EXISTENCE OF KIRILLOV–RESHETIKHIN CRYSTALS FOR NONEXCEPTIONAL TYPES

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Dedicated to Professor Masaki Kashiwara on his sixtieth birthday

ABSTRACT. Using the methods of Kang et al. and recent results on the characters of Kirillov–Reshetikhin modules by Nakajima and Hernandez, the existence of Kirillov–Reshetikhin crystals $B^{r,s}$ is established for all nonexceptional affine types. We also prove that the crystals $B^{r,s}$ of type $B_n^{(1)}$, $D_n^{(1)}$, and $A_{2n-1}^{(2)}$ are isomorphic to recently constructed combinatorial crystals for r not a spin node.

1. INTRODUCTION

The theory of crystal bases by Kashiwara [15] provides a remarkably powerful tool to study the representations of quantum algebras $U_q(\mathfrak{g})$. For instance, the calculation of tensor product multiplicities reduces to counting the number of crystal elements having certain properties. Although crystal bases are bases at $q = 0$, one can “melt” them to get actual bases, called global crystal bases, for integrable highest weight representations of $U_q(\mathfrak{g})$. It turns out that the global crystal basis agrees with Lusztig’s canonical basis [22], and it has many applications in representation theory.

The main focus of this paper is affine finite crystals, that is, crystal bases of finite-dimensional modules for quantum groups corresponding to affine Kac–Moody algebras $\mathfrak{g}$. These crystal bases were first developed by Kang et al. [13, 14], where it was also shown that integrable highest-weight $U_q(\mathfrak{g})$ -modules of arbitrary level can be realized as semi-infinite tensor products of perfect crystals. This is known as the path realization. Many perfect crystals were proven to exist and explicitly constructed in [14].

Irreducible finite-dimensional $U'_q(\mathfrak{g})$ -modules were classified by Chari and Pressley [4, 5] in terms of Drinfeld polynomials. It was conjectured by Hatayama et al. [7, 8] that a certain subset of such modules known as Kirillov–Reshetikhin (KR) modules $W_s^{(r)}$ have a crystal basis $B^{r,s}$. Here the index r corresponds to a node of the Dynkin diagram of $\mathfrak{g}$ except the prescribed 0 and s is an arbitrary positive integer. This conjecture was confirmed in many instances [2, 13, 14, 17, 19, 26, 34], but a proof for general r and s has not been available except type $A_n^{(1)}$ in [14]. Only recently the existence proof was completed in [27] for type $D_n^{(1)}$. Using the methods of [14] and recent results on the characters of KR modules [9, 10, 24], we

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establish the existence of Kirillov–Reshetikhin crystals $B^{r,s}$ for all nonexceptional affine types in this paper:

Theorem 1.1. *The Kirillov–Reshetikhin module $W_s^{(r)}$ associated to any nonexceptional affine Kac–Moody algebra has a crystal basis $B^{r,s}$.*

In addition we prove that for type $B_n^{(1)}$, $D_n^{(1)}$, and $A_{2n-1}^{(2)}$ these crystals coincide with the combinatorial crystals of [30, 32]. Throughout the paper we denote by $B^{r,s}$ the KR crystal associated with the KR module $W_s^{(r)}$. The combinatorial crystal of [30] is called $\tilde{B}^{r,s}$. Our second main result is the following theorem:

Theorem 1.2. *For $1 \leq r \leq n-2$ for type $D_n^{(1)}$, $1 \leq r \leq n-1$ for type $B_n^{(1)}$, $1 \leq r \leq n$ for type $A_{2n-1}^{(2)}$ and $s \in \mathbb{Z}_{>0}$, the crystals $B^{r,s}$ and $\tilde{B}^{r,s}$ are isomorphic.*

The key to the proof of Theorem 1.1 is Proposition 2.1 below, which is due to Kang et al. [14] and states that a finite-dimensional $U'_q(\mathfrak{g})$ -module having a prepolarization and certain $\mathbb{Z}$ -form has a crystal basis if the dimensions of some particular weight spaces are not greater than the weight multiplicities of a fixed module and the values of the prepolarization of certain vectors in the module have some special properties. Using the fusion construction it is established that the KR modules have a prepolarization and $\mathbb{Z}$ -form. The requirements on the dimensions follow from recent results by Nakajima [24] and Hernandez [9, 10]. Necessary values of the prepolarization are calculated explicitly in Propositions 4.1, 4.4, and 4.6.

The isomorphism between the KR crystal $B^{r,s}$ and the combinatorial crystal $\tilde{B}^{r,s}$ is established by showing that isomorphisms as crystals with index sets $\{1, 2, 3, \dots, n\}$ and $\{0, 2, 3, \dots, n\}$ already uniquely determine the whole crystal.

Before presenting our results, let us offer some speculations on combinatorial realizations for the KR crystals. For type $A_n^{(1)}$ the crystals $B^{r,s}$ were constructed combinatorially by Shimozono [31] using the promotion operator. The promotion operator pr is the crystal analogue of the Dynkin diagram automorphism that maps node i to node $i+1$ modulo $n+1$. The affine crystal operator $\tilde{f}_0$ is then given by $\tilde{f}_0 = \text{pr}^{-1} \circ \tilde{f}_1 \circ \text{pr}$. Similarly, the main tool used in [30] to construct the combinatorial crystals $\tilde{B}^{r,s}$ of type $B_n^{(1)}$, $D_n^{(1)}$, and $A_{2n-1}^{(2)}$ is the crystal analogue of the Dynkin diagram automorphism that interchanges nodes 0 and 1. For type $C_n^{(1)}$ and $D_{n+1}^{(2)}$, there exists a Dynkin diagram automorphism $i \mapsto n-i$. It is our intention to exploit this symmetry to construct $\tilde{B}^{r,s}$ of type $C_n^{(1)}$ and $D_{n+1}^{(2)}$ explicitly in a future publication. For type $A_{2n}^{(2)}$ no Dynkin diagram automorphism exists. However, it should still be possible to construct these crystals by looking at the $\{1, 2, \dots, n\}$ and $\{0, 1, 2, \dots, n-1\}$ subcrystals as was done for $r=1$ in [14]. Realizations of $B^{r,s}$ as virtual crystals were given in [28, 29].

The paper is organized as follows. In Section 2 we review necessary background on the quantum algebra $U'_q(\mathfrak{g})$ and the fundamental representations. In particular, we review Proposition 2.1 of [14] which provides a criterion for the existence of a crystal pseudobase. In Section 3 we define KR modules by the fusion construction and show that these modules have a prepolarization. This reduces the existence proof for KR crystals to conditions stated in Proposition 3.7. These conditions are checked explicitly in Section 4 for the various types to prove Theorem 1.1. In Section 5 we review the combinatorial construction of the crystals $\tilde{B}^{r,s}$ of types

$B_n^{(1)}$, $D_n^{(1)}$, and $A_{2n-1}^{(2)}$ and prove in Section 6 that they are isomorphic to $B^{r,s}$, thereby establishing Theorem 1.2.

2. QUANTUM AFFINE ALGEBRA $U'_q(\mathfrak{g})$ AND FUNDAMENTAL REPRESENTATIONS

2.1. Quantum affine algebra. Let $\mathfrak{g}$ be an affine Kac-Moody algebra and $U_q(\mathfrak{g})$ the quantum affine algebra associated to $\mathfrak{g}$. In this section $\mathfrak{g}$ can be any affine algebra. For the notation of $\mathfrak{g}$ or $U_q(\mathfrak{g})$ we follow [17]. For instance, P is the weight lattice, I is the index set of simple roots, and $\{\alpha_i\}_{i \in I}$ (resp. $\{h_i\}_{i \in I}$) is the set of simple roots (resp. coroots). Let $(\ , \)$ be the inner product on P normalized by $(\delta, \lambda) = \langle c, \lambda \rangle$ for any $\lambda \in P$ as in [12], where c is the canonical central element and δ is the generator of null roots. We choose a positive integer d such that $(\alpha_i, \alpha_i)/2 \in \mathbb{Z}d^{-1}$ for any $i \in I$ and set $q_s = q^{1/d}$. Then $U_q(\mathfrak{g})$ is the associative algebra over $\mathbb{Q}(q_s)$ with 1 generated by e_i, f_i ($i \in I$), q^h ($h \in d^{-1}P^*$, $P^* = \text{Hom}_{\mathbb{Z}}(P, \mathbb{Z})$) with certain relations. By convention, we set $q_i = q^{(\alpha_i, \alpha_i)/2}$, $t_i = q_i^{h_i}$, $[m]_i = (q_i^m - q_i^{-m})/(q_i - q_i^{-1})$, $[n]_i! = \prod_{m=1}^n [m]_i$, $e_i^{(n)} = e_i^n / [n]_i!$, $f_i^{(n)} = f_i^n / [n]_i!$.

Let $\{\Lambda_i\}_{i \in I}$ be the set of fundamental weights. Then we have $P = \bigoplus_i \mathbb{Z}\Lambda_i \oplus \mathbb{Z}\delta$. We set

$$P_{cl} = P/\mathbb{Z}\delta.$$

Similar to the quantum algebra $U_q(\mathfrak{g})$ which is associated with P , we can also consider $U'_q(\mathfrak{g})$, which is associated with P_{cl} , namely, the subalgebra of $U_q(\mathfrak{g})$ generated by e_i, f_i, q^h ($h \in d^{-1}(P_{cl})^*$).

Next we introduce two subalgebras (“ $\mathbb{Z}$ -forms”) $U_q(\mathfrak{g})_{K_{\mathbb{Z}}}$ and $U_q(\mathfrak{g})_{\mathbb{Z}}$ of $U_q(\mathfrak{g})$. Let A be the subring of $\mathbb{Q}(q_s)$ consisting of rational functions without poles at $q_s = 0$. We introduce the subalgebras $A_{\mathbb{Z}}$ and $K_{\mathbb{Z}}$ of $\mathbb{Q}(q_s)$ by

$$\begin{aligned} A_{\mathbb{Z}} &= \{f(q_s)/g(q_s) \mid f(q_s), g(q_s) \in \mathbb{Z}[q_s], g(0) = 1\}, \\ K_{\mathbb{Z}} &= A_{\mathbb{Z}}[q_s^{-1}]. \end{aligned}$$

Then we have

$$K_{\mathbb{Z}} \cap A = A_{\mathbb{Z}}, \quad A_{\mathbb{Z}}/q_s A_{\mathbb{Z}} \simeq \mathbb{Z}.$$

We then define $U_q(\mathfrak{g})_{K_{\mathbb{Z}}}$ as the $K_{\mathbb{Z}}$ -subalgebra of $U_q(\mathfrak{g})$ generated by e_i, f_i, q^h ($i \in I, h \in d^{-1}P^*$). $U_q(\mathfrak{g})_{\mathbb{Z}}$ is defined as the $\mathbb{Z}[q_s, q_s^{-1}]$ -subalgebra of $U_q(\mathfrak{g})$ generated by $e_i^{(n)}, f_i^{(n)}, \{t_i\}_i$ ($i \in I, n \in \mathbb{Z}_{>0}$) and q^h ($h \in d^{-1}P^*$). Here we have set $\{t_i\}_i = \prod_{k=1}^n (q_i^{1-k}x - q_i^{k-1}x^{-1})/[n]_i!$. $U_q(\mathfrak{g})_{\mathbb{Z}}$ is a $\mathbb{Z}[q_s, q_s^{-1}]$ -subalgebra of $U_q(\mathfrak{g})_{K_{\mathbb{Z}}}$. We can also introduce subalgebras $U'_q(\mathfrak{g})_{K_{\mathbb{Z}}}$ and $U'_q(\mathfrak{g})_{\mathbb{Z}}$ by replacing q^h ($h \in d^{-1}P^*$) with q^h ($h \in d^{-1}(P_{cl})^*$) in the generators.

