible subspaces $S_{\underline{\mathbf{m}}}(V')$, of signatures $\underline{\mathbf{m}} = (m_1, \dots, m_r)$; we let $P_{\underline{\mathbf{m}}}$ be the orthogonal projection onto the corresponding subspace. The operator $P_{\underline{\mathbf{m}}}\mathcal{D}^m$ thus maps into the sub-bundle $S_{\underline{\mathbf{m}}}(V') \otimes X$. The Shimura invariant differential operators on $C^\infty(D, X)$ are then defined by

$$\mathcal{L}_{\underline{\mathbf{m}}} = (\mathcal{D}_X^m)^* P_{\underline{\mathbf{m}}} \mathcal{D}_X^m,$$

where $(\mathcal{D}_X^m)^*$ is the Hilbert space adjoint. In the present paper we will only consider the operator $\mathcal{L}_{\underline{\mathbf{m}}}$ on the trivial line bundle on D.

The operator $\mathcal{D}$ has been previously studied by Shimura for classical domains. In particular, Shimura [28] has given an easier formula for the operators $P_{\underline{\mathbf{m}}}\mathcal{D}^m$ in terms of the Cayley-Capelli type operators, when $\underline{\mathbf{m}}$ are the fundamental representations $\underline{\mathbf{m}} = \underline{\mathbf{1}^j} = (1, \dots, 1, 0, \dots, 0)$. We specify his results (see Theorem 4.7 loc. cit.) to the special case of scalar-valued functions; it is proved there for classical domains and is generalized in [36] for all bounded symmetric domains.

Theorem 3.1. *Let $1 \le j \le r$ and $\varepsilon = \frac{a}{2}(j-1)$. The operator $P_{\underline{\mathbf{1}^j}}\partial^j$ has the following intertwining property*

$$P_{\underline{\mathbf{1}^j}}\partial^j(J_g^\varepsilon(z)f(gz)) = J_g(z)^{\frac{\varepsilon}{p}}(\otimes^j dg'(z))(P_{\underline{\mathbf{1}^j}}\partial^j f)(gz); \tag{3.2}$$

the operator $P_{\underline{\mathbf{1}^j}}\mathcal{D}^j$ on the space $C^\infty(D)$ with the regular action of G can be expressed in terms of $P_{\underline{\mathbf{1}^j}}\partial^j$ as

$$P_{\underline{\mathbf{1}^j}}\mathcal{D}^j f = h^{\frac{a}{2}(j-1)} P_{\underline{\mathbf{1}^j}}\partial^j(h^{-\frac{a}{2}(j-1)}f). \tag{3.3}$$

Consider the trivial line bundle on D. It has been proved by Shimura [27] that the operators

$$\mathcal{L}_j = \mathcal{L}_{\underline{\mathbf{1}^j}}, \quad j = 1, \dots, r \tag{3.4}$$

form a system of generators of the algebra of all invariant differential operators on $C^\infty(D)$.

Recall the intertwining operator R. Via conjugation by R we get r invariant differential operators

$$\mathcal{L}_{j,\nu} = R^{-1}\mathcal{L}_{\underline{\mathbf{1}^j}}R, \quad j = 1, \dots, r \tag{3.5}$$

on the tensor product $H^\nu \otimes \overline{H^\nu}$.

4. The invariant annihilating differential operators of the spaces H^ν and of the tensor products $H^\nu \otimes \overline{H^\nu}$ at discrete points

We shall prove in this section that the Shimura operators on $C^\infty(D)$ are annihilating differential operators of the image under R of the $H^\nu \otimes \overline{H^\nu}$ at the reducible points $\nu = \frac{a}{2}(j-1)$, $j = 1, \dots, r$. We mention that holomorphic differential operators that annihilate H^ν have been studied in [2] and [32].

Proposition 4.1. *Let $1 \le j_0 \le r$. Consider the intertwining operator $R = R_{\frac{a}{2}(j_0-1)}$ from the tensor product $H^{\frac{a}{2}(j_0-1)} \otimes \overline{H^{\frac{a}{2}(j_0-1)}}$ into $C^\infty(D)$. Its image $R(H^{\frac{a}{2}(j_0-1)} \otimes \overline{H^{\frac{a}{2}(j_0-1)}})$ is annihilated by the Shimura invariant differential operators $\mathcal{L}_j$ for $j_0 \le j \le r$.*

To prove the proposition we need the following

Lemma 4.2. *Let $1 \le j \le r$. The operator $P_{\underline{\mathbf{1}^j}}\partial^j$ annihilates all polynomial spaces $\mathcal{P}_{\underline{\mathbf{m}}}$ with $\underline{\mathbf{m}} = (m_1, \dots, m_r)$ and $m_j = 0$.*

Proof. The space $H^{\frac{a}{2}(j-1)}$ is an irreducible representation of G with the action $\pi(\frac{a}{2}(j-1))$. Consider its subspace of the algebraic sum of all polynomials. By (1.6) it is a direct sum of those $\mathcal{P}_{\underline{\mathbf{m}}}$ so that $m_j = 0$. It forms an irreducible representation of of $\mathfrak{g}^{\mathbb{C}}$. By the intertwining property of $P_{\underline{\mathbf{1}}^j}\partial^j$ stated in Theorem 3.1 we know that the kernel $\operatorname{Ker} P_{\underline{\mathbf{1}}^j}\partial^j$ is an $\mathfrak{g}^{\mathbb{C}}$-invariant subspace. Clearly the constant function $f_0 = 1$ is in the kernel. By irreducibility we know that $\operatorname{Ker} P_{\underline{\mathbf{1}}^j}\partial^j$ is the whole algebraic sum. □

Proof of Proposition 4.1. The function $1\otimes 1$ is a cyclic vector of the $H^\nu \otimes \overline{H^\nu}$ under the action of G, thus the function $R(1\otimes 1) = h^\nu$ is a cyclic vector of $R(H^\nu\otimes\overline{H^\nu})$. Let $\nu = \frac{a}{2}(j_0 - 1)$. We prove $\mathcal{L}_j R(1\otimes 1) = \mathcal{L}_j h^{\frac{a}{2}(j_0-1)} = 0$, the result then follows by the cyclic property of $R(1\otimes 1)$. Now $\mathcal{L}_j = (\mathcal{D}^j)^* P_{\underline{\mathbf{1}}^j}\mathcal{D}^j$ and by Theorem 3.1 we have

$$P_{\underline{\mathbf{1}}^j}\mathcal{D}^j h^{\frac{a}{2}(j_0-1)} = h^{\frac{a}{2}(j-1)} P_{\underline{\mathbf{1}}^j}\partial^j h^{-\frac{a}{2}(j-j_0)}.$$

We claim that $P_{\underline{\mathbf{1}}^j}\partial^j h^{-\frac{a}{2}(j-j_0)} = 0$. Indeed consider the Faraut-Koranyi expansion of $h^{-\frac{a}{2}(j-j_0)}$:

$$h^{-\frac{a}{2}(j-j_0)}(z,z) = \sum_{\underline{\mathbf{m}}: m_{j-j_0+1}=0} (\frac{a}{2}(j-j_0))_{\underline{\mathbf{m}}} K_{\underline{\mathbf{m}}}(z,z).$$

Being the power series expansion of $h^{-\frac{a}{2}(j-j_0)}(z,z)$ it is absolutely convergent in a neighborhood of $z=0$. Each term $K_{\underline{\mathbf{m}}}(z,z)$ is a sum of holomorphic polynomials in $\mathcal{P}_{\underline{\mathbf{m}}}$ with $m_{j-j_0+1} = 0$, consequently $m_j = 0$, with coefficients being anti-holomorphic polynomials. Thus $P_{\underline{\mathbf{1}}^j}\partial^j K_{\underline{\mathbf{m}}}(z,z) = 0$ by Lemma 4.2. Consequently, $P_{\underline{\mathbf{1}}^j}\partial^j h^{-\frac{a}{2}(j-j_0)} = 0$, by the commutativity of the differentiation and summation in a power series expansion. □

Proposition 4.3. *The spherical function $\phi_{\underline{\boldsymbol{\lambda}}}$ is annihilated by all $\mathcal{L}_j$, $j = j_0, \dots, r$, if and only if $\underline{\boldsymbol{\lambda}}$ is in the Weyl group orbit of*

$$(i\rho_1, \dots, i\rho_{r-j_0}, i\rho_{r-j_0+1}, s_{r-j_0+2}, \dots, s_r) \quad \textit{for some } (s_{r-j_0+2}, \dots, s_r) \in \mathbb{C}^{j_0-1}.$$

To prove the result we recall the formula for the eigenvalues of $\mathcal{L}_j$ obtained in [36], Theorem 5.3.

