REPRESENTATION THEORY
An Electronic Journal of the American Mathematical Society
Volume 6, Pages 160–189 (August 7, 2002)
S 1088-4165(02)00134-6

ADMISSIBLE NILPOTENT ORBITS OF REAL AND p-ADIC SPLIT EXCEPTIONAL GROUPS

MONICA NEVINS

ABSTRACT. We determine the admissible nilpotent coadjoint orbits of real and p-adic split exceptional groups of types G_2, F_4, E_6 and E_7. We find that all Lusztig-Spaltenstein special orbits are admissible. Moreover, there exist non-special admissible orbits, corresponding to "completely odd" orbits in Lusztig's special pieces. In addition, we determine the number of, and representatives for, the non-even nilpotent p-adic rational orbits of G_2, F_4 and E_6.

1. INTRODUCTION

Let k denote either $\mathbb{R}$ or a p-adic field (of characteristic zero), and let $\mathbb{G}$ be an algebraic group defined over k. Write $G = \mathbb{G}(k)$ for the k-rational points of $\mathbb{G}$.

The orbit method, introduced by Kirillov [Ki], Moore [M1] and Duflo [Du], among many others, conjectures a deep relationship between irreducible unitary representations of G and the coadjoint orbits of G acting on the dual $\mathfrak{g}^*$ of its Lie algebra. One expects to attach unitary representations of G only to orbits satisfying an appropriate "integrality" condition. In light of the work of Duflo [Du], the best candidate for this condition is "admissibility," as defined in Section 2 below. As a result of much work by Lion-Perrin [LP], Auslander-Kostant [AK] and Vogan [V1, V2], the orbit method has been realized for all but: the orbits of reductive algebraic groups over p-adic fields; and nilpotent orbits of reductive Lie groups over $\mathbb{R}$. (For reductive groups, we can and do identify the adjoint and coadjoint orbits of G in a natural way, which allows us, in particular, to define nilpotent orbits.)

In an effort to understand these remaining cases, Schwarz [Sch], Ohta [O] and the author [N1] determined the admissible nilpotent orbits of most groups of classical type over the real and p-adic fields of characteristic zero. It was found that for all split groups (and some others), the set of admissible orbits coincides exactly with the set of special orbits [L1, Sp]. In this context, we define an orbit of the real or p-adic group to be special if the corresponding algebraic orbit is special. (Alfred Noël has recently addressed the real exceptional groups; see more below.)

In this paper, we consider the nilpotent orbits of k-points of split simply connected algebraic groups of exceptional types G_2, F_4, E_6 and E_7. For each algebraic non-even nilpotent orbit of these groups, we choose a k-rational representative such that the corresponding rational orbit is "split" over k. We then determine the admissibility of that rational orbit. For the groups of types G_2, F_4 and E_6, we go on

Received by the editors July 13, 2001 and, in revised form, April 16, 2002.
2000 *Mathematics Subject Classification.* Primary 20G25; Secondary 17B20, 17B45.
The author was supported in part by the Killam Trust, and by NSERC of Canada grant RGPIN229816.

to compute the number of other k-rational orbits of the given algebraic orbit and determine their admissibility as well. For p-adic groups, this determination of the number of rational orbits is new, as is the explicit construction of representatives of the "additional" orbits given in Appendix A. The list of algebraic non-even nilpotent orbits admitting more than one k-rational orbit under G is given in Table 2 in Section 6. (Even orbits were not considered here because they are all both admissible and special; the same techniques for determining their number, and representatives thereof, apply.)

To state our main theorem most succinctly, let us recall the definition of "special pieces" used by Lusztig in [L2]. Let $\mathbb{G}$ be an algebraic group, and $\mathcal{O}$ a special nilpotent adjoint orbit of $\mathbb{G}$. The special piece $\gamma(\mathcal{O})$ corresponding to $\mathcal{O}$ is the set of all nilpotent orbits contained in the closure of $\mathcal{O}$, but not in the closure of any smaller special orbit. The special pieces form a partition of the set of nilpotent orbits.

In [L2], Lusztig parametrizes the nilpotent orbits in $\gamma(\mathcal{O})$ by the conjugacy classes of a finite group; in particular, the unique special orbit corresponds to the trivial conjugacy class. If $X \in \mathcal{O}$, this group (denote it $\mathcal{G}'_{\mathcal{O}}$) is a subquotient of the component group $A(\mathcal{O})$ of the centralizer of X in $\mathbb{G}$. For classical groups $\mathbb{G}$, the groups $\mathcal{G}'_{\mathcal{O}}$ were determined by Kraft and Procesi in [KP] and are all abelian; for exceptional groups, on the other hand, $\mathcal{G}'_{\mathcal{O}}$ is either trivial or isomorphic to a symmetric group S_r, for some $r \in \{2, 3, 4, 5\}$.

Let us define an orbit $\mathcal{O}' \in \gamma(\mathcal{O})$ to be *completely odd* if either $\mathcal{O}'$ is special, or, when $\mathcal{G}'_{\mathcal{O}} = S_r$, the partition defining the conjugacy class of $\mathcal{O}'$ has no even parts. (These two characterizations coincide in the case that $\mathcal{O} = \mathcal{O}'$ is special and $\mathcal{G}'_{\mathcal{O}} = S_r$.) See Table 1 for a list of the non-special completely odd orbits of exceptional groups.

The following theorem is summarized from Section 6.

Main Theorem. *Let G denote the k-points of a split simply connected algebraic group of one of the exceptional types G_2, F_4, E_6 or E_7. Then*

(i) *the split (see Section 5) admissible orbits arise exactly from completely odd orbits. In particular, every special orbit gives rise to a split admissible orbit.*

Now assume that $k = \mathbb{R}$ or is p-adic with residual characteristic different from 2. Then for the groups of type G_2, F_4 and E_6:

(ii) *admissibility is independent of the choice of rational orbit within a given algebraic orbit, except for the B_2-orbit of F_4. (In this case, the orbit is not completely odd, so the split orbit is not admissible. Depending on the arithmetic of k, either exactly one, or all, of the non-split orbits is admissible.)*

Since for classical groups, the completely odd orbits are exactly the special orbits, this parametrization of admissible orbits is consistent with previous results on split classical groups.

This result clarifies the heretofore mysterious link between the algebraically defined special orbits and the geometrically defined admissible ones. It remains to show this link directly, without case-by-case considerations; proving perhaps a conjecture of Vogan that admissibility is some mod $\mathbb{Z}/2\mathbb{Z}$ reduction of an intrinsic object related to the geometry of the special pieces.

Conjecture. *We expect that the admissible nilpotent orbits of E_8 will be exactly the completely odd orbits.*

TABLE 1. The completely odd non-special orbits of G_2, F_4, E_6, E_7 and E_8.

Group	Orbit	$\mathcal{O}$	$\mathcal{G}'_{\mathcal{O}}$	conjugacy class
G_2	A_1	$G_2(a_1)$	S_3	(3)
F_4	$\widetilde{A_2}+A_1$	$F_4(a_3)$	S_4	$(3,1)$
E_6	$2A_2+A_1$	$D_4(a_1)$	S_3	(3)
E_7	$2A_2+A_1$	$D_4(a_1)$	S_3	(3)
E_7	A_5+A_1	$E_7(a_5)$	S_3	(3)
E_8	$2A_2+A_1$	$D_4(a_1)$	S_3	(3)
E_8	$2A_2+2A_1$	$D_4(a_1)+A_1$	S_3	(3)
E_8	$E_6(a_3)+A_1$	$E_8(a_7)$	S_5	$(3,1,1)$
E_8	A_4+A_3	$E_8(a_7)$	S_5	(5)
E_8	E_6+A_1	$E_8(b_5)$	S_3	(3)

Alfred Noël [No1, No2] has independently computed the admissible nilpotent coadjoint orbits for *all* exceptional groups over $\mathbb{R}$. He applied techniques of Ohta [O] (which do not extend to p-adic fields). His results agree with ours where they overlap, and moreover, they support the above conjecture for E_8 over $\mathbb{R}$. In addition, he has found a handful of orbits for which admissibility is a non-stable criterion (like B_2 of F_4 here). For some non-split exceptional real Lie groups, he has shown that there are non-admissible special orbits (just as for some non-split classical groups over $\mathbb{R}$; see [Sch, O]).

In this paper we have excluded the study of the non-split rational orbits of E_7 due to their sheer number and diversity. Fields of residual characteristic equal to 2 are excluded in the discussion of non-split rational orbits because many of the results used in Appendix B either fail directly in that case or at least would require separate arguments.

The structure of the paper is as follows. In Section 2, we set our notation and recall the definition of an admissible nilpotent orbit. In Section 3, we describe methods for determining the admissibility of "split" nilpotent orbits of the exceptional groups considered here. Our discussion of the occurrence of other rational orbits within (the k-points of) each algebraic orbit begins in Section 4. There, we determine their number using Galois cohomology, and go on to describe a method for obtaining representatives of these additional orbits. In Section 5 we give constraints on the structure of the groups G^ϕ that can arise (for ϕ classifying a non-split orbit in a given algebraic one).

Sections 6 is devoted to studying the orbits individually and recording their admissibility. We relegate our explicit computations with respect to additional rational orbits to the appendices. In Appendix A we give explicit representatives of each rational orbit. In Appendix B we summarize results needed to determine the admissibility of the non-split rational orbits occurring here.

Acknowledgements. Many thanks to Eric Sommers, for pointing out the relation between the admissible non-special orbits and completely odd special pieces, and to Jason Levy, for many fruitful discussions. The determination of the admissible nilpotent orbits of G_2 was part of my thesis [N1], conducted under the generous supervision of David Vogan. Thanks also to the anonymous referee for numerous corrections and helpful suggestions. In particular, they provided the argument for the bijectivity (and not mere injectivity) of the map v in the proof of Proposition 4.1, and helped correct and clarify much of Section 6.

2. Admissibility

In this section, let us set our notation for the remainder of the paper, and recall the definition of admissibility (originally defined by Duflo in [Du] over $\mathbb{R}$).

Let k be a real or p-adic field. Let $\mathbb{G}$ be a linear algebraic group of exceptional type, defined and split over k. Write $G = \mathbb{G}(k)$. Let Φ denote the set of roots of G, and Δ a set of simple roots.

Identify the adjoint orbits of G with its coadjoint orbits via a nondegenerate invariant form $\langle , \rangle$ on $\mathfrak{g}$. For each nilpotent orbit $G \cdot E$ in $\mathfrak{g}$, choose $H, F \in \mathfrak{g}$ so that $\phi = \mathrm{span}\{E, H, F\}$ is a Lie subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2, k)$. Define $\mathfrak{g}^{\phi}$ to be the centralizer in $\mathfrak{g}$ of ϕ (in other words, the span of the trivial subrepresentations of ϕ acting on $\mathfrak{g}$), and G^{ϕ} to be the corresponding subgroup of G. Let $\mathfrak{g}\,[-1]$ denote the subspace of -1 weight vectors of $\mathfrak{g}$ with respect to H. It is a symplectic vector space, endowed with the canonical Kirillov-Kostant symplectic form defined by $\omega_E(X, Y) = \langle E, [X, Y]\rangle$, for all $X, Y \in \mathfrak{g}\,[-1]$. Then G^{ϕ} acts, via the adjoint action, on $\mathfrak{g}\,[-1]$, and preserves ω_E.

Let $Mp(\mathfrak{g}\,[-1]\,, k)$ denote the metaplectic group, that is, the unique two-fold nontrivial covering group of $Sp(\mathfrak{g}\,[-1]\,, k)$. The orbit $G \cdot E$ is *admissible* if the cover $(G^{\phi})^{mp}$ of G^{ϕ} defined by the diagram

$$\begin{array}{ccc} (G^{\phi})^{mp} & \longrightarrow & Mp(\mathfrak{g}\,[-1]\,, k) \\ \big\downarrow & & \big\downarrow \\ G^{\phi} & \xrightarrow{Ad} & Sp(\mathfrak{g}\,[-1]\,, k) \end{array} \tag{2.1}$$

splits (*i.e.* admits a smooth section) over G^{ϕ}_e, where G^{ϕ}_e is the topological identity component when $k = \mathbb{R}$, and an open normal subgroup containing I otherwise (see [N2]).

Remarks. (i) Each even orbit is automatically admissible, since in that case $\mathfrak{g}\,[-1] = \{0\}$; these orbits are also all special. Therefore we need only consider non-even orbits in this paper.

(ii) We will need to assume that the residual characteristic of k is odd to determine the admissibility of non-split rational orbits; see Appendix B.

3. Steinberg cocycles and splitting theorems

In this Section, we describe some particular criteria for admissibility applicable in our setting. Let V be a finite-dimensional symplectic k-vector space and consider the two-fold covering group $Mp(V, k)$ of $Sp(V, k)$. This central extension is defined by an element of $H^2(Sp(V, k), \mu_2)$, where $\mu_2 = \{\pm 1\}$. One cocycle $S \in Z^2(Sp(V, k), \mu_2)$ representing this cohomology class (called the Steinberg 2-cocycle of $Sp(V, k)$) is given as follows (see [R], [LV, appendix]).

Write $\mathbb{C}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. The "usual" cocycle c_l of the metaplectic cover is obtained by restriction of the cocycle of the $\mathbb{C}^1$-cover of the symplectic group. It is defined relative to a choice of lagrangian (*i.e.* maximally isotropic) subspace l of V (see [LV, A.9]). For our purposes, it suffices to note that for $g, g' \in Sp(V, k)$,

$c_l(g, g') = 1$ if either g or g' preserves l. In general, $c_l(g, g')$ takes values in the set of eighth roots of unity $\mu_8 \subset \mathbb{C}^1$.

To define a normalization of c_l taking values in μ_2, we need some definitions. Let $(\cdot/\cdot)_k$ denote the (2-)Hilbert symbol of k (see, for example, [Neu, III §5]). Then the *Weil index* of k ([W, §14]) is a function $\gamma\colon k^*/k^{*2} \to \mu_8$ satisfying in particular the equation $\gamma(1)\gamma(ab) = (a/b)_k\gamma(a)\gamma(b)$ for any $a, b \in k^*$. Next, let $D(g, l) \in k^*/k^{*2}$ be defined as in [LV, A.13]. (Roughly, given an orientation $e \in \wedge^{top} l$ of the lagrangian l, $D(g, l)$ is a measure of the change in orientation between (l, e) and $(g \cdot l, g \cdot e)$, where these orientations may be compared, on the complement of the intersection of l and $g \cdot l$, via the ambient symplectic form.) For our purposes, it suffices to know that if $g \in Sp(V, k)$ is represented by a matrix $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ (with respect to a basis of l and of a complementary lagrangian, in that order), then $D(g, l) = \det(A)$ if $C = 0$, and $D(g, l) = \det(C)$ if $\det(C) \neq 0$. Now define

$$t(g) = \gamma(1)^{1-dim(l)+dim(l\cap g\cdot l)}\gamma(D(g, l))^{-1},$$

and set

$$S(g, h) = \frac{c_l(g, h)t(gh)}{t(g)t(h)} \in \mu_2. \tag{3.1}$$

This is the desired Steinberg 2-cocycle of $Sp(V, k)$.

Moore defined Steinberg cocycles for all simple simply connected Chevalley groups in [M2, Ch.III]. Let $\mathbb{G}$ be such a group, defined and split over k, and set $G = \mathbb{G}(k)$. Let $\mathcal{H}$ be a split Cartan subgroup of G. Then the Steinberg cocycles are representatives of cohomology classes in $H^2(G, A)$ (for A an abelian group) satisfying certain (wonderful) properties. Among these is the following [M2, III, Lemma 8.4], which implies that the covering of G induced by a Steinberg cocycle splits if and only if its restriction to a certain one-parameter subgroup is trivial.

Lemma 3.1 (Moore)**.** *If α is a long root, and $\mathcal{H}_\alpha$ its corresponding one-parameter subgroup in $\mathcal{H}$, then any Steinberg cocycle is determined by its restriction to $\mathcal{H}_\alpha$.*

In the setting of (2.1), when G^ϕ is a (product of) simple simply connected Chevalley group(s), the composition of S and Ad defines the Steinberg cocycle of the extension $(G^\phi)^{mp}$ of G^ϕ, and Lemma 3.1 may be applied. In particular, this lemma reduces the question of splitting of a given covering group for general G^ϕ in this class to the splitting over a root $SL(2, k)$ subgroup.

