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SUBFIELD SYMMETRIC SPACES FOR FINITE SPECIAL LINEAR GROUPS

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Abstract. Let G be a connected algebraic group defined over a finite field $\mathbf{F}_q$. For each irreducible character ρ of $G(\mathbf{F}_{q^r})$, we denote by $m_r(\rho)$ the multiplicity of $1_{G(\mathbf{F}_q)}$ in the restriction of ρ to $G(\mathbf{F}_q)$. In the case where G is reductive with connected center and is simple modulo center, Kawanaka determined $m_2(\rho)$ for almost all cases, and then Lusztig gave a general formula for $m_2(\rho)$. In the case where the center of G is not connected, such a result is not known. In this paper we determine $m_2(\rho)$, up to some minor ambiguity, in the case where G is the special linear group.

We also discuss, for any $r \geq 2$, the relationship between $m_r(\rho)$ with the theory of Shintani descent in the case where G is a connected algebraic group.

0. Introduction

Let G be a connected reductive group defined over a finite field $\mathbf{F}_q$ with Frobenius map F. We consider the finite group G^{F^2} and its subgroup G^F. The quotient space G^{F^2}/G^F is regarded as an analogue of the symmetric space, and is called the subfield symmetric space over a finite field. The determination of spherical functions of G^{F^2}/G^F is almost equivalent to the determination of irreducible characters of the Hecke algebra $H(G^{F^2}, G^F)$. For a class function f on G^{F^2}, we denote by $m_2(f)$ the inner product of f with the induced character $\mathrm{Ind}_{G^F}^{G^{F^2}} 1$. The classification of irreducible characters of $H(G^{F^2}, G^F)$ and the determination of their degrees are equivalent to the determination of $m_2(\rho)$ for all irreducible characters ρ of G^{F^2}.

In [K2], Kawanaka computed $m_2(\rho)$ in the case where G is a classical group with connected center, or in the case where ρ is unipotent and the characteristic is good. Extending Kawanaka's result, Lusztig gave in [L3] a closed formula for $m_2(\rho)$ valid for any G which has the connected center and is simple modulo its center. He expects that his formula is still valid for G with disconnected center. In turn, Henderson studied in [H] the spherical functions of G^{F^2}/G^F by making use of the theory of perverse sheaves, and described them in the case where $G = GL_n$, in which case $H(G^{F^2}, G^F)$ is abelian.

In this paper, we consider $G = SL_n$ with the standard $\mathbf{F}_q$-structure, which is the first example of the disconnected center case. Based on the parametrization of irreducible characters and the description of almost characters in [S3] (which is valid

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under some restriction on p, for example, $p \geq 2n$), we determine $m_2(\rho)$ (Theorem 5.3) for any irreducible characters, up to some minor ambiguity. In particular, we have $m_2(\rho) \in \{0, 1, 2\}$. We discuss the relationship between our result and Lusztig's conjectural formula.

Kawanaka's main idea for the computation of $m_2(\rho)$, beside the use of the results of Lusztig on $m_2(R_T(\theta))$, is to connect it with the twisted Frobenius-Schur indicator through the twisting operator. In Section 1, we generalize Kawanaka's result, and discuss a connection of $m_2(\rho)$ with Shintani descent. This leads to a formula for $m_2(R_x)$ where R_x is an almost character of G^{F^2}, which is regarded as a counterpart of Lusztig's formula for $m_2(\chi_A)$ in [L3, 7], where χ_A is the characteristic function of character sheaves. In Section 1, we also discuss a more general situation. We define $m_r(\rho)$ as the multiplicity of an irreducible character ρ of G^{F^r} with the induced character $\mathrm{Ind}_{G^F}^{G^{F^r}} 1$ for any integer $r \geq 2$. We give some formula (Theorem 1.14) for $m_r(R_x)$, although it is not so effective as the m_2 case.

The subsequent sections are devoted to the computation of $m_2(\rho)$ for the case where $G = SL_n$. We obtain the results by applying the results in Section 1, together with the computation of $m_2(\widetilde{\rho}|_{G^{F^2}})$ for irreducible characters $\widetilde{\rho}$ of $GL_n(\mathbf{F}_{q^2})$.

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Some notation. All the representations considered in this paper are over $\bar{\mathbf{Q}}_l$. For a finite group Γ, we denote by Irr Γ the set of irreducible characters of Γ. If δ is an automorphism of Γ, we denote by $(\mathrm{Irr}\ \Gamma)^\delta$ the set of F-stable irreducible characters of Γ.

Assume that Γ is an abelian group. In this case, we also use the notation $\Gamma^\wedge$ to denote the set of irreducible characters of Γ, which coincides with the group $\mathrm{Hom}(\Gamma, \bar{\mathbf{Q}}_l^*)$. If δ is an automorphism of Γ, we denote by $\Gamma_\delta^\wedge$ the subgroup of $\Gamma^\wedge$ consisting of δ-fixed irreducible characters of Γ. Also in this case, we denote by Γ_δ the largest quotient of Γ on which δ acts trivially, i.e., Γ_δ is the quotient of Γ by the subgroup generated by $g^{-1}\delta(g)$ for $g \in \Gamma$. Don't confuse $\Gamma_\delta^\wedge$ with $(\Gamma^\wedge)_\delta$ for a finite group $\Gamma^\wedge$. The group $(\Gamma^\wedge)_\delta$ does not occur in this paper.

Contents

1. $G(\mathbf{F}_q)$-invariants in $G(\mathbf{F}_{q^r})$-modules and Shintani descent

1.1. For any finite group Γ and an automorphism $F : \Gamma \to \Gamma$, we denote by $\Gamma/{\sim_F}$ the set of F-twisted conjugacy classes in Γ, where $x, y \in \Gamma$ are F-twisted conjugate if there exists $z \in \Gamma$ such that $y = z^{-1}xF(z)$. In the case where F acts trivially on Γ, the set $\Gamma/{\sim_F}$ coincides with the set of conjugacy classes in Γ, which we denote by $\Gamma/{\sim}$.

For a connected algebraic group X defined over $\mathbf{F}_q$, and two Frobenius maps F_1, F_2 on X such that $F_1F_2 = F_2F_1$, we define a norm map

$$N_{F_1/F_2} : X^{F_1}/\sim_{F_2} \to X^{F_2}/\sim_{F_1^{-1}}$$

as follows: for $x \in X^{F_1}$, we choose $\alpha \in X$ such that $x = \alpha^{-1}F_2(\alpha)$, and put $x' = F_1(\alpha)\alpha^{-1}$. Then $x' \in X^{F_2}$ and the correspondence $x \to x'$ induces a bijective map N_{F_1/F_2}, which we call the norm map from $X^{F_1}/\sim_{F_2}$ to $X^{F_2}/\sim_{F_1^{-1}}$.

For a finite set Y, we denote by $C(Y)$ the $\bar{\mathbf{Q}}_l$-space of all $\bar{\mathbf{Q}}_l$-valued functions on Y. Then the norm map N_{F_1/F_2} induces a linear isomorphism

$$Sh_{F_1/F_2} = N^{*-1}_{F_1/F_2} : C(X^{F_1}/\sim_{F_2}) \to C(X^{F_2}/\sim_{F_1^{-1}}),$$

which is called the Shintani descent from X^{F_1} to X^{F_2}.

1.2. Let G be a connected algebraic group defined over a finite field $\mathbf{F}_q$ with Frobenius map F. We fix a positive integer r, and consider the group $H = G\times\cdots\times G$ (r-factors). H is endowed with the natural Frobenius map given by $(g_1, \dots, g_r) \mapsto (F(g_1), \dots, F(g_r))$, which we also denote by F. Let $F' = F\omega : H \to H$ be a twisted Frobenius map on H, where $\omega : H \to H, (g_1, \dots, g_r) \mapsto (g_r, g_1, \dots, g_{r-1})$ is the cyclic permutation of factors. Since $\omega^r = 1$ and $F\omega = \omega F$, we have $(F')^{rm} = F^{rm}$ for any $m \geq 1$.

Lemma 1.3. *The map $G^{F^{rm}} \to H^{F^{rm}}, x \to (x, 1, \dots, 1)$ induces a bijection*

$$f : G^{F^{rm}}/\sim_{F^r} \to H^{F^{rm}}/\sim_{F'} . \tag{1.3.1}$$

Proof. Take $x = (x_1, \dots, x_r), y = (y_1, \dots, y_r) \in H^{F^{rm}}$. If x and y are in the same class, there exists $z = (z_1, \dots, z_r)$ such that $y_i = z_i^{-1}x_iF(z_{i-1})$ for $i \in \mathbf{Z}/r\mathbf{Z}$. Now assume that $x = (x_1, 1, \dots, 1)$. Then $z^{-1}xF'(z) = (y_1, 1, \dots, 1)$ for $z \in G^{F^{rm}}$ if and only if $z = (z_1, F(z_1), \dots, F^{r-1}(z_1))$. Moreover, in this case, $y_1 = z_1^{-1}x_1F^r(z_1)$. This shows that the map f is well defined, and is injective. It is easy to see that each F'-conjugacy class in $H^{F^{rm}}$ contains a representative of the form $(x_1, 1, \dots, 1)$. Hence f is surjective. ☐

1.4. For each $x \in G^{F^{rm}}$, $k \geq 1$, we put $N_k(x) = xF(x)\cdots F^{k-1}(x)$. Then the map $G^{F^{rm}} \to G^{F^{rm}}, x \mapsto N_k(x)$ induces a map $G^{F^{rm}}/\sim_F \to G^{F^{rm}}/\sim_{F^k}$, which we also denote by N_k. Let $\Delta(H) \simeq G$ be the diagonal subgroup of H. The inclusion $\Delta(H)^{F^{rm}} \hookrightarrow H^{F^{rm}}$ induces a map $d : \Delta(H)^{F^{rm}}/\sim_F \to H^{F^{rm}}/\sim_{F'}$. Then we have a commutative diagram

$$\begin{array}{ccc} G^{F^{rm}}/\sim_{F^r} & \xrightarrow{f} & H^{F^{rm}}/\sim_{F'} \\ {\scriptstyle N_r}\uparrow & & \uparrow{\scriptstyle d} \\ G^{F^{rm}}/\sim_F & \xrightarrow{f_0} & \Delta(H)^{F^{rm}}/\sim_F, \end{array} \tag{1.4.1}$$

where f_0 is the bijection induced from the isomorphism $G \xrightarrow{\sim} \Delta(H)$. This follows from the following relation for $x \in G^{F^{rm}}$,

$$(N_r(x), 1, \dots, 1) = y^{-1}(x, x, \dots, x)F'(y)$$

with $y = (1, N_1(x), N_2(x), \dots, N_{r-1}(x))$.

1.5. Concerning the norm maps, we have the following commutative diagram:

$$
\tag{1.5.1}
\begin{array}{ccc}
G^{F^{rm}}/{\sim_{F^r}} & \xrightarrow{N_{F^{rm}/F^r}} & G^{F^r}/{\sim} \\
{\scriptstyle N_r}\big\uparrow & & \big\uparrow{\scriptstyle j} \\
G^{F^{rm}}/{\sim_F} & \xrightarrow{N_{F^{rm}/F}} & G^F/{\sim},
\end{array}
$$

where j is the map induced from the inclusion $G^F \hookrightarrow G^{F^r}$. We show (1.5.1). Let $\hat{x} = G^{F^{rm}}$ and take $\alpha \in G$ such that $\hat{x} = \alpha^{-1}F(\alpha)$. Then $N_{F^{rm}/F}(\hat{x})$ is represented by $x = F^{rm}(\alpha)\alpha^{-1}$. On the other hand, since $\hat{x}' = N_r(\hat{x}) = \alpha^{-1}F^r(\alpha)$, we see that $N_{F^{rm}/F^r}(\hat{x}')$ is represented by $F^{rm}(\alpha)\alpha^{-1}$ which coincides with $j(x)$. This shows the commutativity.

1.6. Let $\sigma' = F'|_{H^{F^{rm}}}$, and $\widetilde{H}^{F^{rm}}$ be the semidirect product of $H^{F^{rm}}$ with the cyclic group $\langle\sigma'\rangle$ of order rm generated by σ'. For a character χ of $G^{F^{rm}}$, we define the character $F(\chi)$ by $F(\chi)(F(g)) = \chi(g)$, and similarly for H. An irreducible character ψ of $H^{F^{rm}}$ is F'-stable if and only if ψ is of the form that

$$
\tag{1.6.1}
\psi = \chi \otimes F(\chi) \otimes \cdots \otimes F^{r-1}(\chi)
$$

for some F^r-stable irreducible character χ on $G^{F^{rm}}$. Let V_i $(1 \le i \le r)$ be an irreducible $G^{F^{rm}}$-module for the irreducible character $F^{i-1}(\chi)$. Then there exists a linear isomorphism $T_i : V_i \to V_{i+1}$ such that $T_i \circ g = F(g) \circ T_i$ for any $g \in G^{F^{rm}}$ with $V_{r+1} = V_1$ and that $(T_rT_{r-1}\cdots T_1)^m = 1$. Let ψ be as in (1.6.1). Then ψ is afforded by the $H^{F^{rm}}$-module $V_1 \otimes V_2 \otimes \cdots \otimes V_r$. Let us define an action of σ' on $V_1 \otimes \cdots \otimes V_r$ by

$$
\sigma' = \omega \circ (T_1 \otimes T_2 \otimes \cdots \otimes T_r),
$$

where ω is the cyclic permutation of factors given by

$$
\omega(x_1 \otimes x_2 \otimes \cdots \otimes x_r) = x_r \otimes x_1 \otimes \cdots \otimes x_{r-1}.
$$

Then we have $\sigma' \circ h = F'(h) \circ \sigma'$ for $h \in H^{F^{rm}}$, and so $V_1 \otimes \cdots \otimes V_r$ can be extended to an $\widetilde{H}^{F^{rm}}$-module. We denote by $\widetilde{\psi}$ the corresponding extension of ψ to $\widetilde{H}^{F^{rm}}$.

Let $\sigma = F|_{G^{F^{rm}}}$, and we consider $G^{F^{rm}}\langle\sigma\rangle$ the semidirect product of $G^{F^{rm}}$ with the cyclic group $\langle\sigma\rangle$ of order rm generated by σ. We define an action of σ^r on V_1 by $\sigma^r = T_rT_{r-1}\cdots T_1$. Then $\sigma^r \circ g = F^r(g) \circ \sigma^r$ for any $g \in G^{F^{rm}}$, and the $G^{F^{rm}}$-module V_1 can be extended to a $G^{F^{rm}}\langle\sigma^r\rangle$-module $\widetilde{V}_1$. We denote by $\widetilde{\chi}$ the corresponding extension of χ to $G^{F^{rm}}\langle\sigma^r\rangle$. We show the following lemma.

Lemma 1.7. *Let $h = (g, 1, \dots, 1) \in H^{F^{rm}}$ with $g \in G^{F^{rm}}$. Let χ be an F^r-stable irreducible character of $G^{F^{rm}}$. Then for $\psi = \chi \otimes F(\chi) \otimes \cdots \otimes F^{r-1}(\chi) \in \operatorname{Irr} H^{F^{rm}}$, we have*

$$
\widetilde{\psi}(h\sigma') = \widetilde{\chi}(g\sigma^r).
$$

Proof. Let $v_1^{(1)}, \dots, v_n^{(1)}$ be a basis of V_1. We define a basis $v_1^{(i+1)}, \dots, v_n^{(i+1)}$ of V_{i+1} inductively by $v_j^{(i+1)} = T_i(v_j^{(i)})$ for $i = 1, 2, \dots r-1$. Then we have

$$
T_r(v_j^{(r)}) = T_r \cdots T_1(v_j^{(1)}) = \sigma^r v_j^{(1)}.
$$

It follows that

$$
h\sigma' \cdot v_{i_1}^{(1)} \otimes v_{i_2}^{(2)} \otimes \cdots \otimes v_{i_r}^{(r)} = (g\sigma^r v_{i_r}^{(1)}) \otimes v_{i_1}^{(2)} \otimes \cdots \otimes v_{i_{r-1}}^{(r)},
$$

and we have

$$\widetilde{\psi}(h\sigma') = \mathrm{Tr}\,(h\sigma', V_1 \otimes \cdots \otimes V_r) = \mathrm{Tr}\,(g\sigma^r, V_1) = \widetilde{\chi}(g\sigma^r).$$

This proves the lemma. □

1.8. Let χ be an F^r-stable irreducible character of $G^{F^{rm}}$, and $\widetilde{\chi}$ be its extension to $G^{F^{rm}}\langle\sigma^r\rangle$ as in the previous lemma. Let $G^{F^{rm}}\sigma^r/\!\sim$ be a set of conjugacy classes in $G^{F^{rm}}\langle\sigma^r\rangle$ contained in the coset $G^{F^{rm}}\sigma^r$. Under the natural bijection $G^{F^{rm}}/\!\sim_{F^r} \simeq G^{F^{rm}}\sigma^r/\!\sim$ via $x \leftrightarrow x\sigma^r$, we have an isomorphism $C(G^{F^{rm}}/\!\sim_{F^r}) \simeq C(G^{F^{rm}}\sigma^r/\!\sim)$. Thus $\widetilde{\chi}|_{G^{F^{rm}}\sigma^r}$ defines an element in the space $C(G^{F^{rm}}/\!\sim_{F^r})$. Put

$$R^{(m)}_{\widetilde{\chi}} = Sh_{F^{rm}/F^r}(\widetilde{\chi}|_{G^{F^{rm}}\sigma^r}).$$

Hence $R^{(m)}_{\widetilde{\chi}}$ is a class function on G^{F^r}. We have the following formula.

Proposition 1.9. *Under the notation as above,*

$$|G^{F^{rm}}|^{-1} \sum_{\hat{g}\in G^{F^{rm}}} \widetilde{\chi}(N_r(\hat{g})\sigma^r) = |G^F|^{-1} \sum_{g\in G^F} R^{(m)}_{\widetilde{\chi}}(g). \tag{1.9.1}$$

Proof. Take $\hat{g} \in G^{F^{rm}}$. Write $\hat{g}$ as $\hat{g} = \alpha^{-1}F(\alpha)$ and put $g = F^{rm}(\alpha)\alpha^{-1}$. Then $g \in G^F$, and we see that $\widetilde{\chi}(N_r(\hat{g})\sigma^r) = R^{(m)}_{\widetilde{\chi}}(g)$ by (1.5.1). Moreover, it is known that

$$\sharp\{x \in G^{F^{rm}} \mid x^{-1}\hat{g}F(x) = \hat{g}\} = \sharp\{y \in G^F \mid y^{-1}gy = g\}.$$

The formula (1.9.1) is immediate from these two facts. □

1.10. Let $c^{(m)}_r(\widetilde{\chi})$ be the left-hand side of (1.9.1), i.e.,

$$c^{(m)}_r(\widetilde{\chi}) = |G^{F^{rm}}|^{-1} \sum_{\hat{g}\in G^{F^{rm}}} \widetilde{\chi}(N_r(\hat{g})\sigma^r). \tag{1.10.1}$$

Then $c^{(m)}_r(\widetilde{\chi})$ is a generalization of the twisted Frobenius-Schur indicator discussed in Kawanaka and Matsuyama [KM]. In the case where $m = 1$, we simply write $c^{(1)}_r(\widetilde{\chi})$ as $c_r(\chi)$. Note that in this case, the extension does not enter the formula, and we have

$$c_r(\chi) = |G^{F^r}|^{-1} \sum_{g\in G^{F^r}} \chi(N_r(g)).$$

If $r = 2$, $c_2(\chi)$ coincides with the Frobenius-Schur indicator defined in [KM].

Let us define, for a class function f of G^{F^r},

$$m_r(f) = \langle f, \mathrm{Ind}_{G^F}^{G^{F^r}} 1\rangle = |G^F|^{-1} \sum_{x\in G^F} f(x). \tag{1.10.2}$$

Then the identity (1.9.1) can be rewritten as

$$c^{(m)}_r(\widetilde{\chi}) = m_r(R^{(m)}_{\widetilde{\chi}}). \tag{1.10.3}$$

We note that (1.10.3) is a generalization of the formula due to Kawanaka [K2, (1.1)]. In fact, in the case where $m = 1$, the Shintani descent Sh_{F^r/F^r} coincides with the inverse of the twisting operator t_1^* on $C(G^{F^r}/\!\sim)$ given in [K2], and so we have $R^{(1)}_{\chi} = t_1^{*-1}\chi$. Then (1.10.3) implies the following.

Corollary 1.11. *Let the notation be as above. Then we have $c_r(\chi) = m_r(t_1^{*-1}\chi)$.*

In the case where $r = 2$, this formula is nothing but the formula (1.1) in [K2].

