

## CUSPIDAL LOCAL SYSTEMS AND GRADED HECKE ALGEBRAS, III

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ABSTRACT. We prove a strong induction theorem and classify the tempered and square integrable representations of graded Hecke algebras.

### INTRODUCTION

Let  $\mathcal{G}$  be the group of rational points of a simple adjoint algebraic group over a  $p$ -adic field, which is an inner form of a split group. Consider the set of isomorphism classes of irreducible admissible representations of  $\mathcal{G}$  whose restriction to some parahoric subgroup contains some irreducible unipotent cuspidal representation of that parahoric subgroup modulo its “unipotent radical”. The classification of such “unipotent” representations of  $\mathcal{G}$  has been established in [L8] (see also [KL] for an earlier special case) in accordance with a conjecture of Langlands (refined in [L1]). In the special case considered in [KL], the tempered and square-integrable representations were also explicitly described; the main tool to do so was an “induction theorem” [KL, 6.2] for affine Hecke algebras with equal parameters. But in the context of [L8] the induction theorem was missing and the problem of describing explicitly the unipotent representations that are tempered or square integrable representations remained open.

One of the techniques used in [L8] was the reduction (see [L5]) of the (equivalent) classification problem for certain affine Hecke algebras with unequal parameters to the problem of classifying the simple modules of certain “graded” Hecke algebras which could be done using methods of equivariant homology. By these methods one can reduce the problem of describing the tempered or square integrable unipotent representations to the analogous problem for graded Hecke algebras. This last problem is solved in the present paper. As in [KL] one of the key ingredients is an “induction theorem”. In fact, we will prove a strong form of the induction theorem (without “denominators”) inspired by [L9, 7.11] which implies the classification of tempered and square integrable representations.

In the case where  $\mathcal{G}$  has small rank, the classification of square integrable unipotent representations of  $\mathcal{G}$  has been given in [R].

After this work was completed, I received the preprint [W] where most results of the present paper (but not the strong induction theorem) are obtained independently in the case  $\mathcal{G} = SO_{2n+1}$ .

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## 1. PRELIMINARIES AND STATEMENT OF RESULTS

**1.1.** Unless otherwise specified, all algebraic varieties are assumed to be over  $\mathbf{C}$ . If  $X$  is a subvariety of  $X'$ , we write  $cl(X)$  for the closure of  $X$  in  $X'$ . For a Lie algebra  $\mathfrak{g}$  let  $\mathfrak{z}_{\mathfrak{g}}$  be the center of  $\mathfrak{g}$ . If  $A$  is a subset of  $\mathfrak{g}$ , we set  $\mathfrak{z}(A) = \{x \in \mathfrak{g}; [x, y] = 0 \ \forall y \in A\}$ ; if  $\mathfrak{a}$  is a Lie subalgebra of  $\mathfrak{g}$ , we set  $\mathfrak{z}_{\mathfrak{a}}(A) = \mathfrak{z}(A) \cap \mathfrak{a}$ .

For any algebraic group  $\mathcal{G}$  let  $\mathcal{G}^0$  be the identity component of  $\mathcal{G}$ , let  $U_{\mathcal{G}}$  be the unipotent radical of  $\mathcal{G}^0$  and let  $Z_{\mathcal{G}}$  be the center of  $\mathcal{G}$ . Let  $\bar{\mathcal{G}} = \mathcal{G}/U_{\mathcal{G}}$  and let  $\pi_{\mathcal{G}} : \mathcal{G} \rightarrow \bar{\mathcal{G}}$  be the canonical homomorphism. Let  $\underline{\mathcal{G}}$  be the Lie algebra of  $\mathcal{G}$ . If  $A$  is a subset of  $\underline{\mathcal{G}}$ , we set  $Z(A) = \{g \in G; \text{Ad}(g)y = y \ \forall y \in A\}$ . Let  $\pi_{\mathcal{G}} : \underline{\mathcal{G}} \rightarrow \bar{\mathcal{G}}$  be the map induced by  $\pi_{\mathcal{G}}$ . Let  $\mathcal{G}_{der}$  be the derived group of  $\mathcal{G}$  (a closed subgroup if  $\mathcal{G}$  is connected). If  $x \in \underline{\mathcal{G}}$  is a semisimple element, we denote by  $\langle x \rangle$  the smallest torus in  $\mathcal{G}$  whose Lie algebra contains  $x$ .

**1.2.** Let  $G$  be a connected reductive algebraic group. Let  $\mathfrak{g} = \underline{G}$ ; let  $\mathfrak{g}_N$  be the variety of nilpotent elements of  $\mathfrak{g}$ . Let  $\mathfrak{g}_{ss}$  be the set of semisimple elements of  $\mathfrak{g}$ . Let  $\mathfrak{P}$  be the variety of parabolic subgroups of  $G$ .

A *cuspidal datum* for  $G$  is a triple  $(\mathcal{P}, \mathbf{c}, \mathcal{L})$  where  $\mathcal{P}$  is a  $G$ -orbit on  $\mathfrak{P}$ ,  $\mathbf{c}$  is a  $G$ -orbit on the set of pairs  $(x, P)$  with  $P \in \mathcal{P}$ ,  $x \in \bar{P}$  is nilpotent, and  $\mathcal{L}$  is an irreducible  $G$ -equivariant local system on  $\mathbf{c}$  such that for some (or any)  $P \in \mathcal{P}$ , the restriction of  $\mathcal{L}$  to the  $\bar{P}$ -orbit

$$\mathbf{c}_P = \{x \in \bar{P}; (x, P) \in \mathbf{c}\}$$

(a local system that is automatically  $\bar{P}$ -equivariant and irreducible) is cuspidal in the sense of [L4, 2.2].

A *cuspidal triple* in  $G$  is a triple  $(L, C, \mathcal{E})$  where  $L$  is a Levi subgroup of a parabolic subgroup of  $G$ ,  $C$  is a nilpotent  $L$ -orbit in  $\underline{L}$  and  $\mathcal{E}$  is an irreducible  $L$ -equivariant local system on  $C$  which is cuspidal in the sense of [L4, 2.2].

To a cuspidal datum  $(\mathcal{P}, \mathbf{c}, \mathcal{L})$  we attach a cuspidal triple as follows: let  $P \in \mathcal{P}$ , let  $L$  be a Levi subgroup of  $P$ , let  $C$  be the nilpotent orbit in  $\underline{L}$  corresponding to  $\mathbf{c}_P$  under the obvious isomorphism  $\underline{L} \xrightarrow{\sim} \bar{P}$  and let  $\mathcal{E}$  be the local system on  $C$  corresponding to  $\mathcal{L}|_{\mathbf{c}_P}$  under the obvious isomorphism  $C \xrightarrow{\sim} \mathbf{c}_P$ . Then  $(L, C, \mathcal{E})$  is a cuspidal triple in  $G$ . Using [L6, 6.8(b), (c)], we see that, conversely, any cuspidal triple is obtained as above from a cuspidal datum  $(\mathcal{P}, \mathbf{c}, \mathcal{L})$  where  $\mathcal{P}, \mathbf{c}$  are unique and  $\mathcal{L}$  is unique up to isomorphism.

**1.3.** Let  $(\mathcal{P}, \mathbf{c}, \mathcal{L})$  be a cuspidal datum for  $G$  and let  $Q \in \mathfrak{P}$  be such that  $Q$  contains some  $P \in \mathcal{P}$ . Then there is an induced cuspidal datum  $(\mathcal{P}', \mathbf{c}', \mathcal{L})$  for  $\bar{Q}$ , defined as follows. Let  $\mathcal{P}'$  be the set of all subgroups  $P'$  of  $\bar{Q}$  such that  $\pi_{\bar{Q}}^{-1}(P') \in \mathcal{P}$ . Let  $\mathbf{c}'$  be the set of all pairs  $(x', P')$  where  $P' \in \mathcal{P}'$  and  $x' \in \bar{P}' = \overline{\pi_{\bar{Q}}^{-1}(P')}$  is such that  $(x', \pi_{\bar{Q}}^{-1}(P')) \in \mathbf{c}$ . The inverse image of  $\mathcal{L}$  under the map  $\mathbf{c}' \rightarrow \mathbf{c}$  given by  $(x', P') \mapsto (x', \pi_{\bar{Q}}^{-1}(P'))$  is denoted again by  $\mathcal{L}$ . Then  $(\mathcal{P}', \mathbf{c}', \mathcal{L})$  is a cuspidal datum for  $\bar{Q}$ .

**1.4.** In the remainder of this paper we fix a cuspidal datum  $(\mathcal{P}, \mathbf{c}, \mathcal{L})$  for  $G$ .

For each  $P \in \mathcal{P}$  we form the torus  $P/P_{\text{der}}$  (resp. the vector space  $\underline{P}/[\underline{P}, \underline{P}]$ ). If  $P, P' \in \mathcal{P}$ , there is a canonical isomorphism  $P'/P'_{\text{der}} \xrightarrow{\sim} P/P_{\text{der}}$  (resp.  $\underline{P}'/[\underline{P}', \underline{P}'] \xrightarrow{\sim} \underline{P}/[\underline{P}, \underline{P}]$ ) induced by  $\text{Ad}(g)$  where  $g \in G$  is such that  $\text{Ad}(g)P' = P$ . This is independent of the choice of  $g$ . Hence we may identify  $P/P_{\text{der}}$  (resp.  $\underline{P}/[\underline{P}, \underline{P}]$ ) for any  $P \in \mathcal{P}$  with a single torus  $\mathbf{T}$  (resp. a single  $\mathbf{C}$ -vector space  $\mathfrak{h}$ ). Thus, for any  $P \in \mathcal{P}$  we have a canonical isomorphism  $P/P_{\text{der}} \xrightarrow{\sim} \mathbf{T}$  (resp.  $\underline{P}/[\underline{P}, \underline{P}] \xrightarrow{\sim} \mathfrak{h}$ ). Since for  $P \in \mathcal{P}$  we have  $\underline{P}/[\underline{P}, \underline{P}] = \underline{P}/P_{\text{der}}$ , we have canonically  $\mathfrak{h} = \underline{\mathbf{T}}$ .

**1.5.** The set

$$\{P' \in \mathfrak{P}; P' \text{ contains strictly some } P \in \mathcal{P} \text{ and is minimal with this property}\}$$

decomposes into  $G$ -orbits  $(\mathcal{P}_i)_{i \in I}$ . Here  $I$  is a finite indexing set.

For any  $J \subset I$  let  $\mathcal{P}_J$  be the set of all  $P' \in \mathfrak{P}$  such that  $P'$  contains some member of  $\mathcal{P}$  and, for  $i \in I$ ,  $P'$  contains some member of  $\mathcal{P}_i$  if and only if  $i \in J$ . Then  $\mathcal{P}_J$  is a  $G$ -orbit on  $\mathfrak{P}$ . We have  $\mathcal{P} = \mathcal{P}_\emptyset$ ,  $\mathcal{P}_i = \mathcal{P}_{\{i\}}$  for  $i \in I$ .

The diagonal action of  $G$  on  $\mathcal{P} \times \mathcal{P}$  has only finitely many orbits; an orbit is said to be *good* if it consists of pairs  $(P, P')$  such that  $P, P'$  have a common Levi subgroup. Let  $W$  be the set of good  $G$ -orbits on  $\mathcal{P} \times \mathcal{P}$ . There is a natural group structure on  $W$  (see [L6, 7.3]).

**1.6.** Let  $P \in \mathcal{P}$  and let  $L$  be a Levi subgroup of  $P$ . For any  $J \subset I$  let  $\mathcal{P}_J$  be the unique member of  $\mathcal{P}_J$  that contains  $P$ . For  $i \in I$  write  $P_i$  instead of  $P_{\{i\}}$ . We have  $P_\emptyset = P$ . Let  $T = Z_L^0$ ; then  $\underline{T} = \mathfrak{z}_{\underline{L}}$ . Let  $N(T)$  be the normalizer of  $T$  in  $G$ . Then  $W(L) = N(T)/L$  acts naturally (and faithfully) on  $\underline{T}$  and on  $\underline{T}^*$ . We have  $\mathfrak{g} = \bigoplus_{\alpha \in \underline{T}^*} \mathfrak{g}^\alpha$  where

$$\mathfrak{g}^\alpha = \{x \in \mathfrak{g}; [y, x] = \alpha(y)x \quad \forall y \in \underline{T}\}.$$

Note that  $\mathfrak{g}^\alpha$  is an  $\underline{L}$ -module by the ad action. Let  $R = \{\alpha \in \underline{T}^*; \alpha \neq 0, \mathfrak{g}^\alpha \neq 0\}$ .

By [L4, 2.5],  $R$  is a (not necessarily reduced) root system in  $\underline{T}^*$  with Weyl group  $W(L)$ . (We do not have to specify the set of coroots since they are determined uniquely by  $R$  and the Weyl group action on  $\underline{T}^*$ .) Let  $L_i$  be the Levi subgroup of  $P_i$  that contains  $L$ . There is a unique  $\alpha_i \in R$  such that  $\mathfrak{g}^{\alpha_i} \subset \underline{U}_P$  and  $\underline{L}_i = \bigoplus_{n \in \mathbf{Z}} \mathfrak{g}^{n\alpha_i}$ . Then  $\{\alpha_i; i \in I\}$  is a set of simple roots for  $R$ .

Let  $\mathcal{C}$  be the nilpotent  $L$ -orbit in  $\underline{L}$  which corresponds to  $\mathbf{c}_P$  under the obvious isomorphism  $\underline{L} \xrightarrow{\sim} \underline{P}$ . Let  $y \in \mathcal{C}$ . For  $i \in I$  let  $c_i$  be the integer  $\geq 2$  such that

$$\begin{aligned} \text{ad}(y)^{c_i-2} : \mathfrak{g}^{\alpha_i} \oplus \mathfrak{g}^{2\alpha_i} &\rightarrow \mathfrak{g}^{\alpha_i} \oplus \mathfrak{g}^{2\alpha_i} \text{ is } \neq 0, \\ \text{ad}(y)^{c_i-1} : \mathfrak{g}^{\alpha_i} \oplus \mathfrak{g}^{2\alpha_i} &\rightarrow \mathfrak{g}^{\alpha_i} \oplus \mathfrak{g}^{2\alpha_i} \text{ is } 0. \end{aligned}$$

Then  $c_i$  is independent of the choice of  $P, L$  and  $y$ .

We identify  $W(L)$  with  $W$  by  $n \mapsto G$ -orbit of  $(P, nPn^{-1})$ . Via the obvious isomorphism  $\underline{T} \xrightarrow{\sim} \underline{P}/[\underline{P}, \underline{P}] = \mathfrak{h}$ , the  $W(L)$  action on  $\underline{T}$  and  $\underline{T}^*$  becomes a  $W$  action on  $\mathfrak{h}$  and  $\mathfrak{h}^*$  (independent of the choice of  $P, L$ ) and the vectors  $\alpha_i (i \in I)$  in  $\underline{T}^*$  become vectors in  $\mathfrak{h}^*$ , denoted again by  $\alpha_i$  (these are also independent of the choice of  $P, L$ ). The action of  $W$  on  $\mathfrak{h}^*$  is denoted by  $w, \xi \mapsto {}^w\xi$ . For  $i \in I$  let  $s_i$  be the unique element of  $W$  which is a reflection in  $\mathfrak{h}^*$  such that  $s_i(\alpha_i) = -\alpha_i$ . Then  $W$  together with  $s_i (i \in I)$  is a Coxeter group. For  $J \subset I$  let  $W_J$  be the subgroup of  $W$  generated by  $\{s_i; i \in J\}$ .

For future use we note the following property:

(a) Let  $i \in I$  and let  $Q \in \mathfrak{P}$  be such that  $P \subset Q$ ,  $\underline{P}_i \not\subset \underline{Q}$ . Then  $\underline{L}_i \cap \underline{Q} = \bigoplus_{n \in \mathbb{N}} \mathfrak{g}^{n\alpha_i}$ . In particular,  $(\mathfrak{g}^{-\alpha_i} \oplus \mathfrak{g}^{-2\alpha_i}) \cap \underline{Q} = 0$ .

**1.7.** Let  $\mathbf{H}$  be the associative  $\mathbf{C}$ -algebra defined by the generators  $\underline{\xi}$  (in 1-1 correspondence with the elements  $\xi \in \mathfrak{h}^*$ ),  $s_i$  (indexed by  $i \in I$ ) and  $\mathbf{r}$ , subject to the following relations:

- (a)  $\underline{a\xi + a'\xi'} = a\underline{\xi} + a'\underline{\xi'}$  for any  $\xi, \xi' \in \mathfrak{h}^*$  and any  $a, a' \in \mathbf{C}$ ;
- (b)  $\underline{\xi\xi'} = \underline{\xi'}\underline{\xi}$  for any  $\xi, \xi' \in \mathfrak{h}^*$ ;
- (c)  $s_i (i \in I)$  satisfy the relations of  $W$ ;
- (d)  $s_i \underline{\xi} - \underline{s_i \xi} s_i = c_i \frac{\xi - s_i \xi}{\alpha_i} \mathbf{r}$  for any  $\xi \in \mathfrak{h}^*$  and any  $i \in I$ .
- (e)  $\mathbf{r}$  is central.

(In (d) we have  $\frac{\xi - s_i \xi}{\alpha_i} \in \mathbf{C}$ .) This is the same as the algebra denoted by  $\mathbf{H}$  in [L4, 6.3].

**1.8.** Let

$$\dot{\mathfrak{g}} = \{(y, P) \in \mathfrak{g} \times \mathcal{P}; y \in \underline{P}; \underline{\pi}_P(y) \in \mathbf{c}_P + \underline{Z_P}\}.$$

Let  $\pi : \dot{\mathfrak{g}} \rightarrow \mathfrak{g}$  be the first projection. Now  $G \times \mathbf{C}^*$  acts on  $\dot{\mathfrak{g}}$  by

$$(g, \lambda) : y \mapsto \lambda^{-2} \text{Ad}(g)y,$$

on  $\mathcal{P}$  by

$$(g, \lambda) : P \mapsto gPg^{-1}$$

and on  $\dot{\mathfrak{g}}$  by

$$(g, \lambda) : (y, P) \mapsto (\lambda^{-2} \text{Ad}(g)y, gPg^{-1}).$$

For  $y \in \mathfrak{g}_N$  we denote by  $M(y)$  or  $M_G(y)$  the stabilizer of  $y$  in  $G \times \mathbf{C}^*$ . Thus,

$$M(y) = \{(g, \lambda) \in G \times \mathbf{C}^*; \text{Ad}(g)y = \lambda^2 y\}.$$

We also have an action of  $G \times \mathbf{C}^*$  on  $\mathbf{c}$  given by  $(g, \lambda) : (x, P) \mapsto (\lambda^{-2} \text{Ad}(g)x, gPg^{-1})$ . Here we regard  $\text{Ad}(g)$  as a map  $\mathbf{c}_P \rightarrow \mathbf{c}_{gPg^{-1}}$ . The local system  $\mathcal{L}$  on  $\mathbf{c}$  is automatically  $G \times \mathbf{C}^*$ -equivariant [L6, 7.15]. Let  $s : \dot{\mathfrak{g}} \rightarrow \mathbf{c}$  be given by  $s(y, P) = (y', P)$  where  $y' \in \mathbf{c}_P$ ,  $\underline{\pi}_P(y) - y' \in \underline{Z_P}$ . Then  $\dot{\mathcal{L}} = s^* \mathcal{L}$  is a  $G \times \mathbf{C}^*$ -equivariant local system on  $\dot{\mathfrak{g}}$  and  $\mathcal{K} = \pi_! (\dot{\mathcal{L}}^*)$  is (up to shift) a  $G \times \mathbf{C}^*$ -equivariant perverse sheaf on  $\mathfrak{g}$ , with a canonical action of  $W$ , [L4, 3.4].

**1.9.** Let  $X$  be an algebraic variety with a given morphism  $X \xrightarrow{m} \mathfrak{g}$ . Define  $X \xrightarrow{m'} \dot{X} \xrightarrow{\dot{m}} \dot{\mathfrak{g}}$  by the cartesian diagram

$$\begin{array}{ccc} \dot{X} & \xrightarrow{\dot{m}} & \dot{\mathfrak{g}} \\ m' \downarrow & & \downarrow \pi \\ X & \xrightarrow{m} & \mathfrak{g} \end{array}$$

Then  $m^*(\mathcal{K})$  is naturally an object of the bounded derived category of constructible sheaves on  $\dot{X}$  with a  $W$ -action inherited from  $\mathcal{K}$ ; hence there is a natural  $W$ -action on the hypercohomology

$$H_c^j(X, m^*(\mathcal{K})) = H_c^j(X, m'_! \dot{\mathcal{L}}^*) = H_c^j(\dot{X}, \dot{\mathcal{L}}^*).$$

(We will often denote various local systems obtained from  $\dot{\mathcal{L}}, \dot{\mathcal{L}}^*$  by some natural construction again by  $\dot{\mathcal{L}}, \dot{\mathcal{L}}^*$ .)

**1.10.** If, in addition,  $X$  has a given action of a closed connected subgroup  $G'$  of  $G \times \mathbf{C}^*$  and  $m$  is compatible with the  $G'$ -actions, and if  $\Gamma$  is a smooth irreducible variety with a free  $G'$ -action, we can form the cartesian diagram

$$\begin{array}{ccc} \Gamma \dot{X} & \xrightarrow{\Gamma \dot{m}} & \Gamma \dot{\mathfrak{g}} \\ \Gamma m' \downarrow & & \Gamma \pi \downarrow \\ \Gamma X & \xrightarrow{\Gamma m} & \Gamma \mathfrak{g} \end{array}$$

where  $Y \mapsto \Gamma Y$  is the functor from algebraic varieties with  $G'$ -action to algebraic varieties given by  $Y \mapsto G' \backslash (\Gamma \times Y)$ .

The local system  $\mathbf{C} \boxtimes \dot{\mathcal{L}}^*$  on  $\Gamma \times \dot{\mathfrak{g}}$  is  $G'$ -equivariant; hence it descends canonically to a local system  $\Gamma \dot{\mathcal{L}}^*$  on  $\Gamma \dot{\mathfrak{g}}$ . Also,  $\mathbf{C} \boxtimes \mathcal{K}$  is (up to shift) a  $G'$ -equivariant perverse sheaf with  $W$ -action on  $\Gamma \times \mathfrak{g}$ ; hence it descends to a perverse sheaf (up to shift)  $\Gamma \mathcal{K}$  with  $W$ -action on  $\Gamma \mathfrak{g}$ . We have canonically  $\Gamma \mathcal{K} = (\Gamma \pi)_!(\Gamma \dot{\mathcal{L}}^*)$ . Then  $(\Gamma m)^*(\Gamma \mathcal{K})$  is naturally an object of the bounded derived category of constructible sheaves on  $\Gamma \dot{X}$ , with a  $W$ -action inherited from  $\Gamma \mathcal{K}$ ; hence there is a natural  $W$ -action on the hypercohomology

$$H_c^{2d-j}(\Gamma X, (\Gamma m)^*(\Gamma \mathcal{K})) = H_c^{2d-j}(\Gamma X, (\Gamma m')_!((\Gamma \dot{m})^*(\Gamma \dot{\mathcal{L}}^*))) = H_c^{2d-j}(\Gamma \dot{X}, \Gamma \dot{\mathcal{L}}^*)$$

where  $d = \dim \dot{X}$ . (We write  $\Gamma \dot{\mathcal{L}}^*$  instead of  $(\Gamma \dot{m})^*(\Gamma \dot{\mathcal{L}}^*)$ .) We can choose  $\Gamma$  so that  $H^n(\Gamma, \mathbf{C}) = 0$  for  $n \in [1, m]$  where  $m$  is large compared with  $j$ . Taking duals, we see that  $W$  acts naturally on the equivariant homology

$$H_j^{G'}(\dot{X}, \dot{\mathcal{L}}) = H_c^{2d-j}(\Gamma \dot{X}, \Gamma \dot{\mathcal{L}}^*)^*$$

[L4, 1.1]. This action is independent of the choice of  $\Gamma$ .

**1.11.** Let  $\mathbf{S} = S(\mathfrak{h}^* \oplus \mathbf{C}) = S(\mathfrak{h}^*) \otimes \mathbf{C}[\mathbf{r}]$  where  $S()$  denotes the symmetric algebra of a  $\mathbf{C}$ -vector space and  $\mathbf{r} = (0, 1) \in \mathfrak{h}^* \oplus \mathbf{C}$ .

For any algebraic group  $G'$  we write  $H_{G'}^*$  instead of  $H_{G'}^*(\text{point}, \mathbf{C})$  (equivariant cohomology). For any surjective homomorphism  $G' \rightarrow G''$  of connected algebraic groups we have a canonical algebra homomorphism  $H_{G'}^* \rightarrow H_{G''}^*$ . (Using the identification [L4, 1.11(a)], this is obtained by associating to a polynomial function  $\underline{G}'' \rightarrow \mathbf{C}$  its composition with the obvious map  $\underline{G}' \rightarrow \underline{G}''$ .) In particular, if  $P \in \mathcal{P}$ , we have a canonical algebra homomorphism

$$H_{P/P_{\text{der}} \times \mathbf{C}^*}^* = H_{\bar{P}/\bar{P}_{\text{der}} \times \mathbf{C}^*}^* \rightarrow H_{\bar{P} \times \mathbf{C}^*}^*.$$

Composing this with the algebra homomorphism  $H_{\bar{P} \times \mathbf{C}^*}^* \rightarrow H_{\bar{P} \times \mathbf{C}^*}^*(\mathbf{c}_P, \mathbf{C})$  (as in [L4, 1.7]) we obtain an algebra homomorphism  $H_{P/P_{\text{der}} \times \mathbf{C}^*}^* \rightarrow H_{\bar{P} \times \mathbf{C}^*}^*(\mathbf{c}_P, \mathbf{C})$ . By [L4, 1.6, 1.4(e), 1.4(h)] we have canonically

$$H_{G \times \mathbf{C}^*}^*(\dot{\mathfrak{g}}, \mathbf{C}) = H_{P \times \mathbf{C}^*}^*(\underline{\mathbb{A}}_P^{-1}(\mathbf{c}_P + \underline{Z}_{\bar{P}}), \mathbf{C}) = H_{P \times \mathbf{C}^*}^*(\mathbf{c}_P, \mathbf{C}) = H_{\bar{P} \times \mathbf{C}^*}^*(\mathbf{c}_P, \mathbf{C}).$$

We obtain an algebra homomorphism  $H_{P/P_{\text{der}} \times \mathbf{C}^*}^* \rightarrow H_{G \times \mathbf{C}^*}^*(\dot{\mathfrak{g}}, \mathbf{C})$ . Using the canonical isomorphism  $P/P_{\text{der}} \xrightarrow{\sim} \mathbf{T}$  we obtain an algebra homomorphism  $H_{\mathbf{T} \times \mathbf{C}^*}^* \rightarrow H_{G \times \mathbf{C}^*}^*(\dot{\mathfrak{g}}, \mathbf{C})$ . This is in fact an algebra isomorphism (a reformulation of [L4, 4.2]). By [L4, 1.10] we have canonically  $H_{\mathbf{T} \times \mathbf{C}^*}^* = S(\mathfrak{h}^* \oplus \mathbf{C}) = \mathbf{S}$ . Thus we have an algebra isomorphism

$$(a) \quad \mathbf{S} \xrightarrow{\sim} H_{G \times \mathbf{C}^*}^*(\dot{\mathfrak{g}}, \mathbf{C}).$$

Assume that  $\tilde{X}$  is an algebraic variety with a given action of a closed connected subgroup  $G'$  of  $G \times \mathbf{C}^*$  and with a given morphism  $\tilde{m} : \tilde{X} \rightarrow \dot{\mathfrak{g}}$  compatible with the  $G'$ -actions. We write  $\dot{\mathcal{L}}$  instead of  $\tilde{m}^* \dot{\mathcal{L}}$  (a local system on  $\tilde{X}$ ). Then  $H_*^{G'}(\tilde{X}, \dot{\mathcal{L}})$  is an  $\mathbf{S}$ -module as follows. Let  $\xi \in \mathbf{S}$ , let  $\xi' \in H_{G \times \mathbf{C}^*}^*(\dot{\mathfrak{g}}, \mathbf{C})$  be the element that corresponds to  $\xi$  under (a) and let  $\xi'' \in H_{G'}^*(\dot{\mathfrak{g}}, \mathbf{C})$  be the image of  $\xi'$  under the homomorphism  $H_{G \times \mathbf{C}^*}^*(\dot{\mathfrak{g}}, \mathbf{C}) \rightarrow H_{G'}^*(\dot{\mathfrak{g}}, \mathbf{C})$  as in [L4, 1.4(f)]. We have  $\tilde{m}^*(\xi'') \in H_{G'}^*(\tilde{X}, \mathbf{C})$ . If  $z \in H_*^{G'}(\tilde{X}, \dot{\mathcal{L}})$ , then  $\xi z$  is defined as the product  $\tilde{m}^*(\xi'') \cdot z \in H_*^{G'}(\tilde{X}, \dot{\mathcal{L}})$  as in [L4, 1.7].

**1.12.** Let  $y \in \mathfrak{g}_N$ . Let

$$\mathcal{P}_y = \{P \in \mathcal{P}; y \in \underline{P}; \pi_P(y) \in \mathbf{c}_P + \underline{Z}_{\bar{P}}\} = \{P \in \mathcal{P}; y \in \underline{P}; \pi_P(y) \in \mathbf{c}_P\}.$$

(The second equality follows from the fact that  $y$  is nilpotent.) The second projection identifies  $\{y\}$  with  $\mathcal{P}_y$ . Note that  $\{y\}$  and  $\{y\}'$  are stable under  $M(y)$ , hence under  $M^0(y)$ . By 1.10 applied to  $X = \{y\}$  and by 1.11 applied to  $\tilde{X} = \{y\}'$  we see that, if  $G'$  is a closed connected subgroup of  $M^0(y)$ , then  $H_*^{G'}(\mathcal{P}_y, \dot{\mathcal{L}})$  has a natural  $W$ -action and a natural  $\mathbf{S}$ -action. It also has a natural  $H_{G'}^*$ -module structure [L4, 1.7]. Now there is a unique  $\mathbf{H}$ -module structure on  $H_*^{G'}(\mathcal{P}_y, \dot{\mathcal{L}})$  such that  $\mathbf{r} \in \mathbf{H}$  acts as  $\mathbf{r} \in \mathbf{S}$ ,  $\underline{\xi} \in \mathbf{H}$  acts as  $\xi \in \mathbf{S}$  (for  $\xi \in \mathfrak{h}^*$ ) and  $s_i \in \mathbf{H}$  acts as  $s_i \in W$  (for  $i \in I$ ). (In the special case when  $G' = M^0(y)$ , this follows from [L4, 8.13]. The case when  $G'$  is not necessarily  $M^0(y)$  can be reduced to the special case using the isomorphism

$$(a) \quad H_{G'}^* \otimes_{H_{M^0(y)}^*} H_*^{M^0(y)}(\mathcal{P}_y, \dot{\mathcal{L}}) \xrightarrow{\sim} H_*^{G'}(\mathcal{P}_y, \dot{\mathcal{L}})$$

as in [L4, 7.5]; that result is applicable in view of [L4, 8.6].) The  $\mathbf{H}$ -module structure commutes with the  $H_{G'}^*$ -module structure on  $H_*^{G'}(\mathcal{P}_y, \dot{\mathcal{L}})$ .

