THE PBW FILTRATION

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ABSTRACT. In this paper we study the PBW filtration on irreducible integrable highest weight representations of affine Kac-Moody algebras $\widehat{\mathfrak{g}}$. The n -th space of this filtration is spanned by the vectors $x_1 \dots x_s v$, where $x_i \in \widehat{\mathfrak{g}}$, $s \leq n$, and v is a highest weight vector. For the vacuum module we give a conjectural description of the corresponding adjoint graded space in terms of generators and relations. For $\mathfrak{g}$ of the type A_1 we prove our conjecture and derive the fermionic formula for the graded character.

INTRODUCTION

Let $\mathfrak{g}$ be a Kac-Moody Lie algebra of finite or affine type and let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. For the dominant integral weight λ let L_λ be the corresponding irreducible highest weight representation with highest weight λ and highest weight vector v_λ . We consider a filtration $U(\mathfrak{g})_s$ on $U(\mathfrak{g})$ defined by

$$U(\mathfrak{g})_0 = \mathbb{C}1, \quad U(\mathfrak{g})_{s+1} = U(\mathfrak{g})_s + \text{span}\{gu : g \in \mathfrak{g}, u \in U(\mathfrak{g})_s\}.$$

This filtration induces a filtration $F_s = U(\mathfrak{g})_s \cdot v_\lambda$ on L_λ . Define an associated graded space

$$L_\lambda^{gr} = F_0 \oplus F_1/F_0 \oplus F_2/F_1 \oplus \dots$$

We define the graded u -character by

$$\text{ch}_u L_\lambda^{gr} = \sum_{s \geq 0} u^s \text{ch}(F_s/F_{s-1}),$$

where we set $F_{-1} = 0$ and ch denotes the usual character with respect to the Cartan subalgebra of $\mathfrak{g}$. For example, for $\mathfrak{g}$ of the type A_1 and its arbitrary finite-dimensional representation, all the spaces F_s/F_{s-1} are one-dimensional. The PBW-filtration for $\mathfrak{g}$ of the type $A_1^{(1)}$ and its level 1 representations were used in [FFJMT] for the study of the $\phi_{1,3}$ -field in Virasoro minimal theories.

Let us briefly describe our approach to the study of the spaces L_λ^{gr} . It is based on the abelianization procedure and the description of L_λ^{gr} in terms of generators and relations (similar techniques can be found, for example, in [AKS, C, CLM, FJKLM, FF, G, P]). Let $\mathfrak{n}_- \hookrightarrow \mathfrak{g}$ be the nilpotent subalgebra of creating operators, i.e.

$$(1) \quad L_\lambda = U(\mathfrak{n}_-) \cdot v_\lambda.$$

The action of $\mathfrak{n}_-$ on L_λ induces the action of the abelian algebra $\mathfrak{n}_-^{ab}$ on L_λ^{gr} ($\mathfrak{n}_-^{ab}$ is isomorphic to $\mathfrak{n}_-$ as a vector space). Therefore the space L_λ^{gr} is generated from the vector v_λ by the action of $U(\mathfrak{n}_-^{ab})$, which is isomorphic to the polynomial algebra.

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This allows us to describe L_λ^{gr} as a quotient of the polynomial algebra on $\mathfrak{n}_-$ by some ideal.

In this paper we study the case of an affine Kac-Moody algebra $\widehat{\mathfrak{g}}$ and its vacuum representations. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra and let $\widehat{\mathfrak{g}}$ be the corresponding affine algebra, i.e., the central extension of $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$. We fix its vacuum level k representation L_k with a highest weight vector v_k . In this case one has $(\mathfrak{g} \otimes \mathbb{C}[t]) \cdot v_k = 0$ and therefore the space L_k^{gr} is generated from the highest weight vector by an action of the universal enveloping algebra of the abelian algebra $\mathfrak{g}^{ab} \otimes t^{-1}\mathbb{C}[t^{-1}]$. This gives

$$(2) \quad L_k^{gr} \simeq U(\mathfrak{g}^{ab} \otimes t^{-1}\mathbb{C}[t^{-1}]) / I_k,$$

where I_k is some ideal. We describe the ideal I_k in the following way. We note that $\mathfrak{g} = \mathfrak{g} \otimes 1$ annihilates the highest weight vector v_k . Therefore the action of $\mathfrak{g}$ on L_k induces a structure of the $\mathfrak{g}$ -module on L_k^{gr} and I_k . (This structure plays an important role in our approach; see the conjecture below. For non-vacuum modules L_λ^{gr} no longer a $\mathfrak{g}$ -module. This means that the ideal of relations for general highest weight must be described in other terms than in the vacuum case. We hope to return to the study of the non-vacuum case elsewhere.) We give a conjectural description of I_k below. The proof is given in the paper for the case $\mathfrak{g} = \mathfrak{sl}_2$.

The adjoint action of $\mathfrak{g}$ on itself endows the space $U(\mathfrak{g}^{ab} \otimes t^{-1}\mathbb{C}[t^{-1}])$ with a structure of the $\mathfrak{g}$ -module. Let θ be the longest root of $\mathfrak{g}$, let $f_\theta \in \mathfrak{g}$ be an element of the weight θ , and let

$$f_\theta(z) = \sum_{n \geq 0} z^n (f_\theta \otimes t^{-n-1})$$

be the corresponding current. We recall (see [P, LP]) that the series $f_\theta(z)^{k+1}$ acts by zero on L_k (the coefficients of the z -expansion of $f_\theta(z)^{k+1}$ are acting by zero).

Conjecture 0.1. *We have an equality*

$$(3) \quad I_k = (U(\mathfrak{g}) \oplus U(\mathfrak{g}^{ab} \otimes t^{-1}\mathbb{C}[t^{-1}])) \cdot \text{span}\{\text{coefficients of } f_\theta(z)^{k+1}\},$$

i.e. I_k is the minimal $U(\mathfrak{g})$ -stable ideal which contains the coefficients of $f_\theta(z)^{k+1}$.

We verify this conjecture for $\mathfrak{g} = \mathfrak{sl}_2$ (the level 1 case was studied in [FFJMT] by a different method). Combining the description of I_k with the vertex operator realization technique we obtain the fermionic formula for the u -character of L_k^{gr} for $\mathfrak{g} = \mathfrak{sl}_2$. (For the fermionic formula for the non-PBW-graded characters of the highest weight modules see [AKS] and references within.)

We note that our proof of Conjecture 0.1 cannot be generalized (at least in the present form) to the case of the higher rank affine algebras. The reason is that the proof uses the simple structure of $\mathfrak{sl}_2$ -modules as well as the existence of the special bases in the representations of the corresponding affine algebra.

The paper is organized in the following way:

In Section 1 we fix our notations and recall some constructions from the representation theory of affine Kac-Moody algebras and bosonic vertex operator algebras.

In Section 2 we study the adjoint graded space L_1^{gr} .

In Section 3 we describe the ideal of relations in L_k^{gr} for the general level and obtain the fermionic formula for the graded character.