Lemma 4.4. *Consider the Shimura operators $\mathcal{L}_j$ on the trivial line bundle on D. Their eigenvalues on the spherical functions $\phi_{\underline{\boldsymbol{\lambda}}}$ are given by*

$$\begin{aligned}\mathcal{L}_j(\underline{\boldsymbol{\lambda}}) = C_j \sum_{k=0}^{j} h_{j-k}\left(\rho_2^2 - \rho_1^2, \cdots, \rho_{r-j+1}^2 - \rho_1^2\right) \\ \times m_k\left(\lambda_1^2 + \rho_1^2, \cdots, \lambda_r^2 + \rho_1^2\right).\end{aligned} \tag{4.1}$$

Here C_j is a positive constant, h_k and m_k are the complete symmetric and elementary symmetric functions of degree k, respectively.

Proof of Proposition 4.3. The result will follow somewhat easier if we follow the proof of Theorem 5.3 in [36]. Suppose that $\underline{\boldsymbol{\lambda}}$ is as in the proposition. Since $r-j+1 \le r-j_0+1$ the set $\{\rho_1, \rho_2, \dots, \rho_{r-j+1}\}$ is a subset of $\{\rho_1, \rho_2, \dots, \rho_{r-j_0+1}\}$ so that $r-j+1$ components of $\underline{\boldsymbol{\lambda}}$ are sign permutations of $i\rho_1, i\rho_2, \dots, i\rho_{r-j+1}$.

We may assume that $\lambda_{j+1} = i\rho_2, \dots, \lambda_r = i\rho_{r-j+1}$. It is proved in [36] (see formula (5.20) and (5.25) there) that in this case the eigenvalue of $\mathcal{L}_j$ is of the form

$$\mathcal{L}_j(\underline{\boldsymbol{\lambda}}) = C_j \prod_{s=1}^{j} (\lambda_s^2 + (\frac{1+b}{2})^2) = C_j \prod_{s=1}^{j} (\lambda_s^2 + \rho_1^2).$$

Thus, if in addition, one of the remaining components, say $\lambda_j = i\rho_1$, then $\mathcal{L}_j(\underline{\boldsymbol{\lambda}}) = 0$.

Conversely, suppose now that $\underline{\boldsymbol{\lambda}}$ is such that $\mathcal{L}_j(\underline{\boldsymbol{\lambda}}) = 0$ for all $j_0 \le j \le r$. We start with the condition $\mathcal{L}_j(\underline{\boldsymbol{\lambda}}) = 0$ for $j = r$ and prove that after a signed permutation we have $\lambda_1 = i\rho_1$, and prove successively by using Lemma 4.4 and the condition $\mathcal{L}_j(\underline{\boldsymbol{\lambda}}) = 0$ for $j = r-1, r-2, \dots, j_0$.

Indeed, the eigenvalue $\mathcal{L}_r(\underline{\boldsymbol{\lambda}})$ is

$$\mathcal{L}_r(\underline{\boldsymbol{\lambda}}) = C_r \prod_{j=1}^{r} (\lambda_j^2 + \rho_1^2).$$

Thus $\mathcal{L}_r(\underline{\boldsymbol{\lambda}}) = 0$ implies that one of the λ_k is $\pm i\rho_1$. We may assume $\lambda_1 = i\rho_1$. Fix this value of λ_1. We study the condition $\mathcal{L}_j(\underline{\boldsymbol{\lambda}}) = 0$ for $j = r-1$. The eigenvalue of $\mathcal{L}_j(\underline{\boldsymbol{\lambda}})$ for $j = r-1$ is then, by Lemma 4.4,

$$\begin{aligned}
\mathcal{L}_j(\underline{\boldsymbol{\lambda}}) &= C_{r-1} \sum_{k=0}^{r-1} h_{r-1-k}(\rho_2^2 - \rho_1^2) m_k(0, \lambda_2^2 + \rho_1^2, \cdots, \lambda_r^2 + \rho_1^2) \\
&= C_{r-1} \sum_{k=0}^{r-1} (\rho_2^2 - \rho_1^2)^{r-1-k} m_k(\lambda_2^2 + \rho_1^2, \cdots, \lambda_r^2 + \rho_1^2) \\
&= C_{r-1} (\rho_2^2 - \rho_1^2)^{r-1} \sum_{k=0}^{r-1} (\rho_2^2 - \rho_1^2)^{-k} m_k(\lambda_2^2 + \rho_1^2, \cdots, \lambda_r^2 + \rho_1^2) \\
&= C_{r-1} (\rho_2^2 - \rho_1^2)^{r-1} \prod_{k=2}^{r} (1 + (\rho_2^2 - \rho_1^2)^{-1} (\lambda_k^2 + \rho_1^2))
\end{aligned} \tag{4.2}$$

where in the last equality we have used the formula

$$\sum_{k=0}^{l} m_k(x_1, \dots, x_l) t^k = \prod_{k=1}^{l} (1 + x_k t)$$

for the generating function of m_k. The product is then, disregarding a positive factor,

$$\prod_{k=2}^{r} (\lambda_k^2 + \rho_2^2).$$

Thus if $\mathcal{L}_j(\underline{\boldsymbol{\lambda}}) = 0$ for $j = r-1$, then one of λ_k, $k = 2, \dots, r$, is $i\rho_2$; we may assume $\lambda_2 = i\rho_2$.

We claim that generally if $\lambda_s = i\rho_s$, for all $1 \le s \le j$, for a fixed $1 \le j \le r$, then the eigenvalue of $\mathcal{L}_{r-j}(\underline{\boldsymbol{\lambda}})$ is

$$C \prod_{k=j+1}^{r} (\lambda_k^2 + \rho_{j+1}^2). \tag{4.3}$$

Accepting temporarily the claim, the eigenvalue of $\mathcal{L}_j(\underline{\boldsymbol{\lambda}}) = 0$ for $j = r-2$ implies that $\lambda_3 = i\rho_3$, and the result is then proved by repeating the argument.

The eigenvalue $\mathcal{L}_{r-j}(\underline{\boldsymbol{\lambda}})$ is, disregarding the nonzero constant,

$$\sum_{k=0}^{r-j} h_{r-j-k}\left(\rho_2^2-\rho_1^2,\ldots,\rho_{j+1}^2-\rho_1^2\right) m_k\left(-\rho_2^2+\rho_1^2,\ldots,-\rho_j^2+\rho_1^2,\lambda_{j+1}^2+\rho_1^2,\lambda_r^2+\rho_1^2\right).$$

To simplify the notation we write $c_k=\rho_k^2-\rho_1^2$ for $k=2,\ldots,j+1$. The above formula is

$$\sum_{k=0}^{r-j} h_{r-j-k}(c_2,\ldots,c_{j+1})\, m_k\left(-c_2,\ldots,-c_j,\lambda_{j+1}^2+\rho_1^2,\ldots,\lambda_r^2+\rho_1^2\right).$$

As a polynomial of λ_{j+1}^2, the point $\lambda_{j+1}^2=-\rho_{j+1}^2$ is its zero. Indeed, denoting temporarily $\mathbf{d}=(\lambda_{j+2}^2+\rho_1^2,\ldots,\lambda_r^2+\rho_1^2)\in\mathbb{C}^{r-j-1}$, at the point $\lambda_{j+1}^2=-\rho_{j+1}^2$ the polynomial is

$$\begin{aligned}
&\sum_{k=0}^{r-j} h_{r-j-k}(c_2,\ldots,c_{j+1})\, m_k(-c_2,\ldots,-c_j,-c_{j+1},\mathbf{d})\\
&=\sum_{k=0}^{r-j} h_{r-j-k}(c_2,\ldots,c_{j+1})\,\sum_{l=0}^{k} m_{k-l}(-c_2,\ldots,-c_j,-c_{j+1})m_l(\mathbf{d})\\
&=\sum_{l=0}^{r-j} m_l(\mathbf{d})\sum_{k=l}^{r-j} h_{r-j-k}(c_2,\ldots,c_{j+1})\, m_{k-l}(-c_2,\ldots,-c_j,-c_{j+1})\\
&=\sum_{l=0}^{r-j} m_l(\mathbf{d})\sum_{k=0}^{r-j-l} h_k(c_2,\ldots,c_{j+1})\, m_{r-j-l-k}(-c_2,\ldots,-c_j,-c_{j+1})\\
&=\sum_{l=0}^{r-j} m_l(\mathbf{d})\delta_{r-j-l,0}=m_{r-j}(\mathbf{d})
\end{aligned} \tag{4.4}$$

where $\delta_{r-j-l,0}$ is the Kronecker symbol; here we use the formula

$$\sum_{l=0}^{k}(-1)^s m_s h_{k-s}=\delta_{k,0}; \tag{4.5}$$

see [15], Section I.2. But the dimensionality of $\mathbf{d}$ is $r-j-1$ which is less than $r-j$, thus $m_{r-j}(\mathbf{d})=0$.