1.12. By Lemma 1.7, $\widetilde{\chi}(N_r(\hat{g})\sigma^r) = \widetilde{\psi}(h\sigma')$ with $h = (N_r(\hat{g}), 1, \dots, 1) \in H^{F^{rm}}$. As in 1.4, h is F'-conjugate to $(\hat{g}, \dots, \hat{g}) \in \varDelta(H)^{F^{rm}}$, and so

$$\widetilde{\psi}(h\sigma') = \widetilde{\psi}((\hat{g}, \dots, \hat{g})\sigma').$$

On the other hand, under the isomorphism $\varDelta(H)^{F^{rm}} \simeq G^{F^{rm}}$, $V_1 \otimes \cdots \otimes V_r$ is an $G^{F^{rm}}$-module, and its character $\chi F(\chi) \cdots F^{r-1}(\chi)$ is F-stable. Moreover, we have $\sigma' \circ g = F(g) \circ \sigma'$ on $V_1 \otimes \cdots \otimes V_r$ for any $g \in G^{F^{rm}}$. This implies that the action of σ' defines a structure of $G^{F^{rm}}\langle\sigma\rangle$-module on $V_1 \otimes \cdots \otimes V_r$, where σ acts by σ' on it. We denote the character of this module by $\widetilde{\psi}_0$, which is an extension of $\chi F(\chi) \cdots F^{r-1}(\chi)$. Thus, we have

$$\widetilde{\psi}((\hat{g}, \dots, \hat{g})\sigma') = \widetilde{\psi}_0(\hat{g}\sigma).$$

Now (1.9.1) can be rewritten as

$$m_r(R^{(m)}_{\widetilde{\chi}}) = |G^{F^{rm}}|^{-1} \sum_{\hat{g} \in G^{F^{rm}}} \widetilde{\psi}_0(\hat{g}\sigma). \tag{1.12.1}$$

1.13. Let k be a positive integer. We define an inner product on $C(G^{F^k}\sigma/{\sim})$ by

$$\langle f, h\rangle_{G^{F^k}\sigma} = |G^{F^k}|^{-1} \sum_{x \in G^{F^k}} f(x\sigma)\overline{h(x\sigma)}$$

for $f, h \in C(G^{F^k}\sigma/\sim)$. Then the following orthogonality relations are known. For any F-stable irreducible characters χ, χ' of G^{F^k} and their extensions $\widetilde{\chi}, \widetilde{\chi}'$ to $G^{F^k}\langle\sigma\rangle$,

$$\langle \widetilde{\chi}, \widetilde{\chi}'\rangle_{G^{F^k}\sigma} = \begin{cases} \theta(\sigma) & \text{if } \widetilde{\chi} = \theta \otimes \widetilde{\chi}' \text{ with } \theta \in \operatorname{Irr}\langle\sigma\rangle, \\ 0 & \text{if } \chi' \neq \chi. \end{cases} \tag{1.13.1}$$

Here in the left-hand side, $\widetilde{\chi}, \widetilde{\chi}'$ are regarded as functions on $G^{F^{rm}}\sigma$ by restriction.

For any $f \in C(G^{F^k}\sigma/{\sim})$, we put

$$\widetilde{M}_k(f) = \langle f, \widetilde{1}\rangle_{G^{F^k}\sigma} = |G^{F^k}|^{-1} \sum_{x \in G^{F^k}} f(x\sigma),$$

where $\widetilde{1}$ means the restriction of the unit character of $G^{F^k}\langle\sigma\rangle$ to $G^{F^k}\sigma$. We also put, for a class function h of G^{F^k},

$$M_k(h) = \langle h, 1\rangle_{G^{F^k}} = |G^{F^k}|^{-1} \sum_{x \in G^{F^k}} h(x).$$

The following statement is immediate from (1.13.1).

(1.13.2) Let ρ be an F-stable character of G^{F^k}, and $\widetilde{\rho}$ its extension to $\widetilde{G}^{F^k}$. Then we have $|\widetilde{M}_k(\widetilde{\rho})| \le M_k(\rho)$. Moreover, if $M_k(\rho) = 1$, then $\widetilde{M}_k(\widetilde{\rho})$ is a k-th root of unity.

We have the following theorem.

Theorem 1.14. *Let χ be an F^r-stable irreducible character of $G^{F^{rm}}$, and $\widetilde{\chi}$ an extension of χ to $G^{F^{rm}}\langle\sigma^r\rangle$. Let $\widetilde{\psi}_0$ be the extension of $\chi F(\chi) \cdots F^{r-1}(\chi)$ to $G^{F^{rm}}\langle\sigma\rangle$ as in 1.12. Put $Sh_{F^{rm}/F^r}(\widetilde{\chi}|_{G^{F^{rm}}\sigma^r}) = R^{(m)}_{\widetilde{\chi}}$.*

(i) *We have* $c_r^{(m)}(\widetilde{\chi}) = m_r(R^{(m)}_{\widetilde{\chi}}) = \widetilde{M}_{rm}(\widetilde{\psi}_0)$. *In particular,*

$$|m_r(R^{(m)}_{\widetilde{\chi}})| \le M_{rm}(\chi F(\chi)\cdots F^{r-1}(\chi)).$$

Furthermore, if $M_{rm}(\chi F(\chi)\cdots F^{r-1}(\chi)) = 1$, *we have* $|m_r(R^{(m)}_{\widetilde{\chi}})| = 1$.

(ii) *Assume that* $r = 2$. *Then there exists a* $2m$*-th root of unity* ζ *such that*

$$m_2(R^{(m)}_{\widetilde{\chi}}) = \begin{cases} \zeta & \text{if } \overline{\chi} = F(\chi), \\ 0 & \text{otherwise}, \end{cases}$$

where $\overline{\chi}$ *is the complex conjugate of the character* χ.

Proof. The equality $m_r(R^{(m)}_{\widetilde{\chi}}) = \widetilde{M}_{rm}(\widetilde{\psi}_0)$ in (i) follows from (1.12.1). The inequality in (i) follows from (1.13.2). Assume that $r = 2$. Then we have

$$M_{2m}(\chi F(\chi)) = \langle \chi F(\chi), 1\rangle_{G^{F^{2m}}} = \langle F(\chi), \overline{\chi}\rangle_{G^{F^{2m}}} = \begin{cases} 1 & \text{if } F(\chi) = \overline{\chi}, \\ 0 & \text{otherwise.} \end{cases}$$

So the assertion (ii) follows from (1.13.2). This proves the theorem. □

1.15. In the case where $r = 2$, we determine the quantity $\zeta = c_2^{(m)}(\widetilde{\chi})$ more explicitly. Let χ be an F^2-stable irreducible character of $G^{F^{2m}}$ and $\widetilde{\chi}$ its extension to $G^{F^{2m}}\langle\sigma^2\rangle$ as in the theorem. Let us assume that $F(\chi) = \overline{\chi}$. We follow the setting in 1.6. In particular, V_1 (resp. V_2) is a $G^{F^{2m}}$-module affording χ (resp. $F(\chi)$). Since $F(\chi) = \overline{\chi}$, the subspace $W = (V_1 \otimes V_2)^{G^{F^{2m}}}$ of $G^{F^{2m}}$-invariant vectors in $V_1 \otimes V_2$ is of dimension 1. The map $\sigma' : V_1 \otimes V_2 \to V_1 \otimes V_2, v_1 \otimes v_2 \mapsto T_2(v_2) \otimes T_1(v_1)$ preserves the space W, and the eigenvalue of σ' on W coincides with $\zeta = c_2^{(m)}(\widetilde{\chi})$. The map $\sigma^2 = T_2T_1 : V_1 \to V_1$ extends the $G^{F^{2m}}$-module V_1 to the $G^{F^{2m}}\langle\sigma^2\rangle$-module $\widetilde{V}_1$ affording the character $\widetilde{\chi}$.

The $G^{F^{2m}}$-module V_2 can be identified with V_1 by replacing the action of $g \in G^{F^{2m}}$ by $F(g)$. Under this identification, we may take $T_1 = \mathrm{Id}_{V_1}$ and $T_2 = \sigma^2$ on V_1. Hence we have $\sigma'(v_1 \otimes v_2) = \sigma^2(v_2) \otimes v_1$. Now the averaging operator $V_1 \otimes V_2 \to W, v \mapsto |G^{F^{2m}}|^{-1}\sum_{g \in G^{F^{2m}}} g \cdot v$ determines a bilinear form $B : V_1 \times V_1 \to \bar{\mathbf{Q}}_l$ (up to scalar) having the following properties:

$$\begin{aligned} B(g\cdot v_1, F(g)\cdot v_2) &= B(v_1, v_2) \quad \text{for } g \in G^{F^{2m}}, v_1, v_2 \in V_1, \\ B(\sigma^2(v_2), v_1) &= \zeta B(v_1, v_2) \quad \text{for } v_1, v_2 \in V_1. \end{aligned} \tag{1.15.1}$$

Conversely, if there exists such a bilinear form on V_1, this form coincides with B up to scalar. Hence ζ determines the value $c_2^{(m)}(\widetilde{\chi})$.

The extension $\widetilde{\chi}$ of χ is determined by the choice of T_1, T_2 such that $(T_2T_1)^m = \mathrm{Id}_{V_1}$. If we replace T_1 by a scalar multiple ξT_1 for an m-th root of unity ξ, it gives a different extension of $\widetilde{\chi}'$ of χ. By changing $\widetilde{\chi}$ by $\widetilde{\chi}'$, the eigenvalue ζ of σ' on W is replaced by $\xi\zeta$. Summing up the above arguments, we have the following refinement of Theorem 1.14, which is a generalization of Theorem 2.1.3 in [K2].

Corollary 1.16. *Let* χ *be an* F^2*-stable irreducible character of* $G^{F^{2m}}$ *and* $\widetilde{\chi}$ *an extension of* χ *to* $G^{F^{2m}}\langle\sigma^2\rangle$.

(i) *We have*

$$c_2^{(m)}(\widetilde{\chi}) = \begin{cases} \zeta & \text{if } F(\chi) = \overline{\chi}, \\ 0 & \text{otherwise}, \end{cases}$$

where ζ is an $2m$-th root of unity.

(ii) *Assume that $F(\chi) = \overline{\chi}$. Let ζ_0 be a primitive $2m$-th root of unity in $\bar{\mathbf{Q}}_l$. Then there exists a unique extension $\widetilde{\chi}$ of χ such that $c_2^{(m)}(\widetilde{\chi}) = 1$ or ζ_0. Let V_1 be the $G^{F^{2m}}\langle\sigma^2\rangle$-module affording $\widetilde{\chi}$. Then $c_2^{(m)}(\widetilde{\chi}) = 1$ (resp ζ_0) if and only if there exists a non-zero bilinear form $B(\cdot,\cdot)$ on V_1 satisfying (1.15.1) with $\zeta = 1$ (resp. $\zeta = \zeta_0$).*

1.17. In the case where G is a connected reductive group with connected center, Lusztig defined in [L1] almost characters of G^F. In the case where G is a special linear group SL_n with F of split type, almost characters are also formulated in [S3]. In either case, the set of almost characters coincides with the set of $Sh_{F^m/F}(\widetilde{\chi}|_{G^{F^m}\sigma})$, up to an m-th root of unity multiple, for sufficiently divisible m, where χ runs over all the F-stable irreducible characters of G^{F^m}. We denote by R_χ the almost character of G^F corresponding to χ. As a corollary to Theorem 1.14, we have the following result.

Corollary 1.18. *Assume that G is either a connected reductive group with connected center, or SL_n with F of split type. Let R_χ be the almost character of G^{F^2} associated to an F^2-stable irreducible character χ of $G^{F^{2m}}$. Then we have*

$$m_2(R_\chi) = \begin{cases} \zeta & \text{if } F(\chi) = \overline{\chi}, \\ 0 & \text{otherwise}, \end{cases} \tag{1.18.1}$$

where ζ is a certain $2m$-th root of unity.

Remark 1.19. In [L3, Prop. 7.2], Lusztig proved a formula concerning the characteristic functions of character sheaves as follows. Let A be an F^2-stable character sheaf of a connected reductive group G. We denote by $\chi_{A,\phi_A} \in C(G^{F^2}/\!\sim)$ the characteristic function of A with respect to an isomorphism $\phi_A : (F^2)^*A \xrightarrow{\sim} A$. Then under the assumption that q is sufficiently large (and that χ_{A,ϕ_A} can be written as a linear combination of irreducible characters with cyclotomic integer coefficients), there exists a choice of ϕ_A such that

$$m_2(\chi_{A,\phi_A}) = \begin{cases} (-1)^{\dim \operatorname{supp} A} & \text{if } F^*(A) \simeq DA, \\ 0 & \text{otherwise}, \end{cases} \tag{1.19.1}$$

where DA is the Verdier dual of A. Since the proof depends on the asymptotic behavior of $q \to \infty$, the condition on q is considerably large. In the case where G has a connected center, using the description of $m_2(\chi)$ for any irreducible character χ of G^{F^2} in [L3], (1.18.1) can be verified directly. In [S2], it was shown that almost characters coincide with the characteristic functions of character sheaves whenever G has a connected center. A similar result was also shown in [S4] for SL_n with F of split type. Hence the formula (1.18.1) is a counter part of (1.19.1) to almost characters, which works without any assumption on q. Also, Theorem 1.14 (ii) is regarded as an extension of (1.19.1) to arbitrary connected algebraic groups.

1.20. As a special case of the situation discussed in Theorem 1.14 (i), we consider the case where $G = GL_n$ with the standard or non-standard Frobenius map F over $\mathbf{F}_q$. Irreducible characters of G^{F^r} are described as follows. Let $G^* \simeq GL_n$ be the dual group of G. For each F^r-stable semisimple class $\{s\}$, choose a representative $s \in G^{*F^r}$. Let T^* be a maximally split maximal torus in $Z_{G^*}(s)$. Let $W = N_{G^*}(T^*)/T^*$ be the Weyl group of G^*, and put $W_s = \{w \in W \mid w(s) = s\}$. Then W_s is the Weyl group of $Z_{G^*}(s)$, and F^r acts naturally on W_s, which we denote by δ. Let $(\mathrm{Irr}\ W_s)^\delta$ be the set of F^r-stable irreducible representations of W_s. For each $E \in (\mathrm{Irr}\ W_s)^\delta$, we fix an extension $\widetilde{E}$ of E to the semidirect group $W_s\langle\delta\rangle$, where $\langle\delta\rangle$ is the infinite cyclic group with generator δ. Put

$$R_{s,\widetilde{E}} = |W_s|^{-1} \sum_{w \in W_s} \mathrm{Tr}\,(w\delta, \widetilde{E}) R_{T^*_w}(s),$$

where $R_{T^*_w}(s)$ denotes the Deligne-Lusztig character $R_{T_w}(\theta)$ under the natural correspondence $(s, T^*_w) \leftrightarrow (\theta, T_w)$.

It is known, under a suitable choice of the extension, $\pm R_{s,\widetilde{E}}$ gives rise to an irreducible character of G^{F^r}, which we denote by $\rho_{s,E}$. Then the set $\mathrm{Irr}\ G^{F^r}$ of irreducible characters of G^{F^r} is given as

$$\mathrm{Irr}\ G^{F^r} = \coprod_{\{s\}} \{\rho_{s,E} \mid E \in (\mathrm{Irr}\ W_s)^\delta\},$$

where $\{s\}$ runs over F^r-stable semisimple conjugacy classes in G^*.

Let (s, T^*) be as above. We choose an F^r-stable maximal torus T of G which is dual to T^*, and let B be a Borel subgroup of G containing T. We choose an integer $m > 0$ such that F^{mr} leaves B invariant. One can find a linear character θ of $T^{F^{rm}}$ corresponding to $s \in T^{*F^{rm}}$. Then we have

$$\mathrm{End}_{G^{F^{rm}}}\big(\mathrm{Ind}_{B^{F^{rm}}}^{G^{F^{rm}}} \widetilde{\theta}\big) \simeq \bar{\mathbf{Q}}_l[W_s],$$

where $\widetilde{\theta}$ is the lift of θ to the linear character of $B^{F^{rm}}$. Let us denote by $\chi_{\theta,E}$ the irreducible constituent of $\mathrm{Ind}_{B^{F^{rm}}}^{G^{F^{rm}}} \widetilde{\theta}$ corresponding to $E \in \mathrm{Irr}\ W_s$. Then $\chi_{\theta,E}$ is F^r-stable if and only if $E \in (\mathrm{Irr}\ W_s)^\delta$, and in which case, $Sh_{F^{rm}/F^r}(\widetilde{\chi}_{\theta,E}|_{G^{F^{rm}}\sigma^r})$ coincides with $\rho_{s,E}$ up to a scalar multiple. Thus under this setting, Theorem 1.14 (i) can be rewritten as follows.

Corollary 1.21. *Let $G = GL_n$ with the standard or non-standard Frobenius map F. Then for each $\rho_{s,E} \in \mathrm{Irr}\ G^{F^r}$, we have*

$$m_r(\rho_{s,E}) \le M_{rm}(\chi_{\theta,E} F(\chi_{\theta,E}) \cdots F^{r-1}(\chi_{\theta,E})).$$

Moreover, if $M_{rm}(\chi_{\theta,E} F(\chi_{\theta,E}) \cdots F^{r-1}(\chi_{\theta,E})) = 1$, we have $m_r(\rho_{s,E}) = 1$.

2. Parametrization of irreducible characters of $SL_n(\mathbf{F}_{q^2})$

2.1. In the remainder of this paper, we assume that $\widetilde{G} = GL_n$ and $G = SL_n$ with Frobenius maps F with respect to the standard $\mathbf{F}_q$-structures. We assume that p is large enough so that the results in [S3] can be applied. (Although there is no assumption for p in [S3], it should be changed. Actually, Kawanaka's construction of generalized Gelfand-Graev characters of GL_n or SL_n requires no assumptions for p. However, our construction (cf. [S3, 2.3]) depends on the Dynkin-Kostant theory, which requires that p is not too small.) For example, $p \ge 2n$ is enough in our case. Let $\widetilde{G}^*$ (resp. G^*) be the dual group of $\widetilde{G}$ (resp. G). Then $\widetilde{G}^* \simeq GL_n$, and $G^* \simeq \widetilde{G}^*/\widetilde{Z}^*$, where $\widetilde{Z}^*$ is the center of $\widetilde{G}^*$. The inclusion map $G \hookrightarrow \widetilde{G}$ induces a natural surjection $\pi : \widetilde{G}^* \to G^*$. As in the case of $\widetilde{G}$, the set Irr G^{F^2} is partitioned as

$$\text{Irr } G^{F^2} = \coprod_{\{s\}} \mathcal{E}(G^{F^2}, \{s\}),$$

where $\{s\}$ runs over F^2-stable semisimple classes in G^*. Take s such that $F^2(s) = s$. Let T^* be an F^2-stable maximal torus of $Z_{G^*}(s)$ such that T^* is contained in an F^2-stable Borel subgroup of $Z_{G^*}(s)$. Let $\widetilde{T}^*$ be an F^2-stable maximal torus of $\widetilde{G}^*$ such that $\pi(\widetilde{T}^*) = T^*$. Then $W = N_{\widetilde{G}^*}(\widetilde{T}^*)/\widetilde{T}^*$ is naturally identified with $N_{G^*}(T^*)/T^*$. Put

$$W_s = N_{Z_{G^*}(s)}(T^*)/T^*, \quad W_s^0 = N_{Z^0_{G^*}(s)}(T^*)/T^*.$$

Then W_s^0 is the Weyl group of $Z^0_{G^*}(s)$. Now W_s can be decomposed as $W_s \simeq W_s^0 \rtimes \Omega_s$, where $\Omega_s = Z_{G^*}(s)/Z^0_{G^*}(s)$ is a cyclic group. If we choose $\dot{s} \in \widetilde{T}^*$ such that $\pi(\dot{s}) = s$, then W_s^0 is naturally identified with $W_{\dot{s}} = \{w \in W \mid w(\dot{s}) = \dot{s}\}$.

F^2 acts naturally on W_s. We denote by δ this action and consider the semidirect product $W_s\langle\delta\rangle$, where $\delta w \delta^{-1} = F^2(w)$. δ stabilizes W_s^0 and Ω_s.

2.2. For each $E \in \text{Irr } W_s^0$, let $\Omega_s(E)$ be the stabilizer of E in Ω_s. Assume that the Ω_s-orbit of E is δ-stable. Put

$$\widetilde{\Omega}_s(E) = \{u \in \Omega_s \mid {}^{u\delta}E = E\}.$$

Then one can write $\widetilde{\Omega}_s(E) = \Omega_s(E)a$ for some $a \in \Omega_s$. Since Ω_s is abelian, $\Omega_s(E)$ is δ-stable, and $\Omega_s(E)$ acts on $\widetilde{\Omega}_s(E)$ by $(z, u) \mapsto z^{-1}u\delta(z)$ for $z \in \Omega_s(E)$ and $u \in \widetilde{\Omega}_s(E)$. We denote by $\widetilde{\Omega}_s(E)_\delta$ the set of equivalent classes under this action. It is easy to see that $\widetilde{\Omega}_s(E)_\delta$ can be identified with the set $\Omega_s(E)_\delta a$, where $\Omega_s(E)_\delta$ is the largest quotient of $\Omega_s(E)$ on which δ acts trivially. Let $\overline{\text{Irr}}\, W_s^0$ be the set of Ω_s-orbits in the set Irr W_s^0. We denote by $(\overline{\text{Irr}}\, W_s^0)^\delta$ the set of δ-stable orbits in Irr W_s^0. By abuse of notation, $(\overline{\text{Irr}}\, W_s^0)^\delta$ means also a set of representatives for the δ-stable Ω_s-orbits in Irr W_s^0.

For each pair (s, E) with $E \in (\overline{\text{Irr}}\, W_s^0)^\delta$, put

$$\overline{\mathcal{M}}_{s,E} = \Omega_s^\delta(E)^\wedge \times \widetilde{\Omega}_s(E)_\delta,$$

where $\Omega_s^\delta = \{u \in \Omega_s, \delta(u) = u\}$ and $\Omega_s^\delta(E)$ is the stabilizer of E in Ω_s^δ, and $\Omega_s^\delta(E)^\wedge$ is the set of irreducible characters of $\Omega_s^\delta(E)$. It is known by [S3] that there exists a natural bijection

$$\mathcal{E}(G^{F^2}, \{s\}) = \coprod_{E \in (\overline{\text{Irr}}\, W_s^0)^\delta} \overline{\mathcal{M}}_{s,E}. \tag{2.2.1}$$

We denote by $\rho_{\eta,z}$ the irreducible character of G^{F^2} corresponding to $(\eta, z) \in \overline{\mathcal{M}}_{s,E}$.

