p -RATIONAL CHARACTERS AND SELF-NORMALIZING SYLOW p -SUBGROUPS

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ABSTRACT. Let G be a finite group, p a prime, and P a Sylow p -subgroup of G . Several recent refinements of the McKay conjecture suggest that there should exist a bijection between the irreducible characters of p' -degree of G and the irreducible characters of p' -degree of $\mathbf{N}_G(P)$, which preserves field of values of correspondent characters (over the p -adics). This strengthening of the McKay conjecture has several consequences. In this paper we prove one of these consequences: If $p > 2$, then G has no non-trivial p' -degree p -rational irreducible characters if and only if $\mathbf{N}_G(P) = P$.

1. INTRODUCTION

If G is a finite group, p is a prime and $P \in \text{Syl}_p(G)$, then the McKay conjecture asserts that $|\text{Irr}_{p'}(G)| = |\text{Irr}_{p'}(\mathbf{N}_G(P))|$, where $\text{Irr}_{p'}(G)$ is the set of complex irreducible characters of G of degree not divisible by p .

Several recent refinements of this conjecture suggest that not only should there exist a bijection $\text{Irr}_{p'}(G) \rightarrow \text{Irr}_{p'}(\mathbf{N}_G(P))$, but that we should be able to find one that preserves the field of values of correspondent characters (over the p -adics) (see [N2] and [T1]). This strengthening of the McKay conjecture has several consequences, and the main purpose of this paper is to prove the following one, which is of interest by itself. Recall that a complex character χ of a finite group G is **p -rational** if there is a positive integer n coprime to p such that $\chi(g)$ is contained in the n th-cyclotomic field $\mathbb{Q}_n := \mathbb{Q}(\exp(2\pi i/n))$ for all $g \in G$.

Theorem A. *Suppose that G is a finite group, p is an odd prime, and let $P \in \text{Syl}_p(G)$. Then G has no non-trivial p' -degree p -rational irreducible characters if and only if $\mathbf{N}_G(P) = P$.*

Therefore, the fact that a finite group G has a self-normalizing Sylow p -subgroup (for p odd) can be read off from the character table of G .

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2. PRELIMINARIES

In the next lemma, we collect several facts that we shall be using throughout this paper. Whenever a group Q acts as automorphisms on G , we write GQ to denote the semidirect product.

Lemma 2.1. *Suppose that a finite p -group Q acts as automorphisms on a finite group G , and let $N \triangleleft G$ be Q -invariant. Let $P/N \in \text{Syl}_p(G/N)$ be stabilized by Q .*

(i) *Let $M \triangleleft G$ be Q -invariant, let $S = PM$ and let $R = M \cap P$. Then*

$$\mathbf{C}_{\mathbf{N}_G(P)/P}(Q) = 1 \quad \text{iff} \quad \mathbf{C}_{\mathbf{N}_G(S)/S}(Q) = 1 \quad \text{and} \quad \mathbf{C}_{\mathbf{N}_M(R)/R}(PQ) = 1.$$

(ii) *$(GQ)/N$ has a self-normalizing Sylow p -subgroup iff $\mathbf{C}_{\mathbf{N}_G(P)/P}(Q) = 1$.*

Proof. (i) We have that $\mathbf{N}_G(S)/S$ is Q -isomorphic to $\mathbf{N}_G(P)/P\mathbf{N}_M(P)$. Also, we have that $P\mathbf{N}_M(P)/P$ and $\mathbf{N}_M(P)/R$ are Q -isomorphic. Since $\mathbf{N}_G(P) \subseteq \mathbf{N}_G(R)$, we have that

$$\mathbf{N}_M(P)/R = \mathbf{C}_{\mathbf{N}_M(R)/R}(P).$$

By coprime action, part (i) easily follows.

(ii) We have that $(PQ)/N$ is a Sylow p -subgroup of $(GQ)/N$. If $H = \mathbf{N}_{GQ}(PQ)$, then

$$H/PQ \cong \mathbf{C}_{\mathbf{N}_G(P)/P}(Q)$$

and the result easily follows. $\square$

The following is a slight variation of Lemma (5.2) in [N1].

Lemma 2.2. *Suppose that a finite p -group Q acts as automorphisms on a finite group G . Suppose that $N \triangleleft G$ is Q -invariant, and let $P/N \in \text{Syl}_p(G/N)$ be stabilized by Q . Suppose that $\chi \in \text{Irr}(G)$ is Q -invariant and has p' -degree. Let $N \subseteq M \triangleleft G$ be Q -invariant. If $\mathbf{C}_{\mathbf{N}_G(P)/P}(Q) = 1$, then χ_M has a unique PQ -invariant irreducible constituent.*

Proof. By Lemma 2.1(i), we may assume that $N = M$. Now, by using Clifford theory and the Frattini argument, it is easy to check that $\mathbf{N}_G(P)/P$ acts transitively on the P -invariant irreducible constituents of χ_N . Also, Q acts on them, because Q stabilizes P and χ is Q -invariant. Now, apply Glauberman's Lemma (13.8) and Corollary (13.9) of [I]. $\square$

3. SIMPLE GROUPS

We shall need to appeal to the classification of finite simple groups in order to prove Theorem A.