We define a total order on $\mathbb{Q}(q_s)$ by

$$f > g \text{ if and only if } f - g \in \bigsqcup_{n \in \mathbb{Z}} \{q_s^n(c + q_s A) \mid c > 0\}$$

and $f \geq g$ if $f > g$ or $f = g$.

Let M and N be $U_q(\mathfrak{g})$ (or $U'_q(\mathfrak{g})$)-modules. A bilinear form $(\ , \) : M \otimes_{\mathbb{Q}(q_s)} N \rightarrow \mathbb{Q}(q_s)$ is called an admissible pairing if it satisfies

$$\begin{aligned} (2.1) \quad (q^h u, v) &= (u, q^h v), \\ (e_i u, v) &= (u, q_i^{-1} t_i^{-1} f_i v), \\ (f_i u, v) &= (u, q_i^{-1} t_i e_i v), \end{aligned}$$

for all $u \in M$ and $v \in N$. Equation (2.1) implies

$$(2.2) \quad (e_i^{(n)}u, v) = (u, q_i^{-n^2}t_i^{-n}f_i^{(n)}v), \quad (f_i^{(n)}u, v) = (u, q_i^{-n^2}t_i^n e_i^{(n)}v).$$

A symmetric bilinear form $(\ , \)$ on M is called a *prepolarization* of M if it satisfies (2.1) for $u, v \in M$. A prepolarization is called a *polarization* if it is positive definite with respect to the order on $\mathbb{Q}(q_s)$.

2.2. Criterion for the existence of a crystal pseudobase. Here we recall the criterion for the existence of a crystal pseudobase given in [14]. We do not review the notion of crystal bases, but refer the reader to [15]. We only note that q in the definition of crystal base in [15] should be replaced by q_s according to the normalization of the inner product $(\ , \)$ on P . We say (L, B) is a crystal pseudobase of an integrable $U_q(\mathfrak{g})$ (or $U'_q(\mathfrak{g})$)-module M , if (i) L is a crystal lattice of M , (ii) $B = B' \sqcup (-B')$ where B' is a $\mathbb{Q}$ -base of $L/q_s L$, (iii) $B = \bigsqcup_{\lambda \in P} B_\lambda$ where $B_\lambda = B \cap (L_\lambda/q_s L_\lambda)$, (iv) $\tilde{e}_i B \subset B \sqcup \{0\}$, $\tilde{f}_i B \subset B \sqcup \{0\}$, and (v) for $b, b' \in B$, $b' = \tilde{f}_i b$ if and only if $b = \tilde{e}_i b'$. Note that only the condition (ii) is replaced from the definition of the crystal base.

Let $\mathfrak{g}_0$ be the finite-dimensional simple Lie algebra whose Dynkin diagram is obtained by removing the 0-vertex from that of $\mathfrak{g}$. In this paper we specify the 0-vertex as in [12] and set $I_0 = I \setminus \{0\}$. Let $\overline{P}_+$ be the set of dominant integral weights of $\mathfrak{g}_0$ and $\overline{V}(\lambda)$ be the irreducible highest weight $U_q(\mathfrak{g}_0)$ -module of highest weight λ for $\lambda \in \overline{P}_+$. The following proposition is easily obtained by combining Proposition 2.6.1 and 2.6.2 of [14].

Proposition 2.1. *Let M be a finite-dimensional integrable $U'_q(\mathfrak{g})$ -module. Let $(\ , \)$ be a prepolarization on M , and $M_{K_{\mathbb{Z}}}$ a $U'_q(\mathfrak{g})_{K_{\mathbb{Z}}}$ -submodule of M such that $(M_{K_{\mathbb{Z}}}, M_{K_{\mathbb{Z}}}) \subset K_{\mathbb{Z}}$. Let $\lambda_1, \dots, \lambda_m \in \overline{P}_+$, and assume that the following conditions hold:*

$$(2.3) \quad \dim M_{\lambda_k} \leq \sum_{j=1}^m \dim \overline{V}(\lambda_j)_{\lambda_k} \text{ for } k = 1, \dots, m.$$

$$(2.4) \quad \begin{aligned} &\text{There exist } u_j \in (M_{K_{\mathbb{Z}}})_{\lambda_j} \ (j = 1, \dots, m) \text{ such that } (u_j, u_k) \in \delta_{jk} + q_s A, \\ &\text{and } (e_i u_j, e_i u_j) \in q_s q_i^{-2(1+\langle h_i, \lambda_j \rangle)} A \text{ for any } i \in I_0. \end{aligned}$$

Set $L = \{u \in M \mid (u, u) \in A\}$ and set $B = \{b \in M_{K_{\mathbb{Z}}} \cap L/M_{K_{\mathbb{Z}}} \cap q_s L \mid (b, b)_0 = 1\}$. Here $(\ , \)_0$ is the $\mathbb{Q}$ -valued symmetric bilinear form on $L/q_s L$ induced by $(\ , \)$. Then we have the following:

- (i) $(\ , \)$ is a polarization on M .
- (ii) $M \simeq \bigoplus_j \overline{V}(\lambda_j)$ as $U_q(\mathfrak{g}_0)$ -modules.
- (iii) (L, B) is a crystal pseudobase of M .

2.3. Fundamental representations. For any $\lambda \in P$, Kashiwara defined a $U_q(\mathfrak{g})$ -module $V(\lambda)$ called extremal weight module [16]. We briefly recall its definition. Let W be the Weyl group associated to $\mathfrak{g}$ and s_i the simple reflection for α_i . Let M be an integrable $U_q(\mathfrak{g})$ -module. A vector u_λ of weight $\lambda \in P$ is called an extremal

vector if there exists a set of vectors $\{u_{w\lambda}\}_{w \in W}$ satisfying

$$(2.5) \quad u_{w\lambda} = u_\lambda \text{ for } w = e,$$

$$(2.6) \quad \text{if } \langle h_i, w\lambda \rangle \geq 0, \text{ then } e_i u_{w\lambda} = 0 \text{ and } f_i^{\langle h_i, w\lambda \rangle} u_{w\lambda} = u_{s_i w\lambda},$$

$$(2.7) \quad \text{if } \langle h_i, w\lambda \rangle \leq 0, \text{ then } f_i u_{w\lambda} = 0 \text{ and } e_i^{\langle h_i, w\lambda \rangle} u_{w\lambda} = u_{s_i w\lambda}.$$

Then $V(\lambda)$ is defined to be the $U_q(\mathfrak{g})$ -module generated by u_λ with the defining relations that u_λ is an extremal vector. For our purpose, we only need $V(\lambda)$ when $\lambda = \varpi_r$ for $r \in I_0$, where ϖ_r is a level 0 fundamental weight

$$(2.8) \quad \varpi_r = \Lambda_r - \langle c, \Lambda_r \rangle \Lambda_0.$$

Then the following facts are known.

Proposition 2.2 ([17, Proposition 5.16]).

- (i) $V(\varpi_r)$ is an irreducible integrable $U_q(\mathfrak{g})$ -module.
- (ii) $\dim V(\varpi_r)_\mu < \infty$ for any $\mu \in P$.
- (iii) $\dim V(\varpi_r)_\mu = 1$ for any $\mu \in W\varpi_r$.
- (iv) $\text{wt } V(\varpi_r)$ is contained in the intersection of $\varpi_r + \sum_{i \in I} \mathbb{Z}\alpha_i$ and the convex hull of $W\varpi_r$.
- (v) $V(\varpi_r)$ has a global crystal base $(L(\varpi_r), B(\varpi_r))$.
- (vi) Any integrable $U_q(\mathfrak{g})$ -module generated by an extremal weight vector of weight ϖ_r is isomorphic to $V(\varpi_r)$.

Let $\lambda \in P^0 = \{\lambda \in P \mid \langle c, \lambda \rangle = 0\}$. $V(\lambda)$ has a $U_q(\mathfrak{g})_{\mathbb{Z}}$ -submodule $V(\lambda)_{\mathbb{Z}}$. Let $\{G(b)\}_{b \in B(\lambda)}$ stand for the global base of $V(\lambda)$. The following result was shown in [33] for $\mathfrak{g}$ simply laced and $\lambda = \varpi_r$, in [23] for $\mathfrak{g}$ simply laced and λ is arbitrary, and in [1] for $\mathfrak{g}$ and λ arbitrary.

Proposition 2.3.

- (i) There exists a prepolarization $(\cdot, \cdot)$ on $V(\lambda)$.
- (ii) $\{G(b)\}_{b \in B(\lambda)}$ is almost orthonormal with respect to $(\cdot, \cdot)$, that is, $(G(b), G(b')) \equiv \delta_{bb'} \pmod{q_s \mathbb{Z}[q_s]}$.

Let d_r be a positive integer such that

$$\{k \in \mathbb{Z} \mid \varpi_r + k\delta \in W\varpi_r\} = \mathbb{Z}d_r.$$

We note that $d_r = \max(1, (\alpha_r, \alpha_r)/2)$ except in the case $d_r = 1$ when $\mathfrak{g} = A_{2n}^{(2)}$ and $r = n$. Then there exists a $U'_q(\mathfrak{g})$ -linear automorphism z_r of $V(\varpi_r)$ of weight $d_r\delta$ sending u_{ϖ_r} to $u_{\varpi_r + d_r\delta}$. Hence we can define a $U'_q(\mathfrak{g})$ -module $W(\varpi_r)$ by

$$W(\varpi_r) = V(\varpi_r)/(z_r - 1)V(\varpi_r).$$

This module is called a fundamental representation.

For a $U'_q(\mathfrak{g})$ -module M let M_{aff} denote the $U'_q(\mathfrak{g})$ -module $\mathbb{Q}(q_s)[z, z^{-1}] \otimes M$ with the actions of e_i and f_i by $z^{\delta_{i0}} \otimes e_i$ and $z^{-\delta_{i0}} \otimes f_i$. For $a \in \mathbb{Q}(q_s)$ we define the $U'_q(\mathfrak{g})$ -module M_a by $M_{\text{aff}}/(z - a)M_{\text{aff}}$.

Proposition 2.4 ([17, Proposition 5.17]).

- (i) $W(\varpi_r)$ is a finite-dimensional irreducible integrable $U'_q(\mathfrak{g})$ -module.
- (ii) For any $\mu \in \text{wt } V(\varpi_r)$, $W(\varpi_r)_{cl(\mu)} \simeq V(\varpi_r)_\mu$. Here the map cl stands for the canonical projection $P \longrightarrow P_{cl}$.
- (iii) $\dim W(\varpi_r)_{cl(\mu)} = 1$ for any $\mu \in W\varpi_r$.

