NILPOTENT ORBITS IN THE DUAL OF CLASSICAL LIE ALGEBRAS IN CHARACTERISTIC 2 AND THE SPRINGER CORRESPONDENCE

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ABSTRACT. Let G be a simply connected algebraic group of type B , C or D over an algebraically closed field of characteristic 2. We construct a Springer correspondence for the dual vector space of the Lie algebra of G . In particular, we classify the nilpotent orbits in the duals of symplectic and orthogonal Lie algebras over algebraically closed or finite fields of characteristic 2.

1. INTRODUCTION

Throughout this paper, let $\mathbf{k}$ be a field of characteristic 2. Let G be an algebraic group of type B , C or D over $\mathbf{k}$ and let $\mathfrak{g}$ be its Lie algebra. Let $\mathfrak{g}^*$ be the dual vector space of $\mathfrak{g}$. We have a natural action of G on $\mathfrak{g}^*$, $g \cdot \xi(x) = \xi(\text{Ad}(g)^{-1}x)$ for $g \in G$, $\xi \in \mathfrak{g}^*$ and $x \in \mathfrak{g}$. Fix a Borel subgroup B of G and let $\mathfrak{b}$ be the Lie algebra of B . Let $\mathfrak{n}' = \{\xi \in \mathfrak{g}^* \mid \xi(\mathfrak{b}) = 0\}$. An element ξ in $\mathfrak{g}^*$ is called nilpotent if there exists $g \in G$ such that $g \cdot \xi \in \mathfrak{n}'$ (see [4]). We classify the nilpotent orbits in $\mathfrak{g}^*$ under the action of G in the cases where $\mathbf{k}$ is algebraically closed and where $\mathbf{k}$ is a finite field $\mathbf{F}_q$. In particular, we obtain the number of nilpotent orbits over $\mathbf{F}_q$ and the structure of component groups of the centralizers of nilpotent elements.

Let G_s be a simply connected algebraic group of type B , C or D defined over $\mathbf{k}$ (assume $\mathbf{k}$ algebraically closed) and let $\mathfrak{g}_s$ be the Lie algebra of G_s . Let $\mathfrak{g}_s^*$ be the dual vector space of $\mathfrak{g}_s$. Let $\mathfrak{A}'_s$ be the set of all pairs $(c', \mathcal{F}')$ where c' is a nilpotent G_s -orbit in $\mathfrak{g}_s^*$ and $\mathcal{F}'$ is an irreducible G_s -equivariant local system on c' (up to isomorphism). We construct a Springer correspondence for $\mathfrak{g}_s^*$ using a similar construction as in [7, 8, 13]. The correspondence is a bijective map from the set of isomorphism classes of irreducible representations of the Weyl group of G_s to the set $\mathfrak{A}'_s$.

2. SYMPLECTIC GROUPS

In this section we study the nilpotent orbits in $\mathfrak{g}^*$ where G is a symplectic group.

2.1. Let V be a vector space of dimension $2n$ over $\mathbf{k}$ equipped with a nondegenerate symplectic form $\beta : V \times V \rightarrow \mathbf{k}$. The symplectic group is defined as $G = \text{Sp}(2n) = \{g \in \text{GL}(V) \mid \beta(gv, gw) = \beta(v, w), \forall v, w \in V\}$ and its Lie algebra is $\mathfrak{g} = \mathfrak{sp}(2n) = \{x \in \mathfrak{gl}(V) \mid \beta(xv, w) + \beta(v, xw) = 0, \forall v, w \in V\}$.

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Let ξ be an element of $\mathfrak{g}^*$. There exists $X \in \mathfrak{gl}(V)$ such that $\xi(x) = \text{tr}(Xx)$ for any $x \in \mathfrak{g}$. We define a quadratic form $\alpha_\xi : V \rightarrow \mathbf{k}$ by

$$\alpha_\xi(v) = \beta(v, Xv).$$

Lemma 2.1. *The quadratic form α_ξ is well-defined.*

Proof. Recall that the space $\text{Quad}(V)$ of quadratic forms on V coincides with the second symmetric power $S^2(V^*)$ of V^* . Consider the following linear mapping

$$\Phi : \text{End}_{\mathbf{k}}(V) \rightarrow S^2(V^*) = \text{Quad}(V), \quad X \mapsto \alpha_X$$

where $\alpha_X(v) = \beta(v, Xv)$. It is easy to see that Φ is $G = Sp(V)$ -equivariant. One can show that $\ker \Phi$ coincides with the orthogonal complement $\mathfrak{g}^\perp$ of $\mathfrak{g} = \mathfrak{sp}(V)$ in $\text{End}_{\mathbf{k}}(V)$ under the nondegenerate trace form. It follows that α_ξ does not depend on the choice of X . $\square$

Remark 2.2. I thank the referee for suggesting the present coordinate free proofs of Lemmas 2.1, 2.6, 3.1, 3.5, 3.6, 3.11 and 4.2. These proofs replace my earlier proofs for which coordinates are used.

Let β_ξ be the symmetric bilinear form associated to α_ξ , namely, $\beta_\xi(v, w) = \alpha_\xi(v + w) - \alpha_\xi(v) - \alpha_\xi(w)$, $v, w \in V$. Define a linear map $T_\xi : V \rightarrow V$ by

$$\beta(T_\xi v, w) = \beta_\xi(v, w).$$

Assume $\xi \in \mathfrak{g}^*$. We denote by $(V_\xi, \beta, \alpha_\xi)$ the vector space V equipped with the symplectic form β and the quadratic form α_ξ .

Definition 2.3. Assume $\xi, \zeta \in \mathfrak{g}^*$. We say that $(V_\xi, \beta, \alpha_\xi)$ is equivalent to $(V_\zeta, \beta, \alpha_\zeta)$ if there exists a vector space isomorphism $g : V_\xi \rightarrow V_\zeta$ such that $\beta(gv, gw) = \beta(v, w)$ and $\alpha_\zeta(gv) = \alpha_\xi(v)$ for all $v, w \in V_\xi$.

Lemma 2.4. *Two elements $\xi, \zeta \in \mathfrak{g}^*$ lie in the same G -orbit if and only if there exists $g \in G$ such that $\alpha_\xi(g^{-1}v) = \alpha_\zeta(v)$, $\forall v \in V$.*

Proof. The two elements ξ, ζ lie in the same G -orbit if and only if there exists $g \in G$ such that $g.\xi(x) = \xi(g^{-1}xg) = \zeta(x)$, $\forall x \in \mathfrak{g}$. Assume $\xi(x) = \text{tr}(X_\xi x)$ and $\zeta(x) = \text{tr}(X_\zeta x)$. A similar argument as in the proof of Lemma 2.1 shows that $g.\xi(x) = \text{tr}(gX_\xi g^{-1}x) = \zeta(x)$ if and only if $\beta(gX_\xi g^{-1}v, v) + \beta(X_\zeta v, v) = 0$ if and only if $\alpha_\xi(g^{-1}v) = \alpha_\zeta(v)$, $\forall v \in V$. $\square$

Corollary 2.5. *Two elements $\xi, \zeta \in \mathfrak{g}^*$ lie in the same G -orbit if and only if $(V_\xi, \beta, \alpha_\xi)$ is equivalent to $(V_\zeta, \beta, \alpha_\zeta)$.*

2.2. From now on we assume that $\xi \in \mathfrak{g}^*$ is nilpotent.

Lemma 2.6. *Let $\xi \in \mathfrak{g}^*$ be nilpotent. Then T_ξ is a nilpotent element in $\mathfrak{gl}(V_\xi)$.*

Proof. Note that the nilpotent elements in $\mathfrak{g}^*$ (resp. $\mathfrak{g}$) are precisely the “unstable” vectors ξ (resp. x), namely, those ξ (resp. x) for which the closure of the G -orbit $G.\xi$ (resp. $\text{Ad}(G)x$) contains 0. By Hilbert’s criterion for instability, there exists a co-character $\phi : \mathbf{G}_m \rightarrow G$ such that $\lim_{a \rightarrow 0} \phi(a).\xi = 0$. To show that T_ξ is nilpotent, it is enough to show that $\lim_{a \rightarrow 0} \text{Ad}(\phi(a))T_\xi = 0$.