1. PRELIMINARIES AND DEFINITIONS

1.1. Kac-Moody Lie algebras, representations and filtrations. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra with a Cartan decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}_-$. We consider the corresponding affine Kac-Moody algebra

$$(4) \quad \widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d,$$

where K is a central element and d is a degree operator, $[d, x \otimes t^i] = -ix \otimes t^i$ for all $x \in \mathfrak{g}$. The affine algebra $\widehat{\mathfrak{g}}$ admits the Cartan decomposition $\widehat{\mathfrak{g}} = \widehat{\mathfrak{n}} \oplus \widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{n}}_-$, where

$$\begin{aligned} \widehat{\mathfrak{n}} &= \mathfrak{n} \otimes 1 \oplus \mathfrak{g} \otimes t\mathbb{C}[t], \\ \widehat{\mathfrak{h}} &= \mathfrak{h} \otimes 1 \oplus \mathbb{C}K \oplus \mathbb{C}d, \\ \widehat{\mathfrak{n}}_- &= \mathfrak{n}_- \otimes 1 \oplus \mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]. \end{aligned}$$

Let $\lambda \in \widehat{\mathfrak{h}}^*$ be an integral dominant weight and let L_λ be the corresponding irreducible highest weight representation of $\widehat{\mathfrak{g}}$ with highest weight λ and highest weight vector v_λ . Then

$$\widehat{\mathfrak{n}} \cdot v_\lambda = 0, \quad hv_\lambda = \lambda(h)v_\lambda \quad \forall h \in \widehat{\mathfrak{h}}, \quad L_\lambda = U(\widehat{\mathfrak{n}}_-) \cdot v_\lambda,$$

where $U(\widehat{\mathfrak{n}}_-)$ is a universal enveloping algebra. The number $\lambda(K)$ is called the level of L_λ . We let L_k to denote the level k vacuum representation (the restriction of the highest weight of L_k to $\mathfrak{h} \otimes 1$ vanishes).

We define an increasing filtration $U(\widehat{\mathfrak{n}}_-)_s$ on the universal enveloping algebra by the following rule:

$$U(\widehat{\mathfrak{n}}_-)_0 = \mathbb{C} \cdot 1, \quad U(\widehat{\mathfrak{n}}_-)_{s+1} = U(\widehat{\mathfrak{n}}_-)_s + \widehat{\mathfrak{n}}_- U(\widehat{\mathfrak{n}}_-)_s.$$

This filtration induces an increasing filtration F_s on L_λ :

$$F_s = U(\widehat{\mathfrak{n}}_-)_s \cdot v_\lambda.$$

The filtration $F_\bullet$ is called the Poincare-Birkhoff-Witt filtration (the PBW filtration for short).

Definition 1.1. Define L_λ^{gr} as an adjoint graded space with respect to the filtration F_s , i.e.

$$(5) \quad L_\lambda^{gr} = F_0 \oplus \bigoplus_{s>0} F_s / F_{s-1}.$$

Define the u -character of L_λ^{gr} as

$$(6) \quad \text{ch}_u L_\lambda^{gr} = \sum_{s \geq 0} u^s \text{ch}(F_s / F_{s-1}),$$

where we set $F_{-1} = 0$ and ch denotes the usual character with respect to $\widehat{\mathfrak{h}}^*$.

Remark 1.1. Let $\widehat{\mathfrak{n}}_-^{ab}$ be an abelian Lie algebra which coincides with $\widehat{\mathfrak{n}}_-$ as a vector space. Then L_λ^{gr} carries a natural structure of representation of $\widehat{\mathfrak{n}}_-^{ab}$.

Remark 1.2. The definition above is valid in the case of finite-dimensional algebras as well, providing a filtration on finite-dimensional modules.

Remark 1.3. For the general weight λ the spaces F_s are not stable under the action of an algebra $\mathfrak{g} = \mathfrak{g} \otimes 1 \hookrightarrow \widehat{\mathfrak{g}}$. This motivates the following modification of the filtration above. Let

$$\widetilde{F}_0 = U(\mathfrak{n}_- \otimes 1) \cdot v_\lambda, \quad \widetilde{F}_{s+1} = \widetilde{F}_s + \widehat{\mathfrak{n}}_- \widetilde{F}_s.$$

Here $F_0 = \mathbb{C}v_\lambda$ is replaced by $\widetilde{F}_0$, which is a finite-dimensional irreducible representation of $\mathfrak{g}$. We denote the corresponding adjoint graded object by $\widetilde{L}_\lambda^{gr}$. It is obvious that for any s the space $\widetilde{F}_s$ is $\mathfrak{g}$ -invariant. We hope to return to the study of this filtration elsewhere.

1.2. The $\mathfrak{sl}_2$ case. Let $\mathfrak{g} = \mathfrak{sl}_2$ and let e, h, f be its standard basis. Then f spans $\mathfrak{n}_-$, h spans $\mathfrak{h}$, and e spans $\mathfrak{n}$. For $x \in \mathfrak{sl}_2$ we set $x_i = x \otimes t^i$. Let v_k be a highest weight vector of the vacuum level k module L_k .

We now recall a construction of the ehf -basis of L_k from [MP] and [FKLMM].

Definition 1.2. A monomial of the form

$$(7) \quad \dots f_{-n}^{a_n} h_{-n}^{b_n} e_{-n}^{c_n} \dots f_{-1}^{a_1} h_{-1}^{b_1} e_{-1}^{c_1}$$

is called an ordered monomial. In addition it is called an ehf -monomial if it satisfies the following conditions:

- (a) $a_i + a_{i+1} + b_{i+1} \leq k$ for $i > 0$,
- (b) $a_i + b_{i+1} + c_{i+1} \leq k$ for $i > 0$,
- (c) $a_i + b_i + c_{i+1} \leq k$ for $i > 0$,
- (d) $b_i + c_i + c_{i+1} \leq k$ for $i > 0$.

Theorem 1.1. The set $\{m \cdot v_k\}$, where m runs over the set of ehf -monomials provides a basis of L_k .

Remark 1.4. The following picture from [FKLMM] illustrates the set of ehf -monomials:

Namely, one considers a set of monomials (7) such that the sum of exponents over any triangle (corresponding to the conditions (a)–(d)) is less than or equal to k .

1.3. Lattice vertex operator algebras and affine Kac-Moody algebras. In this section we recall the main properties of lattice vertex operator algebras (VOA for short) and their principal subspaces. The main references are [K2], [BF], [D], [FK].

Let Q be a lattice of finite rank equipped with a symmetric bilinear form $(\cdot, \cdot) : Q \times Q \rightarrow \mathbb{Z}$ such that $(\alpha, \alpha) > 0$ for all $\alpha \in Q \setminus \{0\}$. Let $\mathfrak{h} = Q \otimes_{\mathbb{Z}} \mathbb{C}$. The form $(\cdot, \cdot)$ induces a bilinear form on $\mathfrak{h}$, for which we use the same notation. Let $\mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$ be the corresponding multi-dimensional Heisenberg algebra with the bracket

$$[\alpha \otimes t^i, \beta \otimes t^j] = i\delta_{i,-j}(\alpha, \beta)K, \quad [K, \alpha \otimes t^i] = 0, \quad \alpha, \beta \in \mathfrak{h}.$$

For $\alpha \in \mathfrak{h}$ define the Fock representation π_α of the Heisenberg algebra generated by a vector $|\alpha\rangle$ such that

$$(\beta \otimes t^n)|\alpha\rangle = 0, \quad n > 0; \quad (\beta \otimes 1)|\alpha\rangle = (\beta, \alpha)|\alpha\rangle; \quad K|\alpha\rangle = |\alpha\rangle.$$

We now define a VOA V_Q associated with Q . We deal only with an even case, i.e. $(\alpha, \alpha) \in 2\mathbb{Z}$ for all $\alpha \in Q$ (in the general case the construction leads to the so-called super VOA). As a vector space

$$V_Q \simeq \bigoplus_{\alpha \in Q} \pi_\alpha.$$

The q -degree on V_Q is defined by

$$(8) \quad \deg_q |\alpha\rangle = \frac{(\alpha, \alpha)}{2}, \quad \deg_q (\beta \otimes t^n) = -n.$$

