

CLASSIFICATION OF UNIPOTENT REPRESENTATIONS OF SIMPLE p -ADIC GROUPS, II

G. LUSZTIG

ABSTRACT. Let $\mathbf{G}(\mathbf{K})$ be the group of \mathbf{K} -rational points of a connected adjoint simple algebraic group over a nonarchimedean local field \mathbf{K} . In this paper we classify the unipotent representations of $\mathbf{G}(\mathbf{K})$ in terms of the geometry of the Langlands dual group. This was known earlier in the special case where $\mathbf{G}(\mathbf{K})$ is an inner form of a split group.

INTRODUCTION

0.1. Let \mathbf{K} be a nonarchimedean local field with a residue field of cardinal q . Let $\mathbf{G}(\mathbf{K})$ be the group of \mathbf{K} -rational points of a connected, adjoint simple algebraic group \mathbf{G} defined over \mathbf{K} which becomes split over an unramified extension of \mathbf{K} . Let $\mathcal{U}(\mathbf{G}(\mathbf{K}))$ be the set of isomorphism classes of unipotent representations of $\mathbf{G}(\mathbf{K})$ (see [L4, 1.21]). Let G be a simply connected almost simple algebraic group over \mathbf{C} of the type dual to that of \mathbf{G} (in the sense of Langlands); let $\vartheta : G \rightarrow G$ be the “graph automorphism” of G associated to the \mathbf{K} -rational structure of \mathbf{G} as in [L4, 8.1]. One of the main results of this paper is the construction of a bijection between $\mathcal{U}(\mathbf{G}(\mathbf{K}))$ and a set of parameters defined in terms of G and ϑ . (See 10.11, 10.12.) This result (or rather a close variant of it) was stated without proof in [L4, 8.1] and was proved in [L4] assuming that $\vartheta = 1$; it supports the Langlands philosophy. See [L4, 0.3] for historical remarks concerning this bijection. One of the main observations of [L4] and the present paper is that the various affine Hecke algebras which arise in connection with unipotent representations of $\mathbf{G}(\mathbf{K})$ can be also found in a completely different way, in terms of G , ϑ and certain cuspidal local systems. Then the problem reduces to classifying the simple modules of these “geometric affine Hecke algebras” with parameter equal to \sqrt{q} . This last problem makes sense in the case where \sqrt{q} is replaced by any $v_0 \in \mathbf{C}^*$. This problem was solved in [L4] assuming that $\vartheta = 1$ and $v_0 \in \mathbf{R}_{>0}$. In the present paper we treat more generally the case where ϑ is arbitrary and v_0 is either 1 or is not a root of 1. Moreover, using results of [L5], we determine which representations are tempered or square integrable.

0.2. Notation. All algebraic groups are assumed to be affine. All algebraic varieties (in particular, all algebraic groups) are assumed to be over \mathbf{C} . If G is an

Received by the editors November 28, 2001 and, in revised form, May 3, 2002.

2000 *Mathematics Subject Classification.* Primary 22E50.

Supported in part by the National Science Foundation. This paper was written while the author enjoyed the hospitality of the Institut des Hautes Études Scientifiques.

algebraic group, G^0 denotes the identity component of G , \bar{G} the group of components of G , U_G the unipotent radical of G^0 , Z_G the center of G , \underline{G} the Lie algebra of G . For $x \in \underline{G}$ let $Z_G(x)$ be the centralizer of x in G . For $x, x' \in \underline{G}$ let $Z_G(x, x') = Z_G(x) \cap Z_G(x')$. If G' is another algebraic group, let $\text{Hom}(G, G')$ be the set of homomorphisms of algebraic groups from G to G' . If G' is a subgroup of G and $g \in G$, we denote by $Z_{G'}(g)$ the centralizer of g in G' ; let $N_G(G')$ be the normalizer of G' in G . Let \mathcal{I}_G be the category of finite dimensional rational representations of G . If $V \in \mathcal{I}_G$, then V is also a \underline{G} -module.

If \mathcal{A} is a subgroup of \mathbf{C}^* , let $G^{\mathcal{A}}$ be the set of all $g \in G$ such that for any $V \in \mathcal{I}_G$, any eigenvalue of $g : V \rightarrow V$ is in \mathcal{A} . If A is a subgroup of \mathbf{C} , let \underline{G}_A be the set of all $x \in \underline{G}$ such that for any $V \in \mathcal{I}_G$, any eigenvalue of $x : V \rightarrow V$ is in A ; let $G_A = G^{\exp(A)}$.

If X is an abelian group, we write $X_{\mathbf{Q}}, X_{\mathbf{C}}$ instead of $X \otimes \mathbf{Q}, X \otimes \mathbf{C}$.

Let $\kappa = 2\pi\sqrt{-1} \in \mathbf{C}$.

Let $z = a + \sqrt{-1}b$ where $a, b \in \mathbf{R}$. We say that $z \geq 0$ if either $a > 0$, or $a = 0, b \geq 0$. We say that $z > 0$ if either $a > 0$, or $a = 0, b > 0$.

0.3. Errata to [L4]. 3.15(b): replace $*R_{K-S}$ by $*\mathcal{R}_{K-S}$.

5.17, line 8: replace C_S by $p(C_S)$.

7.49, line 4: the first and last edge $—, —$ should be replaced by \Leftarrow, \Rightarrow .

8.5, line 3: replace ${}_0\mathbf{1}_{2b}$ by ${}_0\mathbf{1}_{2a}$.

As pointed out to me by Gopal Prasad, the definition of parahoric subgroups in 1.2 is incorrect if the residual characteristic is small. In 1.2 replace the third sentence (“*Note that ... of \mathbf{G} .*”) by:

For $B \in \mathcal{B}$, $(\mathbf{G}, \mathbf{G})B = B(\mathbf{G}, \mathbf{G})$ is a normal subgroup of \mathbf{G} independent of B , of finite index in \mathbf{G} ; we denote it by \mathbf{G}' .

CONTENTS

1. Preliminaries on affine Hecke algebras and graded Hecke algebras.
2. A review of [L2, §8, §9].
3. Some consequences of the first reduction theorem of [L2].
4. Some consequences of the second reduction theorem of [L2].
5. Geometric graded Hecke algebras.
6. The subgroups G_J .
7. The set $\mathfrak{A}(G\tau, G_J, \mathcal{C}, \mathcal{F})$.
8. Geometric affine Hecke algebras.
9. A bijection.
10. The main results.
11. Tables.
- Appendix.

1. PRELIMINARIES ON AFFINE HECKE ALGEBRAS AND GRADED HECKE ALGEBRAS

1.1. (a) A *root system* (R, \check{R}, X, Y) consists of two finitely generated free abelian groups X, Y , a perfect pairing $\langle, \rangle : X \times Y \rightarrow \mathbf{Z}$, finite subsets $R \subset X - \{0\}, \check{R} \subset Y - \{0\}$ and a bijection $\alpha \leftrightarrow \check{\alpha}$ between R and \check{R} such that for any $\alpha \in R$ we have $\langle \alpha, \check{\alpha} \rangle = 2$ and $s_{\alpha} : X \rightarrow X, x \mapsto x - \langle x, \check{\alpha} \rangle \alpha$ (resp. $s_{\alpha} : Y \rightarrow Y, y \mapsto y - \langle \alpha, y \rangle \check{\alpha}$) leaves R (resp. \check{R}) stable.

We sometimes write (R, X) instead of (R, \check{R}, X, Y) .

(b) A **Q-root system** (R, \check{R}, E, E') consists of two finite dimensional **Q**-vector spaces E, E' , a perfect bilinear pairing $\langle \cdot, \cdot \rangle : E \times E' \rightarrow \mathbf{Q}$, finite subsets $R \subset E - \{0\}, \check{R} \subset E' - \{0\}$ and a bijection $\alpha \leftrightarrow \check{\alpha}$ between R and \check{R} such that $\langle \alpha, \check{\alpha}' \rangle \in \mathbf{Z}$ for any $\alpha, \alpha' \in R$ and for any $\alpha \in R$ we have $\langle \alpha, \check{\alpha} \rangle = 2$ and $s_\alpha : E \rightarrow E, e \mapsto e - \langle e, \check{\alpha} \rangle \alpha$ (resp. $s_\alpha : E' \rightarrow E', e' \mapsto e' - \langle \alpha, e' \rangle \check{\alpha}$) leaves R (resp. \check{R}) stable.

We sometimes write (R, E) instead of (R, \check{R}, E, E') .

We set $E_{\mathbf{C}} = E \otimes_{\mathbf{Q}} \mathbf{C}, E'_{\mathbf{C}} = E' \otimes_{\mathbf{Q}} \mathbf{C}$. We denote the **C**-bilinear pairing $E_{\mathbf{C}} \times E'_{\mathbf{C}} \rightarrow \mathbf{C}$ defined by $\langle \cdot, \cdot \rangle$ again by $\langle \cdot, \cdot \rangle$.

Unless otherwise indicated, in both cases (a) and (b) it is assumed that $\alpha \in R \implies 2\alpha \notin R$.

In case (a) (resp. (b)) the Weyl group W_0 is defined as the subgroup of $GL(X)$ or $GL(Y)$ (resp. $GL(E)$ or $GL(E')$) generated by $\{s_\alpha; \alpha \in R\}$. In both cases one has the standard notion of “basis” (or “set of simple roots”) of R and the corresponding notion of positive roots R^+ and positive coroots \check{R}^+ . A basis of R always exists. If a basis of R is given, then W_0 is naturally a (finite) Coxeter group with length function $l : W_0 \rightarrow \mathbf{N}$.

1.2. Assume that we are given a root system (R, \check{R}, X, Y) and a basis Π for it. A *parameter set* consists of a function $\lambda : \Pi \rightarrow \mathbf{N}$ such that $\lambda(\alpha) = \lambda(\alpha')$ whenever $\langle \alpha, \check{\alpha}' \rangle = \langle \alpha', \check{\alpha} \rangle = -1$ together with a function $\lambda^* : \{\alpha \in \Pi; \check{\alpha} \in 2Y\} \rightarrow \mathbf{N}$. If such (λ, λ^*) is given, we define $H_{R,X}^{\lambda, \lambda^*}$ to be the associative algebra over $\mathbf{C}[v, v^{-1}]$ (v is an indeterminate) defined by the generators $T_w, w \in W_0$ and $\theta_x, x \in X$ and by the relations

$$T_w T_{w'} = T_{ww'} \quad \text{for all } w, w' \in W_0 \text{ such that } l(ww') = l(w) + l(w'),$$

$$(T_{s_\alpha} + 1)(T_{s_\alpha} - v^{2\lambda(\alpha)}) = 0 \quad \text{for all } \alpha \in \Pi,$$

$$\theta_{x_1} \theta_{x_2} = \theta_{x_1 + x_2} \quad \text{for all } x_1, x_2 \in X,$$

$$\theta_x (T_{s_\alpha} + 1) - (T_{s_\alpha} + 1) \theta_{s_\alpha(x)} = (\theta_x - \theta_{s_\alpha(x)}) \mathcal{G}(\alpha) \quad \text{for all } x \in X, \alpha \in \Pi$$

where, for $\alpha \in \Pi$, $\mathcal{G}(\alpha)$ equals

$$\frac{\theta_\alpha v^{2\lambda(\alpha)} - 1}{\theta_\alpha - 1} \text{ if } \check{\alpha} \notin 2Y \text{ and } \frac{(\theta_\alpha v^{\lambda(\alpha) + \lambda^*(\alpha)} - 1)(\theta_\alpha v^{\lambda(\alpha) - \lambda^*(\alpha)} + 1)}{\theta_{2\alpha} - 1} \text{ if } \check{\alpha} \in 2Y.$$

(Note that $(\theta_x - \theta_{s_\alpha(x)}) \mathcal{G}(\alpha)$ is a **Z**-linear combination of elements $\theta_{x_1}, x_1 \in X$.)

Now θ_0 is a unit element for $H_{R,X}^{\lambda, \lambda^*}$.

Let $\mathcal{T} = Y \otimes \mathbf{C}^*$. Let \mathcal{O} be the algebra of regular functions $\mathcal{T} \times \mathbf{C}^* \rightarrow \mathbf{C}$. We may identify \mathcal{O} with the $\mathbf{C}[v, v^{-1}]$ -submodule of $H_{R,X}^{\lambda, \lambda^*}$ spanned by $\{\theta_x, x \in X\}$ (a commutative subalgebra): to $v^n \theta_x$ corresponds the regular function $(t, a) \mapsto a^n x(t)$ where $x(t) = \prod_n a_n^{\langle x, y_n \rangle}$ for $t = \sum_n y_n \otimes a_n, y_n \in Y, a_n \in \mathbf{C}^*$. Now W_0 acts naturally on \mathcal{O} and the algebra of invariants \mathcal{O}^{W_0} is the center of $H_{R,X}^{\lambda, \lambda^*}$. For any W_0 -orbit Σ on \mathcal{T} and $v_0 \in \mathbf{C}^*$ let J_{Σ, v_0} be the maximal ideal of \mathcal{O}^{W_0} consisting of the functions in \mathcal{O}^{W_0} that vanish at all points of $\Sigma \times \{v_0\}$. Let $(\mathcal{O}^{W_0})^\wedge$ be the J_{Σ, v_0} -adic completion of \mathcal{O}^{W_0} and let $\hat{H} = H_{R,X}^{\lambda, \lambda^*} \otimes_{\mathcal{O}^{W_0}} (\mathcal{O}^{W_0})^\wedge$.

1.3. For $v_0 \in \mathbf{C}^*$, let $\text{Mod}_{v_0} H_{R,X}^{\lambda, \lambda^*}$ be the category of $H_{R,X}^{\lambda, \lambda^*}$ -modules that are finite dimensional over \mathbf{C} and in which v acts as v_0 times 1. Let $\text{Irr}_{v_0} H_{R,X}^{\lambda, \lambda^*}$ be the set of isomorphism classes of simple objects of $\text{Mod}_{v_0} H_{R,X}^{\lambda, \lambda^*}$.

Let $\text{Mod}_{\Sigma, v_0} H_{R, X}^{\lambda, \lambda^*}$ be the category of $H_{R, X}^{\lambda, \lambda^*}$ -modules $M \in \text{Mod}_{v_0} H_{R, X}^{\lambda, \lambda^*}$ that satisfy $J_{\Sigma, v_0} M = 0$. Let $\text{Irr}_{\Sigma, v_0} H_{R, X}^{\lambda, \lambda^*}$ be the set of isomorphism classes of simple objects of $\text{Mod}_{\Sigma, v_0} H_{R, X}^{\lambda, \lambda^*}$. We have

$$(a) \quad \text{Irr}_{v_0} H_{R, X}^{\lambda, \lambda^*} = \bigsqcup_{\Sigma} \text{Irr}_{\Sigma, v_0} H_{R, X}^{\lambda, \lambda^*}$$

where Σ runs over the W_0 -orbits in \mathcal{T} .

For $M \in \text{Mod}_{v_0} H_{R, X}^{\lambda, \lambda^*}$ and for $t \in \mathcal{T}$ let M_t be the subspace of all $m \in M$ such that for any $x \in X$, m is in the generalized eigenspace of $\theta_x : M \rightarrow M$ corresponding to the eigenvalue $x(t) \in \mathbf{C}^*$. We say that M_t is a weight space of M . We have $M = \bigoplus_t M_t$ where t runs over \mathcal{T} .

Let $\zeta : \mathbf{C}^* \rightarrow \mathbf{R}$ be a group homomorphism such that $\zeta(v_0) \neq 0$. Let X^+ be the set of all $x \in X$ such that $\langle x, \check{\alpha} \rangle \geq 0$ for all $\alpha \in \Pi$. We say that M (as above) is ζ -tempered if the following holds: for any $t \in \mathcal{T}$ such that $M_t \neq 0$ and any $x \in X^+$ we have $\zeta(x(t))/\zeta(v_0) \geq 0$. In the case where R generates a subgroup of finite index of X , we say that M is ζ -square integrable if the following holds: for any $t \in \mathcal{T}$ such that $M_t \neq 0$ and any $x \in X^+ - \{0\}$ we have $\zeta(x(t))/\zeta(v_0) > 0$.

1.4. Assume that we are given a \mathbf{Q} -root system (R, \check{R}, E, E') and a basis Π for it. A *parameter set* is a function $\mu : \Pi \rightarrow \mathbf{Z}$ such that $\mu(\alpha) = \mu(\alpha')$ whenever $\langle \alpha, \check{\alpha}' \rangle = \langle \alpha', \check{\alpha} \rangle = -1$. If such a μ is given, we define $\bar{H}_{R, E}^\mu$ to be the associative algebra over $\mathbf{C}[r]$ (r is an indeterminate) defined by the generators $t_w, w \in W_0$ and $(f), f \in \bar{\mathcal{O}}$ (the algebra of regular functions $E'_\mathbf{C} \oplus \mathbf{C} \rightarrow \mathbf{C}$) and by the relations

$$\begin{aligned} t_w t_{w'} &= t_{ww'} \quad \text{for all } w, w' \in W_0; \\ (f_1)(f_2) &= (f_1 f_2) \quad \text{for all } f_1, f_2 \in \bar{\mathcal{O}}; \\ (a_1 f_1 + a_2 f_2) &= a_1(f_1) + a_2(f_2) \quad \text{for } f_1, f_2 \in \bar{\mathcal{O}} \quad \text{and} \quad a_1, a_2 \in \mathbf{C}[r]; \\ (f) t_{s_\alpha} - t_{s_\alpha}(s_\alpha(f)) &= \mu(\alpha) r \frac{f - s_\alpha(f)}{\alpha} \quad \text{for all } f \in \bar{\mathcal{O}}, \alpha \in \Pi, \alpha \end{aligned}$$

where α is regarded as a linear form on $E'_\mathbf{C} \oplus \mathbf{C}$ (zero on the second factor) so that $\frac{f - s_\alpha(f)}{\alpha} \in \bar{\mathcal{O}}$. (We regard $\bar{\mathcal{O}}$ as a $\mathbf{C}[r]$ -algebra, by identifying r with the second projection $E'_\mathbf{C} \oplus \mathbf{C} \rightarrow \mathbf{C}$.) Now (0) is a unit element for $\bar{H}_{R, E}^\mu$.

We may identify $\bar{\mathcal{O}}$ with the $\mathbf{C}[r]$ -submodule of $\bar{H}_{R, E}^\mu$ consisting of all (f) with $f \in \bar{\mathcal{O}}$ (a commutative subalgebra): to (f) corresponds f . Now W_0 acts naturally on $\bar{\mathcal{O}}$ and the algebra of invariants $\bar{\mathcal{O}}^{W_0}$ is the center of $\bar{H}_{R, E}^\mu$. For any W_0 -orbit $\bar{\Sigma}$ on $E'_\mathbf{C}$ and $r_0 \in \mathbf{C}$ let $\bar{J}_{\bar{\Sigma}, r_0}$ be the maximal ideal of $\bar{\mathcal{O}}^{W_0}$ consisting of functions in $\bar{\mathcal{O}}^{W_0}$ that vanish at all points of $\bar{\Sigma} \times \{r_0\}$.

1.5. For $r_0 \in \mathbf{C}$ let $\text{Mod}_{r_0} \bar{H}_{R, E}^\mu$ be the category of $\bar{H}_{R, E}^\mu$ -modules that are finite dimensional over \mathbf{C} and in which r acts as r_0 times 1. Let $\text{Irr}_{r_0} \bar{H}_{R, E}^\mu$ be the set of isomorphism classes of simple objects of $\text{Mod}_{r_0} \bar{H}_{R, E}^\mu$.

Let $\text{Mod}_{\bar{\Sigma}, r_0} \bar{H}_{R, E}^\mu$ be the category of $\bar{H}_{R, E}^\mu$ -modules $\bar{M} \in \text{Mod}_{r_0} \bar{H}_{R, E}^\mu$ that satisfy $J_{\bar{\Sigma}, r_0} \bar{M} = 0$. Let $\text{Irr}_{\bar{\Sigma}, r_0} \bar{H}_{R, E}^\mu$ be the set of isomorphism classes of simple objects of $\text{Mod}_{\bar{\Sigma}, r_0} \bar{H}_{R, E}^\mu$. We have

$$(a) \quad \text{Irr}_{r_0} \bar{H}_{R, E}^\mu = \bigsqcup_{\bar{\Sigma}} \text{Irr}_{\bar{\Sigma}, r_0} \bar{H}_{R, E}^\mu$$

where $\bar{\Sigma}$ runs over the W_0 -orbits in $E'_\mathbf{C}$.

For $\bar{M} \in \text{Mod}_{r_0} \bar{H}_{R,E}^\mu$ and for $e' \in E'_C$ let $\bar{M}_{e'}$ be the subspace of all $m \in \bar{M}$ such that for any $f \in \bar{\mathcal{O}}$, m is in the generalized eigenspace of $(f) : \bar{M} \rightarrow \bar{M}$ corresponding to the eigenvalue $f(e', r_0)$. We say that $\bar{M}_{e'}$ is a weight space of \bar{M} . We have $\bar{M} = \bigoplus_{e' \in E'_C} \bar{M}_{e'}$.

Let $\tau : \mathbf{C} \rightarrow \mathbf{R}$ be a group homomorphism such that $\tau(r_0) \neq 0$. We say that \bar{M} (as above) is τ -tempered if the following holds: for any $e' \in E'_C$ such that $\bar{M}_{e'} \neq 0$ and any $e \in E$ such that $\langle e, \check{\alpha} \rangle \geq 0$ for all $\alpha \in \Pi$ we have $\tau(\langle e, e' \rangle) / \tau(r_0) \geq 0$. In the case where R generates E as a vector space, we say that \bar{M} is τ -square integrable if the following holds: for any $e' \in E'_C$ such that $\bar{M}_{e'} \neq 0$ and any $e \in E - \{0\}$ such that $\langle x, \check{\alpha} \rangle \geq 0$ for all $\alpha \in \Pi$, we have $\tau(\langle e, e' \rangle) / \tau(r_0) > 0$.

2. A REVIEW OF [L2, §8, §9]

2.1. Let \spadesuit be a \mathbf{Q} -subspace of \mathbf{C} such that $\kappa\mathbf{Q} \cap \spadesuit = 0$. Let $\tilde{\spadesuit}$ be the image of \spadesuit under $\exp : \mathbf{C} \rightarrow \mathbf{C}^*$. Then \exp restricts to a group isomorphism $\spadesuit \xrightarrow{\sim} \tilde{\spadesuit}$.

If T is a torus, we have canonically $T = \mathcal{L} \otimes \mathbf{C}^*$ where \mathcal{L} is the free abelian group $\text{Hom}(\mathbf{C}^*, T)$. We have $T_\spadesuit = \mathcal{L} \otimes \tilde{\spadesuit}$. If $\mathfrak{t} = \underline{T}$ we have canonically $\mathfrak{t} = \mathcal{L}_\mathbf{C}, \mathfrak{t}_\spadesuit = \mathcal{L} \otimes \tilde{\spadesuit}$ and $\exp : \mathfrak{t} \rightarrow T$ (denoted also by \exp_T) induces an isomorphism $\mathfrak{t}_\spadesuit \xrightarrow{\sim} T_\spadesuit$.

2.2. In this section we will refer to a subsection of [L2] such as [L2, 8.13] simply as [8.13].

Now [L2, §6, §9] gives a method which allows one to reduce a number of questions on representations of an affine Hecke algebra to analogous questions on graded Hecke algebras. Here we shall give a variation of this method. We will indicate how to modify §8 and §9 of [L2] (for example [8.13] will become [8.13]') to obtain this variation.

[8.1]'. From now on we assume that

(a) Y is generated by $\check{R} \cup (\frac{1}{2}\check{R} \cap Y)$.

Assume that a W_0 -orbit Σ in \mathcal{T} and an element $v_0 \in \tilde{\spadesuit}$ are given. We define an equivalence relation on Σ as follows: we say that $t, t' \in \Sigma$ are equivalent if $t't^{-1} \in T_\spadesuit$. Let \mathcal{P} be the set of equivalence classes. Note that W_0 acts transitively on \mathcal{P} .

Let $c \in \mathcal{P}$. We choose $t \in c$ and we define

$$R_c = \{\alpha \in R; \alpha(t) \in \tilde{\spadesuit} \text{ if } \check{\alpha} \notin 2Y, \alpha(t) \in \pm \tilde{\spadesuit} \text{ if } \check{\alpha} \in 2Y\}.$$

This clearly does not depend on the choice of t . We set $\check{R}_c = \{\check{\alpha}; \alpha \in R_c\}$. There is a unique subset Π_c of $R_c \cap R^+$ such that $(R_c, \check{R}_c, X, Y, \Pi_c)$ is a root system. Let W_0^c be the Weyl group of this root system (a subgroup of W_0). Using (a) and [L4, 4.5] we see that $W_0^c = \{w \in W_0; w(c) = c\}$. Now c is a W_0^c -orbit in \mathcal{T} .

[8.2]' is empty.

[8.3]' is the same as [8.3] except that the last four lines of [8.3] are replaced by:

Let $T_{w,c}\theta_x (w \in W_0^c, x \in X)$ be the basis elements of H_c analogous to the basis elements $T_w\theta_x (w \in W_0, x \in X)$ of H .

[8.4]' is the same as [8.4] except that the last three lines of [8.4] are deleted.

[8.5]'. If \mathcal{A} is an associative ring with 1, denote by \mathcal{A}_n the ring of all $n \times n$ matrices with entries in \mathcal{A} . We have $\hat{\mathcal{Z}}_c = \hat{\mathcal{O}}_c^{W_0^c}$. Thus \hat{H}_c is a $\hat{\mathcal{O}}_c^{W_0^c}$ -algebra. The identity map $\mathcal{O} \rightarrow \mathcal{O}$ extends continuously to a ring homomorphism $i : \hat{\mathcal{O}} \rightarrow \hat{\mathcal{O}}_c$

(since $J_{\Sigma, v_0} \subset J_{c, v_0}$). This restricts to a ring isomorphism $\hat{O}^{W_0} \xrightarrow{\sim} \hat{O}_c^{W_0^c}$ (since $W_0^c \subset W_0$). Via this isomorphism we can regard \hat{H}_c also as a \hat{O}^{W_0} -algebra.

[8.6]'. **Theorem.** If $c \in \mathcal{P}$, there exists an isomorphism of \hat{O}^{W_0} -algebras $\hat{H} \cong (\hat{H}_c)_n$ where $n = \text{card}(\mathcal{P})$.

[8.7]'=[8.7].

[8.8]' is the same as [8.8] except that the reference to [8.2](b) is deleted.

[8.9]'=[8.9].

[8.10]' remains unchanged except that formula (a) should be replaced by:

$$(a) \quad T_w^c = T_{s_{\alpha_1}}^c T_{s_{\alpha_2}}^{s_{\alpha_1}(c)} \dots T_{s_{\alpha_p}}^{s_{\alpha_{p-1}} \dots s_{\alpha_2} s_{\alpha_1}(c)}.$$

[8.11]'=[8.11], [8.12]'=[8.12].

[8.13]' is deleted except for the line (e) and the three lines following it which are left unchanged.

[8.14]'=[8.14], [8.15]'=[8.15].

[8.16]'. Lines 2 and 3 of 8.16 are replaced by:

For any $c' \in \mathcal{P}$ let $w \in W_0$ be the unique element of minimal length such that $w(c) = c'$. If $s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_p} = w$ is a reduced expression in W_0 , then

$$c \neq s_{\alpha_p}(c) \neq s_{\alpha_{p-1}} s_{\alpha_p}(c) \neq \dots \neq s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_p}(c) = c'.$$

The rest of [8.16] remains unchanged except that $[\Gamma(c)], \gamma, \gamma \in \Gamma(c), T_\gamma^c$ are deleted.

[8.17]' is empty.

[9.1]'. We preserve the setup of §3. Assume that we are given a W_0 -orbit Σ in \mathcal{T} and an element $v_0 \in \tilde{\mathfrak{A}}$ such that for any $t \in \Sigma$ and any $\alpha \in R$ we have

$$\alpha(t) \in \tilde{\mathfrak{A}} \text{ if } \check{\alpha} \notin 2Y, \alpha(t) \in \pm \tilde{\mathfrak{A}} \text{ if } \check{\alpha} \in 2Y.$$

[9.2]'. The text of [9.2] except for the last three lines is replaced by the following:

Define $r_0 \in \mathfrak{A}$ by $\exp(r_0) = v_0$. Let $\mathfrak{t} = \underline{\mathfrak{t}}$. We show that there exists a W_0 -invariant element $t_0 \in \mathcal{T}$ and a W_0 -orbit $\bar{\Sigma}$ in $\mathfrak{t}_{\mathfrak{A}}$ such that $t_0 \exp_{\mathcal{T}}(\bar{\Sigma}) = \Sigma$.

Choose a \mathbf{Q} -subspace \diamond of \mathbf{C} complementary to \mathfrak{A} , and consider the subgroup $\exp(\diamond)$ of \mathbf{C}^* . Then $\mathbf{C}^* = \exp(\diamond) \times \mathfrak{A}$ and we have a W_0 -invariant decomposition $\mathcal{T} = \mathcal{T}_{\diamond} \times \mathcal{T}_{\mathfrak{A}}$. For any $\alpha \in R$ we have $\alpha(\mathcal{T}_{\diamond}) \in \exp(\diamond)$ and $\alpha(\mathcal{T}_{\mathfrak{A}}) \in \tilde{\mathfrak{A}}$. Hence if $pr_1 : \mathcal{T} \rightarrow \mathcal{T}_{\diamond}$ is the first projection, we have for any $t \in \Sigma$,

$$\alpha(pr_1(t)) = 1 \text{ if } \check{\alpha} \notin 2Y, \quad \alpha(pr_1(t)) = \pm 1 \text{ if } \check{\alpha} \in 2Y.$$

It follows that $pr_1(t)$ is W_0 -invariant for any $t \in \Sigma$. Since $pr_1(\Sigma)$ is a single W_0 -orbit, it follows that $pr_1(\Sigma) = \{t_0\}$ for some W_0 -invariant $t_0 \in \mathcal{T}_{\diamond}$. Let $\bar{\Sigma}$ be the unique subset of $\mathfrak{t}_{\mathfrak{A}}$ such that $\exp_{\mathcal{T}}(\bar{\Sigma}) = t_0^{-1}\Sigma$. Then $t_0, \bar{\Sigma}$ are as required. Another choice for $t_0, \bar{\Sigma}$ must be of form $t_0 \exp_{\mathcal{T}}(\xi_0), \bar{\Sigma} - \xi_0$ where $\xi_0 \in \mathfrak{t}_{\mathfrak{A}}$ is W_0 -invariant.

The last three lines of [9.2] remain unchanged.

[9.3]'=[9.3], [9.4]'=[9.4].

[9.5]' is the same as [9.5] except that the last three lines of [9.5] are replaced by the following.

Assume for example that $\check{\alpha} \notin 2Y$ and

$$(c) \quad \alpha(\bar{t}) + 2\lambda(\alpha)r_0 \in \kappa\mathbf{Z} - \{0\}.$$

Since $\bar{t} \in \mathfrak{t}_{\mathfrak{A}}$, we have $\alpha(\bar{t}) \in \mathfrak{A}$. Since $r_0 \in \mathfrak{A}$ and $\lambda(\alpha) \in \mathbf{N}$, it follows that the left-hand side of (c) is contained in \mathfrak{A} . But the right-hand side of (c) is not in \mathfrak{A} and we have a contradiction. Similarly, we see that the other statements (a) and (b) hold.

[9.6]' is the same as [9.6] except that the reference to [9.2](c) is deleted.
 [9.7]' is empty.

3. SOME CONSEQUENCES OF THE FIRST REDUCTION THEOREM OF [L2]

3.1. We place ourselves in the setup of 2.1 and we fix $v_0 \in \tilde{\mathfrak{A}}$. Let

$$X, Y, R, \tilde{R}, \Pi, W_0, \mathcal{O}, \lambda, \lambda^*, \mathcal{T}$$

be as in 1.2. We write H instead of $H_{R,X}^{\lambda, \lambda^*}$. We assume that

(a) Y is generated by $\tilde{R} \cup (\frac{1}{2}\tilde{R} \cap Y)$.