For any G -representation M and $i \in \mathbb{Z}$, we write $M(\phi; i)$ for the i -weight space of the torus $\{\phi(a)\}_{a \in \mathbf{G}_m}$ and $M(\phi; > i) = \bigoplus_{j > i} M(\phi; j)$, and similarly for $M(\phi; \geq i)$, $M(\phi; \leq i)$, etc.

Since $\xi \in \mathfrak{g}^*(\phi, > 0)$, we may choose $X \in \text{End}_{\mathbf{k}}(V)(\phi; > 0)$ such that $\xi(x) = \text{tr}(Xx)$ for all $x \in \mathfrak{g}$. Notice that we have

$$\begin{aligned} \beta((\text{Ad}(\phi(a))T_\xi)v, w) &= \beta(T_\xi\phi(a)^{-1}v, \phi(a)^{-1}w) = \beta_\xi(\phi(a)^{-1}v, \phi(a)^{-1}w) \\ &= \beta(X\phi(a)^{-1}v, \phi(a)^{-1}w) + \beta(\phi(a)^{-1}v, X\phi(a)^{-1}w) \\ &= \beta((\text{Ad}(\phi(a))X)v, w) + \beta(v, (\text{Ad}(\phi(a))X)w). \end{aligned}$$

Since $X \in \text{End}_{\mathbf{k}}(V)(\phi; > 0)$, $\text{Ad}(\phi(a))X \rightarrow 0$ as $a \rightarrow 0$ and thus $\beta(\text{Ad}(\phi(a))T_\xi v, w) \rightarrow 0$ as $a \rightarrow 0$ for any $v, w \in V$. It follows that $\text{Ad}(\phi(a))T_\xi \rightarrow 0$ as $a \rightarrow 0$, since the bilinear form β is nondegenerate. Thus T_ξ is nilpotent. $\square$

Let $A = \mathbf{k}[[t]]$ be the ring of formal power series in the indeterminate t . We consider V_ξ as an A -module by $(\sum a_k t^k)v = \sum a_k T_\xi^k v$. Let E be the vector space spanned by the linear functionals $t^{-k} : A \rightarrow \mathbf{k}$, $\sum a_i t^i \mapsto a_k$, $k \geq 0$. Let E_0 and E_1 be the subspace $\sum \mathbf{k}t^{-2k}$ and $\sum \mathbf{k}t^{-2k-1}$ respectively. Denote $\pi_i : E \rightarrow E_i$, $i = 0, 1$ the natural projections. The vector space E is considered as an A -module by $(au)(b) = u(ab)$ for $a, b \in A, u \in E$. Define $\varphi : V \times V \rightarrow E, \psi : V \rightarrow E_1$ and $\varphi_\xi : V \times V \rightarrow E, \psi_\xi : V \rightarrow E_0$ by

$$\varphi(v, w) = \sum_{k \geq 0} \beta(t^k v, w) t^{-k}, \quad \psi(v) = \sum_{k \geq 0} \beta(t^{k+1} v, t^k v) t^{-2k-1}$$

and

$$\varphi_\xi(v, w) = \sum_{k \geq 0} \beta_\xi(t^k v, w) t^{-k}, \quad \psi_\xi(v) = \sum_{k \geq 0} \alpha_\xi(t^k v) t^{-2k}.$$

Notice that we have $\beta(T_\xi v, v) = \beta_\xi(v, v) = 0$ and $\beta_\xi(T_\xi v, v) = \beta(T_\xi v, T_\xi v) = 0$. By Proposition 2.7 in [2], we can identify $(V_\xi, \alpha = 0, \beta)$ with (V_ξ, φ, ψ) , $(V_\xi, \alpha_\xi, \beta_\xi)$ with $(V_\xi, \varphi_\xi, \psi_\xi)$ and hence $(V_\xi, \beta, \alpha_\xi)$ with $(V_\xi, \varphi, \psi, \varphi_\xi, \psi_\xi)$. The mappings φ, ψ and φ_ξ, ψ_ξ satisfy the following properties ([2]):

- (i) The maps $\varphi(\cdot, w)$ and $\varphi_\xi(\cdot, w)$ are A -linear for every $w \in V_\xi$.
- (ii) $\varphi(v, w) = \varphi(w, v)$, $\varphi_\xi(v, w) = \varphi_\xi(w, v)$ for all $v, w \in V_\xi$.
- (iii) $\varphi(v, v) = \psi(v)$, $\varphi_\xi(v, v) = 0$ for all $v \in V_\xi$.
- (vi) $\psi(v + w) = \psi(v) + \psi(w)$, $\psi_\xi(v + w) = \psi_\xi(v) + \psi_\xi(w) + \pi_0(\varphi_\xi(v, w))$ for all $v, w \in V_\xi$.
- (v) $\psi(av) = a^2 \psi(v)$, $\psi_\xi(av) = a^2 \psi_\xi(v)$ for all $v \in V_\xi$, $a \in A$.

Following [2], we call $(V_\xi, \beta, \alpha_\xi)$ a form module and $(V_\xi, \varphi, \psi, \varphi_\xi, \psi_\xi)$ an abstract form module. Corollary 2.5 says that classifying the nilpotent G -orbits in $\mathfrak{g}^*$ is equivalent to classifying the equivalence classes of the form modules $(V_\xi, \beta, \alpha_\xi)$. In the following we classify the form modules $(V_\xi, \beta, \alpha_\xi)$ via the identification with $(V_\xi, \varphi, \psi, \varphi_\xi, \psi_\xi)$. We write $V_\xi = (V_\xi, \beta, \alpha_\xi)$.

Since T_ξ is nilpotent, there exists a unique sequence of integers $p_1 \geq \dots \geq p_s \geq 1$ and a family of vectors $v_1, \dots, v_s$ such that $T_\xi^{p_i} v_i = 0$ and the vectors $T_\xi^{q_i} v_i$, $0 \leq q_i \leq p_i - 1$ form a basis of V . We define $p(V_\xi) = p(T_\xi) = (p_1, \dots, p_s)$. Define an index function $\chi_{V_\xi} : \mathbb{Z} \rightarrow \mathbb{N}$ for $(V_\xi, \beta, \alpha_\xi)$ by

$$\chi_{V_\xi}(m) = \min\{i \geq 0 \mid T_\xi^m v = 0 \Rightarrow \alpha_\xi(T_\xi^i v) = 0\}.$$

Define $\mu(V_\xi)$ to be the minimal integer $m \geq 0$ such that $T_\xi^m V_\xi = 0$. For $v \in V_\xi$, we define $\mu(v) = \mu(Av)$. We define $\mu(E)$ for E and $\mu(u)$ for $u \in E$ similarly.

Lemma 2.7. *We have $\psi(v) = 0$ and $\varphi_\xi(v, w) = t\varphi(v, w)$ for all $v, w \in V_\xi$.*

Proof. The first assertion follows from $\beta(T_\xi v, v) = 0, \forall v \in V_\xi$. The second assertion follows from $\beta_\xi(T_\xi^k v, w) = \beta(T_\xi^{k+1} v, w)$. $\square$

We study the orthogonal decomposition of V_ξ with respect to φ , which is also an orthogonal decomposition of V_ξ with respect to φ_ξ since $\varphi(v, w) = 0$ implies $\varphi_\xi(v, w) = 0$ (Lemma 2.7). Recall that an orthogonal decomposition of V is an expression of V as a direct sum $V = \sum_{i=1}^r V_i$ of mutually orthogonal submodules V_i . A form module V is called indecomposable if $V \neq 0$ and for every orthogonal decomposition $V = V_1 \oplus V_2$ we have $V_1 = 0$ or $V_2 = 0$. Every form module V has some orthogonal decomposition $V = \sum_{i=1}^r V_i$ in indecomposable submodules $V_1, V_2, \dots, V_r$.