- (iv) $\text{wt } W(\varpi_r)$ is contained in the intersection of $\text{cl}(\varpi_r + \sum_{i \in I} \mathbb{Z}\alpha_i)$ and the convex hull of $W \text{cl}(\varpi_r)$.
- (v) $W(\varpi_r)$ has a global crystal base.
- (vi) Any irreducible finite-dimensional integrable $U'_q(\mathfrak{g})$ -module with $\text{cl}(\varpi_r)$ as an extremal weight is isomorphic to $W(\varpi_r)_a$ for some $a \in \mathbb{Q}(q_s)$.

We also need the following lemma that ensures the existence of the prepolarization on $W(\varpi_r)$.

Lemma 2.5 ([33, 23]). $(z_r u, z_r v) = (u, v)$ for $u, v \in V(\varpi_r)$.

Remark 2.1. This lemma is given as Proposition 7.3 of [33] and also as Lemma 4.7 of [23]. The lemmas or properties used to prove it hold for any affine algebra $\mathfrak{g}$.

Summing up the above discussions we have

Proposition 2.6. *The fundamental representation $W(\varpi_r)$ has the following properties:*

- (i) $W(\varpi_r)$ has a polarization $(\ , \)$.
- (ii) There exists a $U'_q(\mathfrak{g})_{\mathbb{Z}}$ -submodule $W(\varpi_r)_{\mathbb{Z}}$ of $W(\varpi_r)$ such that

$$(W(\varpi_r)_{\mathbb{Z}}, W(\varpi_r)_{\mathbb{Z}}) \subset \mathbb{Z}[q_s, q_s^{-1}].$$

Before finishing this section, let us mention the Drinfeld polynomials. It is known that irreducible finite-dimensional $U'_q(\mathfrak{g})$ -modules are classified by $|I_0|$ -tuple of polynomials $\{P_j(u)\}_{j \in I_0}$ whose constant terms are 1. See e.g. [4]. The degree of P_j is given by $\langle \lambda, h_j \rangle$ where λ is the highest weight of the corresponding module. Hence we have

Lemma 2.7. $W(\varpi_r)$ has the following Drinfeld polynomials

$$P_r(u) = 1 - a_r^\dagger u, \quad P_j(u) = 1 \text{ for } j \neq r$$

with some $a_r^\dagger \in \mathbb{Q}(q_s)$.

For types $A_n^{(1)}, D_n^{(1)}, E_{6,7,8}^{(1)}$ the explicit value of $a_r^\dagger$ is known [23, Remark 3.3].

3. KR MODULES AND THE EXISTENCE OF CRYSTAL BASES

3.1. Fusion construction. Let V be a $U'_q(\mathfrak{g})$ -module. An R -matrix, denoted by $R(x, y)$, is an element of $\text{Hom}_{U'_q(\mathfrak{g})[x^{\pm 1}, y^{\pm 1}]}(V_x \otimes V_y, V_y \otimes V_x)$. For V we assume the following:

(3.1) $V \otimes V$ is irreducible.

(3.2) There exists $\lambda_0 \in P_{cl}$ such that $\text{wt } V \subset \lambda_0 + \sum_{i \in I_0} \mathbb{Z}_{\leq 0} \alpha_i$ and $\dim V_{\lambda_0} = 1$.

Under these assumptions it is known (see e.g. [13]) that there exists a unique R -matrix up to multiple of a scalar function of x, y . Take a nonzero vector u_0 from V_{λ_0} . We normalize $R(x, y)$ in such a way that $R(x, y)(u_0 \otimes u_0) = u_0 \otimes u_0$. The normalized R -matrix is known to depend only on x/y . Because of the normalization, some matrix elements of $R(x, y)$ may have zeros or poles as a function of x/y . At the points $x/y = x_0/y_0 \in \mathbb{Q}(q_s)$ where there is no zero or pole, $R(x_0, y_0)$ is an isomorphism.

Next we review the fusion construction following section 3 of [14]. Let s be a positive integer and $\mathfrak{S}_s$ the s -th symmetric group. Let s_i be the simple reflection

which interchanges i and $i + 1$, and let $\ell(w)$ be the length of $w \in \mathfrak{S}_s$. Let $R(x, y)$ denote the R -matrix for $V_x \otimes V_y$. For any $w \in \mathfrak{S}_s$ we can construct a well-defined map $R_w(x_1, \dots, x_s) : V_{x_1} \otimes \dots \otimes V_{x_s} \rightarrow V_{x_{w(1)}} \otimes \dots \otimes V_{x_{w(s)}}$ by

$$\begin{aligned} R_1(x_1, \dots, x_s) &= 1, \\ R_{s_i}(x_1, \dots, x_s) &= \left(\bigotimes_{j < i} \text{id}_{V_{x_j}} \right) \otimes R(x_i, x_{i+1}) \otimes \left(\bigotimes_{j > i+1} \text{id}_{V_{x_j}} \right), \\ R_{ww'}(x_1, \dots, x_s) &= R_{w'}(x_{w(1)}, \dots, x_{w(s)}) \circ R_w(x_1, \dots, x_s) \\ &\quad \text{for } w, w' \text{ such that } \ell(ww') = \ell(w) + \ell(w'). \end{aligned}$$

Fix $k \in d^{-1}\mathbb{Z} \setminus \{0\}$. Let us assume that

$$(3.3) \quad \text{the normalized } R\text{-matrix } R(x, y) \text{ does not have a pole at } x/y = q^{2k}.$$

For each $s \in \mathbb{Z}_{>0}$, we put

$$\begin{aligned} R_s &= R_{w_0}(q^{k(s-1)}, q^{k(s-3)}, \dots, q^{-k(s-1)}) : \\ &\quad V_{q^{k(s-1)}} \otimes V_{q^{k(s-3)}} \otimes \dots \otimes V_{q^{-k(s-1)}} \rightarrow V_{q^{-k(s-1)}} \otimes V_{q^{-k(s-3)}} \otimes \dots \otimes V_{q^{k(s-1)}}, \end{aligned}$$

where w_0 is the longest element of $\mathfrak{S}_s$. Then R_s is a $U'_q(\mathfrak{g})$ -linear homomorphism. Define

$$V_s = \text{Im } R_s.$$

Let us denote by W the image of

$$R(q^k, q^{-k}) : V_{q^k} \otimes V_{q^{-k}} \longrightarrow V_{q^{-k}} \otimes V_{q^k}$$

and by N its kernel. Then we have

$$\begin{aligned} (3.4) \quad V_s &\text{ considered as a submodule of } V^{\otimes s} = V_{q^{-k(s-1)}} \otimes \dots \otimes V_{q^{k(s-1)}} \\ &\text{ is contained in } \bigcap_{i=0}^{s-2} V^{\otimes i} \otimes W \otimes V^{\otimes(s-2-i)}. \end{aligned}$$

Similarly, we have

$$(3.5) \quad V_s \text{ is a quotient of } V^{\otimes s} / \sum_{i=0}^{s-2} V^{\otimes i} \otimes N \otimes V^{\otimes(s-2-i)}.$$

In the sequel, following [14] we define a prepolarization on V_s and study necessary properties. First we recall the following lemma.

Lemma 3.1 ([14, Lemma 3.4.1]). *Let M_j and N_j be $U'_q(\mathfrak{g})$ -modules and let $(\ , \)_j$ be an admissible pairing between M_j and N_j ($j = 1, 2$). Then the pairing $(\ , \)$ between $M_1 \otimes M_2$ and $N_1 \otimes N_2$ defined by $(u_1 \otimes u_2, v_1 \otimes v_2) = (u_1, v_1)_1(u_2, v_2)_2$ for all $u_j \in M_j$ and $v_j \in N_j$ is admissible.*

Let V be a finite-dimensional $U'_q(\mathfrak{g})$ -module satisfying (3.1) and (3.2). Suppose V has a polarization. The polarization on V gives an admissible pairing between V_x and $V_{x^{-1}}$. Hence it induces an admissible pairing between $V_{x_1} \otimes \dots \otimes V_{x_s}$ and $V_{x_1^{-1}} \otimes \dots \otimes V_{x_s^{-1}}$.

Lemma 3.2 ([14, Lemma 3.4.2]). *If $x_j = x_{s+1-j}^{-1}$ for $j = 1, \dots, s$, then for any $u, u' \in V_{x_1} \otimes \dots \otimes V_{x_s}$, we have*

$$(u, R_{w_0}(x_1, \dots, x_s)u') = (u', R_{w_0}(x_1, \dots, x_s)u).$$

By taking $x_i = q^{k(s-2i+1)}$, we obtain the admissible pairing $(\ , \)$ between $W = V_{q^{k(s-1)}} \otimes V_{q^{k(s-3)}} \otimes \cdots \otimes V_{q^{-k(s-1)}}$ and $W' = V_{q^{-k(s-1)}} \otimes V_{q^{-k(s-3)}} \otimes \cdots \otimes V_{q^{k(s-1)}}$ that satisfies

$$(3.6) \quad (w, R_s w') = (w', R_s w) \quad \text{for any } w, w' \in W.$$

This allows us to define a prepolarization $(\ , \)_s$ on V_s by

$$(R_s u, R_s u')_s = (u, R_s u')$$

for $u, u' \in V_{q^{k(s-1)}} \otimes V_{q^{k(s-3)}} \otimes \cdots \otimes V_{q^{-k(s-1)}}$.

Assume

$$(3.7) \quad V \text{ admits a } U'_q(\mathfrak{g})_{K_{\mathbb{Z}}}\text{-submodule } V_{K_{\mathbb{Z}}} \text{ such that } (V_{K_{\mathbb{Z}}})_{\lambda_0} = K_{\mathbb{Z}} u_0.$$

Let us further set

$$(V_s)_{K_{\mathbb{Z}}} = R_s((V_{K_{\mathbb{Z}}})^{\otimes s}) \cap (V_{K_{\mathbb{Z}}})^{\otimes s}.$$

Then [14, Proposition 3.4.3] follows:

Proposition 3.3.

- (i) $(\ , \)_s$ is a nondegenerate prepolarization on V_s .
- (ii) $(R_s(u_0^{\otimes s}), R_s(u_0^{\otimes s}))_s = 1$.
- (iii) $((V_s)_{K_{\mathbb{Z}}}, (V_s)_{K_{\mathbb{Z}}})_s \subset K_{\mathbb{Z}}$.

3.2. KR modules. We want to apply the fusion construction with V being the fundamental representation $W(\varpi_r)$. Let us take k to be $(\alpha_r, \alpha_r)/2$ except in the case $k = 1$ when $\mathfrak{g} = A_{2n}^{(2)}$ and $r = n$.