Theorem 3.1. *Let G be a finite quasisimple group, let $p > 2$ be a prime and let Q be a finite p -group acting on G . Suppose that $Z = \mathbf{Z}(G)$ is a p' -group. Assume that P/Z is a Q -invariant Sylow p -subgroup in G and $\mathbf{C}_{\mathbf{N}_G(P)/P}(Q) = 1$. Then, for any Q -invariant $\lambda \in \text{Irr}(Z)$, there is a unique irreducible, Q -invariant, p -rational, p' -degree character of G that lies over λ .*

Proof. 1) By Lemma (2.1), our assumptions imply that GQ/Z has a self-normalizing Sylow p -subgroup. Hence, by the main result of [GMN], $p = 3$ and $G/Z = L_2(q)$ with $q = 3^{3^f} \geq 27$. Since $SL_2(q)$ is the universal cover for G/Z , without loss we

may assume that $G = SL_2(q)$ and so $Z = \mathbb{Z}_2$. We may also identify P/Z with the subgroup

$$\left\{ \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \mid y \in \mathbb{F}_q \right\}.$$

Next, $Q/\mathbf{C}_Q(G)$ induces a 3-subgroup of $\text{Aut}(G) = (G/Z) \cdot \text{Out}(G)$, where the subgroup $\mathbb{Z}_f$ of $\text{Out}(G) \simeq \mathbb{Z}_{2f}$ is generated by the field automorphism σ of G that raises every entry of $g \in G$ to its third power. The action on G of any element $h \in Q$ is equal to the action of $n(h)\sigma^{i(h)}$ for some $n(h) \in G$ and some integer $i(h) \geq 0$. We claim that we can find $h_0 \in Q$ such that $i(h_0) = 1$. Otherwise there is some integer e , $1 \leq e \leq f$, such that $3^e \mid i(h)$ for all $h \in Q$. Since Q and σ normalize P , $n(h) \in \mathbf{N}_G(P)$. Observe that $\mathbf{N}_G(P)/P$ is abelian, so $\mathbf{C}_{\mathbf{N}_G(P)/P}(Q) \geq \mathbf{C}_{\mathbf{N}_G(P)/P}(\sigma^{3^e})$. But the latter subgroup has order $(3^{3^e} - 1)/2 \geq 13$, a contradiction. Conversely, if such an h_0 exists, then $\mathbf{C}_{\mathbf{N}_G(P)/P}(Q) = 1$.

Clearly, the conjugation by $n(h)$ fixes every conjugacy class of G and every complex character of G , and centralizes Z . So in what follows we can ignore $n(h)$ for $h \in Q$, and without loss identify $Q/\mathbf{C}_Q(G)$ with $\langle \sigma \rangle$.

2) We will use the notation for conjugacy classes and complex characters of G as given in [Do]. Here we show that G has exactly two irreducible characters that are Q -invariant, p -rational, and of p' -degree, namely, the trivial character 1_G and $\theta_{(q+1)/4}$ (of degree $q - 1$).

Indeed, assume $\chi \in \text{Irr}(G)$ is Q -invariant, p -rational, and of p' -degree > 1 . The p -rationality excludes the characters of degree $(q \pm 1)/2$. The condition on degree eliminates the Steinberg character. So $\chi = \chi_j$ (of degree $q + 1$) or θ_k (of degree $q - 1$) with $1 \leq j \leq (q - 3)/2$, $1 \leq k \leq (q - 1)/2$.

First we consider the case $\chi = \chi_j$. The Q -invariance of χ is now equivalent to the condition $\chi(a) = \chi(a^\sigma)$, where a is an element of order $q - 1$. Recall ([Do]) that $\chi_j(a^l) = 2 \cos(2\pi jl/(q - 1))$. But a^σ is G -conjugate to a^3 , so $\cos(2\pi j/(q - 1)) = \cos(6\pi j/(q - 1))$. It follows that $4j$ is divisible by $q - 1$, which is impossible as $q \equiv 3 \pmod{4}$ and $1 \leq j \leq (q - 3)/2$.

Next we consider the case $\chi = \theta_k$. The Q -invariance of χ is now equivalent to the condition $\chi(b) = \chi(b^\sigma)$, where b is an element of order $q + 1$. Recall ([Do]) that $\theta_k(b^l) = -2 \cos(2\pi kl/(q + 1))$. But b^σ is G -conjugate to b^3 , so $\cos(2\pi k/(q + 1)) = \cos(6\pi k/(q + 1))$. This can happen exactly when $4k$ is divisible by $q + 1$, which implies $k = (q + 1)/4$ since $1 \leq k \leq (q - 1)/2$. Notice that $\theta_{(q+1)/4}$ is in fact rational-valued. A similar argument shows that $\theta_{(q+1)/7}$ is σ^3 -invariant, p -rational, and trivial on Z .

3) Clearly, $\rho \in \text{Irr}(G)$ lies above $\lambda \in \text{Irr}(Z)$ if and only if $\rho_Z = \rho(1)\lambda$. Now 1_G is trivial on Z , and $(\theta_{(q+1)/4})_Z = (q - 1)\alpha$, where α is the unique non-trivial linear character of Z (since $q \equiv 3 \pmod{8}$). Hence the theorem follows. $\square$

Corollary 3.2. *Let $G = L_2(q)$ with $q = 3^n \geq 9$. Let Q be a finite 3-group acting on G , and let P be a Q -invariant Sylow 3-subgroup in G . Then $\mathbf{C}_{\mathbf{N}_G(P)/P}(Q) = 1$ if and only if G has a unique Q -invariant, 3-rational, $3'$ -degree, irreducible complex character.*

Proof. Write $n = m \cdot 3^f$ with $(m, 3) = 1$. The case $m = 1$ has already been considered in the proof of Theorem 3.1, so we may assume $m > 1$. Arguing as in that proof, we see that $|\mathbf{C}_{\mathbf{N}_G(P)/P}(Q)| \geq (3^m - 1)/2 \geq 4$, and $\theta_{2(q+1)/(3^m+1)}$ (in

the notation of [Do]) is a non-trivial Q -invariant, 3-rational, $3'$ -degree, irreducible complex character of G . $\square$

Lemma 3.3. *Let G be a sporadic finite simple group. Then for any prime p , G has at least two non-trivial rational-valued irreducible complex characters of distinct p' -degrees.*

Proof. Direct (but tedious!) inspection of [Atlas]. $\square$

Now we prove the main result of the section.