The main ingredients of the VOA structure on V_Q are bosonic vertex operators $\Gamma_\alpha(z)$ which correspond to highest weight vectors $|\alpha\rangle$. One sets

$$(9) \quad \Gamma_\alpha(z) = S_\alpha z^{\alpha \otimes 1} \exp\left(-\sum_{n < 0} \frac{\alpha \otimes t^n}{n} z^{-n}\right) \exp\left(-\sum_{n > 0} \frac{\alpha \otimes t^n}{n} z^{-n}\right),$$

where $z^{\alpha \otimes 1}$ acts on π_β by $z^{(\alpha, \beta)}$ and the operator S_α is defined by

$$S_\alpha |\beta\rangle = c_{\alpha, \beta} |\alpha + \beta\rangle; \quad [S_\alpha, \beta \otimes t^n] = 0, \quad \alpha, \beta \in \mathfrak{h},$$

where $c_{\alpha, \beta}$ are some non-vanishing constants. The Fourier decomposition is given by

$$\Gamma_\alpha(z) = \sum_{n \in \mathbb{Z}} \Gamma_\alpha(n) z^{-n - (\alpha, \alpha)/2}.$$

In particular,

$$(10) \quad \Gamma_\alpha(-(\alpha, \alpha)/2 - (\alpha, \beta)) |\beta\rangle = c_{\alpha, \beta} |\alpha + \beta\rangle.$$

One of the main properties of vertex operators is the following commutation relation:

$$(11) \quad [\alpha \otimes t^n, \Gamma_\beta(z)] = (\alpha, \beta) z^n \Gamma_\beta(z).$$

Another important formula describes the product of two vertex operators

$$(12) \quad \Gamma_\alpha(z) \Gamma_\beta(w) = (z - w)^{(\alpha, \beta)} S_\alpha S_\beta z^{\alpha \otimes 1} w^{\beta \otimes 1} \times \\ \exp\left(-\left(\sum_{n < 0} \frac{\alpha \otimes t^n}{n} z^{-n} + \frac{\beta \otimes t^n}{n} w^{-n}\right)\right) \exp\left(-\left(\sum_{n > 0} \frac{\alpha \otimes t^n}{n} z^{-n} + \frac{\beta \otimes t^n}{n} w^{-n}\right)\right).$$

This leads to the following proposition:

Proposition 1.1.

$$(13) \quad (\Gamma_\alpha(z))^{(k)} (\Gamma_\beta(z))^{(l)} = 0 \text{ if } k + l < (\alpha, \beta),$$

where the superscript (k) denotes the k -th derivative of the corresponding series. In addition, if $(\alpha, \beta) = 0$, then

$$\Gamma_\alpha(z) \Gamma_\beta(z) \text{ is proportional to } \Gamma_{\alpha + \beta}(z).$$

We now recall the Frenkel-Kac construction which provides a vertex operator realization of the basic representation of the affine algebra $\widehat{\mathfrak{g}}$ for $\mathfrak{g}$ of the types A , D , and E . Let Δ be the root system of $\mathfrak{g}$, let Q be the root lattice, and let

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta} \mathbb{C} e_\alpha \right)$$

be the weight decomposition. For any $\alpha \in Q$ we have a vertex operator

$$\Gamma_\alpha(z) = \sum_{n \in \mathbb{Z}} \Gamma_\alpha(n) z^{-n-1}$$

acting on the space V_Q . We let $V(\mathfrak{g})$ denote the vertex operator algebra associated with $\widehat{\mathfrak{g}}$.

Theorem 1.2. *The identification $\Gamma_\alpha(n) \mapsto e_\alpha \otimes t^n$ defines an isomorphism $V(\mathfrak{g}) \simeq V_Q$, which sends a highest weight vector of L_1 to $|0\rangle$.*

We finish this section with a description of principal subspaces of vertex operator algebras (see [FS]). Let $\alpha_1, \dots, \alpha_N$ be a set of linearly independent vectors generating the lattice Q . Let $M = (m_{i,j})_{1 \leq i,j \leq N}$ be the non-degenerate matrix of the scalar products of α_i ($m_{i,j} = (\alpha_i, \alpha_j)$) such that $m_{i,i} = 2$ for all i . Consider the principal subspace $W_Q \hookrightarrow V_Q$ generated from the vector $|0\rangle$ by an action of operators $\Gamma_{\alpha_i}(-n_i)$ with $n_i \geq 1$ ($1 \leq i \leq N$). Note that because of (10) one has $\Gamma_{\alpha_i}(-n)|0\rangle = 0$ for $n < 1$.

Our goal is to describe W_Q (in particular to find its character). We first realize this subspace as a quotient of a polynomial algebra. Namely, define W'_Q as a quotient of $\mathbb{C}[a_i(-n)]$ with $1 \leq i \leq N$, $n \geq 1$, by the ideal of relations generated by

$$a_i(z)^{(k)} a_j(z)^{(l)}, \quad k+l < m_{ij},$$

where $a_i(z) = \sum_{n \geq 1} z^n a_i(-n)$. We note that $W'_Q = \bigoplus_{\mathbf{n} \in \mathbb{Z}_{\geq 0}^N} W'_{Q,\mathbf{n}}$, where $W'_{Q,\mathbf{n}}$ is a subspace spanned by monomials in $a_i(k)$ such that the number of factors of the type $a_{i_0}(k)$ with fixed i_0 is exactly n_{i_0} . The q -character of $W'_{Q,\mathbf{n}}$ is naturally defined by $\deg_q a_i(k) = -k$.

The following lemma is standard (see for example [FF]).

Lemma 1.1.

$$(14) \quad \text{ch}_q W'_{Q,\mathbf{n}} = \frac{q^{\mathbf{n}M\mathbf{n}/2}}{(q)_{\mathbf{n}}},$$

where $(q)_{\mathbf{n}} = \prod_{j=1}^N (q)_{n_j}$, $(q)_n = \prod_{j=1}^n (1 - q^j)$.

The next proposition states that the spaces W_Q and W'_Q are isomorphic (see for example [FF]).

Proposition 1.2. *The map $|0\rangle \mapsto 1$, $\Gamma_{\alpha_i}(n) \mapsto a_i(n)$ induces the isomorphism*

$$W_Q \simeq W'_Q.$$

In particular, for any $\mathbf{n} = (n_1, \dots, n_N) \in \mathbb{Z}_{\geq 0}^N$,

$$(15) \quad \text{ch}_q(W_Q \cap \pi_{n_1\alpha_1 + \dots + n_N\alpha_N}) = \frac{q^{\mathbf{n}M\mathbf{n}/2}}{(q)_{\mathbf{n}}}.$$

2. LEVEL 1 CASE

2.1. Algebras A_1 and B_1 . We start with the description of the adjoint graded space L_1^{gr} for the vacuum level 1 $\widehat{\mathfrak{sl}_2}$ -module. Note that this case was considered in [FFJMT] by a different method. We make a connection to their approach at the end of this section.