Proposition 3.4. *Assumptions (3.1), (3.2), (3.3), and (3.7) hold for the fundamental representations.*

Proof. (3.1) is a consequence of Proposition 2.4(v) and the fact that $B(\varpi_r)$ is a “simple” crystal (see [17]). (3.2) is valid by Proposition 2.4(iv) with $\lambda_0 = cl(\varpi_r)$. Noting that $W(\varpi_r)$ is a “good” $U'_q(\mathfrak{g})$ -module, (3.3) is the consequence of Proposition 9.3 of [17]. (3.7) is valid, since $W(\varpi_r)$ admits a $U'_q(\mathfrak{g})_{\mathbb{Z}}$ -submodule $W(\varpi_r)_{\mathbb{Z}}$ induced from $V(\varpi_r)_{\mathbb{Z}}$ such that $(W(\varpi_r)_{\mathbb{Z}})_{cl(\varpi_r)} = \mathbb{Z}[q_s, q_s^{-1}]u_{\varpi_r}$. $\square$

For $r \in I_0$ and $s \in \mathbb{Z}_{>0}$ we define the $U'_q(\mathfrak{g})$ -module $W_s^{(r)}$ to be the module constructed by the fusion construction in section 3.1 with $V = W(\varpi_r)$ and $k = (\alpha_r, \alpha_r)/2$ except in the case $k = 1$ when $\mathfrak{g} = A_{2n}^{(2)}$ and $r = n$.

Proposition 3.5.

- (i) *There exists a prepolarization $(\ , \)$ on $W_s^{(r)}$.*
- (ii) *There exists a $U'_q(\mathfrak{g})_{K_{\mathbb{Z}}}$ -submodule $(W_s^{(r)})_{K_{\mathbb{Z}}}$ of $W_s^{(r)}$ such that*

$$((W_s^{(r)})_{K_{\mathbb{Z}}}, (W_s^{(r)})_{K_{\mathbb{Z}}}) \subset K_{\mathbb{Z}}.$$

- (iii) *There exists a vector u_0 of weight $s\varpi_r$ in $(W_s^{(r)})_{K_{\mathbb{Z}}}$ such that $(u_0, u_0) = 1$.*

Proof. The results follow from Propositions 3.3 and 3.4. $\square$

The following proposition is an easy consequence of the main result of Kashiwara [17]. Note also that his result can be applied not only to KR modules but also to any irreducible modules.

Proposition 3.6. $W_s^{(r)}$ is irreducible and its Drinfeld polynomials are given by

$$P_j(u) = \begin{cases} (1 - a_r^\dagger q_r^{1-s} u)(1 - a_r^\dagger q_r^{3-s} u) \cdots (1 - a_r^\dagger q_r^{s-1} u) & (j = r), \\ 1 & (j \neq r), \end{cases}$$

except when $\mathfrak{g} = A_{2n}^{(2)}$ and $r = n$. If $\mathfrak{g} = A_{2n}^{(2)}$ and $r = n$, they are given by replacing q_r with q in the above formula.

Proof. Let V be a nonzero submodule of $V_s = W_s^{(r)}$. To show the irreducibility, it suffices to show that any vector v in V_s is contained in V . By definition there exists a vector $u \in W(\varpi_r)^{\otimes s}$ such that $v = R_s u$. From Theorem 9.2(ii) of [17] we have $u_0^{\otimes s} \in V$. From Theorem 9.2(i) of loc. cit. there exists $x \in U'_q(\mathfrak{g})$ such that $u = \Delta^{(s)}(x)u_0^{\otimes s}$, where $\Delta^{(s)}$ is the coproduct $U'_q(\mathfrak{g}) \rightarrow U'_q(\mathfrak{g})^{\otimes s}$. Hence we have $v = R_s \Delta^{(s)}(x)u_0^{\otimes s} = \Delta^{(s)}(x)R_s u_0^{\otimes s} = \Delta^{(s)}(x)u_0^{\otimes s} \in V$.

Since $W_s^{(r)}$ is the irreducible module in $(W_1^{(r)})_{q_r^{1-s}} \otimes (W_1^{(s)})_{q_r^{3-s}} \otimes \cdots \otimes (W_1^{(r)})_{q_r^{s-1}}$ generated by $u_0^{\otimes s}$, the latter statement is clear from [4, Corollary 3.5], Lemma 2.7 and the fact that if V corresponds to $\{P_j(u)\}$, then V_a corresponds to $\{P_j(au)\}$. $\square$

This irreducible $U'_q(\mathfrak{g})$ -module $W_s^{(r)}$ is called Kirillov–Reshetikhin (KR) module:

Since the KR module $W_s^{(r)}$ is also a $U_q(\mathfrak{g}_0)$ -module by restriction, we have the following direct sum decomposition as a $U_q(\mathfrak{g}_0)$ -module:

$$(3.8) \quad W_s^{(r)} \simeq \bigoplus_{\lambda \in \overline{P}_+} N_s^{(r)}(\lambda) \cdot \overline{V}(\lambda)$$

Namely, $N_s^{(r)}(\lambda)$ is the multiplicity of the irreducible $U_q(\mathfrak{g}_0)$ -module $\overline{V}(\lambda)$ in $W_s^{(r)}$. Then we have a criterion that the KR module has a crystal pseudobase.

Proposition 3.7. Suppose for any $\lambda \in \overline{P}_+$ such that $N_s^{(r)}(\lambda) > 0$ there exist $u(\lambda)_j \in (W_s^{(r)})_{K_{\mathbb{Z}}}$ of weight λ for $j = 1, \dots, N_s^{(r)}(\lambda)$. If we have $(u(\lambda)_j, u(\lambda)_k) \in \delta_{jk} + q_s A$ and $(e_j u(\lambda)_k, e_j u(\lambda)_k) \in q_s q_j^{-2(1+\langle h_j, \lambda \rangle)} A$ for any $j \in I_0$, then $(\ , \)$ on $W_s^{(r)}$ is a polarization, and $W_s^{(r)}$ has a crystal pseudobase.

Proof. We use Proposition 2.1. All the assumptions except (2.4) are satisfied by Propositions 3.5. Note that $(u(\lambda)_j, u(\mu)_k) = 0$ if $\lambda \neq \mu$. $\square$

Remark 3.1. From the previous proposition it immediately follows that if $W_s^{(r)}$ is irreducible as a $U_q(\mathfrak{g}_0)$ -module, then it has a crystal pseudobase (see also [14, Proposition 3.4.4]). There is another case in which the existence of crystal pseudobase is proven for any l and any $\mathfrak{g}$ except $A_n^{(1)}$ as in [14, Proposition 3.4.5]. It corresponds to $r = 2$ when $\mathfrak{g} = B_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(1)}$, $r = 6$ when $\mathfrak{g} = E_6^{(1)}$, and $r = 1$ in all other cases. Here we follow the labeling of vertices of the Dynkin diagram by [12]. We remark that the crystal base of $W_1^{(r)}$ for such r is treated in [2].

There is an explicit formula of $N_s^{(r)}(\lambda)$ called the $(q = 1)$ fermionic formula. We have [3, 7, 8, 9, 10, 20, 24, 25] for references. To explain it, we introduce t_i and $t_i^\vee$ for $i \in I_0$ by

$$t_i = \begin{cases} \frac{2}{(\alpha_i, \alpha_i)} & \text{if } \mathfrak{g} \text{ is untwisted,} \\ 1 & \text{if } \mathfrak{g} \text{ is twisted,} \end{cases}$$

and $t_i^\vee = (t_i \text{ for } \mathfrak{g}^\vee)$, where $\mathfrak{g}^\vee$ is the dual Kac-Moody algebra to $\mathfrak{g}$. For $p \in \mathbb{Z}$ and $m \in \mathbb{Z}_{\geq 0}$ let $\binom{p+m}{m}$ stand for the binomial coefficient, i.e., $\binom{p+m}{m} = \prod_{k=1}^m \frac{p+k}{k}$. Then, for $r \in I_0$, $s \in \mathbb{Z}_{>0}$ and $\lambda \in \overline{P}_+$ we have

$$N_s^{(r)}(\lambda) = \sum_{\mathbf{m}} \prod_{a \in I_0, j \geq 1} \binom{p_j^{(a)} + m_j^{(a)}}{m_j^{(a)}},$$

where

$$p_j^{(a)} = \delta_{ai} \min(j, s) - \frac{1}{t_a^\vee} \sum_{b \in I_0, k \geq 1} (\alpha_a, \alpha_b) \min(t_b j, t_a k) m_k^{(b)}$$

and the sum $\sum_{\mathbf{m}}$ is taken over all $(m_j^{(a)} \in \mathbb{Z}_{\geq 0} \mid a \in I_0, j \geq 1)$ satisfying

$$\sum_{a \in I_0, j \geq 1} j m_j^{(a)} \alpha_a = s \varpi_r - \lambda.$$

The proof of this formula goes as follows. Set $Q_s^{(r)} = \text{ch } W_s^{(r)}$. It suffices to show that $Q_s^{(r)} = \sum_{\lambda \in \overline{P}_+} N_s^{(r)}(\lambda) \text{ch } \overline{V}(\lambda)$. By Theorem 8.1 of [8] (see also Theorem 6.3 of [7] including the twisted cases), it suffices to show that $\{Q_s^{(r)}\}$ satisfies the conditions (A), (B), (C) in the theorem. (A) is evident by the construction of $W_s^{(r)}$, and (B), (C) were verified in [24, 9, 10] for the simply-laced, untwisted and twisted cases, respectively. Note that condition (C) is replaced with another convergence property (4.15) of [21]. Note also that there is an earlier result by Chari [3] for untwisted cases. It should also be noted that there is another explicit formula $M_s^{(r)}(\lambda)$ for the multiplicities $N_s^{(r)}(\lambda)$ which involves unsigned binomial coefficients, that is, $\binom{p+m}{m} = 0$ if $p < 0$ [8, 7]. It was recently shown by Di Francesco and Kedem [6] that $M_s^{(r)}(\lambda) = N_s^{(r)}(\lambda)$ in the untwisted cases.

For nonexceptional types, the explicit value of $N_s^{(r)}(\lambda)$ can be found in Section 7 of [8] for untwisted cases, and in section 6.2 of [7] for twisted cases. See (4.1).

4. EXISTENCE OF CRYSTAL PSEUDOBASES FOR NONEXCEPTIONAL TYPES

In this section we show that any KR module for nonexceptional type has a crystal pseudobase. For type $A_n^{(1)}$ this fact is established in [14]. So we do not deal with the $A_n^{(1)}$ case.

4.1. Dynkin data. First we list the Dynkin diagrams of all nonexceptional affine algebras except $A_n^{(1)}$ in Table 1. We also list the pair $(\nu, \mathfrak{g}_0)$ in the table with a partition $\nu = \square, \square, \square$ and a simple Lie algebra $\mathfrak{g}_0$ whose Dynkin diagram is the one obtained by removing the 0-vertex. Note that the difference of ν comes from the diagram near the 0-vertex.