Now the finite group  $\bar{M}(y) = M(y)/M^0(y)$  acts on  $H_*^{M^0(y)}(\mathcal{P}_y, \dot{\mathcal{L}})$  by [L4, 1.9(a)]. This action commutes with the  $\mathbf{H}$ -module structure and is compatible with the  $H_{M^0(y)}^*$ -module structure where we regard  $H_{M^0(y)}^*$  as being endowed with the action of  $\bar{M}(y)$  given again by [L4, 1.9(a)].

**1.13.** Let  $G'$  be a closed connected subgroup of  $G \times \mathbf{C}^*$ . Then  $\underline{G}' \subset \mathfrak{g} \oplus \mathbf{C}$ . By [L4, 1.11(a)], we may identify  $H_{G'}^*$  with the space of polynomials  $f : \underline{G}' \rightarrow \mathbf{C}$  that are constant on the cosets by the nil-radical of  $\underline{G}'$  and are constant on the Ad-orbits of  $G'$ . Let  $(\sigma, r) \in \underline{G}'$  be a semisimple element. Let  $\mathcal{J}_{\sigma, r}^{G'}$  be the maximal ideal of  $H_{G'}^*$  consisting of all  $f$  such that  $f(\sigma, r) = 0$ . Let  $\mathbf{C}_{\sigma, r} = H_{G'}^*/\mathcal{J}_{\sigma, r}^{G'}$ . (A one-dimensional  $\mathbf{C}$ -vector space.)

Now assume that  $G' \subset M^0(y)$ . Then  $\underline{G}' \subset \{(x, r) \in \mathfrak{g} \oplus \mathbf{C}; [x, y] = 2ry\} = \underline{M^0(y)}$ . In particular, we have  $[\sigma, y] = 2ry$ .

Let  $f_1 : \underline{G}' \rightarrow \mathbf{C}$  be defined by  $f_1(x, r) = r$ . In the  $H_{G'}^*$ -module structure on  $H_*^{G'}(\mathcal{P}_y, \dot{\mathcal{L}})$ ,  $f_1$  acts as multiplication by  $\mathbf{r} \in \mathbf{H}$ . We form

$$E_{y, \sigma, r} = \mathbf{C}_{\sigma, r} \otimes_{H_{G'}^*} H_*^{G'}(\mathcal{P}_y, \dot{\mathcal{L}}) = H_*^{G'}(\mathcal{P}_y, \dot{\mathcal{L}}) / \mathcal{J}_{\sigma, r}^{G'} H_*^{G'}(\mathcal{P}_y, \dot{\mathcal{L}}).$$

Then  $E_{y, \sigma, r}$  inherits from  $H_*^{G'}(\mathcal{P}_y, \dot{\mathcal{L}})$  an  $\mathbf{H}$ -module structure in which  $\mathbf{r} \in \mathbf{H}$  acts as multiplication by  $r$  (since  $f_1 - r \in \mathcal{J}_{\sigma, r}^{G'}$ ). By 1.12(a),  $E_{y, \sigma, r}$  defined in terms of  $G'$  is the same as  $E_{y, \sigma, r}$  defined in terms of  $M^0(y)$ . For this reason we do not include  $G'$  in the notation for  $E_{y, \sigma, r}$ .

Let  $M(y, \sigma) = M(y) \cap (Z(\sigma) \times \mathbf{C}^*)$ . Let  $\bar{M}(y, \sigma)$  be the group of connected components of  $M(y, \sigma)$ . The obvious map  $\bar{M}(y, \sigma) \rightarrow \bar{M}(y)$  is injective, since  $M^0(y) \cap (Z(\sigma) \times \mathbf{C}^*)$  is connected. Clearly, the restriction of the  $\bar{M}(y)$  action on  $H_{M^0(y)}^*$  to  $\bar{M}(y, \sigma)$  leaves  $\mathcal{J}_{\sigma, r}^{M^0(y)}$  stable; hence the action of  $\bar{M}(y)$  on  $H_*^{M^0(y)}(\mathcal{P}_y, \dot{\mathcal{L}})$  induces an action of  $\bar{M}(y, \sigma)$  on  $E_{y, \sigma, r}$ . This action commutes with the  $\mathbf{H}$ -module structure.

**1.14.** Let  $\text{Irr}\bar{M}(y, \sigma)$  be a set of representatives for the isomorphism classes of irreducible representations of  $\bar{M}(y, \sigma)$ . For  $\rho \in \text{Irr}\bar{M}(y, \sigma)$  let

$$E_{y, \sigma, r, \rho} = \text{Hom}_{\bar{M}(y, \sigma)}(\rho, E_{y, \sigma, r}).$$

Let  $\text{Irr}_0\bar{M}(y, \sigma)$  be the set of those  $\rho \in \text{Irr}\bar{M}(y, \sigma)$  such that  $E_{y, \sigma, r, \rho} \neq 0$  or, equivalently (see [L4, 8.10]) such that  $\rho$  appears in the restriction of the  $\bar{M}(y)$ -module  $H_*(\mathcal{P}_y, \dot{\mathcal{L}})$  to  $\bar{M}(y, \sigma)$ . (Equivariant homology or cohomology in which the group is not specified is understood to be with respect to the group  $\{1\}$ .)

**Theorem 1.15.** (a) *Let  $y, \sigma, r$  be as above; assume that  $r \neq 0$ . Let  $\rho \in \text{Irr}_0\bar{M}(y, \sigma)$ . Then the  $\mathbf{H}$ -module  $E_{y, \sigma, r, \rho}$  has a unique maximal submodule. Let  $\bar{E}_{y, \sigma, r, \rho}$  be the simple quotient of  $E_{y, \sigma, r, \rho}$ .*

(b) *Let  $r \in \mathbf{C}^*$ . The map  $(y, \sigma, \rho) \mapsto \bar{E}_{y, \sigma, r, \rho}$  establishes a bijection between the set of all triples  $(y, \sigma, \rho)$  with  $y \in \mathfrak{g}_N$ ,  $\sigma \in \mathfrak{g}_{ss}$  with  $[\sigma, y] = 2ry$  and  $\rho \in \text{Irr}_0\bar{M}(y, \sigma)$  (modulo the natural action of  $G$ ) and the set of isomorphism classes of simple  $\mathbf{H}$ -modules in which  $\mathbf{r}$  acts as multiplication by  $r$ .*

The proof is given in 3.39, 3.41, 3.42. (A bijection as in (b) has already been obtained in [L6] by other means since (a) was not known in [L6].)

**1.16.** Let  $J \subset I$  and let  $Q \in \mathcal{P}_J$ . Let  $Q^1$  be a Levi subgroup of  $Q$ . Let  $y \in \underline{Q}^1$  be nilpotent. Now  $Q^1$  carries a cuspidal datum  $(\mathcal{P}', \mathbf{c}', \dot{\mathcal{L}})$  analogous to that of  $G$  (see 1.3). Here we identify  $Q^1 = \bar{Q}$  via  $\pi_Q$ . Replacing  $G$  by  $Q^1$  in the definition of  $W, \mathfrak{h}, \mathbf{S}, \mathbf{H}, \mathcal{P}_y, \dot{\mathcal{L}}$  we get  $W_J, \mathfrak{h}, \mathbf{S}, \mathbf{H}', \mathcal{P}'_y, \dot{\mathcal{L}}$ . We use  $P' \mapsto \pi_Q^{-1}(P')$  to identify  $\mathcal{P}'$  with  $\mathcal{P}^* = \{P \in \mathcal{P}; P \subset Q\}$  and  $\mathcal{P}'_y$  with  $\mathcal{P}_y^* = \{P \in \mathcal{P}_y; P \subset Q\}$ .

Let  $C$  be a maximal torus of  $M_{Q^1}^0(y) \subset M^0(y)$ . Then  $H_*^C(\mathcal{P}_y^*, \dot{\mathcal{L}})$  is an  $\mathbf{H}'$ -module (by 1.12 for  $Q^1 = \bar{Q}$  instead of  $G$ ). Using the obvious algebra homomorphism  $\mathbf{H}' \rightarrow \mathbf{H}$  (taking the generators of  $\mathbf{H}'$  to the corresponding generators of  $\mathbf{H}$ ) we can form the  $\mathbf{H}$ -module  $\mathbf{H} \otimes_{\mathbf{H}'} H_*^C(\mathcal{P}_y^*, \dot{\mathcal{L}})$ . Now the closed imbedding  $j : \mathcal{P}_y^* \rightarrow \mathcal{P}_y$  induces a map  $j_! : H_*^C(\mathcal{P}_y^*, \dot{\mathcal{L}}) \rightarrow H_*^C(\mathcal{P}_y, \dot{\mathcal{L}})$ , (see [L4, 1.4(b)]). From the definitions we see that  $j_!$  is  $\mathbf{H}'$ -linear; hence it induces an  $\mathbf{H}$ -linear map

$$(a) \quad \mathbf{H} \otimes_{\mathbf{H}'} H_*^C(\mathcal{P}_y^*, \dot{\mathcal{L}}) \rightarrow H_*^C(\mathcal{P}_y, \dot{\mathcal{L}}).$$

Let  ${}_y U_Q = \text{coker}(\text{ad}(y) : \underline{U}_Q \rightarrow \underline{U}_Q)$ . Define  $\epsilon : \underline{M}_{Q^1}^0(y) \rightarrow \mathbf{C}$  (recall that  $\underline{M}_{Q^1}^0(y) \subset \mathfrak{g} \oplus \mathbf{C}$ ) by

$$\epsilon(x, \lambda) = \det(\text{ad}(x) - 2\lambda : {}_y U_Q \rightarrow {}_y U_Q).$$

(For  $(x, r) \in \underline{M}_{Q^1}^0(y)$  we have  $[x, y] = 2ry$ ; hence  $[\text{ad}(x), \text{ad}(y)] = 2\text{rad}(y)$ ; hence  $\text{ad}(x) : \underline{U}_Q \rightarrow \underline{U}_Q$  induces a map  ${}_y U_Q \rightarrow {}_y U_Q$  denoted again by  $\text{ad}(x)$ .) The restriction of  $\epsilon$  to  $\underline{C}$  is denoted again by  $\epsilon$ . By the identification [L4, 1.11(a)] we

may regard  $\epsilon$  as an element of  $H_C^*$ . Applying  $H_C^*[\epsilon^{-1}] \otimes_{H_C^*}$  to (a) we obtain an  $H_C^*[\epsilon^{-1}]$ -linear map

$$(b) \quad H_C^*[\epsilon^{-1}] \otimes_{H_C^*} (\mathbf{H} \otimes_{\mathbf{H}'} H_*^C(\mathcal{P}_y^*, \dot{\mathcal{L}})) \rightarrow H_C^*[\epsilon^{-1}] \otimes_{H_C^*} H_*^C(\mathcal{P}_y, \dot{\mathcal{L}}).$$

**Theorem 1.17** (Induction theorem). *The map 1.16(b) is an ( $\mathbf{H}$ -linear) isomorphism.*

The proof is given in Section 2 as an application of the “strong induction theorem” 2.16 which states the existence of an isomorphism similar to 1.16(b) but in which no elements of  $H_C^*$  need to be inverted.

**Corollary 1.18.** *Assume that  $(\sigma, r)$  is a semisimple element of  $M_{Q^1}^0(y)$  such that  $\epsilon(\sigma, r) \neq 0$ . Define  $E'_{y, \sigma, r}$  like  $E_{y, \sigma, r}$  but in terms of  $Q^1$  instead of  $G$ . Choose  $C$  as in 1.16 such that  $(\sigma, r) \in \underline{C}$ . Then the map 1.16(b) induces an isomorphism of  $\mathbf{H}$ -modules  $\mathbf{H} \otimes_{\mathbf{H}'} E'_{y, \sigma, r} \xrightarrow{\sim} E_{y, \sigma, r}$ .*

**1.19.** Let  $\mathcal{I}$  be the collection of all simple finite dimensional  $\mathfrak{g}$ -modules  $V$  such that for any  $P \in \mathcal{P}$  there exists a  $\underline{P}$ -stable line  $D_P$  in  $V$  (necessarily unique). An equivalent condition is that, for any  $P \in \mathcal{P}$ ,  $\{v \in V; \underline{U}_P v = 0\}$  is a line in  $V$ . (Clearly,  $\{v \in V; \underline{U}_P v = 0\}$  is  $\underline{P}$ -stable, hence if the second condition holds, then the first condition holds. Conversely, let  $P \in \mathcal{P}$  and let  $\mathfrak{b}$  be a Borel subalgebra of  $\underline{P}$ . If  $\{v \in V; \underline{U}_P v = 0\}$  is nonzero, then it is a  $\underline{P}/\underline{U}_P$ -module with a unique line stable under  $\mathfrak{b}/\underline{U}_P$ , a Borel subalgebra of  $\underline{P}/\underline{U}_P$ , since such a line must be  $\mathfrak{b}$ -stable and  $V$  is simple. It follows that  $\{v \in V; \underline{U}_P v = 0\}$  is simple as a  $\underline{P}/\underline{U}_P$ -module. Hence, if there exists a  $\underline{P}$ -stable line in  $V$  (necessarily contained in  $\{v \in V; \underline{U}_P v = 0\}$ ), then that line must be equal to  $\{v \in V; \underline{U}_P v = 0\}$  so that  $\{v \in V; \underline{U}_P v = 0\}$  is a line.)

Let  $V \in \mathcal{I}$ . Then  $V$  defines an element  $\xi_V \in \mathfrak{h}^*$  as follows. Let  $P \in \mathcal{P}$ . Then  $xv = u(x)v$  for  $x \in \underline{P}$ ,  $v \in D_P$  where  $u : \underline{P} \rightarrow \mathbf{C}$  is a Lie algebra homomorphism. Then  $u$  factors through a linear form on  $\underline{P}/[\underline{P}, \underline{P}] = \mathfrak{h}$  denoted by  $\xi_V$ . It is independent of the choice of  $P$ .

**1.20.** *In the remainder of this section we assume that  $G$  is semisimple.*

Let  $r \in \mathbf{C}^*$  and let  $\tau : \mathbf{C} \rightarrow \mathbf{R}$  be a homomorphism of abstract groups such that  $\tau(r) \neq 0$ . Let  $E$  be an  $\mathbf{H}$ -module of finite dimension over  $\mathbf{C}$ . We say that  $E$  is  $\tau$ -tempered if for any  $V \in \mathcal{I}$ , any eigenvalue  $\lambda$  of  $\xi_V$  on  $E$  satisfies  $\tau(\lambda)/\tau(r) \geq 0$ . We say that  $E$  is  $\tau$ -square integrable if for any  $V \in \mathcal{I}$ , other than  $\mathbf{C}$ , any eigenvalue  $\lambda$  of  $\xi_V$  on  $E$  satisfies  $\tau(\lambda)/\tau(r) > 0$ .

The standard basis of the Lie algebra  $\mathfrak{sl}_2(\mathbf{C})$  is denoted as follows:

$$e_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

**Theorem 1.21.** *Assume that  $r \neq 0$ . Let  $y, \sigma, \rho$  be as in 1.15(b). The following three conditions are equivalent:*

- (i)  $E_{y, \sigma, r, \rho}$  is  $\tau$ -tempered.
- (ii)  $\bar{E}_{y, \sigma, r, \rho}$  is  $\tau$ -tempered.
- (iii) *There exists a homomorphism of Lie algebras  $\phi : \mathfrak{sl}_2(\mathbf{C}) \rightarrow \mathfrak{g}$  such that  $y = \phi(e_0)$ ,  $[\sigma, \phi(h_0)] = 0$ ,  $[\sigma, \phi(f_0)] = -2r\phi(f_0)$  and any eigenvalue  $\lambda$  of  $\text{ad}(\sigma - r\phi(h_0)) : \mathfrak{g} \rightarrow \mathfrak{g}$  satisfies  $\tau(\lambda) = 0$ .*

*If these conditions are satisfied, then  $E_{y, \sigma, r, \rho} = \bar{E}_{y, \sigma, r, \rho}$ .*

The proof is given in 3.43.

**Theorem 1.22.** *Assume that  $r \neq 0$ . Let  $y, \sigma, \rho$  be as in 1.15(b). The following five conditions are equivalent:*

- (i)  $y, \sigma$  are not contained in a Levi subalgebra of a proper parabolic subalgebra of  $\mathfrak{g}$ .
- (ii) There exists a homomorphism of Lie algebras  $\phi : \mathfrak{sl}_2(\mathbf{C}) \rightarrow \mathfrak{g}$  such that  $y = \phi(e_0), \sigma = r\phi(h_0)$ ; moreover,  $y$  is distinguished.
- (iii)  $\bar{E}_{y, \sigma, r, \rho}$  is  $\tau$ -square integrable.
- (iv)  $E_{y, \sigma, r, \rho}$  is  $\tau$ -square integrable.
- (v) For any  $V \in \mathcal{I}$ , other than  $\mathbf{C}$ , any eigenvalue of  $r^{-1}\underline{\xi}_V$  on  $E_{y, \sigma, r, \rho}$  is an integer  $\geq 1$ .

If these conditions are satisfied, then  $E_{y, \sigma, r, \rho} = \bar{E}_{y, \sigma, r, \rho}$ .

The proof is given in 3.44.

## 2. A STRONG INDUCTION THEOREM AND A PROOF OF THEOREM 1.17

**2.1.** In this section we place ourselves in the setup of 1.16. Thus,

$$J, Q, Q^1, \mathbf{H}', y, P', P'_y, P^*, P_y^*, C$$

are defined. We set  $\mathfrak{q} = Q, \mathfrak{q}^1 = Q^1, \mathfrak{n} = U_Q$ . We can find a Lie algebra homomorphism  $\phi : \mathfrak{sl}_2(\mathbf{C}) \rightarrow \mathfrak{q}^1$  such that  $\phi(e_0) = y$  and such that  $C$  is a maximal torus of

$$\{(g, \lambda) \in Q^1 \times \mathbf{C}^*; \text{Ad}(g)\phi(e_0) = \lambda^2\phi(e_0), \text{Ad}(g)\phi(f_0) = \lambda^{-2}\phi(f_0)\}.$$

For any  $P \in \mathcal{P}$  we set

$$P^! = (P \cap Q)U_Q$$

(a parabolic subgroup of  $Q$ ).

Let  $f : \mathfrak{q} \rightarrow \mathfrak{q}^1$  be the projection of  $\mathfrak{q} = \mathfrak{q}^1 \oplus \mathfrak{n}$  onto  $\mathfrak{q}^1$ . Let

$$W_* = \{w \in W; w \text{ has minimal length in } wW_J\}.$$

There are only finitely many orbits for the conjugation action of  $Q$  on  $\mathcal{P}$ . A  $Q$ -orbit  $\mathcal{O}$  on  $\mathcal{P}$  is said to be *good* if any  $P \in \mathcal{O}$  has some Levi subgroup that is contained in  $Q$ . For such  $\mathcal{O}$  and for  $P \in \mathcal{O}$  we have  $P^! \in \mathcal{P}^*$  and the  $G$ -orbit of  $(P, P^!)$  in  $\mathcal{P} \times \mathcal{P}$  is good (see 1.5) and indexed by an element  $w \in W_*$ . Moreover,  $\mathcal{O} \mapsto w$  is a well defined bijection between the set of good  $Q$ -orbits on  $\mathcal{P}$  and  $W_*$ . We denote by  $\mathfrak{o}(w)$  the  $Q$ -orbit on  $\mathcal{P}$  corresponding to  $w \in W_*$ . Note that  $P \mapsto P^!$  is a  $Q$ -equivariant morphism  $\mathfrak{o}(w) \rightarrow \mathcal{P}^*$ .

If  $X$  is a subvariety of  $\mathfrak{g}$ , then  $\dot{X} = \{(z, P) \in \dot{\mathfrak{g}}; z \in X\}$  (see 1.9) is a subvariety of  $\dot{\mathfrak{g}}$ . For any subvariety  $F$  on  $\mathcal{P}$  we set

$$\dot{X}_F = \{(z, P) \in \dot{X}; P \in F\}.$$

For  $w \in W_*$  we will often write  $\dot{X}_w$  instead of  $\dot{X}_{\mathfrak{o}(w)}$ . Define

$$\tilde{f} : \dot{\mathfrak{q}}_w \rightarrow (q^1)_1 = \dot{\mathfrak{q}}_1^1, (z, P) \mapsto (f(z), P^!).$$

Let  $S = Z_{Q^1}^0$ . Let  $w \in W_*$ . Let  $\mathfrak{o}(w)^S$  be the fixed point set of the conjugation action of  $S$  on  $\mathfrak{o}(w)$ . The properties (a) and (b) below are easily verified.

- (a) The map  $\mathfrak{o}(w)^S \rightarrow \mathcal{P}^*$  given by  $P \mapsto P^!$  is an isomorphism.
- (b) Let  $z \in \mathfrak{q}$  and let  $P \in \mathfrak{o}(w)^S$ . Then we have  $(z, P) \in \dot{\mathfrak{g}}$  if and only if  $(z, P^!) \in \dot{\mathfrak{g}}$ .

The fixed point set of the  $S$ -action on  $\dot{\mathfrak{q}}_w$  (conjugation on both factors) is  $\dot{\mathfrak{q}}_w^1$ . The restriction of  $\tilde{f}$  defines an isomorphism  $\dot{\mathfrak{q}}_w^1 \xrightarrow{\sim} \dot{\mathfrak{q}}_1^1$ .

We choose a homomorphism of algebraic groups  $\chi : \mathbf{C}^* \rightarrow S$  such that  $\lambda \mapsto \text{Ad}(\chi(\lambda))$  has weights  $> 0$  on  $\mathfrak{n}$ . We define a  $\mathbf{C}^*$ -action on  $\dot{\mathfrak{q}}_w$  by

$$\lambda : (z, P) \mapsto (\text{Ad}(\chi(\lambda))z, \chi(\lambda)P\chi(\lambda)^{-1}).$$

Then  $\tilde{f} : \dot{\mathfrak{q}}_w \rightarrow \dot{\mathfrak{q}}_1^1$  is  $\mathbf{C}^*$ -equivariant where  $\mathbf{C}^*$  acts on  $\dot{\mathfrak{q}}_1^1$  trivially. Let  $n = \dim \bar{P} - \dim \mathfrak{c}_P - \dim Z_{\bar{P}}$  for any  $P \in \mathcal{P}$ .

**Lemma 2.2.** (a)  $\dot{\mathfrak{q}}_w$  is a smooth variety of pure dimension  $\dim Q - n$ .

(b) Let  $(z', P') \in \dot{\mathfrak{q}}_w$ . Define  $(z, P) \in \dot{\mathfrak{q}}_w^1$  by  $\tilde{f}(z', P') = \tilde{f}(z, P)$ . Then  $\lim_{\lambda \rightarrow 0} \lambda(z', P')$  exists in  $\dot{\mathfrak{q}}_w$  and equals  $(z, P)$ .

(c) The fixed point set of the  $\mathbf{C}^*$ -action on  $\dot{\mathfrak{q}}_w$  coincides with the fixed point set of the  $S$ -action on  $\dot{\mathfrak{q}}_w$ .

Consider the fibration  $pr_2 : \dot{\mathfrak{q}}_w \rightarrow \mathfrak{o}(w)$ . Let  $P \in \mathfrak{o}(w)$ . Let  $P^1$  be a Levi subgroup of  $P$  that is contained in  $Q$ . Let  $c^1$  be the nilpotent orbit in  $\underline{P}^1$  corresponding to  $\mathfrak{c}_P$  under  $P^1 \xrightarrow{\sim} \bar{P}$ . Then  $pr_2^{-1}(P)$  may be identified with

$$(c^1 + Z_{P^1} + \underline{U}_P) \cap \mathfrak{q} = c^1 + Z_{P^1} + (\underline{U}_P \cap \mathfrak{q})$$

and this is smooth irreducible of dimension

$$-n + \dim P^1 + \dim(U_P \cap Q) = -n + \dim(P \cap Q).$$

Now  $\mathfrak{o}(w)$  is smooth, irreducible of dimension  $\dim Q - \dim(P \cap Q)$  and (a) follows.

We prove (b). We have  $z' = z + x$  with  $x \in \mathfrak{n}$ . We have  $P'^1 = P^1$  and  $P', P$  are in  $\mathfrak{o}(w)$ , hence  $P' = uPu^{-1}$  for some  $u \in U_Q$ . Thus,

$$\begin{aligned} \lambda(z', P') &= (\text{Ad}(\chi(\lambda))(z + x), \chi(\lambda)uPu^{-1}\chi(\lambda)^{-1}) \\ &= (z + \text{Ad}(\chi(\lambda))x, \chi(\lambda)u\chi(\lambda^{-1})P\chi(\lambda)u^{-1}\chi(\lambda)^{-1}). \end{aligned}$$

Now  $\lim_{\lambda \rightarrow 0} \chi(\lambda)u\chi(\lambda^{-1}) = 1$  and  $\lim_{\lambda \rightarrow 0} \text{Ad}(\chi(\lambda))x = 0$  by the choice of  $\chi$ . Hence  $\lim_{\lambda \rightarrow 0} \lambda(z', P') = (z, P)$ . This proves (b). Now (c) follows immediately from (b).

**2.3.** Let  $A = y + \mathfrak{z}_{\mathfrak{q}}(\phi(f_0))$ ,  $A^1 = A \cap \mathfrak{q}^1$ .

**Lemma 2.4.** Let  $w \in W_*$ .

(a)  $\dot{A}_w$  is smooth of pure dimension  $-n + \dim \mathfrak{z}_{\mathfrak{q}}(\phi(f_0))$ .

(b)  $\dot{A}_1^1$  is smooth of pure dimension  $-n + \dim \mathfrak{z}_{\mathfrak{q}^1}(\phi(f_0))$ .

(c) The map  $\dot{A}_w \rightarrow \dot{A}_1^1$ ,  $(z, P) \mapsto (f(z), P^1)$  is an affine space bundle with fibres of dimension  $\dim \mathfrak{z}_{\mathfrak{n}}(\phi(f_0))$ .

By [L9, 6.9],

(d) the map  $Q \times A \rightarrow \mathfrak{q}$  (given by the adjoint action of  $Q$ ) is smooth with fibres of pure dimension  $\dim \mathfrak{z}_{\mathfrak{q}}(\phi(f_0))$ .

We have a cartesian diagram

$$\begin{array}{ccc} \dot{A}_w & \xrightarrow{pr_1} & A \\ \downarrow & & \downarrow \\ \dot{\mathfrak{q}}_w & \xrightarrow{pr_1} & \mathfrak{q} \end{array}$$

where the vertical maps are the inclusions. This induces a cartesian diagram

$$\begin{array}{ccc} Q \times \dot{A}_w & \longrightarrow & Q \times A \\ \downarrow & & \downarrow \\ \dot{\mathfrak{q}}_w & \longrightarrow & \mathfrak{q} \end{array}$$

where the vertical maps are given by the obvious action of  $Q$ . Using (d), it follows that  $Q \times \dot{A}_w \rightarrow \dot{\mathfrak{q}}_w$  (adjoint action) is smooth with fibres of pure dimension  $\dim \mathfrak{z}_{\mathfrak{q}}(\phi(f_0))$ . Since  $\dot{\mathfrak{q}}_w$  is smooth of pure dimension  $\dim Q - n$  (see 2.2(a)), it follows that  $Q \times \dot{A}_w$  is smooth of pure dimension  $\dim Q - n + \dim \mathfrak{z}_{\mathfrak{q}}(\phi(f_0))$ . Hence  $\dot{A}_w$  is as in (a). Now  $\dot{A}_1^1$  is the same as  $\dot{A}_1$  (where  $\mathfrak{g}, \mathfrak{q}$  are replaced by  $\mathfrak{q}^1, \mathfrak{q}^1$ ). Hence (b) follows from (a).

We prove (c). Note that  $\dot{A}_w$  is a closed subset of  $\dot{\mathfrak{q}}_w$ , stable under the  $S$ -action (hence under the  $\mathbf{C}^*$ -action) on  $\dot{\mathfrak{q}}_w$  (as in 2.1). By 2.2(c), the fixed point set of the  $\mathbf{C}^*$ -action on  $\dot{A}_w$  is the same as the fixed point set of the  $S$ -action of  $\dot{A}_w$ ; that is,

$$\dot{A}_w^{\mathbf{C}^*} = \{(z, P) \in \dot{\mathfrak{g}}; z \in A^1, P \in \mathfrak{o}(w)^S\}$$

and the map  $\dot{A}_w \rightarrow \dot{A}_1^1$  (restriction of  $\tilde{f} : \dot{\mathfrak{q}}_w \rightarrow \dot{\mathfrak{q}}_1^1$ ) restricts to an isomorphism  $\dot{A}_w^{\mathbf{C}^*} \xrightarrow{\sim} \dot{A}_1^1$ . Using 2.2(b) and the fact that  $\dot{A}_w$  is closed in  $\dot{\mathfrak{q}}_w$ , we see that

(e) for any  $(z', P') \in \dot{A}_w$ ,  $\lim_{\lambda \rightarrow 0} \lambda(z', P')$  exists in  $\dot{A}_w$  and belongs to  $\dot{A}_w^{\mathbf{C}^*}$ .

Let  $Z$  be a connected component of  $\dot{A}_w$ . Then  $Z^{\mathbf{C}^*} = Z \cap \dot{A}_w^{\mathbf{C}^*}$  may be identified with  $\tilde{f}(Z)$ , hence is connected (a connected component of  $\dot{A}_w^{\mathbf{C}^*} \cong \dot{A}_1^1$ ). Let  $Z'$  be a smooth  $\mathbf{C}^*$ -equivariant projective compactification of  $Z$ . We have  $Z^{\mathbf{C}^*} \subset Z'^{\mathbf{C}^*}$ ; let  $Z'_1$  be the connected component of  $Z'^{\mathbf{C}^*}$  that contains  $Z^{\mathbf{C}^*}$ . Let  $Z'_2$  be the set of all  $x \in Z'$  such that  $\lim_{\lambda \rightarrow 0} \lambda x$  exists in  $Z'$  and belongs to  $Z'_1$ . From (e) we see that  $Z \subset Z'_2$ ; hence  $Z'_2$  is dense in  $Z'$ . By a known result of Bialynicki-Birula,  $Z'_2$  is locally closed in  $Z'$  (hence open) and the map  $\tilde{f}' : Z'_2 \rightarrow Z'_1$  given by  $x \mapsto \lim_{\lambda \rightarrow 0} \lambda x$  is an affine space bundle. Let  $x \in Z^{\mathbf{C}^*}$ ,  $Z^x = \{x' \in Z; \tilde{f}(x') = x\}$ . We have

$$\dim \tilde{f}'^{-1}(x) = \dim Z'_2 - \dim Z'_1 = \dim Z - \dim Z'_1 \leq \dim Z - \dim Z^{\mathbf{C}^*} \leq \dim Z^x.$$

Since  $Z^x \subset \tilde{f}'^{-1}(x)$  and  $\tilde{f}'^{-1}(x)$  is irreducible, we see that we must have  $Z^x = \tilde{f}'^{-1}(x)$  and  $\dim Z^{\mathbf{C}^*} = \dim Z'_1$ . Thus we have a cartesian diagram

$$\begin{array}{ccc} Z & \longrightarrow & Z'_2 \\ \downarrow & & \downarrow \\ Z^{\mathbf{C}^*} & \longrightarrow & Z'_1 \end{array}$$

(the horizontal maps are inclusions and the vertical maps are  $\tilde{f}, \tilde{f}'$ ). It follows that  $\tilde{f} : Z \rightarrow Z^{\mathbf{C}^*}$  is an affine space bundle with fibres of dimension

$$\begin{aligned} \dim Z'_2 - \dim Z'_1 &= \dim Z - \dim Z^{\mathbf{C}^*} = \dim \dot{A}_w - \dim \dot{A}_w^{\mathbf{C}^*} \\ &= -n + \dim \mathfrak{z}_{\mathfrak{q}}(\phi(f_0)) - (-n + \dim \mathfrak{z}_{\mathfrak{q}^1}(\phi(f_0))) = \dim \mathfrak{z}_{\mathfrak{n}}(\phi(f_0)). \end{aligned}$$

(c) follows. The lemma is proved.