Let Σ be a W_0 -orbit on \mathcal{T} . Let \mathcal{P} be as in [8.1]' (see 2.2). Let $c \in \mathcal{P}$. Recall that c is a W_0^c -orbit on \mathcal{T} . Let $R_c, \tilde{R}_c, \Pi_c, W_0^c$ be as in [8.1]' (see 2.2). Let H_c be the algebra defined in the same way as H , but in terms of $(X, Y, R_c, \tilde{R}_c, \Pi_c)$ instead of $(X, Y, R, \tilde{R}, \Pi)$; the parameter set (λ_c, λ_c^*) that we use to define H_c is given by $\lambda_c(\alpha) = \lambda(\alpha')$, $\lambda_c^*(\alpha) = \lambda^*(\alpha')$, where $\alpha \in \Pi_c, \alpha' \in \Pi$ are in the same W_0 -orbit. (This does not depend on the choice of α' .) Note that \mathcal{O} is a subalgebra of H_c in the same way as \mathcal{O} is a subalgebra of H .

Let J_{c,v_0} be the maximal ideal of $\mathcal{O}^{W_0^c}$ (the center of H_c) consisting of the functions in $\mathcal{O}^{W_0^c}$ that vanish at all points of $c \times \{v_0\}$. Let $(\mathcal{O}^{W_0^c})^\wedge$ be the J_{c,v_0} -adic completion of $\mathcal{O}^{W_0^c}$ and let $\hat{H}_c = H_c \otimes_{\mathcal{O}^{W_0^c}} (\mathcal{O}^{W_0^c})^\wedge$.

Assume that $M \in \text{Mod}_{c,v_0} H_c$. In particular, in the H_c -module M we have $J_{c,v_0} M = 0$. Hence M extends naturally to an \hat{H}_c -module. The $\mathbf{C}[v, v^{-1}]$ -module $M^{\mathcal{P}} = M \oplus M \oplus \dots \oplus M$ (one summand for each $c' \in \mathcal{P}$) is naturally a module over the algebra of matrices with entries in \hat{H}_c indexed by $\mathcal{P} \times \mathcal{P}$. The first reduction theorem [L2, 8.6], in the variant [8.6]' (see 2.2), gives an explicit isomorphism ι of this algebra of matrices with the algebra \hat{H} (see 1.3). Via this isomorphism, $M^{\mathcal{P}}$ becomes an \hat{H} -module and, by restriction, an H -module in $\text{Mod}_{\Sigma, v_0} H$. From the definition we see that, if $f \in \mathcal{O}$ (regarded as an element of \hat{H}), then the (c', c'') -entry of $\iota^{-1}(f)$ (for c', c'' in \mathcal{P}) is 0 if $c' \neq c''$ and is $w'^{-1}(f)$ if $c' = c''$; here $w' \in W_0$ is the unique element of minimal length of W_0 such that $w'(c) = c'$.

Lemma 3.2. *The rule $M \mapsto M^{\mathcal{P}}$ is a bijection $\text{Irr}_{c,v_0} H_c \xrightarrow{\sim} \text{Irr}_{\Sigma, v_0} H$.*

This is an immediate consequence of the definitions and of [L2, 8.6], in the variant [8.6]' (see 2.2).

Let S be the set of all $w \in W_0$ such that the length of w is minimal in wW_0^c , or equivalently, such that $w(\check{\alpha}) \in \tilde{R}^+$ for any $\alpha \in \Pi_c$.

Lemma 3.3. *Let $x \in X$ be such that $\langle x, \check{\alpha} \rangle \geq 0$ for all $\alpha \in \Pi_c$. Let $w \in W_0$ and $x' \in X$ be such that $x = w^{-1}(x')$, $\langle x', \check{\alpha}' \rangle \geq 0$ for all $\alpha' \in \Pi$ and $w^{-1}(\check{\alpha}') \in \tilde{R}^+$ for any $\alpha' \in \Pi$ for which $\langle x', \check{\alpha}' \rangle = 0$. Then $w \in S$.*

If $w \notin S$, then there exists $\alpha \in \Pi_c$ such that $w(\check{\alpha}) \in -\tilde{R}^+$; that is, $w(\check{\alpha}) = \sum_{\alpha' \in \Pi} n_{\alpha'} \check{\alpha}'$ where $-n_{\alpha'} \in \mathbf{N}$. It follows that

$$0 \leq \langle x, \check{\alpha} \rangle = \langle x', w(\check{\alpha}) \rangle = \sum_{\alpha'} n_{\alpha'} \langle x', \check{\alpha}' \rangle.$$

Since $n_{\alpha'} \langle x', \check{\alpha}' \rangle \leq 0$ for all α' , it follows that $n_{\alpha'} \langle x', \check{\alpha}' \rangle = 0$ for all α' . Hence for any $\alpha' \in \Pi$ such that $\langle x', \check{\alpha}' \rangle \neq 0$ we have $n_{\alpha'} = 0$. In other words,

$$w(\check{\alpha}) = \sum_{\alpha' \in \Pi; \langle x', \check{\alpha}' \rangle = 0} n_{\alpha'} \check{\alpha}'.$$

Hence $\check{\alpha} = \sum_{\alpha' \in \Pi; \langle x', \check{\alpha}' \rangle = 0} n_{\alpha'} w^{-1}(\check{\alpha}')$. For each α' in the sum, we have $w^{-1}(\check{\alpha}') \in \check{R}^+$ and $n_{\alpha'} \leq 0$, hence $\check{\alpha} \in -\check{R}^+$, a contradiction. The lemma is proved.

Lemma 3.4. *Let $M \in \text{Mod}_{c, v_0} H_c$. Assume that $\zeta : \mathbf{C}^* \rightarrow \mathbf{R}$ is a homomorphism such that $\zeta(v_0) \neq 0$. The following two conditions are equivalent:*

- (i) *the H_c -module M is ζ -tempered;*
- (ii) *the H -module $M^{\mathcal{P}}$ is ζ -tempered.*

Let D (resp. D') be the set of all $t \in \mathcal{T}$ such that $M_t \neq 0$ (resp. $M_t^{\mathcal{P}} \neq 0$). By the description of $\iota^{-1}(f)$ given in 3.1, we see that $D' = \bigcup_{w \in S} w(D)$ where S consists of all elements $w \in W_0$ such that the length of w is minimal in wW_0^c , or equivalently, such that $w(\check{\alpha}) \in \check{R}^+$ for any $\alpha \in \Pi_c$. Hence (ii) is equivalent to the following condition:

For any $t \in D$, any $w \in S$ and any $x \in X^+$, we have $\zeta(x(w(t)))/\zeta(v_0) \geq 0$, or equivalently $\zeta((w^{-1}x)(t))/\zeta(v_0) \geq 0$.

We see that it is enough to show that the following two conditions for $x \in X$ are equivalent:

- (iii) $\langle x, \check{\alpha} \rangle \geq 0$ for all $\alpha \in \Pi_c$;
- (iv) there exists $w \in S$ and $x' \in X^+$ such that $x = w^{-1}(x')$.

Assume first that (iv) holds. Write $x = w^{-1}(x')$ as in (iv). Let $\alpha \in \Pi_c$. Since $w \in S$, we have $w(\check{\alpha}) \in \check{R}^+$. Using (iv) we deduce that $\langle x', w(\check{\alpha}) \rangle \geq 0$. Thus $\langle w^{-1}(x'), \check{\alpha} \rangle \geq 0$ so that (iii) holds.

Assume next that (iii) holds. We can write uniquely $x = w^{-1}(x')$ where $x' \in X$ satisfies $\langle x', \check{\beta} \rangle \geq 0$ for all $\beta \in \Pi$ and $w \in W_0$ is such that $w^{-1}(\check{\beta}) \in \check{R}^+$ for any $\beta \in \Pi$ for which $\langle x', \check{\beta} \rangle = 0$. By 3.3 we have $w \in S$. Hence (iv) holds. The lemma is proved.

Lemma 3.5. *Assume that R generates a subgroup of finite index in X . Let $M \in \text{Mod}_{c, v_0} H_c$. Assume that $\zeta : \mathbf{C}^* \rightarrow \mathbf{R}$ is a homomorphism such that $\zeta(v_0) \neq 0$. The following two conditions are equivalent:*

- (i) *R_c generates a subgroup of finite index of X and the H_c -module M is ζ -square integrable;*
- (ii) *the H -module $M^{\mathcal{P}}$ is ζ -square integrable.*

Assume that (ii) holds but R_c generates a subgroup of infinite index of X . Then \check{R}_c generates a subgroup of infinite index of Y , hence we can find $z \in X - \{0\}$ such that $\langle z, \check{\alpha} \rangle = 0$ for any $\alpha \in \Pi_c$. We can write uniquely $z = w'^{-1}(x')$ where $x' \in X - \{0\}$ satisfies $\langle x', \check{\beta} \rangle \geq 0$ for all $\beta \in \Pi$ and $w' \in W_0$ is such that $w'^{-1}(\check{\beta}) \in \check{R}^+$ for any $\beta \in \Pi$ for which $\langle x', \check{\beta} \rangle = 0$. By 3.3, we have $w' \in S$. Thus, $z = w'^{-1}(x')$ where $x' \in X - \{0\}$ satisfies $\langle x', \check{\beta} \rangle \geq 0$ for all $\beta \in \Pi$ and $w' \in S$. The same argument can be applied to $-z$ instead of z . We see that $-z = w''^{-1}(x'')$ where $x'' \in X - \{0\}$ satisfies $\langle x'', \check{\beta} \rangle \geq 0$ for all $\beta \in \Pi$ and $w'' \in S$.

Since (ii) holds, and $D' = \bigcup_{w \in S} w(D)$ (D, D' as in the proof of 3.4), we see that, for any $t \in D$, any $w \in S$ and any $x \in X^+ - \{0\}$, we have $\zeta(x(w(t)))/\zeta(v_0) > 0$,

that is, $\zeta((w^{-1}x)(t))/\zeta(v_0) > 0$. In particular, for any $t \in D$ we have

$$\zeta((w'^{-1}x')(t))/\zeta(v_0) > 0 \text{ and } \zeta((w''^{-1}x'')(t))/\zeta(v_0) > 0.$$

We have $0 = z - z = w'^{-1}(x') + w''^{-1}(x'')$, hence

$$\begin{aligned} 0 &< \zeta((w'^{-1}x')(t))/\zeta(v_0) + \zeta((w''^{-1}x'')(t))/\zeta(v_0) \\ &= \zeta((w'^{-1}x')(t)(w''^{-1}x'')(t))/\zeta(v_0) \\ &= \zeta((w'^{-1}x' + w''^{-1}x'')(t))/\zeta(v_0) = \zeta(0(t))/\zeta(v_0) = \zeta(1)/\zeta(v_0) = 0. \end{aligned}$$

This is a contradiction. We see that (ii) implies the first condition in (i). Now the proof continues exactly as in 3.4; in particular, we see that the equivalence of (i) and (ii) follows from the equivalence of 3.4(iii) and 3.4(iv). The lemma is proved.

4. SOME CONSEQUENCES OF THE SECOND REDUCTION THEOREM OF [L2]

4.1. We place ourselves in the setup of 2.1 and we fix $v_0 \in \check{\spadesuit}$. Define $r_0 \in \spadesuit$ by $\exp(r_0) = v_0$. Let

$$X, Y, R, \check{R}, \Pi, W_0, \lambda, \lambda^*, \mathcal{T}$$

be as in 1.2. We write H instead of $H_{R,X}^{\lambda,\lambda^*}$. Let $\mathfrak{t} = \underline{\mathcal{T}}$. Assume that we are given a W_0 -orbit Σ in \mathcal{T} such that for any $t \in \Sigma$ and any $\alpha \in R$ we have

$$\alpha(t) \in \check{\spadesuit} \text{ if } \check{\alpha} \notin 2Y, \quad \alpha(t) \in \pm\check{\spadesuit} \text{ if } \check{\alpha} \in 2Y.$$

As in [9.2]' (see 2.2) we can find $t_0 \in \mathcal{T}$ (W_0 -invariant) and a W_0 -orbit $\bar{\Sigma}$ in $\mathfrak{t}_{\spadesuit}$ such that $\Sigma = t_0 \exp_{\mathcal{T}}(\bar{\Sigma})$.

Now $(R, \check{R}, X_{\mathbf{Q}}, Y_{\mathbf{Q}})$ is a \mathbf{Q} -root system with basis Π and with a parameter set $\mu : \Pi \rightarrow \mathbf{Z}$ defined by

$$\mu(\alpha) = 2\lambda(\alpha) \text{ if } \check{\alpha} \notin 2Y \text{ and } \mu(\alpha) = \lambda(\alpha) + \alpha(t_0)\lambda^*(\alpha) \text{ if } \check{\alpha} \in 2Y.$$

(In the last equality we have $\alpha(t_0) = \pm 1$ since $s_{\alpha}(t_0) = t_0$.) Let $\bar{H} = \bar{H}_{R,E}^{\mu}$. Let $\bar{\mathcal{O}} \subset \bar{H}$ be as in 1.4. We have $Y_{\mathbf{C}} = \mathfrak{t}$. Define $\Psi : \mathfrak{t} \oplus \mathbf{C} \rightarrow \mathcal{T} \times \mathbf{C}^*$ by $(e', z) \mapsto (t_0 \exp_{\mathcal{T}}(e'), \exp(z))$.

Let $(\bar{\mathcal{O}}^W)^{\wedge}$ be the $J_{\bar{\Sigma}, r_0}$ -adic completion of $\bar{\mathcal{O}}^W$. Let $\bar{M} \in \text{Mod}_{\bar{\Sigma}, r_0} \bar{H}$. Since $J_{\bar{\Sigma}, r_0} \bar{M} = 0$, \bar{M} extends naturally to a module over $\hat{\hat{H}} = \hat{\bar{H}}_{\bar{\mathcal{O}}^W}(\bar{\mathcal{O}}^W)^{\wedge}$. The second reduction theorem [L2, 9.3], in the variant [9.3]' (see 2.2), gives an explicit algebra isomorphism of \hat{H} (as in 1.2) with $\hat{\hat{H}}$. Via this isomorphism, \bar{M} becomes an \hat{H} -module and, by restriction, an H -module $\bar{M}^{\dagger} \in \text{Mod}_{\Sigma, v_0} H$. Note that \bar{M}^{\dagger} and \bar{M} have the same underlying \mathbf{C} -vector space. Let $e' \in Y_{\mathbf{C}} = \mathfrak{t}$. From the definitions we see that the e' -weight space $\bar{M}_{e'}$ of \bar{M} is equal to the t -weight space \bar{M}_t^{\dagger} of \bar{M}^{\dagger} where $t \in \mathcal{T}$ is defined by

$$(a) \quad x(t) = x(t_0) \exp\langle x, e' \rangle$$

for all $x \in X$.

Lemma 4.2. *The rule $\bar{M} \mapsto \bar{M}^{\dagger}$ is a bijection $\text{Irr}_{\bar{\Sigma}, r_0} \bar{H} \xrightarrow{\sim} \text{Irr}_{\Sigma, v_0} H$.*

This is an immediate consequence of the definitions and of [L2, 9.3], in the variant [9.3]' (see 2.2).

Lemma 4.3. *Let $\bar{M} \in \text{Mod}_{\bar{\Sigma}, r_0} \bar{H}$. Assume that $\zeta : \mathbf{C}^* \rightarrow \mathbf{R}$ is a homomorphism such that $\zeta(v_0) \neq 0$. Assume that $t_0 \in \mathcal{T}^{\text{Ker} \zeta}$. Define a homomorphism $\tau : \mathbf{C} \rightarrow \mathbf{R}$ by $\tau(z) = \zeta(\exp(z))$. Then $\tau(r_0) \neq 0$. The following two conditions are equivalent:*

- (i) *the \bar{H} -module \bar{M} is τ -tempered;*
- (ii) *the H -module \bar{M}^\dagger is ζ -tempered.*

In view of 4.1(a) it is enough to show that for $e' \in Y_{\mathbf{C}}$, the following two conditions are equivalent:

- (iii) for any $e \in X_{\mathbf{Q}}$ such that $\langle e, \check{\alpha} \rangle \geq 0$ for all $\alpha \in \Pi$ we have $\zeta(\exp\langle e, e' \rangle) / \zeta(v_0) \geq 0$;
 - (iv) for any $x \in X^+$ we have $\zeta(x(t_0) \exp\langle x, e' \rangle) / \zeta(v_0) \geq 0$.
- Since $t_0 \in \mathcal{T}^{\text{Ker} \zeta}$, for any $x \in X$, we have $\zeta(x(t_0)) = 0$ and in (iv) we have

$$(a) \quad \zeta(x(t_0) \exp\langle x, e' \rangle) = \zeta(\exp\langle x, e' \rangle).$$

Since $X \subset X_{\mathbf{Q}}$ it follows that, if (iii) holds, then (iv) holds.

Assume now that (iv) holds. Let $e \in X_{\mathbf{Q}}$ be such that $\langle e, \check{\alpha} \rangle \geq 0$ for all $\alpha \in \Pi$. We can find $n \in \mathbf{N} - \{0\}$ such that $ne \in X^+$. Since (iv) holds, it follows that $\zeta(\exp\langle ne, e' \rangle) / \zeta(v_0) \geq 0$. (We use (a).) Hence $n\zeta(\exp\langle e, e' \rangle) / \zeta(v_0) \geq 0$ so that $\zeta(\exp\langle e, e' \rangle) / \zeta(v_0) \geq 0$. Thus (iii) holds. The lemma is proved.

Lemma 4.4. *Let $\bar{M} \in \text{Mod}_{\bar{\Sigma}, r_0} \bar{H}$. Assume that $\zeta : \mathbf{C}^* \rightarrow \mathbf{R}$ is a homomorphism such that $\zeta(v_0) \neq 0$. Define a homomorphism $\tau : \mathbf{C} \rightarrow \mathbf{R}$ by $\tau(z) = \zeta(\exp(z))$. Then $\tau(r_0) \neq 0$. Assume that R generates $X_{\mathbf{Q}}$ as a \mathbf{Q} -vector space. The following two conditions are equivalent:*

- (i) *the \bar{H} -module \bar{M} is τ -square integrable;*
- (ii) *the H -module \bar{M}^\dagger is ζ -square integrable.*

In view of 4.1(a) it is enough to show that for $e' \in Y_{\mathbf{C}}$, the following two conditions are equivalent:

- (iii) for any $e \in X_{\mathbf{Q}} - \{0\}$ such that $\langle e, \check{\alpha} \rangle \geq 0$ for all $\alpha \in \Pi$ we have $\zeta(\exp\langle e, e' \rangle) / \zeta(v_0) > 0$;
- (iv) for any $x \in X^+ - \{0\}$ we have $\zeta(x(t_0) \exp\langle x, e' \rangle) / \zeta(v_0) > 0$.

This is shown in the same way as in the proof of 4.3. (In this case we have automatically $\zeta(x(t_0)) = 0$. Indeed, since t_0 is W_0 -invariant and R generates $X_{\mathbf{Q}}$, we see that t_0 has finite order in \mathcal{T} , hence $\zeta(x(t_0))$ has finite order in \mathbf{R} , hence $\zeta(x(t_0)) = 0$.) The lemma is proved.

5. GEOMETRIC GRADED HECKE ALGEBRAS

5.1. If G is an algebraic group, the exponential map $\exp : \underline{G} \rightarrow G$ restricts to a bijection $\underline{G}_\spadesuit \xrightarrow{\sim} G_\spadesuit$.

5.2. In this section we review some results of [L1], [L5] and give some variants of them.

Assume that G is a connected reductive algebraic group. Let $\mathfrak{g} = \underline{G}$. Let L be the Levi subgroup of some parabolic subgroup of G . Let \mathcal{C} be a nilpotent L -orbit in \underline{L} and let \mathcal{F} be an irreducible L -equivariant cuspidal local system (over \mathbf{C}) on \mathcal{C} . Let $T = Z_L^0$. Let $\mathfrak{t} = \underline{T}$. We have $\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{t}^*} \mathfrak{g}^\alpha$ where

$$\mathfrak{g}^\alpha = \{x \in \mathfrak{g}; [y, x] = \alpha(y)x \quad \forall y \in \mathfrak{t}\}.$$

Let $R' = \{\alpha \in \mathfrak{t}^* - \{0\}; \mathfrak{g}^\alpha \neq 0\}$, $R = \{\alpha \in R'; \alpha/2 \notin R'\}$. The group $W = N(T)/L$, where $N(T)$ is the normalizer of T in G , acts naturally on \mathfrak{t} and \mathfrak{t}^* . For any $\alpha \in R$

there is a unique element $s_\alpha \in W$ which acts on \mathfrak{t}^* as a reflection sending α to $-\alpha$; there is a unique element $\check{\alpha} \in \mathfrak{t}$ such that $s_\alpha(x) = x - x(\check{\alpha})\alpha$ for all $x \in \mathfrak{t}^*$. Let $\check{R} = \{\check{\alpha}; \alpha \in R\}$. We have canonically $T = Y \otimes \mathbf{C}^*$, $\mathfrak{t} = Y_{\mathbf{C}}$ where Y is the group of all one-parameter subgroups of T . Let

$$\mathfrak{t}_{\mathbf{Q}} = Y_{\mathbf{Q}}, \mathfrak{t}_{\mathbf{Q}}^* = \{x \in \mathfrak{t}^*; x(z) \in \mathbf{Q} \quad \forall z \in \mathfrak{t}_{\mathbf{Q}}\}.$$

Then $(R, \check{R}, \mathfrak{t}_{\mathbf{Q}}^*, \mathfrak{t}_{\mathbf{Q}})$ is a \mathbf{Q} -root system. Let Π be a basis for it. Let $y_0 \in \mathcal{C}$. For any $\alpha \in \Pi$ we denote by $\underline{c}(\alpha)$ the integer ≥ 2 such that $\text{ad}(y_0)^{\underline{c}(\alpha)-2} : \mathfrak{g}^\alpha \oplus \mathfrak{g}^{2\alpha} \rightarrow \mathfrak{g}^\alpha \oplus \mathfrak{g}^{2\alpha}$ is $\neq 0$ and $\text{ad}(y_0)^{\underline{c}(\alpha)-1} : \mathfrak{g}^\alpha \oplus \mathfrak{g}^{2\alpha} \rightarrow \mathfrak{g}^\alpha \oplus \mathfrak{g}^{2\alpha}$ is 0. (This is independent of the choice y_0 .) Then $\alpha \mapsto \underline{c}(\alpha)$ is a parameter set (see 1.4) for our \mathbf{Q} -root system. The corresponding algebra $\bar{H}_{R, \mathfrak{t}_{\mathbf{Q}}}^{\underline{c}}$ (see 1.4) is denoted by $\bar{H}(G, L, \mathcal{C}, \mathcal{F})$.

5.3. Let $r_0 \in \mathbf{C}$. Let σ be a semisimple element of \mathfrak{g} and let y be a nilpotent element of \mathfrak{g} such that $[\sigma, y] = 2r_0y$. Let P be a parabolic subgroup P of G with Levi subgroup L . Let

$$(a) \quad \mathbf{X}_{\sigma, y} = \{g \in G; \text{Ad}(g^{-1})y \in \mathcal{C} + \underline{U}_P, \text{Ad}(g^{-1})\sigma \in \underline{P}\}.$$

We have an obvious map $\mathbf{X}_{\sigma, y} \rightarrow \mathcal{C}$ which takes g to the image of $\text{Ad}(g^{-1})y$ under $\mathcal{C} + \underline{U}_P \rightarrow \mathcal{C}, a + b \mapsto a$. The inverse image of \mathcal{F} under this map is denoted again by \mathcal{F} . On $\mathbf{X}_{\sigma, y}$ we have a free P -action by right translation and \mathcal{F} is P -equivariant, hence it descends to a local system $\tilde{\mathcal{F}}$ on $\mathbf{X}_{\sigma, y}/P$. The group $Z_G(\sigma, y)$ acts on $\mathbf{X}_{\sigma, y}/P$ by left translation and $\tilde{\mathcal{F}}$ is naturally a $Z_G(\sigma, y)$ -equivariant local system. Then $\bar{Z}_G(\sigma, y)$ acts naturally on the cohomology

$$(b) \quad \bigoplus_n H_c^n(\mathbf{X}_{\sigma, y}/P, \tilde{\mathcal{F}}).$$

The set of irreducible representations (up to isomorphism) of $\bar{Z}_G(\sigma, y)$ which appear in the representation (b) is denoted by $\text{Irr}_0 \bar{Z}_G(\sigma, y)$. (This set is independent of the choice of P since another choice of P is of the form nPn^{-1} where $nLn^{-1} = L, \text{Ad}(n)\mathcal{C} = \mathcal{C}$.)

Let $\mathfrak{S}(G, L, \mathcal{C}, \mathcal{F}, r_0)$ be the set consisting of all triples (σ, y, ρ) (modulo the natural action of G) where σ, y are as above and $\rho \in \text{Irr}_0 \bar{Z}_G(\sigma, y)$.

In [L5] a canonical bijection

$$(c) \quad \text{Irr}_{r_0} \bar{H}(G, L, \mathcal{C}, \mathcal{F}) \leftrightarrow \mathfrak{S}(G, L, \mathcal{C}, \mathcal{F}, r_0)$$

is established using geometric methods (equivariant homology).

5.4. We fix elements e^0, h^0, f^0 in \underline{L} which satisfy the standard relations of \mathfrak{sl}_2 and $e^0 \in \mathcal{C}$. Let

$$\begin{aligned} Z &= \{g \in G; \text{Ad}(g)e^0 = e^0, \text{Ad}(g)h^0 = h^0, \text{Ad}(g)f^0 = f^0\}, \\ \tilde{Z} &= \{(g, a) \in G \times \mathbf{C}^*; \text{Ad}(g)e^0 = a^2e^0, \text{Ad}(g)h^0 = h^0, \text{Ad}(g)f^0 = a^{-2}f^0\}. \end{aligned}$$

We have

$$\begin{aligned} \underline{Z} &= \{x \in G; [x, e^0] = 0, [x, h^0] = 0, [x, f^0] = 0\}, \\ \underline{\tilde{Z}} &= \{(x, a) \in \mathfrak{g} \times \mathbf{C}; [x, e^0] = 2ae^0, [x, h^0] = 0, [x, f^0] = -2af^0\}. \end{aligned}$$

There is a unique isomorphism of algebraic groups $\iota : Z^0 \times \mathbf{C}^* \xrightarrow{\sim} \tilde{Z}^0$ such that the induced Lie algebra isomorphism $\underline{Z} \oplus \mathbf{C} \xrightarrow{\sim} \underline{\tilde{Z}}$ is given by $(x, a) \mapsto x + ah^0$.

Lemma 5.5. *The inclusion $\tilde{Z}^0 \rightarrow G \times \mathbf{C}^*$ induces an injective map from the set of semisimple \tilde{Z}^0 -orbits in $\tilde{\underline{Z}}$ to the set of semisimple $G \times \mathbf{C}^*$ -orbits in $\mathfrak{g} \oplus \mathbf{C}$.*

This has been stated without proof and used in [L1, 14.3(a)], [L3, 8.13]. The proof is given in the appendix.

5.6. Let $\bar{\Sigma}$ be a W -orbit on \mathfrak{t} and let $\bar{M} \in \text{Irr}_{\bar{\Sigma}, r_0} \bar{H}(G, L, \mathcal{C}, \mathcal{F})$. Assume that \bar{M} corresponds to σ, y, ρ under 5.3(c).

The center $\bar{\mathcal{O}}^W$ (as in 1.4) of $\bar{H}(G, L, \mathcal{C}, \mathcal{F})$ acts on \bar{M} via a character which may be identified with the W -orbit $\bar{\Sigma} \times \{r_0\}$ on $\mathfrak{t} \times \mathbf{C}$. By [L3, 8.13], $\bar{\mathcal{O}}^W$ is identified with the equivariant cohomology $H_{Z^0 \times \mathbf{C}^*}^*(point)$ which via ι is identified with $H_{\tilde{Z}^0}^*(point)$; from the definitions, the natural action of $H_{\tilde{Z}^0}^*(point)$ on \bar{M} is via a character that may be identified with a semisimple orbit in $\tilde{\underline{Z}}$ which is contained in the $\text{Ad}(G \times \mathbf{C}^*)$ -orbit of (σ, r_0) in $\mathfrak{g} \oplus \mathbf{C}$.

Thus, σ is related to $\bar{\Sigma}$ as follows: there exists $(\sigma', r_0) \in \tilde{\underline{Z}}$ such that σ, σ' are conjugate under G and $\iota^{-1}(\sigma', r_0) = (\sigma' - r_0 h_0, r_0) \in \bar{\Sigma} \times \{r_0\}$.

We define a map

$$(a) \quad \mathfrak{S}(G, L, \mathcal{C}, \mathcal{F}, r_0) \rightarrow \mathfrak{t}/W.$$

Consider an element u of $\mathfrak{S}(G, L, \mathcal{C}, \mathcal{F}, r_0)$ represented by (σ, y, ρ) . We can find

$$(b) \quad \sigma' \text{ in the } G\text{-orbit of } \sigma \text{ such that } \sigma' - r_0 h^0 \in \mathfrak{t}.$$

Indeed, since $\mathbf{X}_{\sigma, y} \neq \emptyset$, there exists $g \in G$ such that $\sigma_1 = \text{Ad}(g^{-1})\sigma \in \underline{L}$, $y' = \text{Ad}(g^{-1})y \in e^0 + \underline{U}_P$. From $[\sigma_1, y'] = 2r_0 y'$ we deduce that $[\sigma_1, e^0] = 2r_0 e^0$. Now we can find $l \in L$ such that $\text{Ad}(l)e^0 \in \mathbf{C}^* e^0$ and such that $\sigma' = \text{Ad}(l)\sigma_1$ satisfies $\sigma' \in \underline{L}$, $[\sigma', e^0] = 2r_0 e^0$, $[\sigma', h^0] = 0$, $[\sigma', f^0] = -2r_0 f^0$. Since e^0 is distinguished in \underline{L} , it follows that $\sigma' - r_0 h^0 \in \mathfrak{t}$, as required.

Let σ' be as in (b) and let $\bar{\Sigma}$ be the W -orbit of $\sigma' - r_0 h^0$ in \mathfrak{t} . Let $\tilde{\sigma}'$ be another element like σ' and let $\tilde{\Sigma}$ be the W -orbit of $\tilde{\sigma}' - r_0 h^0$ in \mathfrak{t} . Then $(\sigma', r_0), (\tilde{\sigma}', r_0)$ are semisimple elements of $\tilde{\underline{Z}}$ and are in the same $G \times \mathbf{C}^*$ -orbit in $\mathfrak{g} \oplus \mathbf{C}$; hence, by 5.5, there exists $(g', a) \in \tilde{Z}^0$ such that $\text{Ad}(g')\sigma' = \tilde{\sigma}'$. Since h^0 is central in $\tilde{\underline{Z}}$, we have $\text{Ad}(g')\sigma' = \tilde{\sigma}'$. Since $(h^0, 1)$ is central in $\tilde{\underline{Z}}$, we have $\text{Ad}(g')h^0 = h^0$. Hence $\text{Ad}(g')(\sigma' - r_0 h^0) = \tilde{\sigma}' - r_0 h^0$. Since $\sigma' - r_0 h^0, \tilde{\sigma}' - r_0 h^0$ belong to \mathfrak{t} , hence to the Cartan subalgebra $\mathfrak{t} \oplus \mathbf{C}h^0$, they are in the same orbit of the Weyl group of \tilde{Z}^0 with respect to that Cartan subalgebra, which may be identified with W . It follows that $\bar{\Sigma} = \tilde{\Sigma}$. Thus we have a map $u \mapsto \bar{\Sigma}$ as in (a).

For any W -orbit $\bar{\Sigma}$ in \mathfrak{t} let $\mathfrak{S}(G, L, \mathcal{C}, \mathcal{F}, \bar{\Sigma}, r_0)$ be the inverse image of $\bar{\Sigma}$ under the map (a). The previous discussion yields the following result.