The simple roots for type B_n, C_n, D_n are

$$\begin{aligned} \alpha_i &= \epsilon_i - \epsilon_{i+1} && \text{for } 1 \leq i < n, \\ \alpha_n &= \begin{cases} \epsilon_{n-1} + \epsilon_n & \text{for type } D_n, \\ \epsilon_n & \text{for type } B_n, \\ 2\epsilon_n & \text{for type } C_n, \end{cases} \end{aligned}$$

TABLE 1. Dynkin diagrams

and the fundamental weights are

$$\begin{aligned}
 \text{Type } D_n: \quad & \varpi_i = \epsilon_1 + \cdots + \epsilon_i && \text{for } 1 \leq i \leq n-2, \\
 & \varpi_{n-1} = (\epsilon_1 + \cdots + \epsilon_{n-1} - \epsilon_n)/2, \\
 & \varpi_n = (\epsilon_1 + \cdots + \epsilon_{n-1} + \epsilon_n)/2; \\
 \text{Type } B_n: \quad & \varpi_i = \epsilon_1 + \cdots + \epsilon_i && \text{for } 1 \leq i \leq n-1, \\
 & \varpi_n = (\epsilon_1 + \cdots + \epsilon_{n-1} + \epsilon_n)/2; \\
 \text{Type } C_n: \quad & \varpi_i = \epsilon_1 + \cdots + \epsilon_i && \text{for } 1 \leq i \leq n,
 \end{aligned}$$

where ϵ_i ($i = 1, \dots, n$) are vectors in the weight space of each simple Lie algebra. (By convention we set $\varpi_0 = 0$.) These elements can be viewed as those of the weight lattice P of the affine algebra in Table 1. On P we defined the inner product $(\cdot, \cdot)$ normalized as $(\delta, \lambda) = \langle c, \lambda \rangle$ for $\lambda \in P$. This normalization is equivalent to setting $(\epsilon_i, \epsilon_j) = \kappa \delta_{ij}$ with $\kappa = \frac{1}{2}$ for $C_n^{(1)}$, $= 2$ for $D_{n+1}^{(2)}$, and $= 1$ for the other types. However, in this section we renormalize it by $(\epsilon_i, \epsilon_j) = \delta_{ij}$. This is equivalent to setting $(\alpha_i, \alpha_i)/2 = 1$ for i not an end node of the Dynkin diagram. We also note that

$$\alpha_0 = \begin{cases} \delta - \epsilon_1 - \epsilon_2 & \text{if } \nu = \boxplus, \\ \delta - 2\epsilon_1 & \text{if } \nu = \boxplus, \\ \delta - \epsilon_1 & \text{if } \nu = \square. \end{cases}$$

4.2. Existence of crystal pseudobases for KR modules. We first present the branching rule of KR modules of affine type listed in Table 1 with respect to the subalgebra $U_q(\mathfrak{g}_0)$. They can be found in [8, Theorems 7.1 and 8.1] and

[7, Theorems 6.2 and 6.3]. For $i \in I_0$ for $\mathfrak{g}$ we say i is a spin node if the vertex i is filled in Table 1. If $r \in I_0$ is a spin node, then the KR module $W_s^{(r)}$ is irreducible as a $U_q(\mathfrak{g}_0)$ -module:

$$W_s^{(r)} \simeq \overline{V}(s\varpi_r).$$

Suppose now that $r \in I_0$ is not a spin node. Let ω be a dominant integral weight of the form $\omega = \sum_i c_i \varpi_i$. Assume $c_i = 0$ for i a spin node. In the standard way we represent ω by the partition that has exactly c_i columns of height i . Then the KR module $W_s^{(r)}$ decomposes into

$$(4.1) \quad W_s^{(r)} \simeq \bigoplus_{\omega} \overline{V}(\omega)$$

as a $U_q(\mathfrak{g}_0)$ -module, where ω runs over all partitions that can be obtained from the $r \times s$ rectangle by removing pieces of shape ν (with ν as in Table 1).

If $r \in I_0$ is a spin node, the KR module $W_s^{(r)}$ has a crystal pseudobase by Remark 3.1. Suppose r is not a spin node. As we have seen, we have $N_s^{(r)}(\lambda) \leq 1$. Hence, by Proposition 3.7, in order to show the existence of crystal pseudobase, it suffices to define a vector $u(\lambda) \in (W_s^{(r)})_{K_{\mathbb{Z}}}$ of weight λ for any λ such that $N_s^{(r)} = 1$, and show $(u(\lambda), u(\lambda)) \in 1 + q_s A$ and $(e_j u(\lambda), e_j u(\lambda)) \in q_s q_j^{-2(1+\langle h_j, \lambda \rangle)} A$ for $j \in I_0$. In the subsequent subsections, we do this task by dividing it into 3 cases according to the shape of ν .

4.3. Calculation of prepolarization: $D_n^{(1)}, B_n^{(1)}, A_{2n-1}^{(2)}$ cases. We assume $1 \leq r \leq n-2$ for $D_n^{(1)}$, $1 \leq r \leq n-1$ for $B_n^{(1)}$ and $1 \leq r \leq n$ for $A_{2n-1}^{(2)}$. Let $r' = \lceil r/2 \rceil$. Let $\mathbf{c} = (c_1, c_2, \dots, c_{r'})$ be a sequence of integers such that $s \geq c_1 \geq c_2 \geq \dots \geq c_{r'} \geq 0$. For such $\mathbf{c}$ we define a vector u_m ($0 \leq m \leq r'$) in $W_s^{(r)}$ inductively by

$$u_m = (e_{r-2m}^{(c_m)} \cdots e_2^{(c_m)} e_1^{(c_m)})(e_{r-2m+1}^{(c_m)} \cdots e_3^{(c_m)} e_2^{(c_m)}) e_0^{(c_m)} u_{m-1},$$

where u_0 is the vector in (iii) of Proposition 3.5. Set $u(\mathbf{c}) = u_{r'}$. The weight of $u(\mathbf{c})$ is given by

$$\lambda(\mathbf{c}) = \sum_{j=0}^{r'} (c_j - c_{j+1}) \varpi_{r-2j},$$

where we have set $c_0 = s, c_{r'+1} = 0$, and ϖ_0 should be understood as 0. $\lambda(\mathbf{c})$ represents all ω in (4.1) when $\mathbf{c}$ runs over all possible sequences. For $l, m \in \mathbb{Z}_{\geq 0}$ such that $m \leq l$ we define the q -binomial coefficient by

$$(4.2) \quad \begin{bmatrix} l \\ m \end{bmatrix} = \frac{[l]!}{[m]![l-m]!}.$$

The following proposition calculates values of the prepolarization $(\ , \)$ on $W_s^{(r)}$.

Proposition 4.1.

- (1) $(u(\mathbf{c}), u(\mathbf{c})) = \prod_{j=1}^{r'} q^{c_j(2s-c_j)} \begin{bmatrix} 2s \\ c_j \end{bmatrix},$
- (2) $(e_j u(\mathbf{c}), e_j u(\mathbf{c})) = 0$ unless $r-j \in 2\mathbb{Z}_{\geq 0}$. If $r-j \in 2\mathbb{Z}_{\geq 0}$, then setting $p = (r-j)/2 + 1$, $(e_j u(\mathbf{c}), e_j u(\mathbf{c}))$ is given by

$$q^{2s-c_{p-1}-1} [2s-c_{p-1}] \prod_{j=1}^{r'} q^{(c_j-\delta_{j,p})(2s-c_j)} \begin{bmatrix} 2s-\delta_{j,p} \\ c_j-\delta_{j,p} \end{bmatrix}.$$

For type $D_n^{(1)}$ this proposition is proven in [27]. The proof goes completely parallel also for type $B_n^{(1)}$ and $A_{2n-1}^{(2)}$. Note that $q_i = q$ for $i \neq n$, $q_n = q, q^{1/2}, q^2$ for $D_n^{(1)}, B_n^{(1)}, A_{2n-1}^{(2)}$, respectively, and $q_s = q^{1/2}$ for $B_n^{(1)}$, $= q$ for $D_n^{(1)}, A_{2n-1}^{(2)}$. Since $q^{m-1}[m], q^{n(m-n)} \in 1 + qA$ and $\langle h_j, \lambda(\mathbf{c}) \rangle = c_{p-1} - c_p \geq 0$, we have $(u(\mathbf{c}), u(\mathbf{c})) \in 1 + q_s A$ and $(e_j u(\mathbf{c}), e_j u(\mathbf{c})) \in q_s q_j^{-2(1+\langle h_j, \lambda(\mathbf{c}) \rangle)} A$, for $j \in I_0$. This establishes the conditions of Proposition 3.7 and hence proves Theorem 1.1 that $W_s^{(r)}$ has a crystal pseudobase.

We denote the crystal of $W_s^{(r)}$ by $B^{r,s}$. Similar to $\mathfrak{g}_0$ one can consider $\mathfrak{g}_1$, which is another (mutually isomorphic) simple Lie algebra obtained by removing the vertex 1 from the Dynkin diagram of $\mathfrak{g}$. The following proposition will be used to show that $B^{r,s}$ is isomorphic to $\tilde{B}^{r,s}$, which is given combinatorially in the next section.

Proposition 4.2. *Let $1 \leq r \leq n-2$ for $\mathfrak{g} = D_n^{(1)}$, $1 \leq r \leq n-1$ for $\mathfrak{g} = B_n^{(1)}$, $1 \leq r \leq n$ for $\mathfrak{g} = A_{2n-1}^{(2)}$, and $s \in \mathbb{Z}_{>0}$. Then for $i = 0, 1$, $B^{r,s}$ decomposes as $U_q(\mathfrak{g}_i)$ -crystals into*

$$B^{r,s} \simeq \bigoplus_{0 \leq m_1 \leq \dots \leq m_s \leq [r/2]} B^{\mathfrak{g}_i}(\sigma^i(\varpi_{r-2m_1} + \dots + \varpi_{r-2m_s})).$$

Here $B^{\mathfrak{g}_i}(\lambda)$ is the crystal base of the highest weight $U_q(\mathfrak{g}_i)$ -module of highest weight λ , and σ is the automorphism on P such that $\sigma(\Lambda_0) = \Lambda_1, \sigma(\Lambda_1) = \Lambda_0, \sigma(\Lambda_j) = \Lambda_j$ ($j > 1$) and extended linearly.