Consequently, $\mathcal{L}_{r-j}(\underline{\boldsymbol{\lambda}})$ as a polynomial of λ_{j+1}^2 has a factor $\lambda_{j+1}^2+\rho_{j+1}^2$. Being a symmetric polynomial of $\lambda_{j+1}^2,\ldots,\lambda_r^2$, it has a factor $\lambda_k^2+\rho_{j+1}^2$ for all $k=j+1,\ldots,r$; thus it is a nonzero constant multiple of their product, thereby the claim (4.3). □

Remark 4.5. The above proof has in effect solved a system of polynomial equations $\mathcal{L}_j(\underline{\boldsymbol{\lambda}})=0$, $j_0\le j\le r$.

5. Matrix form of the operator $R_{\frac{a}{2}}^{-1}\mathcal{L}_1R_{\frac{a}{2}}$ and explicit formulas for the spherical functions

We consider now that $\nu=\frac{a}{2}$. As proved in the previous section the invariant differential operators $\mathcal{L}_j$ for $j\ge 2$ act as zero operators on the image of the tensor product $H^\nu\otimes\overline{H^\nu}$ under R. We thus consider the spectral decomposition of the operator $\mathcal{L}_{1,\frac{a}{2}}=R^{-1}\mathcal{L}_1R$. Moreover, the representations appearing in the decomposition are spherical; any spherical representation is uniquely determined by the spherical function, which in turn is determined by the corresponding eigenvalues of the Shimura operators $\mathcal{L}_1, \dots, \mathcal{L}_r$. The operators $\mathcal{L}_2, \dots, \mathcal{L}_r$ annihilate the spherical functions appearing in the representation. We need only determine the eigenvalues of $\mathcal{L}_1$ on the spherical functions that appear in the decomposition, which in turn is given by the spectrum of $\mathcal{L}_{1,\frac{a}{2}}$.

We will find the matrix form of the $\mathcal{L}_{j,\frac{a}{2}}$ acting on the basis $K_{\underline{\mathbf{m}}}$ of $H^{\frac{a}{2}} \otimes \overline{H^{\frac{a}{2}}}$. As remarked earlier, the functions $K_{\underline{\mathbf{m}}}$ are polynomials in (z, w), holomorphic in z and anti-holomorphic in w, and so are their actions under $\mathcal{L}_{j,\frac{a}{2}}$, thus they are uniquely determined by their restriction on the diagonal; the restrictions are then K-invariant, and are determined by their restriction on $\exp(\mathfrak{a}^+) \cdot 0 = \{(s_1, \dots, s_r); 0 \le s_1 \le \dots \le s_r < 1\}$.

Write $s_j = \exp(t_j \xi_j) \cdot 0 = \tanh t_j$, $j = 1, \dots, r$. In terms of the coordinates $(t_1, \dots, t_r)$, the radial part of the Laplace Beltrami operator is

$$\begin{aligned}\mathcal{L}_1 = \frac{1}{4}(\sum_{j=1}^r \partial_j^2 + a \sum_{r \ge i > j \ge 1} \coth(t_i \pm t_j)(\partial_i \pm \partial_j) \\ + 2\sum_{j=1}^r \coth 2t_j \partial_j + 2b \sum_{j=1}^r \coth t_j \partial_j);\end{aligned}$$

see [7], Chapter II, Proposition 3.9. However, it is more convenient to use the coordinates $x_j = s_j^2$, it is

$$\begin{aligned}\mathcal{L}_1 = \sum_{j=1}^r x_j(1-x_j)^2\partial_j^2 + \sum_{j=1}^r (1-x_j)^2\partial_j + b\sum_{j=1}^r (1-x_j)^2\partial_j \\ + a\sum_{j \ne k} \frac{(1-x_j)(1-x_k)}{x_j - x_k} x_j \partial_j.\end{aligned} \tag{5.1}$$

We can then find a formula for the operator $\mathcal{L}_{1,\nu} = R_\nu^{-1}\mathcal{L}_1 R_\nu$. When acting on K-invariant functions in (z, w) and restricting to the diagonal (z, z), $z = (s_1, \dots, s_r)$, with coordinates $(x_1, \dots, x_r) = (s_1^2, \dots, s_r^2)$,

$$\begin{aligned}\mathcal{L}_{1,\nu} &= R_\nu^{-1}\mathcal{L}_1 R_\nu \\ &= \sum_{j=1}^r x_j\left((1-x_j)^2\partial_j^2 - 2\nu(1-x_j)\partial_j + \nu(\nu-1)\right) \\ &+ \sum_{j=1}^r \left((1-x_j)^2\partial_j - \nu(1-x_j)\right) + b\sum_{j=1}^r ((1-x_j)\partial_j - \nu) \\ &+ a\sum_{j \ne k} \frac{(1-x_k)}{x_j - x_k} x_j ((1-x_j)\partial_j - \nu).\end{aligned} \tag{5.2}$$

As proved in the previous section the spherical functions that appear in the decomposition of the tensor product satisfy, up to the Weyl group action,

$$\underline{\boldsymbol{\lambda}} = (i\rho_1, \dots, i\rho_{r-1}, \lambda), \tag{5.3}$$

for some (with some abuse of notation) $\lambda = \lambda_r \in \mathbb{C}$. We will find the recurrence relations (2.8) and (2.9) with $\underline{\boldsymbol{\lambda}}$ as above.

Notice first that when $\nu = \frac{a}{2}$ the expansion (1.4) becomes

$$h^{-\frac{a}{2}}(z) = \sum_{m=0}^{\infty} (\frac{a}{2})_{\underline{\mathbf{m}}} K_{\underline{\mathbf{m}}}(z, z)$$

with $\underline{\mathbf{m}} = (m, 0, \dots, 0)$, and for $z = s_1 e_1 + \cdots + s_r e_r$

$$\begin{aligned} K_{\underline{\mathbf{m}}}(z, z) &= \frac{1}{(\frac{a}{2})_m} \sum_{k_1+\cdots+k_r=m} \frac{(\frac{a}{2})_{k_1} \cdots (\frac{a}{2})_{k_r}}{k_1! \cdots k_r!} s_1^{2k_1} \cdots s_r^{2k_r}, \\ E_{\underline{\mathbf{m}}}(z, z) &= \sum_{k_1+\cdots+k_r=m} \frac{(\frac{a}{2})_{k_1} \cdots (\frac{a}{2})_{k_r}}{k_1! \cdots k_r!} s_1^{2k_1} \cdots s_r^{2k_r}. \end{aligned} \tag{5.4}$$

For simplicity we write hereafter $\underline{\mathbf{m}} = m$.

Lemma 5.1. *With the above notation,*

$$\mathcal{L}_{1,\frac{a}{2}} E_m = A_m E_{m-1} + B_m E_m + C_m E_{m+1} \tag{5.5}$$

where

$$A_m = (m + \frac{a}{2} r)(m + 1 + b + \frac{a}{2}(r-1)),$$

$$B_m = -\Big(m(2m + ar + b) + \frac{a}{2} r (1 + b + \frac{a}{2}(r-1))\Big)$$

and

$$C_m = (m+1)(m + \frac{a}{2}).$$

Before performing the calculation we note that by invariance the operator $\mathcal{L}_{1,\frac{a}{2}}$ maps each vector E_m into a unique linear combination of themselves. By the formula (5.2) we see easily that $\mathcal{L}_{1,\frac{a}{2}} E_m$ is actually a linear combination of E_{m-1}, E_m and E_{m+1}. To find the coefficients we rewrite $\mathcal{L}_{1,\nu}$ (for general ν) as a sum of three operators that are rising, lowering and respectively keeping the degree of a homogeneous symmetric polynomial. First consider the last term in (5.2),

$$\sum_{j \neq k} \frac{(1 - x_k)}{x_j - x_k} x_j = \sum_{j<k} (\frac{(1 - x_k)}{x_j - x_k} x_j + \frac{(1 - x_j)}{x_k - x_j} x_k) = \sum_{j<k} \frac{x_j - x_k}{x_j - x_k} = \binom{r}{2},$$

and

$$\sum_{j \neq k} \frac{(1 - x_k)}{x_j - x_k} (1 - x_j) x_j \partial_j = \sum_{j<k} \frac{(1 - x_k)(1 - x_j)}{x_j - x_k} (x_j \partial_j - x_k \partial_k).$$

