REPRESENTATION THEORY
An Electronic Journal of the American Mathematical Society
Volume 5, Pages 437–454 (October 30, 2001)
S 1088-4165(01)00140-6

NONVANISHING OF A CERTAIN SESQUILINEAR FORM IN THE THETA CORRESPONDENCE

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ABSTRACT. Suppose $2n+1 \geq p+q$. In an earlier paper in 2000 we study a certain sesquilinear form $(,)_\pi$ introduced by Jian-Shu Li in 1989. For π in the semistable range of $\theta(MO(p,q) \to MSp_{2n}(\mathbb{R}))$, if $(,)_\pi$ does not vanish, then it induces a sesquilinear form on $\theta(\pi)$. In another work in 2000 we proved that $(,)_\pi$ is positive semidefinite under a mild growth condition on the matrix coefficients of π. In this paper, we show that either $(,)_\pi$ or $(,)_{\pi\otimes\det}$ is nonvanishing. These results combined with one result of Przebinda suggest the existence of certain unipotent representations of $Mp_{2n}(\mathbb{R})$ beyond unitary representations of low rank.

1. Introduction

Let $(G_1, G_2) = (O(p,q), Sp(n,\mathbb{R}))$ be a dual pair in $G = Sp(n(p+q),\mathbb{R})$, and let ω be the metaplectic representation of MG, the metaplectic covering group of G. For a subgroup H of G, we will use MH to denote the preimage of H under the metaplectic covering. Let $\mathcal{P}$ be the Harish-Chandra module of ω. Let $\mathcal{R}(MG_i,\omega)$ be the equivalent classes of irreducible Harish-Chandra modules of MG_i which occur as quotients of $\mathcal{P}$. Theta correspondence defined by Howe (see [11]) is a one-to-one correspondence between $\mathcal{R}(MG_1,\omega)$ and $\mathcal{R}(MG_2,\omega)$. We denote this correspondence by $\theta(MG_1 \to MG_2)$. The sets $\mathcal{R}(MG_i,\omega)$ are not known in general.

Let ϵ be the nontrival element in the preimage of the identity. Then $\omega(\epsilon) = -1$. If $\pi \in \mathcal{R}(MG_i,\omega)$, it is easy to see that $\pi(\epsilon) = -1$. Throughout this paper, we consider only the representations π such that $\pi(\epsilon) = -1$. Consider the dual pair $(O(p,q), Sp(n,\mathbb{R}))$ with $p+q \leq 2n+1$. Fix a maximal compact subgroup $K \subset MSp(n(p+q),\mathbb{R})$. Let Λ be the matrix coefficient of the oscillator representation corresponding to the lowest K-type of the oscillator representation ω. The absolute value of Λ was denoted by Ω in [19]. Let $\Lambda(p,q)$ be the restriction of Λ to $MO(p,q)$. The main result could be stated as follows.

Theorem 1.1 (Main Theorem)**.** *Suppose $p+q \leq 2n+1$. Suppose f is a continuous complex valued function on $MSO(p,q)$ such that*

$$f(\epsilon\tilde{g}) = -f(\tilde{g}) \qquad (\tilde{g} \in MSO(p,q))$$

and

$$f\Lambda(p,q) \in L^1(MSO(p,q)).$$

Received by the editors April 24, 2001 and, in revised form, July 30, 2001.
2000 *Mathematics Subject Classification.* Primary 22E45.

Assume that for all u, v in $\mathcal{P}$,

$$\int_{MSO(p,q)} f(\tilde{g})(u, \omega(\tilde{g})v) d\tilde{g} = 0. \tag{1}$$

Then $f \equiv 0$.

We say that a function on MH is an *odd function* if

$$f(\epsilon x) = -f(x) \qquad (\forall \ \ x \in MH).$$

We may now state an equivalent form of the main theorem.

Theorem 1.2 (Density Theorem)**.** *Suppose $p+q \leq 2n+1$. The linear span of the matrix coefficients of ω restricted to $MSO(p,q)$ is dense in $L^2_{odd}(MSO(p,q))$. Here matrix coefficients are functions of the form*

$$(\omega(\widetilde{g})\phi, \psi) \qquad (\widetilde{g} \in MSO(p,q), \psi, \phi \in \mathcal{P}).$$

Let $\mathcal{M}(p,q)$ be the set of matrix coefficients of ω restricted to $MO(p,q)$. Following [6], we define an analytic compactification

$$\mathcal{H} : O(p,q) \to O(p+q).$$

Our compactification has the following interesting property.

Theorem 1.3. *Let $f \in \mathcal{M}(p,q)$. Then $\frac{f}{\Lambda(p,q)}$ is a function on $O(p,q)$. Furthermore, there exists a function in f^0 in $\mathcal{M}(0,p+q)$ such that*

$$\frac{f^0}{\Lambda(0,p+q)}(\mathcal{H}(g)) = \frac{f}{\Lambda(p,q)}(g) \qquad (g \in O(p,q)).$$

We write

$$\frac{\mathcal{M}(0,p+q)}{\Lambda(0,p+q)} \cong \frac{\mathcal{M}(p,q)}{\Lambda(p,q)}.$$

In fact, as long as the group action is concerned,

$$\frac{\mathcal{M}(0,p+q)}{\Lambda(0,p+q)} = \frac{\mathcal{M}(p+q,0)}{\Lambda(p+q,0)}.$$

However, $\Lambda(0,p+q)$ may differ from $\Lambda(p+q,0)$ by a conjugate. That is the reason we keep the notation $O(0,p+q)$ and $\mathcal{M}(0,p+q)$. This theorem is motivated by a suggestion from the referee.

In [8], the author followed some earlier idea of Jian-Shu Li (see [17]) and constructed the theta correspondence in semistable range. Let π be an irreducible representation in the semistable range of $\theta(MG_1 \to MG_2)$ (see [8]). Then one can define a (real) bilinear form on $\omega \otimes \pi$:

$$(\phi \otimes u, \psi \otimes v)_\pi = \int_{MG_1} (\omega(g)\phi, \psi)(v, \pi(g)u) dg.$$

If $(,)_\pi$ is nonvanishing, $\omega \otimes \pi^c$ modulo the radical of $(,)_\pi$ is irreducible. Furthermore, it produces the theta correspondence. The remaining question is whether $(,)_\pi$ vanishes.

Corollary 1.1. *Let $G_1 = O(p,q)$ and $G_2 = Sp(n,\mathbb{R})$. Let* det *be the central character of $MO(p,q)$ defined to be the lift of the determinant on $O(p,q)$. Suppose $p+q \leq 2n+1$. If π is an irreducible representation in the semistable range of*

$$\theta(MG_1 \to MG_2),$$

then at least one of

$$(,)_\pi, (,)_{\pi\otimes\det}$$

is nonvanishing.

Thus up to a central character, $\pi \in R(MG_1, \omega)$. This gives a partial description of $\mathcal{R}(MG_1, \omega)$.

This paper is structured as follows. In Section 2, we introduce the Segal-Bargmann Model. In Section 3, we review some basic theory of symmetric spaces and orthogonal groups. In Section 4, we study the dual pair $(O(p,q), Sp(n,\mathbb{R}))$ in the oscillator representation of $Sp(n(p+q),\mathbb{R})$. We obtain an analytic compactification $\mathcal{H}$ of $O(p,q)$. In Section 5, we utilize the compactification $\mathcal{H}$ to prove the main theorem and the nonvanishing of $(,)_\pi$. The proofs given here are substantially shorter than the proofs given in the author's thesis ([10]).