We first classify the indecomposable modules (with respect to φ) which appear in the orthogonal decompositions of form modules $(V_\xi, \beta, \alpha_\xi)$. Let $(V_\xi, \varphi, \psi, \varphi_\xi, \psi_\xi)$ be an indecomposable module, where $\xi \in \mathfrak{g}^*$ is nilpotent. Since $\psi(v) = 0$ for all $v \in V_\xi$ (Lemma 2.7), by the classification of modules (V_ξ, φ, ψ) , there exist v_1, v_2 such that $V_\xi = Av_1 \oplus Av_2$ with $\mu(v_1) = \mu(v_2) = m$ and $\varphi(v_1, v_2) = t^{1-m}$ (see [2] section 3.5, notice that β is nondegenerate on V_ξ). Denote $\psi_\xi(v_1) = \Psi_1, \psi_\xi(v_2) = \Psi_2$ and $\varphi_\xi(v_1, v_2) = t^{2-m} = \Phi_\xi$.

2.3. In this subsection assume $\mathbf{k}$ is algebraically closed.

Proposition 2.8. *The indecomposable modules are ${}^*W_l(m) = Av_1 \oplus Av_2, [\frac{m}{2}] \leq l \leq m$, with $\mu(v_1) = \mu(v_2) = m, \psi_\xi(v_1) = t^{2-2l}, \psi_\xi(v_2) = 0$ and $\varphi(v_1, v_2) = t^{1-m}$. We have $\chi_{{}^*W_l(m)} = [m; l]$, where $[m; l] : \mathbb{N} \rightarrow \mathbb{Z}$ is defined by $[m; l](k) = \max\{0, \min\{k - m + l, l\}\}$.*

Proof. Assume $\mu(\Psi_1) \geq \mu(\Psi_2)$. Let $v'_2 = v_2 + av_1$. The equation $\psi_\xi(v'_2) = \Psi_2 + a^2\Psi_1 + \pi_0(a\Phi_\xi) = 0$ has a solution for a , hence we can assume $\Psi_2 = 0$. Assume $\Psi_1 = \sum_{i=0}^l a_i t^{-2i}, a_i \in \mathbf{k}, a_l \neq 0$. Let $v'_1 = av_1, a \in A$. We can take a invertible in A such that $\psi_\xi(v'_1) = t^{-2l}$. Let $v''_2 = a^{-1}v'_2$. One verifies that $\psi_\xi(v'_1) = t^{-2l}, \psi_\xi(v''_2) = 0$ and $\varphi(v'_1, v''_2) = t^{1-m}$. Furthermore, we can assume $[m/2] - 1 \leq l \leq m - 1$. In fact, we have $l \leq m - 1$ since $t^m v = 0, \forall v \in V$; if $l < [\frac{m}{2}] - 1$, let $v'_1 = v_1 + t^{m-2l-2}v_2 + t^{m-2[\frac{m}{2}]}v_2$, then $\psi_\xi(v'_1) = t^{-2([\frac{m}{2}]-1)}$ and $\varphi(v'_1, v_2) = t^{1-m}$. One can verify that the modules ${}^*W_l(m), [m/2] \leq l \leq m$ exist and are not equivalent to each other. $\square$

Lemma 2.9. *Assume $m_1 \geq m_2$.*

- (i) *If $l_1 < l_2$, we have ${}^*W_{l_1}(m_1) \oplus {}^*W_{l_2}(m_2) \cong {}^*W_{l_2}(m_1) \oplus {}^*W_{l_2}(m_2)$.*
- (ii) *If $m_1 - l_1 < m_2 - l_2$, we have ${}^*W_{l_1}(m_1) \oplus {}^*W_{l_2}(m_2) \cong {}^*W_{l_1}(m_1) \oplus {}^*W_{m_2-m_1+l_1}(m_2)$.*

Proof. Assume ${}^*W_{l_1}(m_1) \oplus {}^*W_{l_2}(m_2) = Av_1 \oplus Aw_1 \oplus Av_2 \oplus Aw_2$ with $\psi_\xi(v_i) = t^{2-2l_i}, \psi_\xi(w_i) = 0$ and $\varphi(v_i, w_j) = \delta_{i,j}t^{1-m_i}, \varphi(v_i, v_j) = \varphi(w_i, w_j) = 0, i, j = 1, 2$. Let $\tilde{v}_1 = v_1 + (1+t^{l_2-l_1})v_2, \tilde{w}_1 = w_1, \tilde{v}_2 = v_2, \tilde{w}_2 = w_2 + (t^{m_1-m_2} + t^{m_1-l_1-m_2+l_2})w_1$. Then we have $\psi_\xi(\tilde{v}_i) = t^{2-2l_i}, \psi_\xi(\tilde{w}_i) = 0$ and $\varphi(\tilde{v}_i, \tilde{w}_j) = \delta_{i,j}t^{1-m_i}, \varphi(\tilde{v}_i, \tilde{v}_j) = \varphi(\tilde{w}_i, \tilde{w}_j) = 0, i, j = 1, 2$. This proves (i). One can prove (ii) similarly. $\square$

Remark 2.10. Notice that we do not have a “Krull-Schmidt” type theorem here, namely, the indecomposable summands of a form module V are not uniquely determined by V . (See also Lemma 2.14 and Lemma 3.16.)

By Proposition 2.8 and Lemma 2.9, for every module V , there exists a unique sequence of modules ${}^*W_{l_i}(m_i)$ such that V is equivalent to ${}^*W_{l_1}(m_1) \oplus {}^*W_{l_2}(m_2) \oplus \cdots \oplus {}^*W_{l_s}(m_s)$, $[\frac{m_i}{2}] \leq l_i \leq m_i$, $m_1 \geq m_2 \geq \cdots \geq m_s$, $l_1 \geq l_2 \geq \cdots \geq l_s$ and $m_1 - l_1 \geq m_2 - l_2 \geq \cdots \geq m_s - l_s$. Thus the equivalence class of V is characterized by the symbol

$$(m_1)_{l_1}^2 \cdots (m_s)_{l_s}^2.$$

A symbol of the above form is the symbol of a form module if and only if $[\frac{m_i}{2}] \leq l_i \leq m_i$, $m_i \geq m_{i+1}$, $l_i \geq l_{i+1}$ and $m_i - l_i \geq m_{i+1} - l_{i+1}$, $i = 1, \dots, s$.

It follows that $p(V_\xi) = m_1^2 \cdots m_s^2$ and $\chi_{V_\xi}(k) = \sup_i \chi_{{}^*W_{l_i}(m_i)}(k)$ for all $k \in \mathbb{N}$ and $\chi_V(m_i) = \chi_{{}^*W_{l_i}(m_i)} = l_i$. Thus we have the following proposition.

Proposition 2.11. *Two nilpotent elements $\xi, \zeta \in \mathfrak{g}^*$ lie in the same G -orbit if and only if T_ξ, T_ζ are conjugate in $GL(V)$ and $\chi(V_\xi) = \chi(V_\zeta)$.*

We associate to the orbit $(m_1)_{l_1}^2 \cdots (m_s)_{l_s}^2$ a pair of partitions $(l_1, \dots, l_s)(m_1 - l_1, \dots, m_s - l_s)$. In this way we construct a bijection from the set of nilpotent orbits in $\mathfrak{g}^*$ to the set $\{(\mu, \nu) \mid |\mu| + |\nu| = n, \nu_i \leq \mu_i + 1\}$, which has cardinality $p_2(n) - p_2(n-2)$. Here and afterwards we denote by $p_2(n)$ the number of pairs of partitions (μ, ν) such that $|\mu| + |\nu| = n$.