Let us thus compute the Steinberg cocycle arising through the special case of k-representations V of $SL(2, k)$ such that $SL(2, k) \to Sp(V, k)$.

Theorem 3.2. *Suppose V is an even-dimensional representation of $SL(2, k)$ affording an invariant symplectic form. Then the metaplectic cover of $SL(2, k)$ induced by the map $\varphi\colon SL(2, k) \to Sp(V, k)$ is trivial exactly when the total number of subrepresentations of V having dimension of the form $4n + 2$ (for some n) is even.*

Proof. Since the symplectic form on V is $SL(2, k)$-invariant, the decomposition of V into isotypic components under $SL(2, k)$ is orthogonal, and thus the isotypic subspaces of V are themselves nondegenerate symplectic subspaces. The Steinberg cocycle of $SL(2, k)$ defined by this metaplectic cover will thus be a product of the cocycles obtained through each of these symplectic subspaces.

Denote the unique n-dimensional irreducible representation of $SL(2,k)$ by V^n. (Note that it admits a nondegenerate invariant symplectic form if and only if n is even.)

Let us first consider the special case of $V = V^{2n}$. Denote by $h(t)$ the image of the matrix $\left[\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right]$ in $Sp(V,k)$; then

$$h(t) = diag(t^{2n-1}, t^{2n-3}, \dots, t^1, t^{-2n+1}, t^{-2n+3}, \dots, t^{-1}).$$

We must restrict the Steinberg cocycle to the one-parameter subgroup $\{h(t) \mid t \in k^*\}$ to apply Lemma 3.1.

Since $h(t)$ preserves the lagrangian l spanned by the weight vectors of positive weight, $c_l(h(t), h(s)) = 1$ for all $s, t \in k^*$. We compute $D(h(t), l) = det(h(t)|_l) = t^{n^2}$; and so (3.1) simplifies to

$$S(h(s), h(t)) = \frac{\gamma(s^{n^2})\gamma(t^{n^2})}{\gamma(1)\gamma((st)^{n^2})} = (s^{n^2}/t^{n^2})_k. \tag{3.2}$$

Hence S is trivial (for all $s, t \in k^*$) if and only if n is even.

Now suppose V is isotypic for $SL(2,k)$ and contains an odd-dimensional irreducible subrepresentation V^{2n+1}; then V^{2n+1} occurs with even multiplicity. We may choose the lagrangian l to be a direct sum of half of these irreducibles; again, $h(t)$ preserves l. Since $h(t)|_{V^{2n+1}} = diag(t^{2n}, t^{2n-2}, \dots, t^{-2n})$, it follows that $det(h(t)|_{V^{2n+1}}) = 1$ for all $t \in k^*$. Consequently, the Steinberg cocycle takes value identically 1.

Finally, consider the general case, where we have a decomposition of V into irreducibles under $SL(2,k)$ of the form $V = \bigoplus_{r=1}^{N} m_r V^r$. We deduce from the above that the Steinberg cocycle will be $S(h(s), h(t)) = (s^M/t^M)_k$, where $M = \sum_n m_{2n} n^2$. It is thus trivial exactly when M is even, as we were required to show. □

Of course, in what follows, G^ϕ need not be simply connected or split over k. Let us recall a splitting theorem from [N2] which is sometimes applicable in such cases.

Theorem 3.3. *(a) If G^ϕ preserves a lagrangian subspace of $\mathfrak{g}[-1]$, then the corresponding metaplectic cover splits over G^ϕ_e. (b) If G^ϕ preserves complementary lagrangians and there exists a G^ϕ-invariant intertwining operator between them, then the cover splits over all of G^ϕ.*

Corollary 3.4. *Suppose $\mathfrak{g}[-1] = W \oplus W^*$ as a representation of G^ϕ, with either $W \not\approx W^*$ or W a lagrangian subspace. Then the corresponding orbit is admissible.*

Proof. If W and W^* are inequivalent, it follows that both are lagrangian subspaces of $W \oplus W^*$. Hence under either hypothesis, Theorem 3.3 applies to give admissibility. □

4. Occurrence of other rational orbits

For each adjoint orbit $\mathcal{O}$ of the algebraic group $\mathbb{G}$, its set of k-rational points $\mathcal{O}(k)$ is a union of one or more orbits of G. In this Section, let us give means for determining the number of rational orbits in the k-points of each algebraic orbit. As these numbers are well-known for real groups, we restrict attention to k p-adic here.

The main reference for the arguments in Galois cohomology used in the following is Serre's book [S]. Note that in general $H^1(k, G)$ is not a group, but only a pointed set; thus the fibres of a given map may have different cardinalities. Moreover, it is a delicate matter to define a "long exact sequence in cohomology" arising from a short exact sequence of groups. (Many thanks to the referee for sharing their expertise to refine and improve the argument here.)

Proposition 4.1. *Let $\mathbb{G}$ be a simply connected exceptional algebraic group, E a nilpotent k-rational element of its Lie algebra, and ϕ the corresponding $\mathfrak{sl}(2,k)$-subalgebra of $\mathfrak{g}$. Then the number of k-rational orbits of $G = \mathbb{G}(k)$ in the k-points of the algebraic orbit $\mathbb{G} \cdot E$ is equal to the order of $H^1(k, \mathbb{G}^\phi)$.*

Proof. Note first that by [S, III§4.4], the number of rational p-adic orbits in the k-points of a given algebraic orbit $\mathcal{O} = \mathbb{G} \cdot E$ is finite. To compute this number, we begin with the short exact sequence of sets $1 \to \mathbb{G}^E \to \mathbb{G} \to \mathbb{G}/\mathbb{G}^E \to 1$. Take Galois cohomology [S, I§5.5], to obtain the long exact sequence

$$1 \longrightarrow \mathbb{G}^E(k) \longrightarrow \mathbb{G}(k) \longrightarrow (\mathbb{G}/\mathbb{G}^E)(k) \longrightarrow H^1(k, \mathbb{G}^E) \xrightarrow{\alpha} H^1(k, \mathbb{G}).$$

The k-rational orbits are the G-orbits on $(\mathbb{G}/\mathbb{G}^E)(k)$. Hence their number is measured by the quotient $(\mathbb{G}/\mathbb{G}^E)(k)/G$, which is in bijection with the fibre of α. In our setting, $\mathbb{G}$ is a simply connected linear algebraic group, and hence by [KnI, KnII], we have that $H^1(k, \mathbb{G}) = 0$. In particular, the number of rational orbits is given by the order of $H^1(k, \mathbb{G}^E)$.

We can make a further reduction. Since E is nilpotent, $\mathbb{G}^E$ is the semidirect product of its reductive part $\mathbb{G}^\phi$ and a unipotent part $\mathbb{U}^E$. The short exact sequence of groups $1 \to U^E \to \mathbb{G}^E \to \mathbb{G}^\phi \to 1$ yields a long exact sequence in cohomology

$$\cdots \longrightarrow H^1(k, \mathbb{U}^E) \longrightarrow H^1(k, \mathbb{G}^E) \xrightarrow{v} H^1(k, \mathbb{G}^\phi).$$

It remains to prove that v is a bijection. Let $b \in Z^1(k, \mathbb{G}^E)$ be a cocycle defining a class β in $H^1(k, \mathbb{G}^\phi)$. Since $\mathbb{G}^\phi \subset \mathbb{G}^E$, b can be interpreted as a cocycle in $Z^1(k, \mathbb{G}^E)$ as well. Let its associated class in $H^1(k, \mathbb{G}^E)$ be denoted $\widetilde{\beta}$. On the other hand, the map v on cocycles is given by composition with the projection onto $\mathbb{G}^\phi$; hence $v(\widetilde{\beta}) = \beta$, and v is surjective. Furthermore, the first cohomology group of the unipotent group $\mathbb{U}^E$ is trivial by [S, III§2.1], so the fibre of v is trivial at every point, as required. □

In practice, it is far easier to determine $H^1(k, \mathbb{G}^\phi)$ than $H^1(k, \mathbb{G}^E)$, as $\mathbb{G}^\phi$ is a reductive group. It may not be connected; however, its algebraic component group is well-known (see, for example, [CMcG, Ch.8.4]). We have

$$\mathbb{G}^\phi / \mathbb{G}^\phi_0 \simeq \pi_1(\mathbb{G} \cdot E),$$

where $\mathbb{G}^\phi_0$ is the algebraic connected component of the identity (and known to us from the calculations of the preceding section), and $\pi_1(\mathbb{G} \cdot E)$ is the $\mathbb{G}$-equivariant fundamental group of the orbit.

Corollary 4.2. *In the setting of Proposition 4.1, suppose that $\mathbb{G}^\phi_0$ is a semisimple simply connected algebraic group. Then $|H^1(k, \mathbb{G}^\phi)| \leq |H^1(k, \pi_1(\mathbb{G} \cdot E))|$.*

Proof. The short exact sequence $1 \to \mathbb{G}^\phi_0 \to \mathbb{G}^\phi \to \pi_1(\mathbb{G} \cdot E) \to 1$ gives rise to the long exact sequence

$$\cdots \to H^1(k, \mathbb{G}^\phi_0) \to H^1(k, \mathbb{G}^\phi) \to H^1(k, \pi_1(\mathbb{G} \cdot E)) \tag{4.1}$$

in cohomology. By [KnI, KnII], $H^1(k, \mathbb{G}^\phi_0)$ is trivial, so $H^1(k, \mathbb{G}^\phi)$ injects into $H^1(k, \pi_1(\mathbb{G} \cdot E))$. □

For those cases for which $\mathbb{G}_0^\phi$ is not simply connected, we note the following lemma from [Kn, Ch. IV].

Lemma 4.3 (Kneser)**.** *If $\mathbb{G}$ is a semisimple connected algebraic group defined over k, and $\tilde{\mathbb{G}}$ is its simply connected covering group, with kernel F, then $H^1(k,\mathbb{G}) \simeq H^2(k,F)$.*

The proof of this Lemma is done explicitly, on a case-by-case basis, and in particular it does not speak to the case of either reductive or disconnected groups. For those, we have some isolated results from [S]:

- If k contains all nth roots of unity, then $H^1(k,\mu_n) = k^*/k^{*n}$, and $H^2(k,\mu_n) = \mathbb{Z}/n\mathbb{Z}$.
- If $\mathbb{G}_m$ denotes the multiplicative group of the field, then $H^1(k,\mathbb{G}_m) = 0$.

Proposition 4.1 often gives an effective means of computing the number of rational orbits in the k-points of a given algebraic orbit. Another, more direct, method may also be used; though laborious, it yields representatives of the various rational orbits.

By a theorem of Mal'cev [CMcG, Thm 3.4.12], we know that the stabilizer of H in $\mathbb{G}$ acts transitively on the dense subset $\mathcal{P}$ of orbit representatives of $\mathcal{O}$ in $\mathfrak{g}[2]$. Now $\mathcal{P}(k)$ will decompose into one or more orbits under $\mathbb{G}^H(k) = G^H$.

Proposition 4.4. *The rational orbits of G^H acting on $\mathcal{P}$ are in one-to-one correspondence with the rational orbits of G acting on $\mathcal{O}(k) \subset \mathfrak{g}$.*

Proof. Given a rational Lie triple $\{E,H,F\}$ representing the orbit $\mathcal{O}$ (as in Section 2), suppose $\{E',H',F'\}$ is another rational Lie triple representing an orbit in $\mathcal{O}$. Then these triples are conjugate under an element $g \in \mathbb{G}$. In particular, their neutral elements H and H' are both diagonalizable over k, and hence conjugate under $G = \mathbb{G}(k)$. Without loss of generality, assume $H = H'$, so that E and E' both lie in $\mathcal{P}(k)$. Whence the triples are conjugate under G if and only if E and E' are conjugate under G^H. □

The Lie algebra of G^H is simply $\mathfrak{g}[0]$, the zero-weight space of $\mathfrak{g}$ under H. It contains G^ϕ as a subgroup which fixes E. This lemma thus reduces the problem of determining the rational orbits of an exceptional group acting on its Lie algebra down to determining the rational orbits of a much smaller group acting on a vector space. This is often feasible (see Appendix A), but nonetheless somewhat unsatisfactory—we have "reduced" from a simple group and a specific irreducible representation to a reductive group and a (generally) non-irreducible representation.

Remark. In Sections 6.1 to 6.3, we record the fundamental group of each orbit and the number of real rational classes (obtained from [CMcG]). We compute the number of p-adic rational classes using Proposition 4.1 where possible, and Proposition 4.4 otherwise. We then obtain representatives of each in Appendix A.

5. Possible forms of non-split rational orbits

Let $\mathcal{O}$ be a nilpotent orbit under $\mathbb{G}$. We call a rational orbit in $\mathcal{O}(k)$ *split* if the corresponding reductive group G^ϕ is split over k. Each orbit $\mathcal{O}$ of a split group $\mathbb{G}$ has one or more split rational orbits. All split rational orbits of a given $\mathcal{O}$ have k-isomorphic admissibility data $(G^\phi, \mathfrak{g}[-1])$. For non-split orbits, however, the situation is more complex. We proceed to decide the admissibility of non-split orbits as follows.

If E' is a representative of such an orbit, then the $\mathfrak{sl}(2,k)$-subalgebra ϕ' it defines must be conjugate under $\mathbb{G}$ to ϕ. Hence, in particular, $G^{\phi'}$ must be conjugate to G^{ϕ} under an element of $\mathbb{G}$, and in fact must be a (possibly different) k-form of $\mathbb{G}^{\phi}$. Furthermore, upon tensoring over k with an algebraic closure of k, the action of $G^{\phi'}$ on the corresponding $\mathfrak{g}[-1]$ will be equivalent to the split orbit case. Let us explore this latter observation.

In the following let $\mathbb{V}$ be a finite-dimensional representation of a reductive algebraic group $\mathbb{G}$, all defined over k. (In application, this $\mathbb{G}$ will be $\mathbb{G}^{\phi'}$.)

Lemma 5.1. *Suppose $\mathbb{G}$ acts irreducibly on $\mathbb{V}$. Then $G = \mathbb{G}(k)$ acts irreducibly on $V = \mathbb{V}(k)$.*

Proof. Suppose W is an invariant proper k-subspace of V. Then the subspace $\mathbb{W} = W \otimes_k \overline{k}$ is an invariant proper subspace of the action of G on $\mathbb{V}$. Since k is infinite, $G = \mathbb{G}(k)$ is Zariski-dense in $\mathbb{G}$. The set of all $g \in \mathbb{G}$ that preserve the subspace $\mathbb{W}$ is a Zariski-closed set (since the action is algebraic) and contains $\mathbb{G}(k)$, hence is all of $\mathbb{G}$. Thus $\mathbb{W}$ is an invariant subspace, and so must be $\{0\}$, by irreducibility. Whence the result. $\square$

The converse is not true, in general; an irreducible k-rational representation of a non-split k-form of $\mathbb{G}$ may decompose, upon passage to the algebraic (or even separable) closure, into a direct sum of irreducibles of $\mathbb{G}$. Let k_s denote the separable closure of k. More precisely, we have the following results of Tits [T2, Théorème 3.3, Théorème 7.2 and Lemme 7.4].