2.5. Let

$$\begin{aligned}\mathcal{A} &= y + \mathfrak{z}_n(\phi(f_0)), & \mathcal{A}^1 &= \mathcal{A} \cap \mathfrak{q}^1 = \{y\}, \\ \mathcal{A}' &= y + \mathfrak{z}_n(\phi(f_0)) + \mathfrak{t}, & \mathcal{A}'^1 &= \mathcal{A}' \cap \mathfrak{q}^1 = y + \mathfrak{t}, \\ \mathcal{A}'' &= y + \mathfrak{z}_n(\phi(f_0)) + \mathfrak{t}_r, & \mathcal{A}''^1 &= \mathcal{A}'' \cap \mathfrak{q}^1 = y + \mathfrak{t}_r,\end{aligned}$$

where  $\mathfrak{t} = \mathfrak{z}_{\mathfrak{q}^1}$  and  $\mathfrak{t}_r = \{x \in \mathfrak{t}; \mathfrak{z}_{\mathfrak{g}}(x) = \mathfrak{q}^1\}$ . We have  $\mathcal{A} \subset \mathcal{A}' \supset \mathcal{A}''$ ,  $\mathcal{A}^1 \subset \mathcal{A}'^1 \supset \mathcal{A}''^1$ . Hence  $\dot{\mathcal{A}} \subset \dot{\mathcal{A}}' \supset \dot{\mathcal{A}}''$ ,  $\dot{\mathcal{A}}_1^1 \subset \dot{\mathcal{A}}_1'^1 \supset \dot{\mathcal{A}}_1''^1$ . We have

$$(a) \quad \dot{\mathcal{A}}_1^1 = \{y\} \times \mathcal{P}_y^*, \quad \dot{\mathcal{A}}_1'^1 = (y + \mathfrak{t}) \times \mathcal{P}_y^*, \quad \dot{\mathcal{A}}_1''^1 = (y + \mathfrak{t}_r) \times \mathcal{P}_y^*.$$

From the definitions we see that, for  $w \in W_*$ , we have cartesian diagrams

$$\begin{array}{ccc} \dot{\mathcal{A}}_w & \longrightarrow & \dot{\mathcal{A}} \\ \downarrow & & \downarrow \\ \dot{\mathcal{A}}_1^1 & \longrightarrow & \dot{\mathcal{A}}_1^1 \\ \dot{\mathcal{A}}'_w & \longrightarrow & \dot{\mathcal{A}} \\ \downarrow & & \downarrow \\ \dot{\mathcal{A}}_1'^1 & \longrightarrow & \dot{\mathcal{A}}_1^1 \\ \dot{\mathcal{A}}''_w & \longrightarrow & \dot{\mathcal{A}} \\ \downarrow & & \downarrow \\ \dot{\mathcal{A}}_1''^1 & \longrightarrow & \dot{\mathcal{A}}_1^1 \end{array}$$

where the horizontal maps are the obvious inclusions and the vertical maps are defined by  $(z, P) \mapsto (f(z), P^1)$ . Using this and 2.4(c) we see that

(b) *the maps*

$$\dot{\mathcal{A}}_w \rightarrow \dot{\mathcal{A}}_1^1, \quad \dot{\mathcal{A}}'_w \rightarrow \dot{\mathcal{A}}_1'^1, \quad \dot{\mathcal{A}}''_w \rightarrow \dot{\mathcal{A}}_1''^1$$

defined by  $(z, P) \mapsto (f(z), P^1)$  are affine space bundles with fibres of dimension  $\dim \mathfrak{z}_n(\phi(f_0))$ .

**Lemma 2.6.** *Let  $E$  be a connected algebraic group, let  $U$  be a closed normal unipotent subgroup of  $E$  and let  $\mathcal{T}$  be a torus in  $E$ . Let  $e, h, e'$  be elements of  $\underline{E}$  such that  $[h, e] = 2e$ ,  $[h, e'] = -2e'$ ,  $[e, e'] = h$  and such that  $[d, e] = [d, e'] = [d, h] = 0$  for any  $d \in \underline{\mathcal{T}}$ . Let  $\mathfrak{u} = \underline{U}$ . Define  $\Phi : U \times \mathfrak{z}_{\mathfrak{u}}(e') \times \underline{\mathcal{T}} \rightarrow \underline{\mathcal{T}} \times \mathfrak{u}$  by  $\Phi(u, x, d) = (d, \text{Ad}(u)(e + x + d) - e - d)$ . Then  $\Phi$  is an affine space bundle with fibres isomorphic to  $\mathfrak{z}_{\mathfrak{u}}(e)$ .*

Let  $Z = Z_U$ . We show that there exists a morphism  $d \mapsto L'_d, \underline{\mathcal{T}} \rightarrow \text{Hom}(\underline{Z}, \underline{Z})$  and a morphism  $d \mapsto L''_d, \underline{\mathcal{T}} \rightarrow \text{Hom}(\underline{Z}, \ker(\text{ad}(e') : \underline{Z} \rightarrow \underline{Z}))$  such that

$$(a) \quad z = [L'_d(z), e + d] + L''_d(z)$$

for any  $z \in \underline{Z}, d \in \underline{\mathcal{T}}$ . We can find a direct sum decomposition  $\underline{Z} = \bigoplus_{k=1}^N V_k$  where  $V_k$  are vector subspaces of  $\underline{Z}$  and linear forms  $a_1, \dots, a_N$  on  $\underline{\mathcal{T}}$  such that for any  $k$ ,  $V_k$  is a simple  $\mathfrak{sl}_2(\mathbf{C})$ -submodule of  $\underline{Z}$  under  $\text{ad}(e), \text{ad}(h), \text{ad}(e')$  and  $[d, v] = a_k(d)v$  for any  $d \in \underline{\mathcal{T}}, v \in V_k$ .

Let  $b_{0,k}, b_{1,k}, \dots, b_{n_k,k}$  be a basis of  $V_k$  such that

$$\text{ad}(e')b_{0,k} = 0, \text{ad}(e)b_{0,k} = -b_{1,k}, \text{ad}(e)b_{1,k} = -b_{2,k}, \dots, \text{ad}(e)b_{n_k-1,k} = -b_{n_k,k}.$$

For  $s < 0$  we set  $b_{s,k} = 0$ . For  $s \in [0, n_k]$  we set

$$\begin{aligned} L'_d(b_{s,k}) &= b_{s-1,k} + a_k(d)b_{s-2,k} + a_k(d)^2b_{s-3,k} + \cdots + a_k(d)^{s-1}b_{0,k}, \\ L'_d(b_{s,k}) &= a_k(d)^s b_{0,k}. \end{aligned}$$

This defines uniquely  $L'_d, L''_d$ . It is clear that (a) holds.

Next we construct for any  $d \in \underline{\mathcal{T}}$  an isomorphism,

$$(b) \quad p : \{(z', z) \in \underline{\mathcal{Z}} \times \mathfrak{z}_{\underline{\mathcal{Z}}}(e'); [z', e + d] + z = 0\} \xrightarrow{\sim} \mathfrak{z}_{\underline{\mathcal{Z}}}(e).$$

This is by definition a direct sum over  $k$  of isomorphisms

$$\begin{aligned} p_k : \{(v', v) \in V_k \times \ker(\text{ad}(e') : V_k \rightarrow V_k); -\text{ad}(e + d)(v') + v = 0\} \\ \xrightarrow{\sim} \ker(\text{ad}(e') : V_k \rightarrow V_k) \end{aligned}$$

given by  $p_k(c_0b_{0,k} + c_1b_{1,k} + \cdots + c_{n_k}b_{n_k,k}, c'b_0) = c_{n-k}b_{n-k,k}$ .

We prove the lemma by induction on  $\dim U$ . If  $\dim U = 0$ , the result is trivial. Hence we may assume that  $\dim U > 0$  and that the result is true when  $E, U$  are replaced by  $\bar{E} = E/Z, \bar{U} = U/Z$ ,  $\mathcal{T}$  is replaced by the image  $\bar{\mathcal{T}}$  of  $\mathcal{T}$  in  $\bar{E}$  and  $e, h, e'$  are replaced by their images  $\bar{e}, \bar{h}, \bar{e}'$  under  $\underline{E} \rightarrow \bar{E}$ . (We have  $\dim Z > 0$ .) Let  $\bar{u} = \underline{\bar{U}}$ . The obvious map  $u \rightarrow \bar{u}$  may be regarded as a surjective map of  $\mathfrak{sl}_2(\mathbf{C})$ -modules; by the complete reducibility of such modules, this map admits a cross section as an  $\mathfrak{sl}_2(\mathbf{C})$ -module; in particular, the induced map  $\mathfrak{z}_u(e') \rightarrow \mathfrak{z}_u(\bar{e}')$  is surjective. Let  $\bar{x} \mapsto \bar{\tilde{x}}$  be a linear cross section for this last linear map. Let  $\bar{u} \mapsto \bar{\tilde{u}}$  be an algebraic cross section  $\bar{U} \rightarrow U$  for the obvious map  $U \rightarrow \bar{U}$ .

Let  $(d, \xi) \in \underline{\mathcal{T}} \times u$ ; let  $\bar{\xi}$  be the image of  $\xi$  under  $u \rightarrow \bar{u}$  and let  $\bar{d}$  be the image of  $d$  under the isomorphism  $\mathcal{T} \xrightarrow{\sim} \bar{\mathcal{T}}$  induced by  $E \rightarrow \bar{E}$ . We have

$$\Phi^{-1}(d, \xi) \cong \{(u, x) \in U \times \mathfrak{z}_u(e'); \text{Ad}(u)(e + x + d) = e + d + \xi\}.$$

By the induction hypothesis,

$$X =: \{(\bar{u}, \bar{x}) \in \bar{U} \times \mathfrak{z}_{\bar{u}}(\bar{e}'); \text{Ad}(\bar{u})(\bar{e} + \bar{x} + \bar{d}) = \bar{e} + \bar{d} + \bar{\xi}\}$$

is an affine space isomorphic to  $\mathfrak{z}_{\bar{u}}(\bar{e})$ . We have an obvious map  $\Psi : \Phi^{-1}(\xi) \rightarrow X$ . Its fibre at  $(\bar{u}, \bar{x}) \in X$  is the set of all  $(u, x) \in U \times \mathfrak{z}_u(e') \times \mathcal{T}$  such that  $\text{Ad}(u)(e + x + d) = e + d + \xi$  and  $u = \bar{u}\zeta, x = \bar{\tilde{x}} + z$  for some  $\zeta \in Z, z \in \mathfrak{z}_{\underline{\mathcal{Z}}}(f)$ . Note that  $\text{Ad}(\bar{\tilde{u}})(\bar{e} + \bar{\tilde{x}} + \bar{d}) = \bar{e} + \bar{d} + \bar{\xi} + z_0$  for some  $z_0 \in \underline{\mathcal{Z}}$ . The equation  $\text{Ad}(u)(e + x + d) = e + d + \xi$  can be written as  $\text{Ad}(\bar{u}\zeta)(e + \bar{\tilde{x}} + z + d) = e + d + \xi$ , or as  $\text{Ad}(\zeta)(e + d + \xi + z_0) + z = e + d + \xi$ , or as  $\text{Ad}(\zeta)(e + d) + z = e + d - z_0$ . Setting  $\zeta = \exp(z'), z' \in \underline{\mathcal{Z}}$ , we see that the fibre of  $\Psi$  at  $(\bar{u}, \bar{x})$  may be identified with

$$(c) \quad \{(z', z) \in \underline{\mathcal{Z}} \times \mathfrak{z}_{\underline{\mathcal{Z}}}(e'); \text{Ad}(\exp(z'))(e + d) + z = e + d - z_0\}.$$

Since  $[\underline{\mathcal{Z}}, e + d] \in \underline{\mathcal{Z}}$ , we have  $[z', [z', e + d]] = 0$  for  $z' \in \underline{\mathcal{Z}}$  and (c) becomes

$$\{(z', z) \in \underline{\mathcal{Z}} \times \mathfrak{z}_{\underline{\mathcal{Z}}}(e'); [z', e + d] + z = -z_0\}$$

or

$$\{(z', z) \in \underline{\mathcal{Z}} \times \mathfrak{z}_{\underline{\mathcal{Z}}}(e'); [z', e + d] + z + [L'_d(z_0), e + d] + L''_d(z_0) = 0\}.$$

By the substitution  $\tilde{z}' = z' + L'_d(z_0), \tilde{z} = z + L''_d(z_0)$ , this becomes

$$\{(\tilde{z}', \tilde{z}) \in \underline{\mathcal{Z}} \times \mathfrak{z}_{\underline{\mathcal{Z}}}(e'); [\tilde{z}', e + d + \tilde{z}] = 0\}.$$

By the isomorphism (b) this is identified with the vector space  $\mathfrak{z}_{\underline{\mathcal{Z}}}(e)$ . We see that  $\Phi^{-1}(\xi)$  is a vector bundle over  $X$ . Since  $X$  is an affine space, this vector bundle must be trivial (Quillen-Suslin) and therefore  $\Phi^{-1}(\xi)$  is itself an affine space isomorphic to  $\mathfrak{z}_{\underline{\mathcal{Z}}}(e) \times \mathfrak{z}_{\bar{u}}(\bar{e}) \cong \mathfrak{z}_u(e)$ . The lemma is proved.

**Lemma 2.7.** *Let  $\mathfrak{t}'$  be the Lie algebra of a torus contained in  $Z_{Q^1}$ . Let  $X = y + \mathfrak{z}_n(\phi(f_0)) + \mathfrak{t}'$ . Let  $\mathcal{O}$  be a  $Q$ -orbit on  $\mathcal{P}$ . Assume that  $\mathcal{O}$  is not good. Then  $H_c^i(\dot{X}_{\mathcal{O}}, \dot{\mathcal{L}}^*) = 0$  for any  $j \in \mathbf{Z}$ .*

In this proof all local systems are deduced from  $\mathcal{L}^*$  and we omit them from the notation.

The assignment  $P \mapsto P^\dagger$  defines a morphism  $\pi : \mathcal{O} \rightarrow \mathcal{P}'$  where  $\mathcal{P}'$  is a conjugacy class of parabolic subgroups of  $Q$ . The fibres of  $\pi$  are exactly the  $U_Q$ -orbits on  $\mathcal{O}$ . It is enough to show that  $H_c^i(\dot{X}_F, ) = 0$  for any fibre  $F$  of  $\pi$  and any  $i \in \mathbf{Z}$ , where  $\dot{X}_F = \{(z, P) \in \dot{X}; P \in F\}$  (see 2.1). We fix  $P \in F$ ; let  $\mathfrak{p} = \underline{P}$ . Let  $\mathcal{Y} = \{(z, u) \in \mathfrak{g} \times U_Q; (Ad(u)z, P) \in \dot{\mathfrak{g}}, z \in X\}$ . Define  $\mathcal{Y} \rightarrow \dot{X}_F$  by  $(z, u) \mapsto (z, u^{-1}Pu)$ . This is a fibration with fibres isomorphic to  $U_Q \cap P$ . It is then enough to show that  $H_c^i(\mathcal{Y}, ) = 0$  for any  $i \in \mathbf{Z}$ . Setting  $z - y = x + d$  where  $x \in \mathfrak{z}_n(\phi(f_0))$ ,  $d \in \mathfrak{t}'$  and  $\tilde{\mathbf{c}}_P = \pi_P^{-1}(\mathbf{c}_P + \underline{Z_P})$ , we identify  $\mathcal{Y}$  with

$$\{(x, d, u) \in \mathfrak{z}_n(\phi(f_0)) \times \mathfrak{t}' \times U_Q; Ad(u)(y + x + d) \in \tilde{\mathbf{c}}_P\}.$$

This maps to

$$\mathcal{Y}' = \{(d, \nu) \in \mathfrak{t}' \times \mathfrak{n}; y + d + \nu \in \tilde{\mathbf{c}}_P\}$$

by  $(x, d, u) \mapsto \nu = (d, Ad(u)(y + x + d) - y - d)$ ; this is an affine space bundle by 2.6 applied to  $E = Q, U = U_Q, \underline{\mathcal{I}} = \mathfrak{t}', e = y, h = \phi(h_0), e' = \phi(f_0)$ . Hence it is enough to show that  $H_c^i(\mathcal{Y}', ) = 0$ . For any  $d \in \mathfrak{t}'$  let

$$\mathcal{Y}'_d = \{\nu \in \mathfrak{n}; y + d + \nu \in \tilde{\mathbf{c}}_P\}$$

be the fibre at  $d$  of  $pr_1 : \mathcal{Y}' \rightarrow \mathfrak{t}'$ . By the Leray spectral sequence for  $pr_1$ , it is enough to show that  $H_c^i(\mathcal{Y}'_d, ) = 0$  for any  $d \in \mathfrak{t}'$ . If  $y + d \notin \mathfrak{n} + \mathfrak{p}$ , then  $\mathcal{Y}'_d = \emptyset$  and there is nothing to prove. Thus we may assume that  $y + d + \nu_0 \in \mathfrak{p}$  for some  $\nu_0 \in \mathfrak{n}$ . Setting  $\xi = y + d + \nu_0 \in \mathfrak{p} \cap \mathfrak{q}$ ,  $\nu' = \nu - \nu_0$ , we may identify  $\mathcal{Y}'_d$  with

$$\tilde{\mathcal{Y}}' = \{\nu' \in \mathfrak{n}; \xi + \nu' \in \tilde{\mathbf{c}}_P\} = \{\nu' \in \mathfrak{n} \cap \mathfrak{p}; \xi + \nu' \in \tilde{\mathbf{c}}_P\}.$$

Let  $R = (P \cap Q)U_P$  (a proper parabolic subgroup of  $P$  since  $\mathcal{O}$  is not good). Let  $\bar{R}$  be the image of  $R$  under  $P \mapsto \bar{P}$  (a proper parabolic subgroup of  $\bar{P}$ ). The nil-radical of  $\bar{R}$  is  $\mathfrak{n} \cap \mathfrak{p} + \underline{U_P}$ . Hence the nil-radical of  $\bar{R}$  is

$$\mathfrak{n}_1 = (\mathfrak{n} \cap \mathfrak{p} + \underline{U_P})/\underline{U_P} = (\mathfrak{n} \cap \mathfrak{p})/(\mathfrak{n} \cap \underline{U_P}).$$

Let  $k : \mathfrak{n} \cap \mathfrak{p} \rightarrow \mathfrak{n}_1$  be the canonical map. Let  $\xi_1$  be the image of  $\xi$  under  $\mathfrak{p} \rightarrow \bar{P}$ . We have  $\xi \in \mathfrak{q}$ , hence  $\xi_1 \in \bar{R}$ . Let

$$\mathcal{Y}'' = \{\mu \in \mathfrak{n}_1; \xi_1 + \mu \in \mathbf{c}_P + \underline{Z_P}\}.$$

We have a cartesian diagram

$$\begin{array}{ccc} \tilde{\mathcal{Y}}' & \longrightarrow & \mathfrak{n} \\ k' \downarrow & & \downarrow k \\ \mathcal{Y}'' & \longrightarrow & \mathfrak{n}_1 \end{array}$$

where the horizontal maps are the inclusions and  $k'$  is induced by  $k$ . Since  $k$  is an affine space bundle with fibres isomorphic to  $\mathfrak{n} \cap \underline{U_P}$ , so is  $k'$ . It is therefore enough to show that  $H_c^i(\mathcal{Y}'', ) = 0$ . This follows directly from the fact that  $\mathcal{L}^*$  is a cuspidal local system since  $\mathfrak{n}_1$  is the nil-radical of the proper parabolic algebra  $\bar{R}$  of  $\bar{P}$  and  $\xi_1 \in \bar{R}$ . The lemma is proved.

**Lemma 2.8.** *Let  $\delta = \dim \mathfrak{z}_n(\phi(f_0)), b = \dim \mathfrak{t}$ .*

- (a) We have  $H_c^j(\dot{\mathcal{A}}', \dot{\mathcal{L}}^*) = 0, H_c^j(\dot{\mathcal{A}}_1', \dot{\mathcal{L}}^*) = 0$  for odd  $j$ . For any  $j$  we have  $\dim H_c^j(\dot{\mathcal{A}}', \dot{\mathcal{L}}^*) = \sharp(W_*) \dim H_c^{j-2\delta-2b}(\mathcal{P}_y^*, \dot{\mathcal{L}}^*)$ .
- (b) We have  $H_c^j(\dot{\mathcal{A}}, \dot{\mathcal{L}}^*) = 0, H_c^j(\dot{\mathcal{A}}_1, \dot{\mathcal{L}}^*) = 0$  for odd  $j$ . For any  $j$  we have  $\dim H_c^j(\dot{\mathcal{A}}, \dot{\mathcal{L}}^*) = \sharp(W_*) \dim H_c^{j-2\delta}(\mathcal{P}_y^*, \dot{\mathcal{L}}^*)$ .
- (c) Let  $D = \dim \dot{\mathcal{A}}, D' = \dim \dot{\mathcal{A}}'$ . For any  $j$  we have

$$\dim H_j^C(\dot{\mathcal{A}}', \dot{\mathcal{L}}) = \dim H_{j+2D-2D'+2b}^C(\dot{\mathcal{A}}, \dot{\mathcal{L}}).$$

We can arrange the  $Q$ -orbits on  $\mathcal{P}$  in a sequence  $\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_n$  so that  $R_m = \mathcal{O}_1 \cup \mathcal{O}_2 \cup \dots \cup \mathcal{O}_m$  is closed in  $\mathcal{P}$  for any  $m \in [1, n]$ . We set  $R_0 = \emptyset$ . For any  $m$  we have

$$(d) \quad H_c^j(\dot{\mathcal{A}}'_{\mathcal{O}_m}, \dot{\mathcal{L}}^*) = 0 \text{ for odd } j.$$

(Indeed, if  $\mathcal{O}_m$  is not good, this follows from 2.7; if  $\mathcal{O}_m$  is good, then, using 2.5(a),(b), we see that it is enough to show that  $H_c^j(\mathcal{P}_y^*, \dot{\mathcal{L}}^*) = 0$  for odd  $j$ . This follows from [L4, 8.6].) Using induction on  $m$  we deduce that

$$(e) \quad H_c^j(\dot{\mathcal{A}}'_{R_m}, \dot{\mathcal{L}}^*) = 0 \text{ for odd } j$$

for any  $m$ . Taking  $m = n$  we obtain the first sentence of (a). For  $m \in [1, n]$  we have a cohomology exact sequence

$$(f) \quad 0 \rightarrow H_c^j(\dot{\mathcal{A}}'_{\mathcal{O}_m}, \dot{\mathcal{L}}^*) \rightarrow H_c^j(\dot{\mathcal{A}}'_{R_m}, \dot{\mathcal{L}}^*) \rightarrow H_c^j(\dot{\mathcal{A}}'_{R_{m-1}}, \dot{\mathcal{L}}^*) \rightarrow 0$$

(we use (c) and (d)). Using induction on  $m$  it follows that

$$\dim H_c^j(\dot{\mathcal{A}}'_{R_m}, \dot{\mathcal{L}}^*) = \sum_{m' \in [1, m]} \dim H_c^j(\dot{\mathcal{A}}'_{\mathcal{O}_{m'}}, \dot{\mathcal{L}}^*).$$

Taking  $m = n$  we obtain

$$\dim H_c^j(\dot{\mathcal{A}}', \dot{\mathcal{L}}^*) = \sum_{m' \in [1, n]} \dim H_c^j(\dot{\mathcal{A}}'_{\mathcal{O}_{m'}}, \dot{\mathcal{L}}^*).$$

Using 2.7 and 2.5(a),(b) we obtain the second sentence in (a).

The proof of (b) is entirely similar (it is again based on 2.7 and 2.5(a),(b)).

We prove (c). The last equality in (c), in the nonequivariant case (that is the case where  $C$  is replaced by  $\{1\}$ ) follows immediately from the equation  $\dim H_c^j(\dot{\mathcal{A}}', \dot{\mathcal{L}}^*) = \dim H_c^{j-2b}(\dot{\mathcal{A}}, \dot{\mathcal{L}}^*)$  (see (a) and (b)). The case where  $C$  is present can be deduced from the nonequivariant case using the existence of (noncanonical) isomorphisms of graded vector spaces

$$H_*^C(\dot{\mathcal{A}}', \dot{\mathcal{L}}) \cong H_*^* \otimes H_*(\dot{\mathcal{A}}', \dot{\mathcal{L}}), \quad H_*^C(\dot{\mathcal{A}}, \dot{\mathcal{L}}) \cong H_*^* \otimes H_*(\dot{\mathcal{A}}, \dot{\mathcal{L}})$$

which follows from [L4, 7.2(a)] (which is applicable, by (a) and (b)). The lemma is proved.

- (g) *Remark.* The following six conditions are equivalent:

- (1)  $H_c^*(\mathcal{P}_y, \dot{\mathcal{L}}^*) \neq 0$ ;
- (2)  $H_c^*(\mathcal{P}_y^*, \dot{\mathcal{L}}^*) \neq 0$ ;
- (3)  $H_c^*(\dot{\mathcal{A}}, \dot{\mathcal{L}}^*) \neq 0$ ;
- (4)  $H_c^*(\dot{\mathcal{A}}_1, \dot{\mathcal{L}}^*) \neq 0$ ;
- (5)  $H_c^*(\dot{\mathcal{A}}', \dot{\mathcal{L}}^*) \neq 0$ ;
- (6)  $H_c^*(\dot{\mathcal{A}}_1', \dot{\mathcal{L}}^*) \neq 0$ .

Indeed, we have (5)  $\leftrightarrow$  (2) by (a), (3)  $\leftrightarrow$  (2) by (b), (6)  $\leftrightarrow$  (2) and (4)  $\leftrightarrow$  (2) by 2.5(a),(b). It remains to show that (1)  $\leftrightarrow$  (3). Using [L4, 7.2] (which is applicable by the odd vanishing (b) and [L4, 8.6]) we see that (1)  $\leftrightarrow$  (1') and (3)  $\leftrightarrow$  (3') where

$$(1') H_*^C(\mathcal{P}_y, \dot{\mathcal{L}}) \neq 0;$$

$$(3') H_*^C(\dot{\mathcal{A}}, \dot{\mathcal{L}}) \neq 0.$$

It remains to show that (1')  $\leftrightarrow$  (3'). Let  $\mathcal{P}_y^C, \dot{\mathcal{A}}^C, \mathfrak{n}^C$  be the fixed point sets of the  $C$ -action on  $\mathcal{P}_y, \dot{\mathcal{A}}, \mathfrak{n}$ . Since  $C$  contains  $Z_{Q^1}^0$ , we have  $\mathfrak{n}^C = \{0\}$ , hence  $\mathcal{P}_y^C = \dot{\mathcal{A}}^C$ . By the localization theorem [L6, 4.4] (which is applicable by [L4, 8.6] and (b)), the canonical  $H_C^*$ -linear maps  $H_*^C(\mathcal{P}_y^C, \dot{\mathcal{L}}) \rightarrow H_*^C(\mathcal{P}_y, \dot{\mathcal{L}})$ ,  $H_*^C(\dot{\mathcal{A}}^C, \dot{\mathcal{L}}) \rightarrow H_*^C(\dot{\mathcal{A}}, \dot{\mathcal{L}})$  become isomorphisms after the scalars are extended to the quotient field of  $H_C^*$ . Hence the canonical  $H_C^*$ -linear map  $H_*^C(\mathcal{P}_y, \dot{\mathcal{L}}) \rightarrow H_*^C(\dot{\mathcal{A}}, \dot{\mathcal{L}})$  becomes an isomorphism after the scalars are extended to the quotient field of  $H_C^*$ . Since the  $H_C^*$ -modules  $H_*^C(\mathcal{P}_y, \dot{\mathcal{L}}), H_*^C(\dot{\mathcal{A}}, \dot{\mathcal{L}})$  are finitely generated, projective (by [L4, 7.2] which is applicable by [L4, 8.6] and (b)) it follows that (1')  $\leftrightarrow$  (3').

The previous argument shows also that  $H_*^C(\mathcal{P}_y, \dot{\mathcal{L}}) \rightarrow H_*^C(\dot{\mathcal{A}}, \dot{\mathcal{L}})$  is injective.

For  $x \in \mathfrak{g}$  let  $x_s$  be the semisimple part of  $x$ .

**Lemma 2.9.** *Let  $Y = \{x \in \mathfrak{q}; \mathfrak{z}_{\mathfrak{g}}(x_s) \subset \mathfrak{q}\}$ .*

(a)  *$Y$  is an open dense subset of  $\mathfrak{q}$ .*

(b) *Let  $z$  be an element in the image of  $pr_1 : \dot{Y} \rightarrow Y$ . There exists a Levi subgroup  $L$  of  $Q$  such that the following holds: for any  $P \in \mathcal{P}$  such that  $(z, P) \in \dot{Y}$  we have  $Z_L^0 \subset P$ .*

(c) *We have  $\dot{Y} = \bigcup_{w \in W_*} \dot{Y}_w$ .*

(d) *For any  $w \in W_*$ , there is a well defined isomorphism of algebraic varieties  $f_w : \dot{Y}_w \xrightarrow{\sim} \dot{Y}_1$  given by  $(x, P) \mapsto (x, P^1)$ .*

(e) *For any  $w \in W_*$ ,  $\dot{Y}_w$  is open and closed in  $\dot{Y}$ .*

We prove (a). Let  $\mathcal{S}$  be the variety of semisimple classes of  $\mathfrak{q}/\mathfrak{n}$ . Then  $Y$  is the inverse image under  $\mathfrak{q} \rightarrow \mathfrak{q}/\mathfrak{n} \rightarrow \mathcal{S}$  (composition of canonical maps) of a subset  $\bar{Y}$  of  $\mathcal{S}$ . Let  $\mathfrak{c}$  be a Cartan subalgebra of  $\mathfrak{q}^1$ . It is enough to show that  $\bar{Y}$  is nonempty, open in  $\mathcal{S}$  or that the inverse image  $\tilde{Y}$  of  $\bar{Y}$  under the canonical open map  $\mathfrak{c} \rightarrow \mathcal{S}$  is nonempty open in  $\mathfrak{c}$ . Now

$$\tilde{Y} = Y \cap \mathfrak{c} = \{x \in \mathfrak{c}; \mathfrak{z}_{\mathfrak{g}}(x) \subset \mathfrak{q}\} = \{x \in \mathfrak{c}; \mathfrak{z}_{\mathfrak{g}}(x) \subset \mathfrak{q}^1\}.$$

Let  $R_0$  (resp.  $R'_0$ ) be the set of roots of  $\mathfrak{g}$  (resp. of  $\mathfrak{q}^1$ ) with respect to  $\mathfrak{c}$ . Then  $R'_0 \subset R_0$  and

$$\tilde{Y} = \{x \in \mathfrak{c}; \{\alpha \in R_0; \alpha(x) = 0\} \subset R'_0\} = \{x \in \mathfrak{c}; \alpha(x) \neq 0 \quad \forall \alpha \in R_0 - R'_0\}.$$

This is nonempty, open in  $\mathfrak{c}$ ; (a) is proved.