Thus

$$\mathcal{L}_{1,\nu} = \mathcal{L}^+ + \mathcal{L}^0 + \mathcal{L}^-,$$

where

$$\begin{aligned} \mathcal{L}^+ = &\sum_{j=1}^r x_j (\nu(\nu - 1) + 2\nu x_j \partial_j + x_j^2 \partial_j^2) + \sum_{j=1}^r (\nu x_j + x_j^2 \partial_j) \\ &+ a \sum_{j<k} x_j x_k \frac{x_j \partial_j - x_k \partial_k}{x_j - x_k}, \end{aligned} \tag{5.6}$$

$$\mathcal{L}^0 = -2(\nu+1+b)\sum_{j=1}^r x_j\partial_j - 2\sum_{j=1}^r x_j^2\partial_j^2 - a\sum_{j<k}(x_j+x_k)\frac{x_j\partial_j - x_k\partial_k}{x_j-x_k} - \nu r(1+b) - \nu a\binom{r}{2} \tag{5.7}$$

and

$$\mathcal{L}^- = \sum_{j=1}^r x_j\partial_j^2 + (1+b)\sum_{j=1}^r \partial_j + a\sum_{j<k}\frac{x_j\partial_j - x_k\partial_k}{x_j-x_k}. \tag{5.8}$$

To treat the last term in $\mathcal{L}^-$ we observe further that the operator $\frac{x\partial_x - y\partial_y}{x-y}$ acting on a symmetric polynomial $f(x,y) = x^c y^d + y^c x^d$ $(c\ge d\ge 0)$ is

$$\begin{aligned}\frac{x\partial_x - y\partial_y}{x-y} f(x,y) &= (c-d)(xy)^d\frac{x^{c-d}-y^{c-d}}{x-y}\\ &= (c-d)(xy)^d(x^{c-d-1}+\cdots+y^{c-d-1});\end{aligned} \tag{5.9}$$

if we evaluate the result at $y=0$, it is possibly nonzero only if $c>d=0$, in which case it is cx^{c-1}; if it is further evaluated at $x=y=0$, it is zero unless $c=1$.

Proof. Recall that $\nu = \frac{a}{2}$. We calculate $\mathcal{L}^+E_m$ and evaluate it at $(x_1, x_2, \dots) = (x_1, 0, \dots, 0)$, which we write as $(x_1, 0)$. We find

$$\begin{aligned}\mathcal{L}^+E_m(x_1,0) &= (\frac{a}{2})_m\frac{1}{m!}\left(\frac{a}{2}(\frac{a}{2}-1) + am + m(m-1) + \frac{a}{2} + m\right)x_1^{m+1}\\ &= (\frac{a}{2})_m\frac{1}{(m)!}(\frac{a}{2}+m)^2 x_1^{m+1}\\ &= (m+1)(\frac{a}{2}+m)(\frac{a}{2})_{m+1}\frac{1}{(m+1)!}x_1^{m+1}\\ &= (m+1)(\frac{a}{2}+m)E_{m+1}(x_1,0).\end{aligned}$$

So that

$$\mathcal{L}^+E_m(x_1,\dots,x_r) = (m+1)(\frac{a}{2}+m)E_{m+1}(x_1,\dots,x_r)$$

by the uniqueness of the recurrence (5.5).

Next we calculate $\mathcal{L}^-E_m(x_1,0)$. Clearly

$$(\sum_{j=1}^r x_j\partial_j^2)E_m(x_1,0) = x_1^{m-1}m(m-1)\frac{(\frac{a}{2})_m}{m!} = (m-1)(\frac{a}{2}+m-1)E_{m-1}(x_1,0).$$

To deal with the differentiation $\sum_{j=1}^r \partial_j = \partial_1 + \sum_{j=2}^r \partial_j$ we write

$$E_{\underline{\mathbf{m}}}(x) = \frac{(\frac{a}{2})_m}{m!}x_1^m + \sum_{k_1+\cdots+k_r=m, k_1<m}\frac{(\frac{a}{2})_{k_1}\dots(\frac{a}{2})_{k_r}}{k_1!\dots k_r!}x_1^{k_1}\cdots x_r^{k_r},$$

the differentiation ∂_1 on the second term in E_m, when evaluated at $(x_1,0)$, is clearly zero. Therefore

$$\partial_1 E_m(x_1,0) = \partial_1\frac{(\frac{a}{2})_m}{m!}x_1^m = \frac{(\frac{a}{2})_m}{(m-1)!}x_1^{m-1}.$$

Using the observation (5.9) we see that

$$(\sum_{j=2}^{r}\partial_j)E_m(x_1,0) = (\sum_{j=2}^{r}\partial_j)\left(\sum_{j=2}^{r}\frac{(\frac{a}{2})_{m-1}(\frac{a}{2})_1}{(m-1)!}x_1^{m-1}x_j\right)(x_1,0)$$
$$= (r-1)\frac{\frac{a}{2}(\frac{a}{2})_{m-1}}{(m-1)!}x_1^{m-1}.$$

So that

$$\mathcal{L}^- E_m(x_1,0) = C\frac{(\frac{a}{2})_{m-1}}{(m-1)!}x_1^{m-1} = CE_{m-1}(x_1,0)$$

with the constant C given by

$$\begin{aligned}C &= (m-1)(\frac{a}{2}+m-1)) + (1+b)(\frac{a}{2}+m-1+(r-1)\frac{a}{2}) + a(r-1)(\frac{a}{2}+m-1)\\ &\quad + a\frac{a}{2}\frac{(r-1)(r-2)}{2}\\ &= (m-1+\frac{a}{2}r)(m+b+\frac{a}{2}(r-1)),\end{aligned}$$

which is the A_m claimed in the lemma. Finally $\mathcal{L}^0 E_m(x_1,0)$ can be calculated by a similar method. □

We consider the dual relation of (5.7):

$$-(\lambda^2+\rho_r^2)\mathcal{E}_m(\underline{\boldsymbol{\lambda}}) = A_{m+1}\mathcal{E}_{m+1}(\underline{\boldsymbol{\lambda}}) + B_m\mathcal{E}_m(\underline{\boldsymbol{\lambda}}) + C_{m-1}\mathcal{E}_{m-1}(\underline{\boldsymbol{\lambda}}) \tag{5.10}$$

with $\mathcal{E}_0(\underline{\boldsymbol{\lambda}}) = 1$. This relation (5.10) is exactly (9.3) in the appendix, (with α there being ρ_r, $\beta = \frac{1+b}{2}$ and $\gamma = -\frac{1+b}{2}+\frac{a}{2}$) which along with the given $\mathcal{E}_0(\underline{\boldsymbol{\lambda}})$ uniquely determines the polynomials; therefore

$$\mathcal{E}_m(\underline{\boldsymbol{\lambda}}) = {}_3F_2(-m,\rho_r+i\lambda_r,\rho_r-i\lambda_r;1+b+\frac{a}{2}(r-1),\frac{a}{2}r;1). \tag{5.11}$$

Consequently, by the argument in Remark 2.5, the series

$$\psi(z) = \sum_m \mathcal{E}_m(\underline{\boldsymbol{\lambda}})E_m(z) \tag{5.12}$$

is an eigenfunction of $\mathcal{L}_{1,\frac{a}{2}}$ with eigenvalue $-(\lambda^2+\rho_r^2)$, whenever the power series is uniformly convergent.

Theorem 5.2. *The function $h^{-\frac{a}{2}}(z)\phi_{\underline{\boldsymbol{\lambda}}}(z)$ for $\underline{\boldsymbol{\lambda}}$ as in* (5.3) *has the following expansion*

$$h(z)^{-\frac{a}{2}}\phi_{\underline{\boldsymbol{\lambda}}}(z) = \psi(z) = \sum_m \mathcal{E}_m(\underline{\boldsymbol{\lambda}})E_m(z), \tag{5.13}$$

and the series converges uniformly on the compact subset of D.

To prove the theorem we need a technical estimate.