Theorem 3.4. *Let G be a finite nonabelian simple group and $p > 2$ a prime. Let Q be a finite p -group acting on G , P a Q -invariant Sylow p -subgroup in G . Then $\mathbf{C}_{\mathbf{N}_G(P)/P}(Q) = 1$ if and only if G has a unique Q -invariant, p -rational, p' -degree, irreducible complex character.*

Proof. 1) Assume $\mathbf{C}_{\mathbf{N}_G(P)/P}(Q) = 1$. By Lemma 2.1, GQ has a self-normalizing Sylow p -subgroup. Hence, by the main result of [GMN], $G = L_2(q)$ for some $q = 3^{3^f} \geq 27$, and so we are done by Corollary 3.2.

From now on we assume $\mathbf{C}_{\mathbf{N}_G(P)/P}(Q) > 1$. Again by Corollary 3.2 we may assume that $G \not\cong L_2(q)$ for any q divisible by 3. The case of sporadic groups has been handled in Lemma (3.3), since $|\text{Out}(G)| \leq 2$ if G is sporadic. Assume $G = \mathbb{A}_n$ for some $n \geq 5$. Clearly, Q acts trivially on G . If $n \geq 6$, the Specht modules labeled by the partitions $(n-2, 2)$ and $(n-2, 1^2)$ give rise to rational irreducible characters of degree d and $d+1$ with $d = n(n-3)/2$. Similarly, $\mathbb{A}_5$ has rational irreducible characters of degree d and $d+1$ with $d = 4$. Either d or $d+1$ is obviously coprime to p , so we are done in this case. If G is a finite simple group of Lie type in characteristic $\neq p$, then the Steinberg character of G is rational, of p' -degree, and fixed by all automorphisms of G .

2) Henceforth we may assume G is a finite simple group of Lie type in characteristic p (and $G \not\cong {}^2G_2(3)' = L_2(8)$). We will largely follow the proof of [IMN, Theorem 4.1]. There is a simple simply connected algebraic group $\mathcal{G}$ over the algebraic closure $\mathbb{F}$ of $\mathbb{F}_p$ and a Frobenius map F on $\mathcal{G}$ such that $G = \mathcal{G}^F/\mathbf{Z}(\mathcal{G}^F)$. Let $(\mathcal{G}^*, F^*)$ be dual to $(\mathcal{G}, F)$. For every conjugacy class $[s]$ of a semisimple element $s \in \mathcal{G}^{F^*}$ such that $\mathbf{C}_{\mathcal{G}^*}(s)$ is connected, there is a so-called semisimple irreducible character $\chi_s \in \text{Irr}(\mathcal{G}^F)$ of degree coprime to p (cf. [C, DM, L]). By its definition, χ_s is a $\mathbb{Q}$ -linear combination of certain Deligne-Lusztig characters, hence it is p -rational. Also, χ_s belongs to the Lusztig series corresponding to $[s]$. Clearly, $\mathcal{G}^F$ has a unique irreducible character of degree 1—the trivial character—and it belongs to the Lusztig series corresponding to $[1]$. Since Lusztig series are disjoint, we see that $\chi_s(1) > 1$ if $s \neq 1$. If s is contained in the derived subgroup of $\mathcal{G}^{*F^*}$, then $\mathbf{Z}(\mathcal{G}^F) \leq \ker(\chi_s)$, and so χ_s defines an irreducible character of G . Furthermore, by [L], this character is invariant under $\text{Aut}(G) = \text{Aut}(\mathcal{G}^F)$ provided that the class $[s]$ is invariant under $\text{Aut}(\mathcal{G}^{*F^*})$ (and $\mathbf{C}_{\mathcal{G}^*}(s)$ is connected as we assumed above). Hence it suffices to find a non-trivial semisimple element s in $\mathcal{G}^{*F^*}$, lying in the derived subgroup of $\mathcal{G}^{*F^*}$, having connected centralizer in $\mathcal{G}^*$, and such that its conjugacy class is invariant under $\text{Aut}(\mathcal{G}^{*F^*})$.

Suppose first that G is one of the groups: ${}^2G_2(q)$, $G_2(q)$, ${}^3D_4(q)$, $F_4(q)$, $E_6(q)$, ${}^2E_6(q)$, $E_7(q)$ or $E_8(q)$. Let s be an involution with centralizer of type $A_1(q)$, $A_1(q)^2$, $A_1(q^3)A_1(q)$, $B_4(q)$, $D_5(q)$, ${}^2D_5(q)$, $D_6(q)A_1(q)$ or $D_8(q)$, respectively

[GLS, see Table 4.5.1]. By [GLS], this involution has connected centralizer and lies in the derived group of $\mathcal{G}^{*F^*}$. Moreover, by the classification of involutions in these groups, $[s]$ is invariant $\text{Aut}(\mathcal{G}^{*F^*})$. Similarly, if $\mathcal{G}$ is of type A_n with $n = 2$ or $n \geq 4$, then an involution s with centralizer of type $A_{n-2}A_1$ will have desired properties. (Again, see [GLS, Table 4.5.1].)