We prove (b). We can find a Levi subgroup  $L$  of  $Q$  such that  $\mathfrak{z}_{\mathfrak{g}}(z_s) \subset \underline{L}$ . Let  $P \in \mathcal{P}$  be such that  $(z, P) \in \dot{Y}$ . We have  $z \in \underline{P}, \pi_P(z) \in \mathfrak{c}_P + \mathfrak{z}_{\underline{P}}$ , hence  $z_s \in \underline{P}, \pi_P(z_s) \in \mathfrak{z}_{\underline{P}}$ . Hence  $z_s \in \pi_P^{-1}(\mathfrak{z}_{\underline{P}})$ , that is,  $z_s$  is contained in a Cartan subalgebra of  $\pi_P^{-1}(\mathfrak{z}_{\underline{P}})$ . Equivalently,  $z_s \in c$  where  $c$  is the center of a Levi subalgebra  $\underline{P}^1$  of  $\underline{P}$ . Since  $c$  is abelian, from  $z_s \in c$  we deduce  $c \subset \mathfrak{z}_{\mathfrak{g}}(x_s)$ , hence  $c \subset \underline{L}$ . Hence  $\mathfrak{z}_{\underline{L}}$  is contained in  $\mathfrak{z}_{\mathfrak{g}}(c) = \underline{P}^1$ . Thus,  $\mathfrak{z}_{\underline{L}} \subset \underline{P}$  and (b) follows.

We prove (c). Let  $(z, P) \in \dot{Y}$ . Let  $L, \mathfrak{z}_{\underline{L}}$  be as in the proof of (b). We have  $\mathfrak{z}_{\underline{L}} \subset \underline{P}$ . Since  $z \in \mathfrak{z}_{\mathfrak{q}}(x_s) \subset \underline{L}$ , we have  $z \in \underline{L}$ , hence  $[\mathfrak{z}_{\underline{L}}, z] = 0$ . Then  $[\pi_P(\mathfrak{z}_{\underline{L}}), \pi_P(z)] = 0$ . By a known property of cuspidal local systems, the centralizer in  $\underline{P}$  of an element in

$\mathbf{c}_P + \mathfrak{z}_{\underline{P}}$  (in particular,  $\pi_P(z)$ ) has a unique Cartan subalgebra, namely  $\mathfrak{z}_{\underline{P}}$ . It follows that  $\pi_P(\mathfrak{z}_{\underline{L}})$  (the Lie algebra of a torus) is contained in  $\mathfrak{z}_{\underline{P}}$ . Thus,  $\mathfrak{z}_{\underline{L}} \subset \pi_P^{-1}(\mathfrak{z}_{\underline{P}})$ . Hence  $\mathfrak{z}_{\underline{L}}$  is contained in a Cartan subalgebra of  $\pi_P^{-1}(\mathfrak{z}_{\underline{P}})$ . Equivalently,  $\mathfrak{z}_{\underline{L}} \subset c'$  where  $c'$  is the center of a Levi subalgebra of  $\underline{P}$ . It follows that  $\mathfrak{z}_{\mathfrak{g}}(c')$  (a Levi subalgebra of  $\underline{P}$ ) is contained in  $\underline{L}$ , hence is contained in  $\mathfrak{q}$ . In particular, the  $Q$ -orbit of  $P$  in  $\mathcal{P}$  is good. Thus,  $(z, P) \in \dot{Y}_w$  for some  $w \in W_*$ . This proves (c).

We prove (d). Let  $(z, P) \in \dot{Y}_w$ . Let  $L$  be as in (b). Then  $Z_L^0 \subset P$ . By 2.1(b) (with  $S$  replaced by  $Z_L^0$ ) we have  $(z, P^!) \in \dot{Y}_1$ . Hence the morphism  $f_w : \dot{Y}_w \rightarrow \dot{Y}_1$  as in (d) is well defined.

Assume that  $(z, P), (z', P')$  in  $\dot{Y}_w$  have the same image under  $f_w$ , that is  $z = z'$  and  $P^! = P'^!$ . Let  $L$  be as in (b) (attached to  $z = z'$ ). Then  $Z_L^0 \subset P, Z_L^0 \subset P'$ . Using 2.1(a) (with  $S$  replaced by  $Z_L^0$ ) we see that  $P^! = P'^!$  implies  $P = P'$ . Thus,  $f_w$  is injective.

Now let  $(z, P_1) \in \dot{Y}_1$ . Let  $P$  be the unique subgroup in  $\mathfrak{o}(w)^S$  ( $S$  as in 2.1) such that  $P^! = P_1$ . By 2.1(b) we have  $(z, P) \in \dot{Y}_w$ . We have  $f_w(z, P) = (z, P_1)$ . Thus  $f_w$  is surjective. We see that  $f_w$  is bijective. We omit the proof of the fact that  $f_w^{-1}$  is a morphism.

We prove (e). Let us first replace our cuspidal datum (see 1.4) by the cuspidal datum  $(\mathcal{B}, \{0\}, \mathbf{C})$  where  $\mathcal{B}$  is the variety of Borel subgroups of  $G$ . Let  $Y', W'_*, Y'_{w'}$  be the analogues of  $\dot{Y}, W_*, \dot{Y}_w$  for this new cuspidal datum ( $Y$  is unchanged). Now  $Y'_1 = \{(z, B); z \in Y, B \in \mathcal{B}, B \subset Q, z \in \underline{B}\}$  and  $pr_1 : Y'_1 \rightarrow Y$  is proper since  $\{B; B \in \mathcal{B}, B \subset Q\}$  is projective. Using the isomorphism  $Y'_{w'} \rightarrow Y'_1$  (as in (d)), we deduce that  $pr_1 : Y'_{w'} \rightarrow Y$  is proper for any  $w' \in W'_*$ . Hence in the cartesian diagram

$$\begin{array}{ccc} Y' \times_Y Y'_{w'} & \longrightarrow & Y'_{w'} \\ a \downarrow & & \downarrow pr_1 \\ Y' & \xrightarrow{pr_1} & Y \end{array}$$

the map  $a$  is proper. Hence the image under  $a$  of  $\{(\xi, \xi') \in Y' \times_Y Y'_{w'}; \xi \in Y'_{w'}\}$  (a closed subset of  $Y' \times_Y Y'_{w'}$ ) is closed in  $Y'$ . But this image is just  $Y'_{w'}$ . We see that  $Y'_{w'}$  is closed in  $Y'$ .

We now return to the cuspidal datum in 1.4. Let  $m : \mathcal{B} \rightarrow \mathcal{P}$  be the morphism given by  $m(B) = P$  where  $B \subset P$ . This induces a map from the set of  $Q$ -orbits on  $\mathcal{B}$  to the set of  $Q$ -orbits on  $\mathcal{P}$ , which can be viewed as a map  $\bar{m} : W'_* \rightarrow W_*$ . Let  $Y'' = \{(z, P); z \in Y, P \in \mathcal{P}, z \in \underline{P}\}$ . Let  $m' : Y' \rightarrow Y''$  be given by  $m'(z, B) = (z, m(B))$ . It is clear that  $m'$  is a proper morphism. Now  $\dot{Y}$  is a subvariety of  $Y''$ . The restriction of  $m'$  defines a proper morphism  $m'' : m'^{-1}(\dot{Y}) \rightarrow \dot{Y}$ . Since  $Y'_{w'}$  is closed in  $Y'$ , we see that  $Y'_{w'} \cap m'^{-1}(\dot{Y})$  is closed in  $m'^{-1}(\dot{Y})$  (here  $w' \in W'_*$ ). Let  $w \in W_*$ . Since  $\bigcup_{w' \in W'_*; \bar{m}(w')=w} Y'_{w'} \cap m'^{-1}(\dot{Y})$  is closed in  $m'^{-1}(\dot{Y})$  and  $m''$  is proper, it follows that  $m''(\bigcup_{w' \in W'_*; \bar{m}(w')=w} Y'_{w'} \cap m'^{-1}(\dot{Y}))$  is a closed subset of  $\dot{Y}$ . It is clear that this subset is just  $\dot{Y}_w$ . Thus  $\dot{Y}_w$  is closed in  $\dot{Y}$ . Then  $\bigcup_{w_1 \in W_*} \dot{Y}_{w_1}$  is closed in  $\dot{Y}$  hence its complement  $\dot{Y}_w$  is open in  $\dot{Y}$ . This proves (e). The lemma is proved.

**2.10.** Let  $\tilde{K}$  be the direct image with compact support of  $\dot{\mathcal{L}}^*$  under

$$pr_1 : \{(z, P); z \in \mathfrak{q}/\mathfrak{n}; P \in \mathcal{P}^*, z \in \underline{P}; \pi_P(z) \in \mathbf{c}_P + \underline{Z}_{\underline{P}}\} \rightarrow \mathfrak{q}/\mathfrak{n}.$$

Let  $\tilde{K}'$  be the direct image with compact support of  $\dot{\mathcal{L}}^*$  under

$$pr_1 : \{(z, P); z \in \mathfrak{q}; P \in \mathcal{P}^*, z \in \underline{P}; \pi_P(z) \in \mathbf{c}_P + \underline{Z}_{\bar{P}}\} \rightarrow \dot{\mathfrak{q}}.$$

By [L4, 3.4(a)] applied to  $\mathfrak{q}/\underline{U}_Q$  instead of  $\mathfrak{g}$ ,  $\tilde{K}'$  is an intersection cohomology complex (ICC) supported by

$$\bar{Y}_0 = \{z \in \mathfrak{q}/\underline{U}_Q; \exists P \in \mathcal{P}^*, z \in \underline{P}; \pi_P(z) \in \mathbf{c}_P + \underline{Z}_{\bar{P}}\}$$

(a closed irreducible subset of  $\mathfrak{q}/\underline{U}_Q$ ) with a canonical  $W_J$ -action. Clearly,  $\tilde{K}'$  is the inverse image of  $\tilde{K}$  under the obvious vector bundle  $\mathfrak{q} \rightarrow \mathfrak{q}/\underline{U}_Q$ . Hence  $\tilde{K}'$  is an ICC supported by

$$Y_0 = \{z \in \mathfrak{q}; \exists P \in \mathcal{P}^*, z \in \underline{P}; \pi_P(z) \in \mathbf{c}_P + \underline{Z}_{\bar{P}}\}$$

(a closed irreducible subset of  $\mathfrak{q}$ ) with a canonical  $W_J$ -action.

If  $X$  is a subvariety of  $\mathfrak{q}$ , then  $\tilde{K}'|_{\dot{X}_1}$  (a complex of sheaves on  $\dot{X}_1$ ) has a  $W_J$ -action inherited from  $\tilde{K}'$ ; hence there is a natural  $W_J$ -action on the hypercohomology

$$H_c^j(X, \tilde{K}'|_{\dot{X}_1}) = H_c^j(\dot{X}_1, \dot{\mathcal{L}}^*).$$

From the definitions we see that, if  $\iota : \dot{X}_1 \rightarrow \dot{X}$  is the imbedding, the induced map

$$\iota^* : H_c^j(\dot{X}, \dot{\mathcal{L}}^*) \rightarrow H_c^j(\dot{X}_1, \dot{\mathcal{L}}^*)$$

is compatible with the  $W_J$ -actions ( $W_J$  acts on  $H_c^j(\dot{X}, \dot{\mathcal{L}}^*)$  as the restriction of the  $W$ -action 1.9).

**2.11.** Let

$$Y_1 = \{z \in \mathfrak{q}; \exists P \in \mathcal{P}^*, z \in \underline{P}, \pi_P(z) \in \mathbf{c}_P + \underline{Z}_{\bar{P}}, \mathfrak{z}_{\mathfrak{g}}(z_s) \subset \underline{P}\}.$$

We have  $Y_1 \subset Y_0$ . We show that

(a)  $Y_1$  is open dense in  $Y_0$ .

Clearly,  $Y_1 \neq \emptyset$ . Since  $X_1 = \{z \in \mathfrak{g}; \exists P \in \mathcal{P}, z \in \underline{P}, \pi_P(z) \in \mathbf{c}_P + \underline{Z}_{\bar{P}}, \mathfrak{z}_{\mathfrak{g}}(z_s) \subset \underline{P}\}$  is open in

$$X_0 = \{z \in \mathfrak{g}; \exists P \in \mathcal{P}, z \in \underline{P}, \pi_P(z) \in cl(\mathbf{c}_P) + \underline{Z}_{\bar{P}}\}$$

(see [L6, 7.1]) it follows that  $X_1 \cap Y_0$  is open in  $X_0 \cap Y_0 = Y_0$ . Hence to prove (a) it is enough to show that  $X_1 \cap Y_0 = Y_1$ . The inclusion  $Y_1 \subset X_1 \cap Y_0$  is obvious. Conversely, let  $z \in X_1 \cap Y_0$ . Thus,  $z \in \mathfrak{q}$  and there exist  $P \in \mathcal{P}, P' \in \mathcal{P}^*$  such that

$$z \in \underline{P}, z \in \underline{P}', \pi_P(z) \in \mathbf{c}_P + \underline{Z}_{\bar{P}}, \pi_{P'}(z) \in cl(\mathbf{c}_{P'}) + \underline{Z}_{\bar{P}'}, \mathfrak{z}_{\mathfrak{g}}(z_s) \subset \underline{P}.$$

We have  $\pi_P(z_s) \in \underline{Z}_{\bar{P}}, \pi_{P'}(z_s) \in \underline{Z}_{\bar{P}'}$ ; hence there exists a Levi subalgebra  $\mathfrak{l}$  of  $\mathfrak{p}$  and a Levi subalgebra  $\mathfrak{l}'$  of  $\mathfrak{p}'$  such that  $z_s \in \mathfrak{z}_{\mathfrak{l}}, z_s \in \mathfrak{z}_{\mathfrak{l}'}$ . Then  $\mathfrak{l} \subset \mathfrak{z}_{\mathfrak{g}}(z_s)$ . This, together with  $\mathfrak{z}_{\mathfrak{g}}(z_s) \subset \underline{P}$  implies  $\mathfrak{l} = \mathfrak{z}_{\mathfrak{g}}(z_s)$ . We also have  $\mathfrak{l}' \subset \mathfrak{z}_{\mathfrak{g}}(z_s)$  and  $\dim \mathfrak{l}' = \dim \mathfrak{l} = \dim \mathfrak{z}_{\mathfrak{g}}(z_s)$ , hence  $\mathfrak{l}' = \mathfrak{z}_{\mathfrak{g}}(z_s)$ . Thus,  $\mathfrak{l} = \mathfrak{l}'$ . Let  $c$  be the nilpotent orbit in  $\mathfrak{l}$  that corresponds to  $\mathbf{c}_P$  under  $\mathfrak{l} \xrightarrow{\sim} \bar{P}$  and also to  $\mathbf{c}_{P'}$  under  $\mathfrak{l} \xrightarrow{\sim} \bar{P}'$ . Now  $\mathfrak{l}$  is a Levi subalgebra of  $\mathfrak{p} \cap \mathfrak{p}'$ , hence  $z = z' + z''$  where  $z' \in \mathfrak{l}, z'' \in \underline{U}_{P \cap P'}$  are uniquely determined. We also have  $z = z_1 + z_2 + z_3$  with  $z_1 \in c, z_2 \in \mathfrak{z}_{\mathfrak{l}}, z_3 \in \underline{U}_P$  and  $z = z'_1 + z'_2 + z'_3$  with  $z'_1 \in cl(c), z'_2 \in \mathfrak{z}_{\mathfrak{l}}, z'_3 \in \underline{U}_{P'}$ . It follows that  $z_3 \in \underline{U}_P \cap \underline{P}' \subset \underline{U}_{P \cap P'}$  and  $z'_3 \in \underline{U}_{P'} \cap \underline{P} \subset \underline{U}_{P \cap P'}$ . By uniqueness of the decomposition  $z = z' + z''$  we then have  $z_1 + z_2 = z' = z'_1 + z'_2$ . Taking nilpotent parts we deduce  $z_1 = z'_1$ . In particular,  $z'_1 \in c$ . We see that

$$z \in \underline{P}', P \in \mathcal{P}^*, \pi_{P'}(z) \in \mathbf{c}_{P'} + \underline{Z}_{\bar{P}'}, \mathfrak{z}_{\mathfrak{g}}(z_s) \subset \underline{P}',$$

hence  $z \in Y_1$ . This proves (a).

Since  $Y_0$  is irreducible, from (a) we deduce that  $Y_1$  is irreducible. Now  $Y \cap Y_0$  is closed in  $Y$ . Since  $Y_1 \subset Y$ , from (a) we deduce that  $Y_1$  is open dense in  $Y \cap Y_0$ . In particular,  $Y \cap Y_0$  is irreducible.

The image of  $pr_1 : \dot{Y} \rightarrow Y$  has image contained in  $Y_0$ . (Indeed, by 2.9(c),(d), the image of  $pr_1 : \dot{Y} \rightarrow Y$  is contained in the image of  $pr_1 : \dot{Y}_1 \rightarrow Y$ .) Thus we have maps  $pr_1 : \dot{Y} \rightarrow Y \cap Y_0, pr_1 : \dot{Y}_w \rightarrow Y \cap Y_0$ . Taking the direct image with compact support of  $\dot{\mathcal{L}}^*$  under  $pr_1 : \dot{Y} \rightarrow Y \cap Y_0$  (resp.  $pr_1 : \dot{Y}_w \rightarrow Y \cap Y_0$ ), we get a complex of sheaves  $K'$  (resp.  $K'_w$ ) on  $Y \cap Y_0$ . From 2.9(c),(e) we have canonically  $K' = \bigoplus_{w \in W_*} K'_w$  in the derived category. Since  $K' = K|_{Y \cap Y_0}$  where  $K$  (as in 1.8) has a natural  $W$ -action, we see that  $K'$  has a natural  $W$ -action. On the other hand,  $W$  acts on  $W_*$  by  $w : w_1 \mapsto w * w_1$  where  $w * w_1$  is the element of minimal length in  $ww_1W_J$ .

**Lemma 2.12.** *For  $w \in W$  and  $w_1 \in W_*$  we have  $wK'_{w_1} = K'_{w*w_1}$ .*

Clearly,  $K'_1 = \tilde{K}'_{Y \cap Y_0}$ . (Note that  $Y \cap Y_0$  is an open dense subset of  $Y_0$  since  $Y$  is open in  $\mathfrak{q}$  and  $Y \cap Y_0 \neq \emptyset$ .) It follows that  $K'_1$  is an ICC supported by  $Y \cap Y_0$ . Using 2.9(d) we deduce that for any  $w_1 \in W_*$ ,  $K'_{w_1}$  is isomorphic to  $K'_1$  in the derived category. Hence  $K'_{w_1}$  is an ICC supported by  $Y \cap Y_0$ . Since  $K' = \bigoplus_{w_1 \in W_*} K'_{w_1}$ , we see that  $K'$  is an ICC supported by  $Y \cap Y_0$ . It is therefore sufficient to check the equality  $wK'_{w_1} = K'_{w*w_1}$  over the open dense subset  $Y_1$  of  $Y \cap Y_0$ . Using [L4, 3.2(a)] we see that

$$(Y_1)^\circ = \{(z, P); z \in Y_1; \mathfrak{z}_{\mathfrak{g}}(z_s) \text{ is a Levi subalgebra of } \underline{P}\}.$$

Then  $W$  acts freely on  $(Y_1)^\circ$  by  $w : (z, P) \mapsto (z, P')$  where  $P' \in \mathcal{P}$  is defined by the condition that  $\mathfrak{z}_{\mathfrak{g}}(z_s)$  is a Levi subalgebra of  $\underline{P}'$  and  $(P, P')$  is in the good  $G$ -orbit on  $\mathcal{P} \times \mathcal{P}$  corresponding to  $w$ . The decomposition  $(Y_1)^\circ = \bigsqcup_{w_1 \in W_*} (Y_1)_{w_1}^\circ$  clearly satisfies

$$w(Y_1)_{w_1}^\circ = (Y_1)_{w*w_1}^\circ$$

for any  $w \in W, w_1 \in W_*$ . Using the definition of  $K'$  and of the  $W$ -action on it we see that  $wK'_{w_1} = K'_{w*w_1}$  holds over  $Y_1$ . The lemma is proved.

**Lemma 2.13.** *Assume that  $H_c^*(\mathcal{P}_y, \dot{\mathcal{L}}^*) \neq 0$ . Let  $\iota : \dot{\mathcal{A}}'' \rightarrow \dot{\mathcal{A}}''$  be the inclusion. The linear map*

$$H_c^j(\dot{\mathcal{A}}'', \dot{\mathcal{L}}^*) \rightarrow \mathbf{C}[W] \otimes_{\mathbf{C}[W_J]} H_c^j(\dot{\mathcal{A}}''_1, \dot{\mathcal{L}}^*)$$

*defined by  $\xi \mapsto \sum_{w \in W_*} w \otimes \iota^*(w^{-1}\xi)$  is an isomorphism.*

Let  $z \in \mathcal{A}''$ . We have  $z = y + n + t$  where  $n \in \mathfrak{n}$  and  $t \in \mathfrak{t}$ .

By 2.8(g) we have  $H_c^*(\mathcal{P}_y^*, \dot{\mathcal{L}}^*) \neq 0$  hence by [L4, 8.6] we have  $Eu(\mathcal{P}_y^*) \neq 0$ . The set  $\{P \in \mathcal{P}_y^*, t \in \underline{P}\}$  is the fixed point set of a torus action on  $\mathcal{P}_y^*$ , hence it has the same Euler characteristic as  $\mathcal{P}_y^*$ ; in particular, this set is nonempty.

Let  $P \in \mathcal{P}_y^*$  be such that  $t \in \underline{P}$ . Since  $\mathfrak{n} \subset \underline{P}$ , we have  $z \in \underline{P}$ . Since  $t \in \mathfrak{t}$ , we have  $[t, x] \in \underline{U_Q} \subset \underline{U_P}$  for all  $x \in \mathfrak{q}$ . In particular,  $[t, x] \in \underline{U_P}$  for all  $x \in \underline{P}$ , hence

$$(a) \quad \pi_P(t) \in \underline{Z_{\bar{P}}}.$$

Thus,  $\pi_P(z) = \pi_P(y) + \pi_P(t) \in \mathbf{c}_P + \underline{Z_{\bar{P}}}$  so that  $z \in Y_0$ . Now  $z$  and  $y + t$  have the same image in  $\mathfrak{q}/\mathfrak{n}$ ; hence  $z_s$  and  $t$  have the same image in  $\mathfrak{q}/\mathfrak{n}$ ; hence  $z_s$  and  $t$  are in the same  $\text{Ad}(U_Q)$ -orbit. Since  $t \in \mathfrak{t}_r$  we have  $\mathfrak{z}_{\mathfrak{g}}(t) \subset \mathfrak{q}$  hence  $\mathfrak{z}_{\mathfrak{g}}(z_s) \subset \mathfrak{q}$  so that  $z \in Y$ . Thus,  $z \in Y_0 \cap Y$ . We see that  $\mathcal{A}'' \subset Y_0 \cap Y$ . Taking the direct image with compact support of  $\dot{\mathcal{L}}^*$  under  $pr_1 : \dot{\mathcal{A}}'' \rightarrow \dot{\mathcal{A}}''$  (resp.  $pr_1 : \dot{\mathcal{A}}''_w \rightarrow \dot{\mathcal{A}}''$ )

we obtain a complex of sheaves  $K''$  (resp.  $K''_w$ ) on  $\mathcal{A}''$ . This is the same as  $K'|_{\mathcal{A}''}$  (resp.  $K'_w|_{\mathcal{A}''}$ ). Hence we have canonically  $K'' = \bigoplus_{w \in W_*} K''_w$  and from 2.12 we deduce that the  $W$ -action on  $K''$  satisfies  $wK''_{w_1} = K''_{w_*w_1}$  for any  $w \in W$  and any  $w_1 \in W_*$ . Hence we have  $K'' = \bigoplus_{w \in W_*} wK''_1$ . It follows that we have canonically

$$H_c^j(\mathcal{A}'', K'') = \bigoplus_{w \in W_*} H_c^j(\mathcal{A}'', K''_w) = \bigoplus_{w \in W_*} wH_c^j(\mathcal{A}'', K''_1),$$

that is,

$$H_c^j(\dot{\mathcal{A}}'', \dot{\mathcal{L}}^*) = \bigoplus_{w \in W_*} H_c^j(\dot{\mathcal{A}}''_w, \dot{\mathcal{L}}^*) = \bigoplus_{w \in W_*} wH_c^j(\dot{\mathcal{A}}''_1, \dot{\mathcal{L}}^*).$$

The lemma follows.

**Lemma 2.14.** *Let  $\iota' : \dot{\mathcal{A}}'_1 \rightarrow \dot{\mathcal{A}}'$  be the inclusion. The linear map*

$$H_c^j(\dot{\mathcal{A}}', \dot{\mathcal{L}}^*) \rightarrow \mathbf{C}[W] \otimes_{\mathbf{C}[W_J]} H_c^j(\dot{\mathcal{A}}'_1, \dot{\mathcal{L}}^*)$$

*defined by  $\xi \mapsto \sum_{w \in W_*} w \otimes \iota'^*(w^{-1}\xi)$  is an isomorphism.*

If  $H_c^*(\mathcal{P}_y, \dot{\mathcal{L}}^*) = 0$ , then the linear map above is  $0 \rightarrow 0$ , by 2.8(g) and the result is obvious. Assume now that  $H_c^*(\mathcal{P}_y, \dot{\mathcal{L}}^*) \neq 0$ . It suffices to prove the similar result where the ground field  $\mathbf{C}$  is replaced by an algebraic closure  $\bar{F}_p$  of the finite field  $F_p$  where  $p$  is a large enough prime (local systems and cohomology will be  $l$ -adic, where  $l$  is a prime  $\neq p$ ). We will assume (in this proof) that  $G$  has a fixed  $F_q$ -split rational structure ( $F_q \subset \bar{F}_p$  has  $q$  elements) with Frobenius map  $F$ , that  $y, \mathbf{c}, Q, Q^1$  are  $F$ -stable and that we are given an isomorphism  $F^*\mathcal{L} \xrightarrow{\sim} \mathcal{L}$  which induces on any stalk at a point of  $\mathbf{c}(F_q)$  a map of finite order. Moreover, we assume that any nilpotent orbit in  $\mathfrak{g}$  or  $\mathfrak{q}$  is defined over  $F_q$  and that any irreducible local system over such an orbit is defined over  $F_q$ . Using the  $l$ -adic analogue of Lemma 2.13 and taking pure parts we obtain an isomorphism

$$H_c^j(\dot{\mathcal{A}}'', \dot{\mathcal{L}}^*)_{\text{pure}} \rightarrow \bar{\mathbf{Q}}_l[W] \otimes_{\bar{\mathbf{Q}}_l[W_J]} H_c^j(\dot{\mathcal{A}}''_1, \dot{\mathcal{L}}^*)_{\text{pure}}$$

where  $H_c^j(?, ?)_{\text{pure}}$  is the part of  $H_c^j(?, ?)$  where the Frobenius map acts with eigenvalues  $\lambda$  such that any complex absolute value of  $\lambda$  is  $q^{j/2}$ . It is then enough to show that

$$(a) \quad H_c^j(\dot{\mathcal{A}}'', \dot{\mathcal{L}}^*)_{\text{pure}} = H_c^j(\dot{\mathcal{A}}', \dot{\mathcal{L}}^*),$$

$$(b) \quad H_c^j(\dot{\mathcal{A}}''_w, \dot{\mathcal{L}}^*)_{\text{pure}} = H_c^j(\dot{\mathcal{A}}'_w, \dot{\mathcal{L}}^*)$$

for any  $w \in W_*$ .

We prove (b). Using 2.5(b) (or rather its analogue over  $\bar{F}_p$ ) we see that it is enough to show that

$$H_c^j(\dot{\mathcal{A}}''^1_1, \dot{\mathcal{L}}^*)_{\text{pure}} = H_c^j(\dot{\mathcal{A}}'^1_1, \dot{\mathcal{L}}^*).$$

Using 2.5(a) (or rather its analogue over  $\bar{F}_p$ ) we see that it is enough to show that

$$H_c^j((y + \mathfrak{t}_r) \times \mathcal{P}_y^*, \dot{\mathcal{L}}^*)_{\text{pure}} = H_c^j((y + \mathfrak{t}) \times \mathcal{P}_y^*, \dot{\mathcal{L}}^*).$$

Using Künneth's theorem, we see that it is enough to show that

$$(c) \quad H_c^j(\mathcal{P}_y^*, \dot{\mathcal{L}}^*)_{\text{pure}} = H_c^j(\mathcal{P}_y^*, \dot{\mathcal{L}}^*),$$

$$(d) \quad H_c^j(\mathfrak{t}_r, \bar{\mathbf{Q}}_l)_{\text{pure}} = H_c^j(\mathfrak{t}, \bar{\mathbf{Q}}_l).$$

Since  $\mathfrak{t}_r$  is the complement in  $\mathfrak{t}$  of a finite set of hyperplanes, the eigenvalues of  $F$  on  $H_c^j(\mathfrak{t}_r, \bar{\mathbf{Q}}_l)$  are easily seen to be of the form  $q^{j-\dim \mathfrak{t}}$  (see [LE]) and (d) follows.

Now (c) is a special case of

$$(e) \quad H_c^j(\mathcal{P}_x, \dot{\mathcal{L}}^*)_{pure} = H_c^j(\mathcal{P}_x, \dot{\mathcal{L}}^*),$$

(for any nilpotent element  $x \in \mathfrak{g}(F_q)$ ) obtained by replacing  $G$  by  $Q^1$ . To prove (e) is the same as to prove that the eigenvalues of  $F$  on the stalk at  $x \in \mathfrak{g}(F_q)$  of the  $j$ th cohomology sheaf of  $K$  have complex absolute value  $q^{j/2}$ . Now the restriction of  $K$  to the nilpotent variety of  $\mathfrak{g}$  is of the form  $\bigoplus_{O, \mathcal{E}} V_{O, \mathcal{E}} \otimes ICC(cl(O), \mathcal{E})$  where  $O$  runs over the nilpotent orbits in  $\mathfrak{g}$  and  $\mathcal{E}$  are irreducible local systems on  $O$  that are linked to our fixed cuspidal datum by the generalized Springer correspondence [L2, Sec. 6];  $V_{O, \mathcal{E}}$  are certain multiplicity spaces. It is then enough to prove that for any of these  $ICC(cl(O), \mathcal{E})$ , the eigenvalues of  $F$  on the stalk at  $x \in \mathfrak{g}(F_q)$  of the  $j$ th cohomology sheaf have complex absolute value  $q^{j/2}$  (this holds by [L3, 24.6]) and that  $F$  acts trivially on each multiplicity space  $V_{O, \mathcal{E}}$ . These multiplicity spaces can be viewed as multiplicity spaces of the various irreducible representations of  $W$  in the regular representation of  $W$ , hence  $F$  acts on them trivially. This proves (e), hence (c) and (b).