Lemma 5.3. *Suppose $\alpha,\beta\in\mathbb{C}$ and $\gamma>0,\delta>0$. Let $M\geq 0$ be such that $|\alpha|+M\geq\gamma$, $|\beta|+M\geq\delta$. Then*

$$|{}_3F_2(-m,\alpha,\beta;\delta,\gamma;1)| \leq (m+1)!\frac{(|\alpha|+M)_m(|\beta|+M)_m}{(\delta)_m(\gamma)_m}$$

Proof. By its definition

$$ {}_3F_2\,(-m,\ \alpha\,\beta;\gamma,\delta;1) = \sum_{j=0}^{m}(-1)^j\binom{m}{j}\frac{(\alpha)_j(\beta)_j}{(\gamma)_j(\delta)_j}. $$

We observe that

$$ \binom{m}{j} \le m!, $$

and for any two positive numbers $a \ge b > 0$

$$ 1 \le \frac{a}{b} \le \frac{a(a+1)}{b(b+1)} \le \cdots \le \frac{(a)_j}{(b)_j} $$

for any $j \ge 0$. Using these two inequalities we have

$$ \tag{5.14} \begin{aligned} &|{}_3F_2(-m,\alpha,\beta;\gamma,\delta;1)| \\ &\le m!\sum_{j=0}^{m}\frac{(|\alpha|)_j(|\beta|)_j}{(\gamma)_j(\delta)_j} \\ &\le m!\sum_{j=0}^{m}\frac{(|\alpha|+M)_j(|\beta|+M)_j}{(\gamma)_j(\delta)_j} \\ &\le m!\sum_{j=0}^{m}\frac{(|\alpha|+M)_m(|\beta|+M)_m}{(\gamma)_m(\delta)_m} \\ &= (m+1)!\frac{(|\alpha|+M)_m(|\beta|+M)_m}{(\gamma)_m(\delta)_m}. \end{aligned} $$

□

Proof of Theorem 5.2. We prove that the function $\psi(z)$ defined in (5.12) is convergent and that $\phi_{\underline{\boldsymbol{\lambda}}}(z) = h^{\frac{a}{2}}(z)\psi(z)$. Notice that $E_m(s_1e_1+\cdots+s_re_r) \le \frac{(\frac{a}{2}r)_m}{m!}s^{2m}$, where $s = \max\{s_1,\dots,s_r\}$. Using the previous lemma, with α,β,γ and δ replaced by the corresponding numbers in (5.11), and a fixed large M, we get

$$ |E_m(\underline{\boldsymbol{\lambda}})|E_m(s_1e_1+\cdots+s_re_r) \le (m+1)!\frac{(|\alpha|+M)_m(|\beta|+M)_m}{(\gamma)_m(\delta)_m}\frac{(\frac{a}{2}r)_m}{m!}s^{2m}. $$

By the ratio test the series $\psi(z)$ is convergent if $s<1$, we thus get the uniform convergence on compact sets of D. The function $\psi(z)$ is an eigenfunction of $\mathcal{L}_{1,\frac{a}{2}}$ with eigenvalue $-(\lambda^2+\rho_r^2)$, that is, the function $h^{\frac{a}{2}}(z)\psi(z)$ is an eigenfunction of the Laplace-Beltrami operator $\mathcal{L}_1$. We prove that $h^{\frac{a}{2}}(z)\psi(z)$ is also an eigenfunction of the higher order Shimura operators $\mathcal{L}_j$ for $j=2,\dots,r$ with eigenvalue 0, namely they are K-invariant eigenfunctions of a system of generators. The result follows then by the uniqueness of the spherical function. By its definition and Theorem 3.1, $\mathcal{L}_{\underline{\mathbf{1}}^j} = (\mathcal{D}^j)^*P_{\underline{\mathbf{1}}^j}\mathcal{D}^j$ and $P_{\underline{\mathbf{1}}^j}\mathcal{D}^j = h^{\frac{a}{2}(j-1)}P_{\underline{\mathbf{1}}^j}\partial^j(h^{-\frac{a}{2}(j-1)})$, so that $P_{\underline{\mathbf{1}}^j}\mathcal{D}^j$ acting on the function $h^{\frac{a}{2}}(z)\psi(z)$ is

$$ \begin{aligned} P_{\underline{\mathbf{1}}^j}\mathcal{D}^j(h^{\frac{a}{2}}(z)\psi(z)) &= h^{\frac{a}{2}(j-1)}(z)P_{\underline{\mathbf{1}}^j}\partial^j\left(h^{-\frac{a}{2}(j-1)}h^{\frac{a}{2}}(z)\psi(z)\right) \\ &= P_{\underline{\mathbf{1}}^j}\partial^j\left(h^{-\frac{a}{2}(j-2)}(z)\psi(z)\right). \end{aligned} $$

The function $\psi(z)$ is a sum of $K_{\underline{\mathbf{m}}}$ with $m_2=0$, namely the expansion (5.11), and the convergence is uniform on the compact subset of D, as just proved; the function $h^{-\frac{a}{2}(j-2)}(z)$ a sum of $K_{\underline{\mathbf{m}}'}$ with $m_{j-1}=0$, also with the same uniform convergence (see Theorem 1.1). The product of two such $K_{\underline{\mathbf{m}}}$ and $K_{\underline{\mathbf{m}}'}$ is a finite sum of $K_{\underline{\mathbf{n}}}$ with $n_j=0$, by Lemma 1.2. That is, $h^{-\frac{a}{2}(j-2)}(z)\psi(z)$ is a sum of $K_{\underline{\mathbf{m}}}$ with $m_j=0$, all of which are annihilated by $P_{\underline{\mathbf{1}}^j}\partial^j$, by Lemma 4.2. Thus $P_{\underline{\mathbf{1}}^j}\mathcal{D}^j(h^{\frac{a}{2}}\psi)=0$, consequently $\mathcal{L}_{\underline{\mathbf{1}}^j}(h^{\frac{a}{2}}\psi)=0$. $\square$

Proposition 4.3 and Theorem 5.2 can be then summarized in the following corollary; that (a) implies (c) is in Theorem 5.2, and that (c) implies (a) can be proved using the same method as in the above proof (which will not be used in the present paper).

Corollary 5.4. *The following assertions are equivalent for $\underline{\boldsymbol{\lambda}}\in(\mathfrak{a}^*)^{\mathbb{C}}$:*

(a) *$\phi_{\underline{\boldsymbol{\lambda}}}$ is annihilated by all $\mathcal{L}_j$, $2\le j\le r$.*
(b) *$\underline{\boldsymbol{\lambda}}$ is in the Weyl group orbit of $(i\rho_1, i\rho_2,\dots,i\rho_{r-1},\lambda)$ for some $\lambda\in\mathbb{C}$.*
(c) *In the expansion of $h^{-\frac{a}{2}}(z)\phi_{\underline{\boldsymbol{\lambda}}}(z)$ in terms of $K_{\underline{\mathbf{m}}}(z,z)$ all the coefficients of $K_{\underline{\mathbf{m}}}$ are 0 if $m_2>0$.*

Remark 5.5. By the formula (1.3.16) in [14] we see that for $z=s_1e_1$, the function $\psi(z)$ is

$$\psi(z)=(1-s_1^2)^{-\frac{a}{2}}{}_3F_2\left(\frac{a}{2},\rho_r+i\lambda,\rho_r-i\lambda;\frac{a}{2}r,1+b+\frac{a}{2}(r-1);\frac{s_1^2}{s_1^2-1}\right),$$

and consequently

$$\phi_{\underline{\boldsymbol{\lambda}}}(z)={}_3F_2\left(\frac{a}{2},\rho_r+i\lambda,\rho_r-i\lambda;\frac{a}{2}r,1+b+\frac{a}{2}(r-1);\frac{s_1^2}{s_1^2-1}\right).$$

If we formally put $r=1$ and $a=0$, namely when D is the unit ball in $\mathbb{C}^{1+b}$, then the above formula reduces to

$$\phi_{\underline{\boldsymbol{\lambda}}}(z)={}_2F_1\left(\rho+i\lambda,\rho-i\lambda;1+b;\frac{|z|^2}{|z|^2-1}\right);$$

this is the known formula for the spherical function on rank one domains (see e.g. [7], p. 484 or more explicitly [34]).