Proof. If $i = 0$, the claim is a direct consequence of (4.1). For $i = 1$ note that the Weyl group of $\mathfrak{g}_0$ contains an element w which sends ϖ_j to $\sigma(\varpi_j)$ for any j such that $0 \leq j \leq r$, where by convention $\varpi_0 = 0$. (Using the orthogonal basis $\{\epsilon_i\}$ of section 4.1 of the weight space of $\mathfrak{g}_0$, we can take an element w such that $w(\epsilon_i) = (-1)^{\delta(i)} \epsilon_i$, where $\delta(i) = 1$ if $i = 1, n$ for $\mathfrak{g} = D_n^{(1)}$, $i = 1$ for $\mathfrak{g} = B_n^{(1)}$ and $A_{2n-1}^{(2)}$, and $\delta(i) = 0$ otherwise.) Since $W_s^{(r)}$ is a direct sum also as a $U_q(\mathfrak{g}_1)$ -module, it is enough to show the following equality of characters.

$$(4.3) \quad \text{ch } W_s^{(r)} = \sum_{0 \leq m_1 \leq \dots \leq m_s \leq [r/2]} \text{ch } V^{\mathfrak{g}_1}(\sigma(\varpi_{r-2m_1} + \dots + \varpi_{r-2m_s}))$$

Here $V^{\mathfrak{g}_1}(\lambda)$ denotes the highest weight $U_q(\mathfrak{g}_1)$ -module of highest weight λ . But noting $w(\alpha_0) = \alpha_1, w(\alpha_1) = \alpha_0, w(\alpha_j) = \alpha_j$ ($j > 1$) on P_{cl} , (4.3) is shown from

$$\text{ch } W_s^{(r)} = \sum_{0 \leq m_1 \leq \dots \leq m_s \leq [r/2]} \text{ch } V^{\mathfrak{g}_0}(\varpi_{r-2m_1} + \dots + \varpi_{r-2m_s})$$

since w preserves the weight multiplicity. $\square$

4.4. Calculation of prepolarization: $C_n^{(1)}$ case. We assume $1 \leq r \leq n-1$. Let $\mathbf{c} = (c_1, c_2, \dots, c_r)$ be a sequence of integers such that $[s/2] \geq c_1 \geq c_2 \geq \dots \geq c_r \geq 0$. For such $\mathbf{c}$ we define a vector u_m ($0 \leq m \leq r$) in $W_s^{(r)}$ inductively by

$$u_m = e_{r-m}^{(2c_m)} \dots e_2^{(2c_m)} e_1^{(2c_m)} e_0^{(c_m)} u_{m-1},$$

where u_0 is the vector in (iii) of Proposition 3.5. Set $u(\mathbf{c}) = u_r$. The weight of $u(\mathbf{c})$ is given by

$$\lambda(\mathbf{c}) = \sum_{j=0}^r 2(c_j - c_{j+1}) \varpi_{r-j},$$

where we have set $c_0 = s/2, c_{r+1} = 0$, and ϖ_0 should be understood as 0. $\lambda(\mathbf{c})$ represents all ω in (4.1) when $\mathbf{c}$ runs over all possible sequences. In this subsection, besides (4.2) we also use $\left[\begin{smallmatrix} l \\ m \end{smallmatrix} \right]_0$ defined by (4.2) with q replaced by $q_0 = q^2$. (Recall that we have renormalized the inner product $(\ , \)$ on P in such a way that $(\epsilon_i, \epsilon_j) = \delta_{ij}$.)

We are to calculate the values of $(u(\mathbf{c}), u(\mathbf{c}))$ and $(e_j u(\mathbf{c}), e_j u(\mathbf{c}))$. Since the calculation goes parallel to the case of $D_n^{(1)}$ treated in [27], we only give here intermediate results as a lemma. We write $\|u\|^2$ for (u, u) .

Lemma 4.3.

- (1) $\|u_m\|^2 = q_0^{c_m(s-c_m)} \left[\begin{smallmatrix} s \\ c_m \end{smallmatrix} \right]_0 \|u_{m-1}\|^2$,
- (2) $e_j u(\mathbf{c}) = 0$ if $j > r$,
- (3) $\|e_j u(\mathbf{c})\|^2 = q^{2\beta_j} \|f_j u(\mathbf{c})\|^2 + q^{\beta_j-1} [\beta_j] \|u(\mathbf{c})\|^2$ if $1 \leq j \leq r$, where $\beta_j = -\langle h_j, \lambda(\mathbf{c}) \rangle = 2(c_{r+1-j} - c_{r-j})$,
- (4)

$$\begin{aligned} \|f_j u(\mathbf{c})\|^2 &= \prod_{\substack{1 \leq m \leq r \\ m \neq r-j+1}} q_0^{c_m(s-c_m)} \left[\begin{smallmatrix} s \\ c_m \end{smallmatrix} \right]_0 \\ &\quad \times q_0^{c_{r-j+1}(s-1-c_{r-j+1})} \left[\begin{smallmatrix} s-1 \\ c_{r-j+1} \end{smallmatrix} \right]_0 \times q^{2c_{r-j}-1} [2c_{r-j}]. \end{aligned}$$

From this lemma we have

Proposition 4.4.

- (1) $(u(\mathbf{c}), u(\mathbf{c})) = \prod_{m=1}^r q^{c_m(s-c_m)} \left[\begin{smallmatrix} s \\ c_m \end{smallmatrix} \right]_0$,
 - (2)
- $$(e_j u(\mathbf{c}), e_j u(\mathbf{c})) = \begin{cases} q^{2s-2c_{r-j}-1} [2s-2c_{r-j}] \\ \quad \times \prod_{m=1}^r q_0^{(c_m-\delta_{m,r-j+1})(s-c_m)} \left[\begin{smallmatrix} s-\delta_{m,r-j+1} \\ c_m-\delta_{m,r-j+1} \end{smallmatrix} \right] & \text{if } 1 \leq j \leq r, \\ 0 & \text{if } r < j \leq n. \end{cases}$$

Note that $q_i = q$ for $i \neq 0, n$, $q_n = q^2$, and $q_s = q$ under the renormalization. Since $\langle h_j, \lambda(\mathbf{c}) \rangle = -\beta_j = 2(c_{r-j} - c_{r+1-j}) \geq 0$, we have $(u(\mathbf{c}), u(\mathbf{c})) \in 1 + q_s A$ and $(e_j u(\mathbf{c}), e_j u(\mathbf{c})) \in q_s q_j^{-2(1+\langle h_j, \lambda(\mathbf{c}) \rangle)} A$ for $j \in I_0$. By Proposition 3.7 this proves Theorem 1.1.

4.5. Calculation of prepolarization: $A_{2n}^{(2)}, D_{n+1}^{(2)}$ cases. We assume $1 \leq r \leq n$ for $A_{2n}^{(2)}$ and $1 \leq r \leq n-1$ for $D_{n+1}^{(2)}$. Let $\mathbf{c} = (c_1, c_2, \dots, c_r)$ be a sequence of integers such that $s \geq c_1 \geq c_2 \geq \dots \geq c_r \geq 0$. For such $\mathbf{c}$ we define a vector u_m ($0 \leq m \leq r$) in $W_s^{(r)}$ inductively by

$$u_m = e_{r-m}^{(c_m)} \dots e_1^{(c_m)} e_0^{(c_m)} u_{m-1},$$

where u_0 is the vector in (iii) of Proposition 3.5. Set $u(\mathbf{c}) = u_r$. The weight of $u(\mathbf{c})$ is given by

$$\lambda(\mathbf{c}) = \sum_{j=0}^r (c_j - c_{j+1}) \varpi_{r-j},$$

where we have set $c_0 = s, c_{r+1} = 0$, and ϖ_0 should be understood as 0. $\lambda(\mathbf{c})$ represents all ω in (4.1) when $\mathbf{c}$ runs over all possible sequences. In this subsection, besides (4.2) we also use $\begin{bmatrix} l \\ m \end{bmatrix}_0$ defined by (4.2) with q replaced by $q_0 = q^{1/2}$.

As in the previous subsection, we only give here intermediate results as a lemma. As before we write $\|u\|^2$ for (u, u) .

Lemma 4.5.

- (1) $\|u_m\|^2 = q_0^{c_m(2s-c_m)} \begin{bmatrix} 2s \\ c_m \end{bmatrix}_0 \|u_{m-1}\|^2,$
- (2) $e_j u(\mathbf{c}) = 0$ if $j > r$,
- (3) $\|e_j u(\mathbf{c})\|^2 = q^{2\beta_j} \|f_j u(\mathbf{c})\|^2 + q^{\beta_j-1} [\beta_j] \|u(\mathbf{c})\|^2$ if $1 \leq j \leq r$, where $\beta_j = -\langle h_j, \lambda(\mathbf{c}) \rangle = c_{r+1-j} - c_{r-j}$,
- (4)

$$\begin{aligned} \|f_j u(\mathbf{c})\|^2 &= \prod_{m=1}^r q_0^{c_m(2s-2\delta^{(1)}-c_m)} \begin{bmatrix} 2s-2\delta^{(1)} \\ c_m \end{bmatrix}_0 \times q^{c_{r-j}-1} [c_{r-j}] \\ &\quad + \prod_{m=1}^r q_0^{(c_m+\delta^{(1)}-\delta^{(2)})(2s-\delta^{(1)}+\delta^{(2)}-c_m)} \begin{bmatrix} 2s-2\delta^{(1)} \\ c_m-\delta^{(1)}-\delta^{(2)} \end{bmatrix}_0 \\ &\quad \times [2s-c_{r-j}+1]_0^2, \end{aligned}$$

where $\delta^{(1)} = \delta_{m,r-j+1}, \delta^{(2)} = \delta_{m,r-j}$.

From this lemma we have

Proposition 4.6.

- (1) $(u(\mathbf{c}), u(\mathbf{c})) = \prod_{m=1}^r q^{c_m(2s-c_m)} \begin{bmatrix} 2s \\ c_m \end{bmatrix}_0,$
- (2)

$$(e_j u(\mathbf{c}), e_j u(\mathbf{c})) = \begin{cases} q^{2\beta_j} \|f_j u(\mathbf{c})\|^2 + q^{\beta_j-1} [\beta_j] \|u(\mathbf{c})\|^2 & \text{if } 1 \leq j \leq r, \\ 0 & \text{if } r < j \leq n, \end{cases}$$

where β_j and $\|f_j u(\mathbf{c})\|^2$ are given in the previous lemma.

Note that $q_i = q$ for $i \neq 0, n$, $q_n = q^2$ for $A_{2n}^{(2)}$, $= q^{1/2}$ for $D_{n+1}^{(2)}$, and $q_s = q^{1/2}$ under the renormalization. Since $\langle h_j, \lambda(\mathbf{c}) \rangle = -\beta_j = c_{r-j} - c_{r+1-j} \geq 0$, we have $(u(\mathbf{c}), u(\mathbf{c})) \in 1 + q_s A$ and $(e_j u(\mathbf{c}), e_j u(\mathbf{c})) \in q_s q_j^{-2(1+\langle h_j, \lambda(\mathbf{c}) \rangle)} A$ for $j \in I_0$. By Proposition 3.7 this proves Theorem 1.1.

5. COMBINATORIAL CRYSTAL $\tilde{B}^{r,s}$ OF TYPE $D_n^{(1)}, B_n^{(1)}, A_{2n-1}^{(2)}$

In this section we review the combinatorial crystal $\tilde{B}^{r,s}$ of [30, 32] of type $D_n^{(1)}$, $B_n^{(1)}$, and $A_{2n-1}^{(2)}$ and prove some preliminary results that will be needed in Section 6 to establish the equivalence of $\tilde{B}^{r,s}$ and $B^{r,s}$.

5.1. Type D_n , B_n , and C_n crystals. Crystals associated with a $U_q(\mathfrak{g})$ -module when $\mathfrak{g}$ is a simple Lie algebra of nonexceptional type, were studied by Kashiwara and Nakashima [18]. Here we review the combinatorial structure in terms of tableaux of the crystals of type $X_n = D_n, B_n$, and C_n since these are the finite subalgebras relevant to the KR crystals of type $D_n^{(1)}, B_n^{(1)}$, and $A_{2n-1}^{(2)}$.