Theorem 5.2 (Tits)**.** *Let $\mathbb{G}$ be a reductive group defined over k.*

(i) *Let λ be a dominant integral weight of $\mathbb{G}$ and let k_λ be the extension field of k corresponding to the stabilizer subgroup of λ in $Gal(k_s/k)$. Then λ gives rise to an absolutely irreducible representation ρ_λ of $\mathbb{G}$ over some central simple division algebra D_λ over k_λ.*

(ii) *Each k-rational irreducible k-representation of $\mathbb{G}(k)$ is isomorphic to some ρ_λ^k, where ρ_λ^k is obtained from ρ_λ by restriction of scalars (from D_λ and k_λ to k).*

(iii) *Let d_λ denote the degree of D_λ over k_λ (i.e. the square root of the index), and suppose the orbit of λ under the Galois group $Gal(k_s/k)$ is $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$. Then, upon passage to k_s, ρ_λ^k decomposes into a direct sum of d_λ copies of ρ_{λ_1}, d_λ copies of ρ_{λ_2}, and so on.*

The orbit of λ under the Galois group can be read from the Tits diagram (see [T1]) of $\mathbb{G}$. Thus, whenever the decomposition of a (finite-dimensional) vector space V into irreducibles under $\mathbb{G}$ is known, we can apply (iii) of Theorem 5.2 to deduce whether or not this arises from a decomposition of $V(k)$ into irreducibles under $\mathbb{G}(k)$. In many cases, this allows us to exclude from possibility various groups $\mathbb{G}^{\phi'}(k)$ as occurring in other rational orbits. Where non-split orbits may occur, we compute their admissibility on a case-by-case basis (see Appendix B).

6. Admissibility of non-even nilpotent orbits: tables

Our method for choosing a k-rational representative E of $\mathcal{O}$ such that $G \cdot E$ is split is as follows; see [CMcG] for an overview of the subject. Note that all these computations were made feasible by programming them as functions for use with MATLAB® [M].

First set up a Chevalley basis $\{H_\alpha, X_\beta \mid \alpha \in \Delta, \beta \in \Phi\}$ for $\mathfrak{g}$, where $[X_\beta, X_{-\beta}] = H_\beta$ and the other structure constants are obtained from [GS] (for example). Given the weighted Dynkin Diagram of $\mathcal{O}$, reconstruct the neutral element $H \in \mathfrak{g}$ of a Lie triple classifying $\mathcal{O}$, uniquely chosen to lie in the dominant chamber of the maximal split torus. In fact, H will be a non-negative integral linear combination of the root H_α's. Then identify $\mathfrak{g}[2]$, the 2-weight space of H acting on $\mathfrak{g}$. A Zariski-dense subset $\mathcal{P}$ of $\mathfrak{g}[2]$ will be contained in $\mathcal{O}$; an element thereof is chosen as follows.

The Bala-Carter label [Ca] for $\mathcal{O}$ identifies the semisimple part of a Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$, and $\mathfrak{g}[2]$ contains the span of its simple roots. The orbit $\mathcal{O}$ is the saturation of a distinguished orbit of $\mathfrak{l}$ (also identified by the Bala-Carter label). We can choose a "standard" representative of this orbit using techniques of [CMcG, Ch.5]; this is E, the desired representative of a split orbit in $\mathcal{O}(k)$. Finally, one can deduce the value of $F \in \mathfrak{g}[-2]$ such that $\phi = span\{E, H, F\}$ forms an $\mathfrak{sl}(2, k)$-subalgebra of $\mathfrak{g}$, which in turn classifies the rational orbit through E.

Now compute the subalgebra $\mathfrak{g}^\phi = \{Z \in \mathfrak{g}[0] \mid [E, Z] = 0\}$. (This subalgebra is called $\underline{\mathrm{C}}$ in [E]. Elkington's tables contain some errors, however; among them: F_4: orbit $\widetilde{A_1}$; E_6: orbit $A_2 + 2A_1$; E_7: orbits $(3A_1)'$, $2A_2 + A_1$, and $(A_5)'$.) It is split over k, and we can easily decompose the subspace $\mathfrak{g}[-1]$ into irreducibles under $\mathfrak{g}^\phi$.

One can often use this to deduce the structure of (the algebraic identity component of) the corresponding group G^ϕ. For example, if the weights of the adjoint action of $\mathfrak{g}^\phi$ on $\mathfrak{g}$ generate the weight lattice of $\mathfrak{g}^\phi$, then G_0^ϕ must be simply connected. For another, when $\mathfrak{g}^\phi$ is a direct sum of simple or abelian factors, we can determine the intersection, if any, of the corresponding subgroups of G^ϕ as follows. Let $h_\alpha(t)$ denote the one-parameter subgroup of G corresponding to $H_\alpha \in \mathfrak{g}$. Then the one-parameter subgroup corresponding to a linear combination $\sum a_i H_{\alpha_i}$ is $h(t) = \prod h_{\alpha_i}(t)^{a_i}$. Any intersection of factors of G^ϕ occurs in the Cartan subgroup of G, so it suffices to compare elements of the form $h(t)$. Note, in particular, that $h_\alpha(-1)^2 = 1$ for any α; this is what allows group factors to intersect even where their Lie algebras are clearly disjoint.

In each of the following subsections, we list the non-even nilpotent (algebraic) orbits of the given split simply connected exceptional group over k. Each one is identified by both its Bala-Carter label (see [Ca]) and its weighted Dynkin diagram (see, for example, [CMcG]); we also indicate whether or not it is special (following tables in [CMcG]). We then specify a Lie triple $\{E, H, F\}$ representing a split rational orbit (in the k-points of the given algebraic orbit). The span of the Lie triple is an $\mathfrak{sl}(2, k)$ subalgebra denoted ϕ; its centralizer $\mathfrak{g}^\phi$ is described next, both by an explicit spanning set, and its isomorphism class. Next we give the decomposition into irreducibles of the action of $\mathfrak{g}^\phi$ on the subspace $\mathfrak{g}\,[-1]$ of (-1)-weight vectors under H. We then deduce the structure of G_0^ϕ, and whether or not the orbit is admissible, using results of Section 3.

In Sections 6.1 through 6.3, we go on to specify the fundamental group of the orbit and the number of real rational orbits (also from [CMcG]) and compute the number of p-adic orbits (using Section 4). (See Table 2 for a quick summary.) We elaborate on their admissibility, with reference to relevant parts of Appendices A and B for the case-by-case computations.

Table 2. The non-even nilpotent algebraic orbits of G_2, F_4 and E_6 decomposing into more than one k-rational orbit (k real or p-adic).

Group	Algebraic Orbit	# k-rational orbits
F_4	$\widetilde{A_1}$	$\lvert k^*/k^{*2}\rvert$
F_4	$A_1+\widetilde{A_1}$	2
F_4	B_2	$\lvert k^*/k^{*2}\rvert$
F_4	$C_3(a_1)$	$\lvert k^*/k^{*2}\rvert$
E_6	$2A_2+A_1$	$\lvert k^*/k^{*3}\rvert$
E_6	A_5	$\lvert k^*/k^{*3}\rvert$

We specify the labeling of the simple roots at the beginning of each section. Where this is more legible, we write $X_{abc\cdots}$ in place of $X_{a\alpha_1+b\alpha_2+c\alpha_3+\cdots}$ for positive root vectors (and simply $X_{-abc\cdots}$ for the corresponding negative root vector). We may also abbreviate by $H(a,b,\dots)$ the cumbersome $aH_{\alpha_1}+bH_{\alpha_2}+\cdots$.

An irreducible representation of $\mathfrak{sl}(2,k)$ of dimension n is denoted V^n. A one-dimensional μ-eigenspace of an element of the Cartan subalgebra is denoted k_μ. Other representations are described in words or with obvious notation. Where $\mathfrak{g}^\phi$ decomposes as a sum of two (or more) isomorphic ideals, we use subscripts a, b, $\dots$, to distinguish them and their actions on $\mathfrak{g}\,[-1]$. All direct sums of ideals of $\mathfrak{g}^\phi$ are commutative.

6.1. $G^0_{2,2}$. $\alpha \Rrightarrow \beta$

Bala-Carter Label: A_1 **(not special)**
Weighted Dynkin Diagram: $1 \Rrightarrow 0$
Lie Triple ϕ**:** $X_{2\alpha+3\beta}$,
$2H_\alpha+H_\beta$, $X_{-2\alpha-3\beta}$
$\mathfrak{g}^\phi=\mathrm{span}\{X_\beta,\,X_{-\beta},\,H_\beta\}\simeq\mathfrak{sl}(2,k)$
$\mathfrak{g}\,[-1]\simeq V^4$
We have $G_0^\phi=SL(2,k)$; apply Theorem 3.2. The orbit is **admissible**.
Fundamental Group: 1; **#$\mathbb{R}$-orbits:** 1; **#p-adic orbits:** 1

Bala-Carter Label: $\widetilde{A_1}$ **(not special)**
Weighted Dynkin Diagram: $0 \Rrightarrow 1$
Lie Triple ϕ**:** $X_{\alpha+2\beta}$,
$3H_\alpha+2H_\beta$, $X_{-\alpha-2\beta}$
$\mathfrak{g}^\phi=\mathrm{span}\{X_\alpha,\,X_{-\alpha},\,H_\alpha\}\simeq\mathfrak{sl}(2,k)$
$\mathfrak{g}\,[-1]\simeq V^2$
We have $G_0^\phi=SL(2,k)$; apply Theorem 3.2. The orbit is **not admissible**.
Fundamental Group: 1; **#$\mathbb{R}$-orbits:** 1; **#p-adic orbits:** 1

6.2. $F^0_{4,4}$. $\alpha — \beta \Longrightarrow \gamma — \delta$

Bala-Carter Label: A_1 **(not special)**
Weighted Dynkin Diagram: $1 — 0 \Longrightarrow 0 — 0$
Lie Triple ϕ**:** $X_{2\alpha+3\beta+4\gamma+2\delta}$,

$2H_\alpha+3H_\beta+2H_\gamma+H_\delta$, $X_{-2\alpha-3\beta-4\gamma-2\delta}$
$\mathfrak{g}^\phi = \mathrm{span}\{C_3$ with simple roots β, γ, δ $\} \simeq \mathfrak{sp}(6,k)$
$\mathfrak{g}\,[-1] \simeq V^{14}$ (the representation of highest weight $L_1+L_2+L_3$, notation of [FH, §17])
We have $G_0^\phi = Sp(6,k)$. The $SL(2,k)$-subgroup associated with the long root of $Sp(6,k)$ decomposes $\mathfrak{g}\,[-1]$ into five 2-dimensional irreducibles, and four copies of the trivial representation. Apply Lemma 3.1 and Theorem 3.2. The orbit is **not admissible**.
Fundamental Group: 1; **#$\mathbb{R}$-orbits:** 1; **#p-adic orbits:** 1

Bala-Carter Label: $\widetilde{A_1}$ **(special)**
Weighted Dynkin Diagram: $0 — 0 \Longrightarrow 0 — 1$
Lie Triple ϕ: $X_{\alpha+2\beta+3\gamma+2\delta}$,
$2H_\alpha+4H_\beta+3H_\gamma+2H_\delta$, $X_{-\alpha-2\beta-3\gamma-2\delta}$
$\mathfrak{g}^\phi = \mathrm{span}\{X_\alpha,\ X_\beta,\ X_{\alpha+\beta},\ X_{\beta+2\gamma},\ X_{\alpha+\beta+2\gamma},\ X_{\alpha+2\beta+2\gamma},\ H_\alpha,\ H_\beta,\ H_\gamma,$ and the corresponding negative root spaces$\} \simeq \mathfrak{sl}(4,k)$
$\mathfrak{g}\,[-1] \simeq V_{std} \oplus V_{std}^*$
We have $G_0^\phi = SL(4,k)$. Apply Corollary 3.4. The orbit is **admissible**.
Fundamental Group: S_2; **#$\mathbb{R}$-orbits:** 2; **#p-adic orbits:** $|k^*/{k^*}^2|$ (see Appendix A.1)
Note on rational classes: We see from Appendix B.3 that the only other rational form of $\mathbb{G}^\phi$ that could arise is of a special unitary group, and that the cover of G_0^ϕ would split in that case as well. All rational orbits are therefore admissible.

Bala-Carter Label: $A_1+\widetilde{A_1}$ **(special)**
Weighted Dynkin Diagram: $0 — 1 \Longrightarrow 0 — 0$
Lie Triple ϕ: $X_{1222}+X_{1231}$,
$3H_\alpha+6H_\beta+4H_\gamma+2H_\delta$, $X_{-1222}+X_{-1231}$
$\mathfrak{g}^\phi = \mathrm{span}\{\ X_\alpha,\ X_{-\alpha},\ H_\alpha;\ -2X_\gamma+2X_\delta,\ -X_{-\gamma}+X_{-\delta},\ 2H_\gamma+2H_\delta\ \} \simeq \mathfrak{sl}(2,k)_a \oplus \mathfrak{sl}(2,k)_b$
$\mathfrak{g}\,[-1] \simeq (V_a^2 \otimes V_b^5) \oplus (V_a^2 \otimes V_b^1)$
We have that the first $\mathfrak{sl}(2,k)$ corresponds necessarily to an $SL(2,k)$, since it has an even-dimensional irreducible representation. On the other hand, the second $\mathfrak{sl}(2,k)$ embeds into $\mathfrak{g}$ as a 3-dimensional representation (*i.e.* the image lies irreducibly in the $\mathfrak{sl}(3,k)$ subalgebra corresponding to the roots γ and δ), and hence necessarily lifts to $PGL(2,k)$ as a group. Thus $G_0^\phi = SL(2,k) \times PGL(2,k)$. The metaplectic cover splits over each group individually (using Theorem 3.2 and Theorem 3.3, respectively). The orbit is **admissible**.
Fundamental Group: 1; **#$\mathbb{R}$-orbits:** 2; **#p-adic orbits:** 2 (see Appendix A.2)
Note on rational classes: By Appendix B.1 we see that the metaplectic cover will split over all other possible rational forms of $\mathbb{G}^\phi$, and hence that all rational orbits are admissible.

Bala-Carter Label: $A_2+\widetilde{A_1}$ **(not special)**
Weighted Dynkin Diagram: $0 — 0 \Longrightarrow 1 — 0$
Lie Triple ϕ: $X_{1120}+X_{0122}+X_{1221}$,
$4H_\alpha+8H_\beta+6H_\gamma+3H_\delta$, $2X_{-1120}+2X_{-0122}+X_{-1221}$

$\mathfrak{g}^\phi = \mathrm{span}\{\, -2X_\beta - X_{-\delta} + X_{-\alpha-\beta},\, X_\delta + 2X_{\alpha+\beta} - X_{-\beta},\, -2H_\alpha - H_\delta\} \simeq \mathfrak{sl}(2,k)$
$\mathfrak{g}\,[-1] \simeq V^2 \oplus V^4$
We have $G_0^\phi = SL(2,k)$; apply Theorem 3.2. The orbit is **not admissible**.
Fundamental Group: 1; **#$\mathbb{R}$-orbits:** 1; **#p-adic orbits:** 1

Bala-Carter Label: B_2 **(not special)**
Weighted Dynkin Diagram: $2 — 0 \Longrightarrow 0 — 1$
Lie Triple ϕ: $X_{1110} + X_{0122}$,
$6H_\alpha + 10H_\beta + 7H_\gamma + 4H_\delta$, $3X_{-1110} + 4X_{-0122}$
$\mathfrak{g}^\phi = \mathrm{span}\{X_\beta,\, X_{-\beta},\, H_\beta,\, X_{\beta+2\gamma},\, X_{-\beta-2\gamma},\, H_\beta + H_\gamma\} \simeq \mathfrak{sl}(2,k)_a \oplus \mathfrak{sl}(2,k)_b$
$\mathfrak{g}\,[-1] \simeq (V_a^2 \otimes V_b^1) \oplus (V_a^1 \otimes V_b^2)$
We have that $G_0^\phi(k) \simeq SL(2,k) \times SL(2,k)$ since $\mathfrak{g}\,[-1]$ is a faithful representation. The metaplectic cover over each $SL(2,k)$ fails to split by Theorem 3.2. The orbit is **not admissible**.
Fundamental Group: S_2; **#$\mathbb{R}$-orbits:** 2; **#p-adic orbits:** $|k^*/k^{*2}|$ (see Appendix A.3)
Note on rational classes: Each non-split rational orbit has a corresponding group $G^\phi \simeq SL(2,K)$, for K varying over all the nontrivial quadratic extensions of k. As shown in Appendix B.2, if $-1 \in k^{*2}$, then all non-split orbits are admissible; otherwise, only the non-split orbit for which $K = k(\sqrt{-1})$ is admissible.

