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REALISATION OF LUSZTIG CONES

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Abstract. Let $U_q(\mathfrak{g})$ be the quantised enveloping algebra associated to a simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$. The negative part U^- of $U_q(\mathfrak{g})$ possesses a canonical basis $\mathcal{B}$ with favourable properties. Lusztig has associated a cone to a reduced expression $\mathbf{i}$ for the longest element w_0 in the Weyl group of $\mathfrak{g}$, with good properties with respect to monomial elements of $\mathcal{B}$. The first author has associated a subalgebra $A_{\mathbf{i}}$ of U^-, compatible with the dual basis $\mathcal{B}^*$, to each reduced expression $\mathbf{i}$. We show that, after a certain twisting, the string parametrisation of the adapted basis of this subalgebra coincides with the corresponding Lusztig cone. As an application, we give explicit expressions for the generators of the Lusztig cones.

1. Introduction

Let $U = U_q(\mathfrak{g})$ be the quantum group associated to a semisimple Lie algebra $\mathfrak{g}$. The negative part U^- of U has a canonical basis $\mathcal{B}$ with favourable properties (see Kashiwara [14] and Lusztig [18, §14.4.6]). For example, via action on highest weight vectors it gives rise to bases for all the finite-dimensional irreducible highest weight U-modules. The dual canonical basis $\mathcal{B}^*$ of the positive part U^+ has good multiplicative properties. Two elements of $\mathcal{B}^*$ are said to be multiplicative if their product also lies in $\mathcal{B}^*$ up to a power of q.

The first author has shown that for each reduced expression $\mathbf{i} = (i_1, i_2, \ldots, i_N)$ for the longest element w_0 (see § 2.2) in the Weyl group of $\mathfrak{g}$, there is a corresponding subalgebra $A_{\mathbf{i}}$ of U^+, known as a standard adapted subalgebra, with basis given by $A_{\mathbf{i}} \cap \mathcal{B}^*$, consisting entirely of elements which are pairwise multiplicative. The subalgebras $A_{\mathbf{i}}$ are q-polynomial algebras, i.e., algebras given by generators and q-commuting relations, with GK-dimension $N = l(w_0)$. Note that adapted algebras were introduced for the Berenstein-Zelevinsky conjecture and are connected with the larger theory of cluster algebras [11].

By a Lusztig cone of U, we mean the cone $\mathcal{L}_{\mathbf{i}} \subseteq \mathbb{N}^N$ associated by Lusztig (see [19, §16]) to each reduced expression $\mathbf{i}$ for w_0. In [19] these cones arise naturally from the linear term of a nonhomogeneous quadratic form associated to $\mathbf{i}$ which is used by Lusztig to give a positivity condition for a monomial

$$F_{i_1}^{(a_1)} F_{i_2}^{(a_2)} \cdots F_{i_N}^{(a_N)} \tag{1.1}$$

to lie in the canonical basis. Here the F_i are the standard generators of U^-. Monomials of this form with $(a_1, a_2, \ldots, a_N)$ lying in the Lusztig cone corresponding to $\mathbf{i}$ lie in the canonical basis in types A_1, A_2 and A_3 [19], in type A_4 [21] and in type B_2 [30]; see also [26]. Counterexamples of M. Reineke [26] and N. H. Xi [31]

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show that this fails in general in type A_5. Recently, R. Bedard [2] has analysed the quadratic forms associated to these monomials and, as an application, was able to compute some interesting examples in types D_4, A_5 and affine A_1.

Note that Lusztig cones are used to describe regular functions on a reduced real double Bruhat cell of the corresponding algebraic group [32], they have links with primitive elements in the dual canonical basis (this can be seen using [4]) and therefore with the representation theory of affine Hecke algebras [17], and they are known to correspond to regions of linearity of the Lusztig reparametrisation functions (see [10]).

Given a reduced expression $\mathbf{i}$ for w_0, elements of the dual canonical basis can be parametrised via the string parametrisation in direction $\mathbf{i}$ (see [4, §2], [15] and [24, §2]), which we denote by $c_{\mathbf{i}} : \mathcal{B}^* \to \mathcal{C}_{\mathbf{i}}$, where $\mathcal{C}_{\mathbf{i}} \subseteq \mathbb{N}^N$ is known as the string cone corresponding to $\mathbf{i}$.

Our main result is that the set of string parameters (in direction $\mathbf{i}$) of a certain twisting of the standard adapted subalgebra of the dual canonical basis corresponding to $\mathbf{i}$ coincides with the Lusztig cone corresponding to $\mathbf{i}$. The twisting is done with the help of the Schützenberger involution. This gives a realisation of all Lusztig cones in terms of the dual canonical basis. It also implies that all Lusztig cones are simplicial (generalising results of Bedard [1] and the second author [22]), and enables us to give an explicit description of their spanning vectors; see Theorem 8.10.

The paper is organised as follows. Sections 1 and 2 give preliminary results on quantum groups and the canonical basis, including its parametrisations associated to a reduced word, and adapted algebras. In Section 4, we introduce the Schützenberger involution ϕ and its action on the dual canonical basis. In Section 5, we recall some facts on geometric lifting of the canonical basis in order to give a formula which describes ϕ in terms of the parametrisation of the dual canonical basis. A remarkable property is that, with a good choice of parametrisations, the action of the Schützenberger involution on the dual canonical basis is given by an affine map.

In Sections 6 and 7, we apply the results from previous sections to describe explicitly the twisted standard adapted subalgebra associated to a reduced word $\mathbf{i}$, in terms of $\mathbf{i}$-string parametrisation. By the multiplicative property of the adapted subalgebra and the "affine map" property, this can be provided by an $N \times N$ matrix and a column vector. A combinatorial argument, together with the known PBW-parametrisation of the adapted basis of a standard adapted subalgebra, allows us to prove the main theorem: in Section 8, we realise the Lusztig cones in terms of the string parametrisation of twisted standard adapted subalgebras. As an application, we give an explicit formula for the generators of the cones.

2. Notation and preliminaries

2.1. Let $A = (a_{ij})_{1 \leq i,j \leq n}$ be the Cartan matrix of a finite dimensional semi-simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$. Let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$ be a triangular decomposition, where $\mathfrak{h}$ is a Cartan subalgebra and where $\mathfrak{n}^-$, $\mathfrak{n}$ are opposite maximal nilpotent subalgebras of $\mathfrak{g}$. Let $\{\alpha_i\}_i$ be the set of simple roots of the root system Δ resulting from this decomposition. The set of positive roots is denoted by Δ^+.

Let P be the weight lattice generated by the fundamental weights ϖ_i, $1 \leq i \leq n$. Set $P^+ := \sum_i \mathbb{Z}_{\geq 0}\varpi_i$, endowed with the ordering $\sum_i \lambda_i\varpi_i \leq \sum_i \mu_i\varpi_i \Leftrightarrow \lambda_i \leq \mu_i$. The Weyl group W is generated by the reflections s_i corresponding to the simple roots. We denote by $\langle\,,\,\rangle$ the W-invariant form on P; we have $a_{ij} = \langle \alpha_j, \alpha_i^\vee \rangle$ for all i, j, where the $\alpha_i^\vee$'s are the simple coroots.

For n a nonnegative integer and α a positive root, we set $q_\alpha = q^{\langle \alpha,\alpha\rangle/2}$, $[n]_\alpha = \frac{q_\alpha^n - q_\alpha^{-n}}{q_\alpha - q_\alpha^{-1}}$, $[n]_\alpha! = [n]_\alpha [n-1]_\alpha \dots [1]_\alpha$.

2.2. Let W be the Weyl group of $\mathfrak{g}$, with Coxeter generators $s_1, s_2, \dots, s_n$ and corresponding length function. An expression $s_{i_1} s_{i_2} \cdots s_{i_m}$ for an element of w is called reduced if it is of minimal length; we identify such an expression with the tuple $\mathbf{i} = (i_1, i_2, \dots, i_m)$. Set $N := \dim \mathfrak{n}$. It is known that N is the length of the longest element w_0 of the Weyl group. Let $\mathcal{R}$ be the set of reduced expressions for w_0.

Fix $\mathbf{i}$ in $\mathcal{R}$. Let $\mathcal{L}_{\mathbf{i}}$ be the set of points $(c_1, \dots, c_N) \in \mathbb{Z}_{\geq 0}^N$ with the following property : for any two indices $p < p'$ in $\{1, \dots, N\}$ such that $i_p = i_{p'} = i$ and $i_q \neq i$ whenever $p < q < p'$, we have

$$c_p + c_{p'} + \sum_{p<q<p'} a_{i_p, i_q} c_q \leq 0.$$

The cone $\mathcal{L}_{\mathbf{i}}$ is the so-called Lusztig cone associated to the reduced expression $\mathbf{i}$. This is defined in [19, §16] for the simply-laced case. We use here a natural generalisation to the general case which also appears implicitly (for type B_2) in [30].

2.3. Let d be an integer such that $\langle P, P \rangle \subset (2/d)\mathbb{Z}$. Let q be a indeterminate and set $\mathbb{K} = \mathbb{C}(q^{1/d})$. We define the simply connected quantised enveloping $\mathbb{K}$-algebra $U_q(\mathfrak{g})$ as in [12]. Set $d_i = \langle \alpha_i, \alpha_i \rangle / 2$ and $q_i = q^{d_i}$ for all i. Let $U_q(\mathfrak{n})$, resp. $U_q(\mathfrak{n}^-)$, be the subalgebra generated by the canonical generators $E_i := E_{\alpha_i}$, resp. $F_i := F_{\alpha_i}$, of positive, resp. negative, weights, subject to the quantum Serre relations. For all λ in P, let K_λ be the corresponding element in the algebra $U_q^0 = \mathbb{K}[P]$ of the torus of $U_q(\mathfrak{g})$ and $K_i := K_{\alpha_i}$. We have the triangular decomposition $U_q(\mathfrak{g}) = U_q(\mathfrak{n}^-) \otimes U_q^0 \otimes U_q(\mathfrak{n})$. Set

$$U_q(\mathfrak{b}) = U_q(\mathfrak{n}) \otimes U_q^0, \qquad U_q(\mathfrak{b}^-) = U_q(\mathfrak{n}^-) \otimes U_q^0.$$

The algebra $U_q(\mathfrak{g})$ is endowed with a structure of Hopf algebra with comultiplication Δ, antipode S and augmentation ε given by

$$\Delta E_i = E_i \otimes 1 + K_i \otimes E_i,\ \Delta F_i = F_i \otimes K_i^{-1} + 1 \otimes F_i,\ \Delta K_\lambda = K_\lambda \otimes K_\lambda,$$
$$S(E_i) = -K_i^{-1} E_i,\ S(F_i) = -F_i K_i,\ S(K_\lambda) = K_{-\lambda},$$
$$\varepsilon(E_i) = \varepsilon(F_i) = 0,\ \varepsilon(K_\lambda) = 1.$$