The space L_1^{gr} carries the natural structure of the representation of abelian Lie algebra $\mathfrak{sl}_2^{ab} \otimes t^{-1}\mathbb{C}[t^{-1}]$ (see Remark 1.1), where $\mathfrak{sl}_2^{ab}$ is an abelianization of $\mathfrak{sl}_2$, i.e., the 3-dimensional abelian Lie algebra. For $x \in \mathfrak{sl}_2$ we denote the corresponding element of $\mathfrak{sl}_2^{ab}$ by $\tilde{x}$. We also set

$$\tilde{x}(z) = \sum_{i < 0} \tilde{x}_i z^{-i-1}.$$

The space L_1^{gr} is isomorphic to a quotient of the polynomial algebra

$$\mathbb{C}[\tilde{e}_i, \tilde{h}_i, \tilde{f}_i]_{i < 0}$$

by some ideal. Our goal is to show that the ideal of relations is generated by coefficients of the following series

$$(16) \quad \tilde{e}(z)^2, \tilde{e}(z)\tilde{h}(z), 2\tilde{e}(z)\tilde{f}(z) - \tilde{h}(z)^2, \tilde{h}(z)\tilde{f}(z), \tilde{f}(z)^2,$$

i.e., by the $\mathfrak{sl}_2$ consequences of the relation $\tilde{e}(z)^2$ ($\mathfrak{sl}_2$ acts on $\mathfrak{sl}_2^{ab} \otimes t^{-1}\mathbb{C}[t^{-1}]$ via the adjoint action on the space $\mathfrak{sl}_2^{ab}$).

Definition 2.1. Let A_1 be an algebra generated by the commuting variables $\tilde{x}_i$, $x = e, h, f$, $i < 0$ modulo the relations (16).

In order to make a connection between A_1 and L_1^{gr} we need a modification of A_1 .

Definition 2.2. Let B_1 be an algebra generated by the abelian variables $\tilde{x}_i$, $x = e, h, f$, $i < 0$ modulo the relations

$$(17) \quad \tilde{e}(z)^2, \tilde{e}(z)\tilde{h}(z), \tilde{h}(z)^2, \tilde{h}(z)\tilde{f}(z), \tilde{f}(z)^2.$$

We define the (q, z, u) -characters of A_1 and B_1 assigning the (q, z, u) -degree to each generator $\tilde{x}_i$:

$$\deg_q \tilde{x}_i = -i, \deg_z \tilde{e}_i = 2, \deg_z \tilde{f}_i = -2, \deg_z \tilde{h}_i = 0, \deg_u \tilde{x}_i = 1.$$

Lemma 2.1. $\text{ch}_{q,z,u} L_1^{gr} \leq \text{ch}_{q,z,u} A_1 \leq \text{ch}_{q,z,u} B_1$ (the inequalities are true in each weight component).

Proof. To show that $\text{ch}_{q,z,u} L_1^{gr} \leq \text{ch}_{q,z,u} A_1$, it suffices to verify that relations (16) hold in L_1^{gr} . In fact, $e(z)^2 = 0$ in L_1 and $\mathfrak{sl}_2 = \mathfrak{sl}_2 \otimes 1 \hookrightarrow \widehat{\mathfrak{sl}_2}$ is acting on L_1^{gr} as well as on L_1 . In addition all of the relations (16) can be produced from $\tilde{e}(z)^2$ by applying the operator f .

To prove that $\text{ch}_{q,z,u} A_1 \leq \text{ch}_{q,z,u} B_1$, we introduce a filtration G_s on A_1 by setting G_0 to be the subspace generated by variables $\tilde{e}_i, \tilde{f}_i$ and

$$G_{s+1} = \text{span}\{\tilde{h}_i w : i < 0, w \in G_s\} + G_s.$$

Then the relation $2\tilde{e}(z)\tilde{f}(z) - \tilde{h}(z)^2 = 0$ (which holds in A_1) gives $\tilde{h}(z)^2 = 0$ in the adjoint graded space. Lemma 2.1 is proved. $\square$

In the following subsections we give two proofs of the equality

$$\text{ch}_{q,z,u} L_1 = \text{ch}_{q,z,u} B_1.$$

2.2. **A basis of B_1 .** Recall the ehf -basis of L_1 (see Theorem 1.1):

$$(18) \quad \dots e_{-2}^{c_2} f_{-1}^{a_1} h_{-1}^{b_1} e_{-1}^{c_1}$$

with

- (a) $a_i + a_{i+1} + b_{i+1} \leq 1$,
- (b) $a_i + b_{i+1} + c_{i+1} \leq 1$,
- (c) $a_i + b_i + c_{i+1} \leq 1$,
- (d) $b_i + c_i + c_{i+1} \leq 1$.

We now consider a modified set of restrictions

- (a') $a_i + a_{i+1} + b_{i+1} \leq 1$,
- (b') $a_i + b_i + b_{i+1} \leq 1$,
- (c') $b_i + b_{i+1} + c_{i+1} \leq 1$,
- (d') $b_i + c_i + c_{i+1} \leq 1$,

and add one additional restriction

$$(N) \quad b_i + a_{i+1} + c_{i+2} \leq 2.$$

We refer to the monomials (18) with restrictions (a'), (b'), (c'), (d') and (N) as ehf' -monomials.

Conditions (a')–(d') and (N) can be expressed as follows (the first picture explains (a')–(d') and the second explains (N)):

The ehf' -monomials are monomials (18) such that the sum of exponents over any triangle in the first picture is less than or equal to 1 and the sum of exponents over any triangle in the second picture is less than or equal to 2.

Lemma 2.2. *The characters of ehf -monomials and ehf' -monomials coincide.*

Proof. In order to prove Lemma 2.2 we construct a (q, z, u) -degree preserving bijection ϕ from the set of ehf -monomials to the set of ehf' -monomials. Fix some ehf -monomial m of the form (18). If m satisfies conditions (a'), (b'), (c') and (d'), then we set $\phi(m) = m$. Now suppose that j is the smallest number such that (b') or (c') is violated, i.e.

$$a_j + b_j + b_{j+1} = 2 \text{ or } b_j + b_{j+1} + c_{j+1} = 2.$$

This means

$$b_j = b_{j+1} = 1, \quad a_j = c_{j+1} = 0.$$

We construct a new monomial

$$m' = \dots f_{-n}^{a'_n} h_{-n}^{b'_n} e_{-n}^{c'_n} \dots f_{-1}^{a'_1} h_{-1}^{b'_1} e_{-1}^{c'_1},$$

which differs from m only in the terms $a_j, b_j, b_{j+1}, c_{j+1}$. The new values are given by

$$(19) \quad a'_j = c'_{j+1} = 1, \quad b'_j = b'_{j+1} = 0.$$

The new monomial m' satisfies conditions (a') – (d') for all $i \leq j$. In addition it satisfies the condition (N) for all $i < j$. In fact the violation of (N) means that for some $i < j$, $b'_i = a'_{i+1} = c'_{i+2} = 1$. Therefore $i = j - 1$ and $b_{j-1} = b_j = 1$. This gives the violation of the conditions (b') and (c') for $i = j - 1$, which contradicts the choice of j .

We repeat the procedure until all the conditions (a') – (d') are satisfied for all i . Denote the result by $\phi(m)$. We note that for the same reason as above the condition (N) is satisfied by $\phi(m)$. Therefore $\phi(m)$ is an ehf' -monomial.

We now show that ϕ is a bijection. The inverse map ϕ^{-1} can be constructed in the following way. Fix some m' , which is an ehf' -, but not an ehf -monomial. Then for some j ,

$$a'_j + b'_j + c'_{j+1} = 2 \text{ or } a'_j + b'_{j+1} + c'_{j+1} = 2,$$

which means $a'_j = c'_{j+1} = 1, b'_j = b'_{j+1} = 0$. Then the inverse transformation to (19) is given by

$$a_j = c_{j+1} = 0, \quad b_j = b_{j+1} = 1.$$

Lemma 2.2 is proved. $\square$

Lemma 2.3. *The set of ehf' -monomials span B_1 .*

Proof. For any ordered monomial m of the form

$$(20) \quad \dots f_{-n}^{a_n} h_{-n}^{b_n} e_{-n}^{c_n} \dots f_{-1}^{a_1} h_{-1}^{b_1} e_{-1}^{c_1}$$

we denote by $\deg m$ the sum of all the exponents a_i, b_i and c_i . Following [FKLMM] we define a complete lexicographic ordering on the set of ordered monomials by the following rule. If $\deg m > \deg m'$, then $m > m'$. Suppose $\deg m = \deg m'$. If $c_1 < c'_1$, then $m > m'$. If $\deg m = \deg m'$, $c_1 = c'_1$ and $b_1 < b'_1$, then $m > m'$. Next we compare a_1 and a'_1 and so on.