Bala-Carter Label: $\widetilde{A_2} + A_1$ **(not special)**
Weighted Dynkin Diagram: $0 — 1 \Longrightarrow 0 — 1$
Lie Triple ϕ: $X_{0121} + X_{1111} + X_{1220}$,
$5H_\alpha + 10H_\beta + 7H_\gamma + 4H_\delta$, $2X_{-0121} + 2X_{-1111} + X_{-1220}$
$\mathfrak{g}^\phi = \mathrm{span}\{X_\alpha + X_\gamma,\, X_{-\alpha} + X_{-\gamma},\, H_\alpha + H_\gamma\} \simeq \mathfrak{sl}(2,k)$
$\mathfrak{g}\,[-1] \simeq 2V^2 \oplus V^4$
We have $G_0^\phi = SL(2,k)$; apply Theorem 3.2. The orbit is **admissible**.
Fundamental Group: 1; **#$\mathbb{R}$-orbits:** 1; **#p-adic orbits:** 1

Bala-Carter Label: $C_3(a_1)$ **(not special)**
Weighted Dynkin Diagram: $1 — 0 \Longrightarrow 1 — 0$
Lie Triple ϕ: $X_{0120} + X_{1111} + X_{0122}$,
$6H_\alpha + 11H_\beta + 8H_\gamma + 4H_\delta$, $4X_{-0120} + 3X_{-1111} + X_{-0122}$
$\mathfrak{g}^\phi = \mathrm{span}\{X_\beta,\, X_{-\beta},\, H_\beta\} \simeq \mathfrak{sl}(2,k)$
$\mathfrak{g}\,[-1] \simeq 3V^2$
We have $G_0^\phi = SL(2,k)$; apply Theorem 3.2. The orbit is **not admissible**.
Fundamental Group: S_2; **#$\mathbb{R}$-orbits:** 2; **#p-adic orbits:** $|k^*/k^{*2}|$ (see Appendix A.4)
Note on rational classes: By Appendix B.1, we conclude that all rational orbits are split, and hence none are admissible.

Bala-Carter Label: C_3 **(special)**
Weighted Dynkin Diagram: $1 — 0 \Longrightarrow 1 — 2$
Lie Triple ϕ: $X_\delta + X_{\alpha+\beta+\gamma} + X_{\beta+2\gamma}$,
$10H_\alpha + 19H_\beta + 14H_\gamma + 8H_\delta$, $8X_{-\delta} + 5X_{-\alpha-\beta-\gamma} + 9X_{-\beta-2\gamma}$
$\mathfrak{g}^\phi = \mathrm{span}\{X_\beta,\, X_{-\beta},\, H_\beta\} \simeq \mathfrak{sl}(2,k)$
$\mathfrak{g}\,[-1] \simeq 2V^2$

We have $G_0^\phi = SL(2,k)$; apply Theorem 3.2. The orbit is **admissible**.
Fundamental Group: 1; **#ℝ-orbits:** 1; **#p-adic orbits:** 1

6.3. ${}^1E_{6,6}^0$.

$$\begin{array}{ccccccccc} & & & & \alpha_2 & & & & \\ & & & & | & & & & \\ \alpha_1 & - & \alpha_3 & - & \alpha_4 & - & \alpha_5 & - & \alpha_6 \end{array}$$

Bala-Carter Label: A_1 **(special)**

Weighted Dynkin Diagram:
$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & & | & & & & \\ 0 & - & 0 & - & 0 & - & 0 & - & 0 \end{array}$$

Lie Triple ϕ: X_{122321},
$H(1,2,2,3,2,1)$, $X_{-122321}$
$\mathfrak{g}^\phi = \text{span}\{A_5$ with simple roots $\alpha_1,\ \alpha_3,\ \alpha_4,\ \alpha_5,\ \alpha_6\ \} \simeq \mathfrak{sl}(6,k)$
$\mathfrak{g}\,[-1] \simeq V^{20}$
We have $G_0^\phi = SL(6,k)$, the subgroup of the simply connected group corresponding to this sub-Dynkin diagram of E_6. To determine admissibility, apply Lemma 3.1. Each $SL(2,k)$ root subgroup decomposes $\mathfrak{g}\,[-1]$ into $6V^2 \oplus 8V^1$; apply Theorem 3.2. The orbit is **admissible**.
Fundamental Group: 1; **#ℝ-orbits:** 1; **#p-adic orbits:** 1

Bala-Carter Label: $2A_1$ **(special)**

Weighted Dynkin Diagram:
$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & | & & & & \\ 1 & - & 0 & - & 0 & - & 0 & - & 1 \end{array}$$

Lie Triple ϕ: $X_{112211} + X_{111221}$,
$H(2,2,3,4,3,2)$, $X_{-112211} + X_{-111221}$
$\mathfrak{g}^\phi = \text{span}\{H(2,0,1,0,-1,-2)$ and the split B_3 subalgebra with simple root vectors $X_{\alpha_2},\ X_{\alpha_4},\ X_{\alpha_3} - X_{\alpha_5}$ and Cartan spanned by $H_{\alpha_2},\ H_{\alpha_4}$ and $H_{\alpha_3} + H_{\alpha_5}\} \simeq k \oplus \mathfrak{so}(7,k)$
$\mathfrak{g}\,[-1] \simeq W \oplus W^*$, with $W = k_3 \otimes V_{spin}$
We have the (split) $\mathfrak{so}(7,k)$ admits a spin representation, hence lifts to the simply connected $Spin(7,k)$ as a group. Comparing Cartan elements, we deduce that the two factors intersect in a 2-element central subgroup Z. Hence $G^\phi = (k^* \times Spin(7,k))/Z$. Apply Corollary 3.4. The orbit is **admissible**.
Fundamental Group: 1; **#ℝ-orbits:** 1; **#p-adic orbits:** 1 (see Appendix A.5)

Bala-Carter Label: $3A_1$ **(not special)**

Weighted Dynkin Diagram:
$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & | & & & & \\ 0 & - & 0 & - & 1 & - & 0 & - & 0 \end{array}$$

Lie Triple ϕ: $X_{111211} + X_{112210} + X_{011221}$,
$H(2,3,4,6,4,2)$, $X_{-111211} + X_{-112210} + X_{-011221}$
$\mathfrak{g}^\phi = \text{span}\{X_{\alpha_2},\ X_{-\alpha_2},\ H_{\alpha_2};\ A_2$ with simple root vectors $-X_{\alpha_1} + X_{\alpha_5},\ X_{\alpha_3} + X_{\alpha_6}$ and Cartan spanned by $H_{\alpha_1} + H_{\alpha_5}$ and $H_{\alpha_3} + H_{\alpha_6}\} \simeq \mathfrak{sl}(2,k) \oplus \mathfrak{sl}(3,k)$
$\mathfrak{g}\,[-1] \simeq V^2 \otimes (V^1 \oplus V_{adj})$
We have the image of G^ϕ under the adjoint representation is $SL(2,k) \times PGL(3,k)$. For the simply connected group, note that the $\mathfrak{sl}(3,k)$ subalgebra embeds into

$\mathfrak{sl}(3,k)(\alpha_1,\alpha_3) \oplus \mathfrak{sl}(3,k)(\alpha_5,\alpha_6)$, and each of these $\mathfrak{sl}(3,k)$-subalgebras admit 3-dimensional irreducible representations. Thus our $\mathfrak{sl}(3,k)$ lifts to a copy of $SL(3,k)$ and $G^\phi = SL(2,k) \times SL(3,k)$. The metaplectic cover does not split over the $SL(2,k)$ factor by Theorem 3.2. The orbit is **not admissible**.
Fundamental Group: 1; **#ℝ-orbits:** 1; **#p-adic orbits:** 1

Bala-Carter Label: $A_2 + A_1$ **(special)**

Weighted Dynkin Diagram:
$$\begin{array}{c} 1 \\ 1-0-\dot{0}-0-1 \end{array}$$

Lie Triple ϕ: $X_{111110} + X_{101111} + X_{011211}$,
$H(3,4,5,7,5,3)$, $2X_{-111110} + X_{-101111} + 2X_{-011211}$
$\mathfrak{g}^\phi = \mathrm{span}\{H(1,0,1,1,-1,-1)$, and A_2 with simple root vectors X_{α_3}, $X_{\alpha_4+\alpha_5}$ and Cartan spanned by H_{α_3}, $H_{\alpha_4} + H_{\alpha_5}\} \simeq k \oplus \mathfrak{sl}(3,k)$
$\mathfrak{g}[-1] \simeq W \oplus W^*$, with $W = 2(k_{-1} \otimes V_{std}) \oplus (k_3 \otimes V^1)$
We have $G^\phi = k^* \times SL(3,k)$, since the $\mathfrak{sl}(3,k)$ admits a 3-dimensional irreducible representation, and the two subgroups do not intersect. Apply Corollary 3.4. The orbit is **admissible**.
Fundamental Group: 1; **#ℝ-orbits:** 1; **#p-adic orbits:** 1

Bala-Carter Label: $A_2 + 2A_1$ **(special)**

Weighted Dynkin Diagram:
$$\begin{array}{c} 0 \\ 0-1-\dot{0}-1-0 \end{array}$$

Lie Triple ϕ: $X_{101110} + X_{011111} + X_{011210} + X_{111211}$,
$H(3,4,6,8,6,3)$, $2X_{-011110} + 2X_{-011111} + X_{-011210} + X_{-111211}$
$\mathfrak{g}^\phi = \mathrm{span}\{H(1,0,2,0,-2,-1); X_{\alpha_1} + 2X_{\alpha_2+\alpha_4} - X_{-\alpha_4} - X_{-\alpha_6}, -2X_{\alpha_4} - X_{\alpha_6} + X_{-\alpha_1} + X_{-\alpha_2-\alpha_4}, H_{\alpha_1} + 2H_{\alpha_2} - H_{\alpha_6}\} \simeq k \oplus \mathfrak{sl}(2,k)$
$\mathfrak{g}[-1] \simeq W \oplus W^*$, with $W = k_3 \otimes (V^4 \oplus V^2)$
We have $G^\phi = (k^* \times SL(2,k))/Z$ for some 2-element central subgroup Z contained in the Cartan. Apply Corollary 3.4. The orbit is **admissible**.
Fundamental Group: 1; **#ℝ-orbits:** 1; **#p-adic orbits:** 1 (see Appendix A.6)

Bala-Carter Label: A_3 **(special)**

Weighted Dynkin Diagram:
$$\begin{array}{c} 2 \\ 1-0-\dot{0}-0-1 \end{array}$$

Lie Triple ϕ: $X_{\alpha_2+\alpha_4} + X_{011110} + X_{101111}$,
$H(4,6,7,10,7,4)$, $3X_{-\alpha_2-\alpha_4} + 3X_{-011110} + 4X_{-101111}$
$\mathfrak{g}^\phi = \mathrm{span}\{H(2,0,1,0,-1,-2)$ and B_2 with positive root vectors $X_{\alpha_3+\alpha_4}$, $X_{\alpha_5} + X_{-\alpha_3}$, $-X_{\alpha_4} + X_{\alpha_3+\alpha_4+\alpha_5}$, $X_{\alpha_4+\alpha_5}$, and Cartan spanned by $H_{\alpha_3} + H_{\alpha_4}$, $-H_{\alpha_3} + H_{\alpha_5}\} \simeq k \oplus \mathfrak{so}(5,k)$
$\mathfrak{g}[-1] \simeq (k_3 \otimes V^4) \oplus (k_3 \otimes V^4)^*$
We have the $\mathfrak{so}(5,k)$ factor lifts to a copy of $Sp(4,k)$ in the group, by the existence of a 4-dimensional irreducible representation. Comparing Cartan elements, we deduce $G^\phi = (k^* \times Sp(4,k))/Z$, for some 2-element central subgroup Z. Apply Corollary 3.4. The orbit is **admissible**.
Fundamental Group: 1; **#ℝ-orbits:** 1; **#p-adic orbits:** 1 (see Appendix A.7)

Bala-Carter Label: $2A_2 + A_1$ **(not special)**

Weighted Dynkin Diagram:
$$\begin{matrix} & & 0 & & \\ 1- & 0- & 1 & -0 & -1 \end{matrix}$$

Lie Triple ϕ: $X_{101100} + X_{000111} + X_{111110} + X_{011111} + X_{011210}$,
$H(4,5,7,10,7,4)$, $2X_{-101100} + 2X_{-000111} + 2X_{-111110} + 2X_{-011111} + X_{-011210}$
$\mathfrak{g}^{\phi} = \text{span}\{-X_{\alpha_3} + X_{\alpha_5} + X_{-\alpha_2},\ X_{\alpha_2} - X_{-\alpha_3} + X_{-\alpha_5},\ -H_{\alpha_2} + H_{\alpha_3} + H_{\alpha_5}\} \simeq \mathfrak{sl}(2,k)$
$\mathfrak{g}[-1] \simeq 4V^2 \oplus V^4$
We have $G_0^{\phi} = SL(2,k)$; apply Theorem 3.2. The orbit is **admissible**.
Fundamental Group: $\mathbb{Z}/3\mathbb{Z}$; **#$\mathbb{R}$-orbits:** 1; **#p-adic orbits:** $|k^*/k^{*3}|$ (see Appendix A.8)
Note on rational classes: the existence of a single 4-dimensional rational absolutely irreducible representation implies that all orbits must be split, and hence admissible.

Bala-Carter Label: $A_3 + A_1$ **(not special)**

Weighted Dynkin Diagram:
$$\begin{matrix} & & 1 & & \\ 0- & 1- & 0 & -1 & -0 \end{matrix}$$

Lie Triple ϕ: $X_{011100} + X_{101110} + X_{010111} + X_{001111}$,
$H(4,6,8,11,8,4)$, $3X_{-011100} + 4X_{-101110} + 3X_{-010111} + X_{-001111}$
$\mathfrak{g}^{\phi} = \text{span}\{H(2,0,1,0,-1,1);\ X_{\alpha_4},\ X_{-\alpha_4},\ H_{\alpha_4}\} \simeq k \oplus \mathfrak{sl}(2,k)$
$\mathfrak{g}[-1] \simeq (k_3 \oplus 3k_0 \oplus k_{-3}) \otimes V^2$
We have $G^{\phi}(k) = k^* \times SL(2,k)$. Apply Theorem 3.2. The orbit is **not admissible**.
Fundamental Group: 1; **#$\mathbb{R}$-orbits:** 1; **#p-adic orbits:** 1

Bala-Carter Label: $A_4 + A_1$ **(special)**

Weighted Dynkin Diagram:
$$\begin{matrix} & & 1 & & \\ 1- & 1- & 0 & -1 & -1 \end{matrix}$$

Lie Triple ϕ: $X_{\alpha_1+\alpha_3} + X_{011100} + X_{010110} + X_{001110} + X_{000111}$,
$H(6,8,11,15,11,6)$, $6X_{-\alpha_1-\alpha_3} + 4X_{-011100} + 4X_{-010110} + X_{-001110} + 6X_{-000111}$
$\mathfrak{g}^{\phi} = \text{span}\{H(-2,0,-1,-3,1,2)\} \simeq k$
$\mathfrak{g}[-1] \simeq 4k_3 \oplus 4k_{-3}$
We have $G^{\phi} = k^*$; apply Corollary 3.4. The orbit is **admissible**.
Fundamental Group: 1; **#$\mathbb{R}$-orbits:** 1; **#p-adic orbits:** 1

Bala-Carter Label: A_5 **(not special)**

Weighted Dynkin Diagram:
$$\begin{matrix} & & 1 & & \\ 2- & 1- & 0 & -1 & -2 \end{matrix}$$

Lie Triple ϕ: $X_{\alpha_1} + X_{\alpha_6} + X_{011100} + X_{010110} + X_{001110}$,
$H(8,10,14,19,14,8)$, $8X_{-\alpha_1} + 8X_{-\alpha_6} + 5X_{-011100} + 5X_{-010110} + 9X_{-001110}$
$\mathfrak{g}^{\phi} = \text{span}\{X_{\alpha_4}, X_{-\alpha_4}, H_{\alpha_4}\} \simeq \mathfrak{sl}(2,k)$
$\mathfrak{g}[-1] \simeq 3V^2$
We have $G_0^{\phi} = SL(2,k)$; apply Theorem 3.2. The orbit is **not admissible**.
Fundamental Group: $\mathbb{Z}/3\mathbb{Z}$; **#$\mathbb{R}$-orbits:** 1; **#p-adic orbits:** $|k^*/k^{*3}|$ (see Appendix A.9)
Note on rational classes: By Appendix B.1, all orbits are split, hence none are admissible.