2.4. In this subsection, let $\mathbf{k} = \mathbf{F}_q$. Let $G(\mathbf{F}_q)$, $\mathfrak{g}(\mathbf{F}_q)$ be the fixed points of a Frobenius map $\mathfrak{F}_q$ relative to $\mathbf{F}_q$ on G , $\mathfrak{g}$. We study the nilpotent $G(\mathbf{F}_q)$ -orbits in $\mathfrak{g}(\mathbf{F}_q)^*$. Fix

$$\delta \notin \{x^2 + x \mid x \in \mathbf{F}_q\}.$$

We have the following statements whose proofs are entirely similar to those of [13]. For completeness, we also include the proofs here.

Proposition 2.12. *The indecomposable modules over $\mathbf{F}_q$ are*

- (i) ${}^*W_l^0(m) = Av_1 \oplus Av_2$, $(m-1)/2 \leq l \leq m$ with $\psi_\xi(v_1) = t^{2-2l}$, $\psi_\xi(v_2) = 0$ and $\varphi(v_1, v_2) = t^{1-m}$;
- (ii) ${}^*W_l^\delta(m) = Av_1 \oplus Av_2$, $(m-1)/2 < l < m$ with $\psi_\xi(v_1) = t^{2-2l}$, $\psi_\xi(v_2) = \delta t^{-2(m-1-l)}$ and $\varphi(v_1, v_2) = t^{1-m}$.

Proof. Let $V_\xi = Av_1 \oplus Av_2$ be an indecomposable module as in the last paragraph of subsection 2.2. We have $\Phi_\xi = t^{2-m}$. We can assume that $\mu(\Psi_1) \geq \mu(\Psi_2)$. We have the following cases:

Case 1: $\Psi_1 = \Psi_2 = 0$. Let $\tilde{v}_1 = v_1 + t^{m-2[\frac{m}{2}]}v_2$, $\tilde{v}_2 = v_2$, then we have $\psi_\xi(\tilde{v}_1) = t^{2-2[\frac{m}{2}]}$, $\psi_\xi(\tilde{v}_2) = 0$ and $\varphi(\tilde{v}_1, \tilde{v}_2) = t^{1-m}$.

Case 2: $\Psi_1 \neq 0$, $\Psi_2 = 0$. There exist $a, b \in A$ invertible, such that $\psi_\xi(av_1) = t^{-2k}$, $\psi_\xi(bv_2) = 0$ and $\varphi(av_1, bv_2) = t^{1-m}$. Hence we can assume $\Psi_1 = t^{-2k}$ where $k \leq m-1$. If $k < [\frac{m}{2}] - 1$, let $\tilde{v}_1 = v_1 + t^{m-2[\frac{m}{2}]}v_2 + t^{m-2k-2}v_2$, $\tilde{v}_2 = v_2$; otherwise, let $\tilde{v}_1 = v_1$, $\tilde{v}_2 = v_2$. Then we get $\psi_\xi(\tilde{v}_1) = t^{-2k}$, $[\frac{m}{2}] - 1 \leq k \leq m-1$, $\psi_\xi(\tilde{v}_2) = 0$, $\varphi(\tilde{v}_1, \tilde{v}_2) = t^{1-m}$.

Case 3: $\Psi_1 \neq 0$, $\Psi_2 \neq 0$. There exist $a, b \in A$ invertible, such that $\psi_\xi(av_1) = t^{-2l_1}$ and $\varphi(av_1, bv_2) = t^{1-m}$. Hence we can assume $\Psi_1 = t^{-2l_1}$ and $\Psi_2 = \sum_{i=0}^{l_2} a_i t^{-2i}$ where $l_2 \leq l_1 \leq m-1$. Let $\tilde{v}_2 = v_2 + \sum_{i=0}^{m-1} x_i t^i v_1$. Assume $l_1 < \frac{m-2}{2}$, then $\psi_\xi(\tilde{v}_2) = 0$ has a solution for x_i 's and we get Case 2. Assume $l_1 \geq \frac{m-2}{2}$. If $a_{m-l_1-2} \in \{x^2 + x \mid x \in \mathbf{F}_q\}$, then $\psi_\xi(\tilde{v}_2) = 0$ has a solution for x_i 's and we get Case 2; if $a_{m-l_1-2} \notin \{x^2 + x \mid x \in \mathbf{F}_q\}$, then $\psi_\xi(\tilde{v}_2) = \delta t^{-2(m-l_1-2)}$ has a solution for x_i 's.

Summarizing Cases 1–3, we have normalized $V_\xi = Av_1 \oplus Av_2$ with $\mu(v_1) = \mu(v_2) = m$ as follows:

(i) $(m-1)/2 \leq \chi(m) = l \leq m$, $\psi_\xi(v_1) = t^{2-2l}$, $\psi_\xi(v_2) = 0$, $\varphi(v_1, v_2) = t^{1-m}$, denoted by $*W_l^0(m)$.

(ii) $(m-1)/2 < \chi(m) = l < m$, $\psi_\xi(v_1) = t^{2-2l}$, $\psi_\xi(v_2) = \delta t^{-2(m-l-1)}$, $\varphi(v_1, v_2) = t^{1-m}$, denoted by $*W_l^\delta(m)$.

We show that $*W_l^0(m)$ and $*W_l^\delta(m)$, where $\frac{m-1}{2} < l < m$, are not equivalent. Take v_i, w_i , $i = 1, 2$, such that $*W_l^0(m) = Av_1 \oplus Aw_1$, $*W_l^\delta(m) = Av_2 \oplus Aw_2$, $\mu(v_i) = \mu(w_i) = m$, $\psi_\xi(v_i) = t^{2-2l}$, $\psi_\xi(w_1) = 0$, $\psi_\xi(w_2) = \delta t^{2l-2m+2}$ and $\varphi(v_i, w_i) = t^{1-m}$, $i = 1, 2$. The modules $*W_l^0(m)$ and $*W_l^\delta(m)$ are equivalent if and only if there exists a linear isomorphism $g : *W_l^0(m) \rightarrow *W_l^\delta(m)$ such that $\psi_\xi(gv) = \psi_\xi(v)$ and $\varphi(gv, gw) = \varphi(v, w)$ for all $v, w \in *W_l^0(m)$. Assume $gv_1 = \sum_{i=0}^{m-1} (a_i t^i v_2 + b_i t^i w_2)$, $gw_1 = \sum_{i=0}^{m-1} (c_i t^i v_2 + d_i t^i w_2)$. Then a straightforward calculation shows that if $l = \frac{m}{2}$, among the equations $\psi_\xi(gv_1) = \psi_\xi(v_1)$, $\psi_\xi(gw_1) = \psi_\xi(w_1)$, $\varphi(gv_1, gw_1) = \varphi(v_1, w_1)$, the following equations appear:

$$\begin{aligned} c_0^2 + \delta d_0^2 + c_0 d_0 &= 0, \\ a_0 d_0 + b_0 c_0 &= 1. \end{aligned}$$

It follows that $c_0, d_0 \neq 0$ and thus the first equation becomes an ‘‘Artin-Schreier’’ equation $(\frac{c_0}{d_0})^2 + \frac{c_0}{d_0} = \delta$, which has no solutions over $\mathbf{F}_q$. Similarly, if $\frac{m}{2} < l < m$, an ‘‘Artin-Schreier’’ equation $c_{2l-m}^2 + c_{2l-m} = \delta$ appears. It follows that $*W_l^0(m)$ and $*W_l^\delta(m)$, where $\frac{m-1}{2} < l < m$, are not equivalent. $\square$

Remark 2.13. It follows that the equivalence class of the form module $*W_l(m)$ over $\bar{\mathbf{F}}_q$ remains as one equivalence class over $\mathbf{F}_q$ when $l = \frac{m-1}{2}$ or $l = m$ and decomposes into two equivalence classes $*W_l^0(m)$ and $*W_l^\delta(m)$ over $\mathbf{F}_q$ otherwise.