We now show that any monomial m which violates one of the conditions (a') – (d') , (N) can be rewritten as a sum of smaller monomials. Suppose that for some i the condition (a') is violated, i.e. $a_i + a_{i+1} + b_{i+1} > 1$. This means that at least two of the numbers among a_i, a_{i+1}, b_{i+1} are greater than 0. But each of the products $a_i a_{i+1}, a_i b_{i+1}, a_{i+1} b_{i+1}$ can be written as a linear combination of smaller monomials using the relations

$$\sum_{\alpha+\beta=2i+1} \tilde{f}_\alpha \tilde{f}_\beta = 0, \quad \sum_{\alpha+\beta=2i+1} \tilde{f}_\alpha \tilde{h}_\beta = 0, \quad \sum_{\alpha+\beta=2i+2} \tilde{f}_\alpha \tilde{h}_\beta = 0,$$

respectively. By similar reasoning the violation of (b') , (c') and (d') allows us to rewrite the corresponding monomial in terms of the smaller ones. So we finish the proof by rewriting a monomial which satisfies (a') – (d') but not (N) . This reduces to rewriting a monomial $\tilde{h}_i \tilde{e}_{i+1} \tilde{f}_{i+2}$ as a linear combination of smaller ones. We have

$$(21) \quad \tilde{h}_i \tilde{e}_{i+1} \tilde{f}_{i+2} = -\tilde{e}_i \tilde{h}_{i+1} \tilde{f}_{i+2} + \dots = \tilde{e}_i \tilde{h}_{i+2} \tilde{f}_{i+1} + \dots,$$

where we use $\dots$ denote a linear combination of the monomials, which are smaller than $\tilde{h}_i \tilde{e}_{i+1} \tilde{f}_{i+2}$ (the relations $\tilde{e}(z) \tilde{h}(z) = 0$ and $\tilde{h}(z) \tilde{f}(z) = 0$ are used in (21)). Now it is enough to note that the last expression is smaller than $\tilde{h}_i \tilde{e}_{i+1} \tilde{f}_{i+2}$. $\square$

Corollary 2.1. $\text{ch}_{q,z,u} B_1 = \text{ch}_{q,z,u} L_1^{gr}$.

Proof. From Lemmas 2.2 and 2.3 we know that the character of the set of ehf' -monomials is greater than or equal to the character of B_1 and coincides with a character of L_1 . Now Corollary 2.1 follows from Lemma 2.1. $\square$

2.3. The degeneration procedure. In this section we give an alternative proof of Corollary 2.1. We introduce the degeneration procedure which will be used later in the case of general level k .

Let $p, q, r \in \mathbb{R}^3$ be linearly independent vectors with scalar products

$$(p, p) = (q, q) = (r, r) = 2, (p, q) = (q, r) = 1, (p, r) = 0.$$

Let Q_1 be a lattice generated by p, q, r and let $\Gamma_p(z)$, $\Gamma_q(z)$, $\Gamma_r(z)$ be the corresponding vertex operators.

Lemma 2.4. *The identification*

$$\Gamma_p(n) \rightarrow \tilde{e}_n, \Gamma_q(n) \rightarrow \tilde{h}_n, \Gamma_r(n) \rightarrow \tilde{f}_n$$

provides an isomorphism $W_{Q_1} \simeq B_1, |0\rangle \rightarrow 1$.

Proof. Follows from Proposition 1.2. $\square$

We now consider the Lie algebra $\widehat{\mathfrak{sl}_4}$ and its vertex operator realization provided by the Frenkel-Kac construction. For $1 \leq i, j \leq 4$ we set

$$E_{i,j}(z) = \sum_{n \in \mathbb{Z}} (E_{i,j} \otimes t^n) z^{-n-1}.$$

Using Theorem 1.2 and Lemma 2.4 we obtain the following lemma.

Lemma 2.5. *The identification*

$$\tilde{e}_n \rightarrow E_{2,3} \otimes t^n, \tilde{h}_n \rightarrow E_{2,4} \otimes t^n, \tilde{f}_n \rightarrow E_{1,4} \otimes t^n$$

provides an isomorphism between B_1 and the subspace of $V(\mathfrak{sl}_4)$ generated from the highest weight vector by the fields $E_{2,3}(z)$, $E_{2,4}(z)$ and $E_{1,4}(z)$.

We want to show that the character of B_1 is smaller than or equal to the character of L_1 . In order to do this we construct a deformation of the Lie algebra $\mathfrak{sl}_2 \hookrightarrow \mathfrak{sl}_4$ to the subspace spanned by $E_{2,3}$, $E_{2,4}$ and $E_{1,4}$.

Proposition 2.1. *There exists a continuous family of Lie subalgebras $S(\varepsilon)$ of $\mathfrak{sl}_4$, $0 \leq \varepsilon \leq 1$ such that*

- (a) $S(\varepsilon) \simeq \mathfrak{sl}_2$ for $0 \leq \varepsilon < 1$,
- (b) $S(1)$ is spanned by $E_{2,3}$, $E_{2,4}$ and $E_{1,4}$,
- (c) *There exist standard bases $e(\varepsilon)$, $h(\varepsilon)$, $f(\varepsilon)$ of $S(\varepsilon)$ ($0 \leq \varepsilon < 1$) such that*

$$\lim_{\varepsilon \rightarrow 1} \mathbb{C}e(\varepsilon) = \mathbb{C}E_{1,4}, \lim_{\varepsilon \rightarrow 1} \mathbb{C}h(\varepsilon) = \mathbb{C}E_{2,4}, \lim_{\varepsilon \rightarrow 1} \mathbb{C}f(\varepsilon) = \mathbb{C}E_{2,3}.$$

Proof. Let v_1, v_2, v_3, v_4 be a basis of $\mathbb{C}^4$. For $0 \leq \varepsilon < 1$ we let $S(\varepsilon)$ denote the subalgebra determined by the following conditions:

- $S(\varepsilon)$ preserves $\text{span}\{v_1, v_2\}$,
- $S(\varepsilon)$ annihilates $\text{span}\{(1 - \varepsilon)^{\frac{1}{2}}v_3 + \varepsilon v_1, (1 - \varepsilon)v_4 + \varepsilon v_2\}$,
- $S(\varepsilon)$ contains only traceless matrices.