Bala-Carter Label: $D_5(a_1)$ **(special)**

Weighted Dynkin Diagram:
$$\begin{array}{c} 2 \\ 1-1-0-1-1 \end{array}$$

Lie Triple ϕ**:** $X_{\alpha_2} + X_{\alpha_1+\alpha_3} + X_{101100} + X_{001110} + X_{000111}$,
$H(7,10,13,18,13,7)$, $10X_{-\alpha_2} + 2X_{-\alpha_1-\alpha_3} - 5X_{-\alpha_2-\alpha_4} + 2X_{-\alpha_5-\alpha_6} + 5X_{-101100}$ $+ 6X_{-001110} + 7X_{-000111}$
$\mathfrak{g}^\phi = \mathrm{span}\{H(-1,0,1,0,-1,1)\} \simeq k$
$\mathfrak{g}\,[-1] \simeq 3k_3 \oplus 3k_{-3}$
We have $G^\phi = k^*$; apply Corollary 3.4. The orbit is **admissible**.
Fundamental Group: 1; **#$\mathbb{R}$-orbits:** 1; **#p-adic orbits:** 1

6.4. $E^0_{7,7}$.

$$\begin{array}{c} \alpha_2 \\ \alpha_1 - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6 - \alpha_7 \end{array}$$

Bala-Carter Label: A_1 **(special)**

Weighted Dynkin Diagram:
$$\begin{array}{c} 0 \\ 1-0-0-0-0-0 \end{array}$$

Lie Triple ϕ**:** $X_{2234321}$,
$H(2,2,3,4,3,2,1)$, $X_{-2234321}$
$\mathfrak{g}^\phi = \mathrm{span}\{D_6$ with simple roots $\alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_5,\ \alpha_6,\ \alpha_7\} \simeq \mathfrak{so}(12,k)$
$\mathfrak{g}\,[-1] \simeq V^{32}$
We have $G^\phi = Spin(12,k)$, since it corresponds to the sub-Dynkin diagram of the simply connected group. A root $SL(2,k)$ corresponding to a long root decomposes $\mathfrak{g}\,[-1]$ into 16 copies of the trivial representation and 8 copies of the 2-dimensional representation. Thus the cover splits over each long root by Theorem 3.2, and consequently over G^ϕ by Lemma 3.1. The orbit is **admissible**.

Bala-Carter Label: $2A_1$ **(special)**

Weighted Dynkin Diagram:
$$\begin{array}{c} 0 \\ 0-0-0-0-1-0 \end{array}$$

Lie Triple ϕ**:** $X_{1223221} + X_{1123321}$,
$H(2,3,4,6,5,4,2)$, $X_{-1223221} + X_{-1123321}$
$\mathfrak{g}^\phi = \mathrm{span}\{X_{\alpha_7}, X_{-\alpha_7}, H_{\alpha_7}$; B_4 with simple root vectors $X_{\alpha_1}, X_{\alpha_3}, X_{\alpha_4}, X_{\alpha_5} - X_{\alpha_2}$ and Cartan spanned by $H_{\alpha_1},\ H_{\alpha_3},\ H_{\alpha_4},\ H_{\alpha_2} + H_{\alpha_5}\} \simeq \mathfrak{sl}(2,k) \oplus \mathfrak{so}(9,k)$
$\mathfrak{g}\,[-1] \simeq V^2 \otimes V^{16}$
We have $G^\phi = SL(2,k) \times Spin(9,k)$, since each group admits representations of the simply connected group and they admit no intersection. Applying Lemma 3.1 and Theorem 3.2, we deduce that the cover of G^ϕ splits over each piece individually. The orbit is **admissible**.

Bala-Carter Label: $(3A_1)'$ **(not special)**

Weighted Dynkin Diagram:
$$\begin{array}{c} 0 \\ 0-1-0-0-0-0 \end{array}$$

Lie Triple ϕ**:** $X_{1122221} + X_{1123211} + X_{1223210}$,
$H(3,4,6,8,6,4,2)$, $X_{-1122221} + X_{-1123211} + X_{-1223210}$

$\mathfrak{g}^\phi = \mathrm{span}\{X_{\alpha_1}, X_{-\alpha_1}, H_{\alpha_1};$ C3 with simple roots $X_{\alpha_5}, -X_{\alpha_4} + X_{\alpha_6}, X_{\alpha_2} + X_{\alpha_7}\}$
$\simeq \mathfrak{sl}(2,k) \oplus \mathfrak{sp}(6,k)$
$\mathfrak{g}\,[-1] \simeq W \oplus W^*$, with $W = V^2 \otimes (V^1 \oplus V^{14})$
We have that the $\mathfrak{sl}(2,k)$ lifts to $SL(2,k)$. With the help of explicit calculations to determine which copy of C_3 in E_7 we have, we deduce from [LS, Table 8.6] that it admits a 6-dimensional irreducible representation, and hence lifts to $Sp(6,k)$ at the group level. They do not intersect, so $G^\phi = SL(2,k) \times Sp(6,k)$. The cover does not split over the $SL(2,k)$ piece of G^ϕ by Theorem 3.2. The orbit is **not admissible**.

Bala-Carter Label: $4A_1$ **(not special)**

Weighted Dynkin Diagram:
$$\begin{array}{c} 1 \\ 0-0-0-0-0-1 \end{array}$$

Lie Triple ϕ: $X_{1111111} + X_{0112221} + X_{1123211} + X_{1223210}$,
$H(3,5,6,9,7,5,3), X_{-1111111} + X_{-0112221} + X_{-1123211} + X_{-1223210}$
$\mathfrak{g}^\phi = \mathrm{span}\{H_{\alpha_3}, H_{\alpha_5}, H_{\alpha_1}+H_{\alpha_4}+H_{\alpha_6}, X_{\alpha_6} - X_{1011000}, X_{0001100} - X_{-0011000}, X_{\alpha_3}, X_{\alpha_1} - X_{0001110}, X_{0011100} - X_{-\alpha_4}, X_{\alpha_5}, X_{1010000} + X_{0011110}, X_{0000110} + X_{1011100}, X_{1011110}$, and the corresponding negative root vectors$\} \simeq \mathfrak{sp}(6,k)$
$\mathfrak{g}\,[-1] \simeq 2V^6 \oplus V^{14}$
We have $G^\phi = Sp(6,k)$, by the existence of the 6-dimensional irreducible representation. The restriction of the representation $\mathfrak{g}\,[-1]$ to the $SL(2,k)$ arising from the long root decomposes as seven 2-dimensional representations and twelve trivial representations. Hence the cover does not split over this $SL(2,k)$, and therefore, by Lemma 3.1, it doesn't split over $G^\phi = Sp(6,k)$. The orbit is **not admissible**.

Bala-Carter Label: $A_2 + A_1$ **(special)**

Weighted Dynkin Diagram:
$$\begin{array}{c} 0 \\ 1-0-0-0-1-0 \end{array}$$

Lie Triple ϕ: $X_{1011110} + X_{0112221} + X_{1223211}$,
$H(4,5,7,10,8,6,3)$, $2X_{-1011110} + X_{-0112221} + 2X_{-1223211}$
$\mathfrak{g}^\phi = \mathrm{span}\{H(0,2,1,2,1,0,-2); A_3$ with simple roots $\alpha_3, \alpha_4, \alpha_5\} \simeq\ k \oplus \mathfrak{sl}(4,k)$
$\mathfrak{g}\,[-1] \simeq W \oplus W^*$, with $W = (2k_1 \oplus k_3) \otimes V^4$
We have $G_0^\phi = (k^* \times SL(4,k))/Z$, for some 2-element central subgroup Z. Apply Corollary 3.4. The orbit is **admissible**.

Bala-Carter Label: $A_2 + 2A_1$ **(special)**

Weighted Dynkin Diagram:
$$\begin{array}{c} 0 \\ 0-0-1-0-0-0 \end{array}$$

Lie Triple ϕ: $X_{1112111} + X_{1122110} + X_{0112211} + X_{1112210}$,
$H(4,6,8,12,9,6,3), X_{-1112111} + 2X_{-1122110} + 2X_{-0112211} + X_{-1112210}$
$\mathfrak{g}^\phi = \mathrm{span}\{X_{\alpha_2}, X_{-\alpha_2}, H_{\alpha_2}; 2X_{\alpha_1} + X_{\alpha_3} - X_{\alpha_5} + X_{\alpha_7}, X_{-\alpha_1} + 2X_{-\alpha_3} - X_{-\alpha_5} + X_{-\alpha_7}, 2H_{\alpha_1} + 2H_{\alpha_3} + H_{\alpha_5} + H_{\alpha_7}; X_{0000110} - X_{0000011}, X_{-0000110} - X_{-0000011}, H_{\alpha_5} + 2H_{\alpha_6} + H_{\alpha_7}\} \simeq \mathfrak{sl}(2,k)_a \oplus \mathfrak{sl}(2,k)_b \oplus \mathfrak{sl}(2,k)_c$
$\mathfrak{g}\,[-1] \simeq V_a^2 \otimes (V_b^2 \oplus V_b^4) \otimes V_c^2$
We have that the groups all are $SL(2,k)$s. However, the second and third $SL(2,k)$ share a common center Z. So $G_0^\phi = SL(2,k) \times (SL(2,k) \times SL(2,k))/Z$. By Theorem 3.2, the cover splits over the first $SL(2,k)$; by Corollary 3.4, it splits over

the $(SL(2,k)\times SL(2,k))/Z$ factor as well. Hence the cover splits. The orbit is **admissible**.

Bala-Carter Label: A_3 **(special)**

Weighted Dynkin Diagram:
$$\begin{array}{c} 0 \\ 2-0-\dot{0}-0-1-0 \end{array}$$

Lie Triple ϕ**:** $X_{1111000}+X_{1011100}+X_{0112221}$,
$H(6,7,10,14,11,8,4)$, $3X_{-1111000}+3X_{-1011100}+4X_{-0112221}$
$\mathfrak{g}^{\phi}=\mathrm{span}\{X_{\alpha_7},\ X_{-\alpha_7},\ H_{\alpha_7};\ B_3$ with simple root vectors $X_{\alpha_3},\ X_{\alpha_4},\ X_{\alpha_2}+X_{\alpha_5}$ and Cartan spanned by $H_{\alpha_3},\ H_{\alpha_4},\ H_{\alpha_2}+H_{\alpha_5}\}\simeq\mathfrak{sl}(2,k)\oplus\mathfrak{so}(7,k)$
$\mathfrak{g}\,[-1]\simeq V^2\otimes V_{spin}$
We have $G_0^{\phi}=SL(2,k)\times Spin(7)$, because the split $\mathfrak{so}(7,k)$ admits an 8-dimensional irreducible (spin) representation and the two subgroups do not intersect. The cover splits over the $SL(2,k)$ by Theorem 3.2 and over $Spin(7,k)$ by Corollary 3.4. The orbit is **admissible**.

Bala-Carter Label: $2A_2+A_1$ **(not special)**

Weighted Dynkin Diagram:
$$\begin{array}{c} 0 \\ 0-1-\dot{0}-0-1-0 \end{array}$$

Lie Triple ϕ**:** $X_{0011111}+X_{1111111}+X_{1112110}+X_{1122100}+X_{0112210}$,
$H(5,7,10,14,11,8,4)$, $2X_{-0011111}+2X_{-1111111}+2X_{-1112110}+X_{-1122100}+2X_{-0112210}$
$\mathfrak{g}^{\phi}=\mathrm{span}\{-X_{\alpha_1}+X_{\alpha_5}+X_{-\alpha_2},\ H_{\alpha_1}-H_{\alpha_2}+H_{\alpha_5},\ -X_{-\alpha_1}+X_{-\alpha_5}+X_{\alpha_2};\ X_{\alpha_4}+X_{\alpha_7}+X_{0101100},\ H_{\alpha_2}+2H_{\alpha_4}+H_{\alpha_5}+H_{\alpha_7},\ X_{-\alpha_4}+X_{-\alpha_7}+X_{-0101100}\}\simeq\mathfrak{sl}(2,k)_a\oplus\mathfrak{sl}(2,k)_b$
$\mathfrak{g}\,[-1]\simeq 2(V_a^2\otimes V_b^3)\oplus(V_a^4\oplus 2V_a^2)\otimes V_b^1$
We have that the first A_1 is an $SL(2,k)$; the second embeds inside of a copy of A_5 in E_7, and decomposes the standard representation of $\mathfrak{sl}(6,k)$ into even-dimensional irreducibles. Thus it is an $SL(2,k)$ as well. There is no intersection between them. Thus $G_0^{\phi}=SL(2,k)\times SL(2,k)$. By Theorem 3.2, the cover splits over each piece individually. The orbit is **admissible**.

Bala-Carter Label: $(A_3+A_1)'$ **(not special)**

Weighted Dynkin Diagram:
$$\begin{array}{c} 0 \\ 1-0-\dot{1}-0-0-0 \end{array}$$

Lie Triple ϕ**:** $X_{1011000}+X_{1111111}+X_{0112111}+X_{0112210}$,
$H(6,8,11,16,12,8,4)$, $3X_{-1011000}+3X_{-1111111}+X_{-0112111}+4X_{-0112210}$
$\mathfrak{g}^{\phi}=\mathrm{span}\{X_{\alpha_3},\ X_{-\alpha_3},\ H_{\alpha_3};\ X_{\alpha_6},\ X_{-\alpha_6},\ H_{\alpha_6};\ X_{0000111}+X_{-\alpha_2},\ X_{\alpha_2}+X_{-0000111},\ -H_{\alpha_2}+H_{\alpha_5}+H_{\alpha_6}+H_{\alpha_7}\}\simeq\mathfrak{sl}(2,k)_a\oplus\mathfrak{sl}(2,k)_b\oplus\mathfrak{sl}(2,k)_c$
$\mathfrak{g}\,[-1]\simeq V_a^2\otimes((V_b^2\otimes V_c^2)\oplus(V_b^1\otimes V_c^3)\oplus(V_b^1\otimes V_c^1))$
We have the three $\mathfrak{sl}(2,k)$ subalgebras lift to non-intersecting $SL(2,k)$ subgroups, so $G_0^{\phi}=SL(2,k)\times SL(2,k)\times SL(2,k)$. By Theorem 3.2, the cover does not split over the first copy of $SL(2,k)$. The orbit is **not admissible**.

Bala-Carter Label: A_3+2A_1 **(not special)**

Weighted Dynkin Diagram:
$$\begin{array}{c} 0 \\ 1-0-\dot{0}-1-0-1 \end{array}$$

Lie Triple ϕ**:** $X_{1011100}+X_{0101111}+X_{0011111}+X_{0112210}+X_{1122110}$,
$H(6,8,11,16,13,9,5)$, $3X_{-1011100}+4X_{-0101111}+X_{-0011111}+X_{-0112210}+3X_{-1122110}$

$\mathfrak{g}^\phi = \text{span}\{X_{\alpha_4},\ X_{-\alpha_4},\ H_{\alpha_4};\ X_{0111000} - X_{-\alpha_6},\ -X_{\alpha_6} + X_{-0111000},\ H_{\alpha_2} + H_{\alpha_3} + H_{\alpha_4} - H_{\alpha_6}\ \} \simeq \mathfrak{sl}(2,k)_a \oplus \mathfrak{sl}(2,k)_b$
$\mathfrak{g}\,[-1] \simeq (V_a^2 \otimes V_b^3) \oplus 2(V_a^2 \otimes V_b^1) \oplus 4(V_a^1 \otimes V_b^2)$
We have the $\mathfrak{sl}(2,k)$ subalgebras lift to non-intersecting $SL(2,k)$ subgroups, so $G_0^\phi = SL(2,k) \times SL(2,k)$. By Theorem 3.2, the cover of the first copy of $SL(2,k)$ does not split. The orbit is **not admissible**.