The above parametrization satisfies the following properties. The set of G^{*F^2}-conjugacy classes in the set $\{s\}^{F^2}$ is in bijection with $(\Omega_s)_\delta$. For each $x \in (\Omega_s)_\delta$, take a representative $\dot{x} \in \Omega_s$, and let $\ddot{x} \in Z_{G^*}(s)$ be a representative of $\dot{x} \in \Omega_s = Z_{G^*}(s)/Z^0_{G^*}(s)$. Choose $g_x \in G^*$ such that $g_x^{-1}F^2(g_x) = \ddot{x}$, and put $s_x = g_x s g_x^{-1}$. Then $F^2(s_x) = s_x$, and $g_x T^* g_x^{-1} = T_x^*$ is a maximally split torus in $Z_{G^*}(s_x)$. We define $W_{s_x}^0$ in a similar way as W_s^0. Under the isomorphism $W_s^0 \to W_{s_x}^0$ induced by $\text{ad}\, g_x$, the action of $\dot{x}\delta$ on W_s^0 is transferred to the action of F^2 on $W_{s_x}^0$. Hence each $\dot{x}\delta$-stable irreducible character E' of W_s^0 determines the F^2-stable irreducible character E'' of $W_{s_x}^0$. Take an F^2-stable element $\dot{s}_x$ such that $\pi(\dot{s}_x) = s_x$. We consider the irreducible character $\rho_{\dot{s}_x,E''}$ of $\widetilde{G}^{F2}$ as in 1.20, which we denote by $\rho_{\dot{s}_x,E'}$, by abuse of the notation.

It is known from [S3, (4.4.2)] that there exists a natural bijection

$$f : \coprod_{E\in(\overline{\mathrm{Irr}}\,W_s^0)^\delta} \widetilde{\Omega}_s(E)_\delta \simeq \coprod_{x\in(\Omega_s)_\delta} (\mathrm{Irr}\ W_s^0)^{\dot{x}\delta}/\Omega_s^\delta, \tag{2.2.2}$$

where in the right-hand side, $(\mathrm{Irr}\ W_s^0)^{\dot{x}\delta}/\Omega_s^\delta$ means the set of Ω_s^δ-orbits of $\dot{x}\delta$-stable irreducible characters of W_s^0. The bijection is described as follows. Take E in a δ-stable Ω_s-orbit in $\mathrm{Irr}\ W_s^0$. For each $\dot{y} \in \widetilde{\Omega}_s(E)$, there exists $\dot{x} \in \Omega_s$ and $z \in \Omega_s$ such that $\dot{y} = z^{-1}\dot{x}\delta(z)$, where $\dot{x}$ is one of the representatives of $(\Omega_s)_\delta$ chosen above. Then $E_x = {}^zE \in (\mathrm{Irr}\ W_s^0)^{\dot{x}\delta}$. The correspondence $(E, y) \mapsto (x, E_x)$ gives rise to the required bijection f.

Under the above setting, we have

$$\rho_{\dot{s}_x,E_x}|_{G^{F^2}} = \sum_{\eta\in\Omega_s^\delta(E)^\wedge} \rho_{\eta,y}. \tag{2.2.3}$$

Let $\mathcal{T}_{s_x,E_x}$ be the set of irreducible characters occurring in the restriction of $\rho_{\dot{s}_x,E_x}$ to G^{F^2}. We also denote by $\overline{\mathcal{T}}_{s,E}$ the set of $\rho_{\eta,y}$ for $(\eta, y) \in \overline{\mathcal{M}}_{s,E}$. Then (2.2.2) implies that

$$\overline{\mathcal{T}}_{s,E} = \coprod_{(x,E_x)} \mathcal{T}_{s_x,E_x},$$

where (x, E_x) runs over all the pairs corresponding to (y, E) with $y \in \widetilde{\Omega}_s(E)_\delta$ under the map f.

Remark 2.3. In [S3, 4.5], the parameter set $\overline{\mathcal{M}}_{s,E}$ is defined as $\Omega_s^\delta(E)^\wedge \times \Omega_s(E)_\delta$. Since $\widetilde{\Omega}_s(E)_\delta = \Omega_s(E)_\delta a$, this set is in bijection with $\overline{\mathcal{M}}_{s,E}$ in this paper. However, the bijection depends on the choice of $a \in \Omega_s$, and the definition of $\overline{\mathcal{M}}_{s,E}$ in this paper is more convenient for later applications.

2.4. We describe the decomposition of $\rho_{\dot{s}_x,E_x}|_{G^{F^2}}$ in (2.2.3) more precisely. It is known by [L2] that $\mathcal{T}_{s_x,E_x}$ is in bijective correspondence with $\Omega_s^\delta(E)^\wedge$. This bijection is given as follows. The abelian group $\widetilde{G}^{F^2}/G^{F^2}$ acts transitively on $\mathcal{T}_{s_x,E_x}$ by the conjugation action. Also its dual group $(\widetilde{G}^{F^2}/G^{F^2})^\wedge$ acts on $\mathrm{Irr}\,\widetilde{G}^{F^2}$ by $(\theta, \widetilde{\rho}) \mapsto \theta \otimes \widetilde{\rho}$ for a linear character $\theta \in (\widetilde{G}^{F^2}/G^{F^2})^\wedge$ and $\widetilde{\rho} \in \mathrm{Irr}\,\widetilde{G}^{F^2}$. Then for $\rho_0 \in \mathcal{T}_{s_x,E_x}$, the stabilizer of ρ_0 in $\widetilde{G}^{F^2}/G^{F^2}$ and the stabilizer of $\rho_{\dot{s}_x,E_x}$ in $(\widetilde{G}^{F^2}/G^{F^2})^\wedge$ are orthogonal to each other under the natural duality pairing $\widetilde{G}^{F^2}/G^{F^2} \times (\widetilde{G}^{F^2}/G^{F^2})^\wedge \to \bar{\mathbf{Q}}_l$ (cf. [L2, 9]). Let $I(\rho_{\dot{s}_x,E_x})$ be the stabilizer of $\rho_{\dot{s}_x,E_x}$ in $(\widetilde{G}^{F^2}/G^{F^2})^\wedge$. Then, under the choice of ρ_0, the set $\mathcal{T}_{s_x,E_x}$ is in natural bijection with $I(\rho_{\dot{s}_x,E_x})^\wedge$.

We show that $I(\rho_{\dot{s}_x,E_x})$ is isomorphic to $\Omega_s^\delta(E)$. First note that there exists a natural isomorphism

$$\widetilde{Z}^{*F^2} \simeq \mathrm{Hom}(\widetilde{G}^{F^2}/G^{F^2}, \bar{\mathbf{Q}}_l^*) = (\widetilde{G}^{F^2}/G^{F^2})^\wedge. \tag{2.4.1}$$

If z is an element in $\widetilde{Z}^{*F^2}$ corresponding to $\theta \in (\widetilde{G}^{F^2}/G^{F^2})^\wedge$ under the above isomorphism, then θ maps $\mathcal{E}(\widetilde{G}^{F^2}, \{\dot{s}_x\})$ onto $\mathcal{E}(\widetilde{G}^{F^2}, \{z\dot{s}_x\})$. Put

$$\widetilde{Z}^{*F^2}_{s_x} = \{z \in \widetilde{Z}^{*F^2} \mid \ z\dot{s}_x \text{ is conjugate to } \dot{s}_x \text{ under } \widetilde{G}^*\},$$

which does not depend on the choice of $\dot{s}_x$ for s_x. Then, under the identification in (2.4.1), $I(\rho_{\dot{s}_x,E_x})$ is regarded as a subgroup of $\widetilde{Z}^{*F^2}_{s_x}$. Here we have a natural isomorphism

$$\omega_{s_x} : \Omega_s^\delta = \Omega_s^{x\delta} \simeq Z_{G^*}(s)^{\ddot{x}F^2}/Z^0_{G^*}(s)^{\ddot{x}F^2} \to \widetilde{Z}^{*F^2}_{s_x} \tag{2.4.2}$$

defined as follows. For $\bar{z} \in \Omega_s^{x\delta}$, take a representative $z \in Z_{G^*}(s)^{\ddot{x}F^2}$ of $\bar{z}$, and choose $\dot{z} \in \widetilde{G}^{*\ddot{x}F^2}$ such that $\pi(\dot{z}) = z$. Then ${}^{g_x}(\dot{s}^{-1}\dot{z}\dot{s}\dot{z}^{-1}) \in \widetilde{Z}^{*F^2}_{s_x}$, and the map $\bar{z} \mapsto {}^{g_x}(\dot{s}^{-1}\dot{z}\dot{s}\dot{z}^{-1})$ induces a well-defined isomorphism ω_{s_x} since $\pi(Z_{\widetilde{G}^*}(\dot{s}_x)) = Z^0_{G^*}(s_x)$. Now under the identification in (2.4.1), (2.4.2), we may see that $I(\rho_{\dot{s}_x,E_x})$ is a subgroup of Ω_s^δ, and in fact, $I(\rho_{\dot{s}_x,E_x})$ coincides with the stabilizer of E_x in Ω_s^δ. Thus we have $I(\rho_{\dot{s}_x,E_x}) = \Omega_s^\delta(E_x) = \Omega_s^\delta(E)$.

2.5. The bijection between $\mathcal{T}_{s_x,E_x}$ and $\Omega_s^\delta(E)^\wedge$ given in 2.4 depends on the choice of $\rho_0 \in \mathcal{T}_{s_x,E_x}$. We have to choose a specific ρ_0 for each $\mathcal{T}_{s_x,E_x}$. This problem is reduced to a certain special case, and is solved by the aid of generalized Gelfand-Graev characters.

Let $\mathfrak{g}$ be the Lie algebra of G with Frobenius map F. We have a bijection $\log : G_{\text{uni}} \to \mathfrak{g}_{\text{nil}}$ by $v \mapsto v - 1$, where G_{uni} (resp. $\mathfrak{g}_{\text{nil}}$) is the unipotent variety of G (resp. nilpotent variety of $\mathfrak{g}$). Let N be a nilpotent element in $\mathfrak{g}^F$. By Dynkin-Kostant theory, there exists a natural grading $\mathfrak{g} = \bigoplus_{i\in\mathbf{Z}} \mathfrak{g}_i$ associated to N. Let $\mathfrak{u}_i = \bigoplus_{j\ge i} \mathfrak{g}_j$. Then one can find an F-stable parabolic subgroup $P = LU_1$ associated to N, where L is an F-stable Levi subgroup of P with $\operatorname{Lie} L = \mathfrak{g}_0$, and U_1 is the unipotent radical of P with $\operatorname{Lie} U_1 = \mathfrak{u}_1$. Moreover, we have $N \in \mathfrak{g}_2$. Let k be an algebraic closure of $\mathbf{F}_q$. It is known by Kawanaka (see [K1]), that there exists an F-stable subspace $\mathfrak{u}$ ($\mathfrak{u}_{1.5}$ in the notation of [S3]) of $\mathfrak{u}_1$ containing $\mathfrak{u}_2$ and an F-equivariant linear map $\lambda : \mathfrak{u} \to k$ satisfying the following. There exists an F-stable connected unipotent subgroup U of U_1 such that $\log(U) = \mathfrak{u}$ and that the map $\lambda \circ \log : U \to k$ turns out to be an F-stable homomorphism of U. We define a linear character $\varLambda_N$ of U^{F^2} by $\varLambda_N = \psi_2 \circ \lambda \circ \log$, where $\psi_2 : \mathbf{F}_{q^2} \to \bar{\mathbf{Q}}_l^*$ is the additive character defined by $\psi_2 = \psi \circ \operatorname{Tr}_{\mathbf{F}_{q^2}/\mathbf{F}_q}$ for a non-trivial additive character $\psi : \mathbf{F}_q \to \bar{\mathbf{Q}}_l^*$. The generalized Gelfand-Graev character $\varGamma_N$ of G^{F^2} associated to N is defined as

$$\varGamma_N = \operatorname{Ind}_{U^{F^2}}^{G^{F^2}} \varLambda_N.$$

The character $\varGamma_N$ depends only on the G^{F^2}-conjugacy class of N.

We now consider the following special setting for the set $\overline{\mathcal{M}}_{s,E}$ determined by the pair (s, E).

(2.5.1) W_s^0 is isomorphic to $S_b \times \cdots \times S_b$ (t-times) with $b = n/t$, and $\Omega_s \simeq \langle w_0 \rangle$, where w_0 is an element of order t in W_s permuting the factors of W_s^0 transitively. Moreover, $E \in \operatorname{Irr} W_s^0$ is of the form

$$E = E_1 \boxtimes E_1 \boxtimes \cdots \boxtimes E_1 \quad \text{with} \quad E_1 \in \operatorname{Irr} S_b.$$

Then E is Ω_s-stable, and we have $\Omega_s = \Omega_s(E)$.

Now it is known from Lusztig ([L1, 13.4], see [S3, 2.9] for a brief description) that there exists a map $\rho \mapsto \mathcal{O}_\rho$ from Irr $\widetilde{G}^F$ to the set of nilpotent orbits in Lie $\widetilde{G}$. We take $N \in \mathfrak{g}^F$, a nilpotent element (in Jordan normal form) contained in $\mathcal{O}_\rho$, for $\rho = \rho_{\dot{s},E}$. (To be more explicit, $\mathcal{O}_\rho$ is a nilpotent orbit corresponding to the partition of n dual to $\mu \cup \cdots \cup \mu$ (t-times), where μ is a partition of n/t corresponding to E_1). Then it is known that there exists a unique irreducible character ρ_0 such that ρ_0 occurs both in $\mathit{\Gamma}_N$ and in $\rho_{\dot{s}_x,E_x}|_{G^{F^2}}$. By using this ρ_0, one obtains a bijection $\mathcal{T}_{s_x,E_x} \leftrightarrow \Omega_s^\delta(E)^\wedge$ as in 2.4. This is the parametrization given in (2.2.3), where if (x, E_x) corresponds to (E, y) by (2.2.2), then $\rho_{\eta,y}$ corresponds to $\eta \in \Omega_s^\delta(E)^\wedge$.

By the arguments in [S3, 4.5], the parametrization of $\mathcal{T}_{s_x,E_x}$ in the general case is reduced to the case given in (2.5.1). Accordingly, ρ_0 is determined for each $\mathcal{T}_{s_x,E_x}$. However, note that this parametrization still depends on the choice of a nilpotent element N in $\mathfrak{g}$. In what follows, we assume that

(2.5.2) Each nilpotent element $N \in \mathfrak{g}^F$ is taken to be a Jordan normal form.

2.6. In order to apply the results in Section 1, we need to know the condition when $F(\rho) = \overline{\rho}$ for an irreducible character ρ of G^{F^2}. We return to the setting in 2.2, and further assume that $F(s) = s^{-1}$. Then F acts on W_s, preserving W_s^0 and Ω_s. We denote this action by γ, so that $\gamma^2 = \delta$. Note that if ρ' belongs to $\overline{\mathcal{M}}_{s,E}$, then $\overline{\rho}'$ belongs to $\overline{\mathcal{M}}_{s^{-1},E}$ since $E \in \text{Irr } W_s^0$ is self dual. Also, $F(\rho')$ belongs to $\overline{\mathcal{M}}_{F(s),F(E)}$. Hence if ρ as above belongs to $\overline{\mathcal{M}}_{s,E}$, the Ω_s-orbit of E turns out to be γ-stable. It follows that γ leaves $\widetilde{\Omega}_s(E)$ invariant, and induces an action on $\widetilde{\Omega}_s(E)_\delta$. We denote by $\widetilde{\Omega}_s(E)_\delta^\gamma$ the set of γ-fixed points in $\widetilde{\Omega}_s(E)_\delta$. γ acts also on the set $\Omega_s^\delta(E)^\wedge$. We denote by $\Omega_s^\delta(E)^\wedge_{-\gamma}$ the set of $\eta \in \Omega_s^\delta(E)^\wedge$ such that $\gamma(\eta) = \overline{\eta}$. We put, for $E \in (\text{Irr}\, W_s^0)^\gamma$,

$$\overline{\mathcal{M}}^0_{s,E} = \Omega_s^\delta(E)^\wedge_{-\gamma} \times \widetilde{\Omega}_s(E)^\gamma_\delta.$$

We have the following proposition.

Proposition 2.7. *Let $\rho_{\eta,y}$ be the irreducible character of G^{F^2} corresponding to $(\eta, y) \in \overline{\mathcal{M}}_{s,E}$. Assume that $F(s) = s^{-1}$.*

(i) *If the Ω_s-orbit of E is not γ-stable, then $F(\rho_{\eta,y}) \neq \overline{\rho}_{\eta,y}$.*

(ii) *Assume that the Ω_s-orbit of E is γ-stable. Then $F(\rho_{\eta,y}) = \overline{\rho}_{\eta,y}$ if and only if $(\eta, y) \in \overline{\mathcal{M}}^0_{s,E}$.*

The proposition will be proved in 2.11 after some preliminaries. First we note that

Lemma 2.8. *For each N, we have $F(\mathit{\Gamma}_N) = \mathit{\Gamma}_N$, and $\overline{\mathit{\Gamma}_N} = \mathit{\Gamma}_N$.*

Proof. The fact that $F(\mathit{\Gamma}_N) = \mathit{\Gamma}_N$ can be checked directly for any $N \in \mathfrak{g}^F$ since U^{F^2} is F-stable and $\mathit{\Lambda}_N$ is also F-stable. On the other hand, it follows from the definition that we have $\overline{\mathit{\Gamma}_N} = \mathit{\Gamma}_{-N}$. So, in order to show the lemma, it is enough to see that N is conjugate to $-N$ under G^{F^2}. Since N is given by a Jordan normal form, this is reduced to the case where N is regular nilpotent. Assume that N is a regular nilpotent element given in the Jordan normal form. There exists $g = \text{diag}(a, -a, \dots, (-1)^{n-1}a) \in \widetilde{G}$ such that $gNg^{-1} = -N$. Then $g \in G^{F^2}$ if and only if $a \in \mathbf{F}_{q^2}$ and $(-1)^k a^n = 1$ with $k = [n/2]$. We can set $a = 1$ if k is even, and set $a = -1$ if k is odd and n is odd. So, assume that n is even and k is odd, i.e.,

$n = 2k$. In this case, we may take $a \in \mathbf{F}_{q^2}$ such that $a^2 = -1$. Thus we can always find $g \in G^{F^2}$, and the lemma follows. $\square$

As a corollary, we have

Corollary 2.9. *Let $\rho_{\dot{s}_x, E_x} \in \operatorname{Irr} \widetilde{G}^{F^2}$ and $\rho_0 \in \operatorname{Irr} G^{F^2}$ be as in 2.5. Assume that $F(\rho_{\dot{s}_x,E_x})|_{G^{F^2}} = \overline{\rho_{\dot{s}_x,E_x}|}_{G^{F^2}}$. Then we have $F(\rho_0) = \overline{\rho_0}$.*

Proof. The parametrization of $\operatorname{Irr} G^{F^2}$ in terms of the set $\overline{\mathcal{M}}_{s,E}$ is reduced to the special case where $\overline{\mathcal{M}}_{s,E}$ is given by (2.5.1) through the steps (b) and (c) in [S3, 4.5]. Since the steps (b) and (c) are compatible with the F action and with taking duals, the assertion is reduced to the case of (2.5.1). In this case, we have $F(\overline{\varGamma_N}) = \varGamma_N$ by Lemma 2.8. Note that the F-action and taking duals preserve the inner product. Since ρ_0 is the unique irreducible character such that

$$\langle \varGamma_N, \rho_0 \rangle_{G^{F^2}} = \langle \rho_{s_x,E_x}, \rho_0 \rangle_{G^{F^2}} = 1,$$

the corollary follows. $\square$

Lemma 2.10. *Assume that the set $\overline{\mathcal{M}}_{s,E}$ satisfies the assumption of Proposition 2.7 (ii). Take $y \in \widetilde{\Omega}_s(E)_\delta$ and assume that $(E, y) \leftrightarrow (x, E_x)$ under the map in (2.2.2). Then $F(\rho_{\dot{s}_x,E_x})|_{G^{F^2}} = \overline{\rho_{\dot{s}_x,E_x}}|_{G^{F^2}}$ if and only if $y \in \widetilde{\Omega}_s(E)^\gamma_\delta$.*

Proof. We may choose $\dot{y} \in \widetilde{\Omega}_s(E)$ as a representative of $x \in (\Omega_s)_\delta$, so we may assume that $E_x = E$. Then $\mathcal{T}_{s_x,E_x} = \mathcal{T}_{s_y,E}$ corresponds to the set $\Omega^\delta_s(E)^\wedge \times \{y\}$ under the correspondence $\mathcal{T}_{s,E} \leftrightarrow \overline{\mathcal{M}}_{s,E}$ (cf. (2.2.3)). It is easy to see, for any pair (s_1, E_1), that $F(\rho_{\dot{s}_1,E_1}) = \rho_{F(\dot{s}_1),F(E_1)}$, where $F(E_1)$ is the character of $W_{F(s_1)}$ corresponding to E_1 under the isomorphism $W_{s_1} \simeq W_{F(s_1)}$. On the other hand, we have $\overline{\rho_{\dot{s}_1,E_1}} = \rho_{\dot{s}_1^{-1},E_1}$ since $W_{s_1} = W_{s_1^{-1}}$ and E_1 is self dual. It follows that $F(\overline{\rho_{\dot{s}_y,E}}) = \rho_{F(\dot{s}_y^{-1}),F(E)}$. By our assumption, $F(s^{-1}) = s$. Hence we have $F(s_y^{-1}) \in T^*_{\gamma(y)}$ and $F(E) \in (\operatorname{Irr} W^0_s)^{\gamma(y)\delta}$. This implies that $F(\overline{\mathcal{T}_{s_y,E}}) = \mathcal{T}_{s_{\gamma(y)},F(E)} = \mathcal{T}_{s_{\gamma(y)},{}^uE}$ for some $u \in \Omega_s$ since the Ω_s-orbit of E is F-stable. Since $\mathcal{T}_{s_{\gamma(y)},{}^uE} = \mathcal{T}_{s_y,E}$ if and only if $\gamma(y) = y$ in $\widetilde{\Omega}_s(E)_\delta$, the lemma is proved. $\square$

2.11. We shall prove Proposition 2.7. The assertion (i) follows from 2.6. We show (ii). Take $(\eta, y) \in \overline{\mathcal{M}}_{s,E}$. If $\gamma(y) \neq y$, then $F(\rho_{\eta,y}) \neq \overline{\rho_{\eta,y}}$ by Lemma 2.10. So, assume that $y \in \widetilde{\Omega}_s(E)^\gamma_\delta$. Let $\rho_{\dot{s}_x,E_x}$ be the character of $\widetilde{G}^{F^2}$ containing $\rho_{\eta,y}$. Again by Lemma 2.10, we have $F(\rho_{\dot{s}_x,E_x}) = \overline{\rho_{\dot{s}_x,E_x}}$. Let $\rho_0 \in G^{F^2}$ be as in 2.5. Then by Corollary 2.9, we have $F(\rho_0) = \overline{\rho}_0$. If we write $\rho_{\eta,y} = {}^g\rho_0$ with $g \in \widetilde{G}^{F^2}$, we have $F(\overline{\rho}_{\eta,y}) = {}^{F(g)}\rho_0$. Now the action of F induces an action on $(\widetilde{G}^{F^2}/G^{F^2})^\wedge$ which is compatible with the natural pairing $\widetilde{G}^{F^2}/G^{F^2} \times (\widetilde{G}^{F^2}/G^{F^2})^\wedge \to \bar{\mathbf{Q}}_l^*$. Then F stabilizes the subgroup $I(\rho_{\dot{s}_x,E_x})$.