3) For the remaining classical groups, we can assume that $\mathcal{G}$ is defined over the prime field $\mathbb{F}_p$ with corresponding Frobenius map F_0 . Then $\mathcal{G}^F$ is the fixed group of the map $F = F_0^a \tau$ for some $a \geq 1$ and τ is either trivial or a graph automorphism. Here we consider the case $p > 3$. Let $\mathcal{H}$ be the centralizer in $\mathcal{G}^*$ of the group of graph automorphisms of $\mathcal{G}^*$. Then $\mathcal{H}$ is again a reductive group of positive rank defined over $\mathbb{F}_p$ (notice that type A_2 is excluded). Let $\mathcal{H}_0$ be an F_0 -stable A_1 -type subgroup of $\mathcal{H}$. Then the fixed group $\mathcal{H}_0^{F_0}$ (which is isomorphic to $SL_2(p)$ or $PGL_2(p)$) contains a semisimple element s of order 3 which lies in the derived subgroup S of $\mathcal{G}^{*F^*}$.

Obviously, any automorphism of $\mathcal{G}^{*F^*}$ induces an automorphism of S . But S is the Chevalley group of adjoint type corresponding to $(\mathcal{G}^*, F^*)$, and so $\text{Aut}(S)$ is the split extension of $\text{Inndiag}(S)$ by $\Phi_S \Gamma_S$, where $\text{Inndiag}(S)$ is induced by the action of $\mathcal{G}^{*F^*}$ on S , and Φ_S , resp. Γ_S , is the group of field automorphisms, resp. graph automorphisms, of S [GLS, Theorem 2.5.12]. By its construction, the class $[s]$ is invariant under $\text{Inndiag}(S) \cdot \Phi_S$. We claim that $[s]$ is also fixed by any $\sigma \in \Gamma_S$. In the cases under our consideration, $\Phi_S \Gamma_S = \Phi_S \times \Gamma_S$ and so it is abelian [GLS]. Since $\Gamma_S = 1$ if $\mathcal{G}^F$ is twisted, we may assume $\mathcal{G}^F$ is untwisted. In this case τ is trivial. For any $g \in \mathcal{G}^{*F^*}$, $F^*(g^\sigma) = F^*(g)^\sigma = g^\sigma$ and so $g^\sigma \in \mathcal{G}^{*F^*}$. Since $s^\sigma = s$ by its construction, $[s]$ is σ -invariant. Thus $[s]$ is invariant under $\text{Aut}(\mathcal{G}^{*F^*})$.

Since $\mathcal{G}$ is simple of type A_3 , B_n , C_n , or D_n , we see that 2 is the only bad prime (if any) for $\mathcal{G}$, and $\mathbf{Z}(\mathcal{G})$ is a 2-group. Thus s has connected centralizer in $\mathcal{G}^*$ by [S, Corollary 2.25]. Therefore, the corresponding semisimple character χ_s is as required. (One can show that, in all the above cases, χ_s is in fact rational-valued as s is rational.)

4) Now assume $p = 3$; in particular, $\mathcal{G}$ is not of type A_1 . Then the centralizer of the group of graph automorphisms of $\mathcal{G}^*$ is of type B_m or C_m (for some $m \geq 2$) or G_2 , where the latter possibility may occur when $\mathcal{G}$ is of type D_4 . This centralizer therefore contains a subgroup of type B_2 or of type G_2 in all cases. Now $SO_5(3)$ contains a semisimple element of order 5 and $G_2(3)$ contains a semisimple element of order 7 in their derived subgroups. Arguing as in part 3) we see that $[s]$ is invariant under $\text{Aut}(\mathcal{G}^{*F^*})$, and the centralizer of s in $\mathcal{G}^*$ is connected. This yields the desired semisimple character χ_s . $\square$

4. DIRECT AND CENTRAL PRODUCT OF SIMPLE GROUPS

Our aim in this section is to slightly improve on Theorem 3.4. We begin with a lemma.

Lemma 4.1. *Suppose that a finite p -group Q acts on a finite group G , stabilizing $N \triangleleft G$. Suppose that $P/N \in \text{Syl}_p(G/N)$ is Q -invariant, and assume that $G/N = T_1/N \times \cdots \times T_a/N$, where the T_i 's are transitively permuted by Q . Let $P_1 = P \cap T_1$, and let Q_1 be the stabilizer of T_1 in Q and let $U = \{u_1, \dots, u_a\}$ be a transversal for the right cosets of Q_1 in Q with $T_1^{u_j} = T_j$.*

(i)

$$\mathbf{C}_{\mathbf{N}_G(P)/P}(Q) = 1 \quad \text{iff} \quad \mathbf{C}_{\mathbf{N}_{T_1}(P_1)/P_1}(Q_1) = 1.$$

(ii) Suppose that $[T_i, T_j] = 1$ if $i \neq j$, and $[N, Q] = 1$. Given $\theta \in \text{Irr}(N)$, then there is a natural bijection $(\psi_1, \dots, \psi_a) \mapsto \psi_1 \dots \psi_a$ from $\text{Irr}(T_1|\theta) \times \dots \times \text{Irr}(T_a|\theta)$ onto $\text{Irr}(G|\theta)$. Furthermore, if $\psi \in \text{Irr}(T_1|\theta)$, then ψ is Q_1 -invariant if and only if $\psi^{u_1} \dots \psi^{u_a}$ is Q -invariant.