For $\mathfrak{g} = D_n^{(1)}, B_n^{(1)}$, or $A_{2n-1}^{(2)}$, any $\mathfrak{g}_0$ dominant weight ω without a spin component can be expressed as $\omega = \sum_i c_i \varpi_i$ for nonnegative integers c_i and the sum

TABLE 2. KR crystal $B^{1,1}$

runs over all $i = 1, 2, \dots, n$ not a spin node. As explained earlier we represent ω by the partition that has exactly c_i columns of height i . For type D_n , this can be extended by associating a column of height $n-1$ with $\varpi_{n-1} + \varpi_n$ and a column of height n with $2\varpi_n$. For type B_n one may associate a column of height n with $2\varpi_n$. Conversely, if ω is a partition, we write $c_i(\omega)$ for the number of columns of ω of height i . From now on we identify partitions and dominant weights in this way.

The crystal graph $B(\varpi_1)$ of the vector representation for type D_n , B_n , and C_n is given in Table 2 by removing the 0 arrows in the crystal $B^{1,1}$ of type $D_n^{(1)}$, $B_n^{(1)}$, and $A_{2n-1}^{(2)}$, respectively. The crystal $B(\varpi_\ell)$ for ℓ not a spin node can be realized as the connected component of $B(\varpi_1)^{\otimes \ell}$ containing the element $\ell \otimes (\ell-1) \otimes \dots \otimes 1$, where we use the anti-Kashiwara convention for tensor products. Similarly, the crystal $B(\omega)$ labeled by a dominant weight $\omega = \varpi_{\ell_1} + \dots + \varpi_{\ell_k}$ with $\ell_1 \geq \ell_2 \geq \dots \geq \ell_k$ not containing spin nodes can be realized as the connected component in $B(\varpi_{\ell_1}) \otimes \dots \otimes B(\varpi_{\ell_k})$ containing the element $u_{\varpi_{\ell_1}} \otimes \dots \otimes u_{\varpi_{\ell_k}}$, where u_{ϖ_i} is the highest weight element in $B(\varpi_i)$. As shown in [18], the elements of $B(\omega)$ can be labeled by tableaux of shape ω in the alphabet $\{1, 2, \dots, n, \bar{n}, \dots, \bar{1}\}$ for types D_n and C_n and the alphabet $\{1, 2, \dots, n, 0, \bar{n}, \dots, \bar{1}\}$ for type B_n . For the explicit rules of type D_n , B_n , and C_n tableaux we refer the reader to [18]; see also [11].

5.2. Definition of $\tilde{B}^{r,s}$. Let $\mathfrak{g}$ be of type $D_n^{(1)}$, $B_n^{(1)}$, or $A_{2n-1}^{(2)}$ with the underlying finite Lie algebra $\mathfrak{g}_0$ of type $X_n = D_n, B_n$, or C_n , respectively. The combinatorial crystal $\tilde{B}^{r,s}$ is defined as follows. As an X_n -crystal, $\tilde{B}^{r,s}$ decomposes into the following irreducible components

$$(5.1) \quad \tilde{B}^{r,s} \cong \bigoplus_{\omega} B(\omega),$$

for $1 \leq r \leq n$ not a spin node. Here $B(\omega)$ is the X_n -crystal of highest weight ω and the sum runs over all dominant weights ω that can be obtained from $s\varpi_r$ by

the removal of vertical dominoes, where ϖ_i are the fundamental weights of X_n as defined in section 5.1. The additional operators $\tilde{e}_0$ and $\tilde{f}_0$ are defined as

$$(5.2) \quad \begin{aligned} \tilde{f}_0 &= \sigma \circ \tilde{f}_1 \circ \sigma, \\ \tilde{e}_0 &= \sigma \circ \tilde{e}_1 \circ \sigma, \end{aligned}$$

where σ is the crystal analogue of the automorphism of the Dynkin diagram that interchanges nodes 0 and 1. The involution σ is defined in Definition 5.1.

5.3. Definition of σ . To define σ we first need the notion of $\pm$ diagrams. A $\pm$ diagram P of shape Λ/λ is a sequence of partitions $\lambda \subset \mu \subset \Lambda$ such that Λ/μ and μ/λ are horizontal strips. We depict this $\pm$ diagram by the skew tableau of shape Λ/λ in which the cells of μ/λ are filled with the symbol $+$ and those of Λ/μ are filled with the symbol $-$. Write $\Lambda = \text{outer}(P)$ and $\lambda = \text{inner}(P)$ for the outer and inner shapes of the $\pm$ diagram P . For type $A_{2n-1}^{(2)}$ and $r = n$, the inner shape λ is not allowed to be of height n . When drawing partitions or tableaux, we use the French convention where the parts are drawn in increasing order from top to bottom.

There is a bijection $\Phi : P \mapsto b$ from $\pm$ diagrams P of shape Λ/λ to the set of X_{n-1} -highest weight vectors b of X_{n-1} -weight λ in $B_{X_n}(\Lambda)$. Here X_{n-1} is the subalgebra whose Dynkin diagram is obtained from that of X_n by removing node 1. There is a natural homomorphism of the weight lattices $\pi : P(X_n) \rightarrow P(X_{n-1})$, where $\pi(\alpha_i^{X_n}) = \alpha_{i-1}^{X_{n-1}}$ and $\pi(\varpi_i^{X_n}) = \varpi_{i-1}^{X_{n-1}}$, and the partition λ is identified with the X_{n-1} weights under π . We identify the Kashiwara operators $\tilde{f}_i^{X_{n-1}}$ with $\tilde{f}_i^{X_n}$ under the embedding.

Explicitly, the bijection Φ is constructed as follows. Define a string of operators $\tilde{f}_{\vec{a}} := \tilde{f}_{a_1} \tilde{f}_{a_2} \cdots \tilde{f}_{a_\ell}$ such that $\Phi(P) = \tilde{f}_{\vec{a}} u$, where u is the highest weight vector in $B_{X_n}(\Lambda)$, where $\tilde{f}_i$ is the Kashiwara crystal operator corresponding to f_i . Start with $\vec{a} = ()$. Scan the columns of P from right to left. For each column of P for which a $+$ can be added, append $(1, 2, \dots, h)$ to $\vec{a}$, where h is the height of the added $+$. Next scan P from left to right and for each column that contains a $-$ in P , append to $\vec{a}$ the string $(1, 2, \dots, n, n-2, n-3, \dots, h)$ for type D_n , $(1, 2, \dots, n-1, n, n, n-1, \dots, h)$ for type B_n , and $(1, 2, \dots, n-1, n, n-1, \dots, h)$ for type C_n , where h is the height of the $-$ in P . Note that for type C_n the strings $(1, 2, \dots, h)$ and $(1, 2, \dots, n-1, n, n-1, \dots, h)$ are the same for $h = n$, which is why empty columns of height n are excluded for $\pm$ diagrams of type $A_{2n-1}^{(2)}$.

By construction the automorphism σ commutes with $\tilde{f}_i$ and $\tilde{e}_i$ for $i = 2, 3, \dots, n$. Hence it suffices to define σ on X_{n-1} highest weight elements. Because of the bijection Φ between $\pm$ diagrams and X_{n-1} -highest weight elements, it suffices to define the map on $\pm$ diagrams.

Let P be a $\pm$ diagram of shape Λ/λ . Let $c_i = c_i(\lambda)$ be the number of columns of height i in λ for all $1 \leq i < r$ with $c_0 = s - \lambda_1$. If $i \equiv r-1 \pmod{2}$, then in P , above each column of λ of height i , there must be a $+$ or a $-$. Interchange the number of such $+$ and $-$ symbols. If $i \equiv r \pmod{2}$, then in P , above each column of λ of height i , either there are no signs or a $\mp$ pair. Suppose there are $p_i \mp$ pairs above the columns of height i . Change this to $(c_i - p_i) \mp$ pairs. The result is $\mathfrak{S}(P)$, which has the same inner shape λ as P but a possibly different outer shape.

Definition 5.1. Let $b \in \tilde{B}^{r,s}$ and $\tilde{e}_{\mathbf{a}} := \tilde{e}_{a_1} \tilde{e}_{a_2} \cdots \tilde{e}_{a_\ell}$ be such that $\tilde{e}_{\mathbf{a}}(b)$ is a X_{n-1} highest weight crystal element. Define $\tilde{f}_{\mathbf{a}} := \tilde{f}_{a_\ell} \tilde{f}_{a_{\ell-1}} \cdots \tilde{f}_{a_1}$. Then

$$(5.3) \quad \sigma(b) := \tilde{f}_{\mathbf{a}} \circ \Phi \circ \mathfrak{S} \circ \Phi^{-1} \circ \tilde{e}_{\mathbf{a}}(b).$$

It was shown in [30] that $\tilde{B}^{r,s}$ is regular.

5.4. Properties of $\tilde{B}^{r,s}$. For the proof of uniqueness we will require the action of $\tilde{e}_1$ on X_{n-2} highest weight elements, where X_{n-2} is the Dynkin diagram obtained by removing nodes 1 and 2 from X_n . As we have seen in section 5.3, the X_{n-1} -highest weight elements in the branching $X_n \rightarrow X_{n-1}$ can be described by $\pm$ diagrams. Similarly the X_{n-2} -highest weight elements in the branching $X_{n-1} \rightarrow X_{n-2}$ can be described by $\pm$ diagrams. Hence each X_{n-2} -highest weight vector is uniquely determined by a pair of $\pm$ diagrams (P, p) such that $\text{inner}(P) = \text{outer}(p)$. The diagram P specifies the X_{n-1} -component $B_{X_{n-1}}(\text{inner}(P))$ in $B_{X_n}(\text{outer}(P))$, and p specifies the X_{n-2} component inside $B_{X_{n-1}}(\text{inner}(P))$. Let Υ denote the map $(P, p) \mapsto b$ from a pair of $\pm$ diagrams to a X_{n-2} highest weight vector.

To describe the action of $\tilde{e}_1$ on an X_{n-2} highest weight element or by Υ equivalently on (P, p) perform the following algorithm:

- (1) Successively run through all $+$ in p from left to right and, if possible, pair it with the leftmost yet unpaired $+$ in P weakly to the left of it.
- (2) Successively run through all $-$ in p from left to right and, if possible, pair it with the rightmost yet unpaired $-$ in P weakly to the left.
- (3) Successively run through all yet unpaired $+$ in p from left to right and, if possible, pair it with the leftmost yet unpaired $-$ in p .

Lemma 5.1 ([30, Lemma 5.1]). *If there is an unpaired $+$ in p , $\tilde{e}_1$ moves the rightmost unpaired $+$ in p to P . Otherwise, if there is an unpaired $-$ in P , $\tilde{e}_1$ moves the leftmost unpaired $-$ in P to p . Otherwise, $\tilde{e}_1$ annihilates (P, p) .*

In this paper, we will only require the case of Lemma 5.1 when a $-$ from P moves to p . Schematically, if a $-$ from a $\mp$ pair in P moves to p , then the following happens:

$$\begin{array}{|c|c|c|} \hline & \text{blue } - & - \\ \hline + & + & \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|} \hline & + & - \\ \hline & \text{red } - & + \\ \hline \end{array} \quad \text{or} \quad \begin{array}{|c|c|c|c|} \hline - & - & \text{blue } - & - \\ \hline & & + & + \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|c|} \hline + & - & - & - \\ \hline & & \text{red } - & + \\ \hline \end{array},$$

where the blue minus is the minus in P that is being moved and the red minus is the new minus in p . Similarly, schematically if a $-$ not part of a $\mp$ pair in P moves to p , then

$$\begin{array}{|c|c|c|c|} \hline & \text{blue } - & - & - \\ \hline & & & \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|c|} \hline & \text{red } - & - & - \\ \hline & & & \\ \hline \end{array} \quad \text{or} \quad \begin{array}{|c|c|c|c|c|} \hline + & + & \text{blue } - & - & - \\ \hline & & & & \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|c|c|} \hline \text{red } - & + & + & - & - \\ \hline & & & & \\ \hline \end{array}.$$

For any $b \in \tilde{B}^{r,s}$, let $\text{inner}(b)$ be the inner shape of the $\pm$ diagram corresponding to the X_{n-1} highest weight element in the component of b . Furthermore, recall that $\tilde{B}^{r,s}$ is regular, so that in particular $\tilde{e}_0$ and $\tilde{e}_1$ commute. We can now state the lemma needed in the next section.