Then $S(\varepsilon)$ consists of the matrices of the form

$$\begin{pmatrix} x & y & \frac{-x\varepsilon}{\sqrt{1-\varepsilon}} & \frac{-y\varepsilon}{1-\varepsilon} \\ z & -x & \frac{-z\varepsilon}{\sqrt{1-\varepsilon}} & \frac{x\varepsilon}{1-\varepsilon} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let $e(\varepsilon)$, $h(\varepsilon)$, $f(\varepsilon)$ be the standard basis of $S(\varepsilon) \simeq \mathfrak{sl}_2$ (we fix the identification via the upper left 2×2 corner of $S(\varepsilon)$). Then

$$\lim_{\varepsilon \rightarrow 1} (\varepsilon - 1)e(\varepsilon) = E_{1,4}, \quad \lim_{\varepsilon \rightarrow 1} (1 - \varepsilon)h(\varepsilon) = E_{2,4}, \quad \lim_{\varepsilon \rightarrow 1} (1 - \varepsilon)^{\frac{1}{2}}f(\varepsilon) = -E_{2,3}.$$

This finishes the proof of Proposition 2.1. $\square$

Corollary 2.2. *There exists a continuous family of subspaces $M(\varepsilon) \hookrightarrow V(\mathfrak{sl}_4)$, $0 \leq \varepsilon \leq 1$ such that $M(\varepsilon) \simeq L_1$ for $0 \leq \varepsilon < 1$ and $M(1) \simeq B_1$.*

Corollary 2.3. *L_1^{gr} and A_1 are isomorphic as $\mathfrak{sl}_2^{ab} \otimes t^{-1}\mathbb{C}[t^{-1}]$ -modules.*

Proof. Follows from Lemma 2.1 and Corollary 2.2. $\square$

Define the (q, z, u) -character of $L_{0,1}^{gr}$ by

$$(22) \quad \text{ch}_{q,z,u} L_1^{gr} = \sum_{s=0}^{\infty} u^s \text{ch}_{q,z}(F_s/F_{s-1}).$$

Proposition 2.2.

$$(23) \quad \text{ch}_{q,z,u} L_1^{gr} = \sum_{n^+, n^0, n^- \geq 0} u^{n^+ + n^0 + n^-} z^{2(n^+ - n^-)} \frac{q^{(n^+)^2 + (n^0)^2 + (n^-)^2 + n^+ n^0 + n^0 n^-}}{(q)_{n^+} (q)_{n^0} (q)_{n^-}},$$

where $(q)_n = \prod_{i=1}^n (1 - q^i)$.

Proof. We first calculate the character of B_1 . Consider the map ϕ from B_1^* to the space of polynomials in three groups of variables:

$$\begin{aligned} (\phi(\theta))(x_1^-, \dots, x_{n^-}^-, x_1^0, \dots, x_{n^0}^0, x_1^+, \dots, x_{n^+}^+) = \\ \theta(\tilde{f}(x_1^-) \dots \tilde{f}(x_{n^-}^-) \tilde{h}(x_1^0) \dots \tilde{h}(x_{n^0}^0) \tilde{e}(x_1^+) \dots \tilde{e}(x_{n^+}^+)). \end{aligned}$$

Because of the relations (17) the image of ϕ coincides with the space of polynomials of the form

$$(24) \quad \prod_{\substack{\alpha=+, -, 0 \\ 1 \leq i \leq n^\alpha}} x_i^\alpha \prod_{\substack{\alpha=+, -, 0 \\ 1 \leq i < j \leq n^\alpha}} (x_i^\alpha - x_j^\alpha)^2 \prod_{\substack{1 \leq i \leq n^+ \\ 1 \leq j \leq n^0}} (x_i^+ - x_j^0) \prod_{\substack{1 \leq i \leq n^0 \\ 1 \leq j \leq n^-}} (x_i^0 - x_j^-) \times \\ F(x_1^-, \dots, x_{n^-}^-, x_1^0, \dots, x_{n^0}^0, x_1^+, \dots, x_{n^+}^+),$$

where F is an arbitrary polynomial in x_i^α , symmetric in each group $(+, 0, -)$ of variables. The natural (q, z, u) -grading on the space B_1^* comes from the grading on B_1 . It is easy to see that the corresponding character of the space of polynomials (24) is given by

$$u^{n^+ + n^0 + n^-} z^{2(n^+ - n^-)} \frac{q^{(n^+)^2 + (n^0)^2 + (n^-)^2 + n^+ n^0 + n^0 n^-}}{(q)_{n^+} (q)_{n^0} (q)_{n^-}}.$$

Now Proposition 2.2 follows from Corollary 2.1. $\square$

Corollary 2.4.

$$(25) \quad \text{ch}_{q,z} L_1 = \sum_{n^+, n^0, n^- \geq 0} z^{2(n^+ - n^-)} \frac{q^{(n^+)^2 + (n^0)^2 + (n^-)^2 + n^+ n^0 + n^0 n^-}}{(q)_{n^+} (q)_{n^0} (q)_{n^-}}.$$

In particular the right-hand side equals the well-known expression for the character of L_1 :

$$\text{ch}_{q,z} L_1 = \frac{1}{(q)_\infty} \sum_{n \in \mathbb{Z}} z^{2n} q^{n^2},$$

where $(q)_\infty = \prod_{i \geq 1} (1 - q^i)$.

We now compare our formula with one from [FFJMT], which is given in terms of supernomial coefficients (see [SW]).

Corollary 2.5.

$$(26) \quad \text{ch}_{q,z,u} L_1^{gr} = \sum_{m \geq 0} \frac{u^m q^{m^2}}{(q)_m} \sum_{-m \leq l \leq m} z^{2l} S_{m,l}(q),$$

where

$$S_{m,l}(q) = \sum_{m \geq \nu \geq m-l-\nu} q^{(\nu+l-m)(\nu+l)+\nu(\nu-m)} \binom{m}{\nu}_q \binom{\nu}{m-l-\nu}_q$$

$$\text{and } \binom{n}{m}_q = \frac{(q)_n}{(q)_m (q)_{n-m}}.$$

Proof. We note that

$$\frac{1}{(q)_m} \binom{m}{\nu}_q \binom{\nu}{m-l-\nu}_q = \frac{1}{(q)_{m-\nu} (q)_{2\nu-m+l} (q)_{m-l-\nu}}.$$

The change of variables

$$m - \nu = n_+, \quad 2\nu - m + l = n_0, \quad m - l - \nu = n_-$$

identifies formulas (26) and (23). $\square$

3. THE GENERAL CASE

In this section we consider the PBW-filtration on the vacuum level k representation L_k and the corresponding adjoint graded space L_k^{gr} .

3.1. Algebras A_k and B_k . We introduce two series of algebras A_k and B_k generated by Fourier coefficients of the abelian currents $\tilde{e}(z)$, $\tilde{f}(z)$ and $\tilde{h}(z)$, where

$$\tilde{x}(z) = \sum_{i < 0} z^{-i-1} \tilde{x}_i.$$

Definition 3.1. An algebra A_k is a quotient of the polynomial algebra in variables $\tilde{e}_i, \tilde{h}_i, \tilde{f}_i, i < 0$, by the ideal generated by Fourier coefficients of $\tilde{e}(z)^{k+1}$ and all its $\mathfrak{sl}_2$ consequences, i.e. by coefficients of the $2k+3$ series:

$$(27) \quad \tilde{e}(z)^{k+1}, \tilde{e}(z)^k \tilde{h}(z), k \tilde{e}(z)^{k-1} \tilde{h}^2(z) - 2 \tilde{e}(z)^k \tilde{f}(z), \dots, \tilde{f}(z)^{k+1}.$$

In other words, these relations can be described as follows. Identify $\mathfrak{sl}_2$ with its 3-dimensional irreducible representation π_2 with e being the highest weight vector. Then we have an embedding

$$\iota : \pi_{2k+2} \hookrightarrow \pi_2^{\otimes(k+1)}$$

of the $(2k+3)$ -dimensional irreducible $\mathfrak{sl}_2$ module π_{2k+2} into the tensor power $\pi_2^{\otimes(k+1)}$ (the image of π_{2k+2} is generated from $e^{\otimes(k+1)} \in \pi_2^{\otimes(k+1)}$ by the action of the universal enveloping algebra of $\mathfrak{sl}_2$). Define an affinization map α , which sends an element of the tensor power $\pi_2^{\otimes(k+1)}$ to the product of the corresponding series:

$$\alpha(x^1 \otimes \dots \otimes x^{k+1}) = \tilde{x}^1(z) \dots \tilde{x}^{k+1}(z), \quad x^i = e, h, f.$$

Then the defining relations of A_k are coefficients of $\alpha(\iota(\pi_{2k+2}))$.