The argument in 2.4 show that the condition $F(\rho_{\eta,y}) = \overline{\rho}_{\eta,y}$ is described by investigating the action of F on $I(\rho_{\dot{s}_x,E_x})$. We follow the notation in 2.4. $I(\rho_{\dot{s}_x,E_x})$ is regarded as a subgroup of $\widetilde{Z}^{*F^2}$. If we denote by $\widetilde{\omega}_{s_x}$ the map $\Omega^\delta_s \to \widetilde{Z}^{*F^2}$ obtained as the composite of ω_{s_x} and the inclusion $\widetilde{Z}^{*F^2}_{s_x} \hookrightarrow \widetilde{Z}^{*F^2}$, then $\widetilde{\omega}_{s_x}(\Omega^\delta_s(E_x))$ coincides with $I(\rho_{\dot{s}_x,E_x})$. We note the following.

(2.11.1) Assume that $x \in (\Omega_s)_\delta$ is γ-stable. Then the following diagram commutes:

$$\begin{array}{ccc} \Omega_s^\delta & \xrightarrow{\widetilde{\omega}_{s_x}} & \widetilde{Z}^{*F^2} \\ {\scriptstyle -\gamma}\big\downarrow & & \big\downarrow{\scriptstyle F=\gamma} \\ \Omega_s^\delta & \xrightarrow{\widetilde{\omega}_{s_x}} & \widetilde{Z}^{*F^2}, \end{array}$$

where $-\gamma : \Omega_s^\delta \to \Omega_s^\delta$ is the map defined by $z \mapsto \gamma(z)^{-1}$.

We show (2.11.1). Take $z \in \Omega_s^\delta$. Since $F(s) = s^{-1}$, we have

$$\gamma(\widetilde{\omega}_{s_x}(z)) = F({}^{g_x}(\dot{s}^{-1}\dot{z}\dot{s}\dot{z}^{-1})) = {}^{F(g_x)}(\dot{s}F(\dot{z})\dot{s}^{-1}F(\dot{z})^{-1}).$$

On the other hand, since x is γ-stable, $\widetilde{\omega}_{s_x}$ coincides with $\widetilde{\omega}_{s_{\gamma(x)}}$, and we have

$$\widetilde{\omega}_{s_x}(-\gamma(z)) = {}^{F(g_x)}(\dot{s}^{-1}F(\dot{z})^{-1}\dot{s}F(\dot{z})) = {}^{F(g_x)}(\dot{s}F(\dot{z})\dot{s}^{-1}F(\dot{z})^{-1}).$$

since $\dot{s}^{-1}F(\dot{z})^{-1}\dot{s}F(\dot{z})$ is in the center $\widetilde{Z}^*$ of $\widetilde{G}^*$. Hence (2.11.1) holds.

Now (2.11.1) shows that the F-action on $I(\rho_{\dot{s}_x,E_x})$ is transferred to the $-\gamma$ action on $\Omega_s^\delta(E)$. Hence under the parametrization $\mathcal{T}_{s_x,E_x} \leftrightarrow \Omega_s^\delta(E) \times \{y\}$ given by $\rho_{\eta,y} \leftrightarrow (\eta, y)$, we see that $F(\rho_{\eta,y}) = \overline{\rho_{\eta,y}}$ if and only if η is $-\gamma$ stable, i.e., $\eta \in \Omega_s^\delta(E)^\wedge_{-\gamma}$. This proves the proposition.

3. Almost characters of $SL_n(\mathbf{F}_{q^2})$

3.1. We shall parametrize F^2-stable irreducible characters of $G^{F^{2m}}$ for a sufficiently divisible integer m. Let s be an F^2-stable semisimple element in G^*. We assume that m is large enough so that F^{2m} acts trivially on W_s^0 and Ω_s. We denote by $\overline{\mathcal{M}}_{s,E}^{(m)}$ the set which parametrizes irreducible characters of $G^{F^{2m}}$ corresponding to $\overline{\mathcal{M}}_{s,E}$ in the previous section. Hence, $\overline{\mathcal{M}}_{s,E}^{(m)} = \Omega_s(E)^\wedge \times \Omega_s(E)$. Since s is F^2-stable, one can define a map $\delta = F^2 : W_s \to W_s$ as before. If the Ω_s-orbit of E is F^2-stable, then δ stabilizes $\Omega_s(E)$. For a pair (s, E) such that $E \in (\overline{\mathrm{Irr}}\, W_s^0)^\delta$, we define a subset $\mathcal{M}_{s,E}$ of $\overline{\mathcal{M}}_{s,E}^{(m)}$ by

$$\mathcal{M}_{s,E} = \Omega_s(E)^\wedge_\delta \times \Omega_s(E)^\delta,$$

where $\Omega_s(E)^\wedge_\delta$ is the set of δ-stable irreducible characters in $\Omega_s(E)^\wedge$. We denote by $\mathcal{E}(G^{F^{2m}}, \{s\})^{F^2}$ the subset of F^2-stable irreducible characters in $\mathcal{E}(G^{F^{2m}}, \{s\})$. Then by [S3, (4.6.1)] it is known that under the parametrization in (2.2.1) for $G^{F^{2m}}$, we have

$$\mathcal{E}(G^{F^{2m}}, \{s\})^{F^2} = \coprod_{E \in (\overline{\mathrm{Irr}}\, W_s)^\delta} \mathcal{M}_{s,E}.$$

We denote by $\rho_{\eta,z}^{(m)}$ the F^2-stable irreducible character of $G^{F^{2m}}$ corresponding to $(\eta, z) \in \mathcal{M}_{s,E}$.

In the case where $F(s) = s^{-1}$, one can define a map $\gamma = F : W_s \to W_s$ preserving Ω_s and W_s^0, and such that $\delta = \gamma^2$ as before. We denote by $\Omega_s(E)^\gamma$ the γ-fixed point subgroup of $\Omega_s(E)$, and by $\Omega_s(E)^\wedge_{-\gamma}$ the set of $\eta \in \Omega_s(E)^\wedge$ such that $\gamma(\eta) = \overline{\eta}$. Then we define a subset $\mathcal{M}_{s,E}^0$ of $\mathcal{M}_{s,E}$ by

$$\mathcal{M}_{s,E}^0 = \Omega_s(E)^\wedge_{-\gamma} \times \Omega_s(E)^\gamma.$$

The following proposition can be proved in a similar way as in Proposition 2.7.

Proposition 3.2. *Let $\rho_x^{(m)}$ be an F^2-stable irreducible character of $G^{F^{2m}}$ corresponding to $x \in \mathcal{M}_{s,E}$. Assume that $F(s) = s^{-1}$.*

(i) *If the Ω_s-orbit of E is not F-stable, then $F(\rho_x^{(m)}) \neq \overline{\rho}_x^{(m)}$.*

(ii) *Assume that the Ω_s-orbit of E is F-stable. Then $F(\rho_x^{(m)}) = \overline{\rho}_x^{(m)}$ if and only if $x \in \mathcal{M}_{s,E}^0$.*

3.3. Following [S3, 4.6], we define almost characters of G^{F^2}. For a given $\widetilde{\Omega}_s(E)_\delta$, we choose $a = a_E \in \widetilde{\Omega}_s(E)$ and write it as $\widetilde{\Omega}_s(E)_\delta = \Omega_s(E)_\delta a_E$. For $x = (\eta, z) \in \mathcal{M}_{s,E}$ and $y = (\eta', z' a_E) \in \overline{\mathcal{M}}_{s,E}$, we define a pairing $\{x, y\} \in \bar{\mathbf{Q}}_l^*$ by

$$\{x, y\} = |\Omega_s(E)^\delta|^{-1} \eta(z')\eta'(z).$$

Then we define a class function R_x of G^{F^2} by

$$R_x = \sum_{y \in \overline{\mathcal{M}}_{s,E}} \{x, y\}\rho_y. \tag{3.3.1}$$

R_x are called almost characters of G^{F^2}. Note that the definition of the pairing $\{\ ,\ \}$ depends on the choice of $a_E \in \widetilde{\Omega}_s(E)_\delta$. If a_E is replaced by $a' = b^{-1}a_E$ with $b \in \Omega_s(E)_\delta$, then R_x is replaced by $\eta(b)R_x$. Hence the almost character R_x is determined uniquely up to a root of unity multiple.

It is easy to see that (3.3.1) can be converted to the form

$$\rho_y = |\Omega_s(E)^\delta|^{-2} \sum_{x \in \mathcal{M}_{s,E}} \{x, y\}^{-1} R_x. \tag{3.3.2}$$

The following result describes the Shintani descent of irreducible characters of $G^{F^{2m}}$. Here we write the restriction of F^2 on $G^{F^{2m}}$ as δ instead of σ^2, in connection with the previous section.

Theorem 3.4 ([S3, Theorem 4.7])**.** *Let $\rho_x^{(m)}$ be an F^2-stable irreducible character of $G^{F^{2m}}$ corresponding to $x \in \mathcal{M}_{s,E}$, and choose an extension $\widetilde{\rho}_x^{(m)}$ to $G^{F^{2m}}\langle\delta\rangle$. Then*

$$Sh_{F^{2m}/F^2}(\widetilde{\rho}_x^{(m)}|_{G^{F^{2m}}\delta}) = \mu_x R_x,$$

where μ_x is a root of unity depending on the extension $\widetilde{\rho}_x^{(m)}$ and on the choice of a_E.

Combining Theorem 3.4 with Proposition 3.2, we have the following refinement of Corollary 1.18.

Corollary 3.5. *Let $\mathcal{M}_{s,E}$ be such that $F(s) = s^{-1}$ and that the Ω_s-orbit of E is F-stable. Then*

$$m_2(R_x) = \begin{cases} \zeta_x & \text{if } x \in \mathcal{M}_{s,E}^0, \\ 0 & \text{otherwise}, \end{cases}$$

where ζ_x is a certain root of unity.

The following result describes the action of twisting operators on almost characters. In the special case where F^2 acts trivially on the center, this was proved by Bonnafé [B, Théorème 5.5.4]. We note that this result can also be derived from the property of character sheaves, by making use of Lusztig's conjecture for SL_n, which will be discussed in [S4].

Theorem 3.6. *For any $x = (\eta, z) \in \mathcal{M}_{s,E}$, we have*

$$t_1^*(R_x) = \eta(z)^{-1} R_x.$$

The theorem will be proved in 3.15 after some preliminaries. First we recall some general properties of twisting operators.

Lemma 3.7. *Let Γ be a connected algebraic group defined over $\mathbf{F}_q$ with Frobenius map F, and H a connected F-stable subgroup of Γ. Then the twisting operator t_1^* commutes with the induction $\mathrm{Ind}_{H^F}^{\Gamma^F}$.*

Proof. It is clear that t_1^* commutes with the restriction functor $\mathrm{Res}_{H^F}^{\Gamma^F}$. Moreover, t_1^* is an isometry with respect to the inner product on $C(H^F/\sim)$ and $C(\Gamma^F/\sim)$. The lemma follows from these two facts. $\square$

3.8. Let Γ be as in the lemma. For each integer $m > 0$, we consider the group Γ^{F^m}, and its semidirect product $\widetilde{\Gamma}^{F^m} = \Gamma^{F^m}\langle\sigma\rangle$, where σ is the restriction of F on Γ^{F^m}, and $\langle\sigma\rangle$ is the cyclic group of order m with generator σ. Then the twisting operator $t_1^* : C(\Gamma^F/\sim) \to C(\Gamma^F/\sim)$ can be lifted to the operator

$$\tau_1^{*-1} : C(\Gamma^{F^m}\sigma/\sim) \to C(\Gamma^{F^m}\sigma/\sim)$$

in the following way. We define a map $\tau_1 : \Gamma^{F^m}\sigma/\sim \to \Gamma^{F^m}\sigma/\sim$ by $\tau_1(x\sigma) = (x\sigma)^{1-m}$, and define τ_1^* by its transpose. It is shown in [S1, Lemma 4.2] that, under the condition that m is sufficiently divisible, τ_1^* is an isomorphism and satisfies the following commutative diagram:

$$\begin{array}{ccc} C(\Gamma^{F^m}\sigma/\sim) & \xrightarrow{\tau_1^*} & C(\Gamma^{F^m}\sigma/\sim) \\ {\scriptstyle Sh_{F^m/F}}\big\downarrow & & \big\downarrow{\scriptstyle Sh_{F^m/F}} \\ C(\Gamma^F/\sim) & \xleftarrow{t_1^*} & C(\Gamma^F/\sim). \end{array} \tag{3.8.1}$$

We have the following result.

Theorem 3.9 ([S1, Theorem 4.7]). *Let $\widetilde{\rho}$ be an extension of an F-stable irreducible character of Γ^{F^m} to $\widetilde{\Gamma}^{F^m}$. Then for an appropriate choice of (sufficiently divisible) m, there exists a root of unity λ such that*

$$\tau_1^*(\widetilde{\rho}|_{\Gamma^{F^m}\sigma}) = \lambda(\widetilde{\rho}|_{\Gamma^{F^m}\sigma}).$$

The following related result seems to be worth mentioning, although it is not used later. In [S1], under some condition on p, the notion of almost characters was established for any connected algebraic group Γ. Then in view of (3.8.1) together with Theorem 3.9, we have

Corollary 3.10. *For each almost character R_x of Γ^F, there exists a root of unity λ_x such that*

$$t_1^*(R_x) = \lambda_x R_x.$$

The following result was proved by Digne and Michel, which holds for any connected reductive groups.

Proposition 3.11 ([DM]). *Let H be an F-stable Levi subgroup of a parabolic subgroup of a connected reductive group Γ. Then the Lusztig induction $R_H^\Gamma : C(H^F/\sim) \to C(\Gamma^F/\sim)$ commutes with the twisting operator t_1^*.*

3.12. We now return to our original setting, and consider $G = SL_n$. The modified generalized Gelfand-Graev characters were introduced by Kawanaka (see [K1]), which is a refinement of generalized Gelfand-Graev characters. The modified generalized Gelfand-Graev characters are used in [S3] to parametrize irreducible characters of SL_n. Here we discuss the action of twisting operators on the modified generalized Gelfand-Graev characters. We follow the notation in 2.5 (but replacing F^2 by F).

By [S3, 2.6], we may choose $\mathfrak{u}$ so that $\mathfrak{u}$ is L-stable. Let $A_\lambda = Z_L(\lambda)/Z_L^0(\lambda)$. Then by [S3, 2.7], we have

$$A_\lambda \simeq A_G(N) = Z_G(N)/Z_G^0(N).$$

In particular, A_λ is an abelian group. F acts naturally on A_λ, and we consider the quotient group $(A_\lambda)_F$ of A_λ. Put

$$\overline{\mathcal{M}} = (A_\lambda)_F \times (A_\lambda^F)^\wedge.$$

For each pair $(c, \xi) \in \overline{\mathcal{M}}$ one can define a modified generalized Gelfand-Graev character $\varGamma_{c,\xi}$ as follows. For $c \in A_\lambda$, we choose a representative $\dot{c} \in Z_L(\lambda)$. Then we find $\alpha_c \in L$ such that $\alpha_c^{-1}F(\alpha_c) = \dot{c}$. We define a linear map $\lambda_c : \mathfrak{u} \to k$ by $\lambda_c = \lambda \circ \operatorname{Ad}\alpha_c^{-1}$, and define a linear character $\varLambda_c = \psi \circ \lambda_c \circ \log$ on U^F. Since $Z_L(\lambda_c)^F = Z_L(\varLambda_c)^F$, the linear character $\varLambda_c$ can be extended to a linear character on $Z_L(\lambda_c)^F U^F$ trivial on $Z_L^0(\lambda_c)^F$, which we denote also by $\varLambda_c$. On the other hand, since A_λ is abelian, we can define a linear character $\xi^\natural$ of $Z_L(\lambda_c)^F$ trivial on $Z_L^0(\lambda_c)^F$ by

$$\xi^\natural : Z_L(\lambda_c)^F \to (Z_L(\lambda_c)/Z_L^0(\lambda_c))^F \simeq A_\lambda^{\dot{c}F} = A_\lambda^F \to \bar{\mathbf{Q}}_l^*,$$

where the last step is given by $\xi : A_\lambda^F \to \bar{\mathbf{Q}}_l^*$. We denote by the same symbol $\xi^\natural$ the lift of $\xi^\natural$ to $Z_L(\lambda_c)^F U^F$ under the homomorphism $Z_L(\lambda_c)^F U^F \to Z_L(\lambda_c)^F$. Under these settings we define $\varGamma_{c,\xi}$ by

$$\varGamma_{c,\xi} = \operatorname{Ind}_{Z_L(\lambda_c)^F U^F}^{G^F}(\xi^\natural \otimes \varLambda_c).$$

3.13. We choose m large enough so that F^m acts trivially on A_λ. Replacing F by F^m, we have a modified generalized Gelfand-Graev character $\varGamma_{(c,\xi)}^{(m)}$ on G^{F^m}. Now the parameter set $\overline{\mathcal{M}}$ is replaced by $A_\lambda \times (A_\lambda)^\wedge$. We denote by $\mathcal{M}$ the subset of $A_\lambda \times (A_\lambda)^\wedge$ defined by

$$\mathcal{M} = A_\lambda^F \times (A_\lambda)_F^\wedge,$$

where $(A_\lambda)_F^\wedge$ is the set of F-stable irreducible characters of A_λ. Following [S3, 1.8], we construct, for each $(c, \xi) \in \mathcal{M}$, an F-stable modified generalized Gelfand-Graev character $\varGamma_{c,\xi}^{(m)}$, and its extension to $G^{F^m}\langle\sigma\rangle$, where $\sigma = F|_{G^{F^m}}$. For $c \in A_\lambda^F$, we choose $\dot{c} \in L^F$. We construct the linear character $\varLambda_c^{(m)}$ of U^{F^m} as in 3.12, i.e., we choose $\beta_c \in L$ such that $\beta_c^{-1}F^m(\beta_c) = \dot{c}$, and define λ_c by $\lambda_c = \lambda \circ \operatorname{Ad}\beta_c^{-1}$, and put $\varLambda_c^{(m)} = \psi_m \circ \lambda_c \circ \log$, where $\psi_m = \psi \circ \operatorname{Tr}_{\mathbf{F}_{q^m}/\mathbf{F}_q}$. Put $\hat{c} = \beta_c F(\beta_c^{-1}) \in L^{F^m}$. Then $\varLambda_c^{(m)}$ turns out to be $\hat{c}F$-stable.

On the other hand, it can be checked that $\hat{c}F$ acts on $Z_L(\lambda_c)$ commuting with F^m, and that under the isomorphism

$$\operatorname{ad}\beta_c^{-1} : Z_L(\lambda_c)^{F^m}/Z_L^0(\lambda_c)^{F^m} \simeq Z_L(\lambda)^{\dot{c}F^m}/Z_L^0(\lambda)^{\dot{c}F^m} \simeq A_\lambda,$$

the action of $\hat{c}F$ on $Z_L(\lambda_c)^{F^m}$ is transferred to the action of F on A_λ. Hence if we take $\xi \in (A_\lambda)^\wedge_F$, it produces a $\hat{c}F$-stable linear character $\xi^\natural$ on $Z_L(\lambda_c)^{F^m}$. It follows that $\xi^\natural \otimes \Lambda_c^{(m)}$ is $\hat{c}F$-stable for $(c, \xi) \in \mathcal{M}$, and we conclude that $\Gamma_{c,\xi}^{(m)}$ is F-stable.