Acknowledgments. The author wishes to thank the referee for his suggestions and for pointing out the Kashiwara-Vergne paper which shortens the proof of the main theorem. Most of this paper comes out of the author's thesis. The author wishes to thank his advisor David Vogan for guidance.

1.1. **Convention.** Let G be a reductive group. We use $\mathcal{R}(G)$ to denote the set of equivalence classes of *irreducible admissible representations* of G. An irreducible admissible representation is an irreducible Harish-Chandra module equipped with a pre-Hilbert structure which is invariant under the action of a fixed maximal compact subgroup K. When we speak of the group $O(p,q)$ or $SO(p,q)$, we assume $p \leq q$ unless stated otherwise.

Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{C}^n$. For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$, let

$$x^\alpha = \prod_1^n x_i^{\alpha_i}.$$

We use the "overline" to denote complex conjugation. Throughout this paper, $\mathbb{N}$ will be the set of natural numbers including zero.

2. Segal-Bargmann Model

The oscillator representation is a *projective representation* of the *symplectic group*. It was discovered by Segal, Shale and Weil in the early sixties. The most well-known model was the Schrödinger model. In this model, the underlying Hilbert space is the L^2-space on real Euclidean space $\mathbb{R}^n$. The *infinitesimal generators* of the symplectic group act as skew-adjoint differential operators. However, the group action is hard to describe due to the fact that the Fourier integration operator is involved. On the other hand, there is the Bargmann-Segal model. The underlying Hilbert space $\mathcal{F}_n$ is the space of square integrable analytic functions on $\mathbb{C}^n$ with respect to the Gaussian measure. Bargmann computed the isometry between $\mathcal{F}_n$ and $L^2(\mathbb{R}^n)$ (see [1]). This model was later studied by many people. One advantage of the Segal-Bargmann model is that the group action can be represented by integral operators consistently.

The Segal-Bargmann model enables us to study some real analysis problem using complex analytic tools. In this paper, we will first review the Segal-Bargmann model. Then we convert problems on noncompact Lie groups into problems on compact Lie groups in the spirit of the compactification of classical groups from [6] and [7].

2.1. **Vector Space V.** In this paper, V is regarded as

1. an n-dimensional complex Hilbert space $(V, (,))$ with a fixed orthonormal basis
$$\{e_1, e_2, \ldots, e_n\}$$
(sometimes vectors in V are indicated by $\mathbb{C}$);
2. a $2n$-dimensional real vector space $(V, \Re(,))$ with fixed orthonormal basis
$$\{ie_1, ie_2, \ldots, ie_n, e_1, e_2, \ldots, e_n\};$$
3. a $2n$-dimensional real symplectic vector space $(V, \Im(,))$ with fixed standard basis
$$\{ie_1, ie_2, \ldots, ie_n, e_1, e_2, \ldots, e_n\}.$$

We refer to $\{e_1, e_2, \ldots, e_n\}$ as the complex basis, and $\{ie_1, \ldots, ie_n, e_1, \ldots, e_n\}$ as the real basis.

We write $\Omega(,) = \Im(,)$. We have
$$\Omega(ie_i, e_j) = \delta_i^j \qquad \Omega(e_i, e_j) = 0 \qquad \Omega(ie_i, ie_j) = 0.$$
Furthermore, under the real basis, the complex multiplication in V is given by the left multiplication by
$$J = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}.$$
The complex conjugation, always denoted by an overline, is given by the left multiplication by
$$S_{-n,n} = \begin{pmatrix} -I_n & 0 \\ 0 & I_n \end{pmatrix}.$$
In this paper, we will use both languages of real vector space and complex vector space when V is involved.

2.2. **Endomorphisms in V.** Let $End_{\mathbb{R}}(V)$ be the space of real endomorphisms on V, and $End_{\mathbb{C}}(V)$ the space of complex endomorphisms on V. Let $g \in End_{\mathbb{C}}(V)$. Suppose $g = A + iB$, A and B are $n \times n$ real matrices. Then the real form of g is given by
$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$$
We denote this matrix by $g^{\mathbb{R}}$. The map $g \to g^{\mathbb{R}}$ produces an injection from $End_{\mathbb{C}}(V)$ to $End_{\mathbb{R}}(V)$.

On the other hand, suppose $g \in End_{\mathbb{R}}(V)$. Then we may decompose g into a sum of a complex-linear endomorphism and a complex conjugate-linear endomorphism as follows:
$$C_g = \frac{1}{2}(g - JgJ) \qquad A_g = \frac{1}{2}(g + JgJ).$$
Then $g = C_g + A_g$. We define $C_g^{\mathbb{C}}$ and $A_g^{\mathbb{C}}$ by
$$C_g^{\mathbb{C}} v = C_g v \qquad A_g^{\mathbb{C}} v = A_g \overline{v} \qquad (\forall\ v \in V).$$
The maps $g \to C_g^{\mathbb{C}}$ and $g \to A_g^{\mathbb{C}}$ are surjections from $End_{\mathbb{R}}(V)$ to $End_{\mathbb{C}}(V)$.

Finally, if C_g is invertible, we write

$$Z_g = C_g^{-1} A_g.$$

Then Z_g is a conjuagate-linear endomorphism. We write

$$Z_g^{\mathbb{C}} v = Z_g \overline{v}.$$

2.3. **Groups attached to V.** There are five interesting groups attached to the vector space V:

1. $K = U(n)$, the group fixing the complex inner product of V;
2. $O = O(2n)$, the group fixing the real inner product of V;
3. $G = Sp_{2n}(\mathbb{R})$, the group fixing the symplectic form Ω;
4. H, the subgroup of G fixing

$$span(e_1, e_2, \ldots, e_n) \text{ and } span(ie_1, ie_2, \ldots, ie_n);$$

5. H_0, the identity component of H.

It is obvious that

$$K = O \cap G$$

and K is a maximal subgroup of G.

Now observe that the fundamental group of K is $\mathbb{Z}$. It follows that

$$\pi_1(G) = \mathbb{Z}.$$

Therefore, there exists a unique double covering of G. This double covering is often called the metaplectic covering. We denote the covering group by MG and the covering by M. For any subgroup P of G, we denote the preimage of P under the metaplectic covering by MP. Let $\{1, \epsilon\}$ be the preimage of the identity in G. The reader should be warned that MG is NOT an algebraic group. Nevertheless, it has a nice analytic presentation (see [20])

$$MG = \{(\xi, g) \mid g \in G, \xi \in \mathbb{C}, \xi^2 \det(C_g^{\mathbb{C}}) = 1\}.$$

Finally, let us look at MK and MH. Since K is already complex linear,

$$MK = \{(\xi, g) \mid g \in K, \xi \in \mathbb{C}, \xi^2 \det g^{\mathbb{C}} = 1\}.$$

MK is the unique double covering of K. For $g \in H$, we assume that g is of the form

$$\begin{pmatrix} A & 0 \\ 0 & (A^{-1})^t \end{pmatrix} \qquad (A \in GL(n, \mathbb{R})).$$

Then

$$C_g^{\mathbb{C}} = \frac{A + (A^{-1})^t}{2} = \frac{1}{2} A (I + (A^t A)^{-1}).$$

We see that $\det A$ and $\det C_g^{\mathbb{C}}$ must both be real and have the same sign. It follows that

$$MH_0 \cong \{(\xi, g) \mid g \in H_0, \xi = \pm (\det A)^{-\frac{1}{2}}\} \cong H_0 \times \mathbb{Z}/2.$$

The isomorphisms are group isomorphisms. Similarly, as topological spaces, we have

$$MH \cong H \times \mathbb{Z}/2.$$

In summary, the metaplectic covering on H (H_0) splits.