We note that the (q, z, u) -character of A_k is naturally defined by

$$(28) \quad \deg_u \tilde{x}_i = 1, \deg_q \tilde{x}_i = -i, \deg_z \tilde{e}_i = 2, \deg_z \tilde{h}_i = 0, \deg_z \tilde{f}_i = -2.$$

Lemma 3.1. $\text{ch}_{q,z,u} L_k^{gr} \leq \text{ch}_{q,z,u} A_k$.

Proof. Follows from the equality $e(z)^{k+1} = 0$ in L_k . $\square$

Definition 3.2. An algebra B_k is a quotient of the polynomial algebra in variables $\tilde{e}_i, \tilde{h}_i, \tilde{f}_i, i \leq -1$ by the ideal generated by Fourier coefficients of the $2k+3$ series

$$(29) \quad \tilde{e}(z)^i \tilde{h}(z)^{k+1-i}, \quad 1 \leq i \leq k+1; \quad \tilde{h}(z)^i \tilde{f}(z)^{k+1-i}, \quad 0 \leq i \leq k+1.$$

We note that the (q, z, u) -character of B_k is naturally defined by (28).

Lemma 3.2. $\text{ch}_{q,z,u} A_k \leq \text{ch}_{q,z,u} B_k$.

Proof. We introduce a filtration G_s on A_k by setting G_0 to be the subspace generated by variables $\tilde{e}_i, \tilde{f}_i$ (but not $\tilde{h}_i$) and

$$G_{s+1} = \text{span}\{\tilde{h}_i w : i < 0, w \in G_s\} + G_s.$$

Then for $0 \leq i \leq k+1$, the relation

$$\alpha(\iota(f^i \cdot e^{\otimes(k+1)})) = 0,$$

which holds in A_k , contains a term $\tilde{e}(z)^{k+1-i} \tilde{h}(z)^i$ and the coefficients of the difference

$$\alpha(\iota(f^i \cdot e^{\otimes(k+1)})) - \tilde{e}(z)^{k+1-i} \tilde{h}(z)^i$$

belongs to G_{i-1} . This means that the relation $\tilde{e}(z)^{k+1-i} \tilde{h}(z)^i = 0$ holds in the adjoint graded space $G_0 \oplus \bigoplus_{s>0} (G_s/G_{s-1})$. Similarly, the rest of the relations (29) are true in the adjoint graded space. Lemma 3.2 is proved. $\square$

Corollary 3.1. $\text{ch}_{q,z,u} L_k^{gr} \leq \text{ch}_{q,z,u} B_k$.

3.2. A quadratic algebra C_k and principal subspace D_k . We consider a set of commuting variables

$$\tilde{x}_i^{[l]}, \quad x = e, h, f, \quad 1 \leq l \leq k, \quad i \leq -l$$

and the corresponding currents $\tilde{x}^{[l]}(z) = \sum_{i < 0} z^{-i-l} \tilde{x}_i^{[l]}$ (we set $\tilde{x}_i^{[l]} = 0$ for $i > -l$). For the series $p(z)$, let $p(z)^{(r)}$ be the r -th derivative.

Definition 3.3. Let C_k be the quotient of the polynomial algebra in commuting variables $\tilde{x}_i^{[l]}$, $1 \leq l \leq k$, $i \leq -l$ by the ideal of relations generated by coefficients of currents

$$(30) \quad \tilde{x}^{[l]}(z)^{(\alpha)} \tilde{x}^{[m]}(z)^{(\beta)} \text{ for } x = e, h, f, \alpha + \beta < 2 \min(l, m),$$

$$(31) \quad \tilde{e}^{[l]}(z)^{(\alpha)} \tilde{h}^{[m]}(z)^{(\beta)} \text{ for } \alpha + \beta < \max(0, l + m - k),$$

$$(32) \quad \tilde{h}^{[l]}(z)^{(\alpha)} \tilde{f}^{[m]}(z)^{(\beta)} \text{ for } \alpha + \beta < \max(0, l + m - k).$$

We define the (q, z, u) -degree on C_k by the formulas

$$\deg_q \tilde{x}_i^{[l]} = -i, \deg_z \tilde{e}_i^{[l]} = 2l, \deg_z \tilde{h}_i^{[l]} = 0, \deg_z \tilde{f}_i^{[l]} = -2l, \deg_u \tilde{x}_i^{[l]} = l.$$

In what follows we show that the (q, z, u) -characters of B_k and C_k coincide.

We will need the following lemma from [FS, FJKLM].

Lemma 3.3. Consider the quotient of the polynomial algebra $\mathbb{C}[a_0, a_{-1}, \dots]$ by the ideal generated by the coefficients of the series $a(z)^{k+1}$. Then there exists a filtration F_μ of this quotient (labeled by Young diagrams μ) such that the adjoint graded algebra is generated by the coefficients of the series $a^{[i]}(z)$, which are the images of powers $a(z)^i$, $1 \leq i \leq k$. In addition, defining relations in the adjoint graded algebra are of the form

$$(33) \quad a^{[i]}(z)^{(l)} a^{[j]}(z)^{(r)} = 0 \text{ if } l + r < 2 \min(i, j).$$

Lemma 3.4. $\text{ch}_{q,z,u} B_k \leq \text{ch}_{q,z,u} C_k$.

Proof. Using Lemma (3.3) we define a filtration on B_k such that the adjoint graded space is generated by the images of coefficients of the series $\tilde{x}(z)^l$, $x = e, h, f$, $1 \leq l \leq k$. Denote the corresponding series by $\tilde{x}^{[l]}(z)$ and the corresponding Fourier coefficients by $\tilde{x}_i^{[l]}$. Then from Lemma (3.3) we obtain the relations (30). The relations (31), (32) in the adjoint graded space follow from the relations

$$(34) \quad (\tilde{e}(z)^l)^{(\alpha)} (\tilde{h}(z)^m)^{(\beta)} = 0 \text{ for } \alpha + \beta < \max(0, l + m - k),$$

$$(35) \quad (\tilde{h}(z)^l)^{(\alpha)} (\tilde{f}(z)^m)^{(\beta)} = 0 \text{ for } \alpha + \beta < \max(0, l + m - k),$$

which hold in B_k . We thus obtain that all of the relations of C_k are true in the adjoint graded space of B_k with respect to the filtration above. Lemma 3.4 is proved. $\square$

We want to show that $\text{ch}_{q,z,u} B_k \geq \text{ch}_{q,z,u} C_k$. We use the vertex operator technique. Fix an integer N such that there exists a set of linearly independent vectors $p_i, q_i, r_i \in \mathfrak{h} = \mathbb{R}^N$, $1 \leq i \leq k$ with scalar products

$$(36) \quad (p_i, p_j) = (q_i, q_j) = (r_i, r_j) = 2\delta_{i,j}, \quad (p_i, q_j) = (q_i, r_j) = \delta_{i,k+1-j}, \quad (p_i, r_j) = 0.$$

For example, setting $N = 3k$ and fixing some orthonormal basis e_i with respect to $(\cdot, \cdot)$, one can define

$$p_i = \sqrt{2}e_i, \quad q_i = \frac{1}{\sqrt{2}}e_{k+1-i} + \sqrt{\frac{3}{2}}e_{k+i}, \quad r_i = \sqrt{\frac{2}{3}}e_{2k+1-i} + \sqrt{\frac{4}{3}}e_{2k+i}.$$