We prove (a). We can arrange the  $Q$ -orbits on  $\mathcal{P}$  in a sequence  $\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_n$  as in the proof of 2.8. Thus,  $R_m = \mathcal{O}_1 \cup \mathcal{O}_2 \cup \dots \cup \mathcal{O}_m$  is closed in  $\mathcal{P}$  for any  $m \in [1, n]$ . We set  $R_0 = \emptyset$ .

To prove (a) it is enough to show that

$$H_c^j(\dot{\mathcal{A}}''_{R_m}, \dot{\mathcal{L}}^*)_{pure} = H_c^j(\dot{\mathcal{A}}'_{R_m}, \dot{\mathcal{L}}^*),$$

for any  $m \in [0, n]$ . We argue by induction on  $m$ . For  $m = 0$  the result is trivial. Assume now that  $m \geq 1$  and that the result is known for  $m - 1$ . We have a cohomology exact sequence

$$0 \rightarrow H_c^j(\dot{\mathcal{A}}''_{\mathcal{O}_m}, \dot{\mathcal{L}}^*) \rightarrow H_c^j(\dot{\mathcal{A}}''_{R_m}, \dot{\mathcal{L}}^*) \rightarrow H_c^j(\dot{\mathcal{A}}''_{R_{m-1}}, \dot{\mathcal{L}}^*) \rightarrow 0$$

(we use the fact that  $\dot{\mathcal{A}}''_{\mathcal{O}_m}$  is empty if  $\mathcal{O}_m$  is not good (see 2.9(c)), and is both open and closed in  $\dot{\mathcal{A}}''$  if  $\mathcal{O}_m$  is good; see 2.9(e). Taking pure parts in this exact sequence gives again an exact sequence

$$0 \rightarrow H_c^j(\dot{\mathcal{A}}''_{\mathcal{O}_m}, \dot{\mathcal{L}}^*)_{pure} \rightarrow H_c^j(\dot{\mathcal{A}}''_{R_m}, \dot{\mathcal{L}}^*)_{pure} \rightarrow H_c^j(\dot{\mathcal{A}}''_{R_{m-1}}, \dot{\mathcal{L}}^*)_{pure} \rightarrow 0.$$

This exact sequence together with the exact sequence 2.8(f) are the rows of the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_c^j(\dot{\mathcal{A}}''_{\mathcal{O}_m}, )_{pure} & \longrightarrow & H_c^j(\dot{\mathcal{A}}''_{R_m}, )_{pure} & \longrightarrow & H_c^j(\dot{\mathcal{A}}''_{R_{m-1}}, )_{pure} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H_c^j(\dot{\mathcal{A}}'_{\mathcal{O}_m}, ) & \longrightarrow & H_c^j(\dot{\mathcal{A}}'_{R_m}, ) & \longrightarrow & H_c^j(\dot{\mathcal{A}}'_{R_{m-1}}, ) \longrightarrow 0 \end{array}$$

where the vertical maps are induced by the obvious open imbeddings and the symbol  $\dot{\mathcal{L}}^*$  is omitted in the notation. Now the left vertical map is an isomorphism. (If  $\mathcal{O}_m$  is good, this follows from (b); if  $\mathcal{O}_m$  is not good, this follows from the fact that  $H_c^j(\dot{\mathcal{A}}'_{\mathcal{O}_m}, \dot{\mathcal{L}}^*) = 0$  (see 2.7) and that  $\dot{\mathcal{A}}''_{\mathcal{O}_m} = \emptyset$ ; see 2.9(c).) The right vertical map is an isomorphism by the induction hypothesis. It follows automatically that the middle vertical map is an isomorphism. This proves (a). The lemma is proved.

**Lemma 2.15.** *Let  $\iota' : \dot{\mathcal{A}}'_1 \rightarrow \dot{\mathcal{A}}'$  be the inclusion. The  $H_C^*$ -linear map*

$$(a) \quad \mathbf{C}[W] \otimes_{\mathbf{C}[W_J]} H_*^C(\dot{\mathcal{A}}'_1, \dot{\mathcal{L}}) \rightarrow H_*^C(\dot{\mathcal{A}}', \dot{\mathcal{L}})$$

*defined by  $w \otimes \xi \mapsto w\iota'_!(\xi)$  is an isomorphism.*

By [L4, 3.8] (which is applicable in view of 2.8(a)), the two sides of (a) are finitely generated projective  $H_C^*$ -modules which, after applying  $H_{\{1\}}^* \otimes_{H_C^*}$ , become the analogous objects with  $C$  replaced by  $\{1\}$ . Now (a) is an isomorphism when  $C$  is replaced by  $\{1\}$  (we take the transpose of the isomorphism in Lemma 2.14). We see that the lemma can be deduced from the following statement which is easily verified.

Let  $\mathcal{R}$  be the polynomial algebra over  $\mathbf{C}$  in the indeterminates  $x_1, x_2, \dots, x_n$  graded by  $\deg(x_i) = 2$  for all  $i$ . Let  $\mathcal{I}$  be the ideal of  $\mathcal{R}$  generated by  $x_1, x_2, \dots, x_n$ . Let  $M, M'$  be two  $\mathbf{N}$ -graded free  $\mathcal{R}$ -modules and let  $f : M \rightarrow M'$  be an  $\mathcal{R}$ -linear map compatible with the gradings such that  $f$  induces an isomorphism  $M/\mathcal{I}M \xrightarrow{\sim} M'/\mathcal{I}M'$ . Then  $f$  is an isomorphism.

The lemma is proved.

Let  $\iota : \dot{\mathcal{A}}_1 \rightarrow \dot{\mathcal{A}}$  be the inclusion. Consider the  $H_C^*$ -linear map

$$(b) \quad \mathbf{H} \otimes_{\mathbf{H}'} H_*^C(\mathcal{P}_y^*, \dot{\mathcal{L}}) \rightarrow H_*^C(\dot{\mathcal{A}}, \dot{\mathcal{L}})$$

given by the composition

$$(c) \quad \begin{aligned} \mathbf{H} \otimes_{\mathbf{H}'} H_*^C(\mathcal{P}_y^*, \dot{\mathcal{L}}) &= \mathbf{C}[W] \otimes_{\mathbf{C}[W_J]} H_*^C(\mathcal{P}_y^*, \dot{\mathcal{L}}) \\ &\xrightarrow{1 \otimes p^*} \mathbf{C}[W] \otimes_{\mathbf{C}[W_J]} H_*^C(\dot{\mathcal{A}}_1, \dot{\mathcal{L}}) \xrightarrow{a} H_*^C(\dot{\mathcal{A}}, \dot{\mathcal{L}}) \end{aligned}$$

where  $a$  is given by  $w \otimes \xi \mapsto w\iota_!(\xi)$  and  $p : \dot{\mathcal{A}}_1 \rightarrow \mathcal{B}_y^*$  is the affine space bundle  $(z, P) \mapsto P$ .

**Theorem 2.16** (Strong induction theorem). *The  $W$ -module structure and  $\mathbf{S}$ -module structure on  $H_*^C(\dot{\mathcal{A}}, \dot{\mathcal{L}})$  define an  $\mathbf{H}$ -module structure on  $H_*^C(\dot{\mathcal{A}}, \dot{\mathcal{L}})$ . Moreover, the map 2.15(b) is an  $\mathbf{H}$ -linear isomorphism.*

By the argument in 2.8(g), we have a natural  $H_C^*$ -linear imbedding  $H_*^C(\mathcal{P}_y, \dot{\mathcal{L}}) \rightarrow H_*^C(\dot{\mathcal{A}}, \dot{\mathcal{L}})$  which becomes an isomorphism after the scalars are extended to the quotient field of  $H_C^*$ . Since the  $W$ -module structure and  $\mathbf{S}$ -module structure on  $H_*^C(\mathcal{P}_y, \dot{\mathcal{L}})$  are known to define an  $\mathbf{H}$ -module structure, the same must then hold for  $H_*^C(\dot{\mathcal{A}}, \dot{\mathcal{L}})$  (which is projective over  $H_C^*$ ). This proves the first assertion of the theorem.

To prove the second assertion we may assume by 2.8(g) that  $H_c^*(\mathcal{P}_y, \dot{\mathcal{L}}^*) \neq 0$ . The composition of  $C \subset G \times \mathbf{C}^*$  with  $G \times \mathbf{C}^* \xrightarrow{pr_2} \mathbf{C}^*$  is surjective, as we see from the Morozov-Jacobson theorem for  $y \in \mathfrak{q}^1$ . Hence the image under the induced homomorphism  $H_{\mathbf{C}^*}^* \rightarrow H_C^*$  of the generator  $\mathbf{r}$  is a nonzero element of  $H_C^*$  denoted again by  $\mathbf{r}$ . It is enough to show that  $a$  in 2.15(c) is an isomorphism. Recall from 2.8 that  $b = \dim \mathfrak{t}$ . We show that

$$(a) \quad \text{the map } (\iota_1)_! : H_j^C(\dot{\mathcal{A}}_1, \dot{\mathcal{L}}) \rightarrow H_{j+2b}^C(\dot{\mathcal{A}}'_1, \dot{\mathcal{L}}) \text{ induced by the inclusion } \iota_1 : \dot{\mathcal{A}} \rightarrow \dot{\mathcal{A}}' \text{ is injective and its image equals } \mathbf{r}^b H_j^C(\dot{\mathcal{A}}'_1, \dot{\mathcal{L}}).$$

Recall that

$$\begin{aligned} \dot{\mathcal{A}}'_1 &= \{(z, P) \in \dot{\mathfrak{g}}; z \in y + \mathfrak{z}_{\mathbf{n}}(\phi(f_0)) + \mathfrak{t}, P \subset Q\}, \\ \dot{\mathcal{A}}_1 &= \{(z, P) \in \dot{\mathfrak{g}}; z \in y + \mathfrak{z}_{\mathbf{n}}(\phi(f_0)), P \subset Q\}. \end{aligned}$$

We have an isomorphism

$$(b) \quad k : \dot{\mathcal{A}}_1 \times \mathfrak{t} \xrightarrow{\sim} \dot{\mathcal{A}}'_1$$

given by  $((y+n, P), t) \mapsto (y+n+t, P)$  where  $n \in \mathfrak{z}_{\mathfrak{n}}(\phi(f_0))$ ,  $t \in \mathfrak{t}$ . (We use the fact that  $\mathfrak{t} \subset \pi^{-1}(\underline{Z}_{\bar{P}}) + \underline{U}_P$  for any  $P \in \mathcal{P}^*$ ; see 2.13(a).) Note that  $k$  is  $C$ -equivariant where the action  $C \times \mathfrak{t} \rightarrow \mathfrak{t}$  is  $((g, \lambda), t) \mapsto \lambda^{-2}t$ .

This implies, by [L4, 1.10(b)] that the image of  $(\iota_1)_!$  is  $\mathbf{r}^b H_j^C(\dot{\mathcal{A}}'_1, \dot{\mathcal{L}})$ . Since  $\mathbf{r}^b \neq 0$  in  $H_C^*$  and  $H_*^C(\dot{\mathcal{A}}'_1, \dot{\mathcal{L}})$  is projective over  $H_C^*$ , we have

$$\dim \mathbf{r}^b H_j^C(\dot{\mathcal{A}}'_1, \dot{\mathcal{L}}) = \dim H_j^C(\dot{\mathcal{A}}'_1, \dot{\mathcal{L}}) = \dim H_j^C(\dot{\mathcal{A}}_1, \dot{\mathcal{L}}).$$

(The second equality follows from (b) and [L4, 1.4(e)].) Hence  $(\iota_1)_!$  must be an isomorphism onto  $\mathbf{r}^b H_j^C(\dot{\mathcal{A}}'_1, \dot{\mathcal{L}})$ . This proves (a).

As in 2.8(c), let  $D = \dim \dot{\mathcal{A}}$ ,  $D' = \dim \dot{\mathcal{A}}'$ . Let  $D_1 = \dim \dot{\mathcal{A}}_1$ ,  $D'_1 = \dim \dot{\mathcal{A}}'_1$  so that  $D'_1 = D_1 + b$ .

From 2.15 we have  $H_j^C(\dot{\mathcal{A}}', \dot{\mathcal{L}}) = \sum_{w \in W} w H_{j-2D'+2D'_1}^C(\dot{\mathcal{A}}'_1, \dot{\mathcal{L}})$  and  $\iota_! : H_*^C(\dot{\mathcal{A}}, \dot{\mathcal{L}}) \rightarrow H_*^C(\dot{\mathcal{A}}', \dot{\mathcal{L}})$  (induced by the inclusion  $\iota : \dot{\mathcal{A}} \rightarrow \dot{\mathcal{A}}'$ ) is  $W$ -equivariant. Hence from (a) we can deduce that

$$\begin{aligned} \mathbf{r}^b H_j^C(\dot{\mathcal{A}}', \dot{\mathcal{L}}) &= \sum_{w \in W} w \mathbf{r}^b H_{j-2D'+2D'_1}^C(\dot{\mathcal{A}}'_1, \dot{\mathcal{L}}) \subset \sum_{w \in W} w (\iota_1)_! H_{j-2D'+2D'_1}^C(\dot{\mathcal{A}}_1, \dot{\mathcal{L}}) \\ &\subset \sum_{w \in W} w \iota_! H_{j-2D'+2D'_1+2D-2D_1}^C(\dot{\mathcal{A}}, \dot{\mathcal{L}}) \subset \iota_! H_{j-2D'+2D'_1+2D-2D_1}^C(\dot{\mathcal{A}}, \dot{\mathcal{L}}). \end{aligned}$$

Thus,

$$\mathbf{r}^b H_j^C(\dot{\mathcal{A}}', \dot{\mathcal{L}}) \subset \iota_! H_{j-2D'+2b+2D}^C(\dot{\mathcal{A}}, \dot{\mathcal{L}}).$$

It follows that

$$\begin{aligned} \dim \mathbf{r}^b H_j^C(\dot{\mathcal{A}}', \dot{\mathcal{L}}) &= \dim H_j^C(\dot{\mathcal{A}}', \dot{\mathcal{L}}) \leq \dim \iota_! H_{j-2D'+2b+2D}^C(\dot{\mathcal{A}}, \dot{\mathcal{L}}) \\ &\leq \dim H_{j-2D'+2b+2D}^C(\dot{\mathcal{A}}, \dot{\mathcal{L}}). \end{aligned}$$

These inequalities must be equalities since

$$\dim H_j^C(\dot{\mathcal{A}}', \dot{\mathcal{L}}) = \dim H_{j-2D'+2b+2D}^C(\dot{\mathcal{A}}, \dot{\mathcal{L}})$$

(see 2.8(c)). It follows that

$$(c) \quad \mathbf{r}^b H_j^C(\dot{\mathcal{A}}', \dot{\mathcal{L}}) = \iota_! H_{j-2D'+2b+2D}^C(\dot{\mathcal{A}}, \dot{\mathcal{L}}) \text{ and } \iota_! \text{ is an isomorphism onto its image.}$$

From 2.15(a) we deduce

$$\mathbf{C}[W] \otimes_{\mathbf{C}[W_J]} \mathbf{r}^b H_j^C(\dot{\mathcal{A}}'_1, \dot{\mathcal{L}}) \xrightarrow{\sim} \mathbf{r}^b H_{j+2D'-2D'_1}^C(\dot{\mathcal{A}}', \dot{\mathcal{L}})$$

which by (a) and (c) becomes

$$\mathbf{C}[W] \otimes_{\mathbf{C}[W_J]} H_j^C(\dot{\mathcal{A}}_1, \dot{\mathcal{L}}) @>\sim>> H_{j-2D_1+2D}^C(\dot{\mathcal{A}}, \dot{\mathcal{L}}).$$

The theorem is proved.

**2.17.** Let  $V$  be a finite dimensional  $\mathbf{C}$ -vector space with an algebraic action of  $C$ . Let  $[V] \in H_C^*$  be the element corresponding (as in [L4, 1.11]) to the regular function  $\underline{C} \rightarrow \mathbf{C}$ ,  $\xi \mapsto \det(\xi, V)$ . Here  $\xi : V \rightarrow V$  is given by the associated Lie algebra representation of  $\underline{C}$  on  $V$ . Now  $E = \mathfrak{z}_{\mathbf{n}}(\phi(f_0))$  is a  $C$ -module for the restriction of the  $G \times \mathbf{C}^*$ -action on  $\mathfrak{g}$ . Hence  $[E]$  is a well defined element of  $H_C^*$ .

Note that  $E = \ker(ad(f_0) : \mathfrak{n} \rightarrow \mathfrak{n}) \cong \text{coker}(ad(e_0) : \mathfrak{n} \rightarrow \mathfrak{n})$  canonically. (Indeed, we have  $\mathfrak{n} = \ker(ad(f_0) : \mathfrak{n} \rightarrow \mathfrak{n}) \oplus \text{Im}(ad(e_0) : \mathfrak{n} \rightarrow \mathfrak{n})$  since  $\mathfrak{n}$  is an  $\mathfrak{sl}_2(\mathbf{C})$ -module.)

**Lemma 2.18.** (a) *The homomorphism  $H_*^C(\mathcal{P}_y^*, \dot{\mathcal{L}}) \rightarrow H_*^C(\dot{\mathcal{A}}_1, \dot{\mathcal{L}})$  induced by the inclusion  $\mathcal{P}_y^* \subset \dot{\mathcal{A}}_1$  is injective.*  
 (b) *The homomorphism  $H_*^C(\mathcal{P}_y, \dot{\mathcal{L}}) \rightarrow H_*^C(\dot{\mathcal{A}}, \dot{\mathcal{L}})$  induced by the inclusion  $\mathcal{P}_y \subset \dot{\mathcal{A}}$  is injective.*  
 (c) *We have  $[E]H_*^C(\dot{\mathcal{A}}_1, \dot{\mathcal{L}}) \subset H_*^C(\mathcal{P}_y^*, \dot{\mathcal{L}})$ .*  
 (d) *We have  $[E]H_*^C(\dot{\mathcal{A}}, \dot{\mathcal{L}}) \subset H_*^C(\mathcal{P}_y, \dot{\mathcal{L}})$ .*

(b) has already been noted at the end of 2.8. An entirely similar proof yields (a).

We prove (c). We have an isomorphism

$$\mathfrak{z}_{\mathbf{n}}(\phi(f_0)) \times \mathcal{P}_y^* \xrightarrow{\sim} \dot{\mathcal{A}}_1$$

given by  $(n, P) \mapsto (y + n, P)$  (we use the fact that  $\mathfrak{z}_{\mathbf{n}}(\phi(f_0)) \subset \mathfrak{n} \subset \underline{U}_P$  for any  $P \in \mathcal{P}^*$ ). Under this isomorphism, the inclusion  $\mathcal{P}_y^* \subset \dot{\mathcal{A}}_1$  corresponds to the map  $\mathcal{P}_y^* \rightarrow \mathfrak{z}_{\mathbf{n}}(\phi(f_0)) \times \mathcal{P}_y^*$  given by  $P \mapsto (0, P)$ . Now the result follows using [L4, 1.10(b)].

We prove (d). Using 2.16, (c) and the fact that the homomorphism in (b) is  $W$ -equivariant, we see that

$$\begin{aligned} [E]H_*^C(\dot{\mathcal{A}}, \dot{\mathcal{L}}) &= [E] \sum_{w \in W} wH_*^C(\dot{\mathcal{A}}_1, \dot{\mathcal{L}}) = \sum_{w \in W} w[E]H_*^C(\dot{\mathcal{A}}_1, \dot{\mathcal{L}}) \\ &\subset \sum_{w \in W} wH_*^C(\mathcal{P}_y, \dot{\mathcal{L}}) \subset H_*^C(\mathcal{P}_y, \dot{\mathcal{L}}). \end{aligned}$$

The lemma is proved.

**2.19.** From 2.18 we see that we have a natural isomorphism

$$H_C^*[[E]^{-1}] \otimes_{H_C^*} H_*^C(\mathcal{P}_y, \dot{\mathcal{L}}) \xrightarrow{\sim} H_C^*[[E]^{-1}] \otimes_{H_C^*} H_*^C(\dot{\mathcal{A}}, \dot{\mathcal{L}}).$$

We combine this with 2.16; Theorem 1.17 follows.

### 3. PROOF OF THEOREMS 1.15, 1.21, 1.22

**3.1.** Until the end of 3.8, let  $P, L, T, W(L), R, \mathcal{C}$  be as in 1.6. Let  $R_1 = \{\alpha \in R; 2\alpha \notin R\}$ . Then  $R_1$  is (reduced) root system in  $\underline{T}^*$  with Weyl group  $W(L)$ . For  $i \in I$ , the set  $R_1 \cap \{\alpha_i, 2\alpha_i\}$  consists of a single element; we call it  $\alpha'_i$ . Then  $\{\alpha'_i; i \in I\}$  is a set of simple roots for  $R_1$ . We have  $\mathfrak{g}^{\alpha'_i} \subset \underline{U}_P$ .

**3.2.** Let  $\phi_0 : \mathfrak{sl}_2(\mathbf{C}) \rightarrow \underline{L}$  be a Lie algebra homomorphism such that  $\phi_0(e_0) \in \mathcal{C}$ . (Such a  $\phi_0$  exists by the Morozov-Jacobson theorem.) Let  $Z = Z(\text{Im}(\phi_0))^0$  (a connected reductive subgroup of  $G$ ). Then  $\underline{Z} = \mathfrak{z}(\text{Im}(\phi_0))$ .

**Lemma 3.3.** (a)  *$T$  is a maximal torus of  $Z$ .*

- (b) Let  $N_Z(T)$  be the normalizer of  $T$  in  $Z$  so that  $N_Z(T)/T$  is the Weyl group of  $Z$ . The obvious homomorphism  $N_Z(T)/T \rightarrow N(T)/L = W(L)$  is an isomorphism.
- (c) We have  $\mathfrak{g}^0 \cap \underline{Z} = \underline{L} \cap \underline{Z} = \underline{T}$ . For any  $\alpha \in R_1$  we have  $\dim(\mathfrak{g}^\alpha \cap \underline{Z}) = 1$ . Hence  $R_1$  is the root system of  $\underline{Z}$  with respect to  $\underline{T}$ .
- (d)  $\underline{P} \cap \underline{Z}$  is a Borel subalgebra of  $\underline{Z}$ .
- (e) The map  $J \mapsto \underline{P}_J \cap \underline{Z}$  is a bijection between  $\{J; J \subset I\}$  and the set of parabolic subalgebras of  $\underline{Z}$  that contain  $\underline{P} \cap \underline{Z}$ .
- (f)  $\mathfrak{z}_Z = \mathfrak{z}_g$ .

For (a) see [L4, 2.6(a)]. For (b) see [L7, 11.7(b)]. For (c) see [L4, 2.9]. For (d) see [L7, 11.7(a)].

We prove (e). The map in (e) is well defined by (d). This is a map between two finite sets of the same cardinal,  $2^{\sharp(I)}$ . To show that it is bijective, it is enough to show that it is injective. Let  $J, J'$  be two subsets of  $I$  that satisfy  $\underline{P}_J \cap \underline{Z} = \underline{P}_{J'} \cap \underline{Z}$ . We must show that  $J = J'$ . We have  $\underline{P}_J \cap \underline{P}_{J'} = \underline{P}_{J \cap J'}$ . Then  $\underline{P}_{J \cap J'} \cap \underline{Z} = \underline{P}_J \cap \underline{Z} = \underline{P}_{J'} \cap \underline{Z}$  and it is enough to show that  $J \cap J' = J$  and  $J \cap J' = J'$ . Thus we are reduced to the case where  $J \subset J'$ . Assume that  $J \neq J'$ . Let  $i \in J' - J$ . Then  $\underline{P}_i \subset \underline{P}_{J'}$ ,  $\underline{P}_i \not\subset \underline{P}_J$ . Using 1.6(a) with  $Q = P_J$  we see that  $E \cap \underline{P}_J = 0$  where  $E = \mathfrak{g}^{-\alpha_i} \oplus \mathfrak{g}^{-2\alpha_i}$ . Note that  $E \subset \underline{P}_{J'}$ . Let  $E' = E \cap \underline{Z}$ . We have  $E' \cap (\underline{P}_J \cap \underline{Z}) = 0$ ,  $E' \subset \underline{P}_{J'} \cap \underline{Z}$ . Since  $\dim(E') = 1$  (by (c)), we deduce that  $\underline{P}_J \cap \underline{Z} \neq \underline{P}_{J'} \cap \underline{Z}$ , a contradiction. This proves (e).

We prove (f). Since  $\mathfrak{z}_g \subset \mathfrak{z}_Z$ , it is enough to show that the two centers have the same dimension. From (a) and (c) we see that  $\dim \mathfrak{z}_Z = \dim(\underline{T}) - \sharp(I)$ . It is easy to see that  $\dim \mathfrak{z}_g = \dim(\underline{T}) - \sharp(I)$ . This proves (f). The lemma is proved.

**3.4.** Let  $V \in \mathcal{I}$  and let  $D$  be the unique  $\underline{P}$ -stable line in  $V$ . For any  $v \in D, x \in \underline{T}$  we have  $xv = \xi_V(x)v$  where  $\xi_V \in \underline{T}^*$  corresponds under the obvious isomorphism  $\underline{T} \xrightarrow{\sim} \underline{P}/[\underline{P}, \underline{P}] = \mathfrak{h}$  to the vector of  $\mathfrak{h}^*$  denoted in 1.19 again by  $\xi_V$ . Now  $\{x \in \mathfrak{g}; xD \subset D\} = \underline{P}_K$  for a well defined  $K \subset I$ . We then say that  $V \in \mathcal{I}_K$ . We say that  $V \in \mathcal{I}_K^0$  if  $V \in \mathcal{I}_K$  and  $\mathfrak{z}_g$  acts as 0 on  $V$ .

Let  $i \in I$ . Then  $\{\xi_V; V \in \mathcal{I}_{I-\{i\}}^0\} = \{\varpi_i, 2\varpi_i, 3\varpi_i, \dots\}$  where  $\varpi_i \in \underline{T}^*$  (or  $\varpi_i \in \mathfrak{h}^*$ ) is well defined; we have  $\varpi_i = \xi_{\Lambda^i}$  where  $\Lambda^i \in \mathcal{I}_{I-\{i\}}^0$  is well defined up to isomorphism.

**3.5.** Let  $\mathcal{I}'^0$  be the collection of all simple finite dimensional  $\underline{Z}$ -modules on which  $\mathfrak{z}_Z$  acts as 0. Let  $V' \in \mathcal{I}'^0$  and let  $D'$  be the unique  $(\underline{P} \cap \underline{Z})$ -stable line in  $V'$ . There is a unique vector  $\xi'_{V'} \in \underline{T}^*$  such that  $xv = \xi'_{V'}(x)v$  for any  $v \in D', x \in \underline{T}$ . Also  $\{x \in \underline{Z}; xD' \subset D'\} = \underline{P}_K \cap \underline{Z}$  for a well defined  $K \subset I$  (see 3.3(e)). We then say that  $V' \in \mathcal{I}'^0_K$ .

Let  $i \in I$ . We have  $\{\xi'_{V'}; V' \in \mathcal{I}'^0_{I-\{i\}}\} = \{\varpi'_i, 2\varpi'_i, 3\varpi'_i, \dots\}$  where  $\varpi'_i \in \underline{T}^*$  is well defined; we have  $\varpi'_i = \xi'_{\Lambda'_i}$  where  $\Lambda'_i \in \mathcal{I}'^0_{I-\{i\}}$  is well defined up to isomorphism.

**Lemma 3.6.** *Let  $i \in I$ . We have  $\varpi_i = n_i \varpi'_i$  for some  $n_i \in \mathbf{N} - \{0\}$ .*

Let  $D$  be the unique  $\underline{P}$ -stable line in  $\Lambda^i$ . Then  $\underline{P}_{I-\{i\}} = \{x \in \mathfrak{g}; xD \subset D\}$ . We may regard  $\Lambda^i$  as a  $\underline{Z}$ -module by restriction. In this  $\underline{Z}$ -module,  $\mathfrak{z}_Z$  acts as 0 (see 3.3(f)) and  $D$  is stable under the Borel subalgebra  $\underline{P} \cap \underline{Z}$  (see 3.3(d)) of  $\underline{Z}$ . Hence the  $\underline{Z}$ -submodule  $V'$  generated by  $D$  is simple. Clearly,  $\{x \in \underline{Z}; xD \subset D\} = \underline{P}_{I-\{i\}} \cap \underline{Z}$ .

Thus,  $V' \in \mathcal{I}'^0_{I-\{i\}}$ . From the definition, we then have  $\varpi_i \in \{\varpi'_i, 2\varpi'_i, 3\varpi'_i, \dots\}$ . The lemma is proved.

**3.7.** Let  $X = \text{Hom}(\mathbf{T}, \mathbf{C}^*)$  (homomorphisms of algebraic groups). The differential gives an imbedding  $d : X \rightarrow \mathfrak{h}^*$  whose image  $\mathcal{X}$  is a free abelian group such that  $\mathbf{C} \otimes \mathcal{X} \xrightarrow{\sim} \mathfrak{h}^*$ . Let  $\mathfrak{h}_{\mathbf{R}} = \{x \in \mathfrak{h}; \xi(x) \in \mathbf{R} \ \forall \xi \in \mathcal{X}\}$ .

Under the obvious isomorphism  $\underline{T} \xrightarrow{\sim} \underline{P}/[\underline{P}, \underline{P}] = \mathfrak{h}$ ,  $\mathfrak{h}_{\mathbf{R}}$  corresponds to a subset  $\underline{T}_{\mathbf{R}}$  of  $\underline{T}$ .

We shall need the following variant of a lemma of Langlands.

**Lemma 3.8.** *Assume that  $G$  is semisimple.*

- (a) *For any  $f \in \mathfrak{h}_{\mathbf{R}}$  there is a subset  $J$  of  $I$  and a decomposition  $f = {}^0f + {}^1f$  with  ${}^0f, {}^1f \in \underline{T}_{\mathbf{R}}$  such that*

$$\alpha_i({}^0f) < 0 \text{ if } i \in I - J, \quad \alpha_i({}^0f) = 0 \text{ if } i \in J,$$

$$\varpi_i({}^1f) \geq 0 \text{ if } i \in J, \quad \varpi_i({}^1f) = 0 \text{ if } i \in I - J.$$

*Moreover,  $J, {}^0f, {}^1f$  are uniquely determined by  $f$ .*

- (b) *If  $f, f' \in \mathfrak{h}_{\mathbf{R}}$  satisfy  $\varpi_i(f) \leq \varpi_i(f')$  for any  $i \in I$ , then  $\varpi_i({}^0f) \leq \varpi_i({}^0f')$  for any  $i \in I$ .*

Using the isomorphism  $\underline{T} \xrightarrow{\sim} \mathfrak{h}$  in 3.7, we see that the statement above is equivalent to the one where  $\mathfrak{h}, \mathfrak{h}_{\mathbf{R}}$  are replaced by  $\underline{T}, \underline{T}_{\mathbf{R}}$ . Moreover, if  $\alpha_i, \varpi_i$  are replaced by  $\alpha'_i, \varpi'_i$ , then these statements hold by Langlands' lemma [BW, IV, 6.11-6.13] applied to the root system  $R_1$  of  $\underline{Z}$  with respect to  $\underline{T}$ . However, this replacement does not affect the statements since  $\alpha'_i, \varpi'_i$  differ from  $\alpha_i, \varpi_i$  only by rational  $> 0$  factors (see 3.1 and 3.6). The lemma is proved.