6. Irreducible decomposition of the tensor product $H^{\frac{a}{2}}\otimes\overline{H^{\frac{a}{2}}}$

In terms of the orthonormal basis e_m of $(H^{\frac{a}{2}}\otimes\overline{H^{\frac{a}{2}}})_0$, the matrix form of $\mathcal{L}_{1,\frac{a}{2}}$ is

$$\mathcal{L}_{1,\frac{a}{2}}e_m=A_m\left(\frac{d_{m-1}}{d_m}\right)^{\frac{1}{2}}e_{m-1}+B_me_m+C_m\left(\frac{d_{m+1}}{d_m}\right)^{\frac{1}{2}}e_{m+1}. \tag{6.1}$$

On the other hand, recalling Proposition 2.5 with $\varepsilon_m(\underline{\boldsymbol{\lambda}})$ defined in (2.5), we have

(6.2)

$$-(\lambda^2+\rho_r^2)\varepsilon_m(\underline{\boldsymbol{\lambda}})=A_m\left(\frac{d_{m-1}}{d_m}\right)^{\frac{1}{2}}\varepsilon_{m-1}(\underline{\boldsymbol{\lambda}})+B_m\varepsilon_m(\underline{\boldsymbol{\lambda}})+C_m\left(\frac{d_{m+1}}{d_m}\right)^{\frac{1}{2}}\varepsilon_{m+1}(\underline{\boldsymbol{\lambda}}).$$

(This can also be proved by direct calculation using (5.10), noticing that $\frac{A_{m+1}}{C_m}=\frac{d_{m+1}}{d_m}$.) Namely the operator $\mathcal{L}_{1,\frac{a}{2}}$ on e_m and the multiplication operator $-(\lambda^2+\rho_r^2)$ on ε_m has the same matrix form. However, in view of the formula (5.11) and Theorem 9.1 in Section 9, ε_m are of orthonormal basis for a L^2-space with respect to certain measure, and consequently it gives the spectrum of the multiplication operator by $-(\lambda^2+\rho_r^2)$ and further the spectrum of $\mathcal{L}_{1,\frac{a}{2}}$. To justify this heuristic argument we need the following result.

Lemma 6.1. *The operator* $\mathcal{L}_{1,\frac{a}{2}}$ *on* $\{K_{\underline{\mathbf{m}}}, m_2 = 0\} \subset (H^{\frac{a}{2}} \otimes \overline{H^{\frac{a}{2}}})_0$ *has a self-adjoint extension and has a cyclic vector* e_0*. Thus its spectral decomposition is multiplicity free.*

Proof. The cyclicity follows from Lemma 5.1. We will follow the proof of Theorem 2.2 in [16]. We claim first that $\mathcal{L}_{1,\frac{a}{2}}$ on $(H^{\frac{a}{2}}\otimes\overline{H^{\frac{a}{2}}})_0$ is indeed the Casimir operator. Take $\mathcal{V}$ the Gårding subspace of $H^{\frac{a}{2}} \otimes \overline{H^{\frac{a}{2}}}$, and $\mathcal{V}_0$ its subspace of K-invariant elements; $\mathcal{V}$ is stable under the action of G and $\mathfrak{g}$, whereas $\mathcal{V}_0$ is stable under $\mathcal{L}_{1,\frac{a}{2}}$ and the Casimir operator. The operator R is defined on the whose space $H^{\frac{a}{2}}\otimes\overline{H^{\frac{a}{2}}}$ and injective, its image is stable under the regular action $\pi(0)$ of G on $C^\infty(D)$ by the intertwining property of R. Let (say) $\mathcal{C} = X_1^2+\cdots+X_n^2-Y_1^2-\cdots-Y_q^2$ be the Casimir operator where $\{X_j, j = 1,\dots,n\}$ and $\{Y_j, j = 1,\dots,q\}$ are orthogonal bases of $\mathfrak{p}$ and $\mathfrak{k}$ with the Killing form on $\mathfrak{g}$. Thus the regular action $\pi(0)(C)$ of $\mathcal{C}$ on $C^\infty(D)$ is the Laplace-Beltrami operator $\mathcal{L}$. For each $X \in \mathfrak{g}$, we have, for $F \in \mathcal{V}$,

$$\pi(0)(\exp(tX))RF = R\,\pi(\frac{a}{2})\otimes\overline{\pi(\frac{a}{2})}(\exp(tX))F$$

by the intertwining property of R. Consequently,

$$\pi(0)(X)RF = R\,\pi(\frac{a}{2})\otimes\overline{\pi(\frac{a}{2})}(X)F$$

and

$$\pi(0)(X^2)RF = R\,\pi(\frac{a}{2})\otimes\overline{\pi(\frac{a}{2})}(X^2)F.$$

Therefore, summing over the basis vectors,

$$\mathcal{L}(RF) = \pi(0)(\mathcal{C})(RF) = R\pi(\frac{a}{2})\otimes\overline{\pi(\frac{a}{2})}(\mathcal{C})F$$

and

$$\mathcal{L}_{1,\frac{a}{2}}F = (R^{-1}\mathcal{L}R)F = \pi(\frac{a}{2})\otimes\overline{\pi(\frac{a}{2})}(\mathcal{C})F,$$

so that $\mathcal{L}_{1,\frac{a}{2}}$ and $\pi(\frac{a}{2})\otimes\overline{\pi(\frac{a}{2})}(C)$ coincide on the Gårding subspace $\mathcal{V}$; taking $F \in \mathcal{V}_0$ we see that they coincide on $\mathcal{V}_0$. The proof of Theorem 2.2, loc. cit., relies on two observations, the nonnegativity of $-\mathcal{L}_{1,\frac{a}{2}}$ on $(H^{\frac{a}{2}}\otimes\overline{H^{\frac{a}{2}}})_0$ and that $-\mathcal{L}_{1,\frac{a}{2}}+1$ has dense image. Indeed, the inner product $(-\mathcal{L}_{1,\frac{a}{2}}x, x)$ for $x \in \mathcal{V}_0$ is the same as considered in $\mathcal{V}$ since $\mathcal{V}_0$ is invariant under $\mathcal{L}_{1,\frac{a}{2}}$, and is nonnegative (by the Segal theorem that iX for $X \in \mathfrak{g}$ is essentially self-adjoint). The second fact can be proved by using exactly the same lines as in [16] except the space $C_0^\infty(G)$ is replaced by $C_0^\infty(D)^K$ of K-invariant functions on D. (The density can also be proved by using the above matrix form of $\mathcal{L}_{1,\frac{a}{2}}$.) □

This implies that the spherical representations appear with multiplicity free in the decomposition of $H^{\frac{a}{2}}\otimes\overline{H^{\frac{a}{2}}}$.

Remark 6.2. The above lemma on the self-adjointness and multiplicity can be proved for the systems of the conjugates $R^{-1}\mathcal{L}_jR$ of the Shimura operators $\mathcal{L}_j$ on the tensor product $(H^\nu\otimes\overline{H^\nu})_0$ for $\nu \in W(D)$. However, for the sake of conciseness we confine ourself with the above case.

We summarize our results in the following theorem. Observe that discrete parts appear if $a < 1+b$; this happens precisely when D is the Type I non-tube domain $SU(r,s)$ $(r \le s)$ with $b = s-r \ge 2$. Thus for other irreducible domains there are only continuous spectra. We let $\Sigma = \mathbb{R}^+$ for other domains and

$$\Sigma = \mathbb{R}^+ \cup \{i(\frac{a}{2} - \frac{1+b}{2} + k);\, k = 0, \dots, k_0\}$$

if D is the type I domain $SU(r, r+b)/S(U(r) \times U(r+b))$, and where k_0 is the largest nonnegative integer such that $\frac{a}{2} - \frac{1+b}{2} + k < 0$. Let μ be the measure given in Section 9 with

$$\alpha = \rho_r = \frac{1}{2}(1+b+a(r-1)),\ \beta = \frac{1}{2}(1+b),\ \gamma = \frac{a}{2} - \frac{1}{2}(1+b). \tag{6.3}$$

Theorem 6.3. *With the above notation the map $e_{\underline{\mathbf{m}}}$ to ε_m extends to a unitary operator from $(H^{\frac{a}{2}} \otimes \overline{H^{\frac{a}{2}}})_0$ onto $L^2(\Sigma, d\mu)$, and the r-tuple $(\mathcal{L}_{1,\frac{a}{2}}, \mathcal{L}_{2,\frac{a}{2}}, \dots, \mathcal{L}_{r,\frac{a}{2}})$ is unitarily equivalent to the the r-tuple of multiplication operators $(\mathcal{L}_1(\underline{\boldsymbol{\lambda}}), 0, \dots, 0)$; the vector $e_0 = 1 \otimes 1$ and the function $\varepsilon_0(\underline{\boldsymbol{\lambda}}) = 1$ are cyclic vectors in the corresponding spaces. The spectrum of the r-tuple $(\mathcal{L}_{1,\frac{a}{2}}, \mathcal{L}_{2,\frac{a}{2}}, \dots, \mathcal{L}_{r,\frac{a}{2}})$ of generators of invariant differential operators on $H^{\frac{a}{2}} \otimes \overline{H^{\frac{a}{2}}}$ is*

$$\{(-(\lambda^2 + \rho_r^2), 0, \dots, 0);\, \lambda \in \Sigma\}.$$

7. Quantization of the minimal complementary series representations

Let $\mathcal{C}_{\frac{a}{2} - \frac{1+b}{2} + k}$ be the discrete part appearing in the decomposition in Theorem 6.1 and we refer to it as a complementary series representation. In this section we construct directly an intertwining operator from the complementary series $\mathcal{C}_{\frac{a}{2} - \frac{1+b}{2} + k}$ into the space $\mathfrak{S}_2(H^{\frac{a}{2}})$ of Hilbert-Schmidt operators on $H^{\frac{a}{2}}$ using the expansion (5.13). Here k are nonnegative integers and $\frac{a}{2} - \frac{1+b}{2} + k < 0$. The intertwining operator can formally be defined on all spherical functions $\phi_{\underline{\boldsymbol{\lambda}}}$ for all k. However, we prove that it maps $\phi_{\underline{\boldsymbol{\lambda}}}$ into a Hilbert-Schmidt operator if k satisfies the above condition. This gives an alternative proof that those spherical representations appear discretely in the decomposition of the tensor product $H^{\frac{a}{2}} \otimes \overline{H^{\frac{a}{2}}} = \mathfrak{S}_2(H^{\frac{a}{2}})$, and a quantization of the spherical representations as operators on the minimal representation $H^{\frac{a}{2}}$.