2.4. **Segal-Bargmann Model.** Let dx be the Euclidean measure on V and

$$d\mu(x) = \exp(-\frac{1}{2}(x,x))dx$$

the Gaussian measure. Let $\mathcal{P}_n$ be the polynomial ring on $V^{\mathbb{C}}$. We define an inner product $(,)$ on $\mathcal{P}_n$ by

$$(f,g) = \int_V f(x)\overline{g(x)}d\mu(x) \qquad (f, g \in \mathcal{P}_n).$$

This makes $\mathcal{P}_n$ a pre-Hilbert space. Let $\mathcal{F}_n$ be the completion of $\mathcal{P}_n$. Then $\mathcal{F}_n$ is exactly the space of square-Gaussian-integrable analytic functions on $V^{\mathbb{C}}$ (see [1]).

Theorem 2.1 (Segal-Bargmann Model)**.** *Let $G = Sp(n,\mathbb{R})$. For $(\xi, g) \in MG$, we define a unitary operator*

$$\omega(\xi,g)f(z) = \int_V \xi \exp\frac{1}{4}(2(C_g^{-1}z,w) - (z, Z_{g^{-1}}z) - (Z_g w, w))f(w)d\mu(w).$$

Then ω yields a faithful unitary representation of MG. Furthermore, $\mathcal{P}_n$ is the Harish-Chandra module of ω.

We write

$$\mathcal{H}(g,z,w) = 2(C_g^{-1}z,w) - (z, Z_{g^{-1}}z) - (Z_g w, w).$$

We call it the Bargmann-Segal kernel. Under the Cartan decomposition of $Sp_{2n}(\mathbb{R})$, this kernel can be written as

$$\mathcal{H}(g,z,w) = 2(\mathrm{sech}(H)k_1^{-1}z, k_2 w) + (k_1^{-1}z, \tanh(H)k_1^{-1}z) - (\tanh(H)k_2 w, k_2 w)$$

for $g = k_1\exp(H(g))k_2$. On the Harish-Chandra module $\mathcal{P}_n$, the infinitesimal generators of G act as differential operators of degree less or equal to 2 (see [11]).

In Segal-Bargmann model, the lowest K-type is the space of constant functions. Let $\phi_0(z) \equiv 1$. Fix $(\xi, g) \in Mp(n,\mathbb{R})$. From (3) of [6],

$$\mathcal{H}(g,z,w) = (iz^t, \overline{w^t})\mathcal{H}(g)\begin{pmatrix} iz \\ \overline{w} \end{pmatrix}$$

with $\mathcal{H}(g)$ a $2n \times 2n$ symmetric unitary matrix. Write $\mathcal{H}(g) = UU^t$ with $U \in U(n)$. Then

$$\begin{aligned}
&(\omega(\xi,g)\phi_0, \phi_0) \\
&\quad = \xi \int_{V\times V} \exp\frac{1}{4}(iz^t, \overline{w^t})UU^t\begin{pmatrix} iz \\ \overline{w} \end{pmatrix} d\mu(z,w) \\
&\quad = \xi \int_{V\times V} \exp\frac{1}{4}(iz^t, \overline{w^t})\begin{pmatrix} iz \\ \overline{w} \end{pmatrix} d\mu(z,w) \\
&\quad = \xi.
\end{aligned} \tag{2}$$

We denote the function

$$(\xi, g) \to \xi$$

by Λ. Λ is the diagonal matrix coeffient of ω corresponding to the lowest K-type.

3. Groups—Structure Theory

Our understanding of representation theory relies on our understanding of the structure of the group. In this section, we will introduce some standard material which we will use in subsequent sections. Our aim is to do analysis on $O(p,q)$ and $O(p+q)$. Only in this section, will we abandon the notations from the previous section.

3.1. **Symmetric Pairs.**

We recall some definition and basic facts about symmetric spaces from [3, Ch. 4.3]

Definition 3.1 (symmetric pair)**.** Let G be a connected reductive Lie group and H a closed subgroup. Let σ be an involution of G such that

$$(G^\sigma)_0 \subseteq H \subseteq G^\sigma$$

where G^σ is the fix point set of σ and $(G^\sigma)_0$ the identity component of G^σ. The pair (G,H) is called a reductive symmetric pair. If $Ad_G(H)$ is compact, (G,H) is said to be a Riemannian reductive symmetric pair.

We will only be interested in Riemannian symmetric pairs and Riemannian symmetric spaces. According to [3, Ch. 4.3], a Riemannian reductive symmetric pair yields a Riemannian globally reductive symmetric space G/K, and every Riemannian globally symmetric space can be obtained from a Riemannian reductive symmetric pair.

Definition 3.2 (Weyl group)**.** Let (G,K) be a Riemannian reductive symmetric pair. Let $\langle\, ,\, \rangle$ be an invariant real symmetric bilinear form on $\mathfrak{g}$ such that $(,)_{\mathfrak{k}}$ is negative definite. Let $\mathfrak{p} = \mathfrak{k}^\perp$. Let $\mathfrak{h}_{\mathfrak{p}}$ be a maximal Abelian subspace of $\mathfrak{p}$. Let $H = \exp \mathfrak{h}_{\mathfrak{p}}$ be the corresponding Abelian subgroup. Let M, M' be the centralizer and normalizer of $\mathfrak{h}_{\mathfrak{p}}$ in K respectively. In other words,

$$M = \{k \in K \mid Ad(k)h = h \ \forall h \in \mathfrak{h}_{\mathfrak{p}}\},$$

$$M' = \{k \in K \mid Ad(k)\mathfrak{h}_{\mathfrak{p}} \subseteq \mathfrak{h}_{\mathfrak{p}}\}.$$

The quotient group $W(G,K) = M'/M$ is called the Weyl group of (G,K).

Theorem 3.1 (Symmetric decomposition)**.** *Every Riemannian reductive symmetric pair (G,K) induces a decomposition of G into KHK. For an arbitrary $x \in G$, $H(x)$ is unique up to a conjugation of $W(G,K)$ and a multiplication of $K \cap H$.*

Theorem 3.2. *For every reductive symmetric pair (G,K), there exists a G-invariant measure*

$$d_G g = \Delta(H(g)) dk_1 dH dk_2 \qquad (k_1 \exp(H(g))k_2 = g).$$

If $\Delta(H(g)) \neq 0$, we say that g is regular; if $\Delta(H(g)) = 0$, we say that g is singular. The singular set is of codimension at least 2.

Most of the proof can be found in [4, Ch. 1.5, section 2], [3, Ch. 7.3], and [5, Ch. 7.8]. Notice for G noncompact, this decomposition is nothing more than Cartan decomposition, and the results are well-known. In all cases, we will use $d_G g$ to denote the Haar measure of G, and $d_X x$ to denote the G-invariant measure of X. When different groups are involved, we may use $\Delta_G(H)$ to specify the group invariance.

The orthogonal groups are not connected. We can still define symmetric pairs, Weyl groups and symmetric decomposition for reductive groups with a finite number of components. But Theorem 3.2 will not be valid due to the change of orientation. Nevertheless, Theorem 3.2 remains valid for the special orthogonal groups.