We consider a lattice Q generated by the vectors p_i, q_i, r_i and the corresponding VOA V_Q . Set

$$(37) \quad V_e^{(\varepsilon)}(z) = \sum_{i=1}^k \varepsilon^i \Gamma_{p_i}(z), \quad V_h^{(\varepsilon)}(z) = \sum_{i=1}^k \varepsilon^i \Gamma_{q_i}(z), \quad V_f^{(\varepsilon)}(z) = \sum_{i=1}^k \varepsilon^i \Gamma_{r_i}(z).$$

Note that due to Proposition 1.1 one has

$$\Gamma_{p_i}(z)\Gamma_{q_{k+1-i}}(z) = 0 = \Gamma_{q_i}(z)\Gamma_{r_{k+1-i}}(z).$$

Therefore for any $0 \leq i \leq k+1$,

$$(38) \quad (V_e^{(\varepsilon)}(z))^i (V_h^{(\varepsilon)}(z))^{k+1-i} = (V_h^{(\varepsilon)}(z))^i (V_f^{(\varepsilon)}(z))^{k+1-i} = 0.$$

We let $D_k^{(\varepsilon)}$ denote the space generated by the Fourier coefficients of $V_e^{(\varepsilon)}(z)$, $V_h^{(\varepsilon)}(z)$, $V_f^{(\varepsilon)}(z)$ from the highest weight vector. We set $D_k = D_k^{(1)}$.

Lemma 3.5. $\text{ch}_{q,z,u} B_k \geq \text{ch}_{q,z,u} D_k \geq \text{ch}_{q,z,u} C_k$.

Proof. The equation (38) proves the first part of Lemma 3.5. To show the second inequality we use the degeneration procedure. Namely, the deformation $D_k^{(\varepsilon)}$ of D_k satisfies $D_k^{(\varepsilon)} \simeq D_k$ for $1 \geq \varepsilon > 0$ and $D_k^{(0)}$ contains C_k . In fact, the limit $\lim_{\varepsilon \rightarrow 0} D_k^{(\varepsilon)}$ contains the Fourier coefficients of the series

$$\begin{aligned} \Gamma_{p_1+\dots+p_l}(z) &= c_1 \lim_{\varepsilon \rightarrow 0} \varepsilon^{\frac{-l(l+1)}{2}} (V_e^{(\varepsilon)}(z))^l, \\ \Gamma_{q_1+\dots+q_l}(z) &= c_2 \lim_{\varepsilon \rightarrow 0} \varepsilon^{\frac{-l(l+1)}{2}} (V_h^{(\varepsilon)}(z))^l, \\ \Gamma_{r_1+\dots+r_l}(z) &= c_3 \lim_{\varepsilon \rightarrow 0} \varepsilon^{\frac{-l(l+1)}{2}} (V_f^{(\varepsilon)}(z))^l, \end{aligned}$$

where c_1, c_2, c_3 are some non-vanishing constants. We note that

$$\begin{aligned} (p_1 + \dots + p_i, p_1 + \dots + p_j) &= (q_1 + \dots + q_i, q_1 + \dots + q_j) = 2 \min(i, j), \\ (r_1 + \dots + r_i, r_1 + \dots + r_j) &= 2 \min(i, j), \\ (p_1 + \dots + p_i, q_1 + \dots + q_j) &= \max(0, i + j - k), \\ (r_1 + \dots + r_i, q_1 + \dots + q_j) &= \max(0, i + j - k). \end{aligned}$$

Let Q_k be the lattice generated by the vectors

$$p_1 + \dots + p_l, \quad q_1 + \dots + q_l, \quad r_1 + \dots + r_l, \quad 1 \leq l \leq k.$$

Then, using Proposition 1.2 we obtain that the principal subspace W_{Q_k} is isomorphic to C_k . The isomorphism is given by the identification

$$\Gamma_{p_1+\dots+p_l}(i) \mapsto \tilde{e}_i^{[l]}, \quad \Gamma_{q_1+\dots+q_l}(i) \mapsto \tilde{h}_i^{[l]}, \quad \Gamma_{r_1+\dots+r_l}(i) \mapsto \tilde{f}_i^{[l]}.$$

This gives

$$\text{ch}_{q,z,u} D_k \geq \text{ch}_{q,z,u} C_k.$$

Lemma 3.5 is proved. $\square$

Lemmas 3.4 and 3.5 give the following corollary:

Corollary 3.2. $\text{ch}_{q,z,u} B_k = \text{ch}_{q,z,u} C_k = \text{ch}_{q,z,u} D_k$.

Proposition 3.1. $\text{ch}_{q,z,u} L_k \geq \text{ch}_{q,z,u} D_k$.

Proof. We recall that there exists an embedding $L_k \hookrightarrow L_1^{\otimes k}$ such that the highest weight vector v_k of L_k maps to the tensor power $v_1^{\otimes k}$ and for any $x \in \mathfrak{sl}_2$ the current $x(z)$ on L_k corresponds to the sum $\sum_{i=1}^k x^{(i)}(z)$, where

$$x^{(i)}(z) = \text{Id} \otimes \dots \otimes x(z) \otimes \dots \otimes \text{Id}$$

($x(z)$ on the i -th place). From definition of D_k we also have an embedding $D_k \hookrightarrow D_1^{\otimes k}$ (see the definition (37)). Now the degeneration from Corollary 2.2 gives the

degeneration of L_k . From part (c) of Proposition (2.1) we conclude that the limit of this degeneration contains D_k . Proposition 3.1 is proved. $\square$

Theorem 3.1.

- The (q, z, u) characters of L_k^{gr} , A_k , B_k , C_k and D_k coincide.
- $L_k^{gr} \simeq A_k$ as the modules over the abelian algebra with a basis $\tilde{e}_i$, $\tilde{h}_i$, $\tilde{f}_i$, $i \leq -1$.

Proof. Follows from Lemmas 3.1, 3.2, Corollary 3.2 and Proposition 3.1. $\square$

3.3. The character formula. In this section we compute the (q, z, u) -character of L_k^{gr} .

Proposition 3.2.

$$(39) \quad \text{ch}_{q,z,u} C_k = \sum_{\mathbf{n}^-, \mathbf{n}^0, \mathbf{n}^+ \in \mathbb{Z}_{\geq 0}^k} u^{|\mathbf{n}^+| + |\mathbf{n}^0| + |\mathbf{n}^-|} z^{2(|\mathbf{n}^+| - |\mathbf{n}^-|)} \times \frac{q^{\frac{1}{2}(\mathbf{n}^+ A \mathbf{n}^+ + \mathbf{n}^0 A \mathbf{n}^0 + \mathbf{n}^- A \mathbf{n}^-) + \mathbf{n}^+ B \mathbf{n}^0 + \mathbf{n}^0 B \mathbf{n}^-}}{(q)_{\mathbf{n}^+} (q)_{\mathbf{n}^0} (q)_{\mathbf{n}^-}},$$

where for $\mathbf{n} \in \mathbb{Z}_{\geq 0}^k$ we set $|\mathbf{n}| = \sum_{i=1}^k i n_i$ and matrixes A and B are defined by

$$A_{i,j} = 2 \min(i, j), \quad B_{i,j} = \max(0, i + j - k).$$

Proof. Follows from the vertex operator realization of C_k constructed in Corollary 3.2 and Proposition 1.2. $\square$

Theorem 3.2. The (q, z, u) -character of L_k^{gr} is given by the right-hand side of (39).

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