Let $(\,,\,)$ be the Hopf bilinear form, [27], on $U_q(\mathfrak{b}) \times U_q(\mathfrak{b}^-)$, uniquely defined by

$$(E_i, F_j) = \delta_{ij}(1 - q_i^2)^{-1},\ 1 \leq i, j \leq n,$$
$$(XK_\lambda, YK_\mu) = q^{\langle \lambda, \mu \rangle}(X, Y),\ X \in U_q(\mathfrak{n}),\ Y \in U_q(\mathfrak{n}^-).$$

2.4. In this section, we define automorphisms of the quantised enveloping algebra and the Poincaré-Birkhoff-Witt basis. The automorphisms T_i, $1 \leq i \leq n$, as in [18], are given by

$$T_i(E_i) = -K_i^{-1}F_i,$$

$$T_i(E_j) = \sum_{k+l=-a_{ij}} (-1)^k \frac{q_{\alpha_i}^{-k}}{[k]_{\alpha_i}![l]_{\alpha_i}!} E_i^k E_j E_i^l,\ 1 \le i, j \le n,\ i \neq j,$$

$$T_i(F_i) = -E_i K_i,$$

$$T_i(F_j) = \sum_{k+l=-a_{ij}} \frac{q_{\alpha_i}^{k}}{[l]_{\alpha_i}![k]_{\alpha_i}!} F_i^l F_j F_i^k,\ 1 \le i, j \le n,\ i \neq j,$$

$$T_i(K_{\alpha_j}) = K_{s_i(\alpha_j)},\ 1 \le i, j \le n.$$

It is known that the T_i's define a braid action on $U_q(\mathfrak{g})$. Fix $\mathbf{i} = (i_1, \dots, i_N)$ in $\mathcal{R}$. For each k, $1 \le k \le N$, set $\beta_k := s_{i_1} \dots s_{i_{k-1}}(\alpha_k)$. It is well known that $\{\beta_k,\ 1 \le k \le N\}$ is the set of positive roots and that

$$\beta_1 < \beta_2 < \ldots < \beta_N$$

defines a so-called convex ordering on Δ^+. This ordering identifies the semigroup $\mathbb{Z}_{\ge 0}^{\Delta^+}$ with the semigroup $\mathbb{Z}_{\ge 0}^N$. In the sequel, we denote by $\{e_k, 1 \le k \le N\}$ the natural basis of this semigroup.

For all k, define $E_{\beta_k}^{\mathbf{i}} = E_{\beta_k} = T_{i_1} \dots T_{i_{k-1}}(E_{\alpha_{i_k}})$. For all $t = (t_i) \in \mathbb{Z}_{\ge 0}^N$, set $E^{\mathbf{i}}(t) = E(t) := E_{\beta_1}^{(t_1)} \dots E_{\beta_N}^{(t_N)}$, where $E_{\beta_k}^{(t_k)} := \frac{1}{[t_k]_{\beta_k}!} E_{\beta_k}^{t_k}$. It is known that $\{E(t),\ t \in \mathbb{Z}_{\ge 0}^N\}$ is a basis of $U_q(\mathfrak{n})$ called the Poincaré-Birkhoff-Witt basis, in short PBW-basis, associated to the reduced expression $\mathbf{i}$. In the same way, we can define the PBW-basis $\{F(t),\ t \in \mathbb{Z}_{\ge 0}^N\}$ of $U_q(\mathfrak{n}^-)$.

We define now the automorphisms $^-$ (over $\mathbb{C}$), ω (over $\mathbb{K}$), and the antiautomorphism σ (over $\mathbb{K}$) of $U_q(\mathfrak{g})$ by

$$\bar{E}_i = E_i,\ \bar{K}_i = K_i^{-1},\ \bar{F}_i = F_i,\ \bar{q} = q^{-1},\ 1 \le i \le n,$$

$$\omega(E_i) = F_i,\ \omega(K_i) = K_i^{-1},\ \omega(F_i) = E_i,\ \omega(q) = q,\ 1 \le i \le n,$$

$$\sigma(E_i) = E_i,\ \sigma(K_i) = K_i^{-1},\ \sigma(F_i) = F_i,\ \sigma(q) = q,\ 1 \le i \le n.$$

Note that ω is a coalgebra antiautomorphism.

2.5. For any (left) $U_q(\mathfrak{g})$-module M, and any weight μ in P, let $M_\mu = \{m \in M : K_\lambda.m = q^{\langle \lambda, \mu \rangle} m, \forall \lambda \in P\}$ be the subspace of M of weight μ. For all λ in P^+ let $V_q(\lambda)$ be the simple $U_q(\mathfrak{g})$-module with highest weight λ and highest weight vector v_λ. The module $V_q(\lambda)$ satisfies the Weyl character formula. For λ in P^+, $V_q(\lambda)^*$ is naturally endowed with a structure of right $U_q(\mathfrak{g})$-module. Let η_λ be its weight element such that $\eta_\lambda(v_\lambda) = 1$. For λ, μ in P^+, the module $V_q(\lambda) \otimes V_q(\mu)$, endowed with the diagonal action of $U_q(\mathfrak{g})$, has a unique component of type $V_q(\lambda+\mu)$. Hence the restriction map provides a map $r_{\lambda,\mu} : V_q(\lambda)^* \otimes V_q(\mu)^* \simeq (V_q(\lambda) \otimes V_q(\mu))^* \to V_q(\lambda+\mu)^*$. Since P^+ is a free abelian semigroup, one may assume that the v_λ are normalised such that $r_{\lambda,\mu}(\eta_\lambda \otimes \eta_\mu) = \eta_{\lambda+\mu}$, λ, $\mu \in P^+$; see [12, 9.1.10].

Set $R^+ := \bigoplus_{\lambda \in P^+} V_q(\lambda)^* \otimes v_\lambda$. The space R^+ can be equipped with a structure of algebra by the multiplication rule: $(\xi \otimes v_\lambda).(\xi' \otimes v_{\lambda'}) = r_{\lambda,\lambda'}(\xi \otimes \xi') \otimes v_{\lambda+\lambda'}$, $\xi \in V_q(\lambda)^*$, $\xi' \in V_q(\lambda')^*$.

Let j be the map $R^+ \to U_q(\mathfrak{b}^-)^*$, $\xi \otimes v_\lambda \mapsto \xi(?v_\lambda)$, $\xi \in V_q(\lambda)^*$. The following lemma is standard, we give a proof for completion.

Lemma 2.1. *The map j is an embedding of algebras.*

Proof. The map j is a morphism of vector spaces. Now, for λ, λ' in P^+, η in $V_q(\lambda)$, η' in $V_q(\lambda')$, and b in $U_q(\mathfrak{b}^-)$:

$$\begin{aligned} j((\eta\otimes v_\lambda)(\eta'\otimes v_{\lambda'}))(b) &= r_{\lambda,\lambda'}(\eta\otimes\eta')(b.v_{\lambda+\lambda'}) = (\eta\otimes\eta')(b.(v_\lambda\otimes v_{\lambda'})) \\ &= \eta(b_{(1)}.v_\lambda)\eta'(b_{(2)}.v_{\lambda'}) = (j(\eta\otimes v_\lambda).j(\eta'\otimes v_{\lambda'}))(b), \end{aligned}$$

using the Sweedler notation. Hence, j is an algebra morphism.

Now, we prove that j is an embedding. Suppose that $j(\sum_k \eta_k\otimes v_{\lambda_k}) = 0$ for a nonzero element $\sum_k \eta_k\otimes v_{\lambda_k}$ in R^+. One may assume that the λ_k are distinct in P^+ and that η_1 is nonzero. Let $v\in V_q(\lambda_1)$ be such that $\eta_1(v)\neq 0$. Since $\bigoplus_k V_q(\lambda_k)$ is a semisimple $U_q(\mathfrak{g})$-module, the Jacobson density theorem asserts that there exists an a in $U_q(\mathfrak{g})$ such that $a.v_{\lambda_1} = v$ and $a.v_{\lambda_k} = 0$ for $k\neq 1$. By the triangular decomposition, one may assume that a is in $U_q(\mathfrak{b}^-)$. We have $j(\sum_k \eta_k\otimes v_{\lambda_k})(a) = \eta_1(v)\neq 0$. This yields a contradiction. □

Define the map

$$\beta\,:\,U_q(\mathfrak{b})\to U_q(\mathfrak{b}^-)^*,\qquad \beta(u)(v) = (u,v).$$

Then we have:

Theorem 2.2 ([7])**.** *The map β is an injective antihomomorphism of algebras which maps K_λ to $\eta_\lambda\otimes v_\lambda\in R^+\subseteq U_q(\mathfrak{b}^-)^*$. There exists a unique subspace E_λ of $U_q(\mathfrak{n})$ such that $\beta(E_\lambda K_\lambda) = V_q(\lambda)^*\otimes v_\lambda$.*

3. Recollection on canonical bases and adapted algebras

The recollection on canonical bases is from [5]. Results on adapted algebras can be found in [8] and [9].

3.1. The following theorem defines the canonical basis and its Lusztig parametrisation; see [20, Proposition 8.2].

Theorem 3.1 ([20])**.** *Fix $\mathbf{i}$ in $\mathcal{R}$. For all t in $\mathbb{Z}^N_{\geq 0}$, there exists a unique element $b = b_{\mathbf{i}}(t)$ in $U_q(\mathfrak{n}^-)$ such that $\bar{b} = b$ and $b - F^{\mathbf{i}}(t)\in q^{-1}\sum\mathbb{Z}[q^{-1}]F^{\mathbf{i}}(t')$. The map $t\mapsto b_{\mathbf{i}}(t)$ defines a bijection from $\mathbb{Z}^N_{\geq 0}$ to a basis $\mathcal{B}$ of $U_q(\mathfrak{n}^-)$. The basis $\mathcal{B}$ does not depend on the choice of* $\mathbf{i}$.

The basis $\mathcal{B}$ is called the canonical (or global) basis and the map $b_{\mathbf{i}}\,:\,t\mapsto b_{\mathbf{i}}(t)$ is the Lusztig parametrisation of $\mathcal{B}$ associated to the reduced expression $\mathbf{i}$. We can define the action of the Kashiwara operators on the canonical basis as follows: for $1\leq i\leq n$, there exists a unique injective map $\tilde{f}_i\,:\,\mathcal{B}\to\mathcal{B}$, such that for all $\mathbf{i}$ with $i_1 = i$, we have

$$\tilde{f}_i(b_{\mathbf{i}}(t_1,t_2,\ldots,t_N)) = b_{\mathbf{i}}(t_1+1,t_2,\ldots,t_N).$$

For $1\leq i\leq n$, let $\tilde{e}_i\,:\,\mathcal{B}\to\mathcal{B}\cup\{0\}$ be such that $\tilde{e}_i(b) = b'$ if there exists b' such that $\tilde{f}_i(b') = b$ and $\tilde{e}_i(b) = 0$ if not.