3.2. **Noncompact Orthogonal Groups.** Let $G = O(p,q)$ be the group fixing the quadratic form

$$Q(x_1,\dots,x_p,y_1,\dots,y_q)=\sum_{i=1}^{p}x_i^2-\sum_{j=1}^{q}y_j^2.$$

Let $K = O(p)\times O(q)$ be the subgroup of G fixing both

$$Q_1(x_1,\dots,x_p)=\sum_{i=1}^{p}x_i^2;\qquad Q_2(y_1,\dots,y_q)=\sum_{j=1}^{q}y_j^2.$$

Then K is a maximal compact subgroup of G and (G,K) is a symmetric pair. We fix a maximal split Abelian subgroup A consisting of the following elements:

$$\exp H(\lambda)=\begin{pmatrix}\cosh\lambda & \sinh\lambda & 0\\ \sinh\lambda & \cosh\lambda & 0\\ 0 & 0 & I_{q-p}\end{pmatrix}.$$

The corresponding Lie algebra $\mathfrak{a}$ consists of

$$\left\{H(\lambda)=\begin{pmatrix}0_p & \lambda & 0_{p,q-p}\\ \lambda & 0_p & 0_{p,q-p}\\ 0_{q-p,p} & 0_{q-p,p} & 0_{q-p}\end{pmatrix}\mid \lambda=diag(\lambda_1,\dots,\lambda_p)\right\}.$$

The open positive Weyl chamber $\mathfrak{a}^+$ is given by those λ such that

$$\lambda_1>\lambda_2>\cdots>\lambda_p>0.$$

I should remark that for $O(p,p)$, we do need the disconnectedness of $O(p,q)$ in order to produce such a Weyl Chamber. In this formulation, the group G has a $K\overline{A^+}K$ decomposition, namely Every $g\in G$ can be written as

$$k_1\exp H(g)k_2 \qquad (k_1,k_2\in K, H(g)\in\overline{\mathfrak{a}^+}).$$

3.3. **Compact Orthogonal Groups.** Let us consider $G = O(p+q)$. Then $(O(p+q), O(p)\times O(q))$ is a Riemannian symmetric pair. Let $\mathbb{T}_p$ be a compact torus consisting of elements of the form

$$T(\theta)=\begin{pmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & I_{q-p}\end{pmatrix}\qquad (\theta\in(-\pi,\pi]^p).$$

For each θ_i, we may define an element

$$\begin{pmatrix}\cos\theta_i & \sin\theta_i\\ -\sin\theta_i & \cos\theta_i\end{pmatrix}\in\mathbb{T}_1.$$

Then $\mathbb{T}_p$ can be identified with the direct product of p copies of $\mathbb{T}_1$. If we choose $H=\mathbb{T}_p$, then the symmetric decomposition holds for $(O_{p+q}, O_p\times O_q)$.

Observe that

- $H\cap K\cong(\mathbb{Z}/2\mathbb{Z})^p$. More explicitly, we have

$$H\cap K=\{diag(A,A,I_{q-p})\mid A=diag(\pm1,\pm1,\dots,\pm1)\in O_p\}.$$

- $W(O_{p+q}, O_p \times O_q)$ acts on $\mathbb{T}_p \cong \mathbb{T}_1 \times \mathbb{T}_1 \times \ldots \times \mathbb{T}_1$ by permutations and transposes on the copies of $\mathbb{T}_1$. Transpose on
$$\begin{pmatrix} \cos\theta_i & \sin\theta_i \\ -\sin\theta_i & \cos\theta_i \end{pmatrix}$$
can be identified with $\theta_i \to -\theta_i$.

Theorem 3.3 ($K\mathbb{T}_pK$ decomposition)**.** *Every $g \in O_{p+q}$ can be decomposed into $k_1T(\theta(g))k_2$, such that*
$$\pi/2 \geq \theta_1(g) \geq \theta_2(g) \geq \ldots \theta_p(g) \geq 0.$$

Proof. First of all g can be decomposed into $k_1T(\theta)k_2$. We know that $T(\theta)$ is unique modulo the action of $W(O(p+q), O(p)\times O(q))$ and multiplication by $H\cap K$. Applying a multiplication of $H \cap K$ on $T(\theta)$, we may choose $\cos(\theta_i) \geq 0$, i.e.,
$$\pi/2 \geq \theta_i \geq -\pi/2 \qquad (i = 1\ldots, p).$$
Again, applying a conjugation by $W(O_{p+q}, O_p \times O_q)$, we may choose
$$\pi/2 \geq \theta_1 \geq \theta_2 \geq \ldots \geq \theta_p \geq 0.$$
Therefore, $g \in O_{p+q}$ can be decomposed into $k_1T(\theta(g))k_2$, such that
$$\pi/2 \geq \theta_1(g) \geq \theta_2(g) \geq \ldots \theta_p(g) \geq 0.$$
□

4. Dual Pair $(O(p, q), Sp_{2n}(\mathbb{R}))$

The dual pair we are interested is $(O(p, q), Sp_{2n}(\mathbb{R}))$ in $Sp_{2n(p+q)}(\mathbb{R})$. Under a proper setting which we will discuss now, $O(p, q)$ will be the left multiplication on some matrix space, and $Sp_{2n}(\mathbb{R})$ will be the right multiplication. From now on, we return to the notation set in Section 1.

4.1. Vector space V.

1. Let $V = Mat(p, n, \mathbb{C}) \oplus Mat(q, n, \mathbb{C})$. Now V is regarded as a complex linear space. For $v \in V$, we may either write $v = (v_1, v_2)$ or $v = v_1 + v_2$ where $v_1 \in Mat(p, n, \mathbb{C})$ and $v_2 \in Mat(q, n, \mathbb{C})$. Let $v_i = \Re(v_i) + i\Im(v_i)$. We define
$$(u, v) = Tr(u_1\overline{v_1^t}) + Tr(u_2\overline{v_2^t}).$$
An easy computation shows that
$$\Re(u, v) = Tr(\Re(u_1)\Re(v_1^t) + \Im(u_1)\Im(v_1^t) + \Re(u_2)\Re(v_2^t) + \Im(u_2)\Im(v_2^t)),$$
$$\Im(u, v) = Tr(\Im(u_1)\Re(v_1^t) - \Re(u_1)\Im(v_1^t) + \Im(u_2)\Re(v_2^t) - \Re(u_2)\Im(v_2^t)).$$
2. We identify V with $Mat(p + q, 2n, \mathbb{R})$ as follows:
$$C : (u_1, u_2) \to \begin{pmatrix} \Re(u_1) & \Im(u_1) \\ \Im(u_2) & \Re(u_2) \end{pmatrix}.$$
We let
$$\Im(u_1) = X_{12} \qquad \Re(v_1) = Y_{11} \qquad \Im(v_1) = Y_{12} \qquad \Re(u_1) = X_{11},$$
$$\Im(u_2) = X_{21} \qquad \Re(v_2) = Y_{22} \qquad \Im(v_2) = Y_{21} \qquad \Re(u_2) = X_{22}.$$

Then the inverse of C is given by

$$C^{-1}:\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \to (X_{11}+iX_{12}, X_{22}+iX_{21}).$$

We call X_{ij} the real coordinates, and u_i the complex coordinates.