Put $\hat{c}_0 = (\hat{c}\sigma)^m \in L^{F^m}$. We note that $\hat{c}_0 \in Z_L(\lambda_c)^{F^m} = Z_L(\Lambda_c^{(m)})^{F^m}$. In fact, since $\Lambda_c^{(m)}$ is $\hat{c}F$-stable, it is stable by $(\hat{c}\sigma)^m = \hat{c}_0$. We also note that

$$\beta_c^{-1}\widehat{c}_0\beta_c \equiv \dot{c}^{-1} \pmod{Z_L^0(\lambda)^{\dot{c}F^m}}$$

since $(\hat{c}\sigma)^m = \beta_c F^m(\beta_c^{-1})$ and $\dot{c} = \beta_c^{-1}F^m(\beta_c)$. In particular, we have

$$\xi^\natural(\hat{c}_0) = \xi(c^{-1}). \tag{3.13.1}$$

Put $M_c = Z_L(\lambda_c)^{F^m}$ and $M_c^0 = Z_L^0(\lambda_c)^{F^m}$. We consider a subgroup $M_cU^{F^m}\langle\hat{c}\sigma\rangle$ of $G^{F^m}\langle\sigma\rangle$ generated by $M_cU^{F^m}$ and $\hat{c}\sigma$. Since $\xi^\natural \in M_c^\wedge$ is $\hat{c}F$-stable, and $(\hat{c}\sigma)^m = \hat{c}_0 \in M_c$, $\xi^\natural$ may be extended to a linear character $\widetilde{\xi^\natural}$ of $M_c\langle\hat{c}\sigma\rangle$ in m distinct ways. The extension $\widetilde{\xi^\natural}$ is determined by the value $\widetilde{\xi^\natural}(\hat{c}\sigma) = \mu_{c,\xi}$, where $\mu_{c,\xi}$ is any m-th root of $\xi^\natural(\hat{c}_0)$.

We fix an extension $\widetilde{\xi^\natural}$ of $\xi^\natural$ to $M_c\langle\hat{c}\sigma\rangle$. Since $M_cU^{F^m}\langle\hat{c}\sigma\rangle$ is the semidirect product of $M_c\langle\hat{c}\sigma\rangle$ with U^{F^m}, $\widetilde{\xi^\natural}$ may be regarded as a character of $M_cU^{F^m}\langle\hat{c}\sigma\rangle$. On the other hand, since $\widetilde{\Lambda}_c^{(m)}$ is $\hat{c}\sigma$-stable, it can be extended to a linear character on $M_cU^{F^m}\langle\hat{c}\sigma\rangle$ by $\widetilde{\Lambda}_c^{(m)}(\hat{c}\sigma) = 1$. Thus we have a character $\widetilde{\xi^\natural} \otimes \widetilde{\Lambda}_c^{(m)}$ of $M_cU^{F^m}\langle\hat{c}\sigma\rangle$ which is an extension of $\xi^\natural \otimes \Lambda_c^{(m)}$ on $M_cU^{F^m}$. We put

$$\widetilde{\Gamma}_{c,\xi}^{(m)} = \operatorname{Ind}_{M_cU^{F^m}\langle\hat{c}\sigma\rangle}^{G^{F^m}\langle\sigma\rangle}(\widetilde{\xi^\natural} \otimes \widetilde{\Lambda}_c^{(m)}).$$

Then $\widetilde{\Gamma}_{c,\xi}^{(m)}$ gives rise to an extension of $\Gamma_{c,\xi}^{(m)}$ to $G^{F^m}\langle\sigma\rangle$. Note that $\mu_{c,\xi}^{-1}\widetilde{\Gamma}_{c,\xi}^{(m)}|_{G^{F^m}\sigma}$ depends only on the choice of (c, ξ).

Now we have the following result.

Proposition 3.14. *Let the notation be as above. We have*

$$\tau_1^*(\widetilde{\Gamma}_{c,\xi}^{(m)}|_{G^{F^m}\sigma}) = \xi(c)(\widetilde{\Gamma}_{c,\xi}^{(m)}|_{G^{F^m}\sigma})$$

for an appropriate choice of (sufficiently divisible) m.

Proof. The following proof is an analogy of the argument in [S1, Corollary 5.10]. Put $H = LU$. Then H is an F-stable connected subgroup of G. For each $(c, \xi) \in \mathcal{M}$, we denote by θ the linear character $\xi^\natural \otimes \Lambda_c^{(m)}$ of $M_cU^{F^m}$, and by $\widetilde{\theta}$ its extension $\widetilde{\xi^\natural} \otimes \widetilde{\Lambda}_c^{(m)}$ to $M_cU^{F^m}\langle\hat{c}\sigma\rangle$. We put $\widetilde{H}^{F^m} = H^{F^m}\langle\sigma\rangle$, $V = M_cU^{F^m}$, and $\widetilde{V} = M_cU^{F^m}\langle\hat{c}\sigma\rangle$. We consider the induced characters

$$\rho_{c,\xi} = \operatorname{Ind}_V^{H^{F^m}} \theta, \quad \widetilde{\rho}_{c,\xi} = \operatorname{Ind}_{\widetilde{V}}^{\widetilde{H}^{F^m}} \widetilde{\theta}.$$

Then $\rho_{c,\xi}$ is an F-stable character of H^{F^m}, and $\widetilde{\rho}_{c,\xi}$ is an extension of $\rho_{c,\xi}$ to $\widetilde{H}^{F^m}$. Moreover, $\rho_{c,\xi}$ is irreducible by [S3, Lemma 1.7]. Note that

$$\widetilde{\Gamma}_{c,\xi}^{(m)}|_{G^{F^m}\sigma} = \operatorname{Ind}_{H^{F^m}\sigma}^{G^{F^m}\sigma}(\widetilde{\rho}_{c,\xi}|_{H^{F^m}\sigma}).$$

In order to prove the proposition, we have only to show the following formula since τ_1^* commutes with the induction $\operatorname{Ind}_{H^{F^m}\sigma}^{G^{F^m}\sigma}$ by Lemma 3.7 and (3.8.1).

$$\tau_1^*(\widetilde{\rho}_{c,\xi}|_{H^{F^m}\sigma}) = \xi(c)(\widetilde{\rho}_{c,\xi}|_{H^{F^m}\sigma}). \tag{3.14.1}$$

We show (3.14.1). We choose m so that m is a multiple of some fixed integer A, where A is divisible by $|A_\lambda|p$, and that $m-1$ is prime to the order of $\widetilde{H}^{F^m}$. The existence of such m is shown in [S1, Lemma 4.8]. Then the map $f : \widetilde{H}^{F^m} \to \widetilde{H}^{F^m}, g \mapsto g^{1-m}$ is a bijection, and τ_1 is obtained by restricting f to $H^{F^m}\sigma$. Since f stabilizes the conjugacy classes, it induces an isomorphism $f^* : C(\widetilde{H}^{F^m}/\sim) \to C(\widetilde{H}^{F^m}/\sim)$. The map f^* stabilizes the space $C(\widetilde{V}/\sim)$, and we denote by $f^*_{\widetilde{V}}$ the restriction of f^* on $\widetilde{V}$. We note that

(3.14.2) $f^*_{\widetilde{V}}(\widetilde{\theta})$ is a linear character of $\widetilde{V}$ such that $f^*_{\widetilde{V}}(\widetilde{\theta})|_V = \theta$.

In fact, since f induces a homomorphism on $\widetilde{H}^{F^m}$ modulo the commutator subgroup, $f^*_{\widetilde{V}}$ maps linear characters to linear characters. We show that the restriction of $f^*_{\widetilde{V}}(\widetilde{\theta})$ on V coincides with θ. Since $\lambda \circ \log : U \to k$ is a homomorphism of algebraic groups and m is divisble by p, $\lambda(g^m) = 0$ for $g \in U$. This implies that $\varLambda^{(m)}_c(g^{1-m}) = \varLambda^{(m)}_c(g)$ for $g \in U^{F^m}$. On the other hand, since m is divisible by $|A_\lambda|$, $\xi^\natural(g^{1-m}) = \xi^\natural(g)$ for $g \in M_c$. It follows that $\theta(g^{1-m}) = \theta(g)$ for any $g \in V = M_cU^{F^m}$, and the claim follows.

Now it is easy to see that f^* commutes with the induction

$$\operatorname{Ind}_{\widetilde{V}}^{\widetilde{H}^{F^m}} : C(\widetilde{V}/\sim) \to C(\widetilde{H}^{F^m}/\sim).$$

Thus $f^*(\widetilde{\rho}_{c,\xi})$ is also an extension of $\rho_{c,\xi}$ to $\widetilde{H}^{F^m}$. Now the extension of θ to $\widetilde{\theta}$ is characterized by the value $\widetilde{\theta}(\hat{c}\sigma)$, and it determines the extension $\widetilde{\rho}_{c,\xi}$. Since $f(\hat{c}\sigma) = (\hat{c}\sigma)^{1-m} = \hat{c}\sigma \cdot \hat{c}_0^{-1}$, we see that

$$\begin{aligned} f^*_{\widetilde{V}}(\widetilde{\xi^\natural} \otimes \widetilde{\varLambda}^{(m)}_c)(\hat{c}\sigma) &= \xi^\natural(\hat{c}_0^{-1}) \cdot \widetilde{\xi^\natural} \otimes \widetilde{\varLambda}^{(m)}_c(\hat{c}\sigma) \\ &= \xi(c) \cdot \widetilde{\xi^\natural} \otimes \widetilde{\varLambda}^{(m)}_c(\hat{c}\sigma) \end{aligned}$$

by (3.13.1). This proves (3.14.1), and so the proposition follows. □

3.15. We are in a position to prove Theorem 3.6. We apply the previous results to our situation by replacing F by F^2. By [S3, 4.5], the parametrization of irreducible characters, $\rho_{\eta',z'} \leftrightarrow (\eta', z') \in \overline{\mathcal{M}}_{s,E}$ are divided into three steps. Accordingly, the parametrization of almost characters $R_{\eta,z} \leftrightarrow (\eta, z) \in \mathcal{M}_{s,E}$ are divided similarly. The cases (b) and (c) in [loc. cit.] are reduced to the case (a) via Harish-Chandra induction and Lusztig induction. Since the twisting operator commutes with Lusztig induction by Proposition 3.11, the proof of Theorem 3.6 is reduced to the case (a), i.e., the case where (s, E) satisfies the condition (2.5.1).

So, assume that (s, E) is as above. In this case, irreducible characters belonging to $\overline{\mathcal{M}}_{s,E}$ and almost characters belonging to $\mathcal{M}_{s,E}$ are characterized by modified generalized Gelfand-Graev characters as follows. Let $\rho_{\dot{s},E}$ be an irreducible character of $\widetilde{G}^{F^2}$ for some $\dot{s} \in \widetilde{G}^*$ such that $\pi(\dot{s}) = s$, and let N be a nilpotent element such that the nilpotent orbit $\mathcal{O}_N$ containing N coincides with the orbit $\mathcal{O}$ associated to $\rho_{\dot{s},E}$ (see e.g., [S3, 2.9]). Let A_λ be the finite group given in 3.12. For a certain quotient group $\overline{A}_\lambda$ of A_λ with F^2-action, we put

$$\mathcal{M}_{s,N} = \overline{A}_\lambda^\delta \times (\overline{A}_\lambda)^\wedge_\delta,$$

where δ is the action of F^2 on $\overline{A}_\lambda$ and on $(\overline{A}_\lambda)^\wedge$ as before. Let $(\mathcal{T}^{(m)}_{s,E})^{F^2}$ be the set of F^2-stable irreducible characters of $G^{F^{2m}}$ belonging to $\mathcal{M}_{s,E}$. Then there exists a parametrization $\mathcal{M}_{s,N} \leftrightarrow (\mathcal{T}_{s,E}^{(m)})^{F^2}$ via $(c,\xi) \leftrightarrow \rho_{c,\xi}^{(m)}$ satisfying the following properties. Put $\widetilde{\mathcal{M}}_{s,N} = A_\lambda^\delta \times (\overline{A}_\lambda)_\delta^\wedge$. Since $\widetilde{\mathcal{M}}_{s,N}$ is a subset of $A_\lambda^\delta \times (A_\lambda)_\delta^\wedge$, one can define an F^2-stable character $\varGamma_{c,\xi}^{(m)}$ of $G^{F^{2m}}$ for each pair $(c,\xi) \in \mathcal{M}_{s,N}$ (see 3.13). Let $\varphi : \widetilde{\mathcal{M}}_{s,N} \to \mathcal{M}_{s,N}$ be the natural projection. Then for $(c,\xi) \in \widetilde{\mathcal{M}}_{s,N}$ and $(c',\xi') \in \mathcal{M}_{s,N}$, we have the following (cf. [S3, Corollary 2.21]).

$$\langle \varGamma_{c,\xi}^{(m)}, \rho_{c',\xi'}^{(m)} \rangle = \begin{cases} 1 & \text{if } \varphi(c,\xi) = (c',\xi'), \\ 0 & \text{otherwise.} \end{cases} \tag{3.15.1}$$

Assume that $\varphi(c,\xi) = (c',\xi')$, i.e., $\xi = \xi'$ and c' is the image of c under the map $A_\lambda \to \overline{A}_\lambda$. Let $\widetilde{\varGamma}_{c,\xi}^{(m)}$ and $\widetilde{\rho}_{c',\xi'}^{(m)}$ be extensions of $\varGamma_{c,\xi}^{(m)}$ and $\rho_{c',\xi'}^{(m)}$ to $G^{F^{2m}}\langle\delta\rangle$, respectively. Now by Proposition 3.14, $\widetilde{\varGamma}_{c,\xi}^{(m)}|_{G^{F^{2m}}\delta}$ is an eigenfunction for τ_1^* with eigenvalue $\xi(c)$. Since $\rho_{c',\xi'}^{(m)}$ occurs in the decomposition of $\varGamma_{c,\xi}^{(m)}$ with multiplicity 1 by (3.15.1), by applying Theorem 3.9 we see that

$$\tau_1^*(\widetilde{\rho}_{c',\xi'}^{(m)}|_{G^{F^{2m}}\delta}) = \xi'(c')(\widetilde{\rho}_{c',\xi'}^{(m)}|_{G^{F^{2m}}\delta}). \tag{3.15.2}$$

(Note that $\xi(c) = \xi'(c')$). The set $(\mathcal{T}_{s,E}^{(m)})^{F^2}$ is also parametrized by the set $\mathcal{M}_{s,E}$ via $\rho_{\eta,z}^{(m)} \leftrightarrow (\eta,z) \in \mathcal{M}_{s,E}$. Here we need the following fact.

Lemma 3.16. *The bijection $\mathcal{M}_{s,N} \leftrightarrow \mathcal{M}_{s,E}$, $(c',\xi') \leftrightarrow (\eta,z)$ parametrizing the set $(\mathcal{T}_{s,E}^{(m)})^{F^2}$ satisfies the property that $\xi'(c') = \eta(z)$.*

Assuming Lemma 3.16, we continue the proof of the theorem. By the lemma, we have

$$\tau_1^*(\widetilde{\rho}_{\eta,z}^{(m)}|_{G^{F^{2m}}\delta}) = \eta(z)(\widetilde{\rho}_{\eta,z}^{(m)}|_{G^{F^{2m}}\delta}).$$

Now the theorem follows from Theorem 3.4, in view of the commutativity of t_1^{*-1} and τ_1^* given in (3.8.1). This completes the proof of Theorem 3.6 modulo Lemma 3.16.

3.17. We shall prove Lemma 3.16. Here we use the results from [S3]. See 2.20, Corollary 2.21 and 4.11 in [S3] for details. $\mathcal{M}_{s,E}$ is a subset of $\overline{\mathcal{M}}_{s,E}^{(m)} = \Omega_s^\wedge \times \Omega_s$, and $\mathcal{M}_{s,N}$ is a subset of $\overline{\mathcal{M}}_{s,N}^{(m)} = \overline{A}_\lambda \times \overline{A}_\lambda^\wedge$, where $\overline{\mathcal{M}}_{s,N}^{(m)}$ is a set parametrizing $\mathcal{T}_{s,E}^{(m)}$. Hence it is enough to show a similar fact for $\overline{\mathcal{M}}_{s,E}$ and $\overline{\mathcal{M}}_{s,N}$ with respect to G^F, assuming that F acts trivially on Ω_s and on $\overline{A}_\lambda$. Thus we have $\overline{\mathcal{M}}_{s,E} = \Omega_s^\wedge \times \Omega_s$, $\overline{\mathcal{M}}_{s,N} = \overline{A}_\lambda \times \overline{A}_\lambda^\wedge$, and there exists a bijection $\overline{\mathcal{M}}_{s,E} \simeq \overline{\mathcal{M}}_{s,N}$ through bijections $\Omega_s^\wedge \simeq \overline{A}_\lambda$, $\Omega_s \simeq \overline{A}_\lambda^\wedge$. Now for $x \in \Omega_s$, we have an irreducible character $\rho_{\dot{s}_x,E}$ of $\widetilde{G}^F$, and it is decomposed in terms of the parameter set $\overline{\mathcal{M}}_{s,N}$,

$$\rho_{\dot{s}_x,E}|_{G^F} = \sum_{c \in \overline{A}_\lambda} \rho_{c,\xi_x}. \tag{3.17.1}$$

(Compare this with (2.2.3). Here $E_x = E$ by our assumption that $\Omega_s(E) = \Omega_s$). $\xi_x \in \overline{A}_\lambda^\wedge$ is uniquely determined by $x \in \Omega_s$. The map $h : \Omega_s \to \overline{A}_\lambda^\wedge, x \mapsto \xi_x$, is described as follows (cf. [S3, 2.13]). There exists an F-stable Levi subgroup $\widetilde{M}$ of a parabolic subgroup of $\widetilde{G}$ (depending on the pair (s,E)) such that $Z_{\widetilde{L}}(\lambda) \subset \widetilde{M}$, and that $\dot{s}_x$ is contained in the center $Z(\widetilde{M}^*)$ of the dual group $\widetilde{M}^*$. We choose m large enough so that $\dot{s}_x \in Z(\widetilde{M}^*)^{F^m}$, and let $\hat{\theta}'_x$ be the corresponding linear character of $\widetilde{M}^{F^m}$. Then the restriction $\hat{\theta}_x$ of $\hat{\theta}'_x$ to $Z_{\widetilde{L}}(\lambda)^{F^m}$ is F-stable, and we put $\theta_x = Sh_{F^m/F}(\hat{\theta}_x)$, which is a linear character of $Z_{\widetilde{L}}(\lambda)^F$. Then the restriction of θ_x to $Z_L^0(\lambda)^F$ is trivial, and the restriction of θ_x to $Z_L(\lambda)^F$ induces a character of $A_\lambda^F = A_\lambda$ (see Lemma 2.18 and Theorem 2.16 in [S3]). This character factors through $\overline{A}_\lambda$, which gives $\xi_x \in \overline{A}_\lambda^\wedge$. The map h gives a bijection $\Omega_s \simeq \overline{A}_\lambda^\wedge$. Now by our assumption, s is F-stable, and F acts trivially on Ω_s. This implies, by [S3, Lemma 2.18] that ξ_1 (the case where $x = 1$) is the trivial character of $\overline{A}_\lambda$. Let us write $\dot{s}_x = \dot{s}z_x$ with $z_x \in \widetilde{Z}^*$. Since $\dot{s}_x$ is xF-stable, we have

$$\dot{s}^{-1}\dot{x}\dot{s}\dot{x}^{-1} = z_x F(z_x)^{-1}, \tag{3.17.2}$$

where $\dot{x} \in N_{G^*}(T^*)$ is a representative of $x \in \Omega_s \subset W_s$. We may assume that $z_x \in \widetilde{Z}^{*F^m}$. Let $\hat{\psi}'_x$ be the linear character of $\widetilde{G}^{*F^m}$ corresponding to z_x. Then it follows from the above discussion that the restriction $\hat{\psi}_x$ of $\hat{\psi}'_x$ to $Z_{\widetilde{L}}(\lambda)^{F^m}$ is F-stable, and we obtain a linear character $\psi_x = Sh_{F^m/F}(\hat{\psi}_x)$ of $Z_{\widetilde{L}}(\lambda)^F$. Since $\xi_1 = 1$, we see that ξ_x is obtained from the restriction of ψ_x to $Z_L(\lambda)^F$.

Next, we shall describe the bijection between $\overline{A}_\lambda$ and $\Omega_s^\wedge$. There exists a surjective homomorphism $f_1 : \widetilde{G}^F/G^F \to \overline{A}_\lambda$ defined as follows (cf. [S3, 2.19]). For $g \in \widetilde{G}^F$, we can write $g = g_1 z$, with $g_1 \in G, z \in \widetilde{Z}$. Then $g_1^{-1}F(g_1) \in Z$, and it determines an element in $A_\lambda = Z_L(\lambda)/Z_L^0(\lambda)$, and so an element in $\overline{A}_\lambda$, which is unique up to F-conjugacy. Since F acts trivially on $\overline{A}_\lambda$, this gives a well-defined map $f_1 : \widetilde{G}^F/G^F \to \overline{A}_\lambda$. On the other hand, we construct $f_2 : \widetilde{G}^F/G^F \to \Omega_s^\wedge$ as follows. From (3.17.2), we have $\dot{s}^{-1}\dot{x}\dot{s}\dot{x}^{-1} \in \widetilde{Z}^{*F}$ (we may choose $\dot{x} \in N_{G^*}(T^*)^F$), and this defines a well-defined injective homomorphism $f_2^* : \Omega_s \to \widetilde{Z}^{*F}, x \mapsto \dot{s}^{-1}\dot{x}\dot{s}\dot{x}^{-1}$. Since $\widetilde{Z}^{*F} \simeq (\widetilde{G}^F/G^F)^\wedge$, we have a surjective map f_2 as the transpose of f_2^*. Then $\operatorname{Ker} f_1 = \operatorname{Ker} f_2$, and these maps induce the bijection $\overline{A}_\lambda \simeq \Omega_s^\wedge$.