Let $\varepsilon_i(b)$ =max$\{k,\,\tilde{e}_i^k(b)\neq 0\}$ and let $\mathcal{E}\,:\,\mathcal{B}\to P^+$, $b\mapsto\sum_i\varepsilon_i(b)\varpi_i$. Now, the basis $\mathcal{B}$ is stable under σ. For all λ in P^+, set $\mathcal{B}(\lambda) = \{b\in\mathcal{B},\,\mathcal{E}(\sigma(b))\leq\lambda\}$.

A nice theorem of compatibility of the canonical basis with the Weyl modules $V_q(\lambda)$ can be stated as follows:

Theorem 3.2 ([13])**.** *Fix λ in P^+. Then, for b in $\mathcal{B}$, we have $b\cdot v_\lambda\neq 0$ if and only if $b\in\mathcal{B}(\lambda)$. Moreover, $\mathcal{B}(\lambda)\cdot v_\lambda$ is a basis of $V_q(\lambda)$.*

In the sequel, we will identify $\mathcal{B}(\lambda)$ with its image in $V_q(\lambda)$.

3.2. We now introduce the string parametrisation of the canonical basis and the various transition maps.

Fix a reduced expression $\mathbf{i}$ in $\mathcal{R}$ and b in $\mathcal{B}$. The string of b in the direction $\mathbf{i}$ is the sequence of integers $c_{\mathbf{i}}(b) := (t_1, \ldots, t_N)$ defined recursively by

$$t_1 = \varepsilon_{i_1}(b),\ t_2 = \varepsilon_{i_2}(\tilde{e}_{i_1}^{t_1}(b)), \ldots, t_N = \varepsilon_{i_N}(\tilde{e}_{i_{N-1}}^{t_{N-1}} \ldots \tilde{e}_{i_1}^{t_1}(b)).$$

The map $c_{\mathbf{i}}$ defines a bijection from $\mathcal{B}$ onto the set of integral points of a rational convex polyhedral cone $\mathcal{C}_{\mathbf{i}}$ in $\mathbb{R}^N$.

We can now define

$$\begin{aligned} R_{\mathbf{i}}^{\mathbf{i}'} &= (b_{\mathbf{i}'})^{-1} \circ b_{\mathbf{i}} \ : \ \mathbb{Z}_{\geq 0}^N \to \mathbb{Z}_{\geq 0}^N, \\ R_{-\mathbf{i}}^{-\mathbf{i}'} &= c_{\mathbf{i}'} \circ (c_{\mathbf{i}})^{-1} \ : \ \mathcal{C}_{\mathbf{i}} \to \mathcal{C}_{\mathbf{i}'}, \\ R_{-\mathbf{i}}^{\mathbf{i}'} &= (b_{\mathbf{i}'})^{-1} \circ (c_{\mathbf{i}})^{-1} \ : \ \mathcal{C}_{\mathbf{i}} \to \mathbb{Z}_{\geq 0}^N, \\ R_{\mathbf{i}}^{-\mathbf{i}'} &= c_{\mathbf{i}'} \circ b_{\mathbf{i}} \ : \ \mathbb{Z}_{\geq 0}^N \to \mathcal{C}_{\mathbf{i}'}. \end{aligned}$$

3.3. Let $\mathcal{B}^* \subset U_q(\mathfrak{n})$ be the basis dual to $\mathcal{B}$ with respect to the form $(\ ,\ )$ on $U_q(\mathfrak{n}) \times U_q(\mathfrak{n}^-)$. We call it the dual canonical basis. For b in $\mathcal{B}$, we denote by b^* the corresponding element in $\mathcal{B}^*$. Since we work with the dual canonical basis, in the sequel we shall regard $b_{\mathbf{i}}$ as a map from $\mathbb{Z}_{\geq 0}^N$ to $\mathcal{B}^*$ (rather than $\mathcal{B}$), using the identification $b \leftrightarrow b^*$. Similarly, we shall regard $c_{\mathbf{i}}$ as a map from $\mathcal{B}^*$ to $\mathbb{Z}_{\geq 0}^N$.

The set $\mathcal{B}^*$ is stable under σ "up to a power of q". To be more precise, for b in $\mathcal{B}$, there exists an integer m such that $\sigma(b^*) = q^m \sigma(b)^*$.

For λ in P^+ and b in $\mathcal{B}(\lambda)$, let $\pi_\lambda(b)^*$ be the element of $V_q(\lambda)^*$ such that $\pi_\lambda(b)^*(b'.v_\lambda) = \delta_{b,b'}$, for all $b' \in \mathcal{B}(\lambda)$, where δ is the Kronecker symbol. It is easily seen from the definitions that:

Lemma 3.3. *For all λ in P^+ and b in $\mathcal{B}(\lambda)$, we have $\beta(b^* K_\lambda) = \pi_\lambda(b)^* \otimes v_\lambda$.*

In the notation, we will sometimes omit π_λ.

The lemma implies that the spaces E_λ defined by Theorem 2.2 are compatible with the dual canonical basis. By the Weyl character formula, for all w in W and λ in P^+, there is a unique element of $\mathcal{B}^* \cap E_\lambda$ with weight $\lambda - w\lambda$. We denote this element by $b_{w,\lambda}^*$ and the corresponding element in the canonical basis by $b_{w,\lambda}$.

In the sequel, $\{\pi_\lambda(b)^* \otimes v_\lambda,\ b \in \mathcal{B}(\lambda),\ \lambda \in P^+\}$ will be called dual canonical basis of R^+. By a misuse of language, its Lusztig, resp. string, parametrisation will be the Lusztig, resp. string, parametrisation of the corresponding element b in $\mathcal{B}(\lambda)$.

3.4. Two elements of the dual canonical basis are called multiplicative if their product is an element of the dual canonical basis, up to a power of q. By [25], multiplicative elements q-commute. We start with the definition of adapted algebras.

Definition 3.4. A subalgebra A of $U_q(\mathfrak{n})$, resp. R^+, is called *adapted* if

1) the intersection of A and the dual canonical basis is a basis of A, called adapted basis,

2) the elements of this basis are pairwise multiplicative.

We define standard adapted subalgebras of R^+, associated to a fixed reduced expression $\mathbf{i}$ for w_0; see [8].

Set $y_\lambda := \eta_\lambda \otimes v_\lambda$. Let $A_{\mathbf{i}}$ be the subalgebra of R^+ generated by $y_i := y_{\varpi_i}$, $1 \leq i \leq n$, and $c_k^{\mathbf{i}} := b^*_{s_{i_1}\dots s_{i_k},\varpi_{i_k}} \otimes v_{\varpi_{i_k}}$, $1 \leq k \leq N$. (This notation is standard, but shouldn't be confused with $c_{\mathbf{i}}$ which is used for the string parametrisation). Using the antihomomorphism β, we obtain from [8] the following proposition:

Proposition 3.5. *Fix $\mathbf{i} \in \mathcal{R}$. The algebra $A_{\mathbf{i}}$ is an adapted subalgebra of R^+ and the adapted basis is given by monomials in the y_i and the $c_k^{\mathbf{i}}$, up to a power of q.*

As σ is an antiautomorphism which preserves the dual canonical basis up to a power of q, we can now define another family of adapted algebras by twisting the standard ones.

Note that $\sigma(b_{s_{i_1}\dots s_{i_k},\varpi_{i_k}}) \in \mathcal{B}(\mu_k)$, where $\mu_k := \mathcal{E}(b_{s_{i_1}\dots s_{i_k},\varpi_{i_k}})$. Let $A_{\mathbf{i}}^\sigma$ be the subalgebra generated by the y_i, $1 \leq i \leq n$, and the $c_k^{\mathbf{i}\sigma} := \sigma(b_{s_{i_1}\dots s_{i_k},\varpi_{i_k}})^* \otimes v_{\mu_k}$, $1 \leq k \leq N$. These elements q-commute by 2.2. Moreover, by 3.3 and [8, 2.2], we have

Proposition 3.6. *Fix $\mathbf{i} \in \mathcal{R}$. Then,*

(i) *the algebra $A_{\mathbf{i}}^\sigma$ is an adapted subalgebra of R^+,*
(ii) *the adapted basis is given by monomials in the y_i and the $c_k^{\mathbf{i}\sigma}$, up to a power of q,*
(iii) *the Lusztig parametrisation of y_i is zero,*
(iv) *the Lusztig parametrisation of $c_k^{\mathbf{i}\sigma}$ is $\sum e_l$ where l runs over $\{l \leq k,\ i_l = i_k\}$.*

3.5. In this section, we are concerned with the lowest weight vectors in R^+. For all λ in P^+, set $\lambda^* := -w_0\lambda$. Let $v_{w_0\lambda}$ be the unique element of weight $w_0\lambda$ in the canonical basis of $V_q(\lambda)$. Then, $z_\lambda := v^*_{w_0\lambda} \otimes v_\lambda$ is an element of the dual canonical basis of R^+. It is known (see [8]) that these elements belong to all of the standard adapted algebras defined in the section above.

Lemma 3.7. *The elements z_λ, $\lambda \in P^+$ satisfy $z_\lambda z_\mu = z_{\lambda+\mu}$, $\lambda,\ \mu \in P^+$.*

Proof. Let b, b' be two elements of the canonical basis of $U_q(\mathfrak{n}^-)$, with $b \in \mathcal{B}(\lambda)$. By [9, Proposition 3.1], we have $b'^* b^* \in q^{-\langle \lambda, \nu' \rangle}\mathcal{B}^*$, where ν' is the weight of b'^*.

Applying this formula when b^*, resp. b'^*, is the element of $\mathcal{B}(\lambda)^*$, resp. $\mathcal{B}(\mu)^*$, of weight $\lambda + \lambda^*$, resp. $\mu + \mu^*$, and using Theorem 2.2, we obtain that

$$z_\mu z_\lambda = q^{-\langle \lambda, \mu+\mu^* \rangle} q^{\langle \lambda, \mu+\mu^* \rangle} z_{\lambda+\mu} = z_{\lambda+\mu}.$$

This proves the lemma. □

In the sequel, we set $z_i = z_{\varpi_i}$.

4. The Schützenberger involution

We can now define an involution ϕ of R^+ by twisting the dual Weyl modules by the automorphism ω. This involution generalises the Schützenberger involution ([28]), up to a diagram automorphism.