3. Under the real basis, the imaginary part of (X,Y) is given by

$$\Omega(X,Y) = Tr(X_{12}Y_{11}^t - X_{11}Y_{12}^t + X_{21}Y_{22}^t - X_{22}Y_{21}^t).$$

Let

$$S_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \qquad W = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}.$$

Then Ω can be written as

$$\Omega(X,Y) = Tr(-WY^tS_{p,q}X).$$

4.2. **Endomorphisms.**

1. The group $O(p,q)$ acts on $V = Mat(p+q, 2n, \mathbb{R})$ by left multiplication and the group $Sp_{2n}(\mathbb{R})$ acts on V by right multiplication. It is clear that the two group actions preserve $\Omega(,)$ and commute. Therefore

$$(O(p,q), Sp_{2n}(\mathbb{R})) \to Sp(V,\Omega)$$

is a dual pair. We denote the left multiplication by L and right multiplication by R. L and R are interpreted as the embedding of $O(p,q)$ and $Sp_{2n}(\mathbb{R})$ into $Sp_{2n(p+q)}(\mathbb{R})$ to produce the dual pair.

2. The complex scalar multiplication of i as a linear transform in $Mat(p+q, 2n, \mathbb{R})$ can be written as

$$\begin{aligned}\begin{pmatrix} \Re(u_1) & \Im(u_1) \\ \Im(u_2) & \Re(u_2) \end{pmatrix} \to & \begin{pmatrix} -\Im(u_1) & \Re(u_1) \\ \Re(u_2) & -\Im(u_2) \end{pmatrix} \\ & = S_{p,q}\begin{pmatrix} \Re(u_1) & \Im(u_1) \\ \Im(u_2) & \Re(u_2) \end{pmatrix} W.\end{aligned}$$

We denote iX (under the complex basis) by $J(X)$ (under the real basis):

$$J(X) = S_{p,q}XW.$$

Here J is regarded as a real linear endomorphism on V. It follows that

$$J(L(g)J(X)) = S_{p,q}(gS_{p,q}XW)W = -S_{p,q}gS_{p,q}X \qquad (X \in V).$$

3. Based on the above equation, we compute

$$J(L(\exp H(\lambda)))J = L\left(\begin{pmatrix} -\cosh\lambda & \sinh\lambda & 0 \\ \sinh\lambda & -\cosh\lambda & 0 \\ 0 & 0 & -I_{q-p} \end{pmatrix}\right),$$

$$C_{L(\exp H(\lambda))} = L\left(\begin{pmatrix} \cosh\lambda & 0 & 0 \\ 0 & \cosh\lambda & 0 \\ 0 & 0 & I_{q-p} \end{pmatrix}\right),$$

$$A_{L(\exp H(\lambda))} = L\left(\begin{pmatrix} 0 & \sinh\lambda & 0 \\ \sinh\lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right).$$

We denote $C_{L(\exp H(\lambda))}$ by $L(\cosh\lambda)$, and $A_{L(\exp H(\lambda))}$ by $L(\sinh\lambda)$.

4. Immediately, we obtain

$$Z_{L(\exp H(\lambda))} = L\left(\begin{pmatrix} 0 & \tanh(\lambda) & 0 \\ \tanh(\lambda) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right),$$

$$Z_{L(\exp(-\lambda))} = -Z_{L(\exp H(\lambda))}.$$

We denote $Z_{L(\exp H(\lambda))}$ by $L(\tanh\lambda)$.

5. The standard maximal compact subgroup of $O(p,q)$ is $O(p) \times O(q)$. For $(U_1, V_1) \in O(p) \times O(q)$, and $(u_1, v_1) \in V$, we have

$$L(U_1, V_1)(u_1, v_1) = (U_1 u_1, V_1 v_1).$$

4.3. **Segal-Bargmann model.** Let us consider the oscillator representation of $Mp_{2n(p+q)}(\mathbb{R})$. The group $L(O(p,q))$ is a subgroup of $Sp_{2n(p+q)}(\mathbb{R})$. We wish to compute the Segal-Bargmann kernel $\mathcal{H}(g,z,w)$ for $L(O(p,q))$. Let $g = k_1 \exp H(\lambda) k_2$ be the Cartan decomposition of $O_{p,q}$. Let $k_i = (U_i, V_i) \in O_p \times O_q$. Then U_i, V_i are real and

$$U_i^{-1} = U_i^t \qquad V_i^{-1} = V_i^t.$$

We will rely on the real coordinates to navigate. But our final goal is to obtain $\mathcal{H}(g,z,w)$ in complex coordinates.

Let $z = (z_1, z_2), w = (w_1, w_2)$ be the complex coordinates of z and w. Recall that under the Cartan decomposition, the Bargmann-Segal kernel is, in general, defined as

$$\mathcal{H}(g,z,w) = 2(\mathrm{sech}(H) k_1^{-1} z, k_2 w) + (k_1^{-1} z, \tanh(H) k_1^{-1} z) - (\tanh(H) k_2 w, k_2 w).$$

1. In our setting, we compute the first term

$$\begin{aligned} &(L(\cosh\lambda)^{-1} k_1^{-1} z, k_2 w) \\ &= (\mathrm{sech}(\lambda) U_1^{-1} z_1, U_2 w_1) + (\begin{pmatrix} \mathrm{sech}\lambda & 0 \\ 0 & I_{q-p} \end{pmatrix} V_1^{-1} z_2, V_2 w_2) \\ &= Tr(\overline{w_1}^t U_2^t \mathrm{sech}\lambda U_1^t z_1) + Tr(\overline{w_2^t} V_2^t \begin{pmatrix} \mathrm{sech}\lambda & 0 \\ 0 & I_{q-p} \end{pmatrix} V_1^t z_2). \end{aligned} \tag{3}$$

2. To compute $L(\tanh(\lambda))$, we first consider $u = (u_1, u_2)$. Under the real coordinates, we have

$$\begin{aligned} &L(\tanh(\lambda)) \begin{pmatrix} \Re(u_1) & \Im(u_1) \\ \Im(u_2) & \Re(u_2) \end{pmatrix} \\ &= \begin{pmatrix} 0 & \tanh(\lambda) & 0 \\ \tanh(\lambda) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \Re(u_1) & \Im(u_1) \\ \Im(u_2) & \Re(u_2) \end{pmatrix} \\ &= \begin{pmatrix} (\tanh\lambda, 0)\Im(u_2) & (\tanh\lambda, 0)\Re(u_2) \\ \begin{pmatrix} \tanh\lambda \\ 0 \end{pmatrix} \Re(u_1) & \begin{pmatrix} \tanh\lambda \\ 0 \end{pmatrix} \Im(u_1) \end{pmatrix}. \end{aligned} \tag{4}$$

Thus in terms of the complex coordinates, we have

$$L(\tanh\lambda)(u) = (i(\tanh\lambda, 0)\overline{u_2}, i \begin{pmatrix} \tanh\lambda \\ 0 \end{pmatrix} \overline{u_1}).$$

3. We compute the second term of the Segal-Bargmann kernel

$$
\begin{aligned}
&(k_1^{-1}z, L(\tanh\lambda)k_1^{-1}z)\\
&\quad= (U_1^{-1}z_1, i(\tanh\lambda,0)V_1^{-1}\overline{z_2}) + (V_1^{-1}z_2, i\begin{pmatrix}\tanh\lambda\\ 0\end{pmatrix}U_1^{-1}\overline{z_1})\\
&\quad= -2iTr(z_2^tV_1\begin{pmatrix}\tanh\lambda\\ 0\end{pmatrix}U_1^tz_1).
\end{aligned}
\tag{5}
$$