Lemma 2.14. Assume $l_1 \geq l_2$ and $m_1 - l_1 \geq m_2 - l_2$.

(i) If $l_1 + l_2 < m_1$, we have that $*W_{l_1}^0(m_1) \oplus *W_{l_2}^0(m_2)$, $*W_{l_1}^\delta(m_1) \oplus *W_{l_2}^\delta(m_2)$, $*W_{l_1}^0(m_1) \oplus *W_{l_2}^\delta(m_2)$ and $*W_{l_1}^\delta(m_1) \oplus *W_{l_2}^0(m_2)$ are not equivalent to each other.

(ii) If $l_1 + l_2 \geq m_1$, we have $*W_{l_1}^0(m_1) \oplus *W_{l_2}^0(m_2) \cong *W_{l_1}^\delta(m_1) \oplus *W_{l_2}^\delta(m_2)$ and $*W_{l_1}^0(m_1) \oplus *W_{l_2}^\delta(m_2) \cong *W_{l_1}^\delta(m_1) \oplus *W_{l_2}^0(m_2)$. The two pairs are not equivalent to each other.

Proof. We show that $*W_{l_1}^0(m_1) \oplus *W_{l_2}^\delta(m_2)$ and $*W_{l_1}^\delta(m_1) \oplus *W_{l_2}^0(m_2)$ are equivalent if and only if $l_1 + l_2 \geq m_1$. The other statements are proved similarly. Assume $*W_{l_1}^0(m_1) \oplus *W_{l_2}^\delta(m_2)$ and $*W_{l_1}^\delta(m_1) \oplus *W_{l_2}^0(m_2)$ correspond to ξ and ξ' , respectively. Take v_1, w_1 and v_2, w_2 such that $*W_{l_1}^0(m_1) \oplus *W_{l_2}^\delta(m_2) = Av_1 \oplus Aw_1 \oplus Av_2 \oplus Aw_2$ and $\psi_\xi(v_1) = t^{2-2l_1}$, $\psi_\xi(w_1) = 0$, $\varphi(v_1, w_1) = t^{1-m_1}$, $\psi_\xi(v_2) = t^{2-2l_2}$, $\psi_\xi(w_2) = \delta t^{-2(m_2-l_2-1)}$, $\varphi(v_2, w_2) = t^{1-m_2}$, $\varphi(v_1, v_2) = \varphi(v_1, w_2) = \varphi(w_1, v_2) = \varphi(w_1, w_2) = 0$. Similarly, take v'_1, w'_1 and v'_2, w'_2 such that $*W_{l_1}^\delta(m_1) \oplus *W_{l_2}^0(m_2) = Av'_1 \oplus Aw'_1 \oplus Av'_2 \oplus Aw'_2$ and $\psi_{\xi'}(v'_1) = t^{2-2l_1}$, $\psi_{\xi'}(w'_1) = t^{-2(m_1-l_1-1)}$, $\varphi(v'_1, w'_1) = t^{1-m_1}$, $\psi_{\xi'}(v'_2) = t^{2-2l_2}$, $\psi_{\xi'}(w'_2) = 0$, $\varphi(v'_2, w'_2) = t^{1-m_2}$, $\varphi(v'_1, v'_2) = \varphi(v'_1, w'_2) = \varphi(w'_1, v'_2) = \varphi(w'_1, w'_2) = 0$.

The form modules $*W_{l_1}^0(m_1) \oplus *W_{l_2}^\delta(m_2)$ and $*W_{l_1}^\delta(m_1) \oplus *W_{l_2}^0(m_2)$ are equivalent if and only if there exists an A -module isomorphism $g : V \rightarrow V$ such that

$\psi_{\xi'}(gv) = \psi_{\xi}(v)$, $\varphi(gv, gw) = \varphi(v, w)$ for any $v, w \in V$. Assume

$$\begin{aligned} gv_j &= \sum_{i=0}^{m_1-1} (a_{j,i}t^i v'_1 + b_{j,i}t^i w'_1) + \sum_{i=0}^{m_2-1} (c_{j,i}t^i v'_2 + d_{j,i}t^i w'_2), \\ gw_j &= \sum_{i=0}^{m_1-1} (e_{j,i}t^i v'_1 + f_{j,i}t^i w'_1) + \sum_{i=0}^{m_2-1} (g_{j,i}t^i v'_2 + h_{j,i}t^i w'_2), j = 1, 2. \end{aligned}$$

Then $*W_{l_1}^0(m_1) \oplus *W_{l_2}^{\delta}(m_2)$ and $*W_{l_1}^{\delta}(m_1) \oplus *W_{l_2}^0(m_2)$ are equivalent if and only if the equations $\psi_{\xi'}(gv_i) = \psi_{\xi}(v_i)$, $\psi_{\xi'}(gw_i) = \psi_{\xi}(w_i)$, $\varphi(gv_i, gv_j) = \varphi(v_i, v_j)$, $\varphi(gv_i, gw_j) = \varphi(v_i, w_j)$, $\varphi(gw_i, gw_j) = \varphi(w_i, w_j)$, $i, j = 1, 2$, have solutions.

If $l_1 + l_2 < m_1$, some equations are $e_{1,2l_1-m_1}^2 + e_{1,2l_1-m_1} = \delta$ (if $l_1 \neq \frac{m_1}{2}$) or $e_{1,0}^2 + e_{1,0}f_{1,0} + \delta f_{1,0}^2 = 0$, $a_{1,0}f_{1,0} + b_{1,0}e_{1,0} = 1$ (if $l_1 = \frac{m_1}{2}$). As in the proof of Proposition 2.12, we get “Artin-Schreier” equations which have no solutions for $e_{1,2l_1-m_1}$ or $e_{1,0}, f_{1,0}$ in $\mathbf{F}_q$. Hence $*W_{l_1}^0(m_1) \oplus *W_{l_2}^{\delta}(m_2)$ and $*W_{l_1}^{\delta}(m_1) \oplus *W_{l_2}^0(m_2)$ are not equivalent.

If $l_1 + l_2 \geq m_1$, let $gv_1 = v'_1$, $gw_1 = w'_1 + \sqrt{\delta}t^{l_1+l_2-m_1}v'_2$, $gv_2 = v'_2$, $gw_2 = w'_2 + \sqrt{\delta}t^{l_1+l_2-m_2}v'_1$, then this is a solution for the equations. It follows that $*W_{l_1}^0(m_1) \oplus *W_{l_2}^{\delta}(m_2) \cong *W_{l_1}^{\delta}(m_1) \oplus *W_{l_2}^0(m_2)$.