Proof. We have that Q acts on U , and we use the notation $u \cdot q$ to indicate the unique element of U such that

$$Q_1(uq) = Q_1(u \cdot q)$$

for $u \in U$ and $q \in Q$.

(i) In this part, we may certainly assume that $N = 1$. If $P_i = P \cap T_i$, then $P_i \in \text{Syl}_p(T_i)$,

$$P = P_1 \times \dots \times P_a,$$

and

$$\mathbf{N}_G(P) = \mathbf{N}_{T_1}(P_1) \times \dots \times \mathbf{N}_{T_a}(P_a).$$

Notice that if $(T_i)^u = T_j$, then $\mathbf{N}_{T_i}(P_i)^u = \mathbf{N}_{T_j}(P_j) \subseteq \mathbf{N}_G(P)$ for $u \in U$. Assume first that $\mathbf{C}_{\mathbf{N}_G(P)/P}(Q) = 1$ and let $x \in \mathbf{N}_{T_1}(P_1)$ be such that xP_1 is centralized by Q_1 . Denote

$$z = \prod_{u \in U} x^u \in \mathbf{N}_G(P).$$

Now, if $q \in Q$, then

$$z^q P = \prod_{u \in U} x^{uq} P = \prod_{u \in U} (x^{(uq)(u \cdot q)^{-1}})^{u \cdot q} P = \prod_{u \in U} x^{(u \cdot q)} P = zP.$$

By hypothesis, we deduce that $z \in P$. Therefore, $x \in P_1$. The converse is proved analogously.

(ii) Let $\hat{G} = T_1 \times \dots \times T_a$ and consider the group homomorphism $\mu : \hat{G} \rightarrow G$ given by $(x_1, \dots, x_a) \mapsto x_1 \dots x_a$. Now, let $K = \ker(\mu)$ and let $\hat{N} = \mu^{-1}(N) = N \times \dots \times N \triangleleft \hat{G}$. Denote by $\bar{\mu} : \hat{G}/K \rightarrow G$ the natural isomorphism. This isomorphism provides a natural bijection (which we also call $\bar{\mu}$) between the irreducible characters of $\hat{H}/K$ and H , where H is any subgroup of G and $\bar{\mu}(\hat{H}/K) = H$. Since $K \subseteq \ker(\delta)$, where $\delta = \theta \times \dots \times \theta$, and the image of $\delta \in \text{Irr}(\hat{N}/K)$ under $\bar{\mu}$ is θ , then the first part of (ii) easily follows.

Now, notice that Q naturally acts on $\hat{G}$ (and therefore on $\text{Irr}(\hat{G})$). This action is given by the following: if $q \in Q$ and $T_i^q = T_j$ (i.e., if $u_i \cdot q = u_j$), then $(x_1, \dots, x_a)^q = (y_1, \dots, y_a)$, where $y_j = x_i^q$. It is clear that μ is a Q -homomorphism and K is Q -invariant. Now, suppose that $\psi \in \text{Irr}(T_1|\theta)$ is Q_1 -invariant. We claim that $\psi^{u_1} \times \dots \times \psi^{u_a}$ is Q -invariant. Let $(x_1, \dots, x_a) \in \hat{G}$ and $q \in Q$. Suppose that if $u_i \cdot q = u_j$, then $y_j = x_i^q$. Then

$$\begin{aligned} (\psi^{u_1} \times \dots \times \psi^{u_a})^{q^{-1}}(x_1, \dots, x_a) &= \psi^{u_1}(y_1) \dots \psi^{u_a}(y_a) \\ &= \psi^{u_1 q}(y_1^{q^{-1}}) \dots \psi^{u_a q}(y_a^{q^{-1}}) \\ &= \psi^{u_1 \cdot q}(y_1^{q^{-1}}) \dots \psi^{u_a \cdot q}(y_a^{q^{-1}}) \\ &= \psi^{u_1}(x_1) \dots \psi^{u_a}(x_a) \\ &= (\psi^{u_1} \times \dots \times \psi^{u_a})(x_1, \dots, x_a), \end{aligned}$$

as claimed. Now, since $K \subseteq \ker(\psi^{u_1} \times \dots \times \psi^{u_a})$, it follows that $\psi^{u_1} \dots \psi^{u_a}$ is Q -invariant. The converse is easily proved. $\square$

Theorem 4.2. *Let G be a direct product of nonabelian simple groups and let $p > 2$ be a prime. Let Q be a finite p -group acting on a finite group G , and let P be a Q -invariant Sylow p -subgroup in G . Then $\mathbf{C}_{\mathbf{N}_G(P)/P}(Q) = 1$ if and only if G has a unique Q -invariant, p -rational, p' -degree, irreducible complex character.*

Proof. We argue by induction on $|G|$. Suppose that G is the direct product of the set of simple normal subgroups $\mathcal{U}$. We can write $G = X_1 \times \dots \times X_a$, each X_i is the direct product of the simple groups in the Q -orbit of some $S \in \mathcal{U}$. Thus X_i is Q -invariant, and by induction, we easily may assume that Q transitively permutes the elements in $\mathcal{U}$. Now, Lemma 4.1 and Theorem 3.4 prove the theorem. $\square$