Now the parametrization is given as follows. There exists a unique $\rho_0 \in \operatorname{Irr} G^F$ such that ρ_0 occurs in $\rho_{\dot{s}_x,E}|_{G^F}$ and in $\mathit{\Gamma}_N$. In our parametrization, $\rho_0 = \rho_{1,\xi_x} = \rho_{1,x}$ $((1,\xi_x) \in \overline{\mathcal{M}}_{s,N}, (1,x) \in \overline{\mathcal{M}}_{s,E})$. Then any ρ contained in $\rho_{\dot{s}_x,E}|_{G^F}$ is obtained as ${}^g\rho_0$ with $g \in \widetilde{G}^F/G^F$. We then have $\rho = \rho_{c,\xi_x} = \rho_{\eta,x}$ with $c = f_1(g)$ and $\eta = f_2(g)$. Thus, in order to prove the lemma, it is enough to show

$$f_1^* \circ h = f_2^*, \tag{3.17.3}$$

where $f_1^* : \overline{A}_\lambda^\wedge \to \widetilde{Z}^{*F}$ is the transpose of f_1.

Take $x \in \Omega_s$, and let θ be the linear character of $\widetilde{G}^F$ corresponding to $f_2^*(x) \in \widetilde{Z}^{*F}$. For $g = g_1 z \in \widetilde{G}^F$, put $y = g_1^{-1}F(g_1) \in Z \subset \widetilde{Z}^F$. Take $\hat{y} \in \widetilde{Z}^{F^m}$ such that $N_{F^m/F}(\hat{y}) = y$. Then (3.17.3) is deduced from the formula

$$\theta(g) = \hat{\psi}'_x(\hat{y}) \tag{3.17.4}$$

for $g \in \widetilde{G}^F/G^F$. We show (3.17.4). Since $\widetilde{G}^F/G^F \simeq \widetilde{T}^F/T^F$, we may assume that $g \in \widetilde{T}^F$ and $g_1 \in T$. Take $\alpha \in \bar{\mathbf{F}}_q^*$ such that $\det g = \alpha^n$. Then $z = \operatorname{Diag}(\alpha, \dots, \alpha)$ and $g_1 = z^{-1}g$. Since $f_2^*(g) \in \widetilde{Z}^{*F}$, by considering the restriction of θ on $\widetilde{T}^F$, we see that there exists a homomorphism $\omega : \mathbf{F}_q^* \to \bar{\mathbf{Q}}_l^*$ such that

$$\theta(g) = \omega(\deg g) = \omega(\alpha^n). \tag{3.17.5}$$

Let $\hat{\theta}$ be a character of $\widetilde{G}^{F^m}$ such that $\theta = Sh_{F^m/F}(\hat{\theta})$. It is checked that for any $\hat{g} \in \widetilde{T}^{F^m}$, one can write $\hat{\theta}(\hat{g}) = \hat{\omega}(\det \hat{g})$, where $\hat{\omega}$ is a character of $\mathbf{F}_{q^m}^*$ such that

$Sh_{F^m/F}(\hat{\omega}) = \omega$. Then by (3.17.2), we have

$$\hat{\theta} = \hat{\psi}'_x F(\hat{\psi}'_x)^{-1}. \tag{3.17.6}$$

On the other hand, $y \in Z^F$ is expressed as

$$y = g_1^{-1}F(g_1) = zF(z^{-1}) = \mathrm{Diag}(\alpha^{1-q}, \dots, \alpha^{1-q}).$$

Take $\beta \in \bar{\mathbf{F}}_q^*$ such that $\alpha = \beta^{q^m-1}$, and put $\hat{\alpha} = \beta^{q-1}$. Then $\hat{y} = \mathrm{Diag}(\hat{\alpha}^{1-q}, \dots, \hat{\alpha}^{1-q})$. By making use of (3.17.6), we have

$$\hat{\psi}'_x(\hat{y}) = \hat{\theta}(\mathrm{Diag}(\hat{\alpha}, \dots, \hat{\alpha})) = \hat{\omega}(\hat{\alpha}^n) = \omega(\alpha^n).$$

Comparing this with (3.17.5), we see that (3.17.4) holds. Thus Lemma 3.16 is proved.

4. Determination of $m_2(\rho_{\dot{s},E}|_{G^{F^2}})$

4.1. Assume that s is F^2-stable, and the pair (s, T^*) is given as in Section 2. Let $\rho_{\dot{s},E}$ be an irreducible character of $\widetilde{G}^{F^2}$ as in 1.20. In this section, we shall compute the value $m_2(\rho_{\dot{s},E}|_{G^{F^2}})$. Now $\rho_{\dot{s},E}$ is given as

$$\rho_{\dot{s},E} = \varepsilon_{\widetilde{G}^*}\varepsilon_{Z_{\widetilde{G}^*}(\dot{s})}|W_{\dot{s}}|^{-1}\sum_{w\in W_{\dot{s}}} \mathrm{Tr}\,(w\delta, \widetilde{E})R_{\widetilde{T}^*_w}(\dot{s}), \tag{4.1.1}$$

where $\varepsilon_H = (-1)^{\mathbf{F}_{q^2}-\mathrm{rank}(H)}$ for any reductive group H, and $\widetilde{E}$ is a certain extension of $E \in (\mathrm{Irr}\ W_{\dot{s}})^\delta$ to $W_{\dot{s}}\langle\delta\rangle$. Therefore we compute the value $m_2(R_{\widetilde{T}^*_w}(\dot{s}))$ for each $w \in W_{\dot{s}}$.

We consider the isomorphism $\widetilde{Z}^{*F^2} \simeq (\widetilde{G}^{F^2}/G^{F^2})^\wedge$ as in (2.4.1), and a similar one by replacing F^2 by F. By the property of the dual torus, we have the following commutative diagram:

$$\begin{array}{ccc} \widetilde{Z}^{*F^2} & \xrightarrow{\sim} & (\widetilde{G}^{F^2}/G^{F^2})^\wedge \\ {\scriptstyle N_{F^2/F}}\downarrow & & \downarrow{\scriptstyle \mathrm{Res}} \\ \widetilde{Z}^{*F} & \xrightarrow{\sim} & (\widetilde{G}^F/G^F)^\wedge, \end{array} \tag{4.1.2}$$

where Res is the restriction of the character of $\widetilde{G}^{F^2}/G^{F^2}$ on $\widetilde{G}^F/G^F$, and $N_{F^2/F}$ is the norm map $z \to zF(z)$. The norm map is also described as in 1.1. By using this, it is easy to see that $\mathrm{Ker}\,N_{F^2/F}$ coincides with the subset $\{z^{-1}F(z) \mid z \in \widetilde{Z}^{*F^2}\}$, and so $\widetilde{Z}^{*F}$ can be identified with $(\widetilde{Z}^{*F^2})_F$ via the map $zF(z) \leftrightarrow z$ for $z \in (\widetilde{Z}^{*F^2})_F$.

First we note the following general fact.

Lemma 4.2. *Let χ be a class function of $\widetilde{G}^{F^2}$. Then*

$$m_2(\chi|_{G^{F^2}}) = \sum_{\theta\in(\widetilde{G}^F/G^F)^\wedge} m_2(\chi \otimes \widetilde{\theta}),$$

where $\widetilde{\theta}$ is a character of $\widetilde{G}^{F^2}/G^{F^2}$, regarded as a linear character of $\widetilde{G}^{F^2}$, which is an extension of θ via the inclusion $\widetilde{G}^F/G^F \hookrightarrow \widetilde{G}^{F^2}/G^{F^2}$.

Proof. By the Frobenius reciprocity, we have

$$\begin{aligned}\langle \chi|_{G^{F^2}}, \mathrm{Ind}_{G^F}^{G^{F^2}} 1\rangle &= \langle \chi, \mathrm{Ind}_{G^F}^{\widetilde{G}^{F^2}} 1\rangle \\ &= \langle \chi, \mathrm{Ind}_{\widetilde{G}^F}^{\widetilde{G}^{F^2}} \big(\mathrm{Ind}_{G^F}^{\widetilde{G}^F} 1\big)\rangle \\ &= \langle \chi, \sum_{\theta \in (\widetilde{G}^F/G^F)^\wedge} \mathrm{Ind}_{\widetilde{G}^F}^{\widetilde{G}^{F^2}} \theta\rangle .\end{aligned}$$

But for any linear character $\widetilde{\theta}$ of $\widetilde{G}^{F^2}$ such that $\widetilde{\theta}|_{\widetilde{G}^F} = \theta$, we have

$$\langle \chi, \mathrm{Ind}_{\widetilde{G}^F}^{\widetilde{G}^{F^2}} \theta\rangle = \langle \chi \otimes \widetilde{\theta}^{-1}, \mathrm{Ind}_{\widetilde{G}^F}^{\widetilde{G}^{F^2}} 1\rangle = m_2(\chi \otimes \widetilde{\theta}^{-1}).$$

Thus the lemma is proved. □

By applying the above formula to the class function $R_{\widetilde{T}^*_w}(\dot{s})$ of $\widetilde{G}^{F^2}$,

Lemma 4.3. *We have*

$$m_2(R_{\widetilde{T}^*_w}(\dot{s})|_{G^{F^2}}) = \sum_{z \in (\widetilde{Z}^{*F^2})_F} m_2(R_{\widetilde{T}^*_w}(\dot{s}\dot{z})), \tag{4.3.1}$$

where $\dot{z}$ is a representative of z in $\widetilde{Z}^{*F^2}$.

Proof. By 4.1, $(\widetilde{Z}^{*F^2})_F$ is isomorphic to $(\widetilde{G}^F/G^F)^\wedge$. We denote by θ the character of $\widetilde{G}^F/G^F$ corresponding to $z \in (\widetilde{Z}^{*F^2})_F$. Then by (4.1.2), the representative $\dot{z} \in \widetilde{Z}^{*F^2}$ corresponds to a linear character $\widetilde{\theta}$ of $\widetilde{G}^{F^2}/G^{F^2}$, which is an extension of θ. Now it is known that $R_{\widetilde{T}^*_w}(\dot{s}) \otimes \widetilde{\theta} = R_{\widetilde{T}^*_w}(\dot{s}\dot{z})$. Hence the lemma follows from Lemma 4.2. □

We have the following proposition.

Proposition 4.4. *Let s be an element in T^*_w such that $F^2(s) = s$.*

- (i) *Assume that the G^{*F^2}-orbit of s does not contain s' such that $F(s') = s'^{-1}$. Then $m_2(R_{\widetilde{T}^*_w}(\dot{s})|_{G^{F^2}}) = 0$ for any $\dot{s} \in \widetilde{T}^*_w$ such that $\pi(\dot{s}) = s$.*
- (ii) *Assume that $F(s) = s^{-1}$. Then there exists $\dot{s} \in \widetilde{T}^*_w$ such that $\pi(\dot{s}) = s$ and that $F(\dot{s}) = \dot{s}^{-1}$, and we have*

$$m_2(R_{\widetilde{T}^*_w}(\dot{s})|_{G^{F^2}}) = \sum_{x \in \Omega_s^{-\gamma}} m_2(R_{\widetilde{T}^*_w}(\dot{s}z_x)), \tag{4.4.1}$$

where $\Omega_s^{-\gamma} = \{x \in \Omega_s \mid \gamma(x) = x^{-1}\}$ is the subgroup of Ω_s^δ, and $z_x \in \widetilde{Z}^{*F^2}$ is a representative of an element in $(\widetilde{Z}^{*F^2})_F$ such that $z_xF(z_x) = \omega_s(x)$ under the map $\omega_s : \Omega_s^\delta \to \widetilde{Z}^{*F^2}_{\dot{s}} \subset \widetilde{Z}^{*F^2}$ (see 2.4).

Proof. First we show (i). It is known that $R_{\widetilde{T}^*_w}(\dot{s})|_{G^{F^2}}$ coincides with the Deligne-Lusztig character $R_{T^*_w}(s)$ of G^{F^2}. Then by [L3, 2.7 (a)], we have $m_2(R_{T^*_w}(s)) = 0$. (Note that in [loc. cit.], it is assumed that the center of G is connected. However, the above fact holds without this assumption, by 2.3 and 2.6 (b) in [loc. cit.]). Thus (i) holds.

Next we show (ii). Assume that $F(s) = s^{-1}$. Take $\dot{s}_1 \in \widetilde{T}^*$ such that $\pi(\dot{s}_1) = s$. Then there exists some $z \in \widetilde{Z}^*$ such that $F(\dot{s}_1) = \dot{s}_1^{-1}z$ for some $z \in \widetilde{Z}^*$. Take

$z_1 \in \widetilde{Z}^* \simeq \mathbf{G}_m$ such that $z = z_1F(z_1) = z_1^{q+1}$, and put $\dot{s} = \dot{s}_1 z_1$. Then $\pi(\dot{s}) = s$ and $F(\dot{s}) = \dot{s}^{-1}$ as asserted.

Take $\dot{s}$ as above, and consider the formula (4.3.1). Again by [L3, Lemma 2.8], we may only consider, in the sum of the right-hand side of (4.3.1), $z \in (\widetilde{Z}^{*F^2})_F$ such that $F(\dot{s}\dot{z})$ is conjugate to $(\dot{s}\dot{z})^{-1}$ in $\widetilde{G}^*$. Here we note that

(4.4.2) $F(\dot{s}\dot{z})$ is conjugate to $(\dot{s}\dot{z})^{-1}$ if and only if there exists $x \in \Omega_s^{-\gamma}$ such that $\dot{z}F(\dot{z}) = \omega_s(x)$.

We show (4.4.2). Assume that $x \in \Omega_s^{-\gamma}$, and let $\dot{x}$ be an element in $\widetilde{G}^*$ such that $\pi(\dot{x})$ is a representative of x. Then by (2.11.1), $\omega_s(x) = \dot{s}^{-1}\dot{x}\dot{s}\dot{x}^{-1} \in \widetilde{Z}^{*F^2}$ is γ-stable, i.e., $\omega_s(x) \in \widetilde{Z}^{*F}$. Hence by (4.1.2) there exists $\dot{z} \in \widetilde{Z}^{*F^2}$ such that $\omega_s(x) = \dot{z}F(\dot{z})$. It follows that $\dot{s}\dot{z} = \dot{x}\dot{s}\dot{x}^{-1}F(\dot{z})^{-1}$, and we have $F(\dot{s}\dot{z}) = \dot{x}^{-1}(\dot{s}\dot{z})^{-1}\dot{x}$. This shows that $F(\dot{s}\dot{z})$ is conjugate to $(\dot{s}\dot{z})^{-1}$ in $\widetilde{G}^*$. Conversely, assume that $F(\dot{s}\dot{z})$ is conjugate to $(\dot{s}\dot{z})^{-1}$ in $\widetilde{G}^*$. Then there exists $\dot{x} \in \widetilde{G}^*$ such that $F(\dot{s}\dot{z}) = \dot{x}^{-1}(\dot{s}\dot{z})^{-1}\dot{x}$. Clearly $\pi(\dot{x}) \in Z_{G^*}(s)$, and its image in Ω_s determines an element $x \in \Omega_s$. Since $\dot{s}(\dot{z}F(\dot{z})) = \dot{x}\dot{s}\dot{x}^{-1}$, we have $\dot{z}F(\dot{z}) \in \widetilde{Z}_s^{*F^2}$. Moreover, $\dot{z}F(\dot{z})$ is γ-stable. Hence by (2.4.2) and (2.11.1), we see that $x \in \Omega_s^{-\gamma}$. This proves (4.4.2).

Since $\omega_s(x) \in \widetilde{Z}^{*F}$, $\dot{z} \in \widetilde{Z}^{*F^2}$ such that $\dot{z}F(\dot{z}) = \omega_s(x)$ has a unique image on $(\widetilde{Z}^{*F^2})_F \simeq \widetilde{Z}^{*F}$ by (4.1.2). We choose z_x from such $\dot{z}$ for each x. Then the formula (4.4.1) is immediate from (4.4.2). □

By using Lusztig's formula in [L3], we shall compute the right-hand side of (4.4.1) explicitly. We show

Lemma 4.5. *Under the notation in Proposition 4.4 (ii), we have*

$$m_2(R_{\widetilde{T}^*_w}(\dot{s}z_x)) = \sharp\{u \in W_{\dot{s}} \mid w = u({}^{x\gamma}u)\}.$$

Proof. Take $\dot{x} \in N_{\widetilde{G}^*}(\widetilde{T}^*)$ whose image in W is a representative of $x \in \Omega_s^{-\gamma}$. We note that one can choose $\dot{x}$ such that $F(\dot{x}) = \dot{x}^{-1}$. In fact, take any $x' \in N_{\widetilde{G}^*}(\widetilde{T}^*)$ in the inverse image of x. Since $\gamma(x) = x^{-1}$, we have $x'F(x') = t \in \widetilde{T}^*$. We can find $t_1 \in \widetilde{T}^*$ such that $t_1^{-1}F^2(t_1) = t$. Then $\dot{x} = t_1x'F(t_1)^{-1}$ satisfies the required condition.

Take $g \in \widetilde{G}^*$ such that $g^{-1}F(g) = \dot{x}$. Put $s' = {}^g(\dot{s}z_x)$, $\widetilde{T}' = {}^g\widetilde{T}^*$ and $W' = N_{\widetilde{G}^*}(\widetilde{T}')/\widetilde{T}'$. Then $F(s') = s'^{-1}$, $F(\widetilde{T}') = \widetilde{T}'$, and $s' \in \widetilde{T}'$. Moreover, we have $g^{-1}F^2(g) = \dot{x}F(\dot{x}) = 1$, and so $g \in \widetilde{G}^{*F^2}$. We have an isomorphism $f : W \xrightarrow{\sim} W'$ via $\operatorname{ad} g$, and we see that the pair $(s', \widetilde{T}'_{f(w)})$ is $\widetilde{G}^{*F^2}$-conjugate to the pair $(\dot{s}z_x, \widetilde{T}^*_w)$, where $\widetilde{T}'_{f(w)}$ is an F^2-stable maximal torus obtained from $\widetilde{T}'$ by twisting by $f(w) \in W'$.

It follows that

$$R_{\widetilde{T}^*_w}(\dot{s}z_x) = R_{\widetilde{T}'_{f(w)}}(s'). \tag{4.5.1}$$

If we put $W'_{s'} = \{w' \in W' \mid w'(s') = s'\}$, f induces an isomorphism $W_{\dot{s}} \xrightarrow{\sim} W'_{s'}$. Now it is known by [L3, Lemma 2.8, (b)] that

$$m_2(R_{\widetilde{T}'_{f(w)}}(s')) = \sharp\{y \in W'_{s'} \mid f(w) = yF(y)\}. \tag{4.5.2}$$

Hence, by (4.5.1) and (4.5.2), we have

$$\begin{aligned} m_2(R_{\widetilde{T}_w}(\dot{s}z_x)) &= \sharp\{u \in W_{\dot{s}} \mid f(w) = f(u)F(f(u))\} \\ &= \sharp\{u \in W_{\dot{s}} \mid w = u({}^{x\gamma}u)\}, \end{aligned}$$

since $F \circ f = f \circ \dot{x}F$. This proves the lemma. □

We are in a position to determine $m_2(\rho_{\dot{s},E}|_{G^{F^2}})$.

Theorem 4.6. *Let $\rho_{\dot{s},E}$ be an irreducible character of $\widetilde{G}^{F^2}$ as before, and put $s = \pi(\dot{s})$.*

(i) *If s is not G^{*F^2}-conjugate to s' such that $F(s') = s'^{-1}$, then $m_2(\rho_{\dot{s},E}|_{G^{F^2}}) = 0$.*

(ii) *Assume that $F(s) = s^{-1}$. Then we have*

$$m_2(\rho_{\dot{s},E}|_{G^{F^2}}) = \begin{cases} |\Omega_s^{-\gamma}(E)| & \text{if there exists } x \in \Omega_s^{-\gamma} \text{ such that } {}^{x\gamma}E = E\ , \\ 0 & \text{otherwise,} \end{cases}$$

where $\Omega_s^{-\gamma}(E)$ is the stabilizer of E in $\Omega_s^{-\gamma}$.