4. Similarly, the third term of the Segal-Bargmann kernel can be computed as follows:

$$
\begin{aligned}
&(L(\tanh\lambda)k_2w, k_2w)\\
&\quad= \overline{(k_2w, L(\tanh\lambda)k_2w)}\\
&\quad= \overline{-2iTr(W_2^tV_2^t\begin{pmatrix}\tanh\lambda\\ 0\end{pmatrix}U_2w_1)}\\
&\quad= 2iTr(\overline{w_2^t}V_2^t\begin{pmatrix}\tanh\lambda\\ 0\end{pmatrix}U_2\overline{w_1})\\
&\quad= 2iTr(\overline{w_1^t}U_2^t(\tanh\lambda,0)V_2\overline{w_2}).
\end{aligned}
\tag{6}
$$

Summarizing our computations, we obtain

$$
\begin{aligned}
&\mathcal{H}(g,z,w)\\
&\quad= 2Tr\left(\left(\overline{w_1^t}U_2^t, -iz_2^tV_1\right)\begin{pmatrix}\mathrm{sech}\lambda & -\tanh\lambda & 0\\ \tanh\lambda & \mathrm{sech}\lambda & 0\\ 0 & 0 & I_{q-p}\end{pmatrix}\begin{pmatrix}U_1^tz_1\\ iV_2\overline{w_2}\end{pmatrix}\right)\\
&\quad= 2Tr\left(\left(z_1^tU_1, i\overline{w_2^t}V_2^t\right)\begin{pmatrix}\mathrm{sech}\lambda & \tanh\lambda & 0\\ -\tanh\lambda & \mathrm{sech}\lambda & 0\\ 0 & 0 & I_{q-p}\end{pmatrix}\begin{pmatrix}U_2\overline{w_1}\\ -iV_1^tz_2\end{pmatrix}\right)\\
&\quad= 2Tr\left(\left(z_1^t, i\overline{w_2^t}\right)\begin{pmatrix}U_1 & 0\\ 0 & V_2^t\end{pmatrix}\begin{pmatrix}\mathrm{sech}\lambda & \tanh\lambda & 0\\ -\tanh\lambda & \mathrm{sech}\lambda & 0\\ 0 & 0 & I_{q-p}\end{pmatrix}\right.\\
&\qquad\qquad\left.\times\begin{pmatrix}U_2 & 0\\ 0 & V_1^t\end{pmatrix}\begin{pmatrix}\overline{w_1}\\ -iz_2\end{pmatrix}\right).
\end{aligned}
\tag{7}
$$

4.4. **Compactification of $O(p,q)$.** We observe that

$$
\begin{pmatrix}U_2 & 0\\ 0 & V_1^t\end{pmatrix}, \begin{pmatrix}U_2 & 0\\ 0 & V_1^t\end{pmatrix}, \begin{pmatrix}\mathrm{sech}\lambda & \tanh\lambda & 0\\ -\tanh\lambda & \mathrm{sech}\lambda & 0\\ 0 & 0 & I_{q-p}\end{pmatrix} \in O_{p+q}.
$$

In the spirit of [6] and [7], we give the following definition.

Definition 4.1. We define $\mathcal{H} : O_{p,q} \to O_{p+q}$ by

$$
\mathcal{H}(g) = \begin{pmatrix}U_1 & 0\\ 0 & V_2^t\end{pmatrix}\begin{pmatrix}\mathrm{sech}\lambda & \tanh\lambda & 0\\ -\tanh\lambda & \mathrm{sech}\lambda & 0\\ 0 & 0 & I_{q-p}\end{pmatrix}\begin{pmatrix}U_2 & 0\\ 0 & V_1^t\end{pmatrix}
$$

where $g = k_1 \exp H(\lambda)k_2$, and $k_i = (U_i, V_i) \in O_p \times O_q$.

Thus for $g \in O_{p,q}$, the Segal-Bargmann kernel can be written conveniently as

$$\mathcal{H}(L(g), z, w) = 2Tr\left(\left(z_1^t, i\overline{w_2^t}\right)\mathcal{H}(g)\begin{pmatrix} \overline{w_1} \\ -iz_2 \end{pmatrix}\right). \tag{8}$$

Since w_1, w_2, z_1, z_2 can all be chosen arbitrarily, $\mathcal{H}(L(g), z, w)$ determines $\mathcal{H}(g)$ uniquely, and vice versa. In addition, since $\mathcal{H}(g, z, w)$ does not depend on the Cartan decomposition of $O(p, q)$, $\mathcal{H}(g)$ does not depend on the Cartan decomposition. Therefore $\mathcal{H}(g)$ is is well-defined. The group action of $(\xi, g) \in MO(p, q)$ on $\mathcal{F}$ is given by

$$\omega(\xi, g)f(z) = \int_V \xi \exp\left(\frac{1}{2}\left(z_1^t, i\overline{w_2^t}\right)\mathcal{H}(g)\begin{pmatrix} \overline{w_1} \\ -iz_2 \end{pmatrix}\right) f(w)d\mu(w).$$

Since $\mathrm{sech}\lambda > 0$, the image of $\mathcal{H}$ consists of

$$\{k_1 T(\theta) k_2 \mid k_1, k_2 \in O(p) \times O(q), \theta \in \prod_1^p (-\frac{\pi}{2}, \frac{\pi}{2})\}.$$

Since the set $\{T(\theta) \mid \theta_i \in [0, \frac{\pi}{2})\}$ is already dense in

$$\{T(\theta) \mid \pi/2 \geq \theta_1(g) \geq \theta_2(g) \geq \ldots \theta_p(g) \geq 0\}$$

according to the $K\mathbb{T}_p K$ decomposition, $\mathcal{H}(O(p, q))$ is dense in $O(p + q)$.

We recite the following theorem from [6].

Theorem 4.1. *The Bargmann-Segal kernel defines a one-to-one analytic embedding $\mathcal{H}$ from $Sp(n, \mathbb{R})$ to the space of quadratic forms on $(z, \overline{w})$. The closure of $\mathcal{H}(Sp(n, \mathbb{R}))$ can be identified with $U(2n)/O(2n)$. Let G be an arbitrary Lie group with a faithful representation into $Sp_{2n}(\mathbb{R})$. Suppose the closure of $\mathcal{H}(G)$, denoted by $\overline{G}$, is a compact smooth submanifold of $\mathcal{S}$. Then $(\mathcal{H}|_G, \overline{G})$ is an analytic compactification of G.*

As a consequence, we obtain the following result.

Theorem 4.2. *$\mathcal{H}$ is an analytic compactification from $O(p, q)$ to $O(p + q)$. Its restriction on $SO(p, q)$ is an analytic compactification of $SO(p, q)$ to $SO(p + q)$. Let dg be the Haar measure of $SO(p, q)$. It follows that*

$$\frac{d_{SO(p+q)}\mathcal{H}(g)}{dg} \neq 0$$

for all $g \in SO(p, q)$.

As indicated in [7], this compactification has a geometric interpretation. That is, $\mathcal{H}$ can be recognized as some natural embedding of $O(p, q)$ into the Lagrangian Grassmannian of $\mathbb{R}^{2(p+q)}$ equipped with a symmetric form of signature $(p+q, p+q)$. Theorem 4.2 enables us to study the integration theory over $SO(p, q)$ through the integration theory over $SO(p + q)$. One should keep in mind that for any $s \in \mathcal{H}(SO(p, q))$, we always have

$$\frac{d_{SO(p,q)}\mathcal{H}^{-1}(s)}{d_{SO(p+q)}s} \neq 0. \tag{9}$$

5. Proof of the Main Theorem

Let $\mathcal{M}(p,q)$ be the set of K-finite matrix coefficients of ω restricted to $MO(p,q)$. For every function f in $\mathcal{M}(p,q)$, $\frac{f}{\Lambda(p,q)}$ descends into a function on $O(p,q)$. Let us restate the main theorem.