Lemma 5.7. *Let $\bar{\Sigma}$ be a W -orbit in \mathfrak{t} . The bijection 5.3(c) restricts to a bijection $\text{Irr}_{\bar{\Sigma}, r_0} \bar{H}(G, L, \mathcal{C}, \mathcal{F}) \leftrightarrow \mathfrak{S}(G, L, \mathcal{C}, \mathcal{F}, \bar{\Sigma}, r_0)$.*

Lemma 5.8. *In the setup of 2.1, assume that $r_0 \in \spadesuit$. The bijection 5.3(c) restricts to a bijection*

$$\text{Irr}_{r_0}^\spadesuit \bar{H}(G, L, \mathcal{C}, \mathcal{F}) \leftrightarrow \mathfrak{S}^\spadesuit(G, L, \mathcal{C}, \mathcal{F}, r_0)$$

where $\text{Irr}_{r_0}^\spadesuit \bar{H}(G, L, \mathcal{C}, \mathcal{F}) = \bigsqcup \text{Irr}_{\bar{\Sigma}, r_0} \bar{H}(G, L, \mathcal{C}, \mathcal{F})$ (union over all W -orbits $\bar{\Sigma}$ in \mathfrak{t}_\spadesuit) and $\mathfrak{S}^\spadesuit(G, L, \mathcal{C}, \mathcal{F}, r_0)$ consists of all (σ, y, ρ) in $\mathfrak{S}(G, L, \mathcal{C}, \mathcal{F}, r_0)$ such that $\sigma \in \mathfrak{g}_\spadesuit$.

Using 5.7 we see that it is enough to show that, if $(\sigma, y, \rho) \in \mathfrak{S}(G, L, \mathcal{C}, \mathcal{F}, r_0)$ and $\bar{\Sigma} \in \mathfrak{t}/W$ correspond to each other under 5.6(a), then we have $\sigma \in \mathfrak{g}_\blacklozenge$ if and only if $\bar{\Sigma} \subset \mathfrak{t}_\blacklozenge$. We may assume that $\sigma - r_0 h^0 \in \bar{\Sigma}$. Now $r_0 h^0$ has eigenvalues in $\mathbf{Z}r_0$ in any $V \in \mathcal{I}_G$ (a property of \mathfrak{sl}_2), hence $r_0 h^0 \in \mathfrak{g}_\blacklozenge$ (since $r_0 \in \blacklozenge$). If we have two elements of $\mathfrak{g}_\blacklozenge$ that commute, then their sum is again in $\mathfrak{g}_\blacklozenge$. Applying this to the commuting elements $\sigma, -r_0 h^0$ and to the commuting elements $\sigma - r_0 h^0, r_0 h^0$ we deduce that $\sigma \in \mathfrak{g}_\blacklozenge$ if and only if $\sigma - r_0 h^0 \in \mathfrak{g}_\blacklozenge$, that is, if and only if $\bar{\Sigma} \subset \mathfrak{g}_\blacklozenge$. It remains to observe that $\mathfrak{g}_\blacklozenge \cap \mathfrak{t} = \mathfrak{t}_\blacklozenge$. The lemma is proved.

The following result is closely related to [L5, 1.21].

Lemma 5.9. *Assume that $\tau : \mathbf{C} \rightarrow \mathbf{R}$ is a group homomorphism such that $\tau(r_0) \neq 0$. Let $\bar{M} \in \text{Irr}_{r_0} \bar{H}(G, L, \mathcal{C}, \mathcal{F})$ and let (σ, y, ρ) correspond to \bar{M} under 5.3(c). The following two conditions are equivalent:*

- (i) \bar{M} is τ -tempered;
- (ii) *there exists $h \in \mathfrak{g}, \tilde{y} \in \mathfrak{g}$ such that $[y, \tilde{y}] = h, [h, y] = 2y, [h, \tilde{y}] = -2\tilde{y}, [\sigma, h] = 0, [\sigma, \tilde{y}] = -2r_0 \tilde{y}$ and such that $\sigma - r_0 h \in \mathfrak{g}^{\text{Ker}\tau}$.*

Assume that the lemma holds for $(G_i, L_i, \mathcal{C}_i, \mathcal{F}_i)$, $i = 1, 2$ (two data such as $(G, L, \mathcal{C}, \mathcal{F})$). Then one checks easily that it also holds also for $(G_1 \times G_2, L_1 \times L_2, \mathcal{C}_1 \times \mathcal{C}_2, \mathcal{F}_1 \boxtimes \mathcal{F}_2)$. Without loss of generality, we can assume that $G = C \times G'$ where G' is semisimple and C is a torus. Hence it is enough to prove the lemma assuming that G is either semisimple or a torus.

Assume first that G is a torus. Then $G = L = T$. In this case, we must verify that for any $e' \in \mathfrak{t}$, the following two conditions are equivalent:

- (iii) For any $e \in \mathfrak{t}_\mathbf{Q}^*$ we have $\tau(\langle e, e' \rangle) / \tau(r_0) \geq 0$;
- (iv) $e' \in \mathfrak{t}^{\text{Ker}\tau} = Y \otimes \text{Ker}\tau$.

Now (iii) is equivalent to the condition that, for any $e \in \mathfrak{t}_\mathbf{Q}^*$, we have $\tau(\langle e, e' \rangle) = 0$. One checks easily that this is equivalent to (iv).

Next, assume that G is semisimple. The condition that \bar{M} is τ -tempered (as given in 1.5) is in this case equivalent to the condition that \bar{M} is τ -tempered according to the definition in [L5, 1.20]. (This follows easily from [L5, 3.6].) Using [L5, 1.21] we are therefore reduced to verifying the following statement.

If $x \in \mathfrak{g}$ and any eigenvalue ν of $\text{ad}(x) : \mathfrak{g} \rightarrow \mathfrak{g}$ satisfies $\tau(\nu) = 0$, then for any $V \in \mathcal{I}_G$,

- (a) *any eigenvalue ν of $x : V \rightarrow V$ satisfies $\tau(\nu) = 0$ (that is, $x \in \mathfrak{g}^{\text{Ker}\tau}$).*

Clearly, if (a) holds for V , then it also holds for $V^{\otimes n}$ for any $n \geq 0$. Since (a) holds for the adjoint representation, it also holds for tensor powers of the adjoint representations, hence for any direct summand of such a tensor power, hence for any irreducible V on which Z_G acts trivially, hence for any V on which Z_G acts trivially. If $V \in \mathcal{I}_G, n \geq 1$ and (a) holds for $V^{\otimes n}$, then it also holds for V . (Indeed, if ν is an eigenvalue of $x : V \rightarrow V$, then $n\nu$ is an eigenvalue of $x : V^{\otimes n} \rightarrow V^{\otimes n}$, hence $\tau(n\nu) = 0$, hence $n\tau(\nu) = 0$, hence $\tau(\nu) = 0$.) If $V \in \mathcal{I}_G$, then there exists $n \geq 1$ such that Z_G acts trivially on $V^{\otimes n}$. As we have seen earlier, (a) holds for $V^{\otimes n}$, hence it holds for V . Thus (a) holds in general. The lemma is proved.

The following result is closely related to [L5, 1.22].

Lemma 5.10. *Assume that G is semisimple. Assume that $\tau : \mathbf{C} \rightarrow \mathbf{R}$ is a group homomorphism such that $\tau(r_0) \neq 0$. Let $\bar{M} \in \text{Irr}_{r_0} \bar{H}(G, L, \mathcal{C}, \mathcal{F})$ and let (σ, y, ρ) correspond to \bar{M} under 5.3(c). The following two conditions are equivalent:*

- (i) \bar{M} is τ -square integrable;

(ii) *there exists $h \in \mathfrak{g}, \tilde{y} \in \mathfrak{g}$ such that $[y, \tilde{y}] = h, [h, y] = 2y, [h, \tilde{y}] = -2\tilde{y}, \sigma = r_0 h$; moreover, y is distinguished.*

Using [L5, 3.6], we see that the condition that \bar{M} is τ -square integrable (as given in 1.5) is in our case equivalent to the condition that \bar{M} is τ -square integrable according to the definition in [L5, 1.20]. Hence the lemma follows from [L5, 1.22].

5.11. For any $\xi \in \mathfrak{g}$ let

$$(a) \quad \mathbf{Y}_\xi = \{g \in G; \text{Ad}(g^{-1})\xi \in \mathcal{C} + \mathfrak{t} + \underline{U_P}\}.$$

We have an obvious map $\mathbf{Y}_\xi \rightarrow \mathcal{C}$ which takes g to the image of $\text{Ad}(g^{-1})y$ under $\mathcal{C} + \mathfrak{t} + \underline{U_P} \rightarrow \mathcal{C}, a + b + c \mapsto a$. The inverse image of \mathcal{F} under this map is denoted again by \mathcal{F} . On \mathbf{Y}_ξ we have a free P -action by right translation and \mathcal{F} is P -equivariant, hence it descends to a local system $\tilde{\mathcal{F}}$ on \mathbf{Y}_ξ/P . The group $Z_G(\xi)$ acts on \mathbf{Y}_ξ/P by left translation and $\tilde{\mathcal{F}}$ is naturally a $Z_G(\xi)$ -equivariant local system. Then $\bar{Z}_G(\xi)$ acts naturally on the cohomology

$$(b) \quad \bigoplus_n H_c^n(\mathbf{Y}_\xi/P, \tilde{\mathcal{F}}).$$

The set of irreducible representations (up to isomorphism) of $\bar{Z}_G(\xi)$ which appear in the representation (b) is denoted by $\text{Irr}_0 \bar{Z}_G(\xi)$.

5.12. Let σ, y be two elements of \mathfrak{g} such that σ is semisimple, y is nilpotent and $[\sigma, y] = 2r_0 y$. We choose (as we may) elements h, \tilde{y} in \mathfrak{g} such that

$$(a) \quad [y, \tilde{y}] = h, [h, y] = 2y, [h, \tilde{y}] = -2\tilde{y}, [\sigma, h] = 0, [\sigma, \tilde{y}] = -2r_0 \tilde{y}$$

and we set $\xi_1 = \sigma - r_0 h$. This is a semisimple element of \mathfrak{g} (since σ, h are commuting semisimple elements) and it commutes with y, h, \tilde{y} . We set $\xi = \xi_1 + y$. If we make another choice h', \tilde{y}' instead of h, \tilde{y} , then, as it is known, there exists an element $g \in Z_G(\sigma, y)$ such that $\text{Ad}(g)h = h', \text{Ad}(g)\tilde{y} = \tilde{y}'$. Let $\xi'_1 = \sigma - r_0 h'$. We have $\text{Ad}(g)\xi_1 = \xi'_1$. Let $\xi' = \xi'_1 + y$. We have $\text{Ad}(g)\xi = \xi'$. Thus, the G -orbit of ξ is well defined by σ, y (in fact, it depends only on the G -orbit of (σ, y)).

Conversely, assume that $\xi \in \mathfrak{g}$ is given. We can write uniquely $\xi = \xi_1 + y$ where $\xi_1 \in \mathfrak{g}$ is semisimple, $y \in \mathfrak{g}$ is nilpotent and $[\xi_1, y] = 0$. We choose (as we may) elements h, \tilde{y} in \mathfrak{g} such that

$$(b) \quad [y, \tilde{y}] = h, [h, y] = 2y, [h, \tilde{y}] = -2\tilde{y}, [\xi_1, h] = 0, [\xi_1, \tilde{y}] = 0.$$

Let $\sigma = \xi_1 + r_0 h$. Then σ is a semisimple element (since ξ_1, h are commuting semisimple elements) and $[\sigma, y] = 2r_0 y, [\sigma, h] = 0, [\sigma, \tilde{y}] = -2r_0 \tilde{y}$. If we make another choice h', \tilde{y}' instead of h, \tilde{y} , then, as it is known, there exists $g' \in Z_G(\xi_1, y)$ such that $\text{Ad}(g')h = h', \text{Ad}(g')\tilde{y} = \tilde{y}'$. Let $\sigma' = \xi_1 + r_0 h'$. We have $\text{Ad}(g')\sigma = \sigma'$. Thus the G -orbit of (σ, y) is well defined by ξ (in fact, it depends only on the G -orbit of ξ .) Thus we have defined a bijection $\sigma, y \leftrightarrow \xi$ between the set of G -orbits of pairs (σ, y) of elements of \mathfrak{g} such that σ is semisimple, y is nilpotent and $[\sigma, y] = 2r_0 y$, on the one hand, and the set of G -orbits in \mathfrak{g} , on the other hand.

Lemma 5.13. *Assume that (σ, y) corresponds to ξ as above. Then the groups $\bar{Z}_G(\sigma, y)$ and $\bar{Z}_G(\xi)$ may be naturally identified so that $\text{Irr}_0 \bar{Z}_G(\sigma, y) = \text{Irr}_0 \bar{Z}_G(\xi)$.*

We may assume that σ, y, ξ are related as follows: there exist h, \tilde{y} in \mathfrak{g} so that 5.12(a) holds and $\xi = \xi_1 + y$ where $\xi_1 = \sigma - r_0 h$. It is known that $Z' = \{g \in G; \text{Ad}(g)y = y, \text{Ad}(g)h = h, \text{Ad}(g)\tilde{y} = \tilde{y}, \text{Ad}(g)\sigma = \sigma\}$ is a maximal reductive

subgroup of $Z_G(\sigma, y)$, hence it has the same group of components as $Z_G(\sigma, y)$. Similarly, since $Z_G(\xi_1)$ is connected, reductive, $Z'' = \{g \in G; \text{Ad}(g)y = y, \text{Ad}(g)h = h, \text{Ad}(g)\tilde{y} = \tilde{y}, \text{Ad}(g)\xi_1 = \xi_1\}$ is a maximal reductive subgroup of $\{g \in G; \text{Ad}(g)y = y, \text{Ad}(g)\xi_1 = \xi_1\} = Z_G(\xi)$, hence Z'' has the same group of components as $Z_G(\xi)$. Now $Z' = Z''$. It follows that $Z_G(\sigma, y)$ and $Z_G(\xi)$ have the same group of components. Note that

$$(a) \quad \mathbf{Y}_\xi = \{g \in G; \text{Ad}(g^{-1})\xi_1 \in \mathfrak{t} + \underline{U}_P, \text{Ad}(g^{-1})y \in \mathcal{C} + \underline{U}_P\}.$$

Now in the presence of the condition $\text{Ad}(g^{-1})y \in \mathcal{C} + \underline{U}_P$, the conditions

$$(b) \quad \text{Ad}(g^{-1})\xi_1 \in \mathfrak{t} + \underline{U}_P,$$

$$(c) \quad \text{Ad}(g^{-1})\xi_1 \in \underline{P},$$

are equivalent. Indeed, it is clear that if (b) holds, then (c) holds. Conversely, assume that (c) holds. Then $\text{Ad}(g^{-1})\xi_1 = l \bmod \underline{U}_P$ where $l \in \underline{L}$. By our assumption we have $\text{Ad}(g^{-1})y = y_0 \bmod \underline{U}_P$ where $y \in \mathcal{C}$. Since $[\xi_1, y] = 0$, we have $[\text{Ad}(g^{-1})\xi_1, \text{Ad}(g^{-1})y] = 0$ and taking images under $\underline{P} \rightarrow \underline{L}$ we deduce $[l, y_0] = 0$. Since y_0 is distinguished in \underline{L} , its centralizer in \underline{L} is \mathfrak{t} . Thus, $l \in \mathfrak{t}$ so that (b) holds. We see that (a) can be rewritten as

$$\mathbf{Y}_\xi = \{g \in G; \text{Ad}(g^{-1})\xi_1 \in \underline{P}, \text{Ad}(g^{-1})y \in \mathcal{C} + \underline{U}_P\}.$$

Let $s \in \text{Hom}(\mathbf{C}^*, G)$ be such that the tangent map $\mathbf{C} \rightarrow \mathfrak{g}$ of s carries 1 to h . We define a \mathbf{C}^* -action of \mathbf{Y}_ξ by $a : g \mapsto s(a)g$. This induces a \mathbf{C}^* -action on \mathbf{Y}_ξ/P whose fixed point set is

$$\mathbf{Y}' = \{g \in G; \text{Ad}(g^{-1})\xi_1 \in \underline{P}, \text{Ad}(g^{-1})y \in \mathcal{C} + \underline{U}_P, \text{Ad}(g^{-1})h \in \underline{P}\}/P.$$

Similarly, we define a \mathbf{C}^* -action on $\mathbf{X}_{\sigma, y}$ by $a : g \mapsto s(a)g$. This induces a \mathbf{C}^* -action on $\mathbf{X}_{\sigma, y}/P$ whose fixed point set is

$$\mathbf{X}' = \{g \in G; \text{Ad}(g^{-1})y \in \mathcal{C} + \underline{U}_P, \text{Ad}(g^{-1})\sigma \in \underline{P}, \text{Ad}(g^{-1})h \in \underline{P}\}/P.$$

In the presence of the condition $\text{Ad}(g^{-1})h \in \underline{P}$, the conditions $\text{Ad}(g^{-1})\sigma \in \underline{P}$ and $\text{Ad}(g^{-1})\xi_1 \in \underline{P}$ are equivalent (since $\xi_1 = \sigma - r_0 h$). It follows that $\mathbf{Y}' = \mathbf{X}'$. Thus the \mathbf{C}^* -actions on \mathbf{Y}_ξ/P and on $\mathbf{X}_{\sigma, y}/P$ have the same fixed point set. They also have the same action of $Z' = Z''$.

The restriction of the local system $\tilde{\mathcal{F}}$ on \mathbf{Y}_ξ/P (see 5.3) to this fixed point set is the same as the restriction of the local system $\tilde{\mathcal{F}}$ on $\mathbf{X}_{\sigma, y}/P$ (see 5.11) to this fixed point set; the restriction is denoted again by $\tilde{\mathcal{F}}$. By the principle of conservation of Euler characteristics by passage to the fixed point set of a torus action, we have

$$\begin{aligned} \sum_n (-1)^n H_c^n(\mathbf{X}_{\sigma, y}/P, \tilde{\mathcal{F}}) &= \sum_n (-1)^n H_c^n(\mathbf{X}', \tilde{\mathcal{F}}) \\ &= \sum_n (-1)^n H_c^n(\mathbf{Y}', \tilde{\mathcal{F}}) = \sum_n (-1)^n H_c^n(\mathbf{Y}_\xi/P, \tilde{\mathcal{F}}) \end{aligned}$$

as virtual representations of $\bar{Z}_G(\sigma, y) = \bar{Z}_G(\xi)$. Hence for an irreducible representation ρ of $\bar{Z}_G(\sigma, y) = \bar{Z}_G(\xi)$, the conditions

$$\rho \text{ appears in } \sum_n (-1)^n H_c^n(\mathbf{X}_{\sigma, y}/P, \tilde{\mathcal{F}}),$$

$$\rho \text{ appears in } \sum_n (-1)^n H_c^n(\mathbf{Y}_\xi/P, \tilde{\mathcal{F}})$$

are equivalent. For n odd we have $H_c^n(\mathbf{X}_{\sigma, y}/P, \tilde{\mathcal{F}}) = 0$ and $H_c^n(\mathbf{Y}_\xi/P, \tilde{\mathcal{F}}) = 0$. (Indeed, both $\mathbf{X}_{\sigma, y}/P$ and \mathbf{Y}_ξ/P can be regarded as fixed point sets of torus actions

on the variety denoted by \mathcal{P}_y in [L1], and it suffices to use the odd vanishing theorem [L1, 8.6] for \mathcal{P}_y together with [L3, 4.4]). It follows that the conditions

$$\begin{aligned} \rho &\text{ appears in } \sum_n H_c^n(\mathbf{X}_{\sigma,y}/P, \tilde{\mathcal{F}}), \\ \rho &\text{ appears in } \sum_n H_c^n(\mathbf{Y}_\xi/P, \tilde{\mathcal{F}}) \end{aligned}$$

are equivalent. The lemma is proved.

5.14. Let $\mathfrak{T}(G, L, \mathcal{C}, \mathcal{F})$ be the set consisting of all pairs (ξ, ρ) (modulo the natural action of G) where $\xi \in \mathfrak{g}$ and $\rho \in \text{Irr}_0 \bar{Z}_G(\xi)$. By 5.12 and 5.13 we have a canonical bijection

$$(a) \quad \mathfrak{S}(G, L, \mathcal{C}, \mathcal{F}, r_0) \leftrightarrow \mathfrak{T}(G, L, \mathcal{C}, \mathcal{F}).$$

Composing this with the bijection 5.3(c) we obtain a bijection

$$(b) \quad \text{Irr}_{r_0} \bar{H}(G, L, \mathcal{C}, \mathcal{F}) \leftrightarrow \mathfrak{T}(G, L, \mathcal{C}, \mathcal{F}).$$

In the setup of 2.1 and assuming that $r_0 \in \mathfrak{g}_\bullet$, let $\mathfrak{T}^\bullet(G, L, \mathcal{C}, \mathcal{F})$ be the set of all (ξ, ρ) in $\mathfrak{T}(G, L, \mathcal{C}, \mathcal{F})$ such that $\xi \in \mathfrak{g}_\bullet$. Then (a) restricts to a bijection

$$(c) \quad \mathfrak{S}^\bullet(G, L, \mathcal{C}, \mathcal{F}, r_0) \leftrightarrow \mathfrak{T}^\bullet(G, L, \mathcal{C}, \mathcal{F}).$$

Indeed, let $\xi \in \mathfrak{g}$ and write $\xi = \xi_1 + y$ where $\xi_1 \in \mathfrak{g}$ is semisimple, $y \in \mathfrak{g}$ is nilpotent and $[\xi_1, y] = 0$. Let h, \tilde{y} in \mathfrak{g} such that 5.12(b) holds and let $\sigma = \xi_1 + r_0 h$. We must show that $\xi \in \mathfrak{g}_\bullet$ if and only if $\sigma \in \mathfrak{g}_\bullet$. Clearly, $\xi \in \mathfrak{g}_\bullet$ if and only if $\xi_1 \in \mathfrak{g}_\bullet$. As in 5.8, we have $r_0 h \in \mathfrak{g}_\bullet$. (The eigenvalues of h in any $V \in \mathcal{I}_G$ are integers.) As in 5.8, if we have two elements of \mathfrak{g}_\bullet that commute, then their sum is again in \mathfrak{g}_\bullet . Applying this to the commuting elements $\xi_1, r_0 h^0$ and to the commuting elements $\sigma, -r_0 h^0$ we deduce that $\sigma \in \mathfrak{g}_\bullet$ if and only if $\xi_1 \in \mathfrak{g}_\bullet$. This yields (c).

Composing (c) with the bijection in 5.8 we obtain a bijection

$$(d) \quad \text{Irr}_{r_0}^\bullet \bar{H}(G, L, \mathcal{C}, \mathcal{F}) \leftrightarrow \mathfrak{T}^\bullet(G, L, \mathcal{C}, \mathcal{F}).$$

(Notation of 5.8.) We can now reformulate Lemmas 5.9 and 5.10 as follows.

Lemma 5.15. *Assume that $\tau : \mathbf{C} \rightarrow \mathbf{R}$ is a group homomorphism such that $\tau(r_0) \neq 0$. Let $\bar{M} \in \text{Irr}_{r_0} \bar{H}(G, L, \mathcal{C}, \mathcal{F})$ and let (ξ, ρ) correspond to \bar{M} under 5.14(b). The following two conditions are equivalent:*

- (i) \bar{M} is τ -tempered;
- (ii) $\xi \in \mathfrak{g}^{\text{Ker}\tau}$.

By 5.9, condition (i) is equivalent to the condition that the semisimple part ξ_1 of ξ satisfies $\xi_1 \in \mathfrak{g}^{\text{Ker}\tau}$. But this is clearly equivalent to condition (ii).

Lemma 5.16. *Assume that G is semisimple. Assume that $\tau : \mathbf{C} \rightarrow \mathbf{R}$ is a group homomorphism such that $\tau(r_0) \neq 0$. Let $\bar{M} \in \text{Irr}_{r_0} \bar{H}(G, L, \mathcal{C}, \mathcal{F})$ and let (ξ, ρ) correspond to \bar{M} under 5.14(b). The following two conditions are equivalent:*

- (i) \bar{M} is τ -square integrable;
- (ii) ξ is a distinguished nilpotent element.

5.17. The local system on $\exp(\mathcal{C})$ (a unipotent class in L) that corresponds to \mathcal{F} under $\exp : \underline{L} \rightarrow L$ is denoted again by \mathcal{F} . For any $f \in G$, let

$$(a) \quad \dot{\mathbf{Y}}_f = \{g \in G; g^{-1}fg \in \exp(\mathcal{C})TU_P\}.$$

Consider the map $\dot{\mathbf{Y}}_f \rightarrow \exp(\mathcal{C})$ which takes g to the image of $g^{-1}fg$ under $\exp(\mathcal{C})TU_P \rightarrow \exp(\mathcal{C}), abc \mapsto a$. The inverse image of \mathcal{F} under this map is denoted again by \mathcal{F} . On $\dot{\mathbf{Y}}_f$ we have a free P -action by right translation and \mathcal{F} is

P -equivariant, hence it descends to a local system $\tilde{\mathcal{F}}$ on $\dot{\mathbf{Y}}_f/P$. The group $Z_G(f)$ acts on $\dot{\mathbf{Y}}_f/P$ by left translation and $\tilde{\mathcal{F}}$ is naturally a $Z_G(f)$ -equivariant local system. Then $\bar{Z}_G(f)$ acts naturally on the cohomology

$$(b) \quad \bigoplus_n H_c^n(\dot{\mathbf{Y}}_f/P, \tilde{\mathcal{F}}).$$

The set of irreducible representations (up to isomorphism) of $\bar{Z}_G(f)$ which appear in the representation (b) is denoted by $\text{Irr}_0 \bar{Z}_G(f)$.

In the setup of 2.1, let $\dot{\mathfrak{T}}^\bullet(G, L, \mathcal{C}, \mathcal{F})$ be the set of all (f, ρ) (modulo the natural action of G) where $f \in G_\bullet$ and $\rho \in \text{Irr}_0 \bar{Z}_G(f)$.

If $f \in G_\bullet$ corresponds to $\xi \in \mathfrak{g}_\bullet$ under the bijection 5.1, we have $Z_G(f) = Z_G(\xi)$, hence $\bar{Z}_G(f) = \bar{Z}_G(\xi)$; we also have $\dot{\mathbf{Y}}_f = \dot{\mathbf{Y}}_\xi$. (We use that $\exp : \underline{P} \rightarrow P$ restricts to a bijection $\mathcal{C} + \mathfrak{t}_\bullet + U_P \xrightarrow{\sim} \exp(\mathcal{C})T_\bullet U_P$.) It follows that $\text{Irr}_0 \bar{Z}_G(f) = \text{Irr}_0 \bar{Z}_G(\xi)$. We see that $(\xi, \rho) \mapsto (\exp(\xi), \rho)$ defines a bijection

$$(c) \quad \mathfrak{T}^\bullet(G, L, \mathcal{C}, \mathcal{F}) \xrightarrow{\sim} \dot{\mathfrak{T}}^\bullet(G, L, \mathcal{C}, \mathcal{F}).$$

Composing this with the bijection 5.14(d) we obtain a bijection

$$(d) \quad \text{Irr}_{r_0}^\bullet \bar{H}(G, L, \mathcal{C}, \mathcal{F}) \leftrightarrow \dot{\mathfrak{T}}^\bullet(G, L, \mathcal{C}, \mathcal{F}).$$

We can now state the following variant of Lemma 5.15.

Lemma 5.18. *Assume that $\zeta : \mathbf{C}^* \rightarrow \mathbf{R}$ is a group homomorphism such that $\zeta(\exp(r_0)) \neq 0$. Let $\tau = \zeta \exp : \mathbf{C} \rightarrow \mathbf{R}$. Let $\bar{M} \in \text{Irr}_{r_0}^\bullet \bar{H}(G, L, \mathcal{C}, \mathcal{F})$ and let (f, ρ) correspond to \bar{M} under 5.17(d). The following two conditions are equivalent:*

- (i) \bar{M} is τ -tempered;
- (ii) $f \in G^{\text{Ker}\zeta}$.

Using 5.15 we see that it is enough to verify the following statement: for $\xi \in \mathfrak{g}$, we have $\xi \in \mathfrak{g}^{\text{Ker}\tau}$ if and only if $\exp(\xi) \in G^{\text{Ker}\zeta}$. This is immediate. The lemma is proved.

6. THE SUBGROUPS G_J

6.1. We fix an algebraic group \hat{G} such that \hat{G}^0 is simply connected, almost simple. We set $G = \hat{G}^0$. We assume that we are given an element $\vartheta \in \hat{G}$ of finite order d such that $G \times \mathbf{Z}/d\mathbf{Z} \rightarrow \hat{G}$, $(g, j) \mapsto g\vartheta^j$ is a bijection and such that the following holds: there exists a set of Chevalley generators $\{e_{i'}, h_{i'}, f_{i'}; i' \in I'\}$ for $\mathfrak{g} = \underline{\hat{G}} = \underline{G}$ (with standard notation) and a bijection $I' \xrightarrow{\sim} I', i' \mapsto \vartheta i'$ of order d , such that

$$\text{Ad}(\vartheta)(e_{i'}) = e_{\vartheta i'}, \text{Ad}(\vartheta)(h_{i'}) = h_{\vartheta i'}, \text{Ad}(\vartheta)(f_{i'}) = f_{\vartheta i'}$$

for all $i' \in I'$. It follows that \hat{G}/G is a cyclic group of order d generated by the image of ϑ . Let G^1 be the connected component of \hat{G} that contains ϑ . The subspace \mathfrak{t}' of \mathfrak{g} spanned by $\{h_{i'}; i' \in I'\}$ is \underline{T}' for a maximal torus T' of G . Let $\mathcal{Y}' = \text{Hom}(\mathbf{C}^*, T')$, $\mathcal{X}' = \text{Hom}(T', \mathbf{C}^*)$. We have canonically $T' = \mathcal{Y}' \otimes \mathbf{C}^*$, $\mathfrak{t}' = \mathcal{Y}'_{\mathbf{C}}$, $\mathfrak{t}'^* = \mathcal{X}'_{\mathbf{C}}$. Hence we may identify \mathcal{Y}' with a subgroup of \mathfrak{t}' and \mathcal{X}' with a subgroup of \mathfrak{t}'^* . Let $\check{R}' \subset \mathcal{Y}' \subset \mathfrak{t}'$ (resp. $R' \subset \mathcal{X}' \subset \mathfrak{t}'^*$) be the set of coroots (resp. roots) of G with respect to T' . For $\alpha \in R'$ let h_α be the corresponding coroot and let \mathfrak{g}_α be the corresponding root subspace of \mathfrak{g} . For $i' \in I'$ define $\alpha_{i'} \in R'$ by $\mathfrak{g}_{\alpha_{i'}} = \mathbf{C}e_{i'}$. Then $(R', \check{R}', \mathcal{X}', \mathcal{Y}')$ is a root system with basis $\{\alpha_{i'}; i' \in I'\}$. Now ϑ normalizes T' . For

any $j \in [0, d-1]$ we have a bijection $R' \rightarrow R', \alpha \mapsto \vartheta^j \alpha$ given by $\vartheta^j \alpha (\vartheta^j t \vartheta^{-j}) = \alpha(t)$ for $t \in T, \alpha \in R'$, that is, $\text{Ad}(\vartheta^j) \mathfrak{g}_\alpha = \mathfrak{g}_{\vartheta^j \alpha}$ for $\alpha \in R'$.

6.2. Let $G^\vartheta = Z_G(\vartheta), T = Z_{T'}(\vartheta) = T' \cap G^\vartheta, \mathfrak{t} = \underline{T} = \{x \in \mathfrak{t}'; \text{Ad}(\vartheta)x = x\}$. From [St] it is known that:

(a) G^ϑ is connected and T is a maximal torus of G^ϑ . Moreover, $T' = Z_G(T)$; in particular, $N_G(T) \subset N_G(T')$.

Let $W' = N_G(T')/T'$. Conjugation by ϑ induces an isomorphism $W' \xrightarrow{\sim} W'$ whose fixed point set is denoted by W'^ϑ . Let $W = N_{G^\vartheta}(T)/T$. From [St] it is known that

(b) the obvious maps $N_G(T)/T' \leftarrow W \rightarrow W'^\vartheta$ are isomorphisms.