Proof. $\rho_{\dot{s},E}$ is given as in (4.1.1). Thus (i) is immediate from Proposition 4.4 (i). We show (ii). So, assume that $F(s) = s^{-1}$. Since $R_{\widetilde{T}^*_w(\dot{s})}|_{G^{F^2}}$ does not depend on the choice of a representative $\dot{s}$ of s, we may assume that $\dot{s}$ satisfies the property that $F(\dot{s}) = \dot{s}^{-1}$. Then $\varepsilon_{\widetilde{G}^*}\varepsilon_{Z_{\widetilde{G}^*}(s)} = 1$ by [L3, 1.5 (b)]. Hence by (4.4.1) together with Lemma 4.5, we have

$$\begin{aligned} m_2(\rho_{\dot{s},E}|_{G^{F^2}}) &= |W_{\dot{s}}|^{-1}\sum_{w \in W_{\dot{s}}} Tr(w\delta, \widetilde{E})m_2(R_{\widetilde{T}^*_w}(\dot{s})|_{G^{F^2}}) \\ &= |W_{\dot{s}}|^{-1}\sum_{w \in W_{\dot{s}}} \mathrm{Tr}\,(w\delta, \widetilde{E}) \sum_{x \in \Omega_s^{-\gamma}} m_2(R_{\widetilde{T}^*_w}(\dot{s}z_x)) \\ &= |W_{\dot{s}}|^{-1}\sum_{x \in \Omega_s^{-\gamma}}\sum_{w \in W_{\dot{s}}} \mathrm{Tr}\,(w\delta, \widetilde{E})\sharp\{u \in W_{\dot{s}} \mid w = u({}^{x\gamma}u)\}. \end{aligned}$$

Now by Lemma 2.11 in [L3] (see also the formula in the proof of Proposition 2.13 there), one can write

$$\sum_{E' \in (W_{\dot{s}})^\wedge_{x\gamma}} \mathrm{Tr}\,(w\delta, \widetilde{E}') = \sharp\{u \in W_{\dot{s}} \mid w = u({}^{x\gamma}u)\},$$

where $(W_{\dot{s}})^\wedge_{x\gamma}$ is the set of $x\gamma$-stable characters of $W_{\dot{s}}$, and the extension $\widetilde{E}'$ of E' is chosen to be realized over $\mathbf{Q}$. (Note that $(x\gamma)^2 = \delta$ since $x \in \Omega_s^{-\gamma}$). It follows that

$$m_2(\rho_{\dot{s},E}|_{G^{F^2}}) = \sum_{x \in \Omega_s^{-\gamma}}\sum_{E' \in (W_{\dot{s}})^\wedge_{x\gamma}} |W_{\dot{s}}|^{-1}\sum_{w \in W_{\dot{s}}} \mathrm{Tr}\,(w\delta, \widetilde{E})\,\mathrm{Tr}\,(w\delta, \widetilde{E}').$$

But since

$$|W_{\dot{s}}|^{-1}\sum_{w \in W_{\dot{s}}} \mathrm{Tr}\,(w\delta, \widetilde{E})\,\mathrm{Tr}\,(w\delta, \widetilde{E}') = \begin{cases} 1 & \text{if } \widetilde{E} = \widetilde{E}', \\ 0 & \text{if } E \neq E', \end{cases}$$

(here the extension $\widetilde{E}$ is chosen to be over $\mathbf{Q}$, see [L1, 3.2]), we have

$$m_2(\rho_{\dot{s},E}|_{G^{F^2}}) = \sharp\{x \in \Omega_s^{-\gamma} \mid {}^{x\gamma}E = E\ \}.$$

If there exists x_1 such that ${}^{x_1\gamma}E = E$, then $\{x \in \Omega_s^{-\gamma} \mid {}^{x\gamma}E = E\} = \Omega_s^{-\gamma}(E)x_1$. Thus the theorem is proved. $\square$

We shall apply the formula in the theorem to the case $\rho_{\dot{s}_x,E_x}$. First we note the following.

Lemma 4.7. *Assume that $F(s) = s^{-1}$. Let s_x be an element corresponding to $x \in (\Omega_s)_\delta$. Then the G^{*F^2}-class of s_x contains an element s' such that $F(s') = {s'}^{-1}$ if and only if there exists $u \in \Omega_s$ such that $u\gamma(u)$ gives a representative of x in Ω_s.*

Proof. Let $\ddot{x} \in Z_{G^*}(s)$ be an element such that its image to Ω_s gives a representative of x. Let s' be an element contained in the G^{*F^2}-class of s_x. Then s' can be obtained as $s' = {}^gs$ for some $g \in G^*$ such that $g^{-1}F^2(g) = \ddot{x}$. It is easy to see that $F(s') = {s'}^{-1}$ if and only if $g^{-1}F(g) \in Z_{G^*}(s)$. Assume that $g^{-1}F(g) \in Z_{G^*}(s)$. Then we have $u\gamma(u) \in \Omega_s$ gives a representative of x if we put u as the image of $g^{-1}F(g)$ in Ω_s. Conversely, assume that there exists $u \in \Omega_s$ as in the lemma, and let $\dot{u}$ be its representative in $Z_{G^*}(s)$. Then $s' = {}^gs$ for $g \in G^*$ such that $g^{-1}F(g) = \dot{u}$ satisfies the property $F(s') = {s'}^{-1}$. $\square$

4.8. We prepare a notation. Let s be a semisimple element such that $F(s) = s^{-1}$, and $E \in \mathrm{Irr}\, W_s^0$ such that the Ω_s-orbit of E is γ-stable. We define a subset $\widetilde{\Omega}_s(E)_\delta^+$ (resp. $\Omega_s(E)_\delta^+$) of $\widetilde{\Omega}_s(E)_\delta$ (resp. of $\Omega_s(E)_\delta$) by

$$\begin{aligned}\widetilde{\Omega}_s(E)_\delta^+ &= \text{ the image of } \{u\gamma(u) \mid u \in \Omega_s, {}^{u\gamma}E = E\} \text{ into } \widetilde{\Omega}_s(E)_\delta,\\ \Omega_s(E)_\delta^+ &= \text{ the image of } \{v\gamma(v) \mid v \in \Omega_s(E)\} \text{ into } \Omega_s(E)_\delta.\end{aligned}$$

Then we can see that there exists $a_E \in \Omega_s$ such that

$$\widetilde{\Omega}_s(E)_\delta^+ = \Omega_s(E)_\delta^+ a_E. \tag{4.8.1}$$

In fact, since the Ω_s-orbit of E is γ-stable, there exists $b \in \Omega_s$ such that ${}^{b\gamma}E = E$. Then $a_E = b\gamma(b)$ is contained in $\widetilde{\Omega}_s(E)$, and we have $\widetilde{\Omega}_s(E) = \Omega_s(E)a_E$. (4.8.1) follows from this.

As a corollary to Theorem 4.6, we have the following.

Corollary 4.9. *Assume that s is semisimple in G^* such that $F(s) = s^{-1}$, and that $E \in (\overline{\mathrm{Irr}}\, W_s^0)^\delta$. Take $y \in \widetilde{\Omega}_s(E)_\delta$ and let $(E, y) \leftrightarrow (x, E_x)$ be as in (2.2.2). Then we have*

(i) *If the Ω_s-orbit of E is not γ-stable, then $m_2(\rho_{\dot{s}_x,E_x}|_{G^{F^2}}) = 0$.*
(ii) *Assume that the Ω_s-orbit of E is γ-stable. Then*

$$m_2(\rho_{\dot{s}_x,E_x}|_{G^{F^2}}) = \begin{cases} |\Omega_s^{-\gamma}(E)| & \text{if } y \in \widetilde{\Omega}_s(E)_\delta^+, \\ 0 & \text{otherwise.}\end{cases}$$

Proof. Assume that $m_2(\rho_{\dot{s}_x,E_x}|_{G^{F^2}}) \neq 0$. Then s_x is G^{*F^2}-conjugate to some s' such that $F(s') = {s'}^{-1}$ by Theorem 4.6. Since x and y are in the same class in $(\Omega_s)_\delta$, there exists $u \in \Omega_s$ such that the image of $uF(u)$ to $(\Omega_s)_\delta$ coincides with y, by Lemma 4.7. Let $\dot{u} \in Z_{G^*}(s)$ be a representative of u. Then there exists $g \in G^*$ such that $g^{-1}F(g) = \dot{u}$. We see that $g^{-1}F^2(g) = \dot{u}F(\dot{u})$ gives a representative of y as its image on Ω_s, which we denote by $\dot{y} \in \Omega_s$. Then as discussed in (2.2.2), there exists $z \in \Omega_s$ such that $\dot{y} = z^{-1}\dot{x}\delta(z)$, and we have $E_x = {}^zE \in (\mathrm{Irr}\, W_s^0)^{\dot{x}\delta}$. Put

$g_1 = g\dot{z}^{-1}$, where $\dot{z} \in Z_{G^*}(s)$ is a representative of $z \in \Omega_s$. Then $g_1^{-1}F^2(g_1)$ gives $\dot{x}$ as its image to Ω_s, and so we may assume that $s_x = {}^{g_1}s$.

Now $\mathrm{ad}\, g_1$ gives an isomorphism $W_s^0 \to W_{s_x}^0$, and by this isomorphism, $E_x \in (\mathrm{Irr}\ W_s^0)^{\dot{x}\delta}$ is sent to $E'' \in (\mathrm{Irr}\ W_{s_x}^0)^{F^2}$. $\rho_{\dot{s}_x, E_x}$ is defined as $\rho_{\dot{s}_x, E''}$ (cf. 2.2). Here $\mathrm{ad}\, g_1 : W_s^0 \to W_{s_x}^0$ is factored as

$$W_s^0 \xrightarrow{\mathrm{ad}\, \dot{z}^{-1}} W_s^0 \xrightarrow{\mathrm{ad}\, g} W_{s_x}^0.$$

In the first isomorphism, $E_x \in (\mathrm{Irr}\ W_s^0)^{\dot{x}\delta}$ is sent to $E \in (\mathrm{Irr}\ W_s^0)^{\dot{y}\delta}$. Moreover, in the second isomorphism, the actions F, F^2 on $W_{s_x}^0$ is transfered to the actions uF, yF^2 on W_s^0. Then by Theorem 4.6, applied to $\rho_{\dot{s},E} \in \mathrm{Irr}\ G^{(\dot{u}F)^2}$ with $(\dot{u}F)(s) = s^{-1}$, $m_2(\rho_{\dot{s}_x,E_x}|_{G^{F^2}}) \neq 0$ is equivalent to the condition that there exists $h \in \Omega_s^{-u\gamma} = \Omega_s^{-\gamma}$ such that ${}^{hu\gamma}E = E$. In particular, the Ω_s-orbit of E is γ-stable. Since $hu\gamma(hu) = u\gamma(u) = y$, this implies that $y \in \widetilde{\Omega}_s(E)_\delta^+$. Moreover, in that case $m_2(\rho_{\dot{s}_x,E_x}|_{G^{F^2}})$ is given by the formula as claimed in the corollary. Conversely, assume that $y \in \widetilde{\Omega}_s(E)_\delta^+$. Then there exists $\dot{u} \in Z_{G^*}(s)$ such that the image of $\dot{u}F(\dot{u})$ to Ω_s gives a representative of y. Take $g \in G^*$ such that $g^{-1}F(g) = \dot{u}$. Then a similar argument as above implies, by Theorem 4.6, that $m_2(\rho_{\dot{s}_x,E_x}|_{G^{F^2}}) \neq 0$. This proves the corollary. □

5. Determination of $m_2(\rho)$ for $\rho \in \mathrm{Irr}\ SL_n(\mathbf{F}_{q^2})$

5.1. In this section, we shall determine $m_2(\rho)$ for all irreducible characters of G^{F^2}. Our strategy is to compute m_2 for almost characters of G^{F^2} first, and then derive the formula for $m_2(\rho)$ from it.

First we prepare some notation. Let s be an F^2-stable semisimple element in G^*, and E an δ-stable irreducible character of W_s^0. We recall two sets $\overline{\mathcal{M}}_{s,E} = \Omega_s^\delta(E)^\wedge \times \widetilde{\Omega}_s(E)_\delta$ and $\mathcal{M}_{s,E} = \Omega_s(E)_\delta^\wedge \times \Omega_s(E)^\delta$. Assuming that $F(s) = s^{-1}$ and that the Ω_s-orbit of E is F-stable, we define subsets $\Omega_s^\delta(E)_-^\wedge \subset \Omega_s^\delta(E)_{-\gamma}^\wedge \subset \Omega_s^\delta(E)^\wedge$ by

$$\Omega_s^\delta(E)_{-\gamma}^\wedge = \{\theta \in \Omega_s^\delta(E)^\wedge \mid \gamma(\theta) = \theta^{-1}\},$$
$$\Omega_s^\delta(E)_-^\wedge = \{\theta^{-1}\gamma(\theta) \mid \theta \in \Omega_s^\delta(E)^\wedge\}.$$

We also consider subsets $\widetilde{\Omega}_s(E)_\delta^+ \subset \widetilde{\Omega}_s(E)_\delta^\gamma \subset \widetilde{\Omega}_s(E)_\delta$, where

$$\widetilde{\Omega}_s(E)_\delta^\gamma = \{u \in \widetilde{\Omega}_s(E)_\delta \mid \gamma(u) = u\},$$

and $\widetilde{\Omega}_s(E)_\delta^+$ is defined as in 4.8. We define subsets $\Omega_s(E)_\delta^+ \subset \Omega_s(E)_\delta^\gamma \subset \Omega_s(E)_\delta$ in a similar way as above.

Put

$$|\Omega_s(E)^\delta| = t, \quad |\Omega_s(E)^\gamma| = d, \quad |\Omega_s(E)^{-\gamma}| = d'.$$

Then we see easily that

$$|\Omega_s^\delta(E)^\wedge| = |\Omega_s(E)_\delta| = t,$$
$$|\Omega_s^\delta(E)_\gamma^\wedge| = |\Omega_s(E)_\delta^\gamma| = d,$$
$$|\Omega_s^\delta(E)_{-\gamma}^\wedge| = |\Omega_s(E)_\delta^{-\gamma}| = d'.$$

Since $\Omega_s(E)^\delta$ is a cyclic group, $\Omega_s(E)^\delta$ is written as a product of $\Omega_s(E)^\gamma$ and $\Omega_s(E)^{-\gamma}$. If $t = |\Omega_s(E)^\delta|$ is even, $\Omega_s(E)^\delta$ contains a unique element of order 2. In that case $\Omega_s(E)^\gamma$ and $\Omega_s(E)^{-\gamma}$ have a non-trivial intersection, and so $t = dd'/2$. If t is odd, then $\Omega_s(E)^\delta = \Omega_s(E)^\gamma \times \Omega_s(E)^{-\gamma}$, and so $t = dd'$.

There is a surjective homomorphism $\Omega_s^\delta(E)^\wedge \to \Omega_s^\delta(E)^\wedge_-$ given by $\theta \to \theta^{-1}\gamma(\theta)$, whose kernel is given by $\Omega_s^\delta(E)^\wedge_\gamma$. It follows that $\Omega_s^\delta(E)^\wedge_-$ is a subgroup of $\Omega_s^\delta(E)^\wedge_{-\gamma}$ of order t/d. Hence we have

$$[\Omega_s^\delta(E)^\wedge_{-\gamma} : \Omega_s^\delta(E)^\wedge_-] = \begin{cases} 1 & \text{if } t = dd', \\ 2 & \text{if } t = dd'/2. \end{cases} \tag{5.1.1}$$

Similarly, we have a surjective homomorphism $\Omega_s(E)_\delta \to \Omega_s(E)_\delta^+$ given by $z \mapsto z\gamma(z)$ with kernel $\Omega_s(E)_\delta^{-\gamma}$. It follows that $\Omega_s(E)_\delta^+$ is a subgroup of $\Omega_s(E)_\delta^\gamma$ of degree t/d'. Hence we have

$$[\Omega_s(E)_\delta^\gamma : \Omega_s(E)_\delta^+] = \begin{cases} 1 & \text{if } t = dd', \\ 2 & \text{if } t = dd'/2. \end{cases} \tag{5.1.2}$$

The following result describes the values of m_2 for almost characters of G^{F^2}.

Theorem 5.2. *Assume that s is a semisimple element in G^{*F^2}, and E is an irreducible character of W_s^0 such that Ω_s-orbit of E is δ-stable. Let $R_{\eta,z}$ be an almost character associated to $(\eta, z) \in \mathcal{M}_{s,E}$. Then*

(i) *Assume that s is not G^*-conjugate to an element s' such that $F(s') = s'^{-1}$. Then $m_2(R_{\eta,z}) = 0$.*
(ii) *Assume that $F(s) = s^{-1}$. If the Ω_s-orbit of E is not γ-stable, then $m_2(R_{\eta,z}) = 0$.*
(iii) *Assume that $F(s) = s^{-1}$, and that the Ω_s-orbit of E is γ-stable.*
  (a) *Assume that $|\Omega_s(E)^\delta|$ is odd. Then we have*

$$m_2(R_{\eta,z}) = \begin{cases} 1 & \textit{if } \eta \in \Omega_s(E)^\wedge_{-\gamma} \textit{ and } z \in \Omega_s(E)^\gamma, \\ 0 & \textit{otherwise.} \end{cases}$$

  (b) *Assume that $|\Omega_s(E)^\delta|$ is even. Then we have*

$$m_2(R_{\eta,z}) = \begin{cases} 1 & \textit{if } \eta \in \Omega_s(E)^\wedge_{-\gamma} \textit{ and } z \in \Omega_s(E)^+, \\ \varepsilon & \textit{if } \eta \in \Omega_s(E)^\wedge_{-\gamma} \textit{ and } z \in \Omega_s(E)^\gamma - \Omega_s(E)^+, \\ 0 & \textit{otherwise,} \end{cases}$$

where $\varepsilon = c_2(\rho_{1,z_0''}) = \pm 1$ for any $z_0'' \in \widetilde{\Omega}_s(E)_\delta^\gamma - \widetilde{\Omega}_s(E)_\delta^+$.

Proof. We show (i). By Theorem 4.6 (i), $m_2(\rho_{\dot{s}_x, E_x}|_{G^{F^2}}) = 0$ for any $x \in (\Omega_s)_\delta$. It follows that $m_2(\rho_{\eta',z'}) = 0$ for any $(\eta', z') \in \overline{\mathcal{M}}_{s,E}$. Hence $m_2(R_{\eta,z}) = 0$ for any $(\eta, z) \in \mathcal{M}_{s,E}$.

A similar proof works for the assertion (ii) since $m_2(\rho_{\dot{s}_x, E_x}) = 0$ for any $x \in (\Omega_s)_\delta$ by Corollary 4.9 (i).

We show (iii) by computing $m_2(R_{\eta,z})$ for $(\eta, z) \in \mathcal{M}_{s,E}$. By Theorem 3.6, we have $m_2(R_{\eta,z}) = \eta(z)^{-1} m_2(t_1^{*-1} R_{\eta,z})$. By the definition of $R_{\eta,z}$, together with Corollary 1.11, applied to the case where $r = 2$, we have

$$m_2(R_{\eta,z}) = \eta(z)^{-1} \sum_{(\eta', z'a) \in \overline{\mathcal{M}}_{s,E}} \{(\eta, z), (\eta', z')\} c_2(\rho_{\eta', z'a}). \tag{5.2.1}$$

with $a = a_E$. Moreover, by Corollary 1.16 (applied to the case where $m = 1$, see also [K2, Theorem 2.1.3]), together with Proposition 2.7, we have

$$(5.2.2) \qquad c_2(\rho_{\eta', z'a}) = \begin{cases} \pm 1 & \text{if } (\eta', z') \in \Omega_s^\delta(E)^\wedge_{-\gamma} \times \Omega_s(E)^\gamma_\delta, \\ 0 & \text{otherwise.} \end{cases}$$

We note the following.

(5.2.3) Assume that $z' \in \Omega_s(E)^+_\delta$. Then $c_2(\rho_{\eta', z'a}) = 1$ for any $\eta' \in \Omega_s^\delta(E)^\wedge_{-\gamma}$.

We show (5.2.3). Let (x, E_x) correspond to (E, z') via f in (2.2.2). Then we have $m_2(\rho_{\dot{s}_x, E_x}|_{G^{F^2}}) = |\Omega_s(E)^{-\gamma}|$ by Corollary 4.9. Since the twisting operator t_1^* acts trivially on $\rho_{\dot{s}_x, E_x}$, we have $c_2(\rho_{\dot{s}_x.E_x}|_{G^{F^2}}) = |\Omega_s(E)^{-\gamma}|$ by Corollary 1.11. On the other hand, $\rho_{\dot{s}_x, E_x}|_{G^{F^2}}$ can be decomposed as in (2.2.3). Hence by (5.2.2), we have

$$c_2(\rho_{\dot{s}_x, E_x}|_{G^{F^2}}) = \sum_{\eta' \in \Omega_s^\delta(E)^\wedge_{-\gamma}} c_2(\rho_{\eta', z'a}).$$

Since $|\Omega_s(E)^{-\gamma}| = |\Omega_s^\delta(E)^\wedge_{-\gamma}| = d'$, we can conclude that $c_2(\rho_{\eta', z'a}) = 1$, and (5.2.3) follows.

We now compute $m_2(R_{\eta, z})$. In view of (5.2.2), the formula (5.2.1) can be written as

$$(5.2.4) \qquad m_2(R_{\eta, z}) = \eta(z)^{-1} |\Omega_s^\delta(E)|^{-1} \sum_{(\eta', z') \in \Omega_s^\delta(E)^\wedge_{-\gamma} \times \Omega_s(E)^\gamma_\delta} \eta(z')\eta'(z) c_2(\rho_{\eta', z'a}).$$

First consider the case where t is odd, i.e., the case where $t = dd'$. Then by (5.1.2), we have $\Omega_s(E)^\gamma_\delta = \Omega_s(E)^+_\delta$. It follows by (5.2.3) that

$$m_2(R_{\eta, z}) = \eta(z)^{-1} |\Omega_s^\delta(E)|^{-1} \sum_{(\eta', z') \in \Omega_s^\delta(E)^\wedge_{-\gamma} \times \Omega_s(E)^\gamma_\delta} \eta(z')\eta'(z).$$

This implies that $m_2(R_{\eta, z}) = 0$ unless η is trivial on $\Omega_s(E)^\gamma_\delta$ and $z \in \Omega_s^\delta(E)$ is such that $\eta'(z) = 1$ for any $\eta' \in \Omega_s^\delta(E)^\wedge_{-\gamma}$. But since $\Omega_s^\delta(E)^\wedge_{-\gamma} = \Omega_s^\delta(E)^\wedge_{-}$, the condition for z is equivalent to the condition that $z \in \Omega_s(E)^\gamma$. Similarly, since $\Omega_s(E)^\gamma_\delta = \Omega_s(E)^+_\delta$, the condition for η is equivalent to the condition that $\eta \in \Omega_s(E)^\wedge_{-\gamma}$. Now assume that $\eta \in \Omega_s^\delta(E)^\wedge_{-\gamma}$ and $z \in \Omega_s(E)^\gamma$. Since $|\Omega_s^\delta(E)| = |\Omega_s^\delta(E)^\wedge_{-\gamma}| \times |\Omega_s(E)^\gamma_\delta|$, and $\eta(z) = 1$, (5.2.4) implies that $m_2(R_{\eta, z}) = 1$. This proves (a) of (iii).