5. CHARACTER TRIPLES

If $N \triangleleft G$ and $\theta \in \text{Irr}(N)$ is G -invariant (i.e., if (G, N, θ) is a **character triple**), sometimes we wish to replace (G, N, θ) by another triple (G^*, N^*, θ^*) such that $N^* \subseteq \mathbf{Z}(G^*)$, G/N and G^*/N^* are isomorphic, and such that the character theory of G over θ is “isomorphic” to the character theory of G^* over θ^* (for definitions see Chapter 11 in [I]). In our special situation, we would like this “replacement” to be done in such a way that we can control the field of values of characters. Here we use $*$ to denote both the group isomorphism $G/N \rightarrow G^*/N^*$ and the character bijection $\text{Irr}(G|\theta) \rightarrow \text{Irr}(G^*|\theta^*)$. For instance, if $\chi \in \text{Irr}(G|\theta)$, then $\chi(1)/\theta(1) = \chi^*(1)/\theta^*(1)$ by Lemma (11.24) of [I].

Notice that if $|G|_p \leq p^a$, then $\text{Gal}(\mathbb{Q}_{p^a})$, the Galois group of the p^a -cyclotomic field, acts naturally on $\text{Irr}(G)$. (If $|G|_{p'} = m$, then we can view $\sigma \in \text{Gal}(\mathbb{Q}_{p^a})$ as an element of $\text{Gal}(\mathbb{Q}_{p^a m}/\mathbb{Q}_m)$.) Then $\chi \in \text{Irr}(G)$ is p -rational iff χ is $\text{Gal}(\mathbb{Q}_{p^a})$ -fixed.

Theorem 5.1. *Suppose that $N \triangleleft G$ and let $\theta \in \text{Irr}(N)$ be G -invariant and p -rational, where p is an odd prime. Assume that M/N is a non-trivial perfect normal subgroup of G/N . Then there exists a character triple (M^*, N^*, θ^*) isomorphic to (M, N, θ) such that:*

- (i) G acts as automorphisms on M^* , centralizing N^* , and such that

$$((mN)^*)^g = (m^g N)^*$$

for $m \in M$ and $g \in G$.

- (ii) M^* is perfect, $N^* \subseteq \mathbf{Z}(M^*)$, and N^* is a p' -group.
- (iii) $(\chi^*)^g = (\chi^g)^*$ for $g \in G$ and $\chi \in \text{Irr}(M|\theta)$.
- (iv) If $\chi \in \text{Irr}(M|\theta)$, then χ is p -rational if and only if χ^* is p -rational.

Proof. Let $F = \mathbb{Q}_m$, where $m = |N|_{p'}$. By Theorem (10.13) of [I], we know that θ can be afforded by an absolutely irreducible FN -representation $\mathcal{Y}$. Arguing as in Theorem (11.2) of [I], there exists a projective representation $\mathcal{X}$ of G such that $\chi(g) \in GL_n(F)$ for $g \in G$ and satisfying conditions (a), (b), (c) of Theorem (11.2) of [I]. If α is the factor set associated to $\mathcal{X}$, we have that $\alpha(g, n) = \alpha(n, g) = 1$ for $n \in N$ and $g \in G$ and $\alpha(gn, hm) = \alpha(g, h)$ for $g, h \in G$ and $n, m \in N$.

Now, let $F^\times$ be the multiplicative group of F . For any subgroup $H \subseteq G$, we define $\tilde{H} = H \times F^\times$. It is straightforward to check that $\tilde{G}$ is a group with multiplication

$$(g_1, f_1)(g_2, f_2) = (g_1 g_2, \alpha(g_1, g_2) f_1 f_2).$$

Also, $\tilde{H}$ is a subgroup of $\tilde{G}$ whenever H is a subgroup of G . We have that the map $\pi : \tilde{G} \rightarrow G$ given by $(g, f) \mapsto g$ is an onto homomorphism with kernel $\tilde{1} = 1 \times F^\times$. Hence, $\tilde{M}$ and $\tilde{N}$ are normal subgroups of $\tilde{G}$. Furthermore, we can check that $N \times 1$ (which we again call N) is a normal subgroup of $\tilde{G}$, $\tilde{1}$ is contained in $\mathbf{Z}(\tilde{G})$ and $\tilde{N} = N \times \tilde{1}$.

Now, $\tilde{N}/N \subseteq \mathbf{Z}(\tilde{G}/N)$. Let $M_1/N = (\tilde{M}/N)'$. Since $\tilde{M}/\tilde{N}$ is perfect, we have that $\tilde{M} = \tilde{N}M_1$. Also, M_1/N is perfect. Also, M_1/N is finite, by Schur's theorem [H, (IV.2.3)]. In particular, M_1 is finite. Let $N_1 = M_1 \cap \tilde{N}$, and $F_1 = N_1 \cap \tilde{1}$, so that $N_1 = F_1 \times N$. Notice that F_1 is a p' -group, since it is a finite subgroup of $F^\times$.

Now, $M_1 \triangleleft \tilde{G}$ and thus $\tilde{G}$ acts on M_1 by conjugation. Since $\tilde{1}$ is in the kernel of the action, we have that G acts on M_1 .