Let $'\mathcal{Y} = \text{Hom}(\mathbf{C}^*, T), '\mathcal{X} = \text{Hom}(T, \mathbf{C}^*)$. For $\beta \in '\mathcal{X}$ we set $\mathfrak{g}_\beta = \{x \in \mathfrak{g}; \text{Ad}(t)x = \beta(t)x \quad \forall t \in T\}$. Then $\mathfrak{g} = \bigoplus_{\beta \in '\mathcal{X}} \mathfrak{g}_\beta$ and $\mathfrak{g}_0 = \mathfrak{t}'$. Let $'R = \{\beta \in '\mathcal{X} - \{0\}; \mathfrak{g}_\beta \neq 0\}$. There is a unique subset $'\check{R}$ of $'\mathcal{Y} - \{0\}$, in bijection $'h_\beta \leftrightarrow \beta$ with $'R$, such that $(\check{R}, '\mathcal{X}, '\mathcal{Y})$ is a (not necessarily reduced) root system whose associated Weyl group is W . We have canonically $T = '\mathcal{Y} \otimes \mathbf{C}^*, \mathfrak{t} = '\mathcal{Y}_{\mathbf{C}}, \mathfrak{t}^* = '\mathcal{X}_{\mathbf{C}}$. Hence we may identify $'\mathcal{Y}$ with a subgroup of \mathfrak{t} and $'\mathcal{X}$ with a subgroup of \mathfrak{t}^* and we may regard $'R \subset \mathfrak{t}^*, '\check{R} \subset \mathfrak{t}$.

Lemma 6.3. Define $\psi: R' \rightarrow \mathfrak{t}^*$ by $\alpha \mapsto \alpha + \vartheta \alpha + \cdots + \vartheta^{d-1} \alpha$. If $\alpha, \alpha' \in R'$ satisfy $\psi(\alpha) = \psi(\alpha')$, then $\alpha' = \vartheta^j \alpha$ for some $j \in [0, d-1]$.

Let R'_0 be a set of representatives for the orbits of bijection $R' \rightarrow R', \alpha \mapsto \vartheta \alpha$. It is enough to show that $\alpha \mapsto \psi(\alpha)$ is an injective map $R'_0 \rightarrow \mathfrak{t}^*$. This can be easily checked in every case (we may assume that $d \geq 2$).

6.4. If $\alpha \in R' \subset \mathfrak{t}^*$, then $\alpha|_{\mathfrak{t}} \in 'R$. We thus obtain a map $R' \rightarrow 'R, \alpha \mapsto \alpha|_{\mathfrak{t}}$ which is constant on the orbits of $\alpha \mapsto \vartheta \alpha$. In fact, using 6.3, we see that this map induces a bijection from the set of orbits of $\alpha \mapsto \vartheta \alpha$ on R' onto $'R$.

For $\beta \in 'R$ let d'_β be the cardinal of the corresponding orbit in R' ; thus $d'_\beta = \dim \mathfrak{g}_\beta$. For $\beta \in 'R$ we set $d''_\beta = 2$ if either $2\beta \in 'R$ or $\frac{1}{2}\beta \in 'R$ and we set $d''_\beta = 1$ if $2\beta \notin 'R, \frac{1}{2}\beta \notin 'R$. We also set $d_\beta = d'_\beta d''_\beta$.

If $\alpha \in R'$ and $\beta = \alpha|_{\mathfrak{t}}$, we have

$$\begin{aligned} 'h_\beta &= h_\alpha \text{ if } d'_\beta = 1 \text{ or if } d''_\beta = 2, \\ 'h_\beta &= h_\alpha + h_{\vartheta \alpha} \text{ if } d'_\beta = 2 \text{ and } d''_\beta = 1, \\ 'h_\beta &= h_\alpha + h_{\vartheta \alpha} + h_{\vartheta^2 \alpha} \text{ if } d_\beta = 3, \\ 'h_\beta &= 2h_\alpha + 2h_{\vartheta \alpha} \text{ if } d_\beta = 4. \end{aligned}$$

Let \bar{I}' be the set of orbits of the bijection $I' \rightarrow I', i' \mapsto \vartheta i'$. For $i \in \bar{I}'$ let $\beta_i = \alpha_{i'}|_{\mathfrak{t}}$ where i' is any element of the orbit i . Then $\{\beta_i; i \in \bar{I}'\}$ is a basis of the root system $(\check{R}, '\mathcal{X}, '\mathcal{Y})$.

Let R be the subset of \mathfrak{t}^* consisting of the vectors $d_\beta \beta$ for various $\beta \in 'R$. For $\gamma \in R$ we set $h_\gamma = \frac{1}{d_\beta} 'h_\beta$ where $\gamma = d_\beta \beta, \beta \in 'R$. Let \check{R} be the subset of \mathfrak{t} consisting of the vectors h_γ for various $\gamma \in R$.

For $i \in \bar{I}'$ let $d'_i = d'_{\beta_i}, d''_i = d''_{\beta_i}, d_i = d'_i d''_i = d_{\beta_i}$ and let $\gamma_i = d_i \beta_i$. Let \mathcal{Y} be the subgroup of \mathfrak{t} generated by $\{h_{\gamma_i}; i \in \bar{I}'\}$. Let \mathcal{X} be the set of all $\xi \in \mathfrak{t}^*$ that take integer values on \mathcal{Y} . Then $R \subset \mathcal{X}, \check{R} \subset \mathcal{Y}$ and $(R, \check{R}, \mathcal{X}, \mathcal{Y})$ is a (reduced) root system with Weyl group W and with basis $\{\gamma_i; i \in \bar{I}'\}$. It is also irreducible. (If $d = 1$ we have $R' = 'R = R$. If $d = 2$ and R' is of type A_{2n-1} , then $'R, R$ are of type C_n, B_n . If $d = 2$ and R' is of type A_{2n} , then $'R, R$ are of type BC_n, C_n . If

$d = 2$ and R' is of type D_n , then $'R, R$ are of type B_{n-1}, C_{n-1} . If $d = 2$ and R' is of type E_6 , then $'R, R$ are of type F_4, F_4 . If $d = 3$ and R' is of type D_4 , then $'R, R$ are of type G_2, G_2 .)

Let $\gamma_0 \in R$ be the negative of the highest root of R relative to $\{\gamma_i; i \in \bar{I}'\}$. Then $\gamma_0 = d_0\beta_0$ for a unique $\beta_0 \in 'R$ such that $2\beta_0 \notin 'R$. Here $d_0 = d_{\beta_0} = d$. Setting

$$I = \bar{I}' \sqcup \{0\},$$

there are unique integers $n_i \in \mathbf{Z}_{>0}$ ($i \in I$) with $n_0 = 1$, such that

$$\sum_{i \in I} n_i \gamma_i = 0.$$

For $i \in I$, we set $h_i = h_{\gamma_i}$.

Let V be a \mathbf{C} -vector space with basis $\{b_i; i \in I\}$. Let V' be the dual vector space with dual basis $\{b'_i; i \in I\}$. The canonical pairing $V \times V' \rightarrow \mathbf{C}$ is denoted by $x, x' \mapsto x(x')$. We imbed \mathfrak{t} into V' by $y \mapsto \sum_{i \in I} \gamma_i(y)b'_i$; we identify \mathfrak{t} with its image, the subspace $\{\sum_i c_i b'_i; c_i \in \mathbf{C}, \sum_i n_i c_i = 0\}$ of V' . In particular, we may regard h_i as a vector in V' with $b_i(h_i) = 2$. We have $b_i(x') = \gamma_i(x')$ for any $x' \in \mathfrak{t}, i \in I$. Let

$$\mathfrak{t}^1 = \left\{ \sum_i c_i b'_i; c_i \in \mathbf{C}, \sum_i n_i c_i = 1 \right\}.$$

For $i \in I$ define $s_i : V \rightarrow V$ by $s_i(x) = x - x(h_i)b_i$ and its contragredient $s_i : V' \rightarrow V'$ by $s_i(x') = x' - b_i(x')h_i$. Let W^a be the subgroup of $GL(V)$ or $GL(V')$ generated by $\{s_i; i \in I\}$ (an affine Weyl group). Note that $\mathfrak{t}, \mathfrak{t}^1$ are W^a -stable subsets of V' . We obtain a homomorphism $W^a \rightarrow GL(\mathfrak{t})$ whose image coincides with W .

6.5. For any $S \subset I, S \neq \emptyset$ let

$$C_S = \{x' \in \mathfrak{t}^1; x' = \sum_{i \in S} c_i b'_i \text{ with } c_i \in \mathbf{C}, c_i > 0 \quad \forall i \in S\}.$$

The sets C_S are disjoint. Let

$$C' = \bigcup_{S \subset I; S \neq \emptyset} C_S.$$

For $J \subset I, J \neq I$, let W_J be the subgroup of W^a generated by $\{s_i; i \in J\}$ (a finite Coxeter group). For S as above and $x' \in C_S$, we have

$$(a) \{w \in W^a; w(x') = x'\} = W_{I-S}.$$

Lemma 6.6. *Let $x' \in \mathfrak{t}^1$. The W^a -orbit $W^a x'$ meets C' in exactly one point.*

Let $V'_{\mathbf{R}} = \sum_{i \in I} \mathbf{R}b'_i, \mathfrak{t}_{\mathbf{R}} = \mathfrak{t} \cap V'_{\mathbf{R}}, \mathfrak{t}_{\mathbf{R}}^1 = \mathfrak{t}^1 \cap V'_{\mathbf{R}}$.

The following \mathbf{R} -analogue of the lemma is well known.

(a) *Let $x'_1 \in \mathfrak{t}_{\mathbf{R}}^1$. The W^a -orbit $W^a x'_1$ meets $C' \cap \mathfrak{t}_{\mathbf{R}}^1$ in exactly one point.*

We can write $x' = x'_1 + \sqrt{-1}x'_2$ where $x'_1 \in \mathfrak{t}_{\mathbf{R}}^1, x'_2 \in \mathfrak{t}_{\mathbf{R}}$. Using (a) we can find $w \in W^a$ and $S \subset I, S \neq \emptyset$ such that $w(x') = x'_3 + \sqrt{-1}x'_4$ where $x'_3 \in C_S \cap \mathfrak{t}_{\mathbf{R}}^1, x'_4 \in \mathfrak{t}_{\mathbf{R}}$. By a well known property of Weyl chambers applied to W_{I-S} , we can find $w' \in W_{I-S}$ such that $w'(x'_4)(b_i) \in \mathbf{R}_{\geq 0}$ for all $i \in I - S$. By 6.5(a) we have $w'(x'_3) = x'_3$. Thus, $w'(w(x'))(b_i)$ is in $\mathbf{R}_{>0} + \sqrt{-1}\mathbf{R}$ if $i \in S$ and is in $\sqrt{-1}\mathbf{R}_{\geq 0}$ if $i \in I - S$. Hence $w'(w(x')) \in C_{S'}$ for some $S', S \subset S' \subset I$. We see that $W^a x' \cap C' \neq \emptyset$.

Now let z, z' be two points of C' that $w(z) = z'$ for some $w \in W^a$. We can write $z = z_1 + \sqrt{-1}z_2, z' = z'_1 + \sqrt{-1}z'_2$ where $z_1, z'_1 \in C' \cap \mathfrak{t}_{\mathbf{R}}^1, z_2, z'_2 \in \mathfrak{t}_{\mathbf{R}}$. Since z_1, z'_1

are in the same W^a -orbit, we see using (a) that $z_1 = z'_1$. We can find $S \subset I, S \neq \emptyset$ such that $z_1 = z'_1 \in C_S \cap \mathfrak{t}_{\mathbf{R}}^1$. Since $z, z' \in C_S$, we have $z_2(b_i) \geq 0, z'_2(b_i) \geq 0$ for all $i \in I - S$. Moreover, we have $w(z_1) = z'_1 = z_1$ (hence $w \in W_{I-S}$, see 6.5(a)) and $w(z_2) = z'_2$. Using a well known property of Weyl chambers applied to W_{I-S} , we deduce that $z_2 = z'_2$. Thus, $z = z'$. The lemma is proved.

6.7. Let $\mathcal{N} = \{g \in G; gT\vartheta g^{-1} = T\vartheta\}$, $\mathcal{N}' = \mathcal{N} \cap T'$. Clearly, $N_{G^\vartheta}(T) \cap \mathcal{N}' = T$.

Lemma 6.8. $\mathcal{N} = N_{G^\vartheta}(T)\mathcal{N}' = \mathcal{N}'N_{G^\vartheta}(T)$.

If $g \in \mathcal{N}$, then g normalizes the subgroup generated by $T\vartheta$, hence it also normalizes the identity component T of that subgroup. Since $T' = Z_G(T)$, it follows that g normalizes T' . Thus, $\mathcal{N} \subset N_G(T')$. It also follows that \mathcal{N}' is normal in \mathcal{N} , hence $N_{G^\vartheta}(T)\mathcal{N}' = \mathcal{N}'N_{G^\vartheta}(T)$.

Let $g \in \mathcal{N}$. We set $a = g^{-1}\vartheta g\vartheta^{-1}, a' = \vartheta^2 g^{-1}\vartheta^{-1}g\vartheta^{-1}$. For $t \in T$ we have $\vartheta g t \vartheta g^{-1} = g t \vartheta g^{-1}\vartheta$, hence $ata' = t$. Taking $t = 1$, we get $aa' = 1$. Hence $ata^{-1} = t$ for all $t \in T$. Thus, $a \in Z_G(T) = T'$. Thus,

(a) $\vartheta g \vartheta^{-1} = ga$ for some $a \in T'$.

Let \bar{g} be the image of g in $W' = N_G(T')/T'$. Then $\bar{g} \in W'^{\vartheta}$ (see (a) and 6.2). Since $W \xrightarrow{\sim} W'^{\vartheta}$ (see 6.2(b)), there exists $g' \in N_{G^\vartheta}(T), t' \in T'$ such that $g = g't'$. We have $t' \in T' \cap \mathcal{N}$, hence $t' \in \mathcal{N}'$. Thus, $\mathcal{N} = N_{G^\vartheta}(T)\mathcal{N}'$. The lemma is proved.

Lemma 6.9. We have $'\mathcal{Y} \subset \mathcal{Y}$ and $\mathcal{Y}/'\mathcal{Y}$ may be identified with \mathcal{N}'/T .

Let $F = \prod_{i \in \bar{I}'} \mathbf{Z}/d'_i \mathbf{Z}$. Let F' be the subgroup of $(\mathbf{C}^*)^{\bar{I}'}$ consisting of all (a_i) such that $a_i^{d'_i} = 1$ for all $i \in \bar{I}'$. Define $F \xrightarrow{\sim} F'$ by $(l_i) \mapsto (\exp(\kappa l_i/d'_i))$.

From the definitions we see that $'\mathcal{Y}$ has a \mathbf{Z} -basis $\{\frac{1}{d'_i} h_{\beta_i}; i \in \bar{I}'\}$ and \mathcal{Y} has a \mathbf{Z} -basis $\{h_i; i \in \bar{I}'\}$. Recall that $h_i = \frac{1}{d_i} h_{\beta_i}$. It follows that $'\mathcal{Y} \subset \mathcal{Y}$ and we have

$$F \xrightarrow{\sim} \mathcal{Y}/'\mathcal{Y}, (l_i) \mapsto '\mathcal{Y}\text{-coset of } \sum_{i \in \bar{I}'} l_i h_i.$$

By definition, $\mathcal{N}' = \{t \in T'; t\vartheta t^{-1}\vartheta^{-1} \in T\}$. The homomorphism $\chi : \mathcal{N}' \rightarrow T, t \mapsto t\vartheta t^{-1}\vartheta^{-1}$ with kernel T induces an isomorphism $\mathcal{N}'/T \xrightarrow{\sim} \text{Im}(\chi)$. Define $(\mathbf{C}^*)^{\bar{I}'} \xrightarrow{\sim} T$ by $(a_i) \mapsto \prod_{i \in \bar{I}'} h_{\beta_i}(a_i)$. Via this isomorphism, $\text{Im}(\chi)$ corresponds to F' . Combining the isomorphisms above yields the lemma.

6.10. Let p be the composition $\mathfrak{t}^1 \rightarrow \mathfrak{t} \rightarrow T\vartheta$ where the first map is $x \mapsto x - b'_0$ and the second map is $x \mapsto \exp_T(\kappa x)\vartheta$.

Lemma 6.11. The map $x' \mapsto p(x')$ defines a bijection between C' and a set of representatives for the orbits of the \mathcal{N} -action on $T\vartheta$ (by conjugation).

Now $x \mapsto \exp_T(\kappa x)\vartheta$ induces $\mathfrak{t}/'\mathcal{Y} \xrightarrow{\sim} T\vartheta$. Using 6.8, 6.9, we see that via this isomorphism the action of \mathcal{N} on $T\vartheta$ corresponds to the action of the obvious semidirect product of W and $\mathcal{Y}/'\mathcal{Y}$ on $\mathfrak{t}/'\mathcal{Y}$ (with $\mathcal{Y}/'\mathcal{Y}$ normal) where the action of W is the obvious one and the action of $\mathcal{Y}/'\mathcal{Y}$ is by translation. It follows that we have an induced bijection

(a) $\{W\text{-orbits on } \mathfrak{t}/'\mathcal{Y}\} \leftrightarrow \{\mathcal{N}\text{-orbits on } T\vartheta\}$.

It is well known that one may regard \mathcal{Y} as a normal subgroup of W^a in such a way that an element $y \in \mathcal{Y}$ acts on \mathfrak{t}^1 (as part of the W^a -action) in the same way as y acts on \mathfrak{t}^1 by $x' \mapsto x' + y$. Hence we have an obvious bijection

(b) $\{W^a - \text{orbits on } \mathfrak{t}^1\} \leftrightarrow \{W^a/\mathcal{Y} - \text{orbits on } \mathfrak{t}^1/\mathcal{Y}\}.$

Now \mathcal{Y} also acts on \mathfrak{t} by translation and $x' \mapsto x' - b'_0$ induces a bijection $\mathfrak{t}^1/\mathcal{Y} \leftrightarrow \mathfrak{t}/\mathcal{Y}$ which is compatible with the action of W^a/\mathcal{Y} on $\mathfrak{t}^1/\mathcal{Y}, \mathfrak{t}/\mathcal{Y}$. (This is because for $w \in W^a$ we have $w(b'_0) - b'_0 \in \mathcal{Y}$.) Thus we have an obvious bijection

(c) $\{W^a/\mathcal{Y} - \text{orbits on } \mathfrak{t}^1/\mathcal{Y}\} \leftrightarrow \{W^a/\mathcal{Y} - \text{orbits on } \mathfrak{t}/\mathcal{Y}\}.$

By the last sentence in 6.4 we have

(d) $\{W^a/\mathcal{Y} - \text{orbits on } \mathfrak{t}/\mathcal{Y}\} = \{W - \text{orbits on } \mathfrak{t}/\mathcal{Y}\}.$

Combining (a)-(d), we obtain a bijection

$$\{W^a - \text{orbits on } \mathfrak{t}^1\} \leftrightarrow \{\mathcal{N} - \text{orbits on } T\vartheta\}.$$

This is induced by $x' \mapsto p(x')$. We now use 6.6. The lemma follows.

Lemma 6.12. *Let Z be a semisimple G -conjugacy class in G^1 . Then $Z \cap (T\vartheta)$ is exactly one \mathcal{N} -orbit in $T\vartheta$.*

This is classical when $d = 1$. This is also known when $d > 1$. It can be deduced for example from [Se] (this reference deals with compact groups but our case can be treated in a similar way).

Combining 6.11, 6.12, we have the following result.

Proposition 6.13. *The map $x' \mapsto p(x')$ defines a bijection between C' and a set of representatives for the G -conjugacy classes of semisimple elements in G^1 .*

Lemma 6.14. *Let $\gamma \in R$ and $n \in \mathbf{Z}$. Let $H = \{y' \in \mathfrak{t}^1; \gamma(y' - b'_0) = n\}$. Let $S \subset I, S \neq \emptyset$ and let $x' \in C_S$. If $x' \in H$, then $\frac{1}{n_k}b'_k \in H$ for any $k \in S$.*

Let

$$V'_{\mathbf{R}} = \sum_{i \in I} \mathbf{R}b'_i, \mathfrak{t}_R = \mathfrak{t} \cap V'_{\mathbf{R}}, \mathfrak{t}_R^1 = \mathfrak{t}^1 \cap V'_{\mathbf{R}},$$

$$H_{\mathbf{R}} = H \cap \mathfrak{t}_R^1, H' = \{y' \in \mathfrak{t}_{\mathbf{R}}; \gamma(y') = 0\}.$$

The following \mathbf{R} -analogue of the lemma is well known.

(a) *Let $x'_1 \in C_S \cap \mathfrak{t}_{\mathbf{R}}^1$. If $x'_1 \in H_{\mathbf{R}}$, then $\frac{1}{n_k}b'_k \in H_{\mathbf{R}}$ for any $k \in S$.*

(This follows from the fact that $C_S \cap \mathfrak{t}_{\mathbf{R}}^1$ is a facet of a configuration of reflection hyperplanes in $\mathfrak{t}_{\mathbf{R}}^1$ (one of which is $H_{\mathbf{R}}$) and that $\frac{1}{n_k}b'_k (k \in S)$ are the vertices of that facet.)

We now write $x' = x'_1 + \sqrt{-1}x'_2$ where $x'_1 \in H_{\mathbf{R}}, x'_2 \in H'$. We can find $S' \subset S$ such that

$$x'_1 = \sum_{i \in S'} c_{1,i}b'_i, \quad x'_2 = \sum_{i \in S} c_{2,i}b'_i,$$

where

(b) $c_{1,i} \in \mathbf{R}_{>0}$ for $i \in S', c_{2,i} \in \mathbf{R}$ for $i \in S, c_{2,i} \in \mathbf{R}_{>0}$ for $i \in S - S'$.

Using (a) for x'_1 we see that

(c) $\frac{1}{n_k}b'_k \in H_{\mathbf{R}}$ for any $k \in S'$.

If $S' = S$, we are done. Assume now that $S - S' \neq \emptyset$. From (c) we see that, for $k \in S'$, we have $\gamma(b'_k - n_k b'_0) = nn_k$. Hence

$$\gamma\left(\sum_{i \in S'} c_{2,i}b'_i - \sum_{i \in S'} c_{2,i}n_i b'_0\right) = \sum_{i \in S'} c_{2,i}nn_i.$$

Since $x'_2 \in H'$, we have $\sum_{i \in S} c_{2,i} n_i = 0$ and $\gamma(\sum_{i \in S} c_{2,i} b'_i) = 0$. It follows that

$$(d) \quad \begin{aligned} & \gamma\left(\sum_{i \in S'} c_{2,i} b'_i - \sum_{i \in S'} c_{2,i} n_i b'_0\right) \\ &= \gamma\left(-\sum_{i \in S-S'} c_{2,i} b'_i + \sum_{i \in S-S'} c_{2,i} n_i b'_0\right) = -\sum_{i \in S-S'} c_{2,i} n_i. \end{aligned}$$

Set $c = \sum_{i \in S-S'} c_{2,i} n_i$. Since $S - S' \neq \emptyset$ we see using (b) that $c \in \mathbf{R}_{>0}$. From (d) we deduce

$$\gamma\left(\sum_{i \in S-S'} c_{2,i} c^{-1} b'_i - b'_0\right) = n$$

so that $\sum_{i \in S-S'} c_{2,i} c^{-1} b'_i \in H_{\mathbf{R}} \cap C_{S-S'}$. Using again (a) we deduce that $\frac{1}{n_k} b'_k \in H_{\mathbf{R}}$ for any $k \in S - S'$. Combining this and (c) we see that the lemma is proved.

6.15. Let \mathfrak{N} be the set of all pairs (β, j) where $\beta \in {}'R, j \in [0, d-1]$ and $j = 0$ if $d'_\beta = 1, d''_\beta = 1, j = 1$ if $d'_\beta = 1, d''_\beta = 2$.

We regard \mathfrak{N} as a subset of the group $\mathfrak{t}^* \times \mathbf{Z}/d\mathbf{Z}$ by identifying $[0, d-1]$ and $\mathbf{Z}/d\mathbf{Z}$ in the obvious way.

For $i \in I$ define p_i by $p_i = 0$ for $i \in I - \{0\}$, $p_0 = 1$.

Lemma 6.16. *Let $S \subset I, S \neq \emptyset$ and let $x' \in C_S$. The following two conditions for $(\beta, j) \in \mathfrak{N}$ are equivalent:*

- (i) $\beta(x' - b'_0) + \frac{j}{d} \in \mathbf{Z}$;
- (ii) $(\beta, j) = \sum_{i \in I-S} c_i(\beta_i, p_i)$ with $c_i \in \mathbf{Z}$.

Assume that (β, j) satisfies (ii). To show that it satisfies (i), we may assume that $(\beta, j) = (\beta_i, p_i)$ for some $i \in I - S$. Then we have $\beta_i(x' - b'_0) = d_i^{-1} b_i(x' - b'_0)$ and this is 0 if $i \neq 0$ and is $-d_0^{-1} = -d^{-1}$ if $i = 0$. If $i \neq 0$, we have $p_i = 0$, hence $\beta_i(x' - b'_0) + \frac{p_i}{d} = 0$. If $i = 0$, we have $p_i = 1$, hence $\beta_i(x' - b'_0) + \frac{p_i}{d} = -\frac{1}{d} + \frac{1}{d} = 0$, so that (i) holds.

Conversely, assume that (β, j) satisfies (i). Thus, we have $\beta(x' - b'_0) + \frac{j}{d} = n$ for some $n \in \mathbf{Z}$. We can write uniquely $\beta = \sum_{k \in I-S} f_k \beta_k$ where $f_k \in \mathbf{Z}$.

Let H be the affine hyperplane $\{y' \in \mathfrak{t}^1; \beta(y' - b'_0) + \frac{j}{d} = n\}$ in \mathfrak{t}^1 . Note that H is of the form $\{y' \in \mathfrak{t}^1; \gamma(y' - b'_0) = n'\}$ for some $\gamma \in R$ and $n' \in \mathbf{Z}$. Indeed, if $\beta \in {}'R, d'_\beta = 1$, we can take $\gamma = d_\beta \beta, n' = d_\beta n - j$; if $\beta \in {}'R, 2\beta \in {}'R$, we can take $\gamma = 4\beta, n' = 4n - 2j$; if $\beta \in {}'R, \frac{1}{2}\beta \in {}'R$, we can take $\gamma = 2\beta, n' = 2n - j$. Since $x' \in H$ and $x' \in C_S$, we have $\frac{1}{n_k} b'_k \in H$ for any $k \in S$. (See 6.14.) Thus, $\beta(\frac{1}{n_k} b'_k - b'_0) + \frac{j}{d} = n$ for any $k \in S$.

If $k \in S, k \neq 0$, we have $\beta(\frac{1}{n_k} b'_k - b'_0) = \frac{f_k}{d_k n_k}$. Thus, $\frac{f_k}{d_k n_k} + \frac{j}{d} = n$. Since $\frac{j d_k}{d} \in \mathbf{Z}$, it follows that $f_k = n_k g_k$ where $g_k \in \mathbf{Z}, g_k + \frac{j d_k}{d} = d_k n$.

If $0 \in S$, we have $\frac{j}{d} = \beta(0) + \frac{j}{d} = n$ so that $j = 0$ and $n = 0$. In this case we deduce that for $k \in S, k \neq 0$ we have $g_k = 0$, hence $f_k = 0$ so that $\beta = \sum_{k \in I-S} f_k \beta_k$. Moreover, $(\beta, j) = \sum_{k \in I-S} f_k(\beta_k, p_k)$ since $j = 0$.

Assume now that $0 \in I - S$. Then

$$\begin{aligned} \beta &= \sum_{k \in I-S-\{0\}} f_k \beta_k + \sum_{k \in S} n_k (d_k n - \frac{j d_k}{d}) \beta_k \\ &= \sum_{k \in I-S-\{0\}} f_k \beta_k + \sum_{j \in I-0} n_k (d_k n - \frac{j d_k}{d}) \beta_k - \sum_{k \in I-S-\{0\}} n_k (d_k n - \frac{j d_k}{d}) \beta_k \\ &= \sum_{k \in I-S-\{0\}} (f_k - n_k (d_k n - \frac{j d_k}{d})) \beta_k - (nd - j) \beta_0, \\ (\beta, j) &= \sum_{k \in I-S-\{0\}} (f_k - n_k (d_k n - \frac{j d_k}{d})) (\beta_k, p_k) - (nd - j) (\beta_0, p_0) \end{aligned}$$

since $j = -(nd - j) \pmod{d\mathbf{Z}}$. The lemma is proved.

6.17. Let $\beta \in {}^R U \setminus \{0\}$. Then \mathfrak{g}_β is stable under $\text{Ad}(\vartheta) : \mathfrak{g} \rightarrow \mathfrak{g}$. For any $j \in [0, d-1]$ we set $\mathfrak{g}_{\beta,j} = \{x \in \mathfrak{g}_\beta; \text{Ad}(\vartheta)x = \exp(\kappa j/d)x\}$. Clearly, $\mathfrak{g}_\beta = \bigoplus_{j \in [0, d-1]} \mathfrak{g}_{\beta,j}$.

Lemma 6.18. *Let $\beta \in {}^R R$ and let $j \in [0, d-1]$. We have $\dim \mathfrak{g}_{\beta,j} = 1$ if $(\beta, j) \in \mathfrak{N}$ and $\mathfrak{g}_{\beta,j} = 0$ if $(\beta, j) \notin \mathfrak{N}$.*

We can assume that $d > 1$. Assume first that $\dim \mathfrak{g}_\beta = d$. Since \mathfrak{g}_β is the direct sum of d (one-dimensional) root spaces of \mathfrak{g} with respect to T' which are cyclically permuted by $\text{Ad}(\vartheta)$, it follows that $\text{Ad}(\vartheta) : \mathfrak{g}_\beta \rightarrow \mathfrak{g}_\beta$ has order d and its n th power has trace 0 for $1 \leq n < d$. It follows that its $\exp(\kappa j/d)$ -eigenspace is one-dimensional for $0 \leq j < d$. Thus the lemma is proved in this case.

Next assume that $\dim \mathfrak{g}_\beta = 1$. Then $\beta = \alpha|_{\mathfrak{t}}$ for a unique $\alpha \in R'$. We may assume that $\alpha \in \sum_{i' \in I'} \mathbf{N} \alpha_{i'}$. We can find a unique partition $I' = I'_1 \sqcup I'_2$ such that whenever $i'_1 \in I'_1, i'_2 \in I'_2$, the vertices i'_1, i'_2 of the Coxeter graph of G are not joined. Let $x_0 \in \mathfrak{g}_{\gamma_0} - \{0\}$. Let B be the canonical basis of \mathfrak{g} (as a left \mathfrak{g} -module) such that $x_0 \in B$. Then $c_{i'} e_{i'} \in B$ for well defined $c_{i'} \in \mathbf{C}^*$. Moreover, it is not difficult to check that there exist $a_1, a_2 \in \mathbf{C}^*$ such that $c_{i'} = a_1$ for all $i' \in I'_1$, $c_{i'} = a_2$ for all $i' \in I'_2$ and $a_1 + a_2 = 0$. Now $\text{Ad}(\vartheta)(B) = uB$ for some $u \in \mathbf{C}^*$. Hence $c_{i'} e_{\vartheta i'} = u c_{\vartheta i'} e_{\vartheta i'}$ for all i' . We see that $c_{i'} = u c_{\vartheta i'}$. We consider two cases.

Case 1. Both I'_1, I'_2 are stable under $i' \mapsto \vartheta i'$. Then $a_1 = u a_1$ and $u = 1$. It follows that, if $x \in B \cap \mathfrak{g}_\beta$, then $\text{Ad}(\vartheta)x = ux = x$. Thus, if $\mathfrak{g}_{\beta,j} \neq 0$, then $j = 0$.

Case 2. I'_1, I'_2 are interchanged by $i' \mapsto \vartheta i'$ (hence $d = 2$). Then $u a_1 = a_2 = -a_1$ and $u = -1$. It follows that, if $x \in B \cap \mathfrak{g}_\beta$, then $\text{Ad}(\vartheta)x = ux = -x$. Thus, if $\mathfrak{g}_{\beta,j} \neq 0$, then $j = 1$.

The lemma follows.

6.19. Let $J \subset I, J \neq I$. Let

$$\mathfrak{g}_J = \mathfrak{t} \oplus \bigoplus_{\beta, j} \mathfrak{g}_{\beta, j}$$

where $(\beta, j) \in \mathfrak{N}$ is subject to the condition $(\beta, j) \in \sum_{i \in J} \mathbf{Z}(\beta_i, p_i)$.