Next we consider the case where t is even, i.e., the case where $t = dd'/2$. In this case, $\Omega_s^\delta(E)^\wedge_{-}$ is an index two subgroup of $\Omega_s^\delta(E)^\wedge_{-\gamma}$, and $\Omega_s(E)^+_\delta$ is an index two subgroup of $\Omega_s(E)^\gamma_\delta$. We fix $\eta'_0 \in \Omega_s^\delta(E)^\wedge_{-\gamma} - \Omega_s^\delta(E)^\wedge_{-}$ and $z'_0 \in \Omega_s(E)^\gamma_\delta - \Omega_s(E)^+_\delta$. Then by using (5.2.3), (5.2.4) can be written as

$$(5.2.5) \qquad m_2(R_{\eta, z}) = \eta(z)^{-1} |\Omega_s^\delta(E)|^{-1} \sum_{(\eta', z') \in \Omega_s^\delta(E)^\wedge_{-} \times \Omega_s(E)^+_\delta} \eta(z')\eta'(z) A_{\eta', z'},$$

where

$$A_{\eta', z'} = 1 + \eta'_0(z) + \eta(z'_0) c_2(\rho_{\eta', z'z'_0a}) + \eta(z'_0)\eta'_0(z) c_2(\rho_{\eta'\eta'_0, z'z'_0a}).$$

It is known by Corollary 3.5 that

$$(5.2.6) \qquad |m_2(R_{\eta, z})| = \begin{cases} 1 & \text{if } (\eta, z) \in \Omega_s(E)^\wedge_{-\gamma} \times \Omega_s(E)^\gamma, \\ 0 & \text{otherwise.} \end{cases}$$

We now assume that $m_2(R_{\eta,z}) \neq 0$. Hence $\eta \in \Omega_s(E)^\wedge_{-\gamma}$ and $z \in \Omega_s(E)^\gamma$. We note that $\eta \in \Omega_s(E)^\wedge$ is contained in $\Omega_s(E)^\wedge_{-\gamma}$ if and only if η is trivial on $\Omega_s(E)^+_\delta$. Similarly, $\eta' \in \Omega^\delta_s(E)^\wedge$ is contained in $\Omega^\delta_s(E)^\wedge_-$ if and only if η' is trivial on $\Omega_s(E)^\gamma$. In particular, we have $\eta(z') = \eta'(z) = 1$ for any $(\eta', z') \in \Omega^\delta_s(E)^\wedge_- \times \Omega_s(E)^+_\delta$ in the sum in (5.2.5). Since $\eta(z), \eta'_0(z), \eta(z'_0)$ take values ± 1, we see that $m_2(R_{\eta,z}) \in \mathbf{Q}$. This implies that $m_2(R_{\eta,z}) = \pm 1$ by (5.2.6).

We shall consider the two cases, whether $z \in \Omega_s(E)^\gamma$ is contained in $\in \Omega_s(E)^+$ or not. First assume that $z \in \Omega_s(E)^+$. Then $\eta(z) = 1$, $\eta'_0(z) = 1$ and $\eta(z'_0) = \pm 1$. Since $|\Omega^\delta_s(E)^\wedge_-| \times |\Omega_s(E)^+_\delta| = t/2$, it follows from (5.2.5) that

$$m_2(R_{\eta,z})t = t + \sum_{(\eta',z')} \eta(z'_0)(c_2(\rho_{\eta',z'z'_0a}) + c_2(\rho_{\eta'e'_0,z'z'_0a})).$$

Let C be the sum part of this formula. Then we have $-t \leq C \leq t$. Since $m_2(R_{\eta,z}) = \pm 1$, this forces that $C = 0$, and we have $m_2(R_{\eta,z}) = 1$.

Next assume that $z \in \Omega_s(E)^\gamma - \Omega_s(E)^+$. Since $z^{-1}z'_0 \in \Omega_s(E)^+_\delta$, we have $\eta(z^{-1}z'_0) = 1$. Moreover, $\eta'_0(z) = -1$. Hence by (5.2.5), we can write

$$m_2(R_{\eta,z})t = \sum_{(\eta',z')} (c_2(\rho_{\eta',z'z'_0a}) - c_2(\rho_{\eta'\eta'_0,z'z'_0a})).$$

But since

$$\sum_{(\eta',z')} |c_2(\rho_{\eta',z'z'_0a}) - c_2(\rho_{\eta'\eta'_0,z'z'_0a})| \leq t = |m_2(R_{\eta,z})t|,$$

we see that $c_2(\rho_{\eta',z'z'_0a}) = -c_2(\rho_{\eta'\eta'_0,z'z'_0a})$ has a common value for any (η', z'), which coincides with $c_2(\rho_{1,z'_0a}) = -c_2(\rho_{\eta'_0,z'_0a})$. This implies that $m_2(R_{\eta,z}) = c_2(\rho_{1,z'_0a}) = \varepsilon$. By putting $z''_0 = z'_0a$, we obtain the theorem. $\square$

We can now easily translate Theorem 5.2 to the form on $m_2(\rho)$ for irreducible characters ρ.

Theorem 5.3. *Assume that s is a semisimple element in G^{*F^2}, and that $E \in \operatorname{Irr} W^0_s$ is such that the Ω_s-orbit of E is δ-stable. Let $\rho_{\eta',\zeta''}$ be an irreducible character of G^{F^2} associated to $(\eta', z'') \in \overline{\mathcal{M}}_{s,E}$. Then*

(i) *Assume that s is not G^*-conjugate to an element s' such that $F(s') = s'^{-1}$. Then $m_2(\rho_{\eta',z''}) = 0$.*

(ii) *Assume that $F(s) = s^{-1}$. If the Ω_s-orbit of E is not γ-stable, then $m_2(\rho_{\eta',z''}) = 0$.*

(iii) *Assume that $F(s) = s^{-1}$ and that the Ω_s-orbit of E is γ-stable.*

(a) *Assume that $|\Omega_s(E)^\delta|$ is odd. Then we have*

$$m_2(\rho_{\eta',z''}) = \begin{cases} 1 & \text{if } \eta' \in \Omega^\delta_s(E)^\wedge_{-\gamma} \text{ and } z'' \in \widetilde{\Omega}_s(E)^\gamma_\delta, \\ 0 & \text{otherwise.} \end{cases}$$

(b) *Assume that $|\Omega_s(E)^\delta|$ is even. Then we have*

$$m_2(\rho_{\eta',z''}) = \begin{cases} 1+\varepsilon & \text{if } \eta' \in \Omega^\delta_s(E)^\wedge_- \text{ and } z'' \in \widetilde{\Omega}_s(E)^+_\delta, \\ 1-\varepsilon & \text{if } \eta' \in \Omega^\delta_s(E)^\wedge_{-\gamma} - \Omega^\delta_s(E)^\wedge_- \text{ and } z'' \in \widetilde{\Omega}_s(E)^+_\delta, \\ 0 & \text{otherwise,} \end{cases}$$

where $\varepsilon = c_2(\rho_{1,z''_0}) = \pm 1$ for any $z''_0 \in \widetilde{\Omega}_s(E)^\gamma_\delta - \widetilde{\Omega}_s(E)^+_\delta$.

Proof. The assertion (i) and (ii) are already shown in the proof of Theorem 5.2. We show (iii). First assume that $|\Omega_s(E)^\delta|$ is odd. Then (3.3.2) implies, in view of Theorem 5.2, that

$$m_2(\rho_{\eta',z'a}) = |\Omega_s(E)^\delta|^{-1} \sum_{(\eta,z)\in\Omega_s(E)^\wedge_{-\gamma}\times\Omega_s(E)^\gamma} \eta(z')^{-1}\eta'(z)^{-1}.$$

It follows that $m_2(\rho_{\eta',z'a}) = 0$ unless η' is trivial on $\Omega_s(E)^\gamma$, and $\eta(z') = 1$ for any $\eta \in \Omega_s(E)^\wedge_{-\gamma}$, and in which case $m_2(\rho_{\eta',z'a}) = 1$. But this condition is equivalent to the condition that $\eta' \in \Omega^\delta_s(E)^\wedge_{-\gamma}$ and $z' \in \Omega_s(E)^+_\delta$. By replacing $z'a$ by z'', we obtain (a).

Next assume that $|\Omega_s(E)^\delta|$ is even. Let us fix $z_0 \in \Omega_s(E)^\gamma - \Omega_s(E)^+$. Again by Theorem 5.2, we have

$$m_2(\rho_{\eta',z'a}) = |\Omega_s(E)^\delta|^{-1} \sum_{(\eta,z)\in\Omega_s(E)^\wedge_{-\gamma}\times\Omega_s(E)^+} \eta(z')^{-1}\eta'(z)^{-1}(1+\eta'(z_0)^{-1}\varepsilon).$$

It follows that $m_2(\rho_{\eta',z'a}) = 0$ unless η' is trivial on $\Omega^\delta_s(E)^+$ and $\eta(z') = 1$ for any $\eta \in \Omega_s(E)^\wedge_{-\gamma}$, and in which case $m_2(\rho_{\eta',z'a}) = 1 + \eta'(z_0)^{-1}\varepsilon$. The condition for z' is the same as before, and η' is trivial on $\Omega^\delta_s(E)^+$ if and only if $\eta' \in \Omega^\delta_s(E)^\wedge_{-\gamma}$. Moreover,

$$\eta'(z_0) = \begin{cases} 1 & \text{if } \eta' \in \Omega^\delta_s(E)^\wedge_-, \\ -1 & \text{if } \eta' \in \Omega^\delta_s(E)^\wedge_{-\gamma} - \Omega^\delta_s(E)^\wedge_-. \end{cases}$$

Hence (b) holds, and the theorem is proved. □

Remark 5.4. In [L3], Lusztig gave a uniform description of $m_2(\rho)$ for any irreducible character ρ of G^{F^2} in the case where G is a connected reductive group with connected center. He expects that his formulation will be extended also to the disconnected center case. We shall compare our results with Lusztig's conjectural description. Take $(\eta, z) \in \overline{\mathcal{M}}_{s,E}$. For $z \in \widetilde\Omega_s(E)_\delta$, take a representative $\dot z \in \Omega_s(E)$ of z, and put

$$\sqrt{z} = \text{the image of } \{y \in \Omega_s \mid {}^{y\gamma}E = E, (y\gamma)^2 = \dot z\delta\} \text{ into } \Omega_s(E)_\delta.$$

Then $\Omega_s(E)^\delta$ acts on $\sqrt{z}$ by the F-twisted conjugation. We denote by $\sqrt{z}$ the corresponding permutation representation also. Let $[\eta : \sqrt{z}]$ the multiplicity of η in this permutation representation. Now assume that $F(s) = s^{-1}$ and that the Ω_s-orbit of E is F-stable. Then we have the following.

(5.4.1) Assume that $(\eta, z) \in \Omega^\delta_s(E)^\wedge_- \times \widetilde\Omega_s(E)^+_\delta$. Then we have

$$[\eta : \sqrt{z}] = \begin{cases} 1 & \text{if } |\Omega_s(E)^\delta| \text{ is odd}, \\ 2 & \text{if } |\Omega_s(E)^\delta| \text{ is even}. \end{cases}$$

If $(\eta, z) \notin \Omega^\delta_s(E)^\wedge_- \times \widetilde\Omega_s(E)^+_\delta$, then we have $[\eta : \sqrt{z}] = 0$.

In fact, in our setting, $\widetilde\Omega_s(E)^+_\delta$ is the set of $z \in \widetilde\Omega_s(E)_\delta$ such that $\sqrt{z} \neq \emptyset$. Hence if $z \notin \widetilde\Omega_s(E)^+_\delta$, then $\sqrt{z} = \emptyset$, and so $[\eta : \sqrt{z}] = 0$. If $z \in \widetilde\Omega_s(E)^+_\delta$, then $\sqrt{z} = \Omega_s(E)^{-\gamma}_\delta y$ for some $y \in \sqrt{z}$. Let χ be the character of the representation $\sqrt{z}$. Then $\chi(u) = |\Omega_s(E)^{-\gamma}_\delta|$ if $u \in \Omega_s(E)^\gamma$ and $\chi(u) = 0$ otherwise. It follows that

$$[\eta : \sqrt{z}] = \begin{cases} |\Omega_s(E)^{-\gamma}_\delta||\Omega_s(E)^\gamma|/|\Omega_s(E)^\delta| & \text{if } \eta \in \Omega^\delta_s(E)^\wedge_-, \\ 0 & \text{otherwise}. \end{cases}$$

(5.4.1) follows from this.

In the connected center case, $[\eta : \sqrt{z}]$ gives the value $m_2(\rho)$. In our situation, by comparing with Theorem 5.3, we see that $[\eta : \sqrt{z}]$ coincides with $m_2(\rho_{\eta,z})$ if and only if $\varepsilon = 1$. As an example of SL_2 shows (see 5.7), Lusztig's formula does not hold in general, as far as we use the parametrization of Irr G^{F^2} based on the choice of $N \in \mathfrak{g}^F$ as in (2.5.2), i.e., N is given by Jordan normal form. However, as the referee pointed out, this discrepancy can be removed (at least for (s, E) treated in Lemma 5.6), by choosing $N \in \mathfrak{g}^{F^2}$ such that $F(N) = -N$ instead of (2.5.2) (see the discussion in Lemma 5.6).

5.5. We shall determine the value ε explicitly in certain special cases. We assume that (s, E) satisfies the property in (2.5.1). In that case, we have a bijection $\overline{\mathcal{M}}_{s,E} \simeq \overline{\mathcal{M}}_{s,N}$ for a nilpotent element $N \in \mathfrak{g}^F$ (cf. [S3, 4.5]). We have $\Omega_s(E) = \Omega_s$, and $\overline{\mathcal{M}}_{s,E} = (\Omega_s^\delta)^\wedge \times (\widetilde{\Omega}_s)_\delta$. Moreover, $\overline{\mathcal{M}}_{s,N} = (\overline{A}_\lambda)_\delta \times (\overline{A}_\lambda^\delta)^\wedge$, and the above bijection is given through bijections $(\Omega_s^\delta)^\wedge \simeq (\overline{A}_\lambda)_\delta$ and $(\widetilde{\Omega}_s)_\delta \simeq (\overline{A}_\lambda^\delta)^\wedge$. (Compare with Lemma 3.7). These bijections can be explicitly described in a similar way as in Lemma 3.7. Now there exists $g \in G$ such that ${}^gN = -N$. Take $\alpha \in G$ such that $\alpha^{-1}F(\alpha) = g$, and put $N' = {}^\alpha N$. Then we have $F(N') = -N'$. Since $N' \in \mathfrak{g}^{F^2}$, one can write $N' = N_{c_1}$ for a $c_1 \in (A_\lambda)_\delta$. By the isomorphism $(\overline{A}_\lambda)_\delta \simeq (\Omega_s^\delta)^\wedge$, c_1 determines a unique element $\eta_1 \in (\Omega_s^\delta)^\wedge$. The following result was inspired by the comment of the referee. We are very grateful to the referee for this.

Lemma 5.6. *Assume that (s, E) is as in (2.5.1), and that we are in a setting in (iii), (b) in Theorem 5.3. Let $\eta_1 \in (\Omega_s^\delta)^\wedge$ be as in 5.5. Then we have $\eta_1 \in (\Omega_s^\delta)^\wedge_{-\gamma}$, and*

$$\varepsilon = \begin{cases} 1 & \text{if } \eta_1 \in (\Omega_s^\delta)^\wedge_-, \\ -1 & \text{if } \eta_1 \in (\Omega_s^\delta)^\wedge_{-\gamma} - (\Omega_s^\delta)^\wedge_-. \end{cases}$$

Proof. We consider the generalized Gelfand-Graev character $\varGamma_{N'} = \mathrm{Ind}_{U^{F^2}}^{G^{F^2}}(\varLambda_{c_1})$. Since $F(N') = -N'$, we see that $\varLambda_{c_1}$ is a linear character of U^{F^2} such that $F(\varLambda_{c_1}) = \overline{\varLambda}_{c_1}$. Then there exists a basis of the representation $\varGamma_{N'}$ such that the corresponding matrix $R(g)$ satisfies the property (*) that $R(F(g))$ coincides with the complex conjugate of $R(g)$ for all $g \in G^{F^2}$. Let ρ be an irreducible character of G^{F^2} such that $F(\rho) = \overline{\rho}$. Then the representation obtained as the ρ-isotypic part of $\varGamma_{N'}$ also satisfies this property. It follows that if ρ has multiplicity one in $\varGamma_{N'}$, then ρ is afforded by a matrix representation satisfying the property (*) as above. Now it is known by [KM], for any $\rho \in \mathrm{Irr}\, G^{F^2}$, that $c_2(\rho) = 1$ if and only if ρ is afforded by a representation satisfying (*). Summing up the above argument, we see that $c_2(\rho) = 1$ if $\langle \varGamma_{N'}, \rho \rangle = 1$.

Now by our parametrization, ρ_{c_1,ξ_x} $((c_1, \xi_x) \in \overline{\mathcal{M}}_{s,N})$ is the unique character of G^{F^2} appearing in $\varGamma_{N'}$ and in $\rho_{\dot{s}_x,E}$ with multiplicity one. By $\overline{\mathcal{M}}_{s,N} \simeq \overline{\mathcal{M}}_{s,E}$, $(c_1, \xi_x) \leftrightarrow (\eta_1, x)$, this implies that $c_2(\rho_{\eta_1,x}) = 1$ for any $x \in (\widetilde{\Omega}_s)_\delta$. We use the notation in the proof of Theorem 5.2. By (5.2.2), we know that $c_2(\rho_{\eta',z'a}) = 0$ unless $\eta' \in (\Omega_s^\delta)^\wedge_{-\gamma}$ for $(\eta', z'a) \in \overline{\mathcal{M}}_{s,E}$. It follows that $\eta_1 \in (\Omega_s^\delta)^\wedge_{-\gamma}$. In the last part of the proof of Theorem 5.2, we know that $c_2(\rho_{\eta',z'z_0'a}) = -c_2(\rho_{\eta'\eta_0',z'z_0'a})$ has a common value for any $(\eta', z') \in (\Omega_s^\delta)^\wedge_- \times (\Omega_s)^+_\delta$, which coincides with ε. So if

$\eta_1 \in (\Omega_s^\delta)^\wedge_-$ (resp. $\eta_1 \notin (\Omega_s^\delta)^\wedge_-$), then $\eta_1 = \eta'$ (resp. $\eta_1 = \eta'\eta'_0$) for some η', and we have $\varepsilon = 1$ (resp. $\varepsilon = -1$), respectively. This proves the lemma. □

5.7. We give an example in the case where $G = SL_2$ and $\widetilde{G} = GL_2$. Assume that p is odd, and let $\dot{s} = \mathrm{Diag}(1, -1)$ be a regular semisimple element in $\widetilde{G}^*$ and put $s = \pi(\dot{s})$. Then we have $W_s = \Omega_s \simeq \mathbf{Z}/2\mathbf{Z}$, on which F acts trivially. We consider the pair (s, E), where E is the trivial character of $W_s^0 = \{1\}$. Then (s, E) satisfies the property (2.5.1), and we have $\overline{\mathcal{M}}_{s,E} = \Omega_s^\wedge \times \Omega_s$. Since $F(s) = s = s^{-1}$, and $|\Omega_s| = 2$, we see that the pair (s, E) falls in the class (iii), (b) in Theorem 5.2. We have $(\Omega_s)^\wedge_{-\gamma} = \Omega_s^\wedge$, and $(\Omega_s)^\wedge_- = \{1\}$.

On the other hand, for $\rho = \rho_{\dot{s},E} \in \mathrm{Irr}\, \widetilde{G}^{F^2}$, $\mathcal{O}_\rho = \mathcal{O}_N$, where $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is a regular nilpotent element in $\mathfrak{g}^F$. We have $A_\lambda \simeq \mathbf{Z}/2\mathbf{Z}$ on which F acts trivially. Put $t = \mathrm{Diag}(\alpha, \alpha^{-1}) \in G$, then ${}^tN = \alpha^2 N$. We choose $\alpha \in \overline{\mathbf{F}}_q$ such that $\alpha^2 = -1$. Take $g \in G$ such that $g^{-1}F(g) = t$. Then $N' = {}^gN$ satisfies the relation $F(N') = -N'$. We can choose $g = \mathrm{Diag}(\beta, \beta^{-1})$ such that $\beta^{q-1} = \alpha$. Then $g^{-1}F^2(g) = \mathrm{Diag}(\gamma, \gamma^{-1})$, where

$$\gamma = \beta^{q^2-1} = \alpha^{q+1} = \begin{cases} 1 & \text{if } q \equiv 3 \pmod 4, \\ -1 & \text{if } q \equiv 1 \pmod 4. \end{cases}$$

It follows that $N' = N_{c_1}$ with $c_1 \in A_\lambda \simeq \{\pm 1\}$, where $c_1 = 1$ (resp. $c_1 = -1$) if $q \equiv 3 \pmod 4$ (resp. $q \equiv 1 \pmod 4$), respectively. Thus under the isomorphism $A_\lambda \simeq \Omega_s^\wedge$, c_1 corresponds to $\eta_1 \in \Omega_s^\wedge = \{\pm 1\}$. In view of Lemma 5.6, we conclude that

$$\varepsilon = \begin{cases} 1 & \text{if } q \equiv 3 \pmod 4, \\ -1 & \text{if } q \equiv 1 \pmod 4. \end{cases} \tag{5.7.1}$$

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