Bala-Carter Label: $D_4(a_1) + A_1$ **(special)**

Weighted Dynkin Diagram:
$$\begin{array}{c} \qquad\quad 1 \\ 0-1-\dot{0}-0-0-1 \end{array}$$
Lie Triple ϕ: $X_{1111000} + X_{0101111} + X_{0011111} + X_{1011111} + X_{0112210}$,
$H(6,9,12,17,13,9,5)$, $-2X_{-0111000} + 4X_{-1111000} + X_{-0101111} + 2X_{-0011111} + 2X_{-1011111} + 4X_{-0112210} - 2X_{-1112210}$
$\mathfrak{g}^\phi = \text{span}\{X_{\alpha_4},\ X_{-\alpha_4},\ H_{\alpha_4};\ X_{\alpha_6},\ X_{-\alpha_6},\ H_{\alpha_6}\ \} \simeq \mathfrak{sl}(2,k)_a \oplus \mathfrak{sl}(2,k)_b$
$\mathfrak{g}\,[-1] \simeq 4(V_a^1 \otimes V_b^2) \oplus 4(V_a^2 \otimes V_b^1)$
We have $G_0^\phi = SL(2,k) \times SL(2,k)$, since $\mathfrak{g}\,[-1]$ is a faithful representation. By Theorem 3.2, the cover splits over each piece of G^ϕ. The orbit is **admissible**.

Bala-Carter Label: $A_3 + A_2$ **(special)**

Weighted Dynkin Diagram:
$$\begin{array}{c} \qquad\quad 0 \\ 0-0-\dot{1}-0-1-0 \end{array}$$
Lie Triple ϕ: $X_{0001110} + X_{0111110} + X_{0101111} + X_{1011111} + X_{1122100}$,
$H(6,9,12,18,14,10,5)$, $3X_{-0001110} + 2X_{-0111110} + 3X_{-0101111} + 2X_{-1011111} + 4X_{-1122100}$
$\mathfrak{g}^\phi = \text{span}\{2H_{\alpha_1} + H_{\alpha_2} - H_{\alpha_7};\ X_{\alpha_5},\ X_{-\alpha_5},\ H_{\alpha_5}\ \} \simeq k \oplus \mathfrak{sl}(2,k)$
$\mathfrak{g}\,[-1] \simeq W \oplus W^*$, with $W = (3k_1 + k_3) \otimes V^2$
We have $G_0^\phi = k^* \times SL(2,k)$. Apply Theorem 3.2 and Corollary 3.4. The orbit is **admissible**.

Bala-Carter Label: $D_4 + A_1$ **(not special)**

Weighted Dynkin Diagram:
$$\begin{array}{c} \qquad\quad 1 \\ 2-1-\dot{0}-0-0-1 \end{array}$$
Lie Triple ϕ: $X_{\alpha_1} + X_{0111000} + X_{0101111} + X_{0011111} + X_{0112210}$,
$H(10,13,18,25,19,13,7)$, $10X_{-\alpha_1} + 6X_{-0111000} + X_{-0101111} + 6X_{-0011111} + 6X_{-0112210}$
$\mathfrak{g}^\phi = \text{span}\{H_{\alpha_4}, H_{\alpha_6},\ X_{-\alpha_6},\ X_{0001110} + X_{-\alpha_5},\ X_{0001100} - X_{-0000110},\ X_{\alpha_4},\ X_{\alpha_6},\ X_{-0001110} + X_{\alpha_5},\ X_{-0001100} - X_{0000110},\ X_{-\alpha_4}\} \simeq \mathfrak{so}(5,k)$
$\mathfrak{g}\,[-1] \simeq 3V^4$
We have $G^\phi = Sp(4,k)$ by the existence of a 4-dimensional irreducible representation. Thus we have $Sp(4,k)$ mapping into three copies of itself, which corresponding cover does not split. The orbit is **not admissible**.

Bala-Carter Label: $A_4 + A_1$ **(special)**

Weighted Dynkin Diagram:
$$\begin{array}{c} \qquad\quad 0 \\ 1-0-\dot{1}-0-1-0 \end{array}$$
Lie Triple ϕ: $X_{1011000} + X_{0101110} + X_{1111100} + X_{0011111} + X_{0112100}$,

$H(8,11,15,22,17,12,6)$, $4X_{-1011000} + 6X_{-0101110} + 4X_{-1111100} + 6X_{-0011111} + X_{-0112100}$
$\mathfrak{g}^{\phi} = \mathrm{span}\{H_{\alpha_2} + H_{\alpha_3} - H_{\alpha_5},\, H_{\alpha_3} - H_{\alpha_7}\} \simeq k \oplus k$
$\mathfrak{g}\,[-1] \simeq V \oplus V^*$, $\dim(V) = 7$
We have $G_0^{\phi} = k^* \times k^*$. Apply Corollary 3.4. The orbit is **admissible**.

Bala-Carter Label: $D_5(a_1)$ **(special)**

Weighted Dynkin Diagram:

$$\begin{array}{c} 0 \\ 2-0-1-0-1-0 \end{array}$$

Lie Triple ϕ: $X_{\alpha_1} + X_{1010000} + 3X_{0001110} + 2X_{0011110} + X_{0101111} + X_{0112100}$,
$H(10,13,18,26,20,14,7)$, $2X_{-\alpha_1} + 8X_{-1010000} + X_{-0001110} + 2X_{-0011110} + 7X_{-0101111} + 6X_{-0112100} + 2X_{-0111111}$
$\mathfrak{g}^{\phi} = \mathrm{span}\{H_{\alpha_2} - H_{\alpha_7};\, X_{\alpha_5},\, X_{-\alpha_5},\, H_{\alpha_5}\} \simeq k \oplus \mathfrak{sl}(2,k)$
$\mathfrak{g}\,[-1] \simeq (k_1 \otimes 3V^2) \oplus (k_1 \otimes 3V^2)^*$
We have $G_0^{\phi} = k^* \times SL(2,k)$; apply Corollary 3.4. The orbit is **admissible**.

Bala-Carter Label: $(A_5)'$ **(not special)**

Weighted Dynkin Diagram:

$$\begin{array}{c} 0 \\ 1-0-1-0-2-0 \end{array}$$

Lie Triple ϕ: $X_{0000110} + X_{0000011} + X_{1011000} + X_{1111100} + X_{0112100}$,
$H(10,14,19,28,22,16,8)$, $8X_{-0000110} + 8X_{-0000011} + 5X_{-1011000} + 5X_{-1111100} + 9X_{-0112100}$
$\mathfrak{g}^{\phi} = \mathrm{span}\{X_{\alpha_3},\, X_{-\alpha_3},\, H_{\alpha_3},\, X_{\alpha_5} + X_{\alpha_7} + X_{-\alpha_2},\, X_{\alpha_2} + X_{-\alpha_5} + X_{-\alpha_7},\, -H_{\alpha_2} + H_{\alpha_5} + H_{\alpha_7}\,\} \simeq \mathfrak{sl}(2,k)_a \oplus \mathfrak{sl}(2,k)_b$
$\mathfrak{g}\,[-1] \simeq V_a^2 \otimes (V_b^3 \oplus 2V_b^1)$
We have that both subalgebras lift to $SL(2,k)$ at the group level, since the second embeds into a subalgebra of type A_5 in E_7 and decomposes its standard representation into even dimensional irreducibles. They commute and admit no intersection, so $G_0^{\phi} = SL(2,k) \times SL(2,k)$. By Theorem 3.2, the cover splits over the second $SL(2,k)$ but not the first. The orbit is **not admissible**.

Bala-Carter Label: $A_5 + A_1$ **(not special)**

Weighted Dynkin Diagram:

$$\begin{array}{c} 0 \\ 1-0-1-0-1-2 \end{array}$$

Lie Triple ϕ: $X_{\alpha_7} + X_{1011000} + X_{0101110} + X_{0011110} + X_{1111100} + X_{0112100}$,
$H(10,14,19,28,22,16,9)$, $9X_{-\alpha_7} + 5X_{-1011000} + 8X_{-0101110} + 8X_{-0011110} + 5X_{-1111100} + X_{-0112100}$
$\mathfrak{g}^{\phi} = \mathrm{span}\{X_{\alpha_5} + X_{-\alpha_2} - X_{-\alpha_3},\, X_{\alpha_2} - X_{\alpha_3} + X_{-\alpha_5}\, -H_{\alpha_2} - H_{\alpha_3} + H_{\alpha_5}\} \simeq \mathfrak{sl}(2,k)$
$\mathfrak{g}\,[-1] \simeq 4V^2 \oplus V^4$
We have $G_0^{\phi} = SL(2,k)$, and the cover splits by Theorem 3.2. The orbit is **admissible**.

Bala-Carter Label: $D_6(a_2)$ **(not special)**

Weighted Dynkin Diagram:

$$\begin{array}{c} 1 \\ 0-1-0-1-0-2 \end{array}$$

Lie Triple ϕ: $X_{\alpha_7} + 2X_{0000011} - X_{0111000} + X_{0101100} + 5X_{0011100} + 2X_{1111000} + 4X_{0101110} + 3X_{0011110} + 4X_{1011110}$,

$H(10,15,20,29,23,16,9), 5X_{-\alpha_7}+2X_{-0000011}+X_{-0101100}+2X_{-0011100}+3X_{-1111000}$ $+ 2X_{-0101110} + X_{-1011110}$
$\mathfrak{g}^\phi = \mathrm{span}\{X_{\alpha_4}, X_{-\alpha_4}, H_{\alpha_4}\} \simeq \mathfrak{sl}(2,k)$
$\mathfrak{g}[-1] \simeq 5V^2$
We have $G_0^\phi = SL(2,k)$; apply Theorem 3.2. The orbit is **not admissible**.

Bala-Carter Label: $D_5 + A_1$ **(special)**

Weighted Dynkin Diagram:

$$\begin{array}{c} \quad\;\; 1 \\ 2-1-0-1-1-0 \end{array}$$

Lie Triple ϕ: $X_{\alpha_1} + X_{0000110} + X_{0111000} + X_{0101100} + X_{0011100} + X_{0001111}$,
$H(14,19,26,37,29,20,10)$, $14X_{-\alpha_1} + 10X_{-0000110} + 18X_{-0111000} + X_{-0101100}$ $+ 8X_{-0011100} + 10X_{-0001111}$
$\mathfrak{g}^\phi = \mathrm{span}\{X_{\alpha_7} + X_{-\alpha_4}, X_{\alpha_4} + X_{-\alpha_7}, -H_{\alpha_4} + H_{\alpha_7}\} \simeq \mathfrak{sl}(2,k)$
$\mathfrak{g}[-1] \simeq 4V^2$
We have $G_0^\phi = SL(2,k)$; apply Theorem 3.2. The orbit is **admissible**.

Bala-Carter Label: $D_6(a_1)$ **(special)**

Weighted Dynkin Diagram:

$$\begin{array}{c} \quad\;\; 1 \\ 2-1-0-1-0-2 \end{array}$$

Lie Triple ϕ: $X_{\alpha_1} + X_{\alpha_7} + X_{0000011} + X_{0111000} + X_{0101100} + 2X_{0011100} - X_{0011110}$,
$H(14,19,26,37,29,20,11), 14X_{-\alpha_1} - 9X_{-\alpha_7} + 20X_{-0000011} + 8X_{-0111000} + 11X_{-0101100}$ $+ 9X_{-0011100} + 20X_{-0101110}$
$\mathfrak{g}^\phi = \mathrm{span}\{X_{\alpha_4}, X_{-\alpha_4}, H_{\alpha_4}\} \simeq \mathfrak{sl}(2,k)$
$\mathfrak{g}[-1] \simeq 4V^2$
We have $G_0^\phi = SL(2,k)$; apply Theorem 3.2. The orbit is **admissible**.

Bala-Carter Label: D_6 **(not special)**

Weighted Dynkin Diagram:

$$\begin{array}{c} \quad\;\; 1 \\ 2-1-0-1-2-2 \end{array}$$

Lie Triple ϕ: $X_{\alpha_1} + X_{\alpha_6} + X_{\alpha_7} + X_{0111000} + X_{0101100} + X_{0011100}$,
$H(18,25,34,49,39,28,15), 18X_{-\alpha_1} + 28X_{-\alpha_6} + 15X_{-\alpha_7} + 10X_{-0111000} + 15X_{-0101100}$ $+ 24X_{-0011100}$
$\mathfrak{g}^\phi = \mathrm{span}\{X_{\alpha_4}, X_{-\alpha_4}, H_{\alpha_4}\} \simeq \mathfrak{sl}(2,k)$
$\mathfrak{g}[-1] \simeq 3V^2$
We have $G_0^\phi = SL(2,k)$; apply Theorem 3.2. The orbit is **admissible**.

Appendix A. Finding representatives for non-split rational orbits

A.1. **The $\widetilde{A_1}$ orbit of $F^0_{4,4}$.** By Corollary 4.2, the number of orbits is at most $|H^1(k, S_2)| = |k^*/k^{*2}|$. Let us use Proposition 4.4 to deduce that this bound is optimal.

Using the weighted Dynkin Diagram of this orbit, we compute that $\mathfrak{g}[0]$ is the direct sum of a split Lie algebra of type B_3 (with simple roots α, β and γ) and kH_δ (notation of Section 6.2). The subspace $\mathfrak{g}[2]$ has basis

$$B = \{X_{0122}, X_{1122}, X_{1222}, X_{1232}, X_{1242}, X_{1342}, X_{2342}\}.$$

The split $\mathfrak{so}(7)$ acts on $\mathfrak{g}[2]$ as the standard representation, preserving a nondegenerate quadratic form. With respect to the basis B above, this form is $Q(\vec{x}) =$

$x_1x_7 + x_2x_6 + x_3x_5 + x_4^2$. The subspace spanned by H_δ acts diagonally with respect to B. At the group level, then, the action of G^H on $\mathfrak{g}[2]$ factors through $SO(Q) \ltimes \mathbb{G}_m$.

The representative E chosen for the split orbit satisfies $Q(E) \neq 0$, whence $\mathcal{P}$ may be identified as the subset of non-isotropic elements of $\mathfrak{g}[2]$. Given a representative E' of a rational orbit, conjugation by $SO(Q)$ will preserve $Q(E')$, and (we check) the action by the one-parameter subgroup corresponding to H_δ will modify $Q(E')$ by a square. Thus under G^H, $\mathcal{P}$ decomposes into exactly $|k^*/k^{*2}|$ orbits. Explicitly, representatives for these orbits are $mX_{0122} + X_{2342}$, where m runs over the classes mod squares in k^*.

A.2. $A_1 + \widetilde{A_1}$ **orbit of** F_4**.** Here, there are exactly two orbits by Lemma 4.3. We have $\mathfrak{g}[0] = \mathfrak{sl}(3,k)_{\gamma,\delta} \oplus \mathfrak{sl}(2,k)_\alpha \oplus kH_\beta$. A basis for $\mathfrak{g}[2]$ is

$$B = \{X_{1220}, X_{1221}, X_{1222}, X_{1231}, X_{1232}, X_{1242}\}.$$

The $\mathfrak{sl}(2,k)$ acts trivially on $\mathfrak{g}[2]$, the kH_β acts diagonally (with respect to B), and the $\mathfrak{sl}(3,k)$ acts on $\mathfrak{g}[2]$ as the symmetric square of the standard representation of $\mathfrak{sl}(3,k)$.

We have two immediate choices of orbit representatives: $E = X_{1222} + X_{1231}$ and $E' = X_{1220} + X_{1222} + X_{1242}$ (obtained by considering the G^H-action on $\mathfrak{g}[2]$). Let ϕ' be the standard Lie triple corresponding to E'; then $\mathfrak{g}^{\phi'} = \mathfrak{sl}(2,k) \oplus \mathfrak{so}(Q)$, where the quadratic form Q is represented by the 3×3 identity matrix. When $k = \mathbb{R}$ or k has residual characteristic 2 and $(-1/-1)_k = -1$, $\mathfrak{so}(Q)$ is not equivalent to $\mathfrak{sl}(2,k)$, which shows that the two orbits are distinct.

Otherwise, the groups G^ϕ and $G^{\phi'}$ are isomorphic; however, one can (laboriously) prove directly that the two orbits are not rationally conjugate under G^H (for any choice of k).