Lemma 6.20. *There is a unique closed connected reductive subgroup G_J of G with Lie algebra \mathfrak{g}_J . If $x' \in C_{I-J}$, then $G_J = Z_G(p(x'))$.*

Recall that $p(x') = \exp_T(\kappa(x' - b'_0))\vartheta \in T\vartheta$. Now $Z_G(p(x'))$ is a closed connected reductive subgroup of G whose Lie algebra is $\mathfrak{h} = \{x \in \mathfrak{g}; \text{Ad}(p(x'))x = x\}$. Clearly, \mathfrak{h} is stable under the Ad action of T and that of ϑ on \mathfrak{g} . Hence \mathfrak{h} is the sum of its

intersections with the various $\mathfrak{g}_{\beta,j}$ where $\beta \in {}'R \cup \{0\}$. Clearly, $\mathfrak{h} \cap \mathfrak{g}_{0,j}$ is 0 if $j \neq 0$ and is \mathfrak{t} if $j = 0$. Using 6.18 we deduce that

$$\mathfrak{h} = \mathfrak{t} \oplus \bigoplus_{\beta,j} \mathfrak{g}_{\beta,j}$$

where $(\beta, j) \in \mathfrak{N}$ is subject to $\beta(\exp_T(\kappa(x' - b'_0))) \exp(\kappa j/d) = 1$ (with β regarded as a character $T \rightarrow \mathbf{C}^*$) or equivalently, to $\beta(x' - b'_0) + \frac{j}{d} \in \mathbf{Z}$ (with β regarded as a form $\mathfrak{t} \rightarrow \mathbf{C}$). Now using 6.16 we see that $\mathfrak{h} = \mathfrak{g}_J$. The lemma is proved.

6.21. For $i_1, i_2 \in I$ let

$$a_{i_1, i_2} = \gamma_{i_2}(h_{i_1}), {}'a_{i_1, i_2} = \beta_{i_2}({}'h_{\beta_{i_1}}) = \frac{d_{i_1}}{d_{i_2}} a_{i_1, i_2}.$$

Then (a_{i_1, i_2}) is an untwisted affine Cartan matrix and $({}'a_{i_1, i_2})$ is a possibly twisted affine Cartan matrix.

Let $J \subset I, J \neq I$. Let $'R_J$ the set of all $\beta \in {}'R$ such that $\beta = w\beta_i$ for some $i \in J$ and some $w \in W_J$. Let $'\check{R}_J$ be the set of all $'h_\beta$ where $\beta \in {}'R_J$.

Lemma 6.22. *'R_J (resp. '\check{R}_J) is exactly the set of roots (resp. coroots) of G_J with respect to T.*

For $i \in J$ we have $(\beta_i, p_i) \in \mathfrak{N}$, hence by 6.18 and 6.19, β_i is a root of G_J . For $i \in J$ there exists $g \in N_{G_J}(T)$ such that $\text{Ad}(g) : \mathfrak{t} \rightarrow \mathfrak{t}$ is a reflection that takes β_i to $-\beta_i$. By 6.2(b), there exists $g' \in N_{G^\vartheta}(T)$ and $t' \in T'$ such that $g = g't'$. Hence $\text{Ad}(g) : T \rightarrow T$ coincides with $\text{Ad}(g') : T \rightarrow T$. Now there is a unique element in W that acts on \mathfrak{t} as a reflection taking β_i to $-\beta_i$, namely s_i . It follows that $\text{Ad}(g) = s_i : \mathfrak{t} \rightarrow \mathfrak{t}$. Hence if $H_i \in \mathfrak{t}$ is the coroot of G_J corresponding to β_i , we have $x - \beta_i(x)H_i = x - \beta_i(x){}'h_{\beta_i}$ for all $x \in \mathfrak{t}$; hence

$$(a) \ H_i = {}'h_{\beta_i}.$$

By 6.19, any root of G_J is of the form β where $\beta \in {}'R$ satisfies $\beta \in \sum_{i \in J} \mathbf{Z}\beta_i$. Thus, $(\beta_i)_{i \in J}$ is a set of simple roots for G_J . By the first part of the argument, W_J coincides with the Weyl group of G_J (both are subgroups of $\text{Aut}(T)$); it follows that R_J is exactly the set of roots of G_J . The claim that $'\check{R}_J$ is exactly the set of coroots of G_J follows from (a). The lemma is proved.

6.23. From 6.22 we see that the Cartan matrix of G_J is $({}'a_{i_1, i_2})_{i_1, i_2 \in J}$.

6.24. Let $\hat{G}_J = \bigcup_{j \in [0, d-1]} G_J \vartheta^j$. Since ϑ normalizes G_J , \hat{G}_J is a (closed) subgroup of \hat{G} , with identity component G_J . Let Z_J (resp. \hat{Z}_J) be the center of G_J (resp. of \hat{G}_J). Let $\mathfrak{z}_J = \underline{Z}_J$ (a subspace of \mathfrak{t} , hence a subspace of V').

Let V_J be the subspace of V spanned by $\{b_i; i \in J\}$. Let $K = I - J$. Let V'_K be the subspace of V' spanned by $\{b'_i; i \in K\}$, that is, the annihilator of V_J in V' . We have $\mathfrak{z}_J = \mathfrak{t} \cap V'_K$ (a hyperplane in \mathfrak{t}). Let $\mathfrak{z}_J^1 = \mathfrak{t}^1 \cap V'_K$ (an affine hyperplane in V'_K).

Let $X_J = p(\mathfrak{z}_J^1)$. Note that $C_K \subset \mathfrak{z}_J^1$, hence $p(C_K) \subset X_J$.

Lemma 6.25. (a) *We have $Z_J \subset \hat{Z}_J$ and $Z_J^0 = \hat{Z}_J^0$.*

(b) *X_J is a connected component of \hat{Z}_J .*

We prove (a). If $g \in Z_J$, then $g \in T$, hence $\vartheta g = g\vartheta$, hence $g \in \hat{Z}_J$. Thus, $Z_J \subset \hat{Z}_J$. It follows that $Z_J^0 \subset \hat{Z}_J^0$. Now $\hat{Z}_J^0 \subset G_J$, hence $\hat{Z}_J^0 \subset Z_J$. Thus, $\hat{Z}_J^0 = G_J$.

We prove (b). Let $x' \in \mathfrak{z}_J$. We set $x^0 = x' - b'_0 \in \mathfrak{z}_J$. Let $i \in J$. Since $x^0 \in V'_K$, we have $b_i(x^0) = 0$. Now $p(x')$ acts on $\mathfrak{g}_{\beta_i, p_i}$ by the scalar

$$\begin{aligned} \beta_i(\exp_T(\kappa x^0)) \exp(\kappa d^{-1} p_i) &= \exp(\kappa \beta_i(x^0)) \exp(\kappa d^{-1} p_i) \\ &= \exp(\kappa d_i^{-1} \gamma_i(x^0) + \kappa d^{-1} p_i) = \exp(\kappa d_i^{-1} b_i(x^0) + \kappa d^{-1} p_i) \\ &= \exp(-\kappa d_i^{-1} b_i(b'_0) + \kappa d^{-1} p_i) = \exp(-\kappa d_i^{-1} \delta_{0,i} + \kappa d^{-1} p_i) = \exp(0) = 1. \end{aligned}$$

It follows that $p(x')$ centralizes G_J . Since $p(x') \in T\vartheta$, it centralizes ϑ , hence also \hat{G}_J . Thus, $X_J \subset \hat{Z}_J$. Clearly, X_J is connected. If $z \in Z_J^0$, we have $z = \exp_T(\kappa(x'_0))$ for some $x'_0 \in \mathfrak{z}_J$, hence for x' as above, $p(x' + x'_0) = p(x')z$. Thus, X_J is stable by multiplication by Z_J^0 . The lemma is proved.

6.26. Let G_1 be the simply connected almost simple algebraic group corresponding to the root system $(R, \tilde{R}, \mathcal{X}, \mathcal{Y})$ (see 6.4). By 6.13 we have a natural bijection between the set of G_1 -conjugacy classes of semisimple elements in G_1 and the set of G -conjugacy classes of semisimple elements in $G\vartheta$.

Our discussion of semisimple G -conjugacy classes in $G\vartheta$ has been influenced by [K] where a connection between the elements of finite order in $G\vartheta$ and (possibly twisted) affine Lie algebras is given.

7. THE SET $\mathfrak{R}(G\vartheta, G_J, \mathcal{C}, \mathcal{F})$

7.1. We preserve the setup of §6. Let $J \subset I, J \neq I$ and let $K = I - J$. Let $G_{(J)}$ be the centralizer of Z_J^0 in G . Now $G_{(J)}$ is the subgroup of G generated by T' and by the root subgroups of G corresponding to various $\alpha \in R'$ such that $\alpha|_{\mathfrak{t}} \in \sum_{i \in J} \mathbf{Q}\beta_i$. We have $Z_{G_{(J)}}^0 \cap G^\vartheta = Z_J^0$.

Lemma 7.2. *Let $g \in Z_J$. Then we have $g = g_1 g_2 \vartheta^{-n}$ (in \hat{G}) where $g_2 \in Z_G \cap G^\vartheta$, n is an integer and g_1 is either the n th power of an element in X_J (if $n \neq 0$) or is an element of Z_J^0 (if $n = 0$).*

We have $g \in T$ and $\beta_i(g) = 1$ for all $i \in J$. Hence $g = \exp_T(\kappa x)$ where $x \in \mathfrak{t}$ satisfies $\beta_i(x) \in \mathbf{Z}$ for all $i \in J$; hence $x' = \sum_{i \in I} c_i b'_i$ with $c_i \in d_i \mathbf{Z}$ for all $i \in J$, $c_i \in \mathbf{C}$ for all $i \in K$. We set $n = \sum_{k \in K} n_k c_k$. We have $n \in \mathbf{Z}$ since $n = -\sum_{i \in J} n_i c_i$. Let

$$x' = \sum_{k \in K} c_k b'_k - n b'_0 \in \mathfrak{t}, \quad x'' = \sum_{i \in J} c_i b'_i + n b'_0 \in \mathfrak{t}.$$

Then $x = x' + x''$. For any $i \in I - \{0\}$ we have $\beta_i(x'') = d_i^{-1} b_i(\sum_{i_1 \in J} c_{i_1} b'_{i_1} + n b'_0)$.

If $0 \in J$, then $\beta_i(x'')$ equals $d_i^{-1} c_i$ if $i \in J - \{0\}$ and equals 0 if $i \in K$. If $0 \in K$, then $\beta_i(x'')$ equals $d_i^{-1} c_i$ if $i \in J$ and equals 0 if $i \in K - \{0\}$. In any case, $\beta_i(x'') \in \mathbf{Z}$ for $i \in I - \{0\}$. It follows that $\exp_T(\kappa x'')$ is in the kernel of $\beta_i : T \rightarrow \mathbf{C}^*$ for any $i \in I - \{0\}$. Hence $g'' = \exp_T(\kappa x'') \in Z_G \cap G^\vartheta$.

Assume first that $0 \in J$. If $n = 0$, we have $x' \in \mathfrak{z}_J$, hence $g' = \exp_T(\kappa x') \in Z_J^0$ and $g = g' g''$. If $n \neq 0$, we set $g' = \exp_T(\kappa \frac{1}{n} x') \vartheta$. Then $g' \in X_J$ and $g = g'^n g'' \vartheta^{-n}$.

Assume next that $0 \in K$. In this case we have $x' \in \mathfrak{z}_J$, hence $g' = \exp_T(\kappa x') \in Z_J^0$ and $g = g' g''$. The lemma is proved.

Lemma 7.3. *If $g \in X_J$, then $Z_{G_{(J)}}(g) = Z_J$.*

Let $n = \sum_{k \in K} n_k$. We have $n \in \mathbf{Z}_{>0}$. If $x' = n^{-1} \sum_{k \in K} b'_k$, then $x' \in \mathfrak{t}^1$ and $z = p(x') \in X_J$. Since $x' \in C_{I-J}$, we have $Z_G(z) = G_J$ (see 6.20). Let $g \in X_J$. We have $g = zt$ where $t \in Z_J^0 = Z_{G(J)}^0$. Hence $Z_{G(J)}(g) = Z_{G(J)}(z) = Z_G(z) \cap G_J = G_J \cap G_{(J)} = G_J$. The lemma is proved.

7.4. We fix J, K as above, a nilpotent G_J -orbit \mathcal{C} in \mathfrak{g}_J and an irreducible G_J -equivariant cuspidal local system \mathcal{F} (over \mathbf{C}) on \mathcal{C} . The local system on $\dot{\mathcal{C}} = \exp(\mathcal{C})$ (a unipotent class in G_J) that corresponds to \mathcal{F} under $\exp : \mathfrak{g}_J \rightarrow G_J$ is denoted again by \mathcal{F} . Let

$$X_{(J)} = \bigcup_{g_1 \in G_{(J)}} g_1 X_J \dot{\mathcal{C}} g_1^{-1},$$

(a locally closed subset of $G\vartheta$, stable under conjugacy by $G_{(J)}$).

(a) *Let C be an $\text{Ad}(G_{(J)})$ -orbit in $G\vartheta$ that is contained in $X_{(J)}$. There exists an irreducible $G_{(J)}$ -equivariant local system \mathcal{F}' on C (unique up to isomorphism) such that the following holds: for any $x \in X_J$ such that $x\dot{\mathcal{C}} \subset C$, the restriction $\mathcal{F}'|_{x\dot{\mathcal{C}}}$ is the local system obtained from \mathcal{F} via $\dot{\mathcal{C}} \xrightarrow{\sim} x\dot{\mathcal{C}}$ (multiplication by x).*

This is shown as follows. Let $x \in X_J$ be such that $x\dot{\mathcal{C}} \subset C$. There exists an irreducible $G_{(J)}$ -equivariant local system $\mathcal{F}(x)$ on C (unique up to isomorphism) such that $\mathcal{F}(x)|_{x\dot{\mathcal{C}}}$ is the local system obtained from \mathcal{F} via $\dot{\mathcal{C}} \xrightarrow{\sim} x\dot{\mathcal{C}}$ (multiplication by x). (We use the fact that, if $u \in \dot{\mathcal{C}}$, then $Z_{G(J)}(xu) = Z_{G(J)}(u)$, see 7.3). We must only show that the isomorphism class of $\mathcal{F}(x)$ is independent of x . If $\text{card} K = 1$, then X_J is a point and there is nothing to prove. Thus we may assume that $\text{card} K \geq 2$.

Let $x' \in X_J$ be a second element such that $x'\dot{\mathcal{C}} \subset C$. We must show that $\mathcal{F}(x), \mathcal{F}(x')$ are isomorphic. It is enough to show that, if $f \in G_{(J)}$ is such that $fxf^{-1} = x'$ (so that $\text{Ad}(f)Z_J = Z_J$, as we see from 7.3), then $\text{Ad}(f)$ carries $(\dot{\mathcal{C}}, \mathcal{F})$ to $(\dot{\mathcal{C}}, \mathcal{F})$. If $d \geq 2$, then $(\dot{\mathcal{C}}, \mathcal{F})$ is uniquely determined by G_J for $\text{card} K \geq 2$ (see the tables in §11) and we are done. Assume now that $d = 1$. Now Z_J acts on \mathcal{F} through a character $\chi : Z_J \rightarrow \mathbf{C}^*$. Since there is at most one pair consisting of a unipotent class of G_J and an irreducible G_J -equivariant cuspidal local system on it with prescribed action of Z_J , it is enough to show that $\chi(z) = \chi(fzf^{-1})$ for any $z \in Z_J$. Since χ is trivial on Z_J^0 it is enough to show that $fzf^{-1} = z \pmod{Z_J^0}$ for any $z \in Z_J$. Using 7.2, we can write z in the form $z = z_0 z_1^n z_2$ where $z_0 \in Z_J^0, z_1 \in X_J, z_2 \in Z_G, n \in \mathbf{Z}$. (In this case the power of ϑ in 7.2 is 1.) It suffices to show that $fz_0 z_1^n z_2 f^{-1} = z_0 z_1^n z_2 \pmod{Z_J^0}$ or that $fz_0 z_1^n f^{-1} = z_0 z_1^n \pmod{Z_J^0}$, or that $fz_1^n f^{-1} = z_1^n \pmod{Z_J^0}$. (We use that $fz_0 = z_0 f$.) We have $z_1 = xa$ where $a \in Z_J^0$, hence

$$fz_1^n f^{-1} = fx^n a^n f^{-1} = fx^n f^{-1} f a^n f^{-1} = x'^n a^n = z_1^n (x^{-1} x')^n = z_1^n \pmod{Z_J^0}$$

since $x^{-1} x' \in Z_J^0$. Thus, (a) is established.

7.5. We can find a parabolic subgroup P of G which has $G_{(J)}$ as a Levi subgroup and satisfies $\vartheta P \vartheta^{-1} = P$. (We can choose a general enough $y \in \text{Hom}(\mathbf{C}^*, Z_J^0)$ such that, setting $\mathfrak{g}(n) = \{x \in \mathfrak{g}; \text{Ad}(y(a))x = a^n x \ \forall a \in \mathbf{C}^*\}$ for $n \in \mathbf{Z}$, we have $\mathfrak{g}(0) = \underline{G_{(J)}}$. Then $\bigoplus_{n \in \mathbf{N}} \mathfrak{g}(n) = \underline{P}$ for a well defined P which satisfies our requirements.) For any $f \in G\vartheta$ we set

$$\mathbf{U}_f = \{g \in G; g^{-1} f g \in X_{(J)} U_P\}.$$

We have an obvious map $\pi : \mathbf{U}_f \rightarrow X_{(J)}$ which takes g to the image of $g^{-1}fg$ under $X_{(J)}U_P \rightarrow X_{(J)}$, $ab \mapsto a$.

The image of π is a disjoint union of finitely many $\text{Ad}(G_{(J)})$ -orbits in $X_{(J)}$ (since the semisimple part of a point in this image is contained in a fixed $\text{Ad}(G)$ -orbit, namely that of the semisimple part of f). This image carries a $G_{(J)}$ -equivariant local system (on each connected component we take the local system in 7.4(a)). Taking inverse image under $\mathbf{U}_f \rightarrow \pi(\mathbf{U}_f)$ of this local system we obtain a local system on \mathbf{U}_f which is P -equivariant for the free P -action on \mathbf{U}_f given by right translation, hence it descends to a local system on \mathbf{U}_f/P denoted by $\tilde{\mathcal{F}}$. Now $Z_G(f)$ acts on \mathbf{U}_f/P by left translation and $\tilde{\mathcal{F}}$ is naturally a $Z_G(f)$ -equivariant local system. Then $\bar{Z}_G(f)$ acts naturally on the cohomology

$$(a) \quad \bigoplus_n H_c^n(\mathbf{U}_f/P, \tilde{\mathcal{F}}).$$

The set of irreducible representations (up to isomorphism) of $\bar{Z}_G(f)$ which appear in the representation (a) is denoted by $\text{Irr}_1 \bar{Z}_G(f)$.

Let $\mathfrak{R}(G\vartheta, G_J, \mathcal{C}, \mathcal{F})$ be the set of all (f, ρ) (modulo the Ad -action of G) where $f \in G\vartheta$ and $\rho \in \text{Irr}_1 \bar{Z}_G(f)$.

Lemma 7.6. *Assume that $S \subset K, S \neq \emptyset$. Let P' be a parabolic subgroup of G which has $G_{(J)}$ as a Levi subgroup and satisfies $\vartheta P' \vartheta^{-1} = P'$. Then*

- (a) $G_{I-S} \cap G_{(J)} = G_J$;
- (b) $G_{I-S} \cap P'$ is a parabolic subgroup of G_{I-S} with Levi subgroup G_J ;
- (c) $G_{I-S} \cap U_{P'} = U_{G_{I-S} \cap P'}$.

The proof is routine. It will be omitted.

8. GEOMETRIC AFFINE HECKE ALGEBRAS

8.1. We preserve the setup of 7.4. In this and the next subsection we assume that $\text{card}(K) \geq 2$.

As in [L4, 5.6], for any $J' \subset I$ such that $J \subset J' \neq I$, conjugation by the longest element $w_0^{J'}$ of $W_{J'}$ leaves stable $\{s_i; i \in J\}$. Hence for any $k \in K$ we have $w_0^{J \cup k} w_0^J w_0^{J \cup k} = w_0^J$ and $\sigma_k = w_0^{J \cup k} w_0^J = w_0^J w_0^{J \cup k}$ is an involution. Now σ_k preserves the subspace V_J of V , hence also the subspace V'_K of V'_K . Hence the subgroup W^* of W^a generated by $\{\sigma_k; k \in K\}$ acts on V'_K . As in [L4, 2.11], W^* is a Coxeter group (an affine Weyl group).

Let $y_0 \in \mathcal{C}$. For $k \in K$, $\text{ad}(y_0) : \mathfrak{g} \rightarrow \mathfrak{g}$ induces a nilpotent endomorphism of $\mathfrak{g}_{J \cup k} / \mathfrak{g}_J$. Let \underline{c}_k be the largest integer ≥ 2 such that the $(\underline{c}_k - 2)$ th power of this nilpotent endomorphism is nonzero. (This does not depend on the choice of y_0 .)

The W^* -action on V'_K leaves stable the subset \mathfrak{z}_J^1 of V'_K . It also leaves stable the subspace \mathfrak{z}_J of V'_K where it acts through a finite quotient $\mathcal{W} \subset GL(\mathfrak{z}_J)$. Let \mathcal{L}' be the set of all $x \in \mathfrak{z}_J$ such that the translation $z \mapsto z + x$ of \mathfrak{z}_J^1 coincides with the automorphism $x \mapsto w(x)$ of \mathfrak{z}_J^1 for some $w \in W^*$. Then \mathcal{L}' is a subgroup of \mathfrak{z}_J such that $\mathcal{L}'_{\mathbf{C}} = \mathfrak{z}_J$. For $k \in K$ there exist nonzero vectors $\tilde{h}_k \in \mathfrak{z}_J$ and $\tilde{\gamma}_k \in \text{Hom}(\mathfrak{z}_J, \mathbf{C})$ such that $\sigma_k(x) = x - \tilde{\gamma}_k(x)\tilde{h}_k$ for all $x \in \mathfrak{z}_J$ and $\tilde{\gamma}_k(\tilde{h}_k) = 2$. These vectors are uniquely determined if we require that $\tilde{\gamma}_k(x) = z_k x(b_k)$ for all $x \in \mathfrak{z}_J$, $\tilde{h}_k \in \mathcal{L}$ and $z_k \in \mathbf{Z}_{>0}$ is maximum possible (see [L4, 2.11]). We have $z_k \in \{1, 2, 3, 4\}$. Let $\mathcal{L} = \{x \in \mathfrak{z}_J^*; x(\mathcal{L}') \in \mathbf{Z}\}$. Then $\tilde{\gamma}_k \in \mathcal{L}$. Clearly, \mathcal{W} acts naturally on $\mathcal{L}', \mathcal{L}$. Let $\tilde{\mathcal{R}}$

(resp. $\tilde{\mathcal{R}}$) be the set of vectors in \mathfrak{z}_J^* (resp. \mathfrak{z}_J) that are of the form $w(\tilde{\gamma}_k)$ (resp. $w(\tilde{h}_k)$) for some $w \in \mathcal{W}$ and some $k \in K$. Then

(a) $(\tilde{\mathcal{R}}, \tilde{\mathcal{R}}, \mathcal{L}, \mathcal{L}')$ is an irreducible root system with Weyl group \mathcal{W} .

See [L4, 2.11]. Moreover, $\{\tilde{h}_k; k \in K\}$ generates \mathcal{L}' (see [L4, 2.14]). Now $\{\tilde{\gamma}_k; k \in K\}$ spans $\mathcal{L}_{\mathbf{C}}$ over \mathbf{C} with a single relation $\sum_{k \in K} \tilde{n}_k \tilde{\gamma}_k = 0$ where $\tilde{n}_k \in \mathbf{Z}_{>0}$ for all k (at least one \tilde{n}_k is 1) and $\sum_{k \in K} \tilde{n}_k$ is the Coxeter number of (a). For $k \in K$ we have

$$z_k = \frac{n_k}{\tilde{n}_k}.$$

We define a subset K^b of K as follows. If W^* is a Coxeter group of type $\tilde{C}_n, n \geq 1$, and k, k' correspond to the two ends of the Coxeter graph, then $K^b = \{k, k'\}$. In any other case, $K^b = \emptyset$. We set $K^\sharp = K - K^b$.

For $k \in K$ we define \bar{z}_k by $\bar{z}_k = z_k/2$ if $k \in K^b$ and $\bar{z}_k = z_k$ if $k \in K^\sharp$.

We have $\bar{z}_k \in \{\frac{1}{2}, 1, 2, 3\}$. We set

$$\hat{\gamma}_k = (\bar{z}_k/z_k)\tilde{\gamma}_k, \hat{h}_k = (z_k/\bar{z}_k)\tilde{h}_k.$$

We have $\hat{\gamma}_k \in \mathcal{L}$. Let \mathcal{R} (resp. \mathcal{R}^\sim) be the set of vectors in \mathfrak{z}_J^* (resp. \mathfrak{z}_J) that are of the form $w(\hat{\gamma}_k)$ (resp. $w(\hat{h}_k)$) for some $w \in \mathcal{W}$ and some $k \in K$.

In the case where $K^b \neq \emptyset$, we have $\{k \in K; \tilde{n}_k = 1\} = K^b$ and we choose $k_0 \in K$ so that $k_0 \in K^b$ and $\underline{c}_{k_0} \bar{z}_{k_0} d_{k_0} \leq \underline{c}_{k'} \bar{z}_{k'} d_{k'}$ where $K^b = \{k_0, k'\}$. In the case where $K^b = \emptyset$ we choose $k_0 \in K$ such that $\tilde{n}_{k_0} = 1$. One can verify that

$$\dim \mathfrak{g}_{I - \{k_0\}} \geq \dim \mathfrak{g}_{I - \{k\}} \text{ for any } k \in K.$$

Now $\{\tilde{\gamma}_k; k \in K - \{k_0\}\}$ is a basis for $(\tilde{\mathcal{R}}, \tilde{\mathcal{R}}, \mathcal{L}, \mathcal{L}')$ and $\tilde{\gamma}_{k_0}$ is the negative of the highest root of $(\tilde{\mathcal{R}}, \tilde{\mathcal{R}}, \mathcal{L}, \mathcal{L}')$ with respect to this basis. Moreover,

(d) $(\mathcal{R}, \mathcal{R}^\sim, \mathcal{L}, \mathcal{L}')$ is a root system and $\Pi = \{\hat{\gamma}_k; k \in K - \{k_0\}\}$ is a basis for it. Its Weyl group is \mathcal{W} .

See [L4, 2.15]. Since $\{\tilde{h}_k; k \in K\}$ generates \mathcal{L}' we see that $\mathcal{R}^\sim \cup ((1/2)\mathcal{R}^\sim \cap \mathcal{L}')$ generates \mathcal{L}' .

8.2. If $k \in K^b$ and $K^b - \{k\} = \{k'\}$, we set

$$(a) \quad \lambda(\hat{\gamma}_k) = (\underline{c}_k \bar{z}_k d_k + \underline{c}_{k'} \bar{z}_{k'} d_{k'})/2, \quad \lambda^*(\hat{\gamma}_k) = |\underline{c}_k \bar{z}_k d_k - \underline{c}_{k'} \bar{z}_{k'} d_{k'}|/2.$$

If $k \in K^\sharp$, we set

$$(b) \quad \lambda(\hat{\gamma}_k) = \underline{c}_k \bar{z}_k d_k/2.$$

Restricting, we get functions $\lambda : \Pi \rightarrow \mathbf{N}$, $\lambda^* : \{\hat{\gamma}_k \in \Pi; \hat{h}_k \in 2\mathcal{L}'\} \rightarrow \mathbf{N}$. (One can check that λ and λ^* indeed have values in \mathbf{N} .) Then (λ, λ^*) is a parameter set for the root system 7.6(d) with its basis Π . Hence the $\mathbf{C}[v, v^{-1}]$ -algebra $H_{\mathcal{R}, \mathcal{L}}^{\lambda, \lambda^*}$ is well defined as in 1.2. We denote this algebra by $H(G\vartheta, G_J, \mathcal{C}, \mathcal{F})$; we call it a *geometric affine Hecke algebra*.

8.3. Assume now that $\text{card}(K) = 1$. Then $\mathfrak{z}_J = 0$. We set $(\mathcal{R}, \mathcal{R}^\sim, \mathcal{L}, \mathcal{L}') = (0, 0, \emptyset, \emptyset)$. (A root system.) Then $H_{\mathcal{R}, \mathcal{L}}^{\lambda, \lambda^*} = \mathbf{C}[v, v^{-1}]$ is again denoted by $H(G\vartheta, G_J, \mathcal{C}, \mathcal{F})$.

8.4. The definition of K^b given in 8.1 differs slightly from the one given in [L4, 2.13]. (There are only three cases where the definitions differ: those in [L4, 7.16, 7.47, 7.56].) In the context of [L4] it does not matter which of the two definitions we adopt. They both lead to the same $H(G\vartheta, G_J, \mathcal{C}, \mathcal{F})$. However, in the more general context of this paper, the present definition should be adopted (it diverges from the definition in [L4]).

9. A BIJECTION

9.1. We preserve the setup of §§6, 7 and, 8. In particular, we fix J, K as in 7.1, \mathcal{C}, \mathcal{F} as in 7.4, P as in 7.5, k_0 as in 8.1. Let $v_0 \in \mathbf{C}^*$ be such that either $v_0 = 1$ or v_0 is not a root of 1. We choose $r_0 \in \mathbf{C}$ such that $\exp(r_0) = v_0$; if $v_0 = 1$, we choose $r_0 = 0$. Let \spadesuit be the \mathbf{Q} -subspace of \mathbf{C} spanned by r_0 . We have $\spadesuit \cap \kappa\mathbf{Q} = 0$. We can choose a \mathbf{Q} -subspace \diamond of \mathbf{C} such that

$$(a) \quad \kappa\mathbf{Q} \subset \diamond, \diamond \oplus \spadesuit = \mathbf{C}.$$

We have $\mathbf{C}^* = \exp(\diamond) \times \exp(\spadesuit)$.

If G' is an algebraic group, any $g \in G'$ can be written uniquely in the form $g = g_\diamond g_\spadesuit = g_\spadesuit g_\diamond$ where $g_\diamond \in G'_\diamond$ is semisimple and $g_\spadesuit \in G'_\spadesuit$.

In this section we will define a bijection

$$(b) \quad \text{Irr}_{v_0} H(G\vartheta, G_J, \mathcal{C}, \mathcal{F}) \leftrightarrow \mathfrak{R}(G\vartheta, G_J, \mathcal{C}, \mathcal{F})$$

in terms of r_0 and \diamond as above.

If $\text{card}(K) = 1$, then both sides of (b) consist of one element, hence there is a unique bijection between them. In the remainder of this section we assume that $\text{card}(K) \geq 2$.

9.2. Let $V'^\diamond = \{x' \in V'; x' = \sum_{i \in I} c_i b'_i, c_i \in \kappa^{-1}\diamond\}$.

For any $S \subset I, S \neq \emptyset$ let $C_S^\diamond = C_S \cap V'^\diamond$. Let $C'^\diamond = \bigsqcup_{S \subset I, S \neq \emptyset} C_S^\diamond = C' \cap V'^\diamond$. Let $D = \bigsqcup_{S \subset K; S \neq \emptyset} C_S, D^\diamond = \bigsqcup_{S \subset K; S \neq \emptyset} C_S^\diamond$.

(a) *The map $x' \mapsto p(x')$ defines a bijection between C'^\diamond and a set of representatives for the orbits of the \mathcal{N} action on $T_\diamond\vartheta = \hat{G}_\diamond \cap T\vartheta$ (by conjugation).*

This is an immediate consequence of 6.11 and its proof.

We set

$$\mathcal{T} = \mathcal{L}' \otimes \mathbf{C}^*.$$

(A torus.) Then $\mathcal{L}'_C = \mathfrak{z}_J = \underline{\mathcal{T}}$, hence $\exp_{\mathcal{T}} : \mathfrak{z}_J \rightarrow \mathcal{T}$ is defined. Let p' be the composition $\mathfrak{z}_J^1 \rightarrow \mathfrak{z}_J \rightarrow \mathcal{T}$ where the first map is $x \mapsto x - \frac{1}{n_{k_0}} b'_{k_0}$ and the second map is $x \mapsto \exp_{\mathcal{T}}(\kappa x)$.