4.1. Let M be a $U_q(\mathfrak{g})$-module. With the help of the automorphism ω, we define a twisted $U_q(\mathfrak{g})$-module structure on M^* via

$$x.\xi(m) = \xi(\omega(x)m),\ \xi \in M^*,\ x \in U_q(\mathfrak{g}),\ m \in M.$$

Let M^*_ω be the space M^* endowed with this $U_q(\mathfrak{g})$-module structure.

Fix λ in P^+. Then, $V_q(\lambda)^*_\omega$ is a simple right module. By dualising [18, Chap XXI], we obtain

Proposition 4.1. *The module $V_q(\lambda)^*_\omega$ is isomorphic to $V_q(\lambda^*)^*$. There exists a unique right $U_q(\mathfrak{g})$-module isomorphism ϕ_λ which sends the dual canonical basis of $V_q(\lambda)^*_\omega$ to the dual canonical basis of $V_q(\lambda^*)^*$. It sends highest weight vectors to lowest weight vectors and conversely.*

4.2. We define a map $\phi : R^+ \to R^+$, such that

$$\phi(\xi \otimes v_\lambda) = \phi_\lambda(\xi) \otimes v_{\lambda^*},\ \xi \in V_q(\lambda)^*,\ \lambda \in P^+,$$

where ϕ_λ is as above. Then

Proposition 4.2. *The map ϕ is an involutive antiautomorphism of the algebra R^+ which preserves the dual canonical basis of R^+.*

Proof. Recall that ω is involutive. So, by Proposition 4.1, $\phi_{\lambda^*}\phi_\lambda$ is the identity and this implies that ϕ is involutive.

Now, let R^+_λ be the λ-component of R^+, which is isomorphic to $V_q(\lambda)^*$ as a right $U_q(\mathfrak{g})$-module. As noted in 2.4, ω is a coalgebra antiautomorphism. Hence, the map m:

$$m : R^+_\lambda \otimes R^+_\mu \to R^+_{\lambda+\mu},\ \ m(a \otimes b) = \phi^{-1}(\phi(b)\phi(a))$$

is a morphism of right $U_q(\mathfrak{g})$-modules, where $R^+_\lambda \otimes R^+_\mu$ is endowed with the diagonal action. Using Lemma 3.7, we obtain

$$m(y_\lambda \otimes y_\mu) = \phi^{-1}(z_{\lambda+\mu}) = y_{\lambda+\mu} = y_\lambda y_\mu.$$

As $\dim\mathrm{Hom}_{U_q(\mathfrak{g})}(R^+_\lambda \otimes R^+_\mu, R^+_{\lambda+\mu}) = 1$, this proves that m is the multiplication of R^+ and thus that ϕ is an algebra antiautomorphism. The last assertion of the proposition is clear by Proposition 4.1. □

Corollary 4.3. *Fix a reduced expression $\mathbf{i}$ in $\mathcal{R}$. Then, $\phi(A^\sigma_{\mathbf{i}})$ is an adapted subalgebra of R^+.*

The aim of the remaining sections is to prove that the Lusztig cone $\mathcal{L}_{\mathbf{i}}$ is the $\mathbf{i}$-string parametrisation of the adapted basis of $\phi(A^\sigma_{\mathbf{i}})$. We note that this is given by monomials in the z_i and the $\phi(c^{\mathbf{i}\sigma}_k)$ up to a power of q.

5. Geometric lifting and parametrisation

We fix a dominant weight λ. In this section, we study the geometric lifting of the morphism ϕ_λ. By using results of [5], we obtain an explicit formula for the morphism $b_{\mathbf{i}}^{-1}\phi_\lambda c_{\mathbf{i}}^{-1}$ which gives the Lusztig parametrisation $t' = (t'_1, \cdots, t'_N)$ of the element $\phi_\lambda(b)$ in terms of the string $t = (t_1, \cdots, t_N)$ of b, where b is in the dual canonical basis of R^+.

5.1. We give here notation and recollection of [5]. Let G be the semisimple simply connected complex Lie group with Lie algebra $\mathfrak{g}$. For all i, $1 \le i \le n$, we denote by $\varphi_i : SL_2 \hookrightarrow G$ the canonical embedding corresponding to the simple root α_i. Consider the one-parameter subgroups of G defined by

$$x_i(t) = \varphi_i \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \qquad y_i(t) = \varphi_i \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \qquad t \in \mathbb{C},$$

and

$$t^{\alpha_i^\vee} = \varphi_i \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \qquad t \in \mathbb{C}^*.$$

The $x_i(t)$, (resp. $y_i(t)$, $t^{\alpha_i^\vee}$) generate subgroups N, (resp. N^-, H). We have the following commutation relations:

$$t^{\alpha_i^\vee} x_j(t') = x_j(t^{a_{ij}}t')t^{\alpha_i^\vee}, \quad t^{\alpha_i^\vee} y_j(t') = y_j(t^{-a_{ij}}t')t^{\alpha_i^\vee}. \tag{5.1}$$

We define two involutive antiautomorphisms of G, $x \to x^T$ and $x \to x^\iota$, by

$$x_i(t)^T = y_i(t), \quad y_i(t)^T = x_i(t), \quad (t^{\alpha_i^\vee})^T = t^{\alpha_i^\vee},$$
$$x_i(t)^\iota = x_i(t), \quad y_i(t)^\iota = y_i(t), \quad (t^{\alpha_i^\vee})^\iota = t^{-\alpha_i^\vee}.$$

The first one is called transposition and the second one is called inversion.

Let $G_0 := N^-HN$ be the set of elements in G which have a (unique) gaussian decomposition; we write $x = [x]_-[x]_0[x]_+$ for the gaussian decomposition of x in G_0.

For all reduced expressions $\mathbf{i} = (i_1, \ldots, i_N)$ and all N-tuples $t = (t_1, \ldots, t_N)$ in $\mathbb{C}^N$, we set:

$$x_{\mathbf{i}}(t) := x_{i_1}(t_1)\cdots x_{i_N}(t_N), \quad \text{and} \quad x_{-\mathbf{i}}(t) := y_{i_1}(t_1)t_1^{-\alpha_{i_1}^\vee}\cdots y_{i_N}(t_N)t_N^{-\alpha_{i_N}^\vee}.$$

The $x_{\mathbf{i}}$ and the $x_{-\mathbf{i}}$ parametrise subvarieties of G.

Theorem 5.1 ([5]). *There exists a subvariety of G, $L^{e,w_0}_{>0}$, resp. $L^{w_0,e}_{>0}$, such that for all $\mathbf{i}$ in $\mathcal{R}$, the map $x_{\mathbf{i}}$, resp. $x_{-\mathbf{i}}$, is a bijection from $\mathbb{R}^N_{>0}$ to $L^{e,w_0}_{>0}$, resp. $L^{w_0,e}_{>0}$.*

We denote by $\tilde{R}^{\mathbf{i}'}_{\mathbf{i}} := x^{-1}_{\mathbf{i}'} \circ x_{\mathbf{i}}$ and $\tilde{R}^{-\mathbf{i}'}_{-\mathbf{i}} := x^{-1}_{-\mathbf{i}'} \circ x_{-\mathbf{i}}$ the transition maps. A remarkable result of [5] asserts that $\tilde{R}^{\mathbf{i}'}_{\mathbf{i}}$ (respectively, $\tilde{R}^{-\mathbf{i}'}_{-\mathbf{i}}$) is a geometric lifting of the map $R^{\mathbf{i}'}_{\mathbf{i}}$ (respectively, $R^{-\mathbf{i}'}_{-\mathbf{i}}$), which was defined in the first section. Let's be more precise.

By using results on semifields (see [3]), the authors define the so-called tropicalisation, denoted by $[.]_{Trop}$. The map $[.]_{Trop}$ is from the semifield $\mathbb{Q}_{>0}(t_1, \ldots, t_N)$ to the set of maps $\mathbb{Z}^N \to \mathbb{Z}$. The elements of $\mathbb{Q}_{>0}(t_1, \ldots, t_N)$ are called *subtraction-free rational expressions* in the $t_1, \ldots, t_N$. Tropicalising a subtraction-free expression means replacing the multiplication by the operation $a \odot b := a + b$ and the sum by the operation $a \oplus b = \min(a, b)$. We give an example from [3].

Example 5.2. Let x, y be two indeterminates and set $f = x^2 - xy + y^2$. Then, f is a subtraction-free expression because $f = \frac{x^3+y^3}{x+y}$. We have $[f]_{Trop} : \mathbb{Z}^2 \to \mathbb{Z}$, with

$$[f]_{Trop}(m, n) = \mathrm{Min}(3m, 3n) - \mathrm{Min}(m, n) = \mathrm{Min}(2m, 2n).$$

A geometric lifting is an element of the inverse image of this map. We can see in the example above that it is in general not unique.

The following theorem is a result of [5]. Recall that the Langlands dual of G is the semisimple Lie group $G^\vee$ corresponding to the transpose Cartan matrix A^T. We can identify the simple roots (resp. coroots) of $G^\vee$ with the simple coroots (resp. roots) of G. The Weyl groups are naturally identified. In the theorem, the notation $(.)^\vee$ means that we consider the applications defined in the same way but for $G^\vee$, and the notation $[.]_{Trop}$ is the componentwise tropicalisation.