A.3. B_2 **orbit of** F_4**.** Here, the number of orbits is bounded by $|H^1(k, S_2)| = |k^*/k^{*2}|$ by Corollary 4.2. We have $\mathfrak{g}[0] = \mathfrak{so}(5) \oplus kH_\alpha \oplus kH_\delta$, where $\mathfrak{so}(5)$ is the split subalgebra with simple roots β and γ. The subspace $\mathfrak{g}[2]$ is 6-dimensional, and decomposes under the $\mathfrak{so}(5)$ as $V_{std} \oplus k$. Consider the basis of $\mathfrak{g}[2]$ given by

$$B = \{X_\alpha, X_{\alpha+\beta}, X_{\alpha+\beta+\gamma}, X_{\alpha+\beta+2\gamma}, X_{\alpha+2\beta+2\gamma}, X_{\beta+2\gamma+2\delta}\}.$$

As before, the subgroup of G^H corresponding to $\mathfrak{so}(5)$ preserves a quadratic form, which is given by $Q(\vec{x}) = x_1x_5 + x_2x_4 + x_3^2$ in coordinates with respect to B. Moreover, the one-parameter subgroups corresponding to H_α and H_δ can only change the value of $Q(\vec{x})$ by a square, whereas they can scale the last coordinate by any value in k^*. Whence there must be $|k^*/k^{*2}|$ orbits. Representatives of these other orbits are $mX_\alpha + X_{\alpha+2\beta+2\gamma} + X_{\beta+2\gamma+2\delta}$, where m runs over the classes mod squares in k^*.

A more careful look at the groups G^ϕ arising from each of these rational orbits shows that in fact each different value $m \notin k^{*2}$ chosen leads to $G^\phi \simeq SL(2, k(\sqrt{m}))$.

A.4. $C_3(a_1)$ **orbit of** F_4**.** Again, the number of orbits is bounded by $|H^1(k, S_2)| = |k^*/k^{*2}|$, by Corollary 4.2. In this case, $\mathfrak{g}[0] = \mathfrak{sl}(2,k)_\beta \oplus \mathfrak{sl}(2,k)_\delta \oplus kH_\alpha \oplus kH_\gamma$, and $\mathfrak{g}[2]$ is spanned by

$$B = \{X_{\alpha+\beta+\gamma}, X_{\beta+2\gamma}, X_{\beta+2\gamma+\delta}, X_{\alpha+\beta+\gamma+\delta}, X_{\beta+2\gamma+2\delta}\}.$$

Hence $\mathfrak{sl}(2,k)_\beta$ acts trivially, whereas $\mathfrak{sl}(2,k)_\delta$ decomposes $\mathfrak{g}[2]$ into irreducibles $V_{std} \oplus V_{adj}$. The one-parameter subgroup of G^H corresponding to H_α acts by scalars (in k^*) on each of these subrepresentations; the subgroup corresponding to H_γ acts diagonalizably.

The element $X = X_{\beta+2\gamma} + X_{\beta+2\gamma+2\delta} \in \mathfrak{g}[2]$ corresponds, under the identification with the adjoint representation, to the matrix $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. Its orbit under G^H consists of all nonzero hyperbolic (diagonalizable over k) matrices. Choosing representatives $X(m) = X_{\beta+2\gamma} + mX_{\beta+2\gamma+2\delta}$, as m ranges over the classes mod squares of k^*, yields representatives for the remaining rational orbits of maximal dimension. We conclude that the elements $E(m) = X_{\alpha+\beta+\gamma+\delta} + X(m)$ are a set of representatives of distinct (hence all) rational orbits in $\mathbb{G} \cdot E$.

A.5. $2A_1$ **orbit of** E_6**.** In this case, $G^\phi = (k^* \times Spin(7,k))/Z$, and there is no simple way using exact sequences to deduce (or bound) the cardinality of $H^1(k, G^\phi)$. We must compute the number of rational orbits using Proposition 4.4.

Note that $\mathfrak{g}[0] \simeq \mathfrak{so}(8,k) \oplus kH_{\alpha_1} \oplus kH_{\alpha_6}$, and that $\mathfrak{g}[2]$ is the 8-dimensional standard representation of (split) $\mathfrak{so}(8,k)$. With respect to the basis

$$B = \{X_{122321}, X_{112321}, X_{112221}, X_{111221}, X_{101111}, X_{111111}, X_{111211}, X_{112211}\}$$

of $\mathfrak{g}[2]$, the quadratic form preserved by the $\mathfrak{so}(8,k)$ factor is given by $Q(\vec{x}) = x_1x_5 + x_2x_6 + x_3x_7 + x_4x_8$. Our representative $E = (0,0,0,1,0,0,0,1)$ with respect to B, hence $Q(E) = 1$, and the orbit of E under the subgroup of G^H corresponding to $\mathfrak{so}(8,k)$ consists of all those vectors X with $Q(X) = 1$. The action of the one-parameter subgroup $h_6(t)$ on E gives $(0,0,0,t,0,0,0,1)$; this scaling implies that one obtains all vectors X with $Q(X) \neq 0$ as $g \cdot E$, for some $g \in G^H$. It follows that there is exactly one open orbit of G^H in $\mathfrak{g}[2]$, and hence a unique rational orbit of G in the k-points of $\mathbb{G} \cdot E$.

A.6. $A_2 + 2A_1$ **orbit of** E_6**.** Here, $G^\phi = (k^* \times SL(2,k))/Z$. We show directly that there is only one rational orbit.

Begin by noting that

$$\mathfrak{g}[0] = \mathfrak{sl}(2,k)_1 \oplus \mathfrak{sl}(2,k)_6 \oplus \mathfrak{sl}(3,k)_{2,4} \oplus kH_{\alpha_3} \oplus kH_{\alpha_5}$$

and the space $\mathfrak{g}[2]$ is precisely the tensor product $V^2 \otimes V^2 \otimes V^3$ of the standard representations of each of the simple subalgebras. A basis B for this space is abstractly the set of vectors $\vec{e}_i \otimes \vec{e}_j \otimes \vec{e}_k$, (where $\vec{e}_i$ denotes the ith standard basis vector of V^n), ordered lexicographically; here we have

$$\begin{aligned} B \;=\; & \{X_{111211}, X_{111111}, X_{101111}; X_{111210} X_{111110}, X_{101110}; \\ & X_{011211}, X_{011111}, X_{001111}; X_{011210}, X_{011110}, X_{001110}\}. \end{aligned}$$

The orbit representative E is given in coordinates with respect to B by the 12-vector $E = (\vec{e}_1, \vec{e}_3, \vec{e}_2, \vec{e}_1)$.

The action of an element $g \in SL(2,k) \times SL(2,k)$ is to take E to an element $g \cdot E = (\mathbf{r}, \mathbf{s}, \mathbf{t}, \mathbf{u})$ such that if J denotes the matrix product $J = \mathbf{r}\mathbf{u}^t - \mathbf{s}\mathbf{t}^t$, then the matrix $Q = J + J^t$ is preserved. Interpret Q as the matrix of a (clearly isotropic) quadratic form. We claim that the orbit of E under G^H is the set of all $(\mathbf{r}, \mathbf{s}, \mathbf{t}, \mathbf{u})$ such that the corresponding quadratic form Q is nondegenerate; note that Q is necessarily isotropic.

An element $h \in SL(3,k)$ acts on $(\mathbf{r}, \mathbf{s}, \mathbf{t}, \mathbf{u})$ to modify Q to hQh^t. The one-parameter subgroups $h_3(a)$ and $h_4(b)$ act on $(\mathbf{r}, \mathbf{s}, \mathbf{t}, \mathbf{u})$ to give $(\mathbf{r}', \mathbf{s}', \mathbf{t}', \mathbf{u}')$ such that the determinant of Q is scaled by the product ab. Given that all nondegenerate isotropic quadratic forms in 3 variables (over real and p-adic fields) are related by scaling, the claim follows. Whence $G^H \cdot E$ is the unique open orbit in $\mathfrak{g}[2]$, as required.

A.7. A_3 **orbit of** E_6**.** Here, $G^\phi = (k^* \times Sp(4,k))/Z$; again, we show directly that there is only one rational orbit in this case.

Now $\mathfrak{g}[0]$ is the subalgebra $\mathfrak{sl}(4,k)$ with simple roots $\alpha_3, \alpha_4, \alpha_5$, along with the part of the Cartan subalgebra spanned by H_{α_1}, H_{α_2} and H_{α_6}. As a representation of $\mathfrak{sl}(4,k)$, the space $\mathfrak{g}[2]$ is isomorphic to $\wedge^2 V_{std} \oplus k$.

First note the following. Writing $\vec{e}_i$ for the standard basis vector of k^4, a basis for $\wedge^2 V_{std}$ is given by the wedge products $\vec{e}_i \wedge \vec{e}_j$, in lexicographic ordering. The orbit of $GL(4,k)$ through $\vec{e}_1 \wedge \vec{e}_3 + \vec{e}_2 \wedge \vec{e}_4$ is a Zariski open set in $\wedge^2 V_{std}$, given as the set of all $\vec{u} \wedge \vec{v} + \vec{w} \wedge \vec{x}$ such that the determinant of the 4×4 matrix with columns given by the coordinates of $\vec{u}, \vec{w}, \vec{v}, \vec{x}$ is nonzero.

The basis of $\mathfrak{g}[2]$ corresponding to the above abstract basis (with the trivial representation listed last) is

$$B = \{X_{011210}, X_{011110}, X_{\alpha_2+\alpha_3+\alpha_4}, X_{\alpha_2+\alpha_4+\alpha_5}, X_{\alpha_2+\alpha_4}, X_{\alpha_2}; X_{101111}\}$$

Our representative E of the orbit thus corresponds the sum $\vec{e}_1 \wedge \vec{e}_3 + \vec{e}_2 \wedge \vec{e}_4 + X_{101111}$. The action of the one-parameter subgroups $h_1(r)$, $h_2(s)$ and $h_6(t)$ on E is

$$h_1(r)h_2(s)h_6(t) \cdot E = s(rt)^{-1}\vec{e}_1 \wedge \vec{e}_3 + s\vec{e}_2 \wedge \vec{e}_4 + s^{-1}rtX_{101111}.$$

It now follows that the orbit of E under G^H is a Zariski open. Namely, let $b \neq 0$ and $X = \vec{u} \wedge \vec{v} + \vec{w} \wedge \vec{x} + bX_{101111} \in \mathfrak{g}[2]$ such that the aforementioned determinant has value $a \neq 0$. Choose $s = ab$, $rt = ab^2$, and $g \in SL(4,k)$ with columns given by the coordinates of $rts^{-1}\vec{u}$, $s^{-1}\vec{w}$, $\vec{v}$, $\vec{x}$ (in that order). Then $gh_1(r)h_2(s)h_6(t) \cdot E = X$. Hence there is a unique open orbit of G^H on $\mathfrak{g}[2]$, as required.

A.8. $2A_2 + A_1$ **orbit of** E_6**.** By Corollary 4.2, the number of orbits is at most $|H^1(k, \mathbb{Z}/3\mathbb{Z})| = |k^*/k^{*3}|$ (when k contains the cube roots of unity). In this case, $\mathfrak{g}[2]$ is a 9-dimensional space, with basis

$$\begin{aligned} B &= \{X_{101100}, X_{101110}, X_{111100}, X_{111110}, \\ &\quad X_{000111}, X_{001111}, X_{010111}, X_{011111}, X_{011211}\} \end{aligned}$$

and

$$\mathfrak{g}[0] = \mathfrak{sl}(2,k)_{\alpha_2} \oplus \mathfrak{sl}(2,k)_{\alpha_3} \oplus \mathfrak{sl}(2,k)_{\alpha_5} \oplus kH_{\alpha_1} \oplus kH_{\alpha_4} \oplus kH_{\alpha_6}.$$

Note that the three $\mathfrak{sl}(2,k)$ subalgebras commute, and that the Cartan pieces will act diagonalizably on $\mathfrak{g}[2]$. Let us compute the rational orbit of the chosen representative E, which equals $(1,0,0,1,1,0,0,1,1)$ in coordinates with respect to the basis B.

First note that the $SL(2,k)_{\alpha_5}$-action on $\mathfrak{g}[2]$ decomposes as $2V^2 + 5V^1$, with the two copies of the standard representation lying on the first four vectors of the basis above. Hence, for $g_5 = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in SL(2,k)_{\alpha_5}$, we have $g_5 \cdot E = (a,c,b,d,1,0,0,1,1)$, with $ad - bc = 1$. Similarly, an element $g_3 = \left[\begin{smallmatrix} e & f \\ g & h \end{smallmatrix}\right] \in SL(2,k)_{\alpha_3}$ acts nontrivially only on the next four coordinates, so we have $g_3 g_5 \cdot E = (a,c,b,d,e,g,f,h,1)$, with $ad - bc = 1$ and $eh - fg = 1$. The action of $SL(2,k)_{\alpha_2}$ does not further enlarge the orbit.

Now action by the one-parameter subgroups corresponding to H_{α_1}, H_{α_4}, and H_{α_6} (denoted by $h_1(r)$, $h_4(s)$ and $h_6(t)$, respectively) scales these values as follows

$$\begin{aligned} &h_1(r)h_4(s)h_6(t)g_3g_5 \cdot E \\ &\qquad = (rsa, rt^{-1}c, rb, rs^{-1}t^{-1}d, ste, r^{-1}tg, tf, r^{-1}s^{-1}th, r^{-1}st^{-1}). \end{aligned}$$

This gives a degree of freedom of r^2t^{-1} in the first four coordinates, of $r^{-1}t^2$ in the second four coordinates, and $r^{-1}st^{-1}$ in the last coordinate.

To solve for an arbitary element of $\mathfrak{g}[2]$ (except possibly some in a subvariety of lower dimension), one can choose g_3 and g_5 to give the correct values up to scaling in the first four, second four, and last coordinate, respectively. However, to solve $r^2t^{-1} = m$ and $r^{-1}t^2 = n$ implies solving $t^3 = n^2m$, which is possible only if $k^{*3} = k^*$.

Whence representatives of the $|k^*/k^{*3}|$ rational orbits in $\mathbb{G} \cdot E$ are, in coordinates with respect to B, given by $(m, 0, 0, 1, 1, 0, 0, 1, 1)$, with m running over the classes mod cubes in k^*.

A.9. A_5 **orbit of** E_6**.** Again by Corollary 4.2, the number of orbits is at most $|H^1(k, \mathbb{Z}/3\mathbb{Z})| = |k^*/k^{*3}|$ (when k contains the cube roots of unity). We have that $\mathfrak{g}[2]$ is spanned by

$$B = \{X_{\alpha_1}, X_{\alpha_6}, X_{\alpha_2+\alpha_3+\alpha_4}, X_{\alpha_2+\alpha_4+\alpha_5}, X_{\alpha_3+\alpha_4+\alpha_5}\}$$

and that $\mathfrak{g}[0]$ is the direct sum of $\mathfrak{sl}(2,k)_{\alpha_4}$ (which acts trivially on $\mathfrak{g}[2]$) and the rest of the Cartan subalgebra. Our orbit representative E is given by $(1, 1, 1, 1, 1)$ with respect to the above basis B for $\mathfrak{g}[2]$; since the action of G^H is diagonal with respect to B, any orbit representative must have all nonzero coordinates. Computing the Cartan action directly and then solving for any possible orbit representative leads to a cubic equation in k^*. Once again, we deduce there are $|k^*/k^{*3}|$ rational orbits. One set of representatives is given in coordinates with respect to B as $\{(1, 1, 1, m, 1)\}$, where m runs over the classes mod cubes in k^*.

Appendix B. Admissibility of non-split rational orbits

Those rational orbits for which we cannot exclude the possibility that non-split forms of G^ϕ arise must be treated on a case-by-case basis. The admissibility of these orbits is known over $\mathbb{R}$ by [No1, No2]. In this section, we discuss the (very few) cases arising in G_2, F_4 and E_6.

B.1. k**-forms of** $SL(2)$**.** Suppose $\mathbb{G}^\phi$ is $SL(2)$. It admits a unique non-split k-form, which we identify with $SL(1, D)$, for D the quaternions over k [T1] when k is a p-adic field, and with $SU(2)$ when $k = \mathbb{R}$. Note that both these groups are compact, each being isomorphic to a two-fold cover of a special orthogonal group preserving an anisotropic quadratic form.