(b) *The map $x' \mapsto p'(x')$ defines a bijection between D and a set of representatives for the \mathcal{W} -orbits in \mathcal{T} and also a bijection between D^\diamond and a set of representatives for the \mathcal{W} -orbits in \mathcal{T}_\diamond .*

Just like 6.11 was proved using 6.6, the proof of (b) is based on the following analogue of 6.6.

Let $x' \in \mathfrak{z}_J^1$. The W^ -orbit W^*x' meets D in exactly one point.*

This is proved exactly like 6.6, by replacing W^a by W^* .

For $\mathbf{d} \in D^\diamond$, let $\mathcal{W}_{\mathbf{d}}$ be the stabilizer of $p'(\mathbf{d})$ in \mathcal{W} . Let Θ be the set consisting of all pairs (\mathbf{d}, δ) where $\mathbf{d} \in D^\diamond$ and δ is a $\mathcal{W}_{\mathbf{d}}$ -orbit in \mathcal{T}_\spadesuit . Using the decomposition $\mathcal{T} = T_\diamond T_\spadesuit$ we see that there is a bijection $\Theta \xrightarrow{\sim} \mathcal{T}/\mathcal{W}$ which associates to $(\mathbf{d}, \delta) \in \Theta$ the \mathcal{W} -orbit of $p'(\mathbf{d})z$ where $z \in \delta$. (This \mathcal{W} -orbit is denoted by $\Sigma_{\mathbf{d}, \delta}$.)

9.3. The partition 1.3(a) becomes in our case

$$(a) \quad \text{Irr}_{v_0} H_{\mathcal{R}, \mathcal{L}}^{\lambda, \lambda^*} = \bigsqcup_{S \subset K, S \neq \emptyset} \bigsqcup_{\mathbf{d} \in C_S^\circ} \bigsqcup_{\delta \in \mathcal{T}_\bullet / \mathcal{W}_\mathbf{d}} \text{Irr}_{\Sigma_{\mathbf{d}, \delta}, v_0} H_{\mathcal{R}, \mathcal{L}}^{\lambda, \lambda^*}.$$

Let $(\mathbf{d}, \delta) \in \Theta$ where $\mathbf{d} \in C_S^\circ$. Then $p'(\mathbf{d})\delta$ is a $\mathcal{W}_\mathbf{d}$ -orbit contained in $\Sigma_{\mathbf{d}, \delta}$. Moreover,

$$\mathcal{R}_{p'(\mathbf{d})\delta}, \mathcal{R}_{p'(\mathbf{d})\delta}, \Pi_{p'(\mathbf{d})\delta}, \lambda_{p'(\mathbf{d})\delta}, \lambda_{p'(\mathbf{d})\delta}^*$$

are defined in terms of $\mathcal{L}, \mathcal{L}', \mathcal{R}, \mathcal{R}', \Pi, p'(\mathbf{d})\delta, \lambda, \lambda^*$ in the same way as $R_c, \tilde{R}_c, \Pi_c, \lambda_c, \lambda_c^*$ were defined in [8.1]' (see 2.2) and 3.1 in terms of $X, Y, R, \tilde{R}, \Pi, c, \lambda, \lambda^*$. Then, by 3.2 we have a natural bijection

$$(b) \quad \text{Irr}_{\Sigma_{\mathbf{d}, \delta}, v_0} H_{\mathcal{R}, \mathcal{L}}^{\lambda, \lambda^*} \leftrightarrow \text{Irr}_{p'(\mathbf{d})\delta, v_0} H_{\mathcal{R}_{p'(\mathbf{d})\delta}, \mathcal{L}}^{\lambda_{p'(\mathbf{d})\delta}, \lambda_{p'(\mathbf{d})\delta}^*}.$$

Let \mathcal{W}_{K-S} be the subgroup of \mathcal{W} generated by the image of $\{\sigma_k; k \in K-S\}$ in \mathcal{W} . Let \mathcal{R}_{K-S} (resp. \mathcal{R}'_{K-S}) be the set of vectors of \mathcal{L} (resp. \mathcal{L}') of the form $w(\hat{\gamma}_k)$ (resp. $w(\hat{h}_k)$) for some $k \in K-S, w \in \mathcal{W}_{K-S}$. Let $\Pi_{K-S} = \{\hat{\gamma}_k; k \in K-S\}$. Then

$$(\mathcal{L}, \mathcal{L}', \mathcal{R}_{p'(\mathbf{d})\delta}, \mathcal{R}'_{p'(\mathbf{d})\delta}, \Pi_{p'(\mathbf{d})\delta}, \mathcal{W}_\mathbf{d}) = (\mathcal{L}, \mathcal{L}', \mathcal{R}_{K-S}, \mathcal{R}'_{K-S}, \Pi_{K-S}, \mathcal{W}_{K-S}).$$

(The main assertion here is that $\mathcal{R}_{p'(\mathbf{d})\delta} = \mathcal{R}_{K-S}$. In other words, for $\alpha \in \mathcal{R}$, with corresponding $\tilde{\alpha} \in \mathcal{R}'$, the condition that $\alpha \in \mathcal{R}_{K-S}$ is equivalent to the condition that $\alpha(\mathbf{d} - \frac{1}{n_{k_0}}b'_{k_0})$ is in \mathbf{Z} if $\tilde{\alpha} \notin 2\mathcal{L}'$ and is in $\frac{1}{2}\mathbf{Z}$ if $\tilde{\alpha} \in 2\mathcal{L}'$. This is an assertion of the same type as 6.16 and has a similar proof. See also [L4, 3.9, 3.10].)

Thus, the bijection (b) can be rewritten as

$$\text{Irr}_{\Sigma_{\mathbf{d}, \delta}, v_0} H_{\mathcal{R}, \mathcal{L}}^{\lambda, \lambda^*} \leftrightarrow \text{Irr}_{p'(\mathbf{d})\delta, v_0} H_{\mathcal{R}_{K-S}, \mathcal{L}}^{\lambda, \lambda^*};$$

the exponents λ, λ^* in both sides are restrictions of the function given by 8.2(a),(b).

Taking union of all d, δ and composing with (a) we obtain a bijection

$$(c) \quad \text{Irr}_{v_0} H_{\mathcal{R}, \mathcal{L}}^{\lambda, \lambda^*} \leftrightarrow \bigsqcup_{S \subset K, S \neq \emptyset} \bigsqcup_{\mathbf{d} \in C_S^\circ} \bigsqcup_{\delta \in \mathcal{T}_\bullet / \mathcal{W}_\mathbf{d}} \text{Irr}_{p'(\mathbf{d})\delta, v_0} H_{\mathcal{R}_{K-S}, \mathcal{L}}^{\lambda, \lambda^*}.$$

Let $(\mathbf{d}, \delta) \in \Theta$. Then $(\mathcal{L}_\mathbf{Q}, \mathcal{L}'_\mathbf{Q}, \mathcal{R}_{K-S}, \mathcal{R}'_{K-S})$ is a \mathbf{Q} -root system with basis Π_{K-S} .

For $k \in K$ we regard $\hat{\gamma}_k$ as a character $\mathcal{T} \rightarrow \mathbf{C}^*$ given by $l' \otimes a \mapsto a^{\hat{\gamma}_k(l')}$ where $l' \in \mathcal{L}', a \in \mathbf{C}^*$. Then for $z \in \mathfrak{z}_J$ we have $\gamma_k(\exp_{\mathcal{T}}(z)) = \exp(\gamma_k(z))$. We show that, if $k \in K-S, k \in K^\flat$, then

$$\hat{\gamma}_k(p'(\mathbf{d})) = -1 \text{ if } k = k_0 \text{ and } \hat{\gamma}_k(p'(\mathbf{d})) = 1 \text{ if } k \neq k_0.$$

Indeed, $\hat{\gamma}_k(\exp_{\mathcal{T}}(\kappa(\mathbf{d} - \frac{1}{n_{k_0}}b'_{k_0}))) = \exp(\kappa \bar{z}_k b_k(\mathbf{d} - \frac{1}{z_{k_0}}b'_{k_0})) = \exp(-\kappa \delta_{k, k_0}/2)$.

It follows that, if we define $\mu : \Pi_{K-S} \rightarrow \mathbf{Z}$ in terms of λ, λ^* as in 4.1 (with $t_0 = p'(\mathbf{d})$), then

$$\mu(\hat{\gamma}_k) = d_k \bar{z}_k \mathcal{C}_k.$$

Define a \mathcal{W}_{K-S} -orbit $\bar{\delta}$ in \mathcal{T}_\bullet by $\exp_{\mathcal{T}}(\bar{\delta}) = \delta$. By 4.2 we have a bijection

$$\text{Irr}_{\bar{\delta}, r_0} \bar{H}_{\mathcal{R}_{K-S}, \mathcal{L}_\mathbf{Q}}^\mu \leftrightarrow \text{Irr}_{p'(\mathbf{d})\delta, v_0} H_{\mathcal{R}_{K-S}, \mathcal{L}}^{\lambda, \lambda^*}.$$

Taking union over all \mathbf{d}, δ and composing with (c) we obtain a bijection

$$(d) \quad \text{Irr}_{v_0} H_{\mathcal{R}, \mathcal{L}}^{\lambda, \lambda^*} \leftrightarrow \bigsqcup_{S \subset K, S \neq \emptyset} \bigsqcup_{\mathbf{d} \in C_S^\circ} \bigsqcup_{\bar{\delta} \in \mathcal{T}_\bullet / \mathcal{W}_{K-S}} \text{Irr}_{\bar{\delta}, r_0} \bar{H}_{\mathcal{R}_{K-S}, \mathcal{L}_\mathbf{Q}}^\mu.$$

For $k \in K$ we set $^*\gamma_k = (1/d_k \bar{z}_k) \hat{\gamma}_k \in \mathfrak{z}_J^*$ (that is, $^*\gamma_k$ is the restriction of β_k to \mathfrak{z}_J where β_k is regarded as an element of \mathfrak{t}^*). We set $^*h_k = d_k \bar{z}_k \hat{h}_k \in \mathfrak{z}_J$.

Let $S \subset K, S \neq \emptyset$. Let ${}^*\mathcal{R}_{K-S}$ (resp. ${}^*\mathcal{R}'_{K-S}$) be the set of vectors of $\mathcal{L}_{\mathbf{Q}}$ (resp. $\mathcal{L}'_{\mathbf{Q}}$) of the form $w({}^*\gamma_k)$ (resp. $w({}^*h_k)$) for some $k \in K-S, w \in \mathcal{W}_{K-S}$. Let ${}^*\Pi_{K-S} = \{{}^*\gamma_k; k \in K-S\}$. Then $(\mathcal{L}_{\mathbf{Q}}, \mathcal{L}'_{\mathbf{Q}}, \mathcal{R}_{K-S}, \mathcal{R}'_{K-S}, {}^*\Pi_{K-S})$ is a \mathbf{Q} -root system. Define $\mu' : {}^*\Pi_{K-S} \rightarrow \mathbf{N}$ by $\mu'({}^*\gamma_k) = \underline{c}_k$. There is an algebra isomorphism

$$\bar{H}_{\mathcal{R}_{K-S}, \mathcal{L}_{\mathbf{Q}}}^{\mu} \xrightarrow{\sim} \bar{H}_{\mathcal{R}'_{K-S}, \mathcal{L}_{\mathbf{Q}}}^{\mu'}$$

which is the identity on the generators. (We use that $\mu(\hat{\gamma}_k)/\hat{\gamma}_k = \mu'({}^*\gamma_k)/{}^*\gamma_k$.) This induces a bijection

$$\text{Irr}_{\bar{\delta}, r_0} \bar{H}_{\mathcal{R}_{K-S}, \mathcal{L}_{\mathbf{Q}}}^{\mu} \leftrightarrow \text{Irr}_{\bar{\delta}, r_0} \bar{H}_{\mathcal{R}'_{K-S}, \mathcal{L}_{\mathbf{Q}}}^{\mu'}$$

for any $\bar{\delta} \in \underline{\mathcal{T}}_{\blacktriangle}/\mathcal{W}_{K-S}$. Taking union over all $\mathbf{d}, \bar{\delta}$ and composing with (d) we obtain a bijection

$$(e) \quad \text{Irr}_{v_0} H_{\mathcal{R}, \mathcal{L}}^{\lambda, \lambda^*} \leftrightarrow \bigsqcup_{S \subset K, S \neq \emptyset} \bigsqcup_{\mathbf{d} \in C_S^{\circ}} \bigsqcup_{\bar{\delta} \in \underline{\mathcal{T}}_{\blacktriangle}/\mathcal{W}_{K-S}} \text{Irr}_{\bar{\delta}, r_0} \bar{H}_{\mathcal{R}'_{K-S}, \mathcal{L}_{\mathbf{Q}}}^{\mu'}.$$

Let $S \subset K, S \neq \emptyset$. If we apply the definitions of 5.2 to G_{I-S}, G_J (instead of G, L), then R, W of 5.2 become ${}^*\mathcal{R}_{K-S}, \mathcal{W}_{K-S}$. We see that $\bar{H}_{\mathcal{R}'_{K-S}, \mathcal{L}_{\mathbf{Q}}}^{\mu'}$ (as above) may be interpreted as $\bar{H}(G_{I-S}, G_J, \mathcal{C}, \mathcal{F})$. Moreover, if $\mathbf{d} \in C_S^{\circ}$, we have $G_{I-S} = Z_G(p(\mathbf{d}))$ (see 6.20). Hence we may rewrite (e) as

$$(f) \quad \text{Irr}_{v_0} H_{\mathcal{R}, \mathcal{L}}^{\lambda, \lambda^*} \leftrightarrow \bigsqcup_{S \subset K, S \neq \emptyset} \bigsqcup_{\mathbf{d} \in C_S^{\circ}} \bigsqcup_{\bar{\delta} \in (\mathfrak{z}_J)_{\blacktriangle}/\mathcal{W}_{K-S}} \text{Irr}_{\bar{\delta}, r_0} \bar{H}(Z_G(p(\mathbf{d})), G_J, \mathcal{C}, \mathcal{F})$$

(We have used that $\underline{\mathcal{T}} = \mathfrak{z}_J$. Moreover, $(\mathfrak{z}_J)_{\blacktriangle}$ defined in terms of Z_J^0 coincides with $\underline{\mathcal{T}}_{\blacktriangle}$ defined in terms of \mathcal{T} .) By 5.17(d), for any $S \subset K, S \neq \emptyset$ and $\mathbf{d} \in C_S^{\circ}$ we have a bijection

$$\bigsqcup_{\bar{\delta} \in (\mathfrak{z}_J)_{\blacktriangle}/\mathcal{W}_{K-S}} \text{Irr}_{\bar{\delta}, r_0} \bar{H}(Z_G(p(\mathbf{d})), G_J, \mathcal{C}, \mathcal{F}) \leftrightarrow \dot{\mathfrak{z}}^{\blacktriangle}(Z_G(p(\mathbf{d})), G_J, \mathcal{C}, \mathcal{F}).$$

(To define $\dot{\mathfrak{z}}^{\blacktriangle}(Z_G(p(\mathbf{d})), G_J, \mathcal{C}, \mathcal{F})$ we use the parabolic subgroup $P \cap G_{I-S}$ of $Z_G(p(\mathbf{d})) = G_{I-S}$ with Levi subgroup G_J , see 7.6.) Taking union over all \mathbf{d} and composing with (f) we obtain a bijection

$$(g) \quad \text{Irr}_{v_0} H_{\mathcal{R}, \mathcal{L}}^{\lambda, \lambda^*} \leftrightarrow \bigsqcup_{S \subset K, S \neq \emptyset} \bigsqcup_{\mathbf{d} \in C_S^{\circ}} \dot{\mathfrak{z}}^{\blacktriangle}(Z_G(p(\mathbf{d})), G_J, \mathcal{C}, \mathcal{F}).$$

Lemma 9.4. *Let $f \in G\vartheta$. The following three conditions are equivalent:*

- (i) *there exists $g \in G$ such that, for some $S \subset K, S \neq \emptyset$ we have $g^{-1}f_{\diamond}g \in p(C_S^{\circ})$ and $g^{-1}f_{\blacktriangle}g \in \dot{C}(Z_J^0)_{\blacktriangle}(U_P \cap G_{I-S})$;*
- (ii) *there exists $g \in G$ such that for some $S \subset K, S \neq \emptyset$ we have $g^{-1}fg \in \dot{C}(C_S^{\circ})(Z_J^0)_{\blacktriangle}(U_P \cap G_{I-S})$;*
- (iii) *$g^{-1}fg \in X_{(J)}U_P$.*

It is clear that (i) and (ii) are equivalent and that (ii) implies (iii). Now assume that (iii) holds. We show that (ii) holds. We may assume that $f \in X_{(J)}U_P$. By replacing f by a P -conjugate, we may assume that $f \in X_J \dot{C}U_P$. Let f_s and f_u be the semisimple and unipotent part of f . Then f_s is P -conjugate to an element of X_J . Hence, replacing f by a P -conjugate, we may assume that $f_s \in X_J, f_u \in \dot{C}U_P$. Let $(\mathfrak{z}_J^1)_{\diamond}$ be the set of all $x' \in \mathfrak{z}_J^1$ such that $x' = \sum_{k \in K} c_k b'_k$ with $c_k \in \kappa^{-1}\diamond$. Then $X_J = p((\mathfrak{z}_J^1)_{\diamond})(Z_J^0)_{\blacktriangle}$. Now the W^* -action on \mathfrak{z}_J^1 restricts to a W^* -action on $(\mathfrak{z}_J^1)_{\diamond}$.

which has $\bigcup_{S \subset K; S \neq \emptyset} C_S^\circ$ as a fundamental domain. Hence by replacing f by nfn^{-1} for some $n \in G^\vartheta$ such that

$$(a) \quad nG_J n^{-1} = G_J, nX_J n^{-1} = X_J, n\dot{C} n^{-1} = \dot{C}, nG_{(J)} n^{-1} = G_{(J)}, nP n^{-1} = P'$$

(P' is another parabolic subgroup normalized by ϑ which has $G_{(J)}$ as a Levi subgroup), we may assume that $f_\diamond \in p(C_S^\circ)$, $f_\spadesuit \in (Z_J^0)_\spadesuit \dot{C} U_{P'}$. Since $f_\spadesuit \in Z_G(f_\diamond)$ and f_\diamond as well as $(Z_J^0)_\spadesuit \dot{C}$ are contained in the Levi subgroup $G_{(J)}$ of P' , we have automatically $f_\spadesuit \in (Z_J^0)_\spadesuit \dot{C}(Z_G(f_\diamond) \cap U_{P'})$. Since $Z_G(f_\diamond) = G_{I-S}$ (see 6.20), we have

$$f \in p(C_S^\circ)(Z_J^0)_\spadesuit \dot{C}(G_{I-S} \cap U_{P'}).$$

Now the parabolic subgroups $G_{I-S} \cap P'$ and $G_{I-S} \cap P$ of G_{I-S} (both with Levi subgroup G_J , see 7.6) are conjugate under an element $z \in G_{I-S}$ which normalizes G_J . Conjugation by z carries $G_{I-S} \cap U_{P'}$ to $G_{I-S} \cap U_P$, $p(C_S^\circ)$ to $p(C_S^\circ)$, Z_J^0 to Z_J^0 and \dot{C} to \dot{C} , hence it carries f to $zfz^{-1} \in p(C_S^\circ)(Z_J^0)_\spadesuit \dot{C}(U_P \cap G_{I-S})$. Thus, f satisfies (ii). The lemma is proved.

9.5. Let $f \in G^\vartheta$. Assume that $f_\diamond \in p(C_S^\circ)$ where $S \subset K, S \neq \emptyset$. Then $f_\spadesuit \in Z_G(f_\diamond) = G_{I-S}$. We want to compare the varieties:

$$A = \{g \in G; g^{-1}fg \in G_{(J)}U_P\}/P,$$

$$A' = \{g' \in G_{I-S}; g'^{-1}f_\spadesuit g' \in Z_J^0 \dot{C}(U_P \cap G_{I-S})\}/(P \cap G_{I-S}).$$

Let P' be any parabolic subgroup of G such that $\vartheta P' \vartheta^{-1} = P'$, $G_{(J)}$ is a Levi subgroup of P' and $P' \cap G_{I-S} = P \cap G_{I-S}$. Let $A(P') = \{g \in G; g^{-1}fg \in G_{(J)}U_{P'}\}/P'$. Define $f_{P'} : A' \rightarrow A(P')$ by $g'(P \cap G_{I-S}) \mapsto g'P'$. This is clearly injective. We can find $n \in G_{I-S}$ such that 9.4(a) holds. Define $F_n : A(P') \xrightarrow{\sim} A$ by $gP' \mapsto gnP$. The composition $F_n f_{P'} : A' \rightarrow A(P')$ is injective. Its image $A_{P'}$ depends only on P' , not on n . By the argument in the proof of 9.4 we see that A is the disjoint union of finitely many subvarieties $A_{P'}$ (for the various P' as above) and each $A_{P'}$ is isomorphic to A' . It follows easily that

$$(a) \quad \text{Irr}_1 \bar{Z}_G(f) = \text{Irr}_0 \bar{Z}_{Z_G(f_\diamond)}(f_\spadesuit).$$

(Note that $Z_G(f) = Z_{Z_G(f_\diamond)}(f_\spadesuit)$.) Using (a) and 9.4 we see that we have a bijection

$$(b) \quad \bigsqcup_{S \subset K, S \neq \emptyset} \bigsqcup_{\mathbf{d} \in C_S^\circ} \dot{\mathfrak{z}}^\spadesuit(Z_G(p(\mathbf{d})), G_J, \mathcal{C}, \mathcal{F}) \leftrightarrow \mathfrak{R}(G^\vartheta, G_J, \mathcal{C}, \mathcal{F})$$

given by $(\mathbf{d}, (f, \rho)) \mapsto (p(\mathbf{d})f, \rho)$. Here $f \in (Z_G(p(\mathbf{d})))_\spadesuit$. Composing (b) with 9.3(g), we obtain a bijection $\text{Irr}_{v_0} H_{\mathcal{R}, \mathcal{L}}^{\lambda, \lambda^*} \leftrightarrow \mathfrak{R}(G^\vartheta, G_J, \mathcal{C}, \mathcal{F})$. This is, by definition, the bijection 9.1(b).

10. THE MAIN RESULTS

10.1. In 10.2-10.7 we preserve the setup of 9.1.

Lemma 10.2. *Assume that $\text{card}(K) \geq 2$. There is a canonical injective map $\iota : \mathcal{T}/\mathcal{W} \rightarrow (T\vartheta)/\mathcal{N}$ whose image is exactly the image of X_J in $(T\vartheta)/\mathcal{N}$.*

By definition, ι sends the \mathcal{W} -orbit of $t \in \mathcal{T}$ to the \mathcal{N} -orbit of $p(x')$ where $x' \in \mathfrak{z}_J^1$ is such that $p'(x') = t$. Assume that x'' is an element of \mathfrak{z}_J^1 such that $p'(x'') = w_1(t)$ where $w_1 \in \mathcal{W}$. Then there exists $w \in W^*$ such that $p'(w(x'')) = t$. Since $p'(w(x'') - x') = 1$, we have $w(x'') - x' \in \mathcal{L}'$. Hence $x' = w'w(x'')$ for some $w' \in W^*$.

In particular, $x' = \tilde{w}(x'')$ for some $w \in W^a$. Hence $p(x') = np(x'')n^{-1}$ for some $n \in \mathcal{N}$. Thus, ι is well defined.

We show that ι is injective. Now C' is a set of representatives for the W^a -orbits on \mathfrak{t}^1 . Similarly, D is a set of representatives for the W^* -orbits on \mathfrak{t}_J^1 . Let $t_1, t_2 \in \mathcal{T}$ be such that the \mathcal{W} -orbit of t_1 and the \mathcal{W} -orbit of t_2 have the same image under ι . Let $x_1, x_2 \in \mathfrak{t}_J^1$ be such that $p'(x_1) = t_1, p'(x_2) = t_2$. We may assume that $x_1 \in D, x_2 \in D$. Since $D \subset C'$, we have $x_1 \in C', x_2 \in C'$. By assumption we have $p(x_1) = np(x_2)n^{-1}$ for some $n \in \mathcal{N}$. Hence $x_1 = w(x_2)$ for some $w \in W^a$. Since $x_1, x_2 \in C'$, it follows that $x_1 = x_2$. Hence $t_1 = t_2$. This shows that ι is injective. The fact that the image of ι is exactly the image of X_J in $(T\vartheta)/\mathcal{N}$ is obvious. The lemma is proved.

10.3. We show that the bijection 9.1(b) does not depend on the choice of \diamond as in 9.1(a). When $v_0 = 1$, this is obvious: we have $\diamond = \mathbf{C}$. Assume now that $v_0 \neq 1$. It is enough to show that one can define a map

$$\text{Irr}_{v_0} H(G\vartheta, G_J, \mathcal{C}, \mathcal{F}) \rightarrow \mathfrak{R}(G\vartheta, G_J, \mathcal{C}, \mathcal{F})$$

purely in terms of \spadesuit and which coincides with the map defined in §9 in terms of any given \diamond . We can assume that $\text{card}(K) \geq 2$.

Let $M \in \text{Irr}_{v_0} H(G\vartheta, G_J, \mathcal{C}, \mathcal{F})$. We want to attach to M a pair (f, ρ) (up to G -conjugacy) where $f \in G\vartheta$ and $\rho \in \text{Irr}_1 \bar{Z}_G(f)$. We will only indicate the definition of the G -conjugacy class of f . (A similar definition applies to ρ .)

By 1.3, we have $M \in \text{Irr}_{\Sigma, v_0} H_{\mathcal{R}, \mathcal{L}}^{\lambda, \lambda^*}$ for a well defined \mathcal{W} -orbit Σ on \mathcal{T} . Let c be a fiber of $\Sigma \rightarrow \mathcal{T}/\mathcal{T}_\spadesuit$ (restriction of $\mathcal{T} \rightarrow \mathcal{T}/\mathcal{T}_\spadesuit$). Define $\mathcal{R}_c, \mathcal{W}^c, H_c$ in terms of \mathcal{T}, \mathcal{R} in the same way as R_c, W_0^c, H_c were defined in [8.1]' and [8.3]' (see 2.2) in terms of \mathcal{T}, R . By 3.2, to M corresponds an object $M' \in \text{Irr}_{c, v_0} H_c$. As in [9.2]', we can find an element $t_0 \in \mathcal{T}$ whose stabilizer in \mathcal{W} equals \mathcal{W}^c and a \mathcal{W}^c -orbit \bar{c} in \mathfrak{t}_\spadesuit such that $t_0 \exp_{\mathcal{T}}(\bar{c}) = c$. By 4.2 (for this t_0), M' corresponds to an object $M'' \in \text{Irr}_{r_0}^\spadesuit \bar{H}$ where \bar{H} is attached to H_c as in 4.1. Let \tilde{t}_0 be an element of X_J such that $\iota(\mathcal{W}t)$ is the \mathcal{N} -orbit of \tilde{t}_0 (see 10.2). Now \bar{H} may be interpreted as the algebra $\bar{H}(G', G_J, \mathcal{C}, \mathcal{F})$ where $G' = Z_G(\tilde{t}_0)$. (Note that G_J is the Levi subgroup of some parabolic subgroup of G' .) Under 5.17(d), to M'' corresponds a pair (f', ρ') where $f' \in G'_\spadesuit$ is well defined up to conjugation in G' and $\rho' \in \text{Irr}_0 \bar{Z}_{G'}(f')$. We set $f = \tilde{t}_0 f' = f' \tilde{t}_0$. Then the $\text{Ad}(G)$ -orbit of f is well defined by M . Note that we have not used \diamond in this definition. Thus we have the following result.

Theorem 10.4. *Assume that $v \in \mathbf{C}^*$ is either 1 or is not a root of 1. Let $r_0 \in \mathbf{C}$ be as in 9.1. There is a bijection*

$$(a) \quad \text{Irr}_{v_0} H(G\vartheta, G_J, \mathcal{C}, \mathcal{F}) \leftrightarrow \mathfrak{R}(G\vartheta, G_J, \mathcal{C}, \mathcal{F})$$

depending only on r_0 , which for any \diamond as in 9.1(a) coincides with the bijection 9.1(b).

It is likely that the bijection (a) is independent of the choice of r_0 . (Some evidence is given in 10.7.)

Theorem 10.5. *We preserve the setup of 10.4. Assume that v_0 is not a root of 1. Let $\zeta : \mathbf{C}^* \rightarrow \mathbf{R}$ be a group homomorphism such that $\zeta(v_0) \neq 0$. Assume that under 10.4(a), $M \in \text{Irr}_{v_0} H(G\vartheta, G_J, \mathcal{C}, \mathcal{F})$ corresponds to $(f, \rho) \in \mathfrak{R}(G\vartheta, G_J, \mathcal{C}, \mathcal{F})$. Then*

- (a) *M is ζ -tempered if and only if $f \in \hat{G}^{\text{Ker}\zeta}$;*
- (b) *M is ζ -square integrable if and only if any torus in $Z_G(f)$ is $\{1\}$.*

Assume first that $\text{card}(K) = 1$. Then the unique $M \in \text{Irr}_{v_0} H(G\vartheta, G_J, \mathcal{C}, \mathcal{F})$ is obviously ζ -square integrable. The unique element of $\mathfrak{R}(G\vartheta, G_J, \mathcal{C}, \mathcal{F})$ may be represented in the form (f, ρ) where $f = su$ with s being the unique element of X_J and $u \in \dot{\mathcal{C}}$. Assume that T_1 is a torus in $Z_G(f) = Z_{G_J}(u)$. In our case, G_J is a semisimple group and u is a distinguished unipotent element of G_J . Hence any torus in $Z_{G_J}(u)$ is trivial. Thus, $T = 1$ hence (b) holds in this case. We show that $f \in \hat{G}^{\text{Ker}\zeta}$. It is enough to show that $s \in \hat{G}^{\text{Ker}\zeta}$. Since $Z_G(s)$ is semisimple, s is of finite order. Since $\text{Ker}\zeta$ contains all roots of 1, we have $s \in \hat{G}^{\text{Ker}\zeta}$. Thus, (a) holds in this case.

In the remainder of the proof we assume that $\text{card}(K) \geq 2$. Let r_0, \spadesuit be as in 9.1.

Case 1. We assume that $\zeta(\mathbf{C}^*) \subset \mathbf{Q}$. In this case, ζ restricts to an isomorphism $\exp(\spadesuit) \xrightarrow{\sim} \mathbf{Q}$. In particular, we have $\mathbf{C}^* = \text{Ker}\zeta \oplus \exp(\spadesuit)$. Let $\diamond = \{a \in \mathbf{C}; \zeta(\exp(a)) = 1\}$. Then \diamond satisfies 9.1(a) and $\text{Ker}\zeta = \exp(\diamond)$. We can use the definition of the bijection 9.1(b) in terms of this \diamond .

We prove (a). Using Lemmas 3.4, 4.3, 5.18 and the definitions, we are reduced to verifying that for $f \in G\vartheta$ the following two conditions are equivalent:

- (i) $f_\spadesuit \in Z_G(f_\diamond)^{\text{Ker}\zeta}$;
- (ii) $f \in \hat{G}^{\text{Ker}\zeta}$.

(When applying 4.3, we can choose t_0 in 4.1 so that $t_0 \in \mathcal{T}_\diamond$; then it is unique. The assumption $t_0 \in \mathcal{T}^{\text{Ker}\zeta}$ of 4.3 is automatically verified since $\mathcal{T}^{\text{Ker}\zeta} = \mathcal{T}_\diamond$.)

Now (i) is equivalent to the condition $f_\spadesuit \in \hat{G}^{\text{Ker}\zeta}$. This is equivalent to (ii) since we have automatically $f_\diamond \in \hat{G}^{\text{Ker}\zeta}$ (since $\text{Ker}\zeta = \exp(\diamond)$).

We prove (b). Using Lemmas 3.5, 4.4, 5.16 and the definitions we are reduced to verifying that for $f \in G$ the following two conditions are equivalent:

- (iii) $Z_G(f_\diamond)$ is semisimple and f_\spadesuit is a distinguished unipotent element of $Z_G(f_\diamond)$;
- (iv) any torus in $Z_G(f)$ is $\{1\}$.