Theorem 5.3. *Fix two reduced expressions $\mathbf{i}$, $\mathbf{i}'$ in $\mathcal{R}$. Then $(\tilde{R}^{\mathbf{i}'}_{\mathbf{i}})^\vee$, resp. $(\tilde{R}^{-\mathbf{i}'}_{-\mathbf{i}})^\vee$, is a geometric lifting of $R^{\mathbf{i}'}_{\mathbf{i}}$, resp. $R^{-\mathbf{i}'}_{-\mathbf{i}}$:*

$$(i) \quad [(\tilde{R}^{\mathbf{i}'}_{\mathbf{i}})^\vee(t)]_{Trop} = R^{\mathbf{i}'}_{\mathbf{i}}(t), \qquad (ii) \quad [(\tilde{R}^{-\mathbf{i}'}_{-\mathbf{i}})^\vee(t)]_{Trop} = R^{-\mathbf{i}'}_{-\mathbf{i}}(t).$$

5.2. Let $\zeta : L_{>0}^{w_0,e} \to L_{>0}^{e,w_0}$ be the map defined by

$$\zeta(x) := [x^{\iota T}]_+.$$

By 5.1, we obtain that the map ζ is well defined and

Proposition 5.4. *Let $\mathbf{i} = (i_1, \cdots, i_N)$ be a reduced expression for w_0, and suppose $(t'_1, \cdots, t'_N) = (x_{\mathbf{i}}^{-1} \circ \zeta \circ x_{-\mathbf{i}})(t_1, \cdots, t_N)$, $t_i \in \mathbb{C}$. Then, we have*

$$t'_k = t_k^{-1} \prod_{j>k} t_j^{-a_{i_j i_k}}.$$

The following theorem and its corollary are a result of [23] but we give here a sketch of the proof. The description of the geometric lifting of ϕ_λ can be given in terms of parametrisations:

Theorem 5.5. *Fix two reduced expressions $\mathbf{i}$, $\mathbf{i}'$ in $\mathcal{R}$. Then, $(x_{\mathbf{i}}^{-1} \circ \zeta \circ x_{-\mathbf{i}'})^\vee(t)$ is a subtraction-free expression and*

$$b_{\mathbf{i}}^{-1} \phi_\lambda c_{\mathbf{i}'}^{-1}(t) = [(x_{\mathbf{i}}^{-1} \circ \zeta \circ x_{-\mathbf{i}'})^\vee(t)]_{Trop} + b_{\mathbf{i}}^{-1}\phi_\lambda(\eta_\lambda).$$

Set $(l_1, \cdots, l_N) := b_{\mathbf{i}}^{-1}\phi_\lambda(\eta_\lambda)$. We obtain the following tropicalised formula:

Corollary 5.6. *For $(t'_1, \cdots, t'_N) = b_{\mathbf{i}}^{-1}\phi_\lambda c_{\mathbf{i}}^{-1}(t_1, \cdots, t_N)$,*

$$t'_k = l_k - t_k - \sum_{j>k} a_{i_k i_j} t_j.$$

Remark 5.7. It is remarkable that this formula is affine. This is only true in the case $\mathbf{i} = \mathbf{i}'$. In general, the tropical term in the right-hand side of Theorem 5.5 is piecewise linear.

Sketch of the proof. By Proposition 5.4, $(x_{\mathbf{i}}^{-1} \circ \zeta \circ x_{-\mathbf{i}})$ is a subtraction-free expression. The first assertion of the theorem is obtained by composing with $\tilde{R}_{\mathbf{i}'}^{\mathbf{i}}$.

Let $\phi_{\mathbf{i},\mathbf{i}'} : \mathcal{C}_{\mathbf{i}'} \to \mathbb{Z}^N$ be a family of maps labelled by two reduced expressions $\mathbf{i}$ and $\mathbf{i}'$ such that the three following conditions are satisfied:

(1) $\phi_{\mathbf{i},\mathbf{i}'}(0, \cdots, 0) = b_{\mathbf{i}}^{-1}\phi_\lambda(\eta_\lambda)$,
(2) $\phi_{\mathbf{i},\mathbf{i}'} = R_{\mathbf{i}''}^{\mathbf{i}} \circ \phi_{\mathbf{i}'',\mathbf{i}'} = \phi_{\mathbf{i},\mathbf{i}''} \circ R_{-\mathbf{i}'}^{-\mathbf{i}''}$,
(3) for $\phi_{\mathbf{i},\mathbf{i}}(t_1, \cdots, t_N) = (t'_1, \cdots, t'_N)$, $t'_1 + t_1$ and t'_k, $k \neq 1$, depend only on $t_2, \cdots, t_N$.

The theorem follows from the proposition:

Proposition 5.8. [23] *We have,*

(i) *$(\phi_{\mathbf{i},\mathbf{i}'})$ is a family satisfying (1), (2), (3) if and only if*

$$\phi_{\mathbf{i},\mathbf{i}'} = b_{\mathbf{i}}^{-1}\phi_\lambda c_{\mathbf{i}'}^{-1}.$$

(ii) *The family $(\phi_{\mathbf{i},\mathbf{i}'})$ defined by*

$$\phi_{\mathbf{i},\mathbf{i}'}(t) = [(x_{\mathbf{i}}^{-1} \circ \zeta \circ x_{-\mathbf{i}'})^\vee(t)]_{Trop} + b_{\mathbf{i}}^{-1}\phi_\lambda(\eta_\lambda)$$

satisfies the conditions (1), (2), (3).

6. From the PBW-parametrisation to the string parametrisation

Let $\mathbf{i} = (i_1, \cdots, i_N)$ be a reduced expression for w_0. Our aim in this section is to describe the map $u \mapsto c_{\mathbf{i}}\phi_\lambda b_{\mathbf{i}}(u)$, using Proposition 5.4 and Theorem 5.5.

Let $\mathcal{S}$ be the complex rational map extending

$$(t_1, t_2, \ldots, t_N) \mapsto (u_1, u_2, \ldots, u_N) = (x_{\mathbf{i}}^{-1} \circ \zeta \circ x_{-\mathbf{i}})^\vee (t_1, t_2, \ldots, t_N).$$

Then Theorem 5.5 states that $\mathcal{S}(t)$ is a subtraction-free expression and that

$$b_{\mathbf{i}}^{-1}\phi_\lambda c_{\mathbf{i}}^{-1}(t) = [\mathcal{S}(t)]_{Trop} + b_{\mathbf{i}}^{-1}\phi_\lambda(\eta_\lambda).$$

Proposition 5.4 gives an explicit expression for $\mathcal{S}(t)$ (although note that we need to take the expression for the dual root system by the definition of $\mathcal{S}$).

Moreover, $\mathcal{S}$ is clearly birational. We first explain how to invert $\mathcal{S}$.

Lemma 6.1. *Let $(u_1, u_2, \ldots, u_N) \in \mathbb{C}^N$. Then $\mathcal{S}^{-1}(u_1, u_2, \ldots, u_N) = (t_1, t_2, \ldots, t_N) \in \mathbb{C}^N$, where, for $1 \leq k \leq N$, we have*

$$t_k = u_k^{-1} \prod_{j>k} u_j^{\langle s_{i_{k+1}} \cdots s_{i_{j-1}} \alpha_{i_j}, \alpha_{i_k}^\vee \rangle}. \tag{6.1}$$

Proof. We note that, by Proposition 5.4, if $(t_1, t_2, \ldots, t_N) \in \mathbb{C}^N$ and $\mathcal{S}(t_1, t_2, \ldots, t_N) = (u_1, u_2, \ldots, u_N) \in \mathbb{C}^N$, then, for $1 \leq k \leq N$, we have

$$u_k = t_k^{-1} \prod_{j>k} t_j^{-a_{i_k i_j}}. \tag{6.2}$$

Since this map is a monomial transformation of $\mathbb{C}^N$, it is sufficient to show that substituting the expression (6.2) for u_k in terms of the t_j into the right-hand side of equation (6.1) reduces to the left-hand side. Noting that

$$\langle s_{i_{k+1}} \cdots s_{i_{j-1}} \alpha_{i_j}, \alpha_{i_k}^\vee \rangle = \langle \alpha_{i_j}, s_{i_{j-1}} \cdots s_{i_{k+1}} \alpha_{i_k}^\vee \rangle,$$

we obtain

$$\left(t_k^{-1} \prod_{l>k} t_l^{-a_{i_k,i_l}} \right)^{-1} \prod_{j>k} \left(t_j^{-1} \prod_{l>j} t_l^{-a_{i_j,i_l}} \right)^{\langle \alpha_{i_j}, s_{i_{j-1}} \cdots s_{i_{k+1}} \alpha_{i_k}^\vee \rangle}.$$

It is clear that the exponent of t_k in this expression is 1, and that, if $l < k$, then the exponent of t_l is zero. So we consider the case where $l > k$. The exponent of t_l is given by

$$a_{i_k,i_l} + \left(\sum_{l>j>k} -a_{i_j,i_l} \langle \alpha_{i_j}, s_{i_{j-1}} \cdots s_{i_{k+1}} \alpha_{i_k}^\vee \rangle \right) - \langle \alpha_{i_l}, s_{i_{l-1}} \cdots s_{i_{k+1}} \alpha_{i_k}^\vee \rangle.$$

Since $s_{i_j}(\alpha_{i_l}) = \alpha_{i_l} - a_{i_j,i_l}\alpha_{i_j}$, this is equal to

$$\begin{aligned} \langle \alpha_{i_l}, \alpha_{i_k}^\vee \rangle \quad &+ \quad \left(\sum_{l>j>k} \langle s_{i_j}(\alpha_{i_l}), s_{i_{j-1}} \cdots s_{i_{k+1}} \alpha_{i_k}^\vee \rangle - \langle \alpha_{i_l}, s_{i_{j-1}} \cdots s_{i_{k+1}} \alpha_{i_k}^\vee \rangle \right) \\ &- \quad \langle \alpha_{i_l}, s_{i_{l-1}} \cdots s_{i_{k+1}} \alpha_{i_k}^\vee \rangle. \end{aligned}$$

The sum telescopes to give

$$\langle \alpha_{i_l}, \alpha_{i_k}^\vee \rangle + \langle \alpha_{i_l}, s_{i_{l-1}} \cdots s_{i_{k+1}} \alpha_{i_k}^\vee \rangle - \langle \alpha_{i_l}, \alpha_{i_k}^\vee \rangle - \langle \alpha_{i_l}, s_{i_{l-1}} \cdots s_{i_{k+1}} \alpha_{i_k}^\vee \rangle = 0,$$

and we are done. $\square$

Proposition 6.2. *Fix a reduced expression* $\mathbf{i}$ *for* w_0*, and let* $u = (u_1, u_2, \dots, u_N) \in \mathbb{Z}_{\geq 0}^N$*. Then*

$$c_{\mathbf{i}}\phi_\lambda b_{\mathbf{i}}(u) = [(\mathcal{S}^{-1})(u)]_{Trop} + c_{\mathbf{i}}\phi_\lambda(\eta_\lambda),$$

where $\mathcal{S}^{-1}(u)$ *is as in Lemma* 6.1.