Theorem 5.1. *Suppose $p+q \leq 2n+1$. Let f be a function of $MSO(p,q)$ such that*

$$f(\epsilon\tilde{g}) = -(\tilde{g}), \qquad f\Lambda(p,q) \in L^1(SO(p,q)).$$

If for every $h \in \mathcal{M}(p,q)$,

$$\int_{MSO(p,q)} f(\tilde{g})h(\tilde{g})d\tilde{g} = 0,$$

then $f \equiv 0$ almost everywhere.

Observe that

$$\int_{MSO(p,q)} f(\tilde{g})h(\tilde{g})d\tilde{g} = 2\int_{SO(p,q)} (f\Lambda(p,q))(g)\frac{h}{\Lambda(p,q)}(g)dg.$$

5.1. **Proof of Theorem 1.3.** Let ϕ, ψ be two polynomial functions on $\mathbb{C}^{n(p+q)}$. Let $\widetilde{g}$ be an element of $MO(p,q)$. We can write $\widetilde{g} = (\xi, g)$ with $\det C^{\mathbb{C}}_{L(g)} = \xi^{-2}$ and $g \in O_{p,q}$. Then

$$\Lambda(p,q)(\tilde{g}) = \xi.$$

Thus

$$\begin{aligned}
&(\omega(\tilde{g})\phi, \psi) \\
&\quad = \Lambda(p,q)(\tilde{g})\int_{z,w\in\mathbb{C}^{n(p+q)}} \exp\left(\frac{1}{2}Tr(z_1^t, i\overline{w_2}^t)\mathcal{H}(g)\begin{pmatrix} \overline{w_1} \\ -iz_2 \end{pmatrix}\right) \\
&\quad\quad \times \phi(w)\overline{\psi(z)}d\mu(w)d\mu(z) \\
&\quad = \Lambda(p,q)(\tilde{g})\int_{z,w\in\mathbb{C}^{n(p+q)}} \exp\left(\frac{1}{2}Tr(z_1^t, w_2{}^t)\mathcal{H}(g)\begin{pmatrix} w_1 \\ z_2 \end{pmatrix}\right) \\
&\quad\quad \times \phi(\overline{w_1}, i\overline{w_2})\overline{\psi(z_1, iz_2)}d\mu(w)d\mu(z)
\end{aligned} \tag{10}$$

The last equation is obtained by the coordinate change

$$w_1 \to \overline{w_1}, \qquad w_2 \to i\overline{w_2}, \qquad z_1 \to z_1, \qquad z_2 \to iz_2.$$

This coordinate change preserves the Gaussian measure.

Now take $\mathcal{M}^0$ to be the set of functions of the form

$$\begin{aligned}
g \in O(p+q) \to \int_{z,w\in\mathbb{C}^{n(p+q)}} &\exp\left(\frac{1}{2}Tr(z_1^t, w_2{}^t)g\begin{pmatrix} w_1 \\ z_2 \end{pmatrix}\right) \\
&\times \phi(\overline{w_1}, \overline{w_2})\overline{\psi(z_1, z_2)}d\mu(w)d\mu(z).
\end{aligned}$$

Apparently, $\frac{\mathcal{M}(p,q)}{\Lambda(p,q)}$ is just the pull back of $\mathcal{M}^0$ under $\mathcal{H} : O(p,q) \to O(p+q)$.

We must use some caution here concerning the group $O(0, p+q)$. The compactification of $O(0, p+q)$ is itself. However, from Definition 4.1, our compactification is not the identity map. In fact,

$$\mathcal{H}(g) = g^t.$$

Nevertheless, we observe that

(11)

$$\begin{aligned}
&\int_{z,w\in\mathbb{C}^{n(p+q)}} \exp\left(\frac{1}{2}Tr(z_1^t, {w_2}^t)g\begin{pmatrix} w_1 \\ z_2 \end{pmatrix}\right)\phi(\overline{w_1},\overline{w_2})\overline{\psi(z_1,z_2)}d\mu(w)d\mu(z) \\
&\quad= \int_{z,w\in\mathbb{C}^{n(p+q)}} \exp\left(\frac{1}{2}Tr(w_1^t, {z_2}^t)g^t\begin{pmatrix} z_1 \\ w_2 \end{pmatrix}\right)\phi(\overline{w_1},\overline{w_2})\overline{\psi(z_1,z_2)}d\mu(w)d\mu(z) \\
&\quad= \int_{z,w\in\mathbb{C}^{n(p+q)}} \exp\left(\frac{1}{2}Tr(z_1^t, {w_2}^t)g^t\begin{pmatrix} w_1 \\ z_2 \end{pmatrix}\right)\phi(\overline{z_1},\overline{z_2})\overline{\psi(w_1,w_2)}d\mu(w)d\mu(z).
\end{aligned}$$

The space $\mathcal{M}^0$ remains unchanged under the convolution $\mathcal{H} : O(p+q) \to O(p+q)$. We obtain

$$\mathcal{M}^0 = \frac{\mathcal{M}(0,p+q)}{\Lambda(0,p+q)}.$$

Thus, Theorem 1.3 is proved.

5.2. **Proof of the Main Theorem for $MSO(p+q)$.** Now $MSO(0,p+q)$ is the direct product of $\mathbb{Z}/2\mathbb{Z}$ and $SO(0,p+q)$. In fact,

$$\Lambda(0,p+q) : MSO(0,p+q) \to \pm 1 \cong \mathbb{Z}/2\mathbb{Z};$$

$$\ker(\Lambda(0,p+q)) = SO(p,q).$$

Thus, $\Lambda(0,p+q)\otimes\omega|_{MSO(0,p+q)}$ descends into a unitary representation of $SO(p+q)$.

According to Corollaries II.6.7 and II.6.12 of [15], every irreducible unitary representation of $SO(p+q)$ occurs as direct summands in ω. The Stone-Weierstrass-Peter-Weyl theorem implies that $\Lambda(0,p+q)\mathcal{M}(0,p+q) = \mathcal{M}^0$ as functions on $SO(p+q)$ are dense in $C(SO(p+q))$ under the *sup norm topology* (see remarks of Theorem 1.12 in [13]).

Lemma 5.1. *The linear span of $\mathcal{M}^0$ is dense in $C(SO(p+q))$ under the sup norm topology.*

The Density Theorem and the Main Theorem for $MSO(p+q)$ follow immediately.