This is immediate.

General case. As in 1.3, let \mathcal{L}^+ be the set of all $x \in \mathcal{L}$ such that $\langle x, \hat{h}_k \rangle \geq 0$ for all $\hat{\gamma}_k \in \Pi$. We can find x_1, \dots, x_N in $\mathcal{L}^+ - \{0\}$ such that $\mathcal{L}^+ = \sum_{k=1}^N \mathbf{N}x_k$. There exists a finite subset \mathcal{T}_0 of \mathcal{T} such that, for $t \in \mathcal{T}$, the weight space M_t is zero unless $t \in \mathcal{T}_0$. Let A be the (finite) subset of \mathbf{C}^* consisting of all numbers of the form $x_k(t)$ with $k \in [1, N], t \in \mathcal{T}_0$. Let B be the (finite) subset of \mathbf{C}^* consisting of the eigenvalues of f in a fixed faithful $V \in \mathcal{I}_{\hat{G}}$. Then (a) and (b) can be restated as (c) and (d) below:

- (c) we have $\zeta(a)/\zeta(v_0) \geq 0$ for all $a \in A$ if and only if $\zeta(b) = 0$ for all $b \in B$;
- (d) we have $\zeta(a)/\zeta(v_0) > 0$ for all $a \in A$ if and only (iv) holds.

Assume first that $\zeta(a)/\zeta(v_0) \geq 0$ for all $a \in A$ and $\zeta(b) \neq 0$ for some $b \in B$. We can find a \mathbf{Q} -linear form $u : \mathbf{R} \rightarrow \mathbf{Q}$ such that $u(\zeta(v_0)) \neq 0, u(\zeta(b)) \neq 0$ and $u(\zeta(a))/u(\zeta(v_0)) \geq 0$ for all $a \in A$. Applying Case 1 to $u\zeta : \mathbf{C}^* \rightarrow \mathbf{Q}$ instead of ζ , we see that $u(\zeta(b)) = 0$, a contradiction.

Assume next that $\zeta(a)/\zeta(v_0) < 0$ for some $a \in A$ and $\zeta(b) = 0$ for all $b \in B$. We can find a \mathbf{Q} -linear form $u : \mathbf{R} \rightarrow \mathbf{Q}$ such that $u(\zeta(v_0)) \neq 0, u(\zeta(a))/u(\zeta(v_0)) < 0$. We have $u(\zeta(b)) = 0$ for all $b \in B$. Applying Case 1 to $u\zeta : \mathbf{C}^* \rightarrow \mathbf{Q}$ instead of ζ , we see that $u(\zeta(a))/u(\zeta(v_0)) \geq 0$, a contradiction. Thus, (c) holds.

Assume now that $\zeta(a)/\zeta(v_0) > 0$ for all $a \in A$. We can find a \mathbf{Q} -linear form $u : \mathbf{R} \rightarrow \mathbf{Q}$ such that $u(\zeta(v_0)) \neq 0$ and $u(\zeta(a))/u(\zeta(v_0)) > 0$ for all $a \in A$. Applying Case 1 to $u\zeta : \mathbf{C}^* \rightarrow \mathbf{Q}$ instead of ζ , we see that (iv) holds.

Conversely, assume that $\zeta(a)/\zeta(v_0) \leq 0$ for some $a \in A$. We can find a \mathbf{Q} -linear form $u : \mathbf{R} \rightarrow \mathbf{Q}$ such that $u(\zeta(v_0)) \neq 0$ and $u(\zeta(a))/u(\zeta(v_0)) \leq 0$. Applying Case 1 to $u\zeta : \mathbf{C}^* \rightarrow \mathbf{Q}$ instead of ζ , we see that (iv) does not hold. Thus, (d) holds. The theorem is proved.

Corollary 10.6. *Let v_0, ζ, M be as in 10.5.*

(a) *M is ζ -tempered if and only if the following holds: for any $t \in \mathcal{T}$ such that $M_t \neq 0$ and any $x \in \mathcal{L}^+$ we have $\zeta(x(t))/\zeta(v_0) \in \mathbf{Q}_{\geq 0}$.*

(b) *M is ζ -square integrable if and only if the following holds: for any $t \in \mathcal{T}$ such that $M_t \neq 0$ and any $x \in \mathcal{L}^+ - \{0\}$ we have $x(t) = av_0^n$ for some $n \in \mathbf{Z}_{>0}$ and some $a \in \mathbf{C}^*$, a root of 1.*

Assume that M is ζ -tempered and there exists $t \in \mathcal{T}$ and $x \in \mathcal{L}^+$ such that $M_t \neq 0$ and $\zeta(x(t))/\zeta(v_0) \notin \mathbf{Q}_{\geq 0}$. Note that $\zeta(x(t))/\zeta(v_0) \in \mathbf{R}_{>0}$. Since $\zeta(x(t)), \zeta(v_0)$ are nonzero real numbers of the same sign and one is not a rational multiple of the other, we can find a \mathbf{Q} -linear form $u : \mathbf{R} \rightarrow \mathbf{Q}$ such that $u(\zeta(v_0)) \neq 0$ and $u(\zeta(x(t)))/u(\zeta(v_0)) \in \mathbf{Q}_{<0}$. Hence M is not $u\zeta$ -tempered. Let f correspond to M as in 10.5. By 10.5(a) we have $f \in \hat{G}^{\text{Ker}\zeta}$. It follows that $f \in \hat{G}^{\text{Ker}(u\zeta)}$. Using again 10.5(a) (for $u\zeta$ instead of ζ) we see that M is $u\zeta$ -tempered, a contradiction. This proves (a).

We prove (b). By the arguments in the proof of 3.4 and 4.4, we are reduced to the analogous statement for the algebras considered in §5, which is proved in [L5, 1.22]. The corollary is proved.

10.7. In the setup of 9.1 (with $\text{card}(K) \geq 2$) we consider an element $f \in G\vartheta$ such that $(f, \rho) \in \mathfrak{R}(G\vartheta, G_J, \mathcal{C}, \mathcal{F})$ for some ρ . We can write uniquely $f = f_\diamond f'_\blacklozenge f_u$ (three commuting factors) where $f_\diamond \in \hat{G}_\diamond$ is semisimple, $f'_\blacklozenge \in G_\blacklozenge$ is semisimple, $f_u \in G$ is unipotent. Replacing (f, ρ) by a G -conjugate, we may assume that $f_\diamond \in X_J$ (see 9.5) so that $G_J \subset Z_G(f_\diamond)$. Let $t \in \mathcal{T}_\diamond$ be such that $\iota(\mathcal{W}t)$ is the \mathcal{N} -orbit of f_\diamond . (See 10.2.) Let \mathcal{W}_t be the stabilizer of t in \mathcal{W} . Let $\phi \in \text{Hom}(SL_2(\mathbf{C}), Z_G(f_\diamond f'_\blacklozenge))$ be such that $\phi\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) = f_u$. Let

$$\tilde{f} = f'_\blacklozenge \phi\left(\begin{smallmatrix} v_0 & 0 \\ 0 & v_0^{-1} \end{smallmatrix}\right) \in Z_G(f_\diamond).$$

Let $\phi^0 \in \text{Hom}(SL_2(\mathbf{C}), G_J)$ be such that $\phi^0\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \dot{\mathcal{C}}$. Using 5.6, we see that there exists $z \in Z_G(f_\diamond)$ such that

$$(a) \quad z\tilde{f}z^{-1}\phi^0\left(\begin{smallmatrix} v_0^{-1} & 0 \\ 0 & v_0 \end{smallmatrix}\right) \in (Z_J^0)_\blacklozenge$$

and that the orbit of the element (a) under the normalizer of Z_J^0 in $Z_G(f_\diamond)$ does not depend on the choice of z . Since $(Z_J^0)_\blacklozenge = \mathcal{T}_\blacklozenge$ (both may be identified with $(\mathfrak{z}_J)_\blacklozenge$ using $\exp_{Z_J^0}, \exp_{\mathcal{T}}$) we may regard this orbit as a \mathcal{W}_t -orbit c in $\mathcal{T}_\blacklozenge$. Let Σ be the \mathcal{W} -orbit in \mathcal{T} that contains tc . Then Σ depends only on the G -conjugacy class of (f, ρ) and $(f, \rho) \mapsto \Sigma$ is a map

$$(b) \quad \mathfrak{R}(G\vartheta, G_J, \mathcal{C}, \mathcal{F}) \rightarrow \mathcal{T}/\mathcal{W}.$$

We now give a second definition of the map (b). Consider (f, ρ) as above. We can write uniquely $f = f_s f_u$ (two commuting factors) where $f_s \in \hat{G}$ is semisimple,

$f_u \in G$ is unipotent. Replacing (f, ρ) by a G -conjugate, we may assume that $f_s \in X_J$ (see 9.5) so that $G_J \subset Z_G(f_s)$. Let $\phi \in \text{Hom}(SL_2(\mathbf{C}), Z_G(f_s))$ be such that $\phi\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) = f_u$. Let

$$\hat{f} = f_s \phi \begin{pmatrix} v_0 & 0 \\ 0 & v_0^{-1} \end{pmatrix} \in Z_G(f_s).$$

Let $\phi^0 \in \text{Hom}(SL_2(\mathbf{C}), G_J)$ be such that $\phi^0\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \mathcal{C}$. Using 5.6, we see that there exists $z \in Z_G(f_s)$ such that

$$a =: z \hat{f} z^{-1} \phi^0 \begin{pmatrix} v_0^{-1} & 0 \\ 0 & v_0 \end{pmatrix} \in X_J.$$

Let Σ be the \mathcal{W} -orbit in \mathcal{T} such that $\iota(\Sigma)$ is the \mathcal{N} -orbit of a (see 10.2). Then Σ depends only on the G -conjugacy class of (f, ρ) and $(f, \rho) \mapsto \Sigma$ coincides with the map (b). In the setup of 10.4, each $M \in \text{Irr}_{v_0} H_{\mathcal{R}, \mathcal{L}}^{\lambda, \lambda^*}$ belongs to $\text{Irr}_{\Sigma, v_0} H_{\mathcal{R}, \mathcal{L}}^{\lambda, \lambda^*}$ for a unique \mathcal{W} -orbit Σ in \mathcal{T} (as in 1.3(a)). Then $M \mapsto \Sigma$ is a well defined map

$$(c) \quad \text{Irr}_{v_0} H_{\mathcal{R}, \mathcal{L}}^{\lambda, \lambda^*} \rightarrow \mathcal{T}/\mathcal{W}.$$

Composing this with the bijection 10.4(a) we obtain a map $\mathfrak{R}(G\vartheta, G_J, \mathcal{C}, \mathcal{F}) \rightarrow \mathcal{T}/\mathcal{W}$. This coincides with the map (b) (in its first form) as one sees using the definitions.

The fact that the map (b) (in its second form) is independent of the choice of r_0, \diamond and that the same is obviously true for the map (c), suggests that the bijection 10.4(a) is also independent of r_0 .

10.8. We fix $v_0 \in \mathbf{C}^*$ which is either 1 or is not a root of 1. Let \mathfrak{J} be the set of all triples $(J, \mathcal{C}, \mathcal{F})$ as in 7.4. (Here \mathcal{F} is given up to isomorphism.) Putting together the bijections 10.4(a) for various $(J, \mathcal{C}, \mathcal{F}) \in \mathfrak{J}$, we obtain a bijection

$$(a) \quad \bigsqcup_{(J, \mathcal{C}, \mathcal{F}) \in \mathfrak{J}} \text{Irr}_{v_0} H(G\vartheta, G_J, \mathcal{C}, \mathcal{F}) \leftrightarrow \bigsqcup_{(J, \mathcal{C}, \mathcal{F}) \in \mathfrak{J}} \mathfrak{R}(G\vartheta, G_J, \mathcal{C}, \mathcal{F}).$$

Let $\mathfrak{R}(G\vartheta)$ be the set of all (f, ρ) (modulo the Ad-action of G) where $f \in G\vartheta$ and $\rho \in \text{Irr} \bar{Z}_G(f)$ (the set of isomorphism classes of irreducible representations of $\bar{Z}_G(f)$). We will show below that

$$(b) \quad \bigsqcup_{(J, \mathcal{C}, \mathcal{F}) \in \mathfrak{J}} \mathfrak{R}(G\vartheta, G_J, \mathcal{C}, \mathcal{F}) = \mathfrak{R}(G\vartheta).$$

Combined with (a), this gives a bijection

$$(c) \quad \bigsqcup_{(J, \mathcal{C}, \mathcal{F}) \in \mathfrak{J}} \text{Irr}_{v_0} H(G\vartheta, G_J, \mathcal{C}, \mathcal{F}) \leftrightarrow \mathfrak{R}(G\vartheta).$$

We prove (b). We fix $f \in G\vartheta$. We may assume that $f_\diamond \in p(C_S^\diamond)$ where $S \subset I, S \neq \emptyset$. Then $Z_G(f_\diamond) = G_{I-S}$. Let $\rho \in \text{Irr} \bar{Z}_G(f)$. We must show that there is a unique $(J, \mathcal{C}, \mathcal{F}) \in \mathfrak{J}$ such that ρ belongs to $\text{Irr}_1 \bar{Z}_G(f)$ (defined in terms of $G, G_J, \mathcal{C}, \mathcal{F}$) or equivalently (see 9.5(a)) to $\text{Irr}_0 \bar{Z}_{G_{I-S}}(f_\spadesuit)$ (defined in terms of $G_{I-S}, G_J, \mathcal{C}, \mathcal{F}$; we have necessarily $J \subset I - S$). Recall that $\bar{Z}_G(f) = \bar{Z}_{G_{I-S}}(f_\spadesuit)$. Define $\xi \in (\mathfrak{g}_{I-S})_\spadesuit$ by $\exp(\xi) = f_\spadesuit$. We are reduced to verifying the following statement:

For any $\rho \in \text{Irr} \bar{Z}_{G_{I-S}}(\xi)$ there exists a unique $(J, \mathcal{C}, \mathcal{F}) \in \mathfrak{J}$ such that $J \subset I - S$ and ρ is in $\text{Irr}_0 \bar{Z}_{G_{I-S}}(\xi)$ (defined in terms of $G_{I-S}, G_J, \mathcal{C}, \mathcal{F}$).

This follows from [L3, §8].

10.9. Let $\zeta : \mathbf{C}^* \rightarrow \mathbf{R}$ be a homomorphism such that $\zeta(v_0) \neq 0$. By 10.5, the bijection 10.8(c) restricts to a

bijection between the set of ζ -tempered representations in the left-hand side of 10.8(c) and $\{(f, \rho) \in \mathfrak{R}(G\vartheta); f \in \hat{G}^{\text{Ker}\zeta}\}$

and to a

bijection between the set of ζ -square integrable representations in the left-hand side of 10.8(c) and the set of all $(f, \rho) \in \mathfrak{R}(G\vartheta)$ such that any torus in $Z_G(f)$ is $\{1\}$.

Special cases of this result can be found in [KL], [R] and, [W].

10.10. Let \mathbf{K}, q be as in 1.1. Let $\tilde{\mathbf{K}}$ be a maximal unramified extension of \mathbf{K} . Let \mathbf{G} be a connected, adjoint simple algebraic group defined over \mathbf{K} which is split over $\tilde{\mathbf{K}}$. We identify \mathbf{G} with $\mathbf{G}(\tilde{\mathbf{K}})$. Assume that \mathbf{G} is of type dual in the sense of Langlands to G . Define \mathbf{I} as in [L4, 1.10]. This is the set of vertices of the affine Dynkin graph of \mathbf{G} . Let $\tilde{S}(\mathbf{I})$ be the set of bijections $\mathbf{I} \xrightarrow{\sim} \mathbf{I}$ that preserve the graph structure. We have a canonical (surjective) homomorphism from $\tilde{S}(\mathbf{I})$ to the group of automorphisms of G modulo the group of inner automorphisms of G (see [L4, 8.1]). Let $\tilde{S}(\mathbf{I})_\vartheta$ be the fiber of this map over the coset of $\text{Ad}(\vartheta) : G \rightarrow G$. For $u \in \tilde{S}(\mathbf{I})_\vartheta$ we can find a \mathbf{K} -rational structure on \mathbf{G} (compatible with the $\tilde{\mathbf{K}}$ -rational structure) with Frobenius map F_u (see [L4, 1.1]) such that the permutation of \mathbf{I} induced by F_u (as in [L4, 1.12]) is equal to u . (The existence of such a \mathbf{K} -rational structure, while not stated explicitly in [T], can be extracted from the tables in [T].) Then $\mathbf{G}(\mathbf{K})$ coincides with the fixed point set \mathbf{G}^{F_u} . Let $\mathcal{U}(\mathbf{G}^{F_u})$ be the set of isomorphism classes of unipotent representations of \mathbf{G}^{F_u} (see [L4, 1.21]).

Theorem 10.11. *There is a natural bijection $\bigsqcup_{u \in \tilde{S}(\mathbf{I})_\vartheta} \mathcal{U}(\mathbf{G}^{F_u}) \leftrightarrow \mathfrak{R}(G\vartheta)$.*

By [L4, 1.22] we have a natural bijection between $\bigsqcup_{u \in \tilde{S}(\mathbf{I})_\vartheta} \mathcal{U}(\mathbf{G}^{F_u})$ and the disjoint union of the sets of irreducible representations (up to isomorphism) of a finite collection of affine Hecke algebras $\mathcal{H}'(\mathbf{I}, \mathbf{J}, u, \mathbf{E})$ given by a presentation of Iwahori-Matsumoto type with explicitly known parameters and with the indeterminate v being specialized to \sqrt{q} . (Here $u \in \tilde{S}(\mathbf{I})_\vartheta$, \mathbf{J} is a proper u -stable subset of \mathbf{I} , \mathbf{E} is a unipotent cuspidal representation of the F_u -fixed points of the parahoric subgroup attached to \mathbf{J} .) The various $\mathbf{J}, u, \mathbf{E}$ are listed in the tables in §11 as “arithmetic diagrams”; the corresponding affine Hecke algebras $\mathcal{H}'(\mathbf{I}, \mathbf{J}, u, \mathbf{E})$ are listed in the same tables as “H.A.” Rather surprisingly, it turns out that these affine Hecke algebras are exactly the same as the geometric affine Hecke algebras attached to $G\vartheta$ (which are also described in the tables of §11). Therefore, the theorem follows from 10.8(c) with $v_0 = \sqrt{q}$.

10.12. For any homomorphism $\chi : Z_G \rightarrow \mathbf{C}^*$, let $\mathfrak{R}(G\vartheta)_\chi$ be the subset of $\mathfrak{R}(G\vartheta)$ consisting of all (f, ρ) such that via the obvious map $Z_G \rightarrow \bar{Z}_G(f)$, Z_G acts on ρ through the character χ . This gives us a partition $\mathfrak{R}(G\vartheta) = \bigsqcup_\chi \mathfrak{R}(G\vartheta)_\chi$. On the other hand, the bijection in 10.11 induces a partition of $\mathfrak{R}(G\vartheta)$ into subsets indexed by the elements of $\tilde{S}(\mathbf{I})_\vartheta$. This coincides with the previous partition of $\mathfrak{R}(G\vartheta)$.

10.13. Let $\mathbf{K}, q, \tilde{\mathbf{K}}, \mathbf{G}$ be as in 10.10 except that \mathbf{G} is no longer assumed to be split over $\tilde{\mathbf{K}}$. We identify \mathbf{G} with $\mathbf{G}(\tilde{\mathbf{K}})$. One can still define the set \mathbf{I} which indexes the maximal parahoric subgroups of \mathbf{G} (see [T]), a Dynkin graph with set of vertices \mathbf{I}

and a bijection $u : \mathbf{I} \rightarrow \mathbf{I}$ (preserving the graph structure, including the orientation of the double or triple edges) which specifies the \mathbf{K} -rational structure on \mathbf{G} . Then $\mathcal{U}(\mathbf{G}(\mathbf{K}))$ can be defined in the same way as in [L4, 1.21]. We can find another connected adjoint simple algebraic group \mathbf{G}' which is split over $\tilde{\mathbf{K}}$ whose associated $\mathbf{I}, u : \mathbf{I} \rightarrow \mathbf{I}$ (as in 10.10) is the same as the \mathbf{I}, u associated to \mathbf{G} and such that the corresponding Dynkin graph (for \mathbf{G}') is the same as that for \mathbf{G} , except possibly for the orientation of the double or triple edges. Then $\mathcal{U}(\mathbf{G}'(\mathbf{K}))$ is defined. Moreover, we have a natural bijection

$$(a) \quad \mathcal{U}(\mathbf{G}(\mathbf{K})) \leftrightarrow \mathcal{U}(\mathbf{G}'(\mathbf{K})).$$

Indeed, each side of (a) is naturally in bijection with the disjoint union of the sets of irreducible representations of a finite collection of affine Hecke algebras given by a presentation of Iwahori-Matsumoto type at $v = \sqrt{q}$. But the affine Hecke algebras associated to the two sides of (a) are the same, since the recipe that describes them is not sensitive to the orientation of the double or triple edges. (This is analogous to the known statement that the sets of unipotent representations of the finite groups $SO_{2n+1}(F_q)$ and $Sp_{2n}(F_q)$ are in bijection.) Since 10.11 and 10.12 are applicable to $\mathcal{U}(\mathbf{G}'(\mathbf{K}))$, they also provide, via (a), a parametrization of $\mathcal{U}(\mathbf{G}(\mathbf{K}))$.

We describe the various pairs $(\mathbf{G}, \mathbf{G}')$ using the names in the tables of [T].

$$(CB_n, C_n), (BC_n, B_n), (CBC_n, C_n), (G_2^I, G_2), (F_4^I, F_4), ({}^2BC_n, {}^2B_n), ({}^2CB_n, {}^2C_n).$$

10.14. It is likely that our results can be extended to the case where the assumption that \mathbf{G} is adjoint simple is weakened to the assumption that \mathbf{G} is semisimple. Indeed, our main technique, that of reducing to the case of graded Hecke algebras is still available in this more general case (see [L2]).

11. TABLES

11.1. In this section we list the various possibilities for $G\vartheta$ and J as in 7.4 assuming that $d \geq 2$. (The cases where $d = 1$ are listed in [L4].) In each case we describe the affine Dynkin graph associated to the affine Cartan matrix (a_{i_1, i_2}) (resp. (a_{i_1, i_2})) in 6.21; we call this the (γ_i) -graph (resp. the (β_i) -graph). Both these graphs have vertices in bijection with I . The vertices of the (β_i) -graph that are inside a box correspond to the subset J of I . The full subgraph with vertices J is the Dynkin graph of G_J (see 6.23). We also describe the affine Dynkin graph associated to the affine Cartan matrix $(\hat{\gamma}_k(\hat{h}_{k'}))_{k, k' \in K}$. We call this the $\flat - \sharp$ diagram; its vertices are in bijection with K and we attach to any vertex the symbol \flat or \sharp according to whether the corresponding element of K is in K^\flat or K^\sharp . For any vertex corresponding to $k \in K$ we specify some data of the form $a \times b \times c$ where $a = \underline{c}_k, b = \bar{z}_k, c = d_k$. From the $\flat - \sharp$ diagram one obtains an affine Hecke algebra as in 8.2, 8.3. This affine Hecke algebra (in a presentation of Iwahori-Matsumoto type) is denoted by H.A. It turns out to be the same as the affine Hecke algebra attached to the arithmetic diagram (see 10.11) which is also given in each case. The notation for affine Hecke algebras follows the conventions of [L4, 6.9, 6.11].

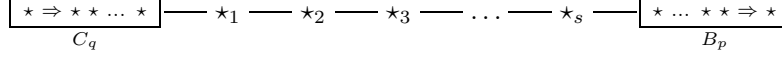
11.2. G is of type A_n , n even, $d = 2$.

$$a \in 4\mathbf{Z}, b \in 1 + 4\mathbf{Z}, n + 1 = 2s - 2 + a(a + 1)/2 + b(b + 1)/2, s \geq 1.$$

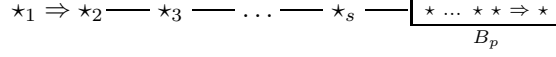
(γ_i) -graph:

$$\star \Rightarrow \star \text{---} \star \text{---} \star \text{---} \star \text{---} \star \text{---} \dots \text{---} \star \text{---} \star \text{---} \star \Leftarrow \star$$

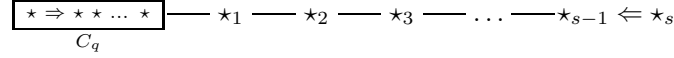
(β_i) -graph:



if $a - b \neq -1, |a + b + 1| \neq 2$; here $2p + 1 = (a + b + 1)^2/4, 2q = (a - b - 1)(a - b + 1)/4$;



if $a - b = -1, |a + b + 1| \neq 2$; here $2p + 1 = (a + b + 1)^2/4$;



if $a - b \neq -1, |a + b + 1| = 2$; here $2q = (a - b - 1)(a - b + 1)/4$;



if $a - b = -1, |a + b + 1| = 2$.

For $s \geq 2$, the $\mathfrak{b} - \mathfrak{h}$ -diagram is

$$\mathfrak{b}_s^{a+b+1 \times 1 \times 2} \Leftarrow \mathfrak{h}_{s-1}^{2 \times 1 \times 2} \text{ --- } \mathfrak{h}_{s-2}^{2 \times 1 \times 2} \text{ --- } \dots \text{ --- } \mathfrak{h}_2^{2 \times 1 \times 2} \Rightarrow \mathfrak{b}_1^{a-b \times 1 \times 2}$$

if $a - b \neq -1, |a + b + 1| \neq 2$;

$$\mathfrak{b}_s^{a+b+1 \times 1 \times 2} \Leftarrow \mathfrak{h}_{s-1}^{2 \times 1 \times 2} \text{ --- } \mathfrak{h}_{s-2}^{2 \times 1 \times 2} \text{ --- } \dots \text{ --- } \mathfrak{h}_2^{2 \times 1 \times 2} \Rightarrow \mathfrak{b}_1^{2 \times \frac{1}{2} \times 2}$$

if $a - b = -1, |a + b + 1| \neq 2$;

$$\mathfrak{b}_s^{2 \times \frac{1}{2} \times 4} \Leftarrow \mathfrak{h}_{s-1}^{2 \times 1 \times 2} \text{ --- } \mathfrak{h}_{s-2}^{2 \times 1 \times 2} \text{ --- } \dots \text{ --- } \mathfrak{h}_2^{2 \times 1 \times 2} \Rightarrow \mathfrak{b}_1^{a-b \times 1 \times 2}$$

if $a - b \neq -1, |a + b + 1| = 2$;

$$\mathfrak{b}_s^{2 \times \frac{1}{2} \times 4} \Leftarrow \mathfrak{h}_{s-1}^{2 \times 1 \times 2} \text{ --- } \mathfrak{h}_{s-2}^{2 \times 1 \times 2} \text{ --- } \dots \text{ --- } \mathfrak{h}_2^{2 \times 1 \times 2} \Rightarrow \mathfrak{b}_1^{2 \times \frac{1}{2} \times 2}$$

if $a - b = -1, |a + b + 1| = 2$.

For $s = 1$, the $\mathfrak{b} - \mathfrak{h}$ -diagram is \emptyset .

H.A.: $\tilde{C}_{s-1}^{sc}[\lfloor 2a+1 \rfloor \lfloor 2b+1 \rfloor]$ if $s \geq 2$ and \emptyset if $s = 1$.

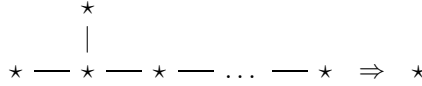
Arithmetic diagram: $\tilde{A}_n, u^2 = 1, u \neq 1, \mathbf{J}$ of type $A_{p'-1} \times A_{q'-1}$ (both components are u -stable), $p' = a(a+1)/2, q' = b(b+1)/2$.

11.3. G is of type A_n, n odd, $d = 2$.

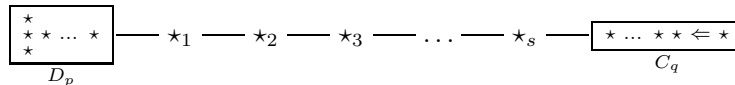
Either $a \in 4\mathbf{Z}, b \in 3 + 4\mathbf{Z}$ or $a \in 2 + 4\mathbf{Z}, b \in 1 + 4\mathbf{Z}$.

$n + 1 = 2s - 2 + a(a+1)/2 + b(b+1)/2, s \geq 1$.

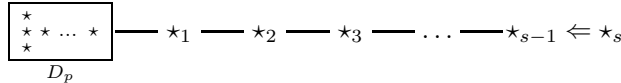
(γ_i) -graph:



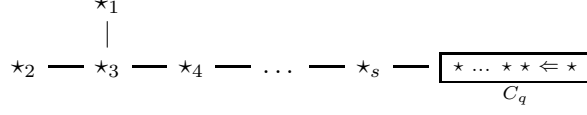
(β_i) -graph:



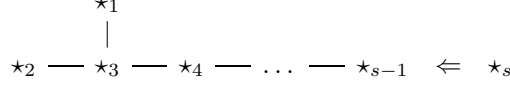
if $a - b \neq 1, a + b \neq -1$; here $2p = (a + b + 1)^2/4, 2q = (a - b - 1)(a - b + 1)/4$;



if $a - b = 1$, $a + b \neq -1$; here $2p = (a + b + 1)^2/4$;



if $a - b \neq 1$, $a + b = -1$; here $2q = (a - b - 1)(a - b + 1)/4$;



if $a - b = 1$, $a + b = -1$.

For $s \geq 2$, the $b - \#$ -diagram is

$$b_s^{a+b+1 \times 1 \times 2} \Leftarrow \#_{s-1}^{2 \times 1 \times 2} \text{ --- } \#_{s-2}^{2 \times 1 \times 2} \text{ --- } \dots \text{ --- } \#_2^{2 \times 1 \times 2} \Rightarrow b_1^{a-b \times 1 \times 2}$$

if $a - b \neq 1$, $a + b \neq -1$;

$$b_1^{a+b+1 \times 1 \times 2} \Leftarrow \#_2^{2 \times 1 \times 2} \text{ --- } \#_3^{2 \times 1 \times 2} \text{ --- } \dots \text{ --- } \#_{s-1}^{2 \times 1 \times 2} \Rightarrow b_s^{2 \times 1 \times 1}$$

if $a - b = 1$, $a + b \neq -1$;

$$\begin{array}{c} \#_s^{a-b \times 1 \times 2} \Leftarrow \#_{s-1}^{2 \times 1 \times 2} \text{ --- } \#_{s-2}^{2 \times 1 \times 2} \text{ --- } \dots \text{ --- } \#_3^{2 \times 1 \times 2} \text{ --- } \#_2^{2 \times 1 \times 2} \\ \#_1^{2 \times 1 \times 2} \end{array}$$

if $a - b \neq 1$, $a + b = -1$;

$$\begin{array}{c} \#_s^{2 \times 1 \times 1} \Leftarrow \#_{s-1}^{2 \times 1 \times 2} \text{ --- } \#_{s-2}^{2 \times 1 \times 2} \text{ --- } \dots \text{ --- } \#_3^{2 \times 1 \times 2} \text{ --- } \#_2^{2 \times 1 \times 2} \\ \#_1^{2 \times 1 \times 2} \end{array}$$

if $a - b = 1$, $a + b = -1$.

For $s = 1$, the $b - \#$ -diagram is \emptyset .

H.A.: $\tilde{C}_{s-1}^{sc} [{}_{|2a+1|}2{}_{|2b+1|}]$ if $s \geq 2$ and $a + b \neq -1$; $\tilde{C}_{s-1} [{}_{|2a+1|}2{}_{|2b+1|}]$ if $s \geq 2$ and $a + b = -1$; \emptyset if $s = 1$.

Arithmetic diagram: \tilde{A}_n , $u^2 = 1$, $u \neq 1$, \mathbf{J} of type $A_{p'-1} \times A_{q'-1}$ (both components are u -stable), $p' = a(a + 1)/2$, $q' = b(b + 1)/2$.

11.4. G is of type D_n , $d = 2$.

$a \geq 1$ odd; $b \geq 0$ even; $n + 1 = s + a^2 + b^2$, $s \geq 1$.