□

Proposition 2.15. *The equivalence class of the module*

$$\begin{aligned} *W_{l_1}(m_1) \oplus \cdots \oplus *W_{l_s}(m_s), [\frac{m_i}{2}] \leq l_i \leq m_i, \\ l_i \geq l_{i+1}, m_i - l_i \geq m_{i+1} - l_{i+1}, i = 1, \dots, s, \end{aligned}$$

over $\bar{\mathbf{F}}_q$ decomposes into at most 2^k equivalence classes over $\mathbf{F}_q$, where

$$k = \#\{1 \leq i \leq s \mid l_i + l_{i+1} < m_i \text{ and } l_i > \frac{m_i - 1}{2}\}.$$

Proof. By Proposition 2.12 and Remark 2.13, it is enough to show that from modules of the form $*W_{l_1}^{\epsilon'_1}(m_1) \oplus \cdots \oplus *W_{l_s}^{\epsilon'_s}(m_s)$, where $\epsilon'_i = 0$ or δ , have at most 2^k equivalence classes. Suppose $i_1, i_2, \dots, i_k$ are such that $1 \leq i_j \leq s$, $l_{i_j} + l_{i_{j+1}} < m_{i_j}$, $l_{i_j} > \frac{m_{i_j} - 1}{2}$, $j = 1, \dots, k$. Using Lemma 2.14 one can easily show that a module of the above form is isomorphic to one of the following modules: $V_1^{\epsilon_1} \oplus \cdots \oplus V_k^{\epsilon_k}$, where $V_t^{\epsilon_t} = *W_{l_{i_t-1}+1}^0(m_{i_t-1}+1) \oplus \cdots \oplus *W_{l_{i_t-1}}^0(m_{i_t-1}) \oplus *W_{m_{i_t}}^{\epsilon_t}(m_{i_t})$, $t = 1, \dots, k-1$, $i_0 = 0$, and $V_k = *W_{l_{i_k-1}+1}^0(m_{i_k-1}+1) \oplus \cdots \oplus *W_{l_{i_k}}^{\epsilon_k}(m_{i_k}) \oplus *W_{l_{i_k}+1}^0(m_{i_k}+1) \oplus \cdots \oplus *W_{l_s}^0(m_s)$, $\epsilon_t = 0$ or δ , $t = 1, \dots, k$. Thus the proposition is proved. □

Corollary 2.16. *The nilpotent orbit $(m_1)_{l_1}^2 \cdots (m_s)_{l_s}^2$ in $\mathfrak{g}(\bar{\mathbf{F}}_q)^*$ splits into at most 2^k $G(\mathbf{F}_q)$ -orbits in $\mathfrak{g}(\mathbf{F}_q)^*$.*

Proposition 2.17. *The number of nilpotent $G(\mathbf{F}_q)$ -orbits in $\mathfrak{g}(\mathbf{F}_q)^*$ is at most $p_2(n)$.*

Proof. Recall that we have mapped the nilpotent orbits in $\mathfrak{g}(\bar{\mathbf{F}}_q)^*$ bijectively to the set $\{(\mu, \nu) \mid |\mu| + |\nu| = n, \nu_i \leq \mu_i + 1\} := \Delta$. By Corollary 2.16, a nilpotent orbit in $\mathfrak{g}(\bar{\mathbf{F}}_q)^*$ corresponding to $(\mu, \nu) \in \Delta$, $\mu = (\mu_1, \mu_2, \dots, \mu_s)$, $\nu = (\nu_1, \nu_2, \dots, \nu_s)$ splits into at most 2^k orbits in $\mathfrak{g}(\mathbf{F}_q)^*$, where $k = \#\{1 \leq i \leq s \mid \mu_{i+1} + 1 \leq \nu_i < \mu_i + 1\}$.

We associate to the orbit 2^k pairs of partitions as follows. Suppose $r_1, r_2, \dots, r_k$ are such that $\mu_{r_i+1} + 1 \leq \nu_{r_i} < \mu_{r_i} + 1, i = 1, \dots, k$ and let

$$\begin{aligned}\mu^{1,i} &= (\mu_{r_{i-1}+1}, \dots, \mu_{r_i}), \nu^{1,i} = (\nu_{r_{i-1}+1}, \dots, \nu_{r_i}), \\ \mu^{2,i} &= (\nu_{r_{i-1}+1} - 1, \dots, \nu_{r_i} - 1), \nu^{2,i} = (\mu_{r_{i-1}+1} + 1, \dots, \mu_{r_i} + 1), i = 1, \dots, k, \\ \mu^{k+1} &= (\mu_{r_k+1}, \dots, \mu_s), \nu^{k+1} = (\nu_{r_k+1}, \dots, \nu_s).\end{aligned}$$

We associate to (μ, ν) the pairs of partitions $(\tilde{\mu}^{\epsilon_1, \dots, \epsilon_k}, \tilde{\nu}^{\epsilon_1, \dots, \epsilon_k})$,

$$\tilde{\mu}^{\epsilon_1, \dots, \epsilon_k} = (\mu^{\epsilon_1, 1}, \mu^{\epsilon_2, 2}, \dots, \mu^{\epsilon_k, k}, \mu^{k+1}), \tilde{\nu}^{\epsilon_1, \dots, \epsilon_k} = (\nu^{\epsilon_1, 1}, \nu^{\epsilon_2, 2}, \dots, \nu^{\epsilon_k, k}, \nu^{k+1}),$$

where $\epsilon_i \in \{1, 2\}, i = 1, \dots, k$. Notice that the pairs of partitions $(\tilde{\mu}^{\epsilon_1, \dots, \epsilon_k}, \tilde{\nu}^{\epsilon_1, \dots, \epsilon_k})$ are distinct and among them only $(\mu, \nu) = (\tilde{\mu}^{1, \dots, 1}, \tilde{\nu}^{1, \dots, 1})$ is in Δ . One can verify that the set of all pairs of partitions constructed as above for all $(\mu, \nu) \in \Delta$ is in bijection with the set $\{(\mu, \nu) \mid |\mu| + |\nu| = n\}$, which has cardinality $p_2(n)$. It follows that the number of nilpotent orbits in $\mathfrak{g}(\mathbf{F}_q)^*$ is less than $p_2(n)$. $\square$

3. ODD ORTHOGONAL GROUPS

In this section we study the nilpotent orbits in $\mathfrak{g}^*$ where G is an odd orthogonal group.

3.1. Let V be a vector space of dimension $2n + 1$ over $\mathbf{k}$ equipped with a non-degenerate quadratic form $\alpha : V \rightarrow \mathbf{k}$. Let $\beta : V \times V \rightarrow \mathbf{k}$ be the bilinear form associated to α . The odd orthogonal group is defined as $G = O(2n + 1) = \{g \in GL(V) \mid \alpha(gv) = \alpha(v), \forall v \in V\}$ and its Lie algebra is $\mathfrak{g} = \mathfrak{o}(2n + 1) = \{x \in \mathfrak{gl}(V) \mid \beta(xv, v) = 0, \forall v \in V \text{ and } \text{tr}(x) = 0\}$. Let ξ be an element of $\mathfrak{g}^*$. There exists $X \in \mathfrak{gl}(V)$ such that $\xi(x) = \text{tr}(Xx)$ for any $x \in \mathfrak{g}$. We define a bilinear form

$$\beta_\xi : V \times V \rightarrow \mathbf{k}, (v, w) \mapsto \beta(Xv, w) + \beta(v, Xw).$$

Lemma 3.1. *The bilinear form β_ξ is well defined.*

Proof. Recall that the space $\text{Alt}(V)$ of alternate bilinear forms on V coincides with the second exterior power $\wedge^2(V^*)$ of V^* . Consider the linear mapping

$$\Phi : \text{End}_{\mathbf{k}}(V) \rightarrow \wedge^2(V^*) = \text{Alt}(V), \quad X \mapsto \beta_X$$

where $\beta_X(v, w) = \beta(Xv, w) + \beta(v, Xw)$ for $v, w \in V$. It is easy to see that Φ is $G = O(V)$ -equivariant. One can show that $\ker \Phi$ coincides with the orthogonal complement $\mathfrak{g}^\perp$ of $\mathfrak{g} = \mathfrak{o}(V)$ in $\text{End}_{\mathbf{k}}(V)$ under the nondegenerate trace form. It follows that β_ξ does not depend on the choice of X . $\square$

Assume $\xi \in \mathfrak{g}^*$. We denote $(V_\xi, \alpha, \beta_\xi)$ the vector space V equipped with the quadratic form α and the bilinear form β_ξ .