5.3. **Proof of the Main Theorem for $MSO(p,q)$.** Suppose that $f\Lambda(p,q) \in L^1(SO(p,q))$. Define an L^1-function on $SO(p+q)$,

$$f^0(g) = (f\Lambda(p,q))(\mathcal{H}^{-1}(g))\frac{d_{SO(p,q)}\mathcal{H}^{-1}(g)}{d_{SO(p+q)}g} \qquad (g \in \mathcal{H}(SO(p,q))).$$

Since $\mathcal{H}(SO(p,q))$ is open and dense in $SO(p+q)$, f^0 is well-defined as an L^1-function. Recall $\mathcal{M}^0$ is the push forward of $\frac{\mathcal{M}(p,q)}{\Lambda(p,q)}$. Equation (1) implies that for any $h(g) \in \mathcal{M}^0$,

$$\int_{SO(p+q)} f^0(g)h(g)d_{SO(p+q)}g = 0.$$

Since $\mathcal{M}^0$ is dense in $C(SO(p+q))$ under the sup norm, $f^0(g) \equiv 0$ almost everywhere. Since

$$\frac{d_{SO(p,q)}\mathcal{H}^{-1}(g)}{d_{SO(p+q)}g} \neq 0 \qquad (\forall \;\; g \in \mathcal{H}(SO(p,q))),$$

$f\Lambda(p,q) \equiv 0$ almost everywhere. But $\Lambda(p,q)(g)$ cannot be zero because

$$\Lambda(p,q)(g)^2 = (\det C^{\mathbb{C}}_{L(g)})^{-1}.$$

Hence $f \equiv 0$ almost everywhere. The Main Theorem follows immediately from the continuity of f.

5.4. **Density theorem on $L^2_{odd}(MSO(p,q))$.** We say a function f on $MSO(p,q)$ is an *odd function* if $f(\epsilon\widetilde{g}) = -f(\widetilde{g})$. We denote the space of odd L^2-functions by $L^2_{odd}(MSO(p,q))$.

Theorem 5.2. *For $p+q \leq 2n+1$, the linear span of $\mathcal{M}(p,q)$ is dense in the space of odd functions in $L^2(MSO(p,q))$.*

Proof. Theorem 3.2 in [17], implies that if $p+q \leq 2n+1$, the matrix coefficient

$$(\omega(\widetilde{g})\phi, \psi) \qquad (\phi, \psi \in \mathcal{P})$$

is in $L^2(MSO(p,q))$. In particular, the function $\Lambda(p,q)$ is in $L^2(MSO(p,q))$. Suppose the linear span of $\mathcal{M}(p,q)$ is not dense in $L^2_{odd}(MSO(p,q))$. Then there must be an odd nonzero L^2-function, say $f(\widetilde{g})$, such that $(f, \mathcal{M}(p,q)) = 0$. Notice that $f\Lambda(p,q) \in L^1(SO(p,q))$. From Theorem 5.1, $f \equiv 0$ almost everywhere. □

5.5. **Nonvanishing Theorems.** Theta correspondence is a one-to-one correspondence between a certain set of projective representations of $O(p,q)$ and certain set of projective representations of $Sp_{2n}(\mathbb{R})$. Studies in the past decade or two find that theta correspondence behaves reasonably well in various perspectives. One unsolved problem is the unitarity conjecture ([18], Conjecture C). In [9], we had some limited success towards this conjecture. However, our approach requires $(,)_\pi$ to be nonvanishing. In this section, we will show that $(,)_\pi$ is indeed nonvanishing up to the central character det.

Let G be the symplectic group $Sp_{2n(p+q)}(\mathbb{R})$. Let $(O(p,q), Sp(n,\mathbb{R}))$ be an irreducible dual pair in G. Let $(\omega, \mathcal{P})$ be the oscillator representation of MG. Let π be an irreducible admissible representation of $MO(p,q)$. Now we may formally define a pairing

$$(\mathcal{P} \otimes \pi, \mathcal{P} \otimes \pi) \to \mathbb{C}$$

as follows: for $\phi \in \mathcal{P}, \psi \in \mathcal{P}, v \in \pi, u \in \pi$.

$$(\phi \otimes v, \psi \otimes u)_\pi = \int_{MO(p,q)} (\phi, \omega(g)\psi)(\pi(g)u, v) dg.$$

In [8], we proved

Theorem 5.3. *Suppose π is in the semistable range of $\theta(O(p,q) \to Sp(n,\mathbb{R}))$. Then $(,)_\pi$ is well-defined. If $(,)_\pi$ is nonvanishing, then $\pi \in \mathcal{R}(MO(p,q), \omega)$.*

The *semistable condition* is a condition on the growth of the matrix coefficients of π at infinity. It may be read off from the Langlands parameter of π. For the sake of this paper, the reader may take the absolute convergence of $(,)_\pi$ as the semistable condition, even though the semistable condition is slightly stronger than the convergence of $(,)_\pi$. Now we may speak of an irreducible representation of $MSO(p,q)$ being in the semistable range if it is a subrepresentation of π in the semistable range of $\theta(MO(p,q) \to MSp_{2n}(\mathbb{R}))$. Again, it is just a growth condition. For a representation π of $MSO(p,q)$ in the semistable range, we can define $(,)_\pi$ in the same fashion.

Notice that the semistable condition implies that

$$\Lambda(p,q)(\pi(g)u,v) \in L^1(SO(p,q)).$$

From the main theorem, we obtain

Corollary 5.1. *Suppose $2n+1 \geq p+q$. Let (π, V_π) be an irreducible admissible representation of $MSO(p,q)$ in the semistable range such that $\pi(\epsilon) = -1$. Then $(,)_\pi$ is nonvanishing.*

Now, regarding $MO(p,q)$, we define a group homomorphism

$$\det : MO(p,q) \to \mathbb{Z}/2$$

such that $ker(\det) = MSO(p,q)$.

Corollary 5.2. *Suppose $2n+1 \geq p+q$. Suppose π is in the semistable range of*

$$\theta(MO(p,q) \to MSp(n,\mathbb{R}))$$

and $\pi(\epsilon) = -1$. If $\pi \cong \pi \otimes \det$, then $(,)_\pi$ does not vanish. If π is not equivalent to $\pi \otimes \det$, then at least one of $(,)_\pi$, $(,)_{\pi\otimes\det}$ does not vanish.

Proof. Let $\widetilde{h}$ be an element in $MO(p,q)$ such that $\det \widetilde{h} = -1$. Then

$$\begin{aligned}
&(\phi \otimes v, \psi \otimes u)_\pi \\
&= \int_{MO(p,q)} (\phi, \omega(\widetilde{g})\psi)(\pi(\widetilde{g})u, v) d\widetilde{g} \\
&= \int_{MSO(p,q)} (\phi, \omega(\widetilde{g})\psi)(\pi(\widetilde{g})u, v) + (\phi, \omega(\widetilde{g})\omega(\widetilde{h})\psi)(\pi(\widetilde{g})\pi(\widetilde{h})u, v) d\widetilde{g}
\end{aligned}$$

(12)

$$\begin{aligned}
&(\phi \otimes v, \psi \otimes u)_{\pi\otimes\det} \\
&= \int_{MO(p,q)} \det(\widetilde{g})(\phi, \omega(\widetilde{g})\psi)(\pi(\widetilde{g})u, v) d\widetilde{g} \\
&= \int_{MSO(p,q)} (\phi, \omega(\widetilde{g})\psi)(\pi(\widetilde{g})u, v) - (\phi, \omega(\widetilde{g})\omega(\widetilde{h})\psi)(\pi(\widetilde{g})\pi(\widetilde{h})u, v) d\widetilde{g}.
\end{aligned}$$

Now Corollary 5.1 will guarantee at least one of the summand is nonvanishing. Therefore, at least one of $(,)_\pi$, $(,)_{\pi\otimes\det}$ is nonvanishing. If π is equivalent to $\pi \otimes \det$, then $(,)_\pi$ is nonvanishing. Because of Theorem 5.3, π occurs in the theta correspondence. If π and $\pi \otimes \det$ are not equivalent, then at least one of them occurs in the theta correspondence. $\square$

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