**3.9.** In the remainder of this section (except in 3.42) we fix  $y \in \mathfrak{g}_N$ ,  $r \in \mathbf{C}$  and  $\sigma \in \mathfrak{g}_{ss}$  with  $[\sigma, y] = 2ry$  such that

$$(a) \quad Eu(\mathcal{P}_y^\sigma) \neq 0$$

where

$$\mathcal{P}^\sigma = \{P \in \mathcal{P}; \sigma \in \underline{P}\}, \mathcal{P}_y^\sigma = \mathcal{P}^\sigma \cap \mathcal{P}_y$$

and  $Eu$  denotes Euler characteristic. Condition (a) is equivalent to each of the following conditions:

$$(b) \quad Eu(\mathcal{P}_y) \neq 0,$$

$$(c) \quad H_*(\mathcal{P}_y, \dot{\mathcal{L}}) \neq 0.$$

The equivalence of (a) and (b) follows from the conservation of  $Eu$  by passage to the fixed point set of a torus action. The equivalence of (b) and (c) follows from the vanishing theorem [L4, 8.6].

If  $r \neq 0$ , conditions (a), (b), and (c) are also equivalent to the condition

$$(d) \quad E_{y, \sigma, r} \neq 0.$$

The equivalence of (c) and (d) follows from [L4, 7.2] (which is applicable in view of [L4, 8.6]).

**Lemma 3.10.** *Let  $Q \in \mathfrak{P}$  and let  $Q^1$  be a Levi subgroup of  $Q$ . Assume that  $y, \sigma$  are contained in  $\underline{Q}^1$ . Then  $\{P \in \mathcal{P}_y^\sigma; P \subset Q\} \neq \emptyset$ .*

We may use the argument in the second paragraph of the proof of 2.13 (replacing  $t$  by  $\sigma$ ).

**3.11.** By a variant of the Morozov-Jacobson theorem (see [KL, 2.4(g)]) we can find  $h, \tilde{y}$  in  $\mathfrak{g}$  such that

$$[\sigma, \tilde{y}] = -2r\tilde{y}, [y, \tilde{y}] = h, [h, y] = 2y, [h, \tilde{y}] = -2\tilde{y}.$$

Then  $[\sigma, h] = 0$ .

We now assume that  $r \neq 0$ . We fix  $\tau : \mathbf{C} \rightarrow \mathbf{R}$  as in 1.20.

Let  $V$  be a finite dimensional  $\mathfrak{g}$ -module. We have

$$V = \bigoplus_{a \in \mathbf{C}} V_a \text{ where } V_a = \{x \in V; \sigma x = ax\},$$

$$V = \bigoplus_{n \in \mathbf{Z}} ({}_n V) \text{ where } {}_n V = \{x \in V; hx = nx\}.$$

Since  $[\sigma, h] = 0$ , the maps  $v \mapsto \sigma v, v \mapsto hv$  from  $V$  to  $V$  commute, hence

$$V = \bigoplus_{n, a; n \in \mathbf{Z}, a \in \mathbf{C}} ({}_n V_a) \text{ where } {}_n V_a = {}_n V \cap V_a.$$

We have

$$V = \bigoplus_{b \in \mathbf{R}} {}^b V \text{ where } {}^b V = \bigoplus_{n, a; \tau(a)/\tau(r)=n+b} ({}_n V_a).$$

These definitions can be applied, in particular, with  $V$  replaced by  $\mathfrak{g}$  with the ad action of  $\mathfrak{g}$ . We have

$$y \in {}_2 \mathfrak{g}_{2r}, h \in {}_0 \mathfrak{g}_0, y' \in {}_{-2} \mathfrak{g}_{-2r}, \sigma \in {}_0 \mathfrak{g}_0.$$

From the definition we have

$$x \in {}_n \mathfrak{g}_a, v \in {}_{n'} V_{a'} \implies xv \in {}_{n+n'} V_{a+a'},$$

$$x \in {}^b \mathfrak{g}, v \in {}^{b'} V \implies xv \in {}^{b+b'} V.$$

**3.12.** We define  $Q \in \mathfrak{P}$  and a Levi subgroup  $Q^1$  of  $Q$  by

$$\underline{Q} = \bigoplus_{n, a; \tau(a)/\tau(r) \leq n} ({}_n \mathfrak{g}_a) = \bigoplus_{b; b \leq 0} {}^b \mathfrak{g},$$

$$\underline{Q}^1 = \bigoplus_{n, a; \tau(a)/\tau(r) = n} ({}_n \mathfrak{g}_a) = \bigoplus_{b; b=0} {}^b \mathfrak{g}.$$

Then

$$\mathfrak{n} = \bigoplus_{n, a; \tau(a)/\tau(r) < n} ({}_n \mathfrak{g}_a) = \bigoplus_{b; b < 0} {}^b \mathfrak{g}$$

is the nil-radical of  $\underline{Q}$ . Also,  $y, h, \tilde{y}, \sigma$  are contained in  $\underline{Q}^1$ . Now  $\text{ad}(\sigma), \text{ad}(y)$  define endomorphisms of  $\mathfrak{n}$  whose commutator is  $2\text{rad}(y)$ . Hence  $\text{ad}(\sigma)$  maps

$${}_y \mathfrak{n} = \text{coker}(\text{ad}(y) : \mathfrak{n} \rightarrow \mathfrak{n})$$

into itself.

**Lemma 3.13.**  $\text{ad}(\sigma) - 2r : {}_y \mathfrak{n} \rightarrow {}_y \mathfrak{n}$  is invertible.

An equivalent statement is that any eigenvalue of  $\text{ad}(\sigma) - 2r$  on

$$(a) \quad \text{coker}(\text{ad}(y) : \bigoplus_{n, a; \tau(a)/\tau(r) < n} ({}_n \mathfrak{g}_a) \rightarrow \bigoplus_{n, a; \tau(a)/\tau(r) < n} ({}_n \mathfrak{g}_a))$$

is  $\neq 0$ . Now (a) is a quotient of  $\bigoplus_{n,a;n \leq 0, \tau(a)/\tau(r) < n} ({}_n\mathfrak{g}_a)$  which is itself a quotient of

$$(b) \quad \bigoplus_{a; \tau(a)/\tau(r) < 0} \mathfrak{g}_a$$

and it is enough to show that any eigenvalue  $\lambda$  of  $\text{ad}(\sigma) - 2r$  on (b) is  $\neq 0$ . We have  $\lambda = a - 2r$  for some  $a \in \mathbf{C}$  such that  $\tau(a)/\tau(r) < 0$ . Then  $\tau(\lambda)/\tau(r) = \tau(a - 2r)/\tau(r) = \tau(a)/\tau(r) - 2 < -2$ . In particular,  $\lambda \neq 0$ .

**Lemma 3.14.** *Let  $z \in G$  be such that  $\text{Ad}(z)y = 2cy$ ,  $\text{Ad}(z)\sigma = \sigma$  for some  $c \in \mathbf{C}$ . Then  $z \in Q$ .*

We must show that  $\text{Ad}(z)x \in \underline{Q}$  for any  $x \in \underline{Q}$ . We may assume that  $x \in {}_n\mathfrak{g}_a$  where  $\tau(a)/\tau(r) \leq n$ . We have  $[\sigma, \text{Ad}(z)x] = \text{Ad}(z)[\sigma, x] = \text{Ad}(z)(ax) = a\text{Ad}(z)x$ . Thus,  $\text{Ad}(z)x \in \underline{G}_a$ . Since  $\text{Ad}(z)y = 2cy$  for some  $c$ ,  $z$  belongs to the parabolic subgroup of  $G$  corresponding to  $\bigoplus_{m \geq 0} ({}_m\mathfrak{g})$ . Hence  $\text{Ad}(z)(x) \in \text{Ad}(z)({}_n\underline{G}) \subset \bigoplus_{m; m \geq n} ({}_m\mathfrak{g})$ . We see that

$$\text{Ad}(z)(x) \in \bigoplus_{m; m \geq n} ({}_m\mathfrak{g}_a) \subset \bigoplus_{m; \tau(a)/\tau(r) \leq m} ({}_m\mathfrak{g}_a) \subset \underline{Q}.$$

The lemma is proved.

**Lemma 3.15.** (a)  $Q$  is independent of the choice of  $h, \tilde{y}$ .  
 (b) We have  $M(y, \sigma) \subset Q \times \mathbf{C}^*$ .

We prove (a). Any other choice of  $h, \tilde{y}$  is of the form  $h', \tilde{y}'$  where  $h' = \text{Ad}(z)h$ ,  $\tilde{y}' = \text{Ad}(z)\tilde{y}$  for some  $z \in G$  such that  $\text{Ad}(z)y = y$ ,  $\text{Ad}(z)\sigma = \sigma$ . (See [KL, 2.4(h)].) Let  $Q'$  be attached to  $h', \tilde{y}'$  in the same way as  $Q$  is attached to  $h, \tilde{y}$ . Then  $Q' = zQz^{-1}$ . By 3.14 we have  $z \in Q$ . Hence  $Q' = Q$ .

We prove (b). Let  $(g, \lambda) \in M(y, \sigma)$ . We can find an element  $g_1$  in the one-parameter subgroup of  $G$  corresponding to  $h$  such that  $\text{Ad}(g_1)y = \lambda^2 y$ . Since  $h \in \underline{Q}^1$ ,  $[\sigma, h] = 0$ , we have  $g_1 \in \underline{Q}^1 \cap Z(\sigma)$ . Replacing  $(g, \lambda)$  by  $(gg_1^{-1}, 1)$  we see that we are reduced to the case where  $\lambda = 1$  and  $g \in Z(y) \cap Z(\sigma)$ .

Since  $g \in Z(\sigma)$  we have  $\text{Ad}(g)(\mathfrak{g}_a) = \mathfrak{g}_a$  for all  $a$ . Since  $g \in Z(y)$ , we have  $\text{Ad}(g)({}_n\mathfrak{g}) \subset \bigoplus_{n'; n' \geq n} ({}_{n'}\mathfrak{g})$ . Hence  $\text{Ad}(g)({}_n\mathfrak{g}_a) \subset \bigoplus_{n'; n' \geq n} ({}_{n'}\mathfrak{g}_a)$  for any  $n, a$ . Using this and the definition of  $\underline{Q}$  (see 3.12) we see that  $\text{Ad}(g)(\underline{Q}) \subset \underline{Q}$ . Hence  $g \in Q$ . The lemma is proved.

**Lemma 3.16.** *Let  $\sigma', y', h', \tilde{y}'$  be another quadruple like  $\sigma, y, h, \tilde{y}$ . Define  $Q'$  in terms of  $\sigma', y', h', \tilde{y}'$  in the same way as  $Q$  was defined in terms of  $\sigma, y, h, \tilde{y}$ . Assume that  $Q = Q' = G$ . Assume that there exist  $P \in \mathcal{P}_y^\sigma$  and  $P' \in \mathcal{P}_{y'}^{\sigma'}$  such that the image of  $\sigma$  in  $\underline{P}/[\underline{P}, \underline{P}] = \mathfrak{h}$  coincides with the image of  $\sigma'$  in  $\underline{P}'/[\underline{P}', \underline{P}'] = \mathfrak{h}$ . Then there exists  $g \in G$  such that  $\text{Ad}(g)$  carries  $(\sigma, y, h, \tilde{y})$  to  $(\sigma', y', h', \tilde{y}')$ .*

Replacing  $(\sigma', y', h', \tilde{y}')$  by a  $G$ -conjugate we may assume that  $P' = P$  and

$$\begin{aligned} \pi_P(\sigma) &= \sigma_1, \pi_P(\sigma_1) = \sigma'_1, \pi_P(y') = \pi_P(y) = y_1, \\ \pi_P(h') &= \pi_P(h) = h_1, \pi_P(\tilde{y}') = \pi_P(\tilde{y}) = \tilde{y}_1, \end{aligned}$$

where  $y_1 \in \mathbf{c}_P$  and

$$\begin{aligned} [y_1, \tilde{y}_1] &= h_1, [h_1, y_1] = 2y_1, [h_1, \tilde{y}_1] = -2\tilde{y}_1, \\ [\sigma_1, y_1] &= 2ry_1, [\sigma_1, \tilde{y}_1] = -2r\tilde{y}_1, \\ [\sigma'_1, y_1] &= 2ry_1, [\sigma'_1, \tilde{y}_1] = -2r\tilde{y}_1. \end{aligned}$$

Moreover,  $\sigma_1$  and  $\sigma'_1$  have the same image in  $\bar{P}/[\bar{P}, \bar{P}]$ . Hence  $x = \sigma_1 - \sigma'_1 \in [\bar{P}, \bar{P}]$ . We have  $[x, y_1] = 0, [x, \tilde{y}_1] = 0$ , hence  $[x, h_1] = 0$ . Since  $y_1$  is a distinguished nilpotent element of  $\bar{P}$ , the centralizer in  $[\bar{P}, \bar{P}]$  of  $y_1, h_1, \tilde{y}_1$  is 0. Thus,  $x = 0$  so that  $\sigma_1 = \sigma'_1$ . Let  $L$  be a Levi subgroup of  $P$ . Since  $\sigma, \sigma' \in \bar{P}$ , there exist  $g, g' \in P$  such that

$$\text{Ad}(g)\sigma \in \underline{L}, \pi_P(\sigma - \text{Ad}(g)\sigma) = 0, \text{Ad}(g')\sigma' \in \underline{L}, \pi_P(\sigma' - \text{Ad}(g')\sigma') = 0.$$

Hence  $\pi_P(\text{Ad}(g)\sigma) = \sigma_1 = \sigma'_1 = \pi_P(\text{Ad}(g')\sigma')$ . Since the restriction of  $\pi_P$  to  $\underline{L}$  is injective, it follows that  $\text{Ad}(g)\sigma = \text{Ad}(g')\sigma'$ . Thus,  $\sigma, \sigma'$  are conjugate in  $G$ .

Replacing  $(\sigma', y', h', \tilde{y}')$  by a  $G$ -conjugate we may assume that  $\sigma' = \sigma$ . We show that

(a)  $y$  belongs to the (unique) open orbit of  $Z(\sigma)$  on  $\mathfrak{g}_{2r}$ .

Let  $G'$  be the connected reductive algebraic subgroup of  $G$  such that

$$\underline{G}' = \bigoplus_{m \in \mathbf{Z}} \mathfrak{g}_{2mr}.$$

Note that  $y, h, \tilde{y}$  are contained in  $\underline{G}'$ . Let  ${}_n\underline{G}' = {}_n\mathfrak{g} \cap \underline{G}'$ . Since in our case,  $Q^1 = \mathfrak{g}$ , any eigenvalue  $b$  of  $\text{ad}(\sigma - rh) : \mathfrak{g} \rightarrow \mathfrak{g}$  satisfies  $\tau(b) = 0$ . Hence  ${}_n\mathfrak{g}_a \neq 0 \implies \tau(a)/\tau(r) = n$ . Hence  $\mathfrak{g}_0 = {}_0\underline{G}'$  and  $\mathfrak{g}_{2r} = {}_2\underline{G}'$ . It follows that  $Z(\sigma)$  is equal to the centralizer  $C(h)$  of  $h$  in  $G'$ . We are reduced to the following known statement about  $\mathfrak{sl}_2$ -triples in  $\underline{G}'$ :  $y$  belongs to the open orbit of  $C(h)$  on  ${}_2\underline{G}'$ . Thus, (a) holds.

Similarly,  $y'$  belongs to the (unique) open orbit of  $Z(\sigma')$  on  $\{x \in \mathfrak{g}; [\sigma', x] = 2rx\}$ . Since  $\sigma = \sigma'$ , we see that both  $y$  and  $y'$  belong to the unique open orbit of  $Z(\sigma)$  on  $\mathfrak{g}_{2r}$ . In particular,  $y, y'$  are conjugate under an element in  $Z(\sigma)$ .

Replacing  $(\sigma, y', h', \tilde{y}')$  by a  $Z(\sigma)$ -conjugate we may therefore assume that  $y = y'$ . As in the proof of 3.15, we can find  $z \in G$  such that  $\text{Ad}(z)y = y, \text{Ad}(z)\sigma = \sigma, h' = \text{Ad}(z)h, \tilde{y}' = \text{Ad}(z)\tilde{y}$ . Replacing  $(\sigma, y, h', \tilde{y}')$  by its  $\text{Ad}(z^{-1})$ -conjugate we may therefore assume that  $(\sigma', y', h', \tilde{y}') = (\sigma, y, h, \tilde{y})$ . The lemma is proved.

**Lemma 3.17.** *We have  $\{P \in \mathcal{P}_y^\sigma; P \subset Q\} \neq \emptyset$ . In particular,  $Q \in \mathcal{P}_K$  for a well defined  $K \subset I$ .*

This follows from 3.10.

**3.18.** Let  $V \in \mathcal{I}_J$  where  $J \subset I$ . For any  $P \in \mathcal{P}$  let  $D_P$  be the unique  $\bar{P}$ -stable line in  $V$ . If  $P \in \mathcal{P}^\sigma$ , we have necessarily  $D_P \subset V_a$  for a unique  $a \in \mathbf{C}$ ; we set  $\nu_V(P) = a$ . Let

$$b_V = \min(b \in \mathbf{R}; {}^bV \neq 0).$$

The following result is closely related to [L1, 2.8, 2.9], [KL, 7.3].

**Lemma 3.19.** *We preserve the setup of 3.18.*

- (a) *If  $P \in \mathcal{P}^\sigma, y \in \bar{P}$ , then  $\tau(\nu_V(P))/\tau(r) \geq b_V$ .*
- (b) *If  $K = J, P \in \mathcal{P}^\sigma, y \in \bar{P}, P \not\subset Q$ , then  $\tau(\nu_V(P))/\tau(r) > b_V$ .*
- (c) *If  $P \in \mathcal{P}, P \subset Q$ , then  $D_P \subset {}^{b_V}V$ .*
- (d) *If  $K = J, P \in \mathcal{P}^\sigma, P \subset Q$ , then  $\tau(\nu_V(P))/\tau(r) = b_V$ .*

From 3.11 we have

$$\mathfrak{n}^b V \subset \bigoplus_{b'; b' < b} {}^{b'} V, \underline{Q}^b V \subset \bigoplus_{b'; b' \leq b} {}^{b'} V.$$

Hence  $\mathfrak{n}^{b_V} V = 0, \underline{Q}^{b_V} V \subset {}^{b_V} V$ .

Let  $P$  be as in (a) and let  $v \in D_P - \{0\}$ . We have  $v \in V_{\nu_V(P)}$ . We write  $v = \sum_m ({}_m v)$  where  ${}_m v \in {}_m V_{\nu_V(P)}$ . Since  $v \neq 0$ , there exists  $n$  such that  ${}_n v \neq 0$ . Since  $y$  is nilpotent in  $\underline{P}$ , we have  $yv = 0$ , hence  $\sum_m y({}_m v) = 0$ . Since  $y \in {}_2 \mathfrak{g}_{2r}$ , we have

$$y({}_m v) \in {}_{m+2} V_{\nu_V(P)+2r}.$$

Since  $\sum_m y({}_m v) = 0$  and the sum  $\sum_m ({}_{m+2} V_{\nu_V(P)+2r})$  is direct, we have  $y({}_m v) = 0$  for all  $m$ . In particular,  $y({}_n v) = 0$ . From  $y({}_n v) = 0$  and  ${}_n v \neq 0$  we see, using the representation theory of  $\mathfrak{sl}_2$  that  $n \geq 0$ . Since  ${}_n v$  is a nonzero vector of  ${}_n V_{\nu_V(P)} \subset \tau(\nu_V(P))/\tau(r) - nV$  and  ${}^{b'} V = 0$  unless  $b' \geq b_V$ , we see that  $\tau(\nu_V(P))/\tau(r) - n \geq b_V$ . Since  $n \geq 0$ , we must have  $\tau(\nu_V(P))/\tau(r) \geq b_V$ . This proves (a).

In the setup of (b), assume that  $\tau(\nu_V(P))/\tau(r) \not\geq b_V$ . Then, by (a), we have  $\tau(\nu_V(P))/\tau(r) = b_V$ . Also, in the proof of (a) we must have  ${}_n v \neq 0 \implies n = 0$  so that  $v = {}_0 v \in {}_0 V_{\nu_V(P)}$  and  $v \in \tau(\nu_V(P))/\tau(r) V = {}^{b_V} V$ . Thus,  $D_P \subset {}^{b_V} V$ . By assumption,  $V$  contains a line  $D$  such that  $\{x \in \mathfrak{g}; xD \subset D\} = \underline{Q}$ . This implies that  $\{v \in V; \mathfrak{n}v = 0\} = D$ . (See the argument in 1.19.) Since  $\mathfrak{n}^{b_V} V = 0$ , it follows that  ${}^{b_V} V \subset D$ . Since  $D_P \subset {}^{b_V} V$ , we have  $D_P \subset D$ , hence  $D_P = D$ . Since  $D_P$  is  $\underline{P}$ -stable, we see that  $D$  is  $\underline{P}$ -stable, hence by the definition of  $D$  we have  $\underline{P} \subset \underline{Q}$  so that  $P \subset Q$ . This proves (b).

Next, assume that  $P$  is as in (c). Since  $\underline{Q}^{b_V} V \subset {}^{b_V} V$ , we have  $\underline{P}^{b_V} V \subset {}^{b_V} V$ . Let  $\mathfrak{b}$  be a Borel subalgebra of  $\underline{P}$ . Then  $\mathfrak{b}^{b_V} V \subset {}^{b_V} V$  and, by Lie's theorem, there exists an  $\mathfrak{b}$ -stable line  $L$  in  ${}^{b_V} V$ . This is necessarily the unique  $\mathfrak{b}$ -stable line in  $V$ . Since  $D_P$  is  $\mathfrak{b}$ -stable we must have  $L = D_P$ , hence  $D_P \subset {}^{b_V} V$ . This proves (c).

In the setup of (d),  $V$  contains a line  $D$  such that  $\{x \in \underline{Q}; xD \subset D\} = \underline{Q}$ . Now  $D$  is  $\underline{P}$ -stable, hence  $D_P = D$ . Since  $h$  is contained in the derived subalgebra of  $\underline{Q}$ , it acts as zero on the  $\underline{Q}$ -stable line  $D$ . Hence  $D \subset {}_0 V$ . We have  $D_P \subset V_{\nu_V(P)}$  hence  $D \subset V_{\nu_V(P)}$ . As in the proof of (b) we have  ${}^{b_V} V \subset D$ ; this must be an equality since  $\dim {}^{b_V} V \geq 1, \dim D = 1$ . From  $D \subset {}_0 V, D \subset V_{\nu_V(P)}, D = {}^{b_V} V$  we deduce  $\tau(\nu_V(P))/\tau(r) = b_V$ . This proves (d). The lemma is proved.

**Lemma 3.20.** *In the setup of 3.18, assume that  $Q = Q^1 = G$ . Let  $P' \in \mathcal{P}_{J'}$  where  $J' \subset J$  and let  $L'$  be a Levi subgroup of  $P'$ . Assume that  $\sigma, y, h, \tilde{y}$  are contained in  $\underline{L}'$ . If  $P \in \mathcal{P}^\sigma, P \subset P'$ , then  $\tau(\nu_V(P))/\tau(r) = b_V$ .*

Clearly,  $\underline{Q}^{1b_V} V \subset {}^{b_V} V$  for any  $b$ . Since  $\underline{Q}^1 = \mathfrak{g}$ , we see that  ${}^{b_V} V$  is a  $\mathfrak{g}$ -submodule of  $V$ . Since  $V$  is simple, we have  $V = {}^{b_V} V$  for some  $b$ . Since  ${}^{b_V} V \neq 0$ , we have  $V = {}^{b_V} V$ . Since  $P' \in \mathcal{P}_{J'}$ , there exists a  $\underline{P}'$ -stable line  $D$  in  $V$ . From our assumptions, we have  $h \in [\underline{P}', \underline{P}']$ . Hence  $hD = 0$  so that  $D \subset {}_0 V$ . Since  $\sigma \in \underline{P}'$ , we have  $\sigma D \subset D$  hence  $D \subset V_a$  for some  $a$ . Thus,  $D \subset {}_0 V_a \subset \tau(a)/\tau(r) V$ . Thus,  $\tau(a)/\tau(r) V \neq 0$ . Now  ${}^{b_V} V = 0$  unless  $b = b_V$ . Hence  $\tau(a)/\tau(r) = b_V$ . Since  $P \subset P'$ , we have  $\underline{P}D \subset D$ . Hence  $D_P = D$  and  $a = \nu_V(P)$ . The lemma is proved.

**Lemma 3.21.**  *$\{P \in \mathcal{P}_y^\sigma; P \subset Q\}$  is open and closed in  $\mathcal{P}_y^\sigma$ .*

(Compare [KL, 7.4].) We can find  $V \in \mathcal{I}_J$  with  $J = K$ . In terms of this  $V$  we define  $\nu_V : \mathcal{P}^\sigma \rightarrow \mathbf{C}$  and  $b_V$  as in 3.18. Since  $\mathcal{P}^\sigma$  is compact,  $\nu_V : \mathcal{P}^\sigma \rightarrow \mathbf{C}$  is

constant on any connected component of  $\mathcal{P}^\sigma$ , hence it is locally constant. Hence its restriction  $\nu_V : \mathcal{P}_y^\sigma \rightarrow \mathbf{C}$  is locally constant. Hence  $\{P \in \mathcal{P}_y^\sigma; \tau(\nu_V(P))/\tau(r) = b_V\}$  is open and closed in  $\mathcal{P}_y^\sigma$ . By 3.19(b),(d), we have

$$\{P \in \mathcal{P}_y^\sigma; \tau(\nu_V(P))/\tau(r) = b_V\} = \{P \in \mathcal{P}_y^\sigma; P \subset Q\}.$$

The lemma follows.

**3.22.** Let  $\psi : \mathcal{P}^\sigma \rightarrow \mathfrak{h}$  be the morphism whose value at  $P$  is the image of  $\sigma \in \underline{P}$  in  $\underline{P}/[\underline{P}, \underline{P}] = \mathfrak{h}$ . This must be locally constant since  $\mathcal{P}^\sigma$  is compact and  $\mathfrak{h}$  is affine. From the definitions, we have

$$\xi_V(\psi(P)) = \nu_V(P)$$

for any  $V \in \mathcal{I}, P \in \mathcal{P}^\sigma$ .

**Lemma 3.23.** *If  $V \in \mathcal{I}_J, J = K, P \in \mathcal{P}^\sigma, y \in \underline{P}, P \not\subset Q$  and  $P' \in \mathcal{P}^\sigma, P' \subset Q$ , then*

$$\tau(\xi_V(\psi(P)))/\tau(r) > \tau(\xi_V(\psi(P')))/\tau(r).$$

In view of 3.22, an equivalent statement is  $\tau(\nu_V(P))/\tau(r) > \tau(\nu_V(P'))/\tau(r)$  and this follows from 3.19(b),(d).

**3.24.** Since  $[\sigma, h] = 0$ , we have  $\sigma - rh \in \mathfrak{g}_{ss}$ . Let  $\mathcal{P}^{\sigma-rh} = \{P \in \mathcal{P}; \sigma - rh \in \underline{P}\}$ .

**Lemma 3.25.** *Let  $A = \{P \in \mathcal{P}^{\sigma-rh}; P \subset Q\}$ .*

- (a) *We have  $A \neq \emptyset$ .*
- (b) *For  $P \in A$ , let  $\mathbf{t}_P$  be the image of  $\sigma - rh$  in  $\underline{P}/[\underline{P}, \underline{P}] = \mathfrak{h}$ . Let  $V \in \mathcal{I}$ . We have  $\tau(\xi_V(\mathbf{t}_P))/\tau(r) = b_V$ .*
- (c)  *$\mathbf{t}_P \in \mathfrak{h}$  is independent of the choice of  $P \in A$ . We denote it by  $\mathbf{t}$ .*

We prove (a). We have  $\sigma - rh \in \underline{Q}$ . Hence there exists a Borel subgroup  $B$  of  $Q$  such that  $\sigma - rh \in \underline{B}$ . By 3.17, we have  $\{P \in \mathcal{P}; P \subset Q\} \neq \emptyset$ . This is a conjugacy class of parabolic subgroups of  $Q$ , hence at least one of its members contains  $B$ . This proves (a).

We prove (b). Let  $D_P$  be as in 3.18. We have  $V = \bigoplus_{c \in \mathbf{C}} {}^{(c)}V$  where

$${}^{(c)}V = \bigoplus_{n, a; a-rn=c} ({}_n V_a) = \{x \in V; (\sigma - rh)x = cx\}.$$

We have  ${}^b V = \bigoplus_{c; \tau(c)/\tau(r)=b} {}^{(c)}V$ . Define  $\nu' : \mathcal{P}^{\sigma-rh} \rightarrow \mathbf{C}$  by  $\nu'(P) = c$  where  $D_P \subset {}^{(c)}V$ . For  $P \in \mathcal{P}^{\sigma-rh}$  we have  $\xi_V(\mathbf{t}_P) = \nu'(P)$ . By 3.19(c), for  $P \in \mathcal{P}, P \subset Q$  we have  $D_P \subset {}^{b_V} V$ , that is,  $D_P \subset \bigoplus_{c; \tau(c)/\tau(r)=b_V} {}^{(c)}V$ . Thus if  $P \in A$ , then  $\nu'(P) = c$  where  $\tau(c)/\tau(r) = b_V$ . Hence  $\xi_V(\mathbf{t}_P) = c$  where  $\tau(c)/\tau(r) = b_V$ . Hence  $\tau(\xi_V(\mathbf{t}_P)) = b_V \tau(r)$ . This proves (b).

We prove (c). Let  $P', P''$  be two members of  $A$ . Let  $V \in \mathcal{I}$ . By (b), we have  $\tau(\xi_V(\mathbf{t}_{P'})) = \tau(\xi_V(\mathbf{t}_{P''}))$ . Since this holds for any  $\tau$ , we have  $\xi_V(\mathbf{t}_{P'}) = \xi_V(\mathbf{t}_{P''})$ . Since  $\xi_V$  with  $V \in \mathcal{I}$  span  $\mathfrak{h}^*$ , it follows that  $\mathbf{t}_{P'} = \mathbf{t}_{P''}$ . The lemma is proved.