Theorem 5.2 implies that these groups do not admit 2-dimensional k-rational representations. Rather, their standard 4-dimensional k-rational representations decompose into two 2-dimensional irreducibles upon passage to the algebraic closure.

Lemma B.1. *The metaplectic cover of a compact k-from of $SL(2)$ arising from its 4-dimensional irreducible k-rational representation splits.*

Proof. First let k denote a p-adic field (of residual characteristic different from 2), and $D = \langle 1, i, j, k \mid i^2 = a, j^2 = b, ij = k = -ji \rangle$ (where $a, b \in k^*$ satisfy $(a/b)_k = -1$) the skew field of quaternions over k. Realize the given representation of $SL(1, D)$ as left multiplication on D. This action preserves the symplectic structure on $D = F^4$ given by $\omega(1, k) = \omega(j, i) = 1$. Whence our map $SL(1, D) \to Sp(4, k)$. The image of $SL(1, D)$ is a compact subgroup of $Sp(4, k)$, which preserves a self-dual lattice in F^4 (namely, the $\mathcal{R}$-span of $\{1, i, j, k\}$, where $\mathcal{R}$ denotes the integer ring of k). Hence by [MVW, Ch2.II.8], the cover splits.

Now let $k = \mathbb{R}$. The standard representation of $U(2)$ preserves a hermitian form on $\mathbb{C}^2$. The imaginary part of this form defines a symplectic form on $\mathbb{R}^4$ preserved by $U(2)$, and hence gives a map $U(2) \to Sp(4, \mathbb{R})$. This realizes $U(2)$ as part of a dual reductive pair in $Sp(4, \mathbb{R})$, whose full $\mathbb{C}^1$ metaplectic cover splits. Restriction to $SU(2)$ gives a splitting over that subgroup; since $SU(2)$ is equal to its group of commutators, the image of this splitting map must lie in $Mp(4, \mathbb{R})$, as required. $\square$

B.2. **k-forms of $SL(2) \times SL(2)$.** Suppose $\mathbb{G}^\phi$ is $SL(2) \times SL(2)$. Its non-split k-forms include not only direct product groups, with one or both factors non-split forms of $SL(2)$, but also $SL(2, K)$, for K a quadratic extension field of k, viewed as a k-group. The former cases may be understood with the help of Appendix B.1.

For $k = \mathbb{R}$, the group $SL(2, \mathbb{C})$ is simply connected, and so any cover of it will split, implying the orbit is admissible. For k a p-adic field, the answer is less straightforward. We compute the metaplectic cover in the fundamental case of the standard representation of $SL(2, K)$ (a k-rational representation by restriction of scalars).

Lemma B.2. *Let k be a p-adic field of residual characteristic different from 2. Let $K = k(\sqrt{\alpha})$ be a quadratic extension field of k (where $\alpha \in k^*$ is not a square). View $SL(2, K)$ as a k-group; then it admits a 4-dimensional irreducible k-rational representation, coming from the standard representation of $SL(2, K)$ on $K^2 \simeq k^4$. This gives a homomorphism*

$$\varphi\colon SL(2, K) \to Sp(4, k).$$

The metaplectic cover of $SL(2, K)$ determined by the lift of φ to $Mp(4, k)$ splits when either $-1 \in k^{*2}$ or $\alpha = -1 \notin k^{*2}$. It does not split otherwise.

Proof. Note that Lemma 3.1 applies to $SL(2, K)$. The calculation of the restriction of the metaplectic cocycle of $Sp(4, k)$ to the diagonal subgroup of $SL(2, K)$ is as follows. Let $h = diag(z, z^{-1}) \in \mathcal{H} \subset SL(2, K)$, with $z \in K^*$. Then $\varphi(h)$ is a block diagonal matrix $(A, {}^tA^{-1})$, where $A = \left[\begin{smallmatrix} a & \alpha b \\ b & a \end{smallmatrix}\right]$ and $z = a + \sqrt{\alpha} b$.

The Steinberg cocycle (3.1) of $Sp(4, k)$, which is restricted to $\varphi(\mathcal{H})$, is $S(h, h') = t(hh')t(h)^{-1}t(h')^{-1}$ since $\varphi(\mathcal{H})$ preserves lagrangian subspaces. Moreover, $t(h) = \gamma(1)\gamma(\det(A))^{-1}$; so, as before, (*cf.* (3.2)), we have $S(h, h') = (\det(A)/\det(A'))_k$. Now $\det(A) = a^2 - \alpha b^2 = N_{K/k}(z)$, the norm of z. The image of these norms in k^*/k^{*2} has index 2. Since k has residual characteristic different from 2, this implies that $\det(A)$ can take on only 2 different values modulo k^{*2}: denote these values $\{1, a\}$.

If $(a/a)_k = (a/-1)_k = 1$, then the Steinberg cocycle is identically 1 on $\varphi(\mathcal{H})$, implying that the cover splits. This occurs whenever $-1 \in k^{*2}$, or when $\alpha = -1 \notin k^{*2}$.

The remaining cases have $k^*/k^{*2} = \{1, -1, \varpi, -\varpi\}$, for $\varpi \in k$ an element of minimal positive valuation, and $\alpha \in \{\varpi, -\varpi\}$. It follows that $a = -\alpha$ and $(a/a)_k =$

-1 and the Steinberg cocycle is nontrivial. (In fact, one can verify directly that the cocycle $(N_{K/k}(z), N_{K/k}(z'))_k$ is equivalent to the nontrivial Steinberg cocycle $(z/z')_K$.) Thus the cover does not split. $\square$

Remark. The (2-)Hilbert symbol is not as simply understood for fields of the residual characteristic equal to 2. Part of the difficulty lies in the 2^{N+2} classes mod squares in k^* (where N is the valuation of 2 in k); the triviality of the Steinberg cocycle is far less likely. For $\mathbb{Q}_2$, for example, one can show that for any nontrivial quadratic extension K, $SL(2, K)$ inherits a nontrivial metaplectic cover from φ.

B.3. **k-forms of $SL(4)$.** Finally, suppose $\mathbb{G}^\phi$ is $SL(4)$. Its non-split rational p-adic forms are (see [T1]): $SL(1, D)$, for D a central simple division algebra over k such that $[D : k] = 16$; $SL(2, D)$, for D the quaternions over k, *i.e.*, $[D : k] = 4$; $SU(K, h)$ for (two kinds of) degree 4 hermitian forms h over a quadratic extension field K of k. Over $k = \mathbb{R}$, its non-split rational forms are $SU(4)$, $SU(1, 3)$, $SU(2, 2)$ and $SL(2, \mathbb{H})$, where $\mathbb{H}$ denotes the quaternions over $\mathbb{R}$.

Denote the three fundamental weights in $SL(4)$ by λ_1, λ_2, and λ_3.

Lemma B.3. *The special unitary group forms of $SL(4)$ admit k-rational representations which decompose into $V_{\lambda_1} \oplus V_{\lambda_3}$ at the level of the algebraic closure, whereas the non-split special linear group forms of $SL(4)$ do not.*

Proof. The special unitary forms of $SL(4)$ all arise with Tits index ${}^2A^{(1)}_{3,r}$, with $r \in \{0, 1, 2\}$ [T1]; this implies that λ_1 and λ_3 are in the same Weyl group orbit. By Theorem 5.2, we deduce that the 8-dimensional standard k-representations of these groups decompose into $V_{\lambda_1} \oplus V_{\lambda_3}$ at the level of the algebraic closure.

In contrast, the non-split special linear group forms of $SL(4)$ arise with Tits index ${}^1A^{(d)}_{3,r}$, with $d(r + 1) = 4$ [T1]; this implies that each Weyl group orbit of a fundamental weight is a singleton. Consequently, the irreducible k-representations $\rho^k_{\lambda_i}$ of Theorem 5.2 decompose into isotypic subspaces at the level of the algebraic closure. Since additionally no non-split form admits a 4-dimensional k-rational irreducible representation, $V_{\lambda_1} \oplus V_{\lambda_3}$ cannot occur. $\square$

Hence, whenever $\mathfrak{g}[-1]$ decomposes as $V_{std} \oplus V^*_{std}$ under $G^\phi = SL(4, k)$, it follows that the only other k-rational forms of $\mathbb{G}^\phi$ that can occur correspond to special unitary groups.

Lemma B.4. *Let K be a quadratic extension field of k, and h a Hermitian form on K^4. The standard representation of $SU(h, K)$ gives a map $\varphi \colon SU(h, K) \to Sp(8, k)$. The metaplectic cover of $SU(h, K)$ determined by φ splits.*

Proof. As in the second part of the proof of Lemma B.1, the unitary group arises as half of a dual reductive pair (for k p-adic, see [MVW, 3.I.1]), and hence admits a splitting over the special unitary group into the metaplectic group. $\square$

References

[AK] Louis Auslander and Bertram Kostant, "Quantization and representations of solvable Lie groups," Bull. Amer. Math. Soc. **73** (1967), 692–695. MR **39:**2910

[Ca] Roger W. Carter, *Simple Groups of Lie Type*, Pure and Applied Mathematics, vol. 28, Wiley, 1972. MR **53:**10946

[CMcG] David H. Collingwood and William M. McGovern, *Nilpotent Orbits in Semisimple Lie Algebras*, Van Nostrand Reinhold Mathematics Series, 1993. MR **94j:**17001

[Do] Dragomir Ž. Doković, "Classification of nilpotent elements in simple exceptional real Lie algebras of inner type and description of their centralizers," Journal of Algebra **112** (1988), 503–524. MR **89h:**17010

[Du] M. Duflo, "Construction de représentations unitaires d'un groupe de Lie," in *Harmonic Analysis and Group Representations*, C.I.M.E. (1980), 129–222. MR **87b:**22028

[E] Gordon Bradley Elkington, "Centralizers of unipotent elements in semisimple algebraic groups," Journal of Algebra **23**, (1972), 137–163. MR **46:**7342

[FH] William Fulton and Joe Harris, *Representation Theory. A First Course*, Graduate Texts in Mathematics **129**, Springer-Verlag, 1991. MR **93a:**20069

[GS] Peter B. Gilkey and Gary M. Seitz, "Some representations of exceptional Lie algebras," Geometriae Dedicata **25** (1988), 407–416. MR **89h:**20056

[Ki] A.A. Kirillov, "Unitary representations of nilpotent Lie groups," [Russian], Uspehi Mat. Nauk **17** no. 4(106) (1962), 57–110. MR **25:**5396

[KnI] Martin Kneser, "Galois-Kohomologie halbeinfacher algebraischer Gruppen über p-adischen Körpern. I" (German) Math. Z. **88** (1965), 40–47. MR **30:**4760

[KnII] ———, "Galois-Kohomologie halbeinfacher algebraischer Gruppen über p-adischen Körpern. II" (German) Math. Z. **89** (1965), 250–272. MR **32:**5658

[Kn] ———, *Lectures on Galois cohomology of classical groups*. With an appendix by T. A. Springer. Notes by P. Jothilingam. Tata Institute of Fundamental Research Lectures on Mathematics, No. 47. Tata Institute of Fundamental Research, Bombay, 1969. MR **49:**5195

[KP] Hanspeter Kraft and Claudio Procesi, "A special decomposition of the nilpotent cone of a classical Lie algebra," Astérisque, 173–174 (1989), pp. 271–279. MR **90m:**17025

[LS] Martin W. Liebeck and Gary M. Seitz, *Reductive Subgroups of Exceptional Algebraic Groups*, Memoirs of the Amer. Math. Soc., vol. 580, 1996. MR **96i:**20059

[LP] Gérard Lion and Patrice Perrin, "Extension des représentations de groupes unipotents p-adiques. Calculs d'obstructions," in *Noncommutative Harmonic Analysis and Lie Groups*, Lecture Notes in Mathematics, **880**, Springer-Verlag, Berlin, Heidelberg, New York, 1981, 337–356. MR **83h:**22032

[LV] Gérard Lion and Michele Vergne, *The Weil representation, Maslov index and theta series*. Progress in Mathematics, **6**, Birkäuser, Boston, MA, 1980. MR **81j:**58075

[L1] George Lusztig, "A class of irreducible representations of a Weyl group," Nederl. Akad. Wetensch. Indag. Math. **41** (1979), 323–335. MR **81a:**20052

[L2] ———, "Notes on unipotent classes," Asian J. Math **1** no 1. (1997), 194–207. MR **98k:**20078

[M] MATLAB® software by The MathWorks, http://www.mathworks.com.

[MVW] Colette Mœglin, Marie-France Vignéras, and Jean-Loup Waldspurger, *Correspondances de Howe sur un corps p-adique*. Lecture Notes in Mathematics, 1291, Springer-Verlag, Berlin, 1987. MR **91f:**11040

[M1] C.C. Moore, "Decomposition of unitary representations defined by discrete subgroups of nilpotent groups," Ann. of Math **82** (1965), 146–182. MR **31:**5928

[M2] ———, *Group extensions of p-adic and adelic linear groups*, Inst. Hautes Études Sci. Publ. Math. No. 35, 1968, 157–222. MR **39:**5575

[Neu] Jürgen Neukirch, *Class Field Theory*, Grundlehren der Mathematischen Wissenschaften **280**, Springer-Verlag, Berlin, 1986. MR **87i:**11005

[N1] Monica Nevins, Ph.D. Thesis, MIT.

[N2] ———, "Admissible nilpotent coadjoint orbits of p-adic reductive groups," Represent. Theory **3** (1999), 105–126 (electronic). MR **2000k:**22023

[No1] Alfred G. Noël, "Classification of admissible nilpotent orbits in simple exceptional real Lie algebras of inner type," Represent. Theory **5** (2001), 455-493.

[No2] ———, "Classification of admissible nilpotent orbits in simple real Lie algebras $E_{6(6)}$ and $E_{6(-26)}$," Represent. Theory **5** (2001), 494-502.

[O] Takuya Ohta, "Classification of admissible nilpotent orbits in the classical real Lie algebras," J. Algebra **136** (1991), no. 2, 290–333. MR **92j:**22032

[P] Patrice Perrin, "Représentations de Schrödinger, indice de Maslov et groupe metaplectique," in *Noncommutative Harmonic Analysis and Lie Groups*, Lecture Notes in Mathematics **880**, Springer-Verlag, Berlin, New York, 1981, 370–407. MR **83m:**22027

[R] R. Ranga Rao, "On some explicit formulas in the theory of Weil representation," Pac. J. of Math. **157**, No. 2 (1993), 335–371. MR **94a:**22037

[Sch] James O. Schwarz, "The Determination of the Admissible Nilpotent Orbits in Real Classical Groups," Ph.D. thesis, MIT, 1987.

[S] Jean-Pierre Serre, *Cohomologie galoisienne*, Fifth edition, Lecture Notes in Mathematics, **5**, Springer-Verlag, Berlin, 1994. MR **96b:**12010

[Sp] N. Spaltenstein, *Classes unipotentes et sous-groupes de Borel*, Lecture Notes in Math, **946**, Springer-Verlag, New York, 1982. MR **94a:**14024

[T1] Jacques Tits, "Classification of Algebraic Semisimple Groups," Proceedings of Symposia in Pure Mathematics, Volume IX, 1966, 33–62. MR **37:**309

[T2] ———, "Représentations linéaires irréductibles d'un groupe réductif sur un corps quelconque," J. Reine Angew. Math. **247** (1971), 196–220. MR **43:**3269

[V1] David A. Vogan, Jr., *Unitary Representations of Reductive Lie Groups*, Annals of Mathematical Studies, **118**, Princeton University Press, 1987. MR **89g:**22024

[V2] ———, "The Method of Coadjoint Orbits for Real Reductive Groups," IAS/Park City Mathematics Series **6** (1998). MR **2001k:**22027

[W] A. Weil, "Sur certains groupes d'opérateurs unitaires," Acta Math. **111** (1964), 143–211. MR **29:**2324

Department of Mathematics and Statistics, University of Ottawa, Ontario, Canada K1N 6N5

E-mail address: `mnevins@uottawa.ca`