Proof. By Theorem 5.5, we have that

$$b_{\mathbf{i}}^{-1}\phi_\lambda c_{\mathbf{i}}^{-1}(t) = [S(t)]_{trop} + b_{\mathbf{i}}^{-1}\phi_\lambda(\eta_\lambda). \tag{6.3}$$

It follows that

$$c_{\mathbf{i}}\phi_{\lambda^*} b_{\mathbf{i}}(u) = [\mathcal{S}^{-1}(u - b_{\mathbf{i}}^{-1}\phi_\lambda(\eta_\lambda))]_{Trop},$$

noting that $\phi_{\lambda^*}\phi_\lambda$ is the identity map. We note that $\mathcal{S}$ is an invertible monomial map, so its tropicalisation is linear. Hence

$$c_{\mathbf{i}}\phi_{\lambda^*} b_{\mathbf{i}}(u) = [\mathcal{S}^{-1}(u)]_{Trop} - [\mathcal{S}^{-1}(b_{\mathbf{i}}^{-1}\phi_\lambda(\eta_\lambda))]_{Trop}.$$

Substituting $t = c_{\mathbf{i}}\phi_{\lambda^*}(\eta_{\lambda^*})$ into (6.3), we obtain

$$0 = b_{\mathbf{i}}^{-1}\eta_{\lambda^*} = [S(c_{\mathbf{i}}\phi_{\lambda^*}(\eta_{\lambda^*}))]_{Trop} + b_{\mathbf{i}}^{-1}\phi_\lambda(\eta_\lambda).$$

Hence

$$[\mathcal{S}^{-1}(b_{\mathbf{i}}^{-1}\phi_\lambda(\eta_\lambda))]_{Trop} = -c_{\mathbf{i}}\phi_{\lambda^*}(\eta_{\lambda^*}).$$

Hence we have

$$c_{\mathbf{i}}\phi_{\lambda^*} b_{\mathbf{i}}(u) = [\mathcal{S}^{-1}(u)]_{Trop} + c_{\mathbf{i}}\phi_{\lambda^*}(\eta_{\lambda^*}),$$

giving the required result (since $\lambda \mapsto \lambda^*$ is an involution). □

We can compute the constant term in this formula as follows:

Lemma 6.3. *Let* $\mathbf{i}$ *be a reduced expression for* w_0*. Then* $c_{\mathbf{i}}(\phi_\lambda\eta_\lambda) = (v_1, v_2, \dots, v_N)$*, where, for* $1 \leq k \leq N$*, we have*

$$v_k = \langle s_{i_{k-1}} \cdots s_{i_1}\lambda, \alpha_{i_k}^\vee \rangle.$$

Proof. This follows from [18, 28.1.4], since we are computing the string of the lowest weight vector in $V_q(\lambda^*)^*$. □

We remark that $\phi_\lambda(\eta_\lambda) = z_{\lambda^*}$, so this lemma is computing the string $c_{\mathbf{i}}z_{\lambda^*}$.

7. The string parametrisation of a twisted standard adapted subalgebra

Recall (see Corollary 4.3) that the monomials in the elements $z_1, z_2, \dots, z_n$ and the elements $\phi(c_k^{\mathbf{i}\sigma})$, $k = 1, 2, \dots, N$, form an adapted basis for the twisted standard adapted subalgebra $\phi A_{\mathbf{i}}^\sigma$ of R^+. Our aim is to compute the string parameters of these elements. We therefore use Proposition 6.2 to apply the map $c_{\mathbf{i}}\phi_{\mu_k} b_{\mathbf{i}}$ to the element $b_{\mathbf{i}}^{-1}(c_k^{\mathbf{i}\sigma})$, for each $1 \leq k \leq N$; see 3.4. These last vectors were described in Proposition 3.6(iv).

7.1. For convenience, we define a matrix V with columns given by these vectors.

Definition 7.1. Let $M_N(\mathbb{Z})$ denote the ring of $N \times N$ matrices with integer entries. Let $V = (V_{jk}) \in M_N(\mathbb{Z})$ be defined by

$$V_{jk} = \begin{cases} 1 & j \leq k, \text{ if } i_j = i_k, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathbf{v}_k$ denote the kth column of V; this is $b_{\mathbf{i}}^{-1}(c_k^{\mathbf{i}\sigma})$ by Proposition 3.6(iv).

7.2. In order to apply Proposition 6.2 to compute $c_{\mathbf{i}}\phi_{\mu_k}b_{\mathbf{i}}(\mathbf{v}_k)$, for each $1 \leq k \leq N$, we first need to apply $(\mathcal{S}^{-1})_{Trop}$ to $\mathbf{v}_k$.

By Lemma 6.1, we have that $(\mathcal{S}^{-1})_{Trop}(u_1, u_2, \ldots, u_N) = (t_1, t_2, \ldots, t_N)$, where

$$\begin{aligned} t_k &= -u_k + \sum_{j>k} \langle s_{i_{k+1}} \cdots s_{i_{j-1}} \alpha_{i_j}, \alpha_{i_k}^\vee \rangle u_j. \\ &= \sum_{j=1}^{N} T_{kj} u_j, \end{aligned}$$

where $T = (T_{jk})$ is the matrix defining the linear map $(\mathcal{S}^{-1})_{Trop}$. We have

$$T_{jk} = \begin{cases} -1 & j = k, \\ \langle s_{i_{j+1}} \cdots s_{i_{k-1}} \alpha_{i_k}, \alpha_{i_j}^\vee \rangle & j < k, \\ 0 & \text{otherwise.} \end{cases}$$

We note that T is the inverse of the matrix S defining the linear map $\mathcal{S}_{Trop}$.

Definition 7.2. Let $C = (C_{jk}) \in M_N(\mathbb{Z})$ be the matrix given by

$$C_{jk} = \begin{cases} \langle s_{i_{j+1}} \cdots s_{i_k} \varpi_{i_k}, \alpha_{i_j}^\vee \rangle & j \leq k, \\ 0 & \text{otherwise.} \end{cases}$$

We have:

Lemma 7.3. *Let S, V and C be the matrices defined above. Then we have $S^{-1}V = -C$.*

Proof. We have that $(S^{-1}V)_{jl} = \sum_{j \leq k \leq l} T_{jk} V_{kl}$, and is zero if $j > l$. If $j \leq l$, then, using that $\langle\, ,\, \rangle$ is W-invariant,

$$(S^{-1}V)_{jl} = T_{jj} V_{jl} + \sum_{j<k\leq l, i_k = i_l} \langle s_{i_{k-1}} \cdots s_{i_{j+1}} \alpha_{i_j}^\vee, \alpha_{i_l} \rangle. \tag{7.1}$$

Note that $T_{jj}V_{jl} = -\delta_{i_j, i_l}$. Now, by calculating explicitly the coefficient of $\alpha_{i_l}^\vee$ in $s_{i_l} \cdots s_{i_j} \alpha_{i_j}^\vee$, we find that it is equal to the right-hand side of 7.1. Hence,

$$(S^{-1}V)_{jl} = \langle \varpi_{i_l}, s_{i_l} \cdots s_{i_j} \alpha_{i_j}^\vee \rangle = -\langle s_{i_{j+1}} \cdots s_{i_l} \varpi_{i_l}, \alpha_{i_j}^\vee \rangle,$$

which is equal to $-C_{jl}$ as required. □

We have the following corollary:

Corollary 7.4. *The entries of C are nonnegative.*

Proof. In proof of Lemma 7.3, we noted that C_{jl} is equal to the negative of the coefficient of $\alpha_{i_l}^\vee$ in the negative coroot $s_{i_l} \cdots s_{i_j} \alpha_{i_j}^\vee$. □

7.3. We now would like to compute $c_{\mathbf{i}}\phi_{\mu_k}b_{\mathbf{i}}(\mathbf{v}_k)$ for each k (note that the $\mathbf{v}_k$ are the columns of V). In the following lemma we set $\varepsilon_l(c_k^{\mathbf{i}}) = \varepsilon_l(b_{s_{i_1} \ldots s_{i_k}, \varpi_{i_k}})$ by a misuse of notation.

Lemma 7.5. *For $1 \leq l \leq n$ and $1 \leq k \leq N$, we have*

$$\varepsilon_l(c_k^{\mathbf{i}}) = \begin{cases} -\langle s_{i_1} \cdots s_{i_k} \varpi_{i_k}, \alpha_l^\vee \rangle & \text{if } \langle s_{i_1} \cdots s_{i_k} \varpi_{i_k}, \alpha_l^\vee \rangle \leq 0, \\ 0 & \text{otherwise.} \end{cases}$$

In particular,

$$\mu_k = \sum_{1 \leq l \leq n, \langle s_{i_1} \cdots s_{i_k} \varpi_{i_k}, \alpha_l^\vee \rangle \leq 0} -\langle s_{i_1} \cdots s_{i_k} \varpi_{i_k}, \alpha_l^\vee \rangle \varpi_l.$$

Proof. This follows from the definition of the $c_k^{\mathbf{i}}$ and [18, 28.1.4] (and sl_2-representation theory). ☐

Let $\mathbf{c}_k$ be the kth column of $C = -S^{-1}V$, and let $\mathbf{p}_k = c_{\mathbf{i}}(\phi_{\mu_k}\eta_{\mu_k})$. Let P be the matrix with columns $\mathbf{p}_k$, $k = 1, 2, \ldots, N$.

Lemma 7.6. *For* $1 \leq k \leq N$, *we have*

$$c_{\mathbf{i}}\phi_{\mu_k} b_{\mathbf{i}}(\mathbf{v}_k) = -\mathbf{c}_k + \mathbf{p_k}.$$

In particular, the entries of $-C + P$ *are nonnegative.*

Proof. This follows immediately from Proposition 6.2 and Lemma 7.3. ☐

As a consequence, we have:

Proposition 7.7. *Let* $\mathbf{i}$ *be any reduced expression for* w_0. *Then* $c_{\mathbf{i}}\phi(A_{\mathbf{i}}^{\sigma})$ *coincides with the nonnegative integer span of the columns of* $-C + P$ *and the strings* $c_{\mathbf{i}}(z_i)$, $1 \leq i \leq n$.

Proof. By Proposition 3.6, $c_{\mathbf{i}}\phi(A_{\mathbf{i}}^{\sigma})$ is the nonnegative integer span of the $c_{\mathbf{i}}\phi_{\mu_k} b_{\mathbf{i}}(\mathbf{v}_k)$ together with the strings $c_{\mathbf{i}}(z_i)$, $1 \leq i \leq n$. Since $c_{\mathbf{i}}(b^* b'^*) = c_{\mathbf{i}}(b^*) + c_{\mathbf{i}}(b'^*)$ when the elements b^* and b'^* of the dual canonical basis are multiplicative [4, Cor. 3.3], the proposition follows from Lemma 7.6 and Corollary 4.3. ☐

7.4. We also note the following formula for the entries of P:

Lemma 7.8. *For* $1 \leq j, k \leq N$, *we have*

$$P_{jk} = \sum_{1 \leq l \leq n, \langle s_{i_1} \cdots s_{i_k} \varpi_{i_k}, \alpha_l^{\vee} \rangle \leq 0} -\langle s_{i_1} \cdots s_{i_k} \varpi_{i_k}, \alpha_l^{\vee} \rangle \langle s_{i_{j-1}} \cdots s_{i_1} \varpi_l, \alpha_{i_j}^{\vee} \rangle.$$

Proof. By definition P_{jk} is the jth entry of $c_{\mathbf{i}}(\phi_{\mu_k}\eta_{\mu_k})$. Hence, the lemma results from Lemma 7.5 and Lemma 6.3. ☐

7.5. We will next show that some of the columns of $-C + P$ are entirely zero, and therefore can be neglected in Proposition 7.7.

Definition 7.9. Given $k \in \{1, 2, \ldots, N\}$, we set $k(1) = \min\{j : j > k, i_j = i_k\}$, i.e., the first occurrence of i_k to the right of i_k in $\mathbf{i}$. If there is no such occurrence, we set $k(1) = N + 1$.