We define

$$\tilde{\mathcal{X}}(g, f) = f\mathcal{X}(g).$$

It is clear that $\tilde{\mathcal{X}}$ is an F -representation of $\tilde{G}$. Also, $\tilde{\mathcal{X}}(n) = \mathcal{Y}(n)$ for $n \in N$. Now, define the linear character $\lambda \in \text{Irr}(N_1)$ by $\lambda(n, f) = \bar{f}$. Let $\theta_1 = 1 \times \theta \in \text{Irr}(N_1)$ (which is p -rational. Let τ be the character of the representation of $\mathcal{X}_{M_1}$. Note that τ is $\tilde{G}$ -invariant and has its values in F . Also, $\tau_{N_1} = \bar{\lambda}\theta_1$ and $\tau_N = \theta$. Thus $\tau \in \text{Irr}(M_1)$.

Now, the map $x \mapsto \pi(x)$ is an onto group homomorphism from M_1 to M with kernel F_1 . Since $F_1 \subseteq \ker(\theta_1)$, by Lemma (11.26) of [I], we have that (M_1, N_1, θ_1) and (M, N, θ) are isomorphic character triples. These isomorphisms commute with G -conjugation and preserve the field of values of characters. Now, by the remark after Lemma (11.27) of [I], we have that (M_1, N_1, λ) and (M_1, N_1, θ_1) are isomorphic character triples. Furthermore, if $\chi \in \text{Irr}(M|\theta)$, we have that $\chi^* = \chi\tau$. Now, $p^a = |M|_p \geq |M/N|_p = |M_1|_p$, and $\text{Gal}(\mathbb{Q}_{p^a})$ naturally acts on $\text{Irr}(M|\theta)$ and $\text{Irr}(M_1|\theta_1)$. Since τ is fixed by $\text{Gal}(\mathbb{Q}_{p^a})$, we see that χ is p -rational if and only if χ^* is p -rational. Also, the bijection $\chi \mapsto \chi^*$ commutes with G -action.

Finally, (M_1, N_1, λ) and $(M_1/N, N_1/N, \lambda)$ are isomorphic by Lemma (11.26) of [I] (with an isomorphism that preserves fields of values and commutes with G -action). Now, write $M^* = M_1/N$, $N^* = N_1/N$ and $\theta^* = \lambda$, and observe that N^* is a central p' -subgroup of M^* centralized by G . $\square$

6. MAIN RESULTS

If $G = XY$ is a central product and $Z = X \cap Y \subseteq \mathbf{Z}(G)$, then it is well known that given $\lambda \in \text{Irr}(Z)$, $\alpha \in \text{Irr}(X|\lambda)$ and $\beta \in \text{Irr}(Y|\lambda)$, then there is a unique $\alpha \cdot \beta \in \text{Irr}(G|\lambda)$. (In fact, Lemma 4.1(ii) provides a proof of this result.) Since α and β uniquely determine $\alpha \cdot \beta$, we have for instance, that $\alpha \cdot \beta$ is p -rational, if α and β are p -rational.

Theorem 6.1. *Suppose that Q is a finite p -group acting on a finite group G , where $p \geq 3$. Let $N \triangleleft G$ be Q -invariant, and suppose that $P/N \in \text{Syl}_p(G/N)$ is Q -invariant. Assume that $\mathbf{C}_{\mathbf{N}_G(P)/P}(Q) = 1$. If $\theta \in \text{Irr}_{p'}(N)$ is p -rational and PQ -invariant, then there exists a unique p' -degree, p -rational, Q -invariant $\chi \in \text{Irr}(G)$ lying over θ .*

Proof. We argue by induction on $|G : N|$. Let T be the stabilizer of θ in G , and note that $P \subseteq T$ and that T is Q -invariant. If $T < G$, then by induction there is a unique $\mu \in \text{Irr}(T|\theta)$ of p' -degree, Q -invariant and p -rational. Now, μ^G is Q -invariant, p' -degree and p -rational, and it is the unique such one lying over θ , by

the uniqueness in the Clifford correspondence. We may assume therefore that θ is G -invariant.

Let $N \subseteq M \triangleleft G$ be Q -invariant. Let $S = PM$ and $R = P \cap M$. Hence $S/M \in \text{Syl}_p(G/M)$ is Q -invariant, and $R/N \in \text{Syl}_p(M/N)$ is PQ -invariant. Now, by applying Lemma 2.1, we have that PQ acts on M and $\mathbf{C}_{\mathbf{N}_M(R)/R}(PQ) = 1$. Also, Q acts on G/M and $\mathbf{C}_{\mathbf{N}_G(S)/S}(Q) = 1$.

Suppose that $N < M < G$. Then by induction there exists a unique p' -degree, PQ -invariant, p -rational character $\psi \in \text{Irr}(M|\theta)$. Also, by induction, there is a unique $\chi \in \text{Irr}(G|\psi)$ of p' -degree, Q -invariant and p -rational. Let $\tau \in \text{Irr}(G)$ be p' -degree, p -rational, Q -invariant over θ . Now, by Lemma 2.2, τ_M has a unique PQ -invariant irreducible constituent ξ . If σ is a Galois automorphism of $\text{Gal}(\mathbb{Q}_{|G|})$ fixing p' -roots of unity, then ξ^σ is a PQ -invariant constituent of τ_M . By uniqueness, it follows that ξ is also p -rational. By Lemma 2.2, we also have that ξ_N has a unique PQ -invariant constituent. Since θ is the unique PQ -invariant constituent of τ_N (by Lemma 2.1), we deduce that ξ lies over θ . By induction, we deduce that $\xi = \psi$ and $\tau = \chi$.