Theorem 7.1. *Let $\underline{\boldsymbol{\lambda}}$ be as (5.3) with $\lambda = -i(\frac{a-1-b}{2} + k)$, and $0 \le k < \frac{a}{2}(r-2) + 1 + b$ being integers. If*

$$q > \frac{1+b+a(r-1)}{(1+b+\frac{a}{2}(r-2)-k)},$$

then the map

$$\begin{aligned}\phi_{\underline{\boldsymbol{\lambda}}}(z) &\mapsto h^{-\frac{a}{2}}(z,w)\phi_{\underline{\boldsymbol{\lambda}}}(z,w)\\ &= \sum_{\underline{\mathbf{m}}} d_{\underline{\mathbf{m}}}^{-\frac{1}{2}} \frac{1}{(1)_m(\frac{a}{2})_m} S_m(\lambda_1^2, \rho_r, \frac{1}{2}(1+b), \frac{a-(1+b)}{2}) e_{\underline{\mathbf{m}}}(z,w)\end{aligned}$$

extends to a linear operator from the linear span of $\{\phi_{\underline{\boldsymbol{\lambda}}}(gz); g \in G\}$ (of all translations of $\phi_{\underline{\boldsymbol{\lambda}}}$ under G) into the Schatten-von Neumann class $\mathfrak{S}_q$ of bounded operators on $H^{\frac{a}{2}}$. Define the norm of an element in the linear span to be the Schatten-von Neumann norm of the corresponding operator. The closure of the linear span of $\{\phi_{\underline{\boldsymbol{\lambda}}}(gz); g \in G\}$ is then a G-invariant Banach space.

The rest of this section is devoted to the proof.

Recall the expansion (5.13). From which we get immediately an expansion for its polarization,

$$h^{-\frac{a}{2}}(z,w)\phi_{\underline{\lambda}}(z,w) = \sum_{\underline{\mathbf{m}}} d_{\underline{\mathbf{m}}}^{-\frac{1}{2}} \frac{1}{(1)_m(\frac{a}{2})_m} S_m(\lambda_1^2, \rho_r, \frac{1}{2}(1+b), \frac{a-(1+b)}{2}) e_{\underline{\mathbf{m}}}(z,w).$$

Here we use the S_m to denote the continuous dual Hahn polynomials; see Section 9.

The operators $(\frac{a}{2})_{\underline{\mathbf{m}}} K_{\underline{\mathbf{m}}}$ are pairwise orthogonal projections, their $\mathfrak{S}_q$ norms satisfy

$$\|(\frac{a}{2})_{\underline{\mathbf{m}}} K_{\underline{\mathbf{m}}}\|_q^q = \dim(\mathcal{P}_{\underline{\mathbf{m}}}) = d_{\underline{\mathbf{m}}},$$

and

$$\|e_{\underline{\mathbf{m}}}\|_q^q = d_{\underline{\mathbf{m}}}^{1-\frac{q}{2}};$$

thus

(7.1)

$$\|h^{-\frac{a}{2}}\phi_{\underline{\lambda}}\|_q^q = \sum_{\underline{\mathbf{m}}} \left(d_{\underline{\mathbf{m}}}^{-\frac{1}{2}} \frac{1}{(1)_m(\frac{a}{2})_m} \right)^q \left(S_m(\lambda^2, \rho_r, \frac{1}{2}(1+b), \frac{a-(1+b)}{2}) \right)^q d_{\underline{\mathbf{m}}}^{1-\frac{q}{2}}.$$

Let us take $\lambda = -i(\frac{a}{2} - \frac{1}{2}(1+b) + k)$. Then

$$\begin{aligned} &S_m(\lambda_1^2, \rho_r, \frac{1}{2}(1+b), \frac{a-(1+b)}{2}) \\ &\quad = (\frac{a}{2}r)_m (1+b+\frac{a}{2}(r-1))_m \\ &\quad\quad \times {}_3F_2\left(-m, \frac{a}{2}r+k, 1+b+\frac{a}{2}(r-2)-k; \frac{a}{2}r, 1+b+\frac{a}{2}(r-1); 1\right). \end{aligned} \tag{7.2}$$

To estimate it for large m we use Thomae's transformation formula (see [6], p. 59),

$$\begin{aligned} &{}_3F_2\left(-m, \frac{a}{2}r+k, 1+b+\frac{a}{2}(r-2)-k; \frac{a}{2}r, 1+b+\frac{a}{2}(r-1); 1\right) \\ &= \frac{(\frac{a}{2}+k)_m}{(1+b+\frac{a}{2}(r-1))_m} {}_3F_2\left(-m, -k, 1+b+\frac{a}{2}(r-2)-k; \frac{a}{2}r, 1-m-\frac{a}{2}-k; 1\right). \end{aligned}$$

The term ${}_3F_2$ in r.h.s. is now bounded for all m, due to the appearance of the parameter $-k$ so that it is a finite sum of k-terms, each of which is bounded. Thus we find

$$\begin{aligned} &|S_m(\lambda_1^2, \rho_r, \frac{1}{2}(1+b), \frac{a-(1+b)}{2})| \\ &\quad \approx (\frac{a}{2}r)_m (1+b+\frac{a}{2}(r-1))_m \frac{(\frac{a}{2}+k)_m}{(1+b+\frac{a}{2}(r-1))_m} \\ &\quad = (\frac{a}{2}r)_m (\frac{a}{2}+k)_m. \end{aligned}$$

Also, the dimension $d_m = d_{\underline{\mathbf{m}}}$ of the space $\mathcal{P}_{\underline{\mathbf{m}}}$ has been calculated by Upmeier [29]:

$$d_{\underline{\mathbf{m}}} = \frac{(\frac{a}{2}r)_m (1+b+\frac{a}{2}(r-1))_m}{(1)_m(\frac{a}{2})_m}.$$

We can find the estimate of each term in (7.1):

$$
\begin{aligned}
&\left(d_{\underline{\mathbf{m}}}^{-\frac{1}{2}}\frac{1}{(1)_m(\frac{a}{2})_m}\right)^q ((\frac{a}{2}r)_m(\frac{a}{2}+k)_m)^q d_{\underline{\mathbf{m}}}^{1-\frac{q}{2}}\\
&= \left(\frac{(\frac{a}{2}r)_m(\frac{a}{2}+k)_m}{(1)_m(\frac{a}{2})_m}\right)^q {d_{\underline{\mathbf{m}}}}^{1-q}\\
&\approx m^{q(\frac{a}{2}r+\frac{a}{2}+k-1-\frac{a}{2})} m^{(1-q)(\frac{a}{2}r+1+b+\frac{a}{2}(r-1)-\frac{a}{2}-1)}\\
&\approx m^{a(r-1)+b+q(k-1-b-\frac{a}{2}(r-2))}.
\end{aligned}
\tag{7.3}
$$

Thus (7.1) is convergent if and only if

$$a(r-1)+b+q(k-1-b-\frac{a}{2}(r-2))<-1$$

or

$$q(1+b+\frac{a}{2}(r-2)-k)>1+b+a(r-1).$$

Since $q>0$, $k=0,1,\dots$, and $r\geq 2$, the above condition is equivalent to

$$0\leq k<\frac{a}{2}(r-2)+1+b$$

and

$$q>\frac{1+b+a(r-1)}{1+b+\frac{a}{2}(r-2)-k}. \tag{7.4}$$

This proves the first part of the theorem, and the remaining part then follows by abstract arguments.