Lemma 7.10. *Suppose that* $k(1) = N + 1$. *Then, for* $j = 1, 2, \ldots, N$, *we have* $P_{jk} = C_{jk}$, *i.e., the* k*th column of* P *coincides with the* k*th column of* C.

Proof. We have

$$C_{jk} = \begin{cases} \langle s_{i_{j+1}} \cdots s_{i_k} \varpi_{i_k}, \alpha_{i_j}^{\vee} \rangle & j \leq k, \\ 0 & j > k. \end{cases}$$

Since $k(1) = N + 1$, we have

$$C_{jk} = \langle s_{i_{j+1}} \cdots s_{i_N} \varpi_{i_k}, \alpha_{i_j}^{\vee} \rangle,$$

(in either case). By Lemma 7.5,

$$\begin{aligned}
\mu_k &= \sum_{1\le l\le n, \langle s_{i_1}\cdots s_{i_k}\varpi_{i_k},\alpha_l^\vee\rangle\le 0} -\langle s_{i_1}\cdots s_{i_k}\varpi_{i_k}\alpha_l^\vee\rangle\varpi_l \\
&= \sum_{1\le l\le n, \langle s_{i_1}\cdots s_{i_N}\varpi_{i_k},\alpha_l^\vee\rangle\le 0} -\langle s_{i_1}\cdots s_{i_N}\varpi_{i_k}\alpha_l^\vee\rangle\varpi_l \\
&= \sum_{1\le l\le n, \langle w_0\varpi_{i_k},\alpha_l^\vee\rangle\le 0} -\langle w_0\varpi_{i_k},\alpha_l^\vee\rangle\varpi_l \\
&= -w_0\varpi_{i_k}.
\end{aligned}$$

Hence, by Lemma 6.3

$$\begin{aligned}
P_{jk} &= -\langle s_{i_{j-1}}\cdots s_{i_1}w_0\varpi_{i_k},\alpha_{i_j}^\vee\rangle \\
&= -\langle s_{i_j}\cdots s_{i_N}\varpi_{i_k},\alpha_{i_j}^\vee\rangle \\
&= \langle s_{i_{j+1}}\cdots s_{i_N}\varpi_{i_k},\alpha_{i_j}^\vee\rangle \\
&= C_{jk}.
\end{aligned}$$

□

7.6. We replace the zero columns in $-C+P$ with vectors which we will see are the strings of the elements $z_1, z_2, \ldots, z_n$, in order to obtain a matrix whose nonnegative integer span is the set of string parameters of the twisted standard adapted subalgebra of R^+ corresponding to $\mathbf{i}$. We call this matrix X:

Definition 7.11. Let $X = (X_{jk}) \in M_N(\mathbb{Z})$ be the matrix defined as follows:

$$X_{jk} = \begin{cases} \langle s_{i_{j-1}}\cdots s_{i_1}\varpi_{i_k},\alpha_{i_j}^\vee\rangle, & k(1) = N+1, \\ -C_{jk}+P_{jk}, & \text{otherwise.}\end{cases}$$

We note that the P_{jk} are given by Lemma 7.8, and that the C_{jk} are given in Definition 7.2. Also, it follows from Lemma 6.3 that if $k(1) = N+1$, then the kth column of X is the string $c_{\mathbf{i}}z_{\varpi_{i_k}^*}$.

We have:

Proposition 7.12. *Let* $\mathbf{i}$ *be any reduced expression for* w_0*. Then* $c_{\mathbf{i}}\phi(A_{\mathbf{i}}^\sigma)$ *coincides with the nonnegative integer span of the columns of* X*. In particular, the entries of* X *are nonnegative.*

Proof. We note that the matrix X is the same as $-C+P$, except that if $1 \le k \le N$ and $k(1) = N+1$, then the kth column (which is zero by Lemma 7.10) is replaced by the string of $z_{\varpi_{i_k}^*}$. The result now follows from Proposition 7.7. □

8. Lusztig cones and twisted standard adapted subalgebras

In this section, we will show that the cone of string parameters of $c_{\mathbf{i}}\phi(A_{\mathbf{i}}^\sigma)$, given by the nonnegative integer span of the columns of X, coincides with the Lusztig cone corresponding to $\mathbf{i}$. At the same time we will show that the Lusztig cones are simplicial.

8.1. First we define a matrix $\widetilde{L}$ whose rows include the defining inequalities of the Lusztig cone corresponding to $\mathbf{i}$ as a subset of $\mathbb{N}^N$. This matrix will later be modified to a matrix defining the Lusztig cone as a subset of $\mathbb{Z}^N$.

Definition 8.1. Let $\widetilde{L} \in M_N(\mathbb{Z})$ be the matrix defined by

$$\widetilde{L}_{jk} = \begin{cases} -1 & k = j \text{ or } k = j(1), \\ -a_{i_j,i_k} & j < k < j(1), \\ 0 & \text{otherwise.} \end{cases}$$

Remark 8.2. Let $\widetilde{\mathbf{r}}_j$ denote the jth row of $\widetilde{L}$. The defining inequalities of the Lusztig cone $\mathcal{L}_{\mathbf{i}}$ (as a subset of $\mathbb{Z}^N_{\geq 0}$) are those inequalities of the form $\widetilde{\mathbf{r}}_j \cdot \mathbf{c} \geq 0$ for those j such that $j(1) \leq N$.

8.2. We will next show how this matrix is related to the matrices S and V already considered, i.e., that $V^{-1}S = \widetilde{L}$. This will have the consequence that $\widetilde{L}S^{-1}V = I$, in particular, showing that the columns of $S^{-1}V$ satisfy the defining inequalities of the Lusztig cone corresponding to $\mathbf{i}$. We recall from Proposition 5.4 that $\mathcal{S}_{Trop}$ is defined by the matrix $S = (S_{jk}) \in M_N(\mathbb{Z})$ where

$$S_{jk} = \begin{cases} -1 & j = k, \\ -a_{i_j,i_k} & j < k, \\ 0 & \text{otherwise.} \end{cases}$$

We next need to compute the inverse of the matrix V.

Lemma 8.3. *Let $W = (W_{jk}) \in M_N(\mathbb{Z})$ be the matrix defined as follows:*

$$W_{jk} = \begin{cases} 1 & j = k, \\ -1 & j < k, k = j(1), \\ 0 & \textit{otherwise.} \end{cases}$$

Then $W = V^{-1}$.

Proof. We show that $WV = I$, the identity matrix. The j,l-entry of WV is given by $Z_{jl} = \sum_{k=1}^{N} W_{jk}V_{kl}$. For this to be nonzero, we must have $j \leq k \leq l$ and $k = j$ or $k = j(1)$. There are 5 cases:

Case (a): If $l < j$, then clearly $Z_{jl} = 0$.

Case (b): If $l = j$, then $Z_{jl} = Z_{jj} = W_{jj}V_{jj} = 1 \cdot 1 = 1$.

Case (c): If $j < l < j(1)$, then $Z_{jl} = W_{jj}V_{jl} = 1 \cdot 0 = 0$.

Case (d): If $l = j(1)$, then $Z_{jl} = W_{jj}V_{j,j(1)} + W_{j,j(1)}V_{j(1),j(1)} = 1\cdot 1 + (-1)\cdot 1 = 0$.

Case (e): If $l > j(1)$, then $Z_{jl} = W_{jj}V_{jl} + W_{j,j(1)}V_{j(1),l} = V_{jl} - V_{j(1),l} = 0$ since $V_{jl} = V_{j(1),l}$. □

Lemma 8.4. *Let V, S and $\widetilde{L}$ be the matrices as defined above. Then $V^{-1}S = \widetilde{L}$.*

Proof. The j,l-entry of $V^{-1}S$ is given by $Y_{jl} = \sum_{k=1}^{N} W_{jk}S_{kl}$. To be nonzero, we must have $j \leq k \leq l$ and $k = j$ or $k = j(1)$. As before, we have the 5 cases:

Case (a): If $l < j$, then clearly $Y_{jl} = 0$.

Case (b): If $l = j$, then $Y_{jl} = Y_{jj} = W_{jj}S_{jj} = 1 \cdot (-1) = -1$.

Case (c): If $j < l < j(1)$, then $Y_{jl} = W_{jj}S_{jl} = S_{jl} = -a_{i_j,i_l}$.

Case (d): If $l = j(1)$, then $Y_{jl} = Y_{j,j(1)} = W_{jj}S_{j,j(1)} + W_{j,j(1)}S_{j(1),j(1)} = 1 \cdot (-2) + (-1) \cdot (-1) = -1$.

Case (e): If $l > j(1)$, then $Y_{jl} = W_{jj}S_{jl} + W_{j,j(1)}S_{j(1),l} = S_{jl} - S_{j(1),l} = -a_{i_j,i_l} + a_{i_{j(1)},i_l} = -a_{i_j,i_l} + a_{i_j,i_l} = 0$.

We see that $Y_{jl} = \widetilde{L}_{jl}$ in every case. $\square$

8.3. We now define a slightly altered version of the matrix $\widetilde{L}$, whose rows will eventually be seen as the defining inequalities of the Lusztig cone as a subset of $\mathbb{Z}^N$. We will also see that this matrix is the inverse of the matrix X.

Definition 8.5. Let $L = (L_{jk}) \in M_N(\mathbb{Z})$ be the matrix defined as follows:

$$L_{jk} = \begin{cases} -1 & k = j \text{ or } k = j(1), \\ -a_{i_j,i_k} & j < k < j(1), \\ 1 & j(1) = N+1,\ s_{i_1}s_{i_2}\cdots s_{i_{k-1}}(\alpha_{i_k}) = \alpha_{i_j}, \\ 0 & \text{otherwise.} \end{cases}$$

First, we show how some rows of L are related to the strings of lowest weight vectors:

Lemma 8.6. *Let*

$$\mathbf{v} = c_{\mathbf{i}}(\phi_\lambda \eta_\lambda) = (v_1, v_2, \ldots, v_N),$$

as in Lemma 6.3. *Suppose that* $1 \le j \le j(1) \le N$. *Let* $\mathbf{r}_j$ *be the* j*th row of* L. *Then we have* $\mathbf{r}_j \cdot \mathbf{v} = 0$.