So we may assume that G/N is a chief factor of GQ . Suppose that G/N is a p -group. In this case, by Theorem (6.30) of [I], θ^G has a unique p -rational irreducible constituent χ . Also, $\chi_N = \theta$, and thus χ has p' -degree. By uniqueness, notice that χ is Q -invariant. Suppose now that G/N is a p' -group. Then $\mathbf{C}_{G/N}(Q) = 1$. By Problem (13.10) of [I], there is a unique Q -invariant $\chi \in \text{Irr}(G|\theta)$. By uniqueness, notice that χ is p -rational. Since G/N is a p' -group, then χ has p' -degree, and we are also done in this case.

So we may assume that G/N is a direct product of nonabelian simple groups of order divisible by p transitively permuted by Q . In particular, G/N is perfect.

Now, (GQ, N, θ) is a character triple. By Theorem 5.1, there exists an isomorphic character triple (G^*, N^*, θ^*) such that Q acts on G^* , N^* is a central p' -group, the actions of Q on G/N and G^*/N^* are isomorphic, and such that the bijection $\text{Irr}(G|\theta) \rightarrow \text{Irr}(G^*|\theta^*)$ commutes with Q -action and preserves p -rational characters. Hence, we see that there is no loss to assume that $N = \mathbf{Z}(G)$ is a p' -subgroup centralized by Q .

Write

$$G/N = T_1/N \times \dots \times T_a/N,$$

where T_i/N is simple and the T_i 's are transitively permuted by Q . Hence, for $i \neq j$, we have that $[T_i, T_j] = 1$. Write $S_i = (T_i)'$, so that S_i is perfect and $T_i = S_i N$. Write $P_i = P \cap T_i$, $R_i = P_i \cap S_i = P \cap S_i$ and $N_i = S_i \cap N$. We have that $R_i/N_i \in \text{Syl}_p(S_i/N_i)$.

Now, if Q_1 is the stabilizer of T_1 in P_1 , we have that

$$\mathbf{C}_{\mathbf{N}_{T_1}(P_1)/P_1}(Q_1) = 1$$

by Lemma 4.1. Since $\mathbf{N}_{T_i}(P_i)/P_i$ and $\mathbf{N}_{S_i}(R_i)/R_i$ are Q_1 -isomorphic, it follows that

$$\mathbf{C}_{\mathbf{N}_{S_1}(R_1)/R_1}(Q_1) = 1.$$

Hence, by Theorem 3.1, there exists a unique p -rational, p' -degree, Q_1 -invariant character η of S_1 lying over $\theta_1 = \theta_{N_1}$. Now, if $\psi = \eta \cdot \theta$, we have that $\psi \in \text{Irr}(T_1|\theta)$ is Q_1 -invariant, p -rational, and of p' -degree. By Lemma 4.1(ii), we have that, if $\{u_1, \dots, u_a\}$ is a transversal for the right cosets of Q_1 in Q , then $\chi = \psi^{u_1} \dots \psi^{u_a}$ is Q -invariant, p -rational of p' -degree lying over θ . Suppose that $\xi = \xi_1 \dots \xi_a$

is another one, where $\xi_i \in \text{Irr}(T_i|\theta)$. Then $\xi_{T_i} = e_i \xi_i$. In particular, $(\xi_1)_{S_1}$ is a p' -degree, p -rational, Q_1 -invariant irreducible character over θ_1 , and thus $\xi_1 = \psi$. Since $\xi_{T_i} = (\xi^{u_i})_{T_i} = e_i \xi_i^{u_i}$, we have that $\xi_i = \psi^{u_i}$, and this finishes the proof of the theorem. $\square$

Finally, we are able to prove Theorem A.

Theorem 6.2. *Suppose that G is a finite group, $p \geq 3$ is a prime, and $P \in \text{Syl}_p(G)$. Then G has no non-trivial p' -degree p -rational irreducible characters if and only if $\mathbf{N}_G(P) = P$.*

Proof. Suppose first that $\mathbf{N}_G(P) = P$. Then, setting $Q = 1$, $N = 1$, and θ to be the trivial character in Theorem 6.1, we see that our theorem holds. For the converse, we assume the theorem is false, and choose a counterexample with $|G|$ as small as possible. Let N be a minimal normal subgroup of G , and let $R = P \cap N$. Since $\text{Irr}(G/N)$ has no non-trivial p' -degree p -rational characters, by minimality, we have that PN/N is self-normalizing in G/N . By Theorem 6.1, we have that N has no non-trivial P -invariant p -rational characters of p' -degree. If N is a p' -group, then $\mathbf{C}_N(P) = 1$ and we have that $\mathbf{N}_G(P) = P$. If N is a p -group, then $\mathbf{N}_G(P) = P$ (because P/N is self-normalizing). If N is a direct product of simple groups, then we apply Theorem 4.2, and we obtain $\mathbf{C}_{\mathbf{N}_G(R)/R}(P) = 1$, and our result follows. $\square$

As we have already said, finite groups with a self-normalizing Sylow p -subgroup are solvable for $p \geq 5$ [GMN]. Using this, one could prove one direction of Theorem A (but only for $p \geq 5$) by using the main result in [T2] (or the natural bijection constructed in [N1]). As pointed out in [N2], Theorem A is not true for $p = 2$. According to Theorem (5.2) of [N2], this case needs a totally different approach.

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