(γ_i) -graph:

$$\star \Rightarrow \star \text{ --- } \star \text{ --- } \star \text{ --- } \star \text{ --- } \star \text{ --- } \dots \text{ --- } \star \text{ --- } \star \text{ --- } \star \Leftarrow \star$$

(β_i) -graph:

$$\boxed{\star \Leftarrow \star \star \dots \star} \text{ --- } \star_1 \text{ --- } \star_2 \text{ --- } \star_3 \text{ --- } \dots \text{ --- } \star_s \text{ --- } \boxed{\star \dots \star \star \Rightarrow \star} \\ B_p \qquad \qquad \qquad B_q$$

if $a + b \neq 1$, $|a - b| \neq 1$; here $2p + 1 = (a + b)^2$, $2q + 1 = (a - b)^2$;

$$\boxed{\star \Leftarrow \star \star \dots \star} \text{ --- } \star_1 \text{ --- } \star_2 \text{ --- } \star_3 \text{ --- } \dots \text{ --- } \star_{s-1} \Rightarrow \star_s \\ B_p$$

if $a + b \neq 1$, $|a - b| = 1$; here $2p + 1 = (a + b)^2$;

$$\star_1 \Leftarrow \star_2 \text{ --- } \star_3 \text{ --- } \dots \text{ --- } \star_{s-1} \Rightarrow \star_s$$

if $a + b = 1, |a - b| = 1$.

For $s \geq 2$, the $\mathfrak{b} - \#$ -diagram is

$$\mathfrak{b}_1^{2(a+b) \times 1 \times 1} \Leftarrow \#_2^{2 \times 1 \times 1} \text{---} \#_3^{2 \times 1 \times 1} \text{---} \dots \text{---} \#_{s-1}^{2 \times 1 \times 1} \Rightarrow \mathfrak{b}_s^{2|a-b| \times 1 \times 1}$$

if $a + b \neq 1, |a - b| \neq 1$;

$$\mathfrak{b}_1^{2(a+b) \times 1 \times 1} \Leftarrow \#_2^{2 \times 1 \times 1} \text{---} \#_3^{2 \times 1 \times 1} \text{---} \dots \text{---} \#_{s-1}^{2 \times 1 \times 1} \Rightarrow \mathfrak{b}_s^{2 \times \frac{1}{2} \times 2}$$

if $a + b \neq 1, |a - b| = 1$;

$$\mathfrak{b}_s^{2 \times \frac{1}{2} \times 2} \Leftarrow \#_{s-1}^{2 \times 1 \times 1} \text{---} \#_{s-2}^{2 \times 1 \times 1} \text{---} \dots \text{---} \#_2^{2 \times 1 \times 1} \Rightarrow \mathfrak{b}_1^{2 \times \frac{1}{2} \times 2}$$

if $a + b = 1, |a - b| = 1$.

For $s = 1$, the $\mathfrak{b} - \#$ -diagram is \emptyset .

H.A.: $\tilde{C}_{s-1}^{sc}[2a12b]$ if $s \geq 2$ and \emptyset if $s = 1$.

Arithmetic diagram: \tilde{D}_n , $u : \mathbf{I} \rightarrow \mathbf{I}$ has exactly $n - 1$ fixed points, \mathbf{J} of type $D_{p'} \times D_{q'}$ (u acts nontrivially on $D_{p'}$), $p' = a^2, q' = b^2$.

11.5. G is of type D_n , $d = 2$.

$a \geq 0, b \geq 0, a = n + 1 \pmod{2}, (b^2 + b)/2 = n \pmod{2}$;

$n + 1 = 2a^2 + (b^2 + b)/2 - 1 + 2s, s \geq 1$.

(γ_i) -graph:

$$\star \Rightarrow \star \text{---} \star \text{---} \star \text{---} \star \text{---} \star \text{---} \dots \text{---} \star \text{---} \star \text{---} \star \Leftarrow \star$$

(β_i) -graph:

$$\boxed{\star \Leftarrow \star \star \dots \star} \text{---} \star_1 \text{---} \boxed{\star} \text{---} \star_2 \text{---} \boxed{\star} \text{---} \dots \text{---} \boxed{\star} \text{---} \star_s \text{---} \boxed{\star \dots \star \star \Rightarrow \star} \\ B_p \qquad \qquad \qquad B_q$$

if $2a + b \neq 1, |4a - 2b - 1| \neq 3$; here

$$2p + 1 = (2a + b)(2a + b + 1)/2, 2q + 1 = (2a - b)(2a - b - 1)/2;$$

$$\boxed{\star \Leftarrow \star \star \dots \star} \text{---} \star_1 \text{---} \boxed{\star} \text{---} \star_2 \text{---} \boxed{\star} \text{---} \dots \text{---} \boxed{\star} \Rightarrow \star_s \\ B_p$$

if $2a + b \neq 1, |4a - 2b - 1| = 3$; here $2p + 1 = (2a + b)(2a + b + 1)/2$;

$$\star_1 \Leftarrow \boxed{\star} \text{---} \star_2 \text{---} \boxed{\star} \text{---} \dots \text{---} \boxed{\star} \Rightarrow \star_s$$

if $2a + b = 1, 2a - b = -1$.

For $s \geq 2$, the $\mathfrak{b} - \#$ -diagram is

$$\mathfrak{b}_1^{(4a+2b+1) \times 1 \times 1} \Leftarrow \#_2^{4 \times 1 \times 1} \text{---} \#_3^{4 \times 1 \times 1} \text{---} \dots \text{---} \#_{s-1}^{4 \times 1 \times 1} \Rightarrow \mathfrak{b}_s^{|4a-2b-1| \times \frac{1}{2} \times 2}$$

if $2a + b \neq 1, |4a - 2b - 1| \neq 3$;

$$\mathfrak{b}_1^{(4a+2b+1) \times 1 \times 1} \Leftarrow \#_2^{4 \times 1 \times 1} \text{---} \#_3^{4 \times 1 \times 1} \text{---} \dots \text{---} \#_{s-1}^{4 \times 1 \times 1} \Rightarrow \mathfrak{b}_s^{3 \times \frac{1}{2} \times 2}$$

if $2a + b \neq 1, |4a - 2b - 1| = 3$;

$$\mathfrak{b}_1^{3 \times \frac{1}{2} \times 2} \Leftarrow \#_2^{4 \times 1 \times 1} \text{---} \#_3^{4 \times 1 \times 1} \text{---} \dots \text{---} \#_{s-1}^{4 \times 1 \times 1} \Rightarrow \mathfrak{b}_s^{3 \times \frac{1}{2} \times 2}$$

if $2a + b = 1, 2a - b = -1$.

For $s = 1$, the $\mathfrak{b} - \#$ -diagram is \emptyset .

H.A.: $\tilde{C}_{s-1}^{sc}[4a22b+1]$ if $s \geq 2$ and \emptyset if $s = 1$.

Arithmetic diagram: \tilde{D}_n , $u : \mathbf{I} \rightarrow \mathbf{I}$ has $< n - 1$ fixed points, \mathbf{J} of type $D_{p'} \times D_{p'} \times A_{r-1}$ where $p' = a^2, r = (b^2 + b)/2$.

11.6. G is of type E_6 , $d = 2$. (2 possible cuspidal local systems.)
 (γ_i) -graph:

$$\star \text{---} \star \text{---} \star \Rightarrow \star \text{---} \star$$

(β_i) -graph:

$$\boxed{\star \text{---} \star} \text{---} \star_1 \Leftarrow \boxed{\star \text{---} \star}$$

H.A.: \emptyset .

Arithmetic diagram: \tilde{E}_6 , $u^2 = 1, u \neq 1$, \mathbf{J} of type E_6 (with two possible unipotent cuspidal representations, one the dual of the other).

11.7. G is of type E_6 , $d = 2$.
 (γ_i) -graph:

$$\star \text{---} \star \text{---} \star \Rightarrow \star \text{---} \star$$

(β_i) -graph:

$$\star_1 \text{---} \boxed{\star \text{---} \star \Leftarrow \star \text{---} \star}$$

H.A.: \emptyset .

Arithmetic diagram: \tilde{E}_6 , $u^2 = 1, u \neq 1$, \mathbf{J} of type E_6 (with a self-dual unipotent cuspidal representation).

11.8. G is of type E_6 , $d = 2$.
 (γ_i) -graph:

$$\star \text{---} \star \text{---} \star \Rightarrow \star \text{---} \star$$

(β_i) -graph:

$$\boxed{\star} \text{---} \star_1 \text{---} \boxed{\star} \Leftarrow \star_2 \text{---} \boxed{\star}$$

$\mathfrak{b} - \sharp$ -diagram:

$$\mathfrak{b}_1^{4 \times 1 \times 2} \xrightarrow{\infty} \mathfrak{b}_2^{5 \times 2 \times 2}$$

H.A.: $\mathbf{1} \xrightarrow{\infty} \mathbf{9}$.

Arithmetic diagram: \tilde{E}_6 , $u^2 = 1, u \neq 1$, \mathbf{J} of type A_5 .

11.9. G is of type E_6 , $d = 2$.
 (γ_i) -graph:

$$\star \text{---} \star \text{---} \star \Rightarrow \star \text{---} \star$$

(β_i) -graph:

$$\star_1 \text{---} \star_2 \text{---} \star_3 \Leftarrow \star_4 \text{---} \star_5$$

$\mathfrak{b} - \sharp$ -diagram:

$$\sharp_1^{2 \times 1 \times 2} \text{---} \sharp_2^{2 \times 1 \times 2} \text{---} \sharp_3^{2 \times 1 \times 2} \Rightarrow \sharp_4^{2 \times 1 \times 1} \text{---} \sharp_5^{2 \times 1 \times 1}$$

H.A.:

$$\mathbf{2} \text{---} \mathbf{2} \Leftarrow \mathbf{1} \text{---} \mathbf{1} \text{---} \mathbf{1}$$

Arithmetic diagram: \tilde{E}_6 , $u^2 = 1, u \neq 1$, $\mathbf{J} = \emptyset$.

11.10. G is of type D_4 , $d = 3$.

(γ_i) -graph:

$$\star \longrightarrow \star \equiv > \star$$

(β_i) -graph:

$$\star_1 \longrightarrow \boxed{\star < \equiv \star}$$

H.A.: \emptyset .

Arithmetic diagram: \tilde{D}_4 , $u^3 = 1, u \neq 1$, \mathbf{J} of type D_4 .

11.11. G is of type D_4 , $d = 3$.

(γ_i) -graph:

$$\star \longrightarrow \star \equiv > \star$$

(β_i) -graph:

$$\boxed{\star} \longrightarrow \star_1 < \equiv \boxed{\star}$$

H.A.: \emptyset .

Arithmetic diagram: \tilde{D}_4 , $u^3 = 1, u \neq 1$, \mathbf{J} of type D_4 with a unipotent cuspidal other than that in 11.10.

11.12. G is of type D_4 , $d = 3$.

(γ_i) -graph:

$$\star \longrightarrow \star \equiv > \star$$

(β_i) -graph:

$$\star_1 \longrightarrow \star_2 < \equiv \star_3$$

$\mathfrak{b} - \sharp$ -diagram:

$$\sharp_1^{2 \times 1 \times 3} \longrightarrow \sharp_2^{2 \times 1 \times 3} \equiv > \sharp^{2 \times 1 \times 1}$$

H.A.: $\mathbf{3} < \equiv \mathbf{1} \longrightarrow \mathbf{1}$.

Arithmetic diagram: \tilde{D}_4 , $u^3 = 1, u \neq 1$, $\mathbf{J} = \emptyset$.

APPENDIX. PROOF OF LEMMA 5.5

A.1. We may assume that \mathfrak{g} is simple. If $\underline{L} = \mathfrak{g}$, there is nothing to prove. If \underline{L} is a Cartan subalgebra, then $h^0 = 0$ and there is nothing to prove. In the rest of the proof we assume that $\underline{L} \neq \mathfrak{g}$ and \underline{L} is not a Cartan subalgebra.

Since $\mathfrak{t} \oplus \mathbf{C}h^0$ is a Cartan subalgebra of $\tilde{\mathbf{Z}}$, it is enough to prove the following statement:

(a) *Let $x, x' \in \mathfrak{t}, z \in \mathbf{C}$ be such that $x + zh^0, x' + zh^0$ are G -conjugate in \mathfrak{g} . Then x, x' are in the same W -orbit.*

If (a) holds for $z = 1$, then it also holds for any $z \neq 0$. (We replace x, x', z by $z^{-1}x, z^{-1}x', 1$.) Thus it is enough to prove (a) for $z \in \{0, 1\}$.

Let $\mathfrak{S} = \{x + ah^0 \in \mathfrak{t} + \mathbf{C}h^0; \alpha(x) \geq 0 \quad \forall \alpha \in \Pi\}$. As in 6.6 we see that \mathfrak{S} is a fundamental domain for the action of W on $\mathfrak{t} \oplus \mathbf{C}h^0$. Hence it is enough to prove the following statement:

Let $x, x' \in \mathfrak{t}, z \in \{0, 1\}$ be such that $x + zh^0, x' + zh^0$ belong to \mathfrak{S} and are G -conjugate in \mathfrak{g} . Then $x = x'$.

We consider the various cases separately.

For a multiset X consisting of finitely many numbers in \mathbf{C} we denote by $\max X$ the complex number $x \in X$ such that $x - x' \geq 0$ for any $x' \in X$.

A.2. Assume that $\mathfrak{g} = \mathfrak{sl}_{ab}(\mathbf{C})$, $\underline{L} = \mathfrak{sl}_a(\mathbf{C})^b \oplus \mathbf{C}^{b-1}$. Here $a > 1, b > 1$. Let V be a \mathbf{C} -vector space with basis e_1, e_2, \dots, e_{ab} . We may assume that $\mathfrak{g} = \mathfrak{sl}(V)$. We may assume that

$$x(e_{ai+l}) = x_i e_{ai+l}, x'(e_{ai+l}) = x'_i e_{ai+l} \quad \text{for } i \in [0, b-1], l \in [0, a-1]$$

where $x_i, x'_i \in \mathbf{C}$ satisfy $\sum_i x_i = \sum_i x'_i = 0$ and $x_i - x_{i+1} \geq 0, x'_i - x'_{i+1} \geq 0$ for $i \in [0, b-2]$ and that

$$h^0(e_{ai+l}) = (a-1-2l)e_{ai+l} \quad \text{for } i \in [0, b-1], l \in [0, a-1].$$

Since $x + zh^0, x' + xh^0$ are conjugate under $SL(V)$, they must have the same eigenvalues in V . Thus, the multisets

$$\begin{aligned} X &= \{x_i + z(a-1-2l)\}_{i \in [0, b-1], l \in [0, a-1]}, \\ X' &= \{x'_i + z(a-1-2l)\}_{i \in [0, b-1], l \in [0, a-1]} \end{aligned}$$

coincide. Clearly, $\max X = x_0 + z(a-1)$ and $\max X' = x'_0 + z(a-1)$. Since $X = X'$ we have $x_0 + z(a-1) = x'_0 + z(a-1)$. Hence $x_0 = x'_0$. Removing

$$x_0 + z(a-1), x_0 + z(a-3), \dots, x_0 + z(-a+1)$$

(resp. $x'_0 + z(a-1), x'_0 + z(a-3), \dots, x'_0 + z(-a+1)$) from X (resp. X') we obtain a multiset X_1 (resp. X'_1). We have $X_1 = X'_1$. Clearly, $\max X_1 = x_1 + z(a-1)$ and $\max X'_1 = x'_1 + z(a-1)$. Since $X_1 = X'_1$ we have $x_1 + z(a-1) = x'_1 + z(a-1)$. Hence $x_1 = x'_1$. Continuing in this way we find $x_i = x'_i$ for $i \in [0, b-1]$. Hence $x = x'$.

A.3. Assume that $\mathfrak{g} = \mathfrak{sp}_{2n+2p}(\mathbf{C})$, $\underline{L} = \mathfrak{sp}_{2n}(\mathbf{C}) \oplus \mathbf{C}^p$. Here $n > 1, p > 1$ and $n = (m^2 + m)/2$.

Let V be a \mathbf{C} -vector space with basis $e_1, e_2, \dots, e_{n+p}, e'_{n+p}, \dots, e'_2, e'_1$ and with a symplectic form $(\cdot, \cdot) : V \times V \rightarrow \mathbf{C}$ such that $(e_i, e'_j) = \delta_{ij}, (e_i, e_j) = (e'_i, e'_j) = 0$ for $i, j \in [1, n+p]$. We may assume that $\mathfrak{g} = \mathfrak{sp}(V)$ and that

$$\begin{aligned} x(e_i) &= x_i e_i, x(e'_i) = -x_i e'_i, x'(e_i) = x'_i e_i, x'(e'_i) = -x'_i e'_i \quad \text{for } i \in [1, p], \\ x(e_i) &= 0, x(e'_i) = 0, x'(e_i) = 0, x'(e'_i) = 0 \quad \text{for } i \in [p+1, p+n], \end{aligned}$$

where $x_i, x'_i \in \mathbf{C}$ satisfy $x_i - x_{i+1} \geq 0, x'_i - x'_{i+1} \geq 0$ for $i \in [1, p-1]$, $x_p \geq 0, x'_p \geq 0$. We may also assume that $h^0(e_i) = 0, h^0(e'_i) = 0$ for $i \in [1, p]$,

$$h^0(e_i) = c_i e_i, h^0(e'_i) = -c_i e'_i \quad \text{for } i \in [p+1, p+n]$$

where $c_i \in \mathbf{Z}$. Since $x + zh^0, x' + xh^0$ are conjugate under $Sp(V)$, they must have the same eigenvalues in V . Thus, the multisets

$$\begin{aligned} Y &= \{x_i, -x_i (i \in [1, p]), c_i, -c_i (i \in [p+1, p+n])\}, \\ Y' &= \{x'_i, -x'_i (i \in [1, p]), c_i, -c_i (i \in [p+1, p+n])\} \end{aligned}$$

coincide. Removing $\{c_i, -c_i (i \in [p+1, p+n])\}$ from Y (resp. Y') we obtain a multiset X (resp. X'). We have $X = X'$. Clearly, $\max X = x_1$ and $\max X' = x'_1$. Since $X = X'$ we have $x_1 = x'_1$. Removing x_1 (resp. x'_1) from X (resp. X') we obtain a multiset X_1 (resp. X'_1). We have $X_1 = X'_1$. Clearly, $\max X_1 = x_2$ and $\max X'_1 = x'_2$. Since $X_1 = X'_1$ we have $x_2 = x'_2$. Continuing in this way we find $x_i = x'_i$ for $i \in [1, p]$. Hence $x = x'$.

A.4. Assume that $\mathfrak{g} = \mathfrak{so}_{n+2p}(\mathbf{C})$, $\underline{L} = \mathfrak{so}_n(\mathbf{C}) \oplus \mathbf{C}^p$. Here $n > 2, p > 1$ and $n = m^2$. This case is completely similar to that in A.3.

A.5. Assume that $\mathfrak{g} = \mathfrak{so}_{2n+4p}(\mathbf{C})$, $\underline{L} = \mathfrak{so}_{2n}(\mathbf{C}) \oplus \mathfrak{sl}_2^p \oplus \mathbf{C}^p$. Here $n > 0, p > 0$ and $2n = (m^2 + m)/2$.

Let V be a \mathbf{C} -vector space with basis $e_1, e_2, \dots, e_{n+2p}, e'_{n+2p}, \dots, e'_2, e'_1$ and with a symmetric bilinear form $(,) : V \times V \rightarrow \mathbf{C}$ such that $(e_i, e'_j) = \delta_{ij}$, $(e_i, e_j) = (e'_i, e'_j) = 0$ for $i, j \in [1, n+2p]$. We may assume that $\mathfrak{g} = \mathfrak{so}(V)$ and that

$$\begin{aligned} x(e_{2i-1}) &= x_i e_{2i-1}, x(e_{2i}) = x_i e_{2i-1}, x(e'_{2i-1}) = -x_i e_{2i-1}, x(e'_{2i}) = -x_i e_{2i-1}, \\ x'(e_{2i-1}) &= x'_i e_{2i-1}, x'(e_{2i}) = x'_i e'_{2i-1}, x'(e'_{2i-1}) = -x'_i e_{2i-1}, x'(e'_{2i}) = -x'_i e'_{2i-1} \end{aligned}$$

for $i \in [1, p]$,

$$x(e_i) = 0, x(e'_i) = 0, x'(e_i) = 0, x'(e'_i) = 0 \quad \text{for } i \in [2p+1, 2p+n],$$

where $x_i, x'_i \in \mathbf{C}$ satisfy

$$x_i - x_{i+1} \geq 0, x'_i - x'_{i+1} \geq 0 \quad \text{for } i \in [1, p-1], \quad x_p \geq 0, x'_p \geq 0.$$

We may also assume that

$$h^0(e_{2i-1}) = e_{2i-1}, h^0(e_{2i}) = -e_{2i}, h^0(e'_{2i-1}) = -e_{2i-1}, h^0(e'_{2i}) = e_{2i}$$

for $i \in [1, p]$ and

$$h^0(e_i) = c_i e_i, h^0(e'_i) = -c_i e_i \quad \text{for } i \in [2p+1, 2p+n]$$

where $c_i \in \mathbf{Z}$. Since $x + zh^0, x' + xh^0$ are conjugate under $SO(V)$, they must have the same eigenvalues in V . Thus, the multisets

$$Y = \{x_i + z, x_i - z, -x_i + z, -x_i - z (i \in [1, p]), c_i, -c_i (i \in [2p+1, 2p+n])\},$$

$$Y' = \{x'_i + z, x'_i - z, -x'_i + z, -x'_i - z (i \in [1, p]), c_i, -c_i (i \in [2p+1, 2p+n])\}$$

coincide. Removing $\{c_i, -c_i (i \in [2p+1, 2p+n])\}$ from Y (resp. Y') we obtain a multiset X (resp. X'). We have $X = X'$.

Clearly, $\max X = x_1 + z$ and $\max X' = x'_1 + z$. Since $X = X'$ we have $x_1 + z = x'_1 + z$, hence $x_1 = x'_1$. Removing $x_1 + z, x_1 - z, -x_1 + z, -x_1 - z$ (resp. $x'_1 + z, x'_1 - z, -x'_1 + z, -x'_1 - z$) from X (resp. X') we obtain a multiset X_1 (resp. X'_1). We have $X_1 = X'_1$. Clearly, $\max X_1 = x_2 + z$ and $\max X'_1 = x'_2 + z$. Since $X_1 = X'_1$ we have $x_2 + z = x'_2 + z$, hence $x_2 = x'_2$. Continuing in this way we find $x_i = x'_i$ for $i \in [1, p]$. Hence $x = x'$.

A.6. Assume that $\mathfrak{g} = \mathfrak{so}_{2n+1+4p}(\mathbf{C})$, $\underline{L} = \mathfrak{so}_{2n+1}(\mathbf{C}) \oplus \mathfrak{sl}_2^p \oplus \mathbf{C}^p$. Here $n \geq 0, p > 0$, $2n+1 = (m^2 + m)/2$. This case is completely similar to that in A.5.

A.7. Assume that \mathfrak{g} is of type E_6 and $\underline{L} \cong \mathfrak{sl}_3(\mathbf{C})^3 \oplus \mathbf{C}^2$. We number the vertices of the Coxeter diagram by $1, 2, \dots, 6$ where the edges are $1-2-3-4-5$ and $3-6$. For $i \in [1, 6]$ define $u_i \in \mathfrak{t}$ by $\alpha_j(u_i) = \delta_{ij}$ for all $j \in [1, 6]$. Then \mathfrak{t} is spanned by u_3, u_6 . We may assume that $h^0 = 2\check{\alpha}_1 + 2\check{\alpha}_2 + 2\check{\alpha}_4 + 2\check{\alpha}_5$.

We have $x = au_3 + bu_6$, $x' = a'u_3 + b'u_6$ where $a, b, a', b' \in \mathbf{C}$ are ≥ 0 . Let Y be the multiset consisting of the numbers

$$\begin{aligned} 2a + b + 2z, 2a + b, 2a + b - 2z, a + b + 2z, \\ a + b, a + b - 2z, a + 2z, a, a - 2z, 4z, 2z, 2z, \end{aligned}$$

their negatives, and of $0, 0, 0$. Let Y' be the multiset consisting of the numbers

$$\begin{aligned} 2a' + b' + 2z, 2a' + b', 2a' + b' - 2z, a' + b' + 2z, a' + b', a' + b' - 2z, \\ a' + 2z, a', a' - 2z, 4z, 2z, 2z, \end{aligned}$$

their negatives, and of $0, 0, 0$. The eigenvalues of $x + zh^0$ (resp. $x' + zh^0$) on a minuscule \mathfrak{g} -module V are the 27 numbers in the multiset Y (resp. Y'). Since $x + zh^0, x' + zh^0$ are in the same G -orbit, they have the same eigenvalues on V . Thus, $Y = Y'$.

Removing $4z, 2z, 2z$ from Y (resp. Y') we obtain a multiset X (resp. X') with 24 elements. We have $X = X'$. Clearly, $\max X = 2a + b + 2z$ and $\max X' = 2a' + b' + 2z$. It follows that

$$(a) \quad 2a + b + 2z = 2a' + b' + 2z.$$

Removing from X (resp. X') the numbers $2a + b + 2z, 2a + b, 2a + b - 2z$ (resp. $2a' + b' + 2z, 2a' + b', 2a' + b' - 2z$) we obtain a multiset X_1 (resp. X'_1). By (a), we have $X_1 = X'_1$. Clearly, $\max X_1 = a + b + 2z$ and $\max X'_1 = a' + b' + 2z$. It follows that

$$(b) \quad a + b + 2z = a' + b' + 2z.$$

From (a) and (b) we deduce that $a = a', b = b'$. Thus $x = x'$ as required.

A.8. Assume that \mathfrak{g} is of type E_7 and $\underline{L} \cong \mathfrak{sl}_2(\mathbf{C})^3 \oplus \mathbf{C}^4$. We number the vertices of the Coxeter diagram by $1, 2, \dots, 7$ where the edges are $1-2-3-4-5-6$ and $3-7$. For $i \in [1, 7]$ define $u_i \in \mathfrak{t}$ by $\alpha_j(u_i) = \delta_{ij}$ for all $j \in [1, 7]$. Then \mathfrak{t} is spanned by u_1, u_2, u_3, u_5 . We may assume that $h^0 = \check{\alpha}_4 + cha_6 + \check{\alpha}_7$.

We have $x = au_1 + bu_2 + cu_3 + du_5$ $x' = a'u_1 + b'u_2 + c'u_3 + d'u_5$ where $a, b, c, d, a', b', c', d' \in \mathbf{C}$ are ≥ 0 . Let Y be the multiset consisting of

$$\begin{aligned} & a + 2b + 3c + 2d + z, a + 2b + 3c + 2d - z, a + 2b + 3c + d + z, a + 2b + 3c + d - z, \\ & a + 2b + 2c + d + z, a + 2b + 2c + d - z, a + b + 2c + d + z, a + b + 2c + d - z, \\ & a + b + c + d + z, a + b + c + d - z, b + 2c + d + z, b + 2c + d - z, \\ & b + c + d + z, b + c + d - z, a + b + c + z, a + b + c - z, \\ & c + d + z, c + d - z, b + c + z, b + c - z, d + z, d - z, c + z, c - z, 3z, z, z, z \end{aligned}$$

and their negatives. Let Y' be the multiset obtained from Y by replacing a, b, c, d, z by a', b', c', d', z .

The eigenvalues of $x + zh^0$ (resp. $x' + zh^0$) on the minuscule \mathfrak{g} -module V are the 56 numbers in the multiset Y (resp. Y'). Since $x + zh^0, x' + zh^0$ are in the same G -orbit, they have the same eigenvalues on V . Thus, $Y = Y'$. Removing $3z, z, z, z$ from Y (resp. Y') we obtain a multiset X (resp. X') with 52 elements. We have $X = X'$. Clearly, $\max X = a + 2b + 3c + 2d + z$ and $\max X' = a' + 2b' + 3c' + 2d' + z$. It follows that

$$(a) \quad a + 2b + 3c + 2d + z = a' + 2b' + 3c' + 2d' + z.$$

Removing from X (resp. X') the numbers $a + 2b + 3c + 2d + z, a + 2b + 3c + 2d - z$ (resp. $a' + 2b' + 3c' + 2d' + z, a' + 2b' + 3c' + 2d' - z$) we obtain a multiset X_1 (resp. X'_1). By (a), we have $X_1 = X'_1$. Clearly, $\max X_1 = a + 2b + 3c + d + z$ and $\max X'_1 = a' + 2b' + 3c' + d' + z$. It follows that

$$(b) \quad a + 2b + 3c + d + z = a' + 2b' + 3c' + d' + z.$$

Removing from X_1 (resp. X'_1) the numbers $a + 2b + 3c + d + z, a + 2b + 3c + d - z$ (resp. $a' + 2b' + 3c' + d' + z, a' + 2b' + 3c' + d' - z$) we obtain a multiset X_2 (resp. X'_2). By (b), we have $X_2 = X'_2$. Clearly, $\max X_2 = a + 2b + 2c + d + z$ and $\max X'_2 = a' + 2b' + 2c' + d' + z$. It follows that

$$(c) \quad a + 2b + 2c + d + z = a' + 2b' + 2c' + d' + z.$$

Removing from X_2 (resp. X'_2) the numbers $a + 2b + 2c + d + z, a + 2b + 2c + d - z$ (resp. $a' + 2b' + 2c' + d' + z, a' + 2b' + 2c' + d' - z$) we obtain a multiset X_3 (resp. X'_3). By (c), we have $X_3 = X'_3$. Clearly, $\max X_3 = a + b + 2c + d + z$ and $\max X'_3 = a' + b' + 2c' + d' + z$. It follows that

(d) $a + b + 2c + d + z = a' + b' + 2c' + d' + z$.

From (a), (b), (c), (d) we deduce that $a = a', b = b', c = c', d = d'$. Thus $x = x'$ as required.

Lemma 5.5 is proved.

REFERENCES

- [K] V. G. Kac, *Automorphisms of finite order of semisimple Lie algebras*, Funktsional. Anal. i Prilozh. **3** (1969), 94-96. MR **40**:4322
- [KL] D. Kazhdan and G. Lusztig, *Proof of the Deligne-Langlands conjecture for Hecke algebras*, Inv. Math. **87** (1987), 153-215. MR **88d**:11121
- [L1] G. Lusztig, *Cuspidal local systems and graded Hecke algebras*, Publ. Math. I.H.E.S. **67** (1988), 145-202. MR **90e**:22029
- [L2] G. Lusztig, *Affine Hecke algebras and their graded version*, Jour. Amer. Math. Soc. **2** (1989), 599-635. MR **90e**:16049
- [L3] G. Lusztig, *Cuspidal local systems and graded Hecke algebras*, Representations of groups, (B. Allison and G. Cliff, eds.), Canad. Math. Soc. Conf. Proc., vol. 16, Amer. Math. Soc., 1995, pp. 217-275. MR **96m**:22038
- [L4] G. Lusztig, *Classification of unipotent representations of simple p -adic groups*, Int. Math. Res. Not. (1995), 517-589. MR **98b**:22034
- [L5] G. Lusztig, *Cuspidal local systems and graded Hecke algebras*, III, Represent. Theory **6** (2002), 202-242.
- [R] M. Reeder, *Formal degrees and L -packets of unipotent discrete series of exceptional p -adic groups*, J. Reine Angew. Math. **520** (2000), 37-93. MR **2001k**:22039
- [Se] G. Segal, *The representation ring of a compact Lie group*, Publ. Math. I.H.E.S. **30** (1968), 113-128. MR **40**:1529
- [St] R. Steinberg, *Endomorphisms of linear algebraic groups*, Mem. Amer. Math. Soc., no. 80 (1968). MR **37**:6288
- [T] J. Tits, *Reductive groups over local fields*, Automorphic forms, representations and L -functions, Proc. Symp. Pure Math., vol. 33, Amer. Math. Soc., 1979, pp. 29-69. MR **80h**:20064
- [W] J.-L. Waldspurger, *Représentations de réduction unipotente pour $SO(2n + 1)$: quelques conséquences d'un article de Lusztig*, 2001, preprint.

DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MASSACHUSETTS 02139

E-mail address: gyuri@math.mit.edu