Remark 7.2. Observe that since $k\geq 0$, the cut-off (7.4) is

$$\frac{1+b+a(r-1)}{1+b+\frac{a}{2}(r-2)-k}>1.$$

Namely those operators are never trace class. When $q=2$, the condition (7.4) amounts to

$$0\leq k<\frac{1+b}{2}-\frac{a}{2}$$

which is our previous condition in Section 6. Notice further that when

$$\frac{1+b}{2}-\frac{a}{2}\leq k<\frac{a}{2}(r-2)+1+b$$

the operator $h^{-\frac{a}{2}}(z,w)\phi_{\underline{\boldsymbol{\lambda}}}(z,w)$ is in $\mathfrak{S}_q$ for $q>2$, since $q-2>0$ in this case.

Remark 7.3. The unitary complementary series $\mathcal{C}_{\frac{a}{2}-\frac{1+b}{2}+k}$ for the group $SU(N,2)$ have also been discovered previously; see [13], Theorem 2.1 (a)(iii). (There, Knapp and Speh classified the unitary principal series representations $Ind_{MAN}(\tau\times\underline{\boldsymbol{\lambda}})$ induced from a minimal parabolic subgroup MAN with τ being a one-dimensional representation of M. In our case τ is the trivial representation; our series $\mathcal{C}_{\frac{a}{2}-\frac{1+b}{2}+k}$ are exactly the all those classified in Theorem 2.1(a)(iii) there. However, our result also gives the K-type structures of the representations.)

8. Positive definiteness of spherical functions

Proposition 8.1. *The spherical function $\phi_{\underline{\boldsymbol{\lambda}}}(z)$ is positive definite for all $\underline{\boldsymbol{\lambda}}$ in (5.3) and $\lambda \in \Sigma$.*

Proof. Fix $\lambda_0 \in \mathbb{R}^+$. For any $0 < \delta < \lambda_0$ take nonnegative C^∞-function f on $(0, \infty)$ with compact support $(\lambda_0 - \delta, \lambda_0 + \delta)$ so that

$$\int_{\lambda_0-\delta}^{\lambda_0+\delta} |f(\lambda)|^2 d\mu(\lambda) = 1.$$

Consider the element

$$F_\delta = \int_{\lambda_0-\delta}^{\lambda_0+\delta} f(\lambda) h^{-\frac{a}{2}}(z, w) \phi_{\underline{\boldsymbol{\lambda}}}(z, w) d\mu(\lambda).$$

By Theorem 6.2 F_δ is a unit vector in $(H^{\frac{a}{2}} \otimes \overline{H^{\frac{a}{2}}})_0$. But

$$(\pi(\frac{a}{2}) \otimes \overline{\pi(\frac{a}{2})}(g) F_\delta, F_\delta) = \int_{\lambda_0-\delta}^{\lambda_0-\delta} f(\lambda) \phi_{\underline{\boldsymbol{\lambda}}}(g \cdot 0) d\mu(\lambda)$$

is positive definite function of $g \in G$, being the matrix coefficients of a unitary representation. Let $\delta \to 0$. We claim that

$$(\pi(\frac{a}{2}) \otimes \overline{\pi(\frac{a}{2})}(g) F_\delta, F_\delta) \to \phi_{\underline{\boldsymbol{\lambda}}}(g \cdot 0),$$

therefore $\phi_{\underline{\boldsymbol{\lambda}}}(g \cdot 0)$ is a point-wise limit of positive definite functions, and consequently is positive definite. The proof of the above claim is a routine method. Indeed for any $\epsilon > 0$ let δ be chosen so that $|\phi_\lambda(g \cdot 0) - \phi_{\lambda_0}(g \cdot 0)| < \epsilon$ when $|\lambda - \lambda_0| \le \delta$. Then

$$(\pi(\frac{a}{2}) \otimes \overline{\pi(\frac{a}{2})}(g) F_\delta, F_\delta) - \phi_{\underline{\boldsymbol{\lambda}}}(g \cdot 0) = \int_{\lambda_0-\delta}^{\lambda_0+\delta} f(\lambda)(\phi_{\underline{\boldsymbol{\lambda}}}(g \cdot 0) - \phi_{\lambda_0}(g \cdot 0)) d\mu(\lambda)$$

and its absolute value is dominated, using Cauchy-Schwarz inequality, by

$$\epsilon \int_{\lambda_0-\delta}^{\lambda_0+\delta} |f(\lambda)| d\mu(\lambda) \le \epsilon \left(\int_{\lambda_0-\delta}^{\lambda_0+\delta} |f(\lambda)|^2 d\mu(\lambda) \right)^{\frac{1}{2}} \mu(\lambda_0 - \delta, \lambda_0 - \delta)^{\frac{1}{2}} \le \epsilon,$$

since μ is a probability measure, $\mu(\lambda_0 - \delta, \lambda_0 - \delta) \le 1$. $\square$

9. Appendix: Orthogonality relation of continuous dual Hahn polynomials

We summarize here some formulas that we used in this paper; see [31] and [14].

The continuous dual Hahn polynomials are defined by

$$S_m(x^2, \alpha, \beta, \gamma) = (\alpha + \beta)_m (\alpha + \gamma)_m \widetilde{S}_m(x^2, \alpha, \beta, \gamma) \tag{9.1}$$

where

$$\widetilde{S}_m(x^2) = \widetilde{S}_m(x^2, \alpha, \beta, \gamma) = {}_3F_2\left(-m, \alpha + ix, \alpha - ix; \alpha + \beta, \alpha + \gamma; 1\right). \tag{9.2}$$

Then the functions $\widetilde{S}_m(x^2, \alpha, \beta, \gamma)$ satisfy the recurrence relation (see [14])

(9.3)

$$-(\alpha^2 + x^2)\widetilde{S}_m(x^2) = A_{m+1}\widetilde{S}_{m+1}(x^2) - (A_{m+1} + C_{m-1})\widetilde{S}_m(x^2) + C_{m-1}\widetilde{S}_{m-1}(x^2),$$

where

$$A_m = (m+\alpha+\beta)(m+\alpha+\gamma), \quad C_m = (m+1)(m+\beta+\gamma).$$

Their orthogonality relation is given in the following

Theorem 9.1. *Let*

$$s_m(x^2,\alpha,\beta,\gamma) = \left(\frac{(\alpha+\beta)_m(\alpha+\gamma)_m}{(1)_m(\beta+\gamma)_m}\right)^{\frac{1}{2}} \widetilde{S}_m(x^2,\alpha,\beta,\gamma). \tag{9.4}$$

If α,β,γ are positive, then

$$\int_0^\infty s_m(x^2,\alpha,\beta,\gamma)s_l(x^2,\alpha,\beta,\gamma)d\mu(x) = \delta_{ml}$$

where $d\mu(x)$ on $(0,\infty)$ is the measure

$$\begin{aligned} d\mu(x) &= d\mu_{\alpha,\beta,\gamma}(x) \\ &= \frac{1}{2\pi}\frac{1}{\Gamma(\alpha+\beta)\Gamma(\alpha+\gamma)\Gamma(\beta+\gamma)}\left|\frac{\Gamma(\alpha+ix)\Gamma(\beta-ix)\Gamma(\gamma+ix)}{\Gamma(2ix)}\right|^2. \end{aligned}$$

If $\gamma<0$, α and β are positive, then

$$\int_{(0,\infty)\cup\{i(\gamma+k);k=0,1,\dots,\gamma+k<0\}} s_m(x^2,\alpha,\beta,\gamma)s_l(x^2,\alpha,\beta,\gamma)d\mu(x) = \delta_{ml}, \tag{9.5}$$

where $d\mu(x)$ is the sum of the above measure on $(0,\infty)$ and the discrete measure on $\{i(\gamma+k); 0\le k<-\gamma\}$:

$$\begin{aligned} &\sum_{0\le k<\gamma}\frac{\Gamma(\gamma+\beta)\Gamma(\gamma+\alpha)\Gamma(\beta-\gamma)\Gamma(\alpha-\gamma)}{\Gamma(-2\gamma)} \\ &\qquad\times\frac{(2\gamma)_k(\gamma+1)_k(\gamma+\beta)_k(\gamma+\alpha)_k}{(\gamma)_k(\gamma-\beta+1)_k(\gamma-\alpha+1)_k k!}(-1)^k\delta_{i(\gamma+k)}. \end{aligned}$$

Here $\delta_{i(\gamma+k)}$ stands for the Dirac measure at the given point.

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Department of Mathematics, Chalmers University of Technology and Göteborg University, S-412 96 Göteborg, Sweden

E-mail address: `genkai@math.chalmers.se`