Definition 3.2. Assume $\xi, \zeta \in \mathfrak{g}^*$. We say that $(V_\xi, \alpha, \beta_\xi)$ and $(V_\zeta, \alpha, \beta_\zeta)$ are equivalent if there exists a vector space isomorphism $g : V_\xi \rightarrow V_\zeta$ such that $\alpha(gv) = \alpha(v)$ and $\beta_\zeta(gv, gw) = \beta_\xi(v, w)$ for any $v, w \in V_\xi$.

Lemma 3.3. *Two elements $\xi, \zeta \in \mathfrak{g}^*$ lie in the same G -orbit if and only if there exists $g \in G$ such that $\beta_\zeta(gv, gw) = \beta_\xi(v, w)$ for any $v, w \in V$.*

Proof. Assume $\xi(x) = \text{tr}(Xx), \zeta(x) = \text{tr}(X'x), \forall x \in \mathfrak{g}$. Using a similar argument as in the proof of Lemma 3.1, one can see that ξ, ζ lie in the same G -orbit if and only if there exists $g \in G$ such that $\beta((gXg^{-1} + X')v, w) + \beta(v, (gXg^{-1} + X')w) = 0, \forall v, w \in V$. $\square$

Corollary 3.4. *Two elements $\xi, \zeta \in \mathfrak{g}^*$ lie in the same G -orbit if and only if $(V_\xi, \alpha, \beta_\xi)$ is equivalent to $(V_\zeta, \alpha, \beta_\zeta)$.*

3.2. From now on we assume that $\xi \in \mathfrak{g}^*$ is nilpotent. Let $(V_\xi, \alpha, \beta_\xi)$ be defined as in subsection 3.1. Let λ be a formal parameter. There exists a smallest integer m such that there exists a set of vectors $v_0, \dots, v_m$ for which $\beta_\xi(\sum_{i=0}^m v_i \lambda^i, v) + \lambda \beta(\sum_{i=0}^m v_i \lambda^i, v) = 0$ for any $v \in V$ (see Lemma 3.5 below). Lemmas 3.5–3.9 in the following extend some results in [6]. (Most parts of the proofs are included in [6]. We add some conditions about the quadratic form α .)

Lemma 3.5. *The vectors $v_0, \dots, v_m$ (up to multiple) and $m \geq 0$ are uniquely determined by β_ξ and β . Moreover, $\beta(v_i, v_j) = \beta_\xi(v_i, v_j) = 0$, $i, j = 0, \dots, m$, $\alpha(v_i) = 0$, $i = 0, \dots, m-1$ and we can assume $\alpha(v_m) = 1$.*

Proof. Since ξ is nilpotent, we can find a cocharacter $\phi : \mathbf{G}_m \rightarrow G$ for which $\xi \in \mathfrak{g}^*(\phi, > 0)$. Moreover, we can find $X \in \text{End}_{\mathbf{k}}(V)(\phi, > 0)$ such that $\xi(x) = \text{tr}(Xx)$ for all $x \in \mathfrak{g}$.

Let w_0 be a nonzero vector such that $\beta(w_0, -) = 0$. Then w_0 is unique up to a multiple. We have that $w_0 \in V(\phi, 0)$. If $\beta_\xi(w_0, v) = 0$ for all $v \in V$, then $m = 0$ and we are done.

Now assume $\beta_\xi(w_0, -)$ does not vanish on V . Fix $v \in V$. It is easy to show that if $\beta_\xi(v, w_0) = 0$, then there is $v' \in V$ for which $\beta(v', -) = \beta_\xi(v, -)$. Moreover, if $\tilde{w}_{i-1} \in V(\phi, \geq i-1)$, then one can show that for all $\tilde{w}_i$ such that $\beta(\tilde{w}_i, -) = \beta_\xi(\tilde{w}_{i-1}, -)$, we have $\tilde{w}_i \in V(\phi, \geq i) + \mathbf{k}w_0$.

We define inductively a set of vectors w_i , $i = 0, \dots, m$, such that $\beta(w_0, -) = 0$, $\beta(w_i, -) = \beta_\xi(w_{i-1}, -)$, $\beta_\xi(w_m, -) = 0$ and m is minimal. We have defined w_0 . Assume $w_{i-1} \in V(\phi, \geq i-1)$ is found. Then $\beta_\xi(w_{i-1}, w_0) = \beta(w_{i/2}, w_{i/2}) = 0$ if i is even and $\beta_\xi(w_{i-1}, w_0) = \beta_\xi(w_{(i-1)/2}, w_{(i-1)/2}) = 0$ if i is odd. We define w_i to be the unique vector such that $\beta(w_i, -) = \beta_\xi(w_{i-1}, -)$ and $w_i \in V(\phi, \geq i)$. One readily sees that we find a unique (up to multiple) set of vectors w_i , $i = 0, \dots, m$, such that $\beta(w_0, -) = 0$, $\beta(w_i, -) = \beta_\xi(w_{i-1}, -)$, $\beta_\xi(w_m, -) = 0$ and m is minimal.

Since all $w_i \in V(\phi, \geq 0)$, we see that $\beta(w_i, w_j) = 0$. Since for $i > 0$, $w_i \in V(\phi, > 0)$, we see that $\alpha(w_i) = 0$ for $i > 0$. Since $X \in \text{End}_{\mathbf{k}}(\phi, > 0)$, it follows that $\beta_\xi(w_i, w_j) = 0$. We take $v_i = w_{m-i}$. Moreover, we can assume $\alpha(v_m) = \alpha(w_0) = 1$. $\square$

Lemma 3.6. *Assume $m \geq 1$. There exist $u_0, u_1, \dots, u_{m-1}$ such that $\beta(v_i, u_j) = \beta_\xi(v_{i+1}, u_j) = \delta_{i,j}$, $\beta(u_i, u_j) = \beta_\xi(u_i, u_j) = 0$, $i, j = 0, \dots, m-1$, $\alpha(u_i) = 0$, $i = 0, \dots, m-1$, and furthermore, $\beta(u_i, v) = \beta_\xi(u_{i-1}, v)$, $i = 1, \dots, m-1$, for all $v \in V$.*

Proof. Choose u_0 such that $\beta(u_0, v_i) = 0$, $i = 1, \dots, m-1$, $\beta(u_0, v_0) = 1$ and $\alpha(u_0) = 0$ (such that u_0 exists). We find inductively a set of vectors u_i , $1 \leq i \leq m-1$ such that $\beta(u_i, -) = \beta_\xi(u_{i-1}, -)$ and $\alpha(u_i) = 0$. Assume u_{i-1} , $1 \leq i \leq m-1$ is found. Since $\beta_\xi(u_{i-1}, v_m) = \beta(u_0, v_{m-i}) = 0$ (note that $m-i \geq 1$), there exist a unique u_i such that $\beta(u_i, -) = \beta_\xi(u_{i-1}, -)$ and $\alpha(u_i) = 0$. (The existence is as in the proof of Lemma 3.5 and the uniqueness is guaranteed by the condition $\alpha(u_i) = 0$.)