**Lemma 3.26.** *Let  $P' \in \mathcal{P}^\sigma$  be such that  $y \in \underline{P'}, P' \subset Q$ . (Such  $P'$  exists by 3.17).*

- (a) *If  $i \in I - K$ , then  $-\tau(\alpha_i(\mathbf{t}))/\tau(r) > 0$ .*
- (b) *If  $i \in K$ , then  $-\tau(\alpha_i(\mathbf{t}))/\tau(r) = 0$ .*
- (c) *Let  $V \in \mathcal{I}$ . Then  $\tau(\xi_V(\psi(P') - \mathbf{t}))/\tau(r) \geq 0$ .*
- (d) *Let  $V \in \mathcal{I}_J$  where  $K \subset J \subset I$ . Then  $\tau(\xi_V(\psi(P') - \mathbf{t}))/\tau(r) = 0$ .*

Pick  $P \in A$  (see 3.25(a)). Since  $\sigma - rh$  is a semisimple element of  $\underline{Q}^1 \cap \underline{P}$  and  $P \subset Q$ , we can find a Levi subgroup  $L$  of  $P$  such that  $L \subset Q^1$  and  $\sigma - rh \in \underline{L}$ . Let  $T = Z_L^0$ . Under the obvious isomorphism  $\underline{T} \xrightarrow{\sim} \underline{P}/[\underline{P}, \underline{P}] = \mathfrak{h}$ ,  $\mathfrak{t} \in \mathfrak{h}$  corresponds to an element  $t \in \underline{T}$  such that  $t = \sigma - rh + x$  where  $x \in [\underline{P}, \underline{P}]$ . Since  $\sigma - rh \in \underline{L}$ ,  $t \in \underline{L}$ , we have  $x \in [\underline{P}, \underline{P}] \cap \underline{L} = [\underline{L}, \underline{L}]$ . Let  $i \in I$ . Now  $\mathfrak{g}^{-\alpha_i}$  (defined as in 1.6 in terms of our  $P, L$ ) is an  $\underline{L}$ -module, hence  $\text{tr}(x, \mathfrak{g}^{-\alpha_i}) = 0$  (since  $x \in [\underline{L}, \underline{L}]$ ). Let  $z_1, z_2, \dots, z_k$  be the eigenvalues of  $\text{ad}(\sigma - rh)$  on  $\mathfrak{g}^{-\alpha_i}$ .

Assume first that  $i \in K$ . Then  $\underline{L}_i \subset \underline{Q}^1$  (notation of 1.6), hence  $\mathfrak{g}^{-\alpha_i} \subset \underline{Q}^1$ . Using this and the definition of  $\underline{Q}^1$ , we have  $\tau(z_j) = 0$  for all  $j \in [1, k]$ . Then  $\text{tr}(\sigma - rh, \mathfrak{g}^{-\alpha_i}) = z_1 + \dots + z_k$  and  $\tau(\text{tr}(\sigma - rh, \mathfrak{g}^{-\alpha_i})) = \tau(z_1) + \dots + \tau(z_k) = 0 + \dots + 0 = 0$ . Now  $t$  acts on  $\mathfrak{g}^{-\alpha_i}$  as  $-\alpha_i(t)$  times the identity, hence  $\text{tr}(t, \mathfrak{g}^{-\alpha_i}) = -k\alpha_i(t)$ . Since  $t = \sigma - rh + x$ , we have  $-\tau(k\alpha_i(t)) = 0 + 0 = 0$  and  $-\tau(\alpha_i(t)) = 0$ .

Assume next that  $i \in I - K$ . Then  $\underline{P}_i \not\subset \underline{Q}$  (notation of 1.6). Using 1.6(a) we see that  $(\mathfrak{g}^{-\alpha_i} \oplus \mathfrak{g}^{-2\alpha_i}) \cap \underline{Q} = 0$ . Hence  $\mathfrak{g}^{-\alpha_i} \cap \underline{Q} = 0$ . This implies that  $\tau(z_j) > 0$  for all  $j \in [1, k]$ . Then  $\text{tr}(\sigma - rh, \mathfrak{g}^{-\alpha_i}) = z_1 + \dots + z_k$  and  $\tau(\text{tr}(\sigma - rh, \mathfrak{g}^{-\alpha_i})) = \tau(z_1) + \dots + \tau(z_k) > 0$ . Now  $\text{tr}(t, \mathfrak{g}^{-\alpha_i}) = -k\alpha_i(t)$ . Since  $t = \sigma - rh + x$  we have  $-\tau(k\alpha_i(t)) > 0$  and  $-\tau(\alpha_i(t)) > 0$ . This proves (a) and (b).

We prove (c). Using 3.25(b) and 3.22 we see that we need to prove that  $\tau(\nu_V(P'))/\tau(r) \geq b_V$ . This follows from 3.19(a).

We prove (d). The subgroup generated by the  $\xi_V$  with  $V \in \mathcal{I}_K$  contains any  $\xi_V$  with  $V \in \mathcal{P}_J$  where  $K \subset J \subset I$ . Hence it suffices to prove (d) for  $V \in \mathcal{I}_K$ . Using 3.25(b) and 3.22 we see that we must prove that  $\tau(\nu_V(P'))/\tau(r) = b_V$  when  $V \in \mathcal{I}_K$ . This follows from 3.19(d). The lemma is proved.

**3.27.** For  $x \in \mathfrak{h}$  define  ${}^\tau x \in \mathfrak{h}_{\mathbf{R}}$  (see 3.7) by  $\gamma({}^\tau x) = \tau(\gamma(x))/\tau(r)$  for all  $\gamma \in \mathcal{X}$ . Then  $x \mapsto {}^\tau x$  is a group homomorphism  $\mathfrak{h} \rightarrow \mathfrak{h}_{\mathbf{R}}$ . We can now reformulate Lemma 3.26 as follows.

**Lemma 3.28.** *Let  $P' \in \mathcal{P}^\sigma$  be such that  $y \in \underline{P}'$ ,  $P' \subset Q$ .*

- (a) *If  $i \in I - K$ , then  $-\alpha_i({}^\tau \mathfrak{t}) > 0$ .*
- (b) *If  $i \in K$ , then  $-\alpha_i({}^\tau \mathfrak{t}) = 0$ .*
- (c) *Let  $V \in \mathcal{I}$ . Then  $\xi_V({}^\tau \psi(P')) - {}^\tau \mathfrak{t} \geq 0$ .*
- (d) *Let  $V \in \mathcal{I}_J$  where  $K \subset J \subset I$ . Then  $\xi_V({}^\tau \psi(P')) - {}^\tau \mathfrak{t} = 0$ .*

**Lemma 3.29.** *Assume that  $G$  is semisimple. Let  $P' \in \mathcal{P}^\sigma$  be such that  $y \in \underline{P}'$ ,  $P' \subset Q$ . Then  ${}^0({}^\tau \psi(P')) = {}^\tau \mathfrak{t} \in \mathfrak{h}_{\mathbf{R}}$  (notation of 3.8(a)).*

This follows immediately from 3.8(a) and 3.28.

**Lemma 3.30.** *Let  $\sigma', y', h', \tilde{y}'$  be another quadruple like  $\sigma, y, h, \tilde{y}$ . Define  $Q', Q'^1, \mathfrak{t}' \in \mathfrak{h}$  in terms of  $\sigma', y', h', \tilde{y}'$  in the same way that  $Q, Q^1, \mathfrak{t} \in \mathfrak{h}$  were defined in terms of  $\sigma, y, h, \tilde{y}$ . Define  $K' \subset I$  by  $Q' \in \mathcal{P}_{K'}$ . Assume that there exist  $P_1, P_2 \in \mathcal{P}_y^\sigma$  with  $P_1 \subset Q$  and  $P'_1, P'_2 \in \mathcal{P}_{y'}^{\sigma'}$  with  $P'_1 \subset Q'$ , such that*

- (i) *the image  $\eta$  of  $\sigma$  in  $\underline{P}_1/[\underline{P}_1, \underline{P}_1] = \mathfrak{h}$  coincides with the image of  $\sigma'$  in  $\underline{P}'_1/[\underline{P}'_1, \underline{P}'_1] = \mathfrak{h}$  and*
- (ii) *the image  $\eta'$  of  $\sigma'$  in  $\underline{P}'_1/[\underline{P}'_1, \underline{P}'_1] = \mathfrak{h}$  coincides with the image of  $\sigma$  in  $\underline{P}_2/[\underline{P}_2, \underline{P}_2] = \mathfrak{h}$ .*

*Then there exists  $g \in G$  such that  $\text{Ad}(g)$  carries  $(\sigma, y, h, \tilde{y})$  to  $(\sigma', y', h', \tilde{y}')$ .*

The general case reduces easily to the case where  $G$  is semisimple. We now assume that  $G$  is semisimple. Applying 3.29 twice (once for  $\sigma, y, h, \tilde{y}, P_1$  and once for  $\sigma', y', h', \tilde{y}', P'_1$ ) we see that

$${}^0(\tau\eta) = \tau\mathbf{t}, {}^0(\tau\eta') = \tau\mathbf{t}'.$$

(Notation of 3.8(a).) Applying 3.19(a) (reformulated with the aid of 3.22 and 3.25(b)) to  $\sigma, y, h, \tilde{y}, P_2$  (where  $P_2$  is not necessarily contained in  $Q$ ) we see that

$$(a) \quad \xi_V(\tau\eta' - \tau\mathbf{t}) \geq 0$$

for any  $V \in \mathcal{I}$ . Using (a) and 3.8(b) with  $f = \tau\mathbf{t}', f' = \tau\eta$  we deduce that  $\xi_{\Lambda_i}({}^0(\tau\eta')) \geq \xi_{\Lambda_i}({}^0(\tau\mathbf{t}))$  for any  $i \in I$ . Hence  $\xi_{\Lambda_i}(\tau\mathbf{t}') \geq \xi_{\Lambda_i}(\tau\mathbf{t})$  for any  $i \in I$ . (We have  ${}^0(\tau\mathbf{t}) = \tau\mathbf{t}$ . Indeed, for any  $f \in \mathfrak{h}_{\mathbf{R}}$  we have  ${}^0({}^0f) = {}^0f$ .) By symmetry we have also  $\xi_{\Lambda_i}(\tau\mathbf{t}) \geq \xi_{\Lambda_i}(\tau\mathbf{t}')$  for any  $i \in I$ . Hence  $\xi_{\Lambda_i}(\tau\mathbf{t}) = \xi_{\Lambda_i}(\tau\mathbf{t}')$  for any  $i \in I$ . Thus, the annihilator of  $\tau\mathbf{t} - \tau\mathbf{t}'$  in  $\mathfrak{h}^*$  contains any  $\xi_{\Lambda_i}$  with  $i \in I$ . Since these elements span  $\mathfrak{h}^*$ , it follows that  $\tau\mathbf{t} = \tau\mathbf{t}'$ . Then for any  $\gamma \in \mathcal{X}$  we have  $\tau(\gamma(\mathbf{t})) = \tau(\gamma(\mathbf{t}'))$ . Since this holds for any  $\tau$ , we deduce that  $\gamma(\mathbf{t}) = \gamma(\mathbf{t}')$  for any  $\gamma \in \mathcal{X}$ . Since  $\mathcal{X}$  generates  $\mathfrak{h}^*$ , it follows that  $\mathbf{t} = \mathbf{t}'$ .

From 3.26 we see that  $K = \{i \in I; \tau(\alpha_i(\mathbf{t})) = 0\}$ . Similarly,  $K' = \{i \in I; \tau(\alpha_i(\mathbf{t}')) = 0\}$ . Since  $\mathbf{t} = \mathbf{t}'$ , it follows that  $K = K'$ . Thus, there exists  $g_1 \in G$  such that  $Q' = g_1 Q g_1^{-1}, Q'^1 = g_1 Q^1 g_1^{-1}$ . Replacing  $(\sigma', y', h', \tilde{y}')$  by a quadruple in the same  $G$ -orbit, we see that we may assume that  $Q = Q', Q^1 = Q'^1$ .

We show that  $P'_2$  is automatically contained in  $Q$ . Assume that  $P'_2 \not\subset Q' = Q$ . Let  $V \in \mathcal{I}_K$ . Applying 3.19(b) (reformulated with the aid of 3.22 and 3.25(b)) to  $\sigma', y', h', \tilde{y}', P'_2$ , we see that  $\xi_V(\tau\eta) > \xi_V(\tau\mathbf{t}')$ . Applying 3.19(d) (reformulated with the aid of 3.22 and 3.25(b)) to  $\sigma, y, h, \tilde{y}, P_1$ , we see that  $\xi_V(\tau\eta) = \xi_V(\tau\mathbf{t})$ . This contradicts the previous inequality since  $\mathbf{t} = \mathbf{t}'$ . Thus,

$$(b) \quad P'_2 \subset Q' = Q.$$

Note that all of  $\sigma, y, h, \tilde{y}, \sigma', y', h', \tilde{y}'$  are contained in  $Q^1$ . We show that these elements satisfy the hypotheses of Lemma 3.16 (with  $G$  replaced by  $Q^1$ ).

The analogues of  $Q, Q'$  (when  $G$  is replaced by  $Q^1$ ) are  $Q^1, Q^1$ . Now in  $Q^1$  we have an analogue of  $\mathcal{P}$ , namely

$$\mathcal{P}' = \{R; R = P \cap Q^1, P \in \mathcal{P}, P \subset Q\}.$$

(See 1.3.) Let  $R_1 = P_1 \cap Q^1, R'_2 = P'_2 \cap Q^1$ . Clearly  $R_1 \in \mathcal{P}'$ ; by (b), we have  $R_1, R'_2 \in \mathcal{P}'$ . Now  $\mathfrak{h}$  defined in terms of  $Q^1, \mathcal{P}'$  is canonically the same as  $\mathfrak{h}$  defined in terms of  $G, \mathcal{P}$ . From (i) and (b) we deduce that the image of  $\sigma$  in  $\underline{R_1}/[\underline{R_1}, \underline{R_1}] = \mathfrak{h}$  coincides with the image of  $\sigma'$  in  $\underline{R'_2}/[\underline{R'_2}, \underline{R'_2}] = \mathfrak{h}$ . Thus, 3.16 is applicable and  $(\sigma, y, h, \tilde{y}), (\sigma', y', h', \tilde{y}')$  are conjugate under an element of  $Q^1$ . The lemma is proved.

**Lemma 3.31.** *Assume that  $G$  is semisimple. The following two conditions on  $(y, \sigma, r)$  are equivalent:*

- (i) *For any  $P \in \mathcal{P}_y^\sigma$  and any  $V \in \mathcal{I}$  we have  $\xi_V(\tau\psi(P)) \geq 0$ .*
- (ii)  *$Q = G$ .*

Assume first that  $Q = G$ . By 3.28(b) we have  $\alpha_i(\tau\mathbf{t}) = 0$  for all  $i \in I$ . Hence  $\tau\mathbf{t} = 0$ . By 3.28(c), for any  $P \in \mathcal{P}_y^\sigma$  and any  $V \in \mathcal{I}$  we have  $\xi_V(\tau\psi(P) - \tau\mathbf{t}) \geq 0$ , hence  $\xi_V(\tau\psi(P)) \geq 0$ . Thus (ii) implies (i).

Next, assume that  $Q \neq G$ , that is,  $K \neq I$ . Let  $i \in I - K$  and let  $V \in \mathcal{I}_{I-\{i\}}$ . Then  $\xi_V = \sum_{j \in I} z_j \alpha_j$  (in  $\mathfrak{h}^*$ ) where  $z_j \geq 0$  for all  $j$  and  $z_i > 0$ . By 3.28(a),(b) we have  $\alpha_j(\tau \mathbf{t}) \leq 0$  for all  $j \in I$  and  $\alpha_i(\tau \mathbf{t}) < 0$ . Hence

$$\xi_V(\tau \mathbf{t}) = \sum_{j \in I} z_j \alpha_j(\tau \mathbf{t}) < 0.$$

Now let  $P \in \mathcal{P}_y^\sigma$  be such that  $P \subset Q$ . By 3.28(d) we have  $\xi_V(\tau \psi(P) - \tau \mathbf{t}) = 0$ , hence  $\xi_V(\tau \psi(P)) < 0$ . Thus (i) implies (ii). The lemma is proved.

**Lemma 3.32.** *Assume that  $G$  is semisimple. The following four conditions on  $(y, \sigma, r)$  are equivalent:*

- (i) *If  $P' \in \mathfrak{P}$  and  $L'$  is a Levi subgroup of  $P'$  such that  $\sigma, y$  are contained in  $\underline{L}'$ , then  $P' = G$ .*
- (ii)  *$y$  is a distinguished nilpotent element of  $\mathfrak{g}$  and there exists  $\hat{y} \in \underline{G}_{-2r}$  such that  $[y, \hat{y}] = r^{-1}\sigma$ .*
- (iii) *For any  $P \in \mathcal{P}_y^\sigma$  and any  $V \in \mathcal{I}, V \neq \mathbf{C}$ , we have  $\xi_V(\psi(P)) = nr$  where  $n \in \mathbf{N} - \{0\}$ .*
- (iv) *For any  $P \in \mathcal{P}_y^\sigma$  and any  $V \in \mathcal{I}, V \neq \mathbf{C}$ , we have  $\xi_V(\tau \psi(P)) > 0$ .*

The fact that (i) implies (ii) is proved in [L1, 2.5].

We show that (ii) implies (iii). Assume that (ii) holds. Let  $\mathfrak{s} = \mathbf{C}y + \mathbf{C}\sigma + \mathbf{C}\hat{y}$  (a homomorphic image of  $\mathfrak{sl}_2(\mathbf{C})$ ). To prove (iii) it is enough to verify the following statement.

*Let  $P \in \mathcal{P}_y^\sigma$  and let  $V \in \mathcal{I}, V \neq \mathbf{C}$ . Let  $D_P$  be the  $\underline{P}$ -stable line in  $V$ . Then  $\sigma$  acts on  $D_P$  as multiplication by  $rn$  where  $n \in \mathbf{N} - \{0\}$ . (Compare [L1, 2.8].)*

From the representation theory of  $\mathfrak{sl}_2$ , we see that  $\sigma$  acts on  $D_P$  as  $rn$  where  $n \in \mathbf{N}$ . (Use that  $y$  acts as 0 on  $D_P$ .) Assume that  $\sigma$  acts on  $D_P$  as 0. Then  $D_P$  must be stable under  $\mathfrak{s}$  which acts on it by 0. Now let  $P' \in \mathfrak{P}$  be such that  $\underline{P}'$  is the stabilizer of  $D_P$  in  $\underline{G}$  (we have  $P' \neq G$ ). Then  $\mathfrak{s} \subset \underline{P}'$ , hence  $\mathfrak{s}$  is contained in a Levi subalgebra of  $\underline{P}'$ . Thus,  $y$  is not distinguished in  $\underline{G}$ , contradicting (ii). Thus (ii) implies (iii).

It is clear that (iii) implies (iv).

Assume now that (iv) holds and (i) does not hold. Then we can find  $P' \in \mathfrak{P}$ ,  $P' \neq G$ , and a Levi subgroup  $L'$  of  $P'$  such that  $\sigma \in \underline{L}', y \in \underline{L}'$ . Replacing if necessary  $P', L'$  by a  $G$ -conjugate, we may assume that  $\sigma, y, h, \hat{y}$  are all contained in  $\underline{L}'$ . Let  $A_1 = \{P \in \mathcal{P}_y^\sigma; P \subset P'\}$ . By 3.10 (applied to  $P'$  instead of  $Q$ ), we have  $A_1 \neq \emptyset$ . We have  $P' \in \mathcal{P}_J, J \neq I$ . Since (iv) holds, we see from 3.31 that  $Q = Q^1 = G$  and from its proof, that  $\tau \mathbf{t} = 0$ . Let  $V \in \mathcal{I}_J$ . Then  $V \neq \mathbf{C}$ . Let  $P \in A_1$ . By 3.20 (reformulated with the aid of 3.22 and 3.25(b)) we have  $\xi_V(\tau \psi(P) - \tau \mathbf{t}) = 0$ . Since  $\tau \mathbf{t} = 0$ , it follows that  $\xi_V(\tau \psi(P)) = 0$ . This contradicts (iv) since  $V \neq \mathbf{C}$ . Thus, (iv) implies (i). The lemma is proved.

**3.33.** Let  $E$  be an  $\mathbf{S}$ -module of finite dimension over  $\mathbf{C}$ . There is a canonical direct sum decomposition  $E = \bigoplus_{\eta \in \mathfrak{h}} \eta E$  where  $\eta E$  (a *weight space*) is the set of all  $x \in E$  such that for any  $\xi \in \mathfrak{h}^*$ ,  $\xi : E \rightarrow E$  is given on  $\eta E$  by multiplication by  $\xi(\eta)$  plus a nilpotent endomorphism of  $\eta E$ .

**3.34.** The torus  $G' = \langle (\sigma, r) \rangle$  in  $G \times \mathbf{C}^*$  is well defined (see 1.1). The fixed point set of the  $G'$ -action on  $\mathfrak{g}$  (restriction of the  $G \times \mathbf{C}^*$  action) is

$$\mathfrak{g}^{G'} = \{(y', P) \in \mathfrak{g}; [\sigma, y'] = 2ry', \sigma \in \underline{P}\}.$$

Let  $pt$  denote a point  $(y', P)$  of  $\mathfrak{g}^{G'}$ . Since  $\sigma \in \underline{P}$ , we have  $\underline{G}' \subset \underline{P} \oplus \mathbf{C}$ .

Let  $k$  be the composition

$$\mathbf{S} = H_{G \times \mathbf{C}^*}^*(\mathfrak{g}, \mathbf{C}) \rightarrow H_{G'}^*(\mathfrak{g}, \mathbf{C}) \rightarrow H_{G'}^*(pt, \mathbf{C}) \rightarrow \mathbf{C}_{\sigma, r}$$

where the first map is as in [L4, 1.4(g)], the second map is  $j^*$  (see [L4, 1.4(a)] attached to the imbedding  $j : pt \rightarrow \mathfrak{g}$  and the third map is the quotient defined as in 1.13. For  $\xi \in \mathfrak{h}^*$  (a subset of  $\mathbf{S}$ ) we have from the definitions:

$$(a) \quad k(\xi) = \xi'(\sigma, r)$$

where  $\xi'$  is the linear form on  $\underline{G}'$  given by the composition  $\underline{G}' \rightarrow \underline{P} \rightarrow \underline{P}/[\underline{P}, \underline{P}] = \mathfrak{h} \xrightarrow{\xi} \mathbf{C}$  (the unspecified maps are the obvious ones).

The image of  $\psi : \mathcal{P}^\sigma \rightarrow \mathfrak{h}$  (as in 3.22) is a finite subset  $D$  of  $\mathfrak{h}$ . From the definitions we have  $\xi'(\sigma, r) = \xi(\psi(P))$ , hence (a) can be rewritten as follows:

$$(b) \quad k(\xi) = \xi(\psi(P)).$$

Now let  $\tilde{X}$  be a subvariety of  $\mathfrak{g}^{G'}$ . By 1.11,

$$H_*^{G'}(\tilde{X}, \dot{\mathcal{L}}) = H_{G'}^*(\tilde{X}, \mathbf{C}) \otimes H_*(\tilde{X}, \dot{\mathcal{L}})$$

(see [L6, 1.21]) is naturally an  $\mathbf{S}$ -module. Hence

$$A = \mathbf{C}_{\sigma, r} \otimes_{H_{G'}^*} H_*^{G'}(\tilde{X}, \dot{\mathcal{L}}) = H_*(\tilde{X}, \dot{\mathcal{L}})$$

(where  $\mathbf{C}_{\sigma, r} = H_{G'}^*/\mathcal{I}_{\sigma, r}^{G'}$  is as in 1.13) is again an  $\mathbf{S}$ -module.

We have a morphism  $\tilde{X} \rightarrow D \subset \mathfrak{h}$  given by  $(y', P) \mapsto \psi(P)$ . Consider the partition  $\tilde{X} = \bigsqcup_{\delta \in \Delta} \tilde{X}^\delta$  of  $\tilde{X}$  into connected components ( $\Delta$  is the set of irreducible components of  $\tilde{X}$ .) Now each connected component  $\tilde{X}^\delta$  is mapped by  $\tilde{X} \rightarrow D$  to a single point of  $D$  denoted  $\psi(\delta)$ . Since  $\tilde{X}^\delta$  is open and closed in  $\tilde{X}$  and is  $G'$ -stable, we may identify canonically  $A = \bigoplus_{\delta \in \Delta} A^\delta$  where  $A^\delta = H_*(\tilde{X}^\delta, \dot{\mathcal{L}})$ . Clearly this direct sum decomposition is compatible with the  $\mathbf{S}$ -module structure.

**Lemma 3.35.** *For any  $\eta \in \mathfrak{h}$  we have  ${}_\eta A = \bigoplus_{\delta \in \Delta; \psi(\delta) = \eta} A^\delta$ .*

We may assume that  $\tilde{X}$  is connected. Let  $d \in D$  be defined by  $d = \psi(P)$  for any  $(y, P) \in \tilde{X}$ . Let  $\xi \in \mathfrak{h}^*$ . We must show that  $\xi - \xi(d)$  acts nilpotently on  $A$ . Let  $\tilde{\xi}$  be the image of  $\xi$  under the composition

$$\begin{aligned} \mathbf{S} &= H_{G \times \mathbf{C}^*}^*(\mathfrak{g}, \mathbf{C}) \rightarrow H_{G'}^*(\mathfrak{g}, \mathbf{C}) \rightarrow H_{G'}^*(\tilde{X}, \mathbf{C}) \\ &= H_{G'}^* \otimes H^*(\tilde{X}, \mathbf{C}) \rightarrow \mathbf{C}_{\sigma, r} \otimes H^*(\tilde{X}, \mathbf{C}) \end{aligned}$$

where the first map is as in [L4, 1.4(g)], the second map is  $\tilde{m}^*$  (see [L4, 1.4(a)] attached to the imbedding  $\tilde{m} : \tilde{X} \rightarrow \mathfrak{g}$ , and the third map is induced by the quotient defined as in 1.13. The action of  $\xi$  on  $A$  is multiplication by  $\tilde{\xi} \in H^*(\tilde{X}, \mathbf{C})$  on  $H_*(\tilde{X}, \dot{\mathcal{L}})$ . We have  $\tilde{\xi} = \tilde{\xi}_0 + \tilde{\xi}_>$  where  $\tilde{\xi}_0 \in H^0(\tilde{X}, \mathbf{C})$  and  $\tilde{\xi}_> \in \bigoplus_{n>0} H^n(\tilde{X}, \mathbf{C})$ . Clearly, multiplication by  $\tilde{\xi}_>$  on  $H_*(\tilde{X}, \dot{\mathcal{L}})$  is nilpotent. Since  $\tilde{X}$  is connected, we have  $\tilde{\xi}_0 = c\mathbf{1}$  where  $c \in \mathbf{C}$  and  $\mathbf{1} \in H^0(\tilde{X}, \mathbf{C})$  is the unit element of the algebra  $H^*(\tilde{X}, \mathbf{C})$  (which acts as the identity on  $H_*(\tilde{X}, \dot{\mathcal{L}})$ ). It is then enough to show that  $c = \xi(d)$ . Let  $pt$  denote a point  $(y', P)$  of  $\tilde{X}$ . Let  $j' : pt \rightarrow \tilde{X}$  be the imbedding.

From the definitions,  $j'^* : H^*(\tilde{X}, \mathbf{C}) \rightarrow H^*(pt, \mathbf{C}) = \mathbf{C}$  carries  $\tilde{\xi}$  to  $k(\xi)$  (as in 4.2), that is, to  $\xi(\psi(P)) = \xi(d)$  (see 4.2(b)). It automatically carries  $\tilde{\xi}_>$  to 0 hence it carries  $\tilde{\xi}_0$  to  $\xi(d)$ . It also preserves unit elements. Hence it carries  $c\mathbf{1}$  to  $c$ . Since  $\tilde{\xi}_0 = c\mathbf{1}$ , it follows that  $c = \xi(d)$ . The lemma is proved.

**3.36.** Let  $G' = \langle(\sigma, r)\rangle \subset G \times \mathbf{C}^*$ . Since  $(\sigma, r) \in \overline{M^0(y)}$ , we have  $G' \subset M^0(y)$ . Let  $\mathcal{M} = E_{y, \sigma, r}$  be as in 1.13 (recall that the choice of  $G'$  in 1.13 is immaterial; in particular, we may take  $G' = \langle(\sigma, r)\rangle$ ). For  $\rho \in \text{Irr}_0 \bar{M}(y, \sigma)$  we set  $\mathcal{M}_\rho = E_{y, \sigma, r, \rho}$ .

The fixed point set of the  $G'$ -action on  $\mathcal{P}_y$  (restriction of the  $M(y)$ -action) is just  $\mathcal{P}_y^\sigma$ . By the localization theorem [L6, 4.4(b)] (which is applicable in view of the odd vanishing theorem [L4, 8.6]), the imbedding  $j : \mathcal{P}_y^\sigma \rightarrow \mathcal{P}_y$  induces an isomorphism

$$j! : \mathbf{C}_{\sigma, r} \otimes_{H_{G'}^*} H_*^{G'}(\mathcal{P}_y^\sigma, \dot{\mathcal{L}}) \xrightarrow{\sim} \mathbf{C}_{\sigma, r} \otimes_{H_{G'}^*} H_*^{G'}(\mathcal{P}_y, \dot{\mathcal{L}})$$

or equivalently

$$(a) \quad A \xrightarrow{\sim} \mathcal{M}.$$

Here  $A$  (as in 3.34) is an  $\mathbf{S}$ -module and  $\mathcal{M}$  is an  $\mathbf{H}$ -module (in particular, an  $\mathbf{S}$ -module via the obvious algebra homomorphism  $\mathbf{S} \rightarrow \mathbf{H}$ ). The isomorphism (a) is compatible with  $\mathbf{S}$ -module structures. Now the direct sum decomposition  $A = \bigoplus_{\delta \in \Delta} A^\delta$  in 3.34 (where  $\Delta$  is the set of irreducible components of  $\mathcal{P}_y^\sigma$ ) corresponds under (a) to a direct sum decomposition

$$(b) \quad \mathcal{M} = \bigoplus_{\delta \in \Delta} \mathcal{M}^\delta$$

and we can reformulate 3.35 as follows:

$$(c) \quad \text{For any } \eta \in \mathfrak{h} \text{ we have } \eta \mathcal{M} = \bigoplus_{\delta \in \Delta; \psi(\delta) = \eta} \mathcal{M}^\delta.$$

Here  $\eta \mathcal{M}$  are the weight spaces of  $\mathcal{M}$ .