Proof. Recall that for $k = 1, 2, \ldots, N$, we have

$$v_k = \langle s_{i_{k-1}} \cdots s_{i_1}\lambda, \alpha_{i_k}^\vee \rangle,$$

by Lemma 6.3. We have

$$\begin{aligned} \mathbf{r}_j \cdot \mathbf{v} &= -v_j - v_{j_{(1)}} - \sum_{j<k<j_{(1)}} a_{i_j,i_k} v_k \\ &= -\langle s_{i_{j-1}} \cdots s_{i_1}\lambda, \alpha_{i_j}^\vee \rangle - \langle s_{i_{j(1)-1}} \cdots s_{i_1}\lambda, \alpha_{i_j}^\vee \rangle - \sum_{j<k<j(1)} a_{i_j,i_k} \langle s_{i_{k-1}} \cdots s_{i_1}\lambda, \alpha_{i_k}^\vee \rangle. \end{aligned}$$

We note that

$$s_{i_k}(\alpha_{i_j}^\vee) = \alpha_{i_j}^\vee - \langle \alpha_{i_k}, \alpha_{i_j}^\vee \rangle \alpha_{i_k}^\vee = \alpha_{i_j}^\vee - a_{i_j,i_k}\alpha_{i_k}^\vee,$$

so

$$\begin{aligned} \mathbf{r}_j \cdot \mathbf{v} &= -\langle s_{i_{j-1}} \cdots s_{i_1}\lambda, \alpha_{i_j}^\vee \rangle - \langle s_{i_{j(1)-1}} \cdots s_{i_1}\lambda, \alpha_{i_j}^\vee \rangle \\ &\quad + \sum_{j<k<j(1)} \langle s_{i_{k-1}} \cdots s_{i_1}\lambda, s_{i_k}(\alpha_{i_j}^\vee) - \alpha_{i_j}^\vee \rangle \\ &= -\langle s_{i_{j-1}} \cdots s_{i_1}\lambda, \alpha_{i_j}^\vee \rangle - \langle s_{i_{j(1)-1}} \cdots s_{i_1}\lambda, \alpha_{i_j}^\vee \rangle \\ &\quad + \langle s_{i_{j(1)-1}} \cdots s_{i_1}\lambda, \alpha_{i_j}^\vee \rangle - \langle s_{i_j} \cdots s_{i_1}\lambda, \alpha_{i_j}^\vee \rangle = 0, \end{aligned}$$

the sum telescoping. $\square$

Next, we show that some entries of $-C + P = S^{-1}V + P$ are zero:

Lemma 8.7. *Suppose that* $1 \le j \le N$ *and that* $s_{i_1}s_{i_2}\cdots s_{i_{j-1}}(\alpha_{i_j}^\vee) = \alpha_r^\vee$ *is a simple coroot. Then*

$$P_{jk} = C_{jk}.$$

Proof. We note that, by Lemma 7.8,

$$P_{jk} = \begin{cases} -\langle s_{i_1} \cdots s_{i_k}\varpi_{i_k}, \alpha_r^\vee \rangle & \langle s_{i_1} \cdots s_{i_k}\varpi_{i_k}, \alpha_r^\vee \rangle \le 0, \\ 0 & \langle s_{i_1} \cdots s_{i_k}\varpi_{i_k}, \alpha_r^\vee \rangle > 0, \end{cases}$$

using the fact that $\langle \varpi_l, \alpha_r^\vee \rangle = \delta_{lr}$. Suppose first that $j \leq k$. Then:

$$\begin{aligned} \langle s_{i_1} \cdots s_{i_k} \varpi_{i_k}, \alpha_r^\vee \rangle &= \langle s_{i_{j+1}} \cdots s_{i_k} \varpi_{i_k}, s_{i_j} \cdots s_{i_1} \alpha_r^\vee \rangle \\ &= \langle s_{i_{j+1}} \cdots s_{i_k} \varpi_{i_k}, s_{i_j} \alpha_{i_j}^\vee \rangle \\ &= -\langle s_{i_{j+1}} \cdots s_{i_k} \varpi_{i_k}, \alpha_{i_j}^\vee \rangle \\ &= -C_{jk} \leq 0, \end{aligned}$$

by Corollary 7.4. So $P_{jk} = C_{jk}$. If $j > k$, then

$$\begin{aligned} \langle s_{i_1} \cdots s_{i_k} \varpi_{i_k}, \alpha_r^\vee \rangle &= \langle \varpi_{i_k}, s_{i_k} \cdots s_{i_1} \alpha_r^\vee \rangle \\ &= \langle s_{i_{j-1}} \cdots s_{i_{k+1}} \varpi_{i_k}, s_{i_{j-1}} \cdots s_{i_{k+1}} s_{i_k} \cdots s_{i_1} \alpha_r^\vee \rangle \\ &= \langle s_{i_{j-1}} \cdots s_{i_{k+1}} \varpi_{i_k}, \alpha_{i_j}^\vee \rangle \\ &= \langle \varpi_{i_k}, s_{i_{k+1}} \cdots s_{i_{j-1}} \alpha_{i_j}^\vee \rangle \geq 0, \end{aligned}$$

since $s_{i_{k+1}} \cdots s_{i_{j+1}} \alpha_{i_j}^\vee$ is a positive coroot. It follows that $P_{jk} = 0 = C_{jk}$. □

Remark 8.8. We note that the condition

$$s_{i_1} \cdots s_{i_k}(\alpha_{i_j}^\vee) = \alpha_r^\vee$$

is equivalent to the condition

$$s_{i_1} \cdots s_{i_k}(\alpha_{i_j}) = \alpha_r.$$

Second, this result shows that $(S^{-1}V + P)_{jk} = 0$ under this assumption, by Lemma 7.3.

8.4. We can now prove the following, as we have all the pieces we need:

Proposition 8.9. *Let L and X be the matrices defined as above, so that the non-negative integer span of the columns of X is the cone of the string parameters of the twisted standard adapted subalgebra corresponding to* $\mathbf{i}$. *Then $LX = I$, the identity matrix.*

Proof. Denote by $\mathbf{x}_k$ the kth column of X. Suppose first that $1 \leq j \leq j(1) \leq N$, and that $1 \leq k \leq k(1) \leq N$. Then $\mathbf{r}_j$ (the jth row of L) is the same as the jth row of $\widetilde{L}$. By Lemma 8.4, $\widetilde{L}S^{-1}V = I$. Hence $\mathbf{r}_j \cdot (-\mathbf{c}_k) = \delta_{jk}$, since $-\mathbf{c}_k$ is the kth column of $S^{-1}V$. By Lemma 8.6, $\mathbf{r}_j \cdot \mathbf{p}_k = 0$ (where $\mathbf{p}_k$ is the kth column of P). It follows (from the definition of X) that $\mathbf{r}_j \cdot \mathbf{x}_k = \delta_{jk}$.

If $1 \leq j \leq j(1) \leq N$ and $k(1) = N + 1$, then $\mathbf{x}_k = c_{\mathbf{i}} z^*_{\varpi_{i_k}}$. By Lemma 8.6, we have that $\mathbf{r}_j \cdot \mathbf{x}_k = 0$ (as required, noting that we must have $j \neq k$).

If $j(1) = N + 1$, then let $1 \leq l \leq N$ be defined by $s_{i_1} \cdots s_{i_{l-1}} \alpha_{i_l} = \alpha_{i_j}$. Then $L_{jl} = 1$ is the only nonzero entry in $\mathbf{r}_j$. It follows that $\mathbf{r}_j.\mathbf{x}_k = X_{lk}$.

Case (a): Suppose that $k(1) \leq N$. Then $X_{lk} = -C_{lk} + P_{lk} = 0$ by Lemma 8.7.

Case (b): Suppose that $k(1) = N + 1$. Then

$$\begin{aligned} X_{lk} &= \langle s_{i_{l-1}} \cdots s_{i_1} \varpi_{i_k}, \alpha_{i_l}^\vee \rangle \\ &= \langle \varpi_{i_k}, s_{i_1} \cdots s_{i_{l-1}} \alpha_{i_l}^\vee \rangle \\ &= \langle \varpi_{i_k}, \alpha_{i_j}^\vee \rangle \\ &= \delta_{j,k}, \end{aligned}$$

as required (noting that $j(1) = k(1) = N + 1$).
The proposition is proved. □

If $\mathbf{c} = (c_1, c_2, \dots, c_N) \in \mathbb{Z}^N$, we write $\mathbf{c} \geq 0$ to denote $c_k \geq 0$ for $k = 1, 2, \dots, N$. We have the following consequences.

Theorem 8.10. *The Lusztig cone $\mathcal{L}_{\mathbf{i}}$ is simplicial, defined by the matrix L:*

$$\mathcal{L}_{\mathbf{i}} = \{\mathbf{c} \in \mathbb{Z}^N \ : \ L\mathbf{c} \geq 0\}.$$

It coincides with the nonnegative integer span of the columns of the matrix X (see Definition 7.11).

Proof. By Proposition 7.12, the entries of X are nonnegative. By Proposition 8.9, $XL = I$, so nonnegative integer combinations of the rows of L are of the form $(0, 0, \dots, 0, 1, 0, \dots, 0)$ (with a 1 in the kth position) and therefore correspond to inequalities of the form $c_k \geq 0$. So

$$\{\mathbf{c} \in \mathbb{Z}^N \ : \ L\mathbf{c} \geq 0\} \subseteq \mathbb{N}^N.$$

Since the inequalities corresponding to rows of L are either defining inequalities of $\mathcal{L}_{\mathbf{i}}$ or inequalities of the form $c_k \geq 0$, the claimed equality follows, and it is then immediate that $\mathcal{L}_{\mathbf{i}}$ is spanned by the columns of $L^{-1} = X$. $\square$

Remark 8.11. The fact that $\mathcal{L}_{\mathbf{i}}$ is simplicial was already known for quiver-compatible reduced expressions for w_0 for $\mathbf{g}$ simply laced [1] and for all reduced expressions for w_0 in type A_n [22].

And we have:

Theorem 8.12. *Let $\mathbf{i}$ be any reduced expression for w_0. Let $\mathcal{L}_{\mathbf{i}}$ denote the Lusztig cone corresponding to $\mathbf{i}$. Let $\phi(A^\sigma_{\mathbf{i}})$ denote the twisted standard adapted subalgebra corresponding to $\mathbf{i}$. Then*

$$c_{\mathbf{i}}(\phi(A^\sigma_{\mathbf{i}})) = \mathcal{L}_{\mathbf{i}}.$$

Proof. This follows from Proposition 7.12 and Theorem 8.10. $\square$

Example 8.13. Suppose that $\mathfrak{g} = sl_4(\mathbb{C})$ (type A_3). Let $\mathbf{i} = (2, 3, 2, 1, 2, 3)$, a reduced expression for w_0. Then we have

$$V = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} -1 & -1 & 1 & 0 & -1 & 1 \\ 0 & -1 & -1 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

$$C = -TV = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The matrix X of spanning vectors of the Lusztig cone corresponding to $\mathbf{i}$ and the defining matrix L are given by:

$$X = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad L = X^{-1} = \begin{pmatrix} -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

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