Lemma 5.2. *Let $b \in \tilde{B}^{r,s}$ be an X_{n-2} highest weight vector corresponding under Υ to the tuple of $\pm$ diagrams (P, p) where $\text{inner}(p) = \text{outer}(P)$. Assume that $\varepsilon_0(b), \varepsilon_1(b) > 0$. Then $\text{inner}(b)$ is strictly contained in $\text{inner}(\tilde{e}_0(b))$, $\text{inner}(\tilde{e}_1(b))$, and $\text{inner}(\tilde{e}_0\tilde{e}_1(b))$.*

Proof. By assumption p does not contain any $-$ and $\tilde{e}_1$ is defined. Hence $\tilde{e}_1$ moves a $-$ in P to p . This implies that the inner shape of b is strictly contained in the inner shape of $\tilde{e}_1(b)$.

The involution σ does not change the inner shape of b (only the outer shape). By the same arguments as before, the inner shape of b is strictly contained in the inner shape of $\tilde{e}_1\sigma(b)$. Since σ does not change the inner shape, this is still true for $\tilde{e}_0(b) = \sigma\tilde{e}_1\sigma(b)$.

Now let us consider $\tilde{e}_0\tilde{e}_1(b)$. For the change in inner shape we only need to consider $\tilde{e}_1\sigma\tilde{e}_1(b)$, since the last σ does not change the inner shape. By the same arguments as before, $\tilde{e}_1$ moves a $-$ from P to p and σ does not change the inner shape. The next $\tilde{e}_1$ will move another $-$ in $\sigma\tilde{e}_1(b)$ to p . Hence p will have grown by two $-$, so that the inner shape of $\tilde{e}_1\sigma\tilde{e}_1(b)$ is increased by two boxes. $\square$

6. EQUIVALENCE OF $B^{r,s}$ AND $\tilde{B}^{r,s}$ OF TYPE $D_n^{(1)}$, $B_n^{(1)}$, AND $A_{2n-1}^{(2)}$

In this section all crystals are of type $D_n^{(1)}$, $B_n^{(1)}$, or $A_{2n-1}^{(2)}$ with corresponding classical subalgebra of type $X_n = D_n, B_n, C_n$, respectively.

Let B and B' be regular crystals of type $D_n^{(1)}$, $B_n^{(1)}$, or $A_{2n-1}^{(2)}$ with index set $I = \{0, 1, 2, \dots, n\}$. We say that $B \simeq B'$ is an isomorphism of J -crystals if B and B' agree as sets and all arrows colored $i \in J$ are the same.

Proposition 6.1. *Suppose that there exist two isomorphisms:*

$$\begin{aligned} \Psi_0 : \tilde{B}^{r,s} &\simeq B && \text{as an isomorphism of } \{1, 2, \dots, n\}\text{-crystals,} \\ \Psi_1 : \tilde{B}^{r,s} &\simeq B && \text{as an isomorphism of } \{0, 2, \dots, n\}\text{-crystals.} \end{aligned}$$

Then $\Psi_0(b) = \Psi_1(b)$ for all $b \in \tilde{B}^{r,s}$ and hence there exists an I -crystal isomorphism $\Psi : \tilde{B}^{r,s} \simeq B$.

Remark 6.1. Note that Ψ_0 and Ψ_1 preserve weights, that is, $\text{wt}(b) = \text{wt}(\Psi_0(b)) = \text{wt}(\Psi_1(b))$ for all $b \in \tilde{B}^{r,s}$. This is due to the fact that if all but one coefficient m_j are known for a weight $\Lambda = \sum_{j=0}^n m_j \Lambda_j$, then the missing m_j is also determined by the level 0 condition.

Proof. If $\Psi_0(b) = \Psi_1(b)$ for a b in a given X_{n-1} -component C , then $\Psi_0(b') = \Psi_1(b')$ for all $b' \in C$ since $\tilde{e}_i\Psi_0(b') = \Psi_0(\tilde{e}_ib')$ and $\tilde{e}_i\Psi_1(b') = \Psi_1(\tilde{e}_ib')$ for $i \in J = \{2, 3, \dots, n\}$. Hence it suffices to prove $\Psi_0(b) = \Psi_1(b)$ for only one element b in each X_{n-1} -component C . We are going to establish the theorem for b corresponding to the pairs of $\pm$ diagrams (P, p) where $\text{inner}(p) = \text{outer}(p)$. Note that this is an X_{n-2} -highest weight vector, but not necessarily an X_{n-1} -highest weight vector.

We proceed by induction on $\text{inner}(b)$ by containment. First suppose that both $\varepsilon_0(b), \varepsilon_1(b) > 0$. By Lemma 5.2, the inner shape of $\tilde{e}_0\tilde{e}_1b$, $\tilde{e}_0b$, and $\tilde{e}_1b$ is bigger than the inner shape of b , so that by induction hypothesis $\Psi_0(\tilde{e}_0\tilde{e}_1b) = \Psi_1(\tilde{e}_0\tilde{e}_1b)$, $\Psi_0(\tilde{e}_0b) = \Psi_1(\tilde{e}_0b)$, and $\Psi_0(\tilde{e}_1b) = \Psi_1(\tilde{e}_1b)$. Therefore we obtain

$$\begin{aligned} \tilde{e}_0\tilde{e}_1\Psi_0(b) &= \tilde{e}_0\Psi_0(\tilde{e}_1b) = \tilde{e}_0\Psi_1(\tilde{e}_1b) = \Psi_1(\tilde{e}_0\tilde{e}_1b) = \Psi_0(\tilde{e}_0\tilde{e}_1b) \\ &= \tilde{e}_1\Psi_0(\tilde{e}_0b) = \tilde{e}_1\Psi_1(\tilde{e}_0b) = \tilde{e}_1\tilde{e}_0\Psi_1(b). \end{aligned}$$

This implies that $\Psi_0(b) = \Psi_1(b)$.

Next we need to consider the cases when $\varepsilon_0(b) = 0$ or $\varepsilon_1(b) = 0$, which comprises the base case of the induction. Let us first treat the case $\varepsilon_1(b) = 0$. Recall that

$\text{inner}(p) = \text{outer}(p)$ so that p contains only empty columns. Hence it follows from the description of the action of $\tilde{e}_1$ of Lemma 5.1, that $\varepsilon_1(b) = 0$ if and only if P consists only of empty columns or columns containing $+$.

Claim. $\Psi_0(b) = \Psi_1(b)$ for all b corresponding to the pair of $\pm$ diagrams (P, p) where P contains only empty columns and columns with $+$, and $\text{inner}(p) = \text{outer}(p)$.

The claim is proved by induction on k , which is defined to be the number of empty columns in P of height strictly smaller than r . For $k = 0$ the claim is true by weight considerations. Now assume the claim is true for all $0 \leq k' < k$ and we will establish the claim for k . Suppose that $\Psi_1(b) = \Psi_0(\tilde{b})$ where $\tilde{b} \neq b$. By weight considerations $\tilde{b}$ must correspond to a pair of $\pm$ diagrams $(\tilde{P}, p)$, where $\tilde{P}$ has the same columns containing $+$ as P , but some of the empty columns of P of height h strictly smaller than r could be replaced by columns of height $h + 2$ containing $\mp$. Denote by k_+ the number of columns of P containing $+$. Then

$$m := \varepsilon_0(b) = k_+ + k,$$

since under σ all empty columns in P become columns with $\pm$ and columns containing $+$ become columns with $-$. By Lemma 5.1, then $\tilde{e}_1$ acts on $(\mathfrak{S}(P), p)$ as often as there are minus signs in $\mathfrak{S}(P)$, which is $k_+ + k$. Set $\hat{b} = \tilde{e}_1^a \tilde{b}$, where $a > 0$ is the number of columns in $\tilde{P}$ containing $\mp$. If $(\hat{P}, \hat{p})$ denotes the tuple of $\pm$ diagrams associated to $\hat{b}$, then compared to $(\tilde{P}, p)$ all $-$ from the $\mp$ pairs in $\tilde{P}$ moved to p . Note that $\hat{P}$ has only $k - a < k$ empty columns of height less than r , so that by induction hypothesis $\Psi_0(\hat{b}) = \Psi_1(\hat{b})$. Hence

$$(6.1) \quad \Psi_1(b) = \Psi_0(\tilde{b}) = \Psi_0(\tilde{f}_1^a \tilde{b}) = \tilde{f}_1^a \Psi_0(\tilde{b}) = \tilde{f}_1^a \Psi_1(\tilde{b}).$$

Note that

$$\varepsilon_0(\hat{b}) = \varepsilon_0(\tilde{b}) = m - a < m.$$

Hence

$$\tilde{e}_0^m \Psi_1(b) = \Psi_1(\tilde{e}_0^m b) \neq 0,$$

but

$$\tilde{e}_0^m \tilde{f}_1^a \Psi_1(\hat{b}) = \tilde{f}_1^a \Psi_1(\tilde{e}_0^m \hat{b}) = 0$$

which contradicts (6.1). This implies that we must have $\tilde{b} = b$ proving the claim.

The case $\varepsilon_0(b) = 0$ can be proven in a similar fashion to the case $\varepsilon_1(b) = 0$. Using the explicit action of $\mathfrak{S}$ on P and Lemma 5.1, it follows that $\varepsilon_0(b) = 0$ if and only if P consists only of columns containing $-$ or $\mp$ pairs.

Claim. $\Psi_0(b) = \Psi_1(b)$ for all b corresponding to the pair of $\pm$ diagrams (P, p) where P contains only columns with $-$ and columns with $\mp$ pairs, and $\text{inner}(p) = \text{outer}(p)$.

By induction on the number of $\mp$ pairs in P , this claim can be proven similarly as before (using the fact that $\mathfrak{S}$ changes columns with $-$ into columns with $+$ and columns with $\mp$ pairs into empty columns). $\square$

Proof of Theorem 1.2. Both crystals $B^{r,s}$ and $\tilde{B}^{r,s}$ have the same classical decomposition (5.1) as X_n crystals with index set $\{1, 2, \dots, n\}$ and $\{0, 2, 3, \dots, n\}$ by

Proposition 4.2. Hence there exist crystal isomorphisms Ψ_0 and Ψ_1 . By Proposition 6.1 there exists an I -isomorphism $\Psi : \tilde{B}^{r,s} \cong B^{r,s}$ which proves the theorem. $\square$

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