Since the  $\bar{M}(y, \sigma)$ -action on  $\mathcal{M}$  commutes with the  $\mathbf{H}$ -module structure, each weight space  $\eta \mathcal{M}$  is  $\bar{M}(y, \sigma)$ -stable. It follows that for  $\rho \in \text{Irr} \bar{M}(y, \sigma)$ , we have

$$(d) \quad \eta(\mathcal{M}_\rho) = \text{Hom}_{\bar{M}(y, \sigma)}(\rho, \eta \mathcal{M}).$$

Now the  $M(y)$ -action on  $\mathcal{P}_y$  restricts to an  $M(y, \sigma)$ -action on  $\mathcal{P}_y^\sigma$ . Hence  $\bar{M}(y, \sigma) = M(y, \sigma)/M(y, \sigma)^0$  acts naturally on  $\Delta$ . Let  $\bar{\Delta}$  be the set of  $\bar{M}(y, \sigma)$ -orbits on  $\Delta$  and let  $\delta \mapsto \bar{\delta}$  be the canonical map  $\Delta \rightarrow \bar{\Delta}$ . Let  $\rho \in \text{Irr} \bar{M}(y, \sigma)$ . We have

$$\mathcal{M}_\rho = \bigoplus_{\epsilon \in \bar{\Delta}} \mathcal{M}_\rho^\epsilon$$

where  $\mathcal{M}_\rho^\epsilon = \text{Hom}_{\bar{M}(y, \sigma)}(\rho, \bigoplus_{\delta \in \Delta; \bar{\delta} = \epsilon} \mathcal{M}^\delta)$ . Now  $\psi : \Delta \rightarrow \mathfrak{h}$  is constant on  $\bar{M}(y, \sigma)$ -orbits hence it induces a map  $\bar{\psi} : \bar{\Delta} \rightarrow \mathfrak{h}$ . Then, from (c) and (d) we deduce:

$$(e) \quad \text{for any } \eta \in \mathfrak{h} \text{ we have } \eta(\mathcal{M}_\rho) = \bigoplus_{\epsilon \in \bar{\Delta}; \bar{\psi}(\epsilon) = \eta} \mathcal{M}_\rho^\epsilon.$$

Let  $\mathcal{P}^* = \{P \in \mathcal{P}; P \subset Q\}$ . Let  $\Delta^1$  (resp.  $\Delta^2$ ) be the set of irreducible components of  $\mathcal{P}_y^\sigma$  that are contained in  $\mathcal{P}^*$  (resp. in  $\mathcal{P} - \mathcal{P}^*$ ). By 3.21, we have a partition  $\Delta = \Delta^1 \sqcup \Delta^2$ . Let

$$\mathcal{M}^1 = \bigoplus_{\delta \in \Delta^1} \mathcal{M}^\delta, \mathcal{M}^2 = \bigoplus_{\delta \in \Delta^2} \mathcal{M}^\delta.$$

Then  $\mathcal{M} = \mathcal{M}^1 \oplus \mathcal{M}^2$ . Now the summands  $\mathcal{M}^1, \mathcal{M}^2$  of  $\mathcal{M}$  are stable under  $\bar{M}(y, \sigma)$  (since  $M(y, \sigma) \subset Q \times \mathbf{C}^*$ , see 3.15). Hence, setting

$$\mathcal{M}_\rho^1 = \text{Hom}_{\bar{M}(y, \sigma)}(\rho, \mathcal{M}^1), \mathcal{M}_\rho^2 = \text{Hom}_{\bar{M}(y, \sigma)}(\rho, \mathcal{M}^2),$$

we have  $\mathcal{M}_\rho = \mathcal{M}_\rho^1 \oplus \mathcal{M}_\rho^2$ .

**Lemma 3.37.** (a) *Each of  $\mathcal{M}^1$  and  $\mathcal{M}^2$  is a sum of weight spaces of  $\mathcal{M}$ .*  
 (b) *Each of  $\mathcal{M}_\rho^1$  and  $\mathcal{M}_\rho^2$  is a sum of weight spaces of  $\mathcal{M}_\rho$ .*

Using the inclusion  $\mathcal{M}^\delta \subset \psi(\delta)\mathcal{M}$  (see 3.36(c)) we see that

(c)  $\mathcal{M}^1 \subset \mathcal{M}^{(1)}, \mathcal{M}^2 \subset \mathcal{M}^{(2)}$  where

$$\mathcal{M}^{(1)} = \sum_{\delta; \delta \in \Delta^1} \psi(\delta)\mathcal{M}, \mathcal{M}^{(2)} = \sum_{\delta; \delta \in \Delta^2} \psi(\delta)\mathcal{M}.$$

We show that

(d)  $\mathcal{M}^{(1)} \cap \mathcal{M}^{(2)} = 0$ .

Since the weight spaces of  $\mathcal{M}$  form a direct sum, it is enough to show that  $\psi(\delta) \neq \psi(\delta')$  for any  $\delta, \delta' \in \Delta$  such that  $\delta \in \mathcal{P}^*, \delta' \in \mathcal{P} - \mathcal{P}^*$  or equivalently, that  $\psi(P) \neq \psi(P')$  for any  $P, P' \in \mathcal{P}_y^\sigma$  such that  $P \subset Q, P' \not\subset Q$ .

Let  $V \in \mathcal{I}_K$ . By 3.19(b),(d) we have  $\tau(\nu_V(P))/\tau(r) = b_V, \tau(\nu_V(P'))/\tau(r) > b_V$ , hence  $\nu_V(P) \neq \nu_V(P')$ . Hence  $\xi_V(\psi(P)) \neq \xi_V(\psi(P'))$ . Hence  $\psi(P) \neq \psi(P')$ , as desired, and (d) is proved.

Since  $\mathcal{M} = \mathcal{M}^1 \oplus \mathcal{M}^2$ , we see from (c) and (d) that  $\mathcal{M}^1 = \mathcal{M}^{(1)}, \mathcal{M}^2 = \mathcal{M}^{(2)}$ . Since each  $\mathcal{M}^{(1)}, \mathcal{M}^{(2)}$  is a sum of weight spaces of  $\mathcal{M}$ , (a) follows.

We prove (b). Let  $k \in \{1, 2\}$ . By (a) we have  $\mathcal{M}^k = \bigoplus_{n=1}^N (\eta_n \mathcal{M})$  where  $\eta_n$  are distinct elements of  $\mathfrak{h}$ . It follows that

$$\mathcal{M}_\rho^k = \text{Hom}_{\bar{M}(y, \sigma)}(\rho, \mathcal{M}^k) = \bigoplus_{n=1}^N \text{Hom}_{\bar{M}(y, \sigma)}(\rho, \eta_n \mathcal{M}) = \bigoplus_{n=1}^N (\eta_n \mathcal{M}_\rho).$$

The lemma is proved.

**3.38.** Replacing  $G, y, \sigma, r$  by  $Q^1, y, \sigma, r$  in the definition of  $W, \mathfrak{h}, \mathbf{S}, \mathbf{H}, \mathcal{P}_y, \mathcal{P}_y^\sigma, \dot{\mathcal{L}}, \mathcal{M}$  we obtain (as in 1.16)  $W_K, \mathfrak{h}, \mathbf{S}, \mathbf{H}', \mathcal{P}'_y, \mathcal{P}'_y^\sigma, \dot{\mathcal{L}}, \mathcal{M}'$ . Since  $M(y, \sigma) \subset Q$ , the analogue of  $M(y, \sigma)$  for  $Q^1$  instead of  $G$  is a quotient of  $M(y, \sigma)$  by a unipotent normal subgroup. Hence  $\bar{M}(y, \sigma)$  defined in terms of  $Q^1$  is the same as that defined in terms of  $G$ . Hence for  $\rho \in \text{Irr} \bar{M}(\sigma, y)$ , we can define  $\mathcal{M}'_\rho$  in terms of  $\mathcal{M}'$  in the same way as  $\mathcal{M}_\rho$  was defined in terms of  $\mathcal{M}$ . By 1.18 we have an isomorphism

$$(a) \quad \Psi : \mathbf{H} \otimes_{\mathbf{H}'} \mathcal{M}' \xrightarrow{\sim} \mathcal{M}.$$

Then  $x \mapsto \Psi(1 \otimes x)$  is a map  $\mathcal{M}' \rightarrow \mathcal{M}$ .

(b) This is an isomorphism of  $\mathcal{M}'$  onto  $\mathcal{M}^1$ .

Indeed, using the localization theorem [L6, 4.4(b)],  $\mathcal{M}' \rightarrow \mathcal{M}$  may be identified with the map

$$\mathbf{C}_{\sigma, r} \otimes_{H_{G'}}^* H_*^{G'}(\mathcal{P}'_y^\sigma, \dot{\mathcal{L}}) \rightarrow \mathbf{C}_{\sigma, r} \otimes_{H_{G'}}^* H_*^{G'}(\mathcal{P}_y^\sigma, \dot{\mathcal{L}})$$

induced by the obvious inclusion  $\mathcal{P}'_y^\sigma \subset \mathcal{P}_y^\sigma$  whose image is the open and closed subset  $\mathcal{P}_y^\sigma \cap \mathcal{P}^*$  of  $\mathcal{P}_y^\sigma$ .

Now for any  $\rho \in \text{Irr} \bar{M}(\sigma, y)$ , (a) induces an isomorphism

$$(c) \quad \mathbf{H} \otimes_{\mathbf{H}'} \mathcal{M}'_\rho \xrightarrow{\sim} \mathcal{M}_\rho$$

and (b) induces an isomorphism

$$(d) \quad \mathcal{M}'_\rho \xrightarrow{\sim} \mathcal{M}_\rho^1.$$

Since  $\mathbf{H}' \rightarrow \mathbf{H}$  is injective and  $\mathbf{H}$  is a free  $\mathbf{H}'$ -module, from (c) we deduce that

$$(e) \quad \mathcal{M}'_\rho \neq 0 \leftrightarrow \mathcal{M}_\rho \neq 0.$$

**3.39. Proof of Theorem 1.15(a).** Note that in the setup of 1.15(a) we have  $E_{y,\sigma,r} \neq 0$ , hence we are also in the setup of 3.9.

By 3.38(e) we have  $\mathcal{M}'_\rho \neq 0$ . As in the proof of 3.16 we see that  $y$  belongs to the unique open orbit of  $Z(\sigma) \cap Q^1$  in  $\mathfrak{g}_{2r} \cap \mathfrak{q}^1$ . Hence we may apply [L4, 8.17(b)] to  $Q^1$  instead of  $G$  and conclude that  $\mathcal{M}'_\rho$  is a simple  $\mathbf{H}'$ -module.

Let  $F$  be a proper  $\mathbf{H}$ -submodule of  $\mathcal{M}_\rho$ . If  $F \cap \mathcal{M}_\rho^1 = \mathcal{M}_\rho^1$ , then, since  $\mathcal{M}_\rho^1$  generates  $\mathcal{M}_\rho$  as a  $\mathbf{H}$ -module (see 3.28(c),(d)), it follows that  $F$  generates  $\mathcal{M}_\rho$  as a  $\mathbf{H}$ -module, hence  $F = \mathcal{M}_\rho$  contradicting the assumption that  $F$  is proper.

Thus we must have  $F \cap \mathcal{M}_\rho^1 \neq \mathcal{M}_\rho^1$ . Since  $\mathcal{M}_\rho^1$  is a simple  $\mathbf{H}'$ -module and  $F \cap \mathcal{M}_\rho^1$  is a proper  $\mathbf{H}'$ -submodule of  $\mathcal{M}_\rho^1$ , it follows that

(a)  $F \cap \mathcal{M}_\rho^1 = 0$ .

For any  $\eta \in \mathfrak{h}$  we have  $_\eta F \subset _\eta(\mathcal{M}_\rho)$  and by 3.37(b),  $_\eta(\mathcal{M}_\rho)$  is contained either in  $\mathcal{M}_\rho^1$  or in  $\mathcal{M}_\rho^2$ . Thus, we have either  $_\eta F \subset \mathcal{M}_\rho^1$  or  $_\eta F \subset \mathcal{M}_\rho^2$ . The first alternative cannot occur if  $_\eta F \neq 0$  by (a). Thus, we have  $_\eta F \subset \mathcal{M}_\rho^2$  for any  $\eta$  hence  $F \subset \mathcal{M}_\rho^2$ . It follows that the sum of all proper  $\mathbf{H}$ -submodules of  $\mathcal{M}_\rho$  is contained in  $\mathcal{M}_\rho^2$  and thus it is itself a proper submodule (since  $\mathcal{M}_\rho^1$  is  $\neq 0$ , being isomorphic to  $\mathcal{M}'_\rho$ ). This proves 1.15(a).

**3.40.** Let  $\rho \in \text{Irr}_0 \bar{M}(\sigma, y)$ . Let  $\mathcal{M}_{\rho, \max}$  be the unique maximal  $\mathbf{H}$ -submodule of  $\mathcal{M}_\rho$ . Recall that  $\mathcal{M}_{\rho, \max} \subset \mathcal{M}_\rho^2$ . It follows that

(a) *The obvious map  $\mathcal{M}'_\rho = \mathcal{M}_\rho^1 \rightarrow \mathcal{M}_\rho / \mathcal{M}_{\rho, \max}$  is injective.*

Let  $V \in \mathcal{I}_K$ . Let

$$X = \{\eta \in \mathfrak{h}; _\eta(\mathcal{M}_\rho / \mathcal{M}_{\rho, \max}) \neq 0, \tau(\xi_V(\eta)) / \tau(r) \text{ is minimum possible}\}.$$

Note that  $X$  is well defined (see the proof of 3.37) and the minimum value in the definition of  $X$  is  $b_V$ . From the proof of 3.37 we see also that

(b) *The image of the map in (a) equals  $\sum_{\eta \in X} _\eta(\mathcal{M}_\rho / \mathcal{M}_{\rho, \max})$ .*

**3.41. Proof of injectivity in Theorem 1.15(b).** Let  $y^!, \sigma^!, h^!, \tilde{y}^!$  be another quadruple like  $y, \sigma, h, \tilde{y}$  (with the same  $r$ ). Let  $\rho \in \text{Irr}_0 \bar{M}(\sigma, y)$ ,  $\rho^! \in \text{Irr}_0 \bar{M}(\sigma^!, y^!)$ . Define

$$Q^!, Q^{!1}, \mathcal{M}_{\rho^!}^1, \mathcal{M}_{\rho^!}^{'1}, \mathcal{M}_{\rho^!}^{!1}, \mathcal{M}_{\rho^!, \max}^1, \psi'$$

in terms of  $y^!, \sigma^!, h^!, \tilde{y}^!, \rho^!$  in the same way as

$$Q, Q^1, \mathcal{M}_\rho, \mathcal{M}'_\rho, \mathcal{M}_\rho^1, \mathcal{M}_{\rho, \max}, \psi$$

were defined in terms of  $y, \sigma, h, \tilde{y}, \rho$ . Assume that

(a)  $\mathcal{M}_\rho / \mathcal{M}_{\rho, \max} \cong \mathcal{M}_{\rho^!}^1 / \mathcal{M}_{\rho^!, \max}^1$

as  $\mathbf{H}$ -modules. We show that there exists  $g \in G$  which conjugates  $y, \sigma, h, \tilde{y}, \rho$  to  $y^!, \sigma^!, h^!, \tilde{y}^!, \rho^!$ .

We can find  $\eta \in \mathfrak{h}$  such that  $_\eta(\mathcal{M}'_\rho) \neq 0$ . By 3.40(a), we have  $_\eta(\mathcal{M}_\rho / \mathcal{M}_{\rho, \max}) \neq 0$ , hence by (a),  $_\eta(\mathcal{M}_{\rho^!}^1 / \mathcal{M}_{\rho^!, \max}^1) \neq 0$ .

It follows that  $_\eta \mathcal{M}^1 \neq 0$ ,  $_\eta(\mathcal{M}_{\rho^!}^1) \neq 0$ .

Using 3.36(c) we deduce that there exist  $P_1 \in \mathcal{P}_y^\sigma$  and  $P_2' \in \mathcal{P}_{y'}^{\sigma'}$  such that  $P_1 \subset Q$  and

$$\psi(P_1) = \eta = \psi^1(P_2').$$

By symmetry, there exist  $P_1' \in \mathcal{P}_{y'}^{\sigma'}$ ,  $P_2 \in \mathcal{P}_y^\sigma$  and  $\eta' \in \mathfrak{h}$  such that  $P_1' \subset Q$  and

$$\psi^1(P_1') = \eta' = \psi(P_2).$$

Then the assumptions of 3.30 are verified and we see that there exists  $g \in G$  which conjugates  $y, \sigma, h, \tilde{y}$  to  $y^!, \sigma^!, h^!, \tilde{y}^!$ . Thus, we may assume that  $y^! = y, \sigma^! = \sigma, h^! = h, \tilde{y}^! = \tilde{y}$ . Then

$$\begin{aligned} Q^! &= Q, Q^{1!} = Q^1, \mathcal{M}_{\rho^!}^! = \mathcal{M}_{\rho^!}, \mathcal{M}_{\rho^!}^{\prime!} = \mathcal{M}_{\rho^!}^{\prime}, \\ \mathcal{M}_{\rho^!}^{1!} &= \mathcal{M}_{\rho^!}^1, \mathcal{M}_{\rho^!, \max}^! = \mathcal{M}_{\rho^!, \max}, \psi' = \psi. \end{aligned}$$

Let  $V \in \mathcal{I}_K$ . Consider an  $\mathbf{H}$ -linear isomorphism  $\mathcal{M}_{\rho}/\mathcal{M}_{\rho, \max} \xrightarrow{\sim} \mathcal{M}_{\rho^!}/\mathcal{M}_{\rho^!, \max}$ . This clearly carries the subspace

$$\sum_{\eta \in X} \eta(\mathcal{M}_{\rho}/\mathcal{M}_{\rho, \max})$$

( $X$  as in 3.40) onto the subspace

$$\sum_{\eta \in X} \eta(\mathcal{M}_{\rho^!}/\mathcal{M}_{\rho^!, \max})$$

(here  $X$  is as in 3.40, and the analogous set for  $\mathcal{M}_{\rho^!}$  is again  $X$ ); hence, by 3.40, it carries the subspace  $\mathcal{M}_{\rho}^1$  isomorphically onto the subspace  $\mathcal{M}_{\rho^!}^1$ . Hence it induces an isomorphism of  $\mathbf{H}'$ -modules  $\mathcal{M}_{\rho}^{\prime} \xrightarrow{\sim} \mathcal{M}_{\rho^!}^1$ . As in the proof of 3.16 we see that  $y$  belongs to the unique open orbit of  $Z(\sigma) \cap Q^1$  in  $\mathfrak{g}_{2r} \cap \mathfrak{q}^1$ . We now apply [L4, 8.17(c)] to  $Q^1$  instead of  $G$  and deduce that  $\rho^! = \rho$ . This completes the proof.

**3.42. Proof of surjectivity in Theorem 1.15(b).** A statement close to the surjectivity in Theorem 1.15(b) was stated in [L4, 8.15] but the proof given there has an error in line -7 of p. 199 (“Since  $H_{M^0(y)}^*/I''$  is an artinian  $\mathbf{C}$ -algebra...”). (I thank David Vogan for pointing out that error). In the part of the proof preceding that line it was shown that:

- (a) if  $\mathcal{N}$  is a simple  $\mathbf{H}$ -module, then there exists  $y \in \mathfrak{g}_N$ , an ideal  $J$  of finite codimension in  $\mathcal{A} = H_{M^0(y)}^*$  and a nonzero  $\mathbf{H}$ -linear map  $X/JX \rightarrow \mathcal{N}$  where  $X = H_*^{M^0(y)}(\mathcal{P}_y, \dot{\mathcal{L}})$ .

We continue the proof starting from (a). We have  $\dim_{\mathbf{C}}(X/JX) < \infty$  (this can be seen from [L4, 7.2] using [L4, 8.6]). Let  $X/JX \rightarrow X'$  be the largest semisimple quotient of the  $\mathbf{H}$ -module  $X/JX$ . Then  $X'$  inherits from  $X/JX$  an  $\mathcal{A}$ -module structure. Clearly, there exists a nonzero  $\mathbf{H}$ -linear map  $X' \rightarrow \mathcal{N}$ . Let  $X'_{\mathcal{N}}$  be the  $\mathcal{N}$ -isotypical part of the  $\mathbf{H}$ -module  $X'$ . Then  $X'_{\mathcal{N}}$  is an  $\mathcal{A}$ -submodule and  $X'_{\mathcal{N}} \neq 0$ . Let  $\text{rad} J = \{x \in \mathcal{A}; x^n \in J \text{ for some } n \geq 1\}$ . The elements of  $\text{rad}(J)$  act on  $X'_{\mathcal{N}}$  as commuting nilpotent elements. Hence  $\text{rad}(J)X'_{\mathcal{N}} \neq X'_{\mathcal{N}}$  (since  $X'_{\mathcal{N}} \neq 0$ ). Hence there exists a nonzero  $\mathbf{H}$ -linear map  $X'_{\mathcal{N}}/\text{rad}(J)X'_{\mathcal{N}} \rightarrow \mathcal{N}$ . Now  $X'_{\mathcal{N}}/\text{rad}(J)X'_{\mathcal{N}}$  is a direct summand of the  $\mathbf{H}$ -module  $X'/\text{rad}(J)X'$ , hence there exists a nonzero  $\mathbf{H}$ -linear map  $X'/\text{rad}(J)X' \rightarrow \mathcal{N}$ . The canonical map  $X/\text{rad}(J)X \rightarrow X'/\text{rad}(J)X'$  is surjective, hence there exists a nonzero  $\mathbf{H}$ -linear map  $X/\text{rad}(J)X \rightarrow \mathcal{N}$ . Since  $\text{rad}(\text{rad}(J)) = \text{rad}(J)$ , we may assume that  $J = \text{rad}(J)$ . Then the commutative algebra  $A/JA$  is a finite direct sum of copies of  $\mathbf{C}$ . Let  $I_1, \dots, I_k$  be the maximal ideals of  $A$  that contain  $J$ . We have  $A/J \xrightarrow{\sim} A/I_1 \oplus \dots \oplus A/I_k$  and  $X/JX = X/I_1X \oplus \dots \oplus X/I_kX$  (as  $\mathbf{H}$ -modules). Hence there exists  $j \in [1, k]$  and a nonzero  $\mathbf{H}$ -linear map  $X/I_jX \rightarrow \mathcal{N}$ . Hence we may assume that  $J$  is a maximal ideal of  $\mathcal{A}$ . Then there exists a semisimple element  $(\sigma, r) \in \overline{M^0(y)}$  such that  $J = \mathcal{J}_{\sigma, r}^{M^0(y)}$  (see 1.13). Then  $X/JX = E_{y, \sigma, r}$  and we see that there exists a nonzero  $\mathbf{H}$ -linear

map  $E_{y,\sigma,r} \rightarrow \mathcal{N}$ . Now  $E_{y,\sigma,r} = \bigoplus_{\rho} \rho \otimes E_{y,\sigma,r,\rho}$  where  $\rho$  runs over  $\text{Irr}_0 \bar{M}(y, \sigma)$ . Hence there exists  $\rho \in \text{Irr}_0 \bar{M}(y, \sigma)$  and a nonzero  $\mathbf{H}$ -linear map  $E_{y,\sigma,r,\rho} \rightarrow \mathcal{N}$ . This map is surjective since the  $\mathbf{H}$ -module  $\mathcal{N}$  is simple. It follows that the induced map  $\bar{E}_{y,\sigma,r,\rho} \rightarrow \mathcal{N}$  is an isomorphism. This completes the proof of Theorem 1.15.

**3.43. Proof of Theorem 1.21.** Assume that 1.21(iii) holds. Replacing  $h, \tilde{y}$  by  $\phi(h_0), \phi(f_0)$ , we see that  $G = Q^1 = Q$ . Using 3.31 we see that 3.31(i) holds. Using 3.36(c) we deduce that for any  $\eta \in \mathfrak{h}$  such that  ${}_{\eta}\mathcal{M} \neq 0$  and any  $V \in \mathcal{I}$  we have  $\tau(\xi_V(\eta))/\tau(r) \geq 0$ . Hence  $\mathcal{M}$  is  $\tau$ -tempered. Hence  $\mathcal{M}_{\rho}$  is  $\tau$ -tempered and 1.21(i) holds.

Clearly, if 1.21(i) holds, then 1.21(ii) holds.

Assume now that 1.21(iii) does not hold. Using 3.31 we see that  $Q \neq G$ , that is,  $i \in I - K$ . Let  $V \in \mathcal{I}_{I-\{i\}}$ . Let  $\eta \in \mathfrak{h}$  be such that  ${}_{\eta}\mathcal{M}'_{\rho} \neq 0$ . Then, by 3.40(a), we have  ${}_{\eta}(\mathcal{M}_{\rho}/\mathcal{M}_{\rho, \max}) \neq 0$ . We can find  $P \in \mathcal{P}_y^{\sigma}$  such that  $P \subset Q$  and  $\eta = \psi(P)$ . By the second paragraph in the proof of 3.31 we have  $\tau(\xi_V(\psi(P)))/\tau(r) < 0$ , hence  $\tau(\xi_V(\eta))/\tau(r) < 0$ . It follows that  $\mathcal{M}_{\rho}/\mathcal{M}_{\rho, \max}$  is not  $\tau$ -tempered. Thus 1.21(ii) does not hold.

Thus the three conditions in 1.21 are equivalent. If these conditions are satisfied then, as we have seen above, we have  $G = Q$ . As in the proof of 3.16 we see that  $y$  belongs to the unique open orbit of  $Z(\sigma)$  in  $\mathfrak{g}_{2r}$ . Hence we may apply [L4, 8.17(b)] and conclude that  $\mathcal{M}_{\rho}$  is a simple  $\mathbf{H}$ -module. This completes the proof of 1.21.

**3.44. Proof of Theorem 1.22.** The equivalence of 1.22(i) and 1.22(ii) follows from 3.32.

Assume that 1.22(ii) holds. Using 3.32 we see that 3.32(iii) holds. Using 3.36(c) we deduce that for any  $\eta \in \mathfrak{h}$  such that  ${}_{\eta}\mathcal{M} \neq 0$  and any  $V \in \mathcal{I}, V \neq \mathbf{C}$  we have  $\xi_V(\eta) = nr$  where  $n \in \{1, 2, 3, \dots\}$ . Hence 1.22(v) holds.

Clearly, if 1.22(v) holds, then 1.22(iv) holds.

By 1.21, conditions 1.22(iii) and 1.22(iv) are equivalent.

Assume that 1.22(iv) holds and that 1.22(i) does not hold. Then we can find  $P' \in \mathfrak{P}$ ,  $P' \neq G$  and a Levi subgroup  $L'$  of  $P'$  such that  $y \in \underline{L}', \sigma \in \underline{L}'$ . We may assume that  $h \in \underline{L}', \tilde{y} \in \underline{L}'$ . By 3.10, we have  $\{P \in \mathcal{P}_y^{\sigma}; P \subset P'\} \neq \emptyset$ . In particular,  $P' \in \mathcal{P}_J$  for some  $J \subset I, J \neq I$ . In particular  $L'$  inherits a natural cuspidal datum (as in 1.3) and in terms of this we can define  $\tilde{\mathbf{H}}, \tilde{\mathcal{M}}$  in the same way as  $\mathbf{H}, \mathcal{M}$  were defined in terms of the cuspidal datum of  $G$ . We have a natural imbedding

$$j_0 : \tilde{\mathcal{M}} \rightarrow \mathcal{M}.$$

Let  $\mathbf{n}' = \underline{U}_{P'}$  and let  ${}_y\mathbf{n}' = \text{coker}(\text{ad}(y) : \mathbf{n}' \rightarrow \mathbf{n}')$ . We show that

(a)  $\text{ad}(\sigma) - 2r : {}_y\mathbf{n}' \rightarrow {}_y\mathbf{n}'$  is invertible.

By 1.21 (and its proof) we have  $Q = Q^1 = G$ . Thus,  ${}_n\mathfrak{g}_a \neq 0 \implies \tau(a)/\tau(r) = n$ . We must show that any eigenvalue of  $\text{ad}(\sigma) - 2r$  on  $\text{coker}(\text{ad}(y) : \mathbf{n}' \rightarrow \mathbf{n}')$  is  $\neq 0$ . Now the last cokernel is a quotient of  $\bigoplus_{n,a;n \leq 0} ({}_n\mathfrak{g}_a)$ . Hence it suffices to show that if  ${}_n\mathfrak{g}_a \neq 0$  and  $n \leq 0$  then  $a - 2r \neq 0$ . But

$$\tau(a - 2r)/\tau(r) = (n\tau(r) - 2\tau(r))/\tau(r) = n - 2 \leq -2,$$

hence  $a - 2r \neq 0$ , as required. This proves (a).

We see that 1.18 is applicable (with  $P', L'$  instead of  $Q, Q^1$ ). We deduce that

(b)  $j_0(\tilde{\mathcal{M}})$  generates the  $\mathbf{H}$ -module  $\mathcal{M}$ .

We can find a surjective  $\mathbf{H}$ -linear map  $p : \mathcal{M} \rightarrow \mathcal{M}_\rho$ . The composition  $\tilde{\mathcal{M}} \xrightarrow{j_0} \mathcal{M} \xrightarrow{p} \mathcal{M}_\rho$  is nonzero. (Otherwise,  $j_0(\tilde{\mathcal{M}})$  would be contained in the proper  $\mathbf{H}$ -submodule  $\text{Ker}(p)$  of  $\mathcal{M}$  contradicting (b).) Since  $pj_0$  is  $\mathbf{S}$ -linear, it follows that there exists  $\eta \in \mathfrak{h}$  such that  ${}_\eta\tilde{\mathcal{M}} \neq 0$  and  ${}_\eta\mathcal{M}_\rho \neq 0$ . Hence there exists  $P \in \mathcal{P}_y^\sigma$  such that  $P \subset P'$  and  $\psi(P) = \eta$ . By the argument in the last paragraph of the proof of 3.32, we have  $\tau(\xi_V(\psi(P)))/\tau(r) = 0$ , that is,  $\tau(\xi_V(\eta))/\tau(r) = 0$  where  $V \in \mathcal{I}_J$ . This contradicts 1.22(iv) since  $V \neq \mathbf{C}$ . We have proved that if 1.22(iv) holds, then 1.22(i) holds.

This completes the proof of 1.22.

#### REFERENCES

- [BW] A. Borel and N. Wallach, *Continuous cohomology, discrete subgroups and representations of reductive groups*, *Ann. Math. Stud.*, vol. 94, Princeton Univ. Press, 1980. MR **83c**:22018
- [KL] D. Kazhdan and G. Lusztig, *Proof of the Deligne-Langlands conjecture for Hecke algebras*, *Inv. Math.* **87** (1987), 153-215. MR **88d**:11121
- [LE] G. I. Lehrer,  *$l$ -adic cohomology of hyperplane complements*, *Bull. Lond. Math. Soc.* **24** (1992), 76-82. MR **92j**:14022
- [L1] G. Lusztig, *Some examples of square integrable representations of semisimple  $p$ -adic groups*, *Trans. Amer. Math. Soc.* **227** (1983), 623-653. MR **84j**:22023
- [L2] G. Lusztig, *Intersection cohomology complexes on a reductive group*, *Inv. Math.* **75** (1984), 205-272. MR **86d**:20050
- [L3] G. Lusztig, *Character Sheaves*, V, *Adv. Math.* **61** (1986), 103-155. MR **87m**:20118c
- [L4] G. Lusztig, *Cuspidal local systems and graded Hecke algebras*, I, *Publ. Math. I.H.E.S.* **67** (1988), 145-202. MR **90e**:22029
- [L5] G. Lusztig, *Affine Hecke algebras and their graded version*, *J. Amer. Math. Soc.* **2** (1989), 599-635. MR **90e**:16049
- [L6] G. Lusztig, *Cuspidal local systems and graded Hecke algebras*, II, *Representations of groups*, (B. Allison and G. Cliff, eds.) *Canad. Math. Soc. Conf. Proc.*, vol. 16, Amer. Math. Soc., 1995, pp. 217-275. MR **96m**:22038
- [L7] G. Lusztig, *Study of perverse sheaves arising from graded Lie algebras*, *Adv. Math.* **112** (1995), 147-217. MR **96j**:17024
- [L8] G. Lusztig, *Classification of unipotent representations of simple  $p$ -adic groups*, *Int. Math. Res. Notices* (1995), 517-589. MR **98b**:22034
- [L9] G. Lusztig, *Bases in equivariant  $K$ -theory*, II, *Represent. Theory* **3** (1999), 281-353. MR **2000h**:20085
- [R] M. Reeder, *Formal degrees and  $L$ -packets of unipotent discrete series of exceptional  $p$ -adic groups*, *J. Reine Angew. Math.* **520** (2000), 37-93. MR **2001k**:22039
- [W] J.-L. Waldspurger, *Représentations de réduction unipotente pour  $SO(2n+1)$ : quelques conséquences d'un article de Lusztig*, preprint 2001.

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