Now it follows that if $i < j$, $\beta(v_i, u_j) = \beta_\xi(v_{i-1}, u_{j-2}) = \beta_\xi(v_0, u_{j-i-1}) = 0$; if $i > j$, $\beta(v_i, u_j) = \beta(v_{i+1}, v_{j+1}) = \beta(v_m, u_{j-i+m}) = 0$; if $i = j$, $\beta(v_i, u_i) = \beta(v_{i-1}, u_{i-1}) = \beta(v_0, u_0) = 1$. Moreover, $\beta(u_i, u_{i+2k}) = \beta(u_{i+k}, u_{i+k}) = 0$,

$\beta(u_i, u_{i+2k+1}) = \beta_\xi(u_{i+k}, u_{i+k}) = 0$. It follows that $\beta(u_i, u_j) = 0$. Similarly, $\beta_\xi(u_i, u_j) = 0$. The u_i 's satisfy the conditions desired. $\square$

Lemma 3.7. *The vectors $v_0, v_1, \dots, v_m, u_0, u_1, \dots, u_{m-1}$ are linearly independent.*

Proof. Assume $\sum_{i=0}^m a_i v_i + \sum_{i=0}^{m-1} b_i u_i = 0$. Then $\beta(\sum_{i=0}^m a_i v_i + \sum_{i=0}^{m-1} b_i u_i, u_j) = a_j = 0$, $\beta(\sum_{i=0}^m a_i v_i + \sum_{i=0}^{m-1} b_i u_i, v_j) = b_j = 0$, $j = 0, \dots, m-1$ and $\beta_\xi(\sum_{i=0}^m a_i v_i + \sum_{i=0}^{m-1} b_i u_i, u_{m-1}) = a_m = 0$. $\square$

Let V_{2m+1} be the vector subspace of V spanned by $v_0, v_1, \dots, v_m, u_0, u_1, \dots, u_{m-1}$. If $m = 0$, let W be a complementary subspace of V_{2m+1} in V . If $m \geq 1$, let $W = \{w \in V_\xi \mid \beta(w, v) = \beta_\xi(w, v) = 0, \forall v \in V_{2m+1}\}$.

Lemma 3.8. *We have $V_\xi = V_{2m+1} \perp_{\beta, \beta_\xi} W$.*

Proof. Assume $m = 0$. The lemma follows since by the definition of v_0 we have $\beta(v_0, v) = \beta_\xi(v_0, v) = 0$ for any $v \in V$. Assume $m \geq 1$. A vector w is in W if and only if $\beta(w, v_i) = \beta_\xi(w, v_i) = 0$, $i = 0, \dots, m$ and $\beta(w, u_i) = \beta_\xi(w, u_i) = 0$, $i = 0, \dots, m-1$. By our choice of v_i and u_i , we have $\beta(v_m, w) = \beta_\xi(v_0, w) = 0$, $\beta(w, v_i) = \beta_\xi(w, v_{i+1})$ and $\beta(w, u_i) = \beta_\xi(w, u_{i-1})$. Hence $w \in W$ if and only if $\beta(w, u_i) = \beta(w, v_i) = 0$, $i = 0, \dots, m-1$ and $\beta_\xi(w, u_{m-1}) = 0$. Thus $\dim W \geq \dim V_\xi - (2m+1)$. Now we show $V_{2m+1} \cap W = \{0\}$. Let $w = \sum_{i=0}^m a_i v_i + \sum_{i=0}^{m-1} b_i u_i \in V_{2m+1} \cap W$. We have $\beta(w, u_j) = a_j = 0$, $\beta(w, v_j) = b_j = 0$, $j = 0, \dots, m-1$, and $\beta_\xi(w, u_{m-1}) = a_m = 0$. Hence together with the dimension condition we get the conclusion. $\square$

Let $V_\xi = V_{2m+1} \oplus W$ be as in Lemma 3.8. Then we get a $2(n-m)$ -dimensional vector space W , equipped with a quadratic form $\alpha|_W$ and a bilinear form $\beta_\xi|_{W \times W}$. It is easily seen that the quadratic form $\alpha|_W$ is nondefective on W , namely, $\beta|_{W \times W}$ is nondegenerate. Define a linear map $T_\xi : W \rightarrow W$ by

$$\beta(T_\xi w, w') = \beta_\xi(w, w'), w, w' \in W.$$

Lemma 3.9. *Assume $V_\xi = V_{2m_\xi+1, \xi} \oplus W_\xi$ is equivalent to $V_\zeta = V_{2m_\zeta+1, \zeta} \oplus W_\zeta$, then $m_\xi = m_\zeta$ and $(W_\xi, \beta, \beta_\xi)$ is equivalent to $(W_\zeta, \beta, \beta_\zeta)$.*

Proof. Assume $V_{2m_\xi+1, \xi} = \text{span}\{v_i^1, u_i^1\}$ and $V_{2m_\zeta+1, \zeta} = \text{span}\{v_i^2, u_i^2\}$, where v_i^1, v_i^2 are as in Lemma 3.5 and u_i^1, u_i^2 are as in Lemma 3.6. By assumption, there exists

$$g : V_{2m_\xi+1, \xi} \oplus W_\xi \rightarrow V_{2m_\zeta+1, \zeta} \oplus W_\zeta$$

such that $\beta(gv, gw) = \beta(v, w)$ and $\beta_\zeta(gv, gw) = \beta_\xi(v, w)$. Since for all $v \in V$, $\beta_\zeta(\sum_{i=0}^{m_\zeta} v_i^2 \lambda^i, v) + \lambda \beta(\sum_{i=0}^{m_\zeta} v_i^2 \lambda^i, v) = 0$, we get

$$\beta_\xi\left(\sum_{i=0}^{m_\zeta} g^{-1} v_i^2 \lambda^i, v\right) + \lambda \beta\left(\sum_{i=0}^{m_\zeta} g^{-1} v_i^2 \lambda^i, v\right) = 0.$$

Hence by Lemma 3.5, $m_\xi = m_\zeta$ and $g^{-1} v_i^2 \in V_{2m_\xi+1, \xi}$.

For $w \in W_\xi$, suppose $gw = \sum a_i v_i^2 + \sum b_i u_i^2 + w'$ where $w' \in W_\zeta$. Since $g^{-1} v_i^2 \in V_{2m_\xi+1, \xi}$, we have $\beta_\xi(g^{-1} v_i^2, w) = 0$, $i = 0, \dots, m_\xi$. It follows that $\beta_\zeta(v_i^2, gw) = b_i = 0$, $i = 0, \dots, m_\xi - 1$. We get $gw = \sum a_i v_i^2 + w'$. Define

$$\varphi : W_\xi \rightarrow W_\zeta, w \mapsto gw \text{ projects to } W_\zeta.$$

Let $w_1, w_2 \in W_\xi$. Assume $gw_1 = \sum a_i^1 v_i^2 + w'_1$, $gw_2 = \sum a_i^2 v_i^2 + w'_2$. We have $\beta(gw_1, gw_2) = \beta(w'_1, w'_2) = \beta(w_1, w_2)$, $\beta_\zeta(gw_1, gw_2) = \beta_\zeta(w'_1, w'_2) = \beta_\xi(w_1, w_2)$,

namely, $\beta(\varphi(w_1), \varphi(w_2)) = \beta(w_1, w_2)$, $\beta_\zeta(\varphi(w_1), \varphi(w_2)) = \beta_\xi(w_1, w_2)$. Now we show that φ is a bijection. Let $w \in W_\xi$ be such that $\varphi(w) = 0$. Then for any $v \in W_\xi$, $\beta(v, w) = \beta(\varphi(v), \varphi(w)) = 0$. Since $\beta|_{W_\xi \times W_\xi}$ is nondegenerate, $w = 0$. Thus φ is injective. On the other hand, we have $\dim W_\xi = \dim W_\zeta$. Hence φ is bijective. $\square$

Corollary 3.10. *Assume $V_\xi = V_{2m_\xi+1, \xi} \oplus W_\xi$ is equivalent to $V_\zeta = V_{2m_\zeta+1, \zeta} \oplus W_\zeta$, then $m_\xi = m_\zeta$ and T_ξ, T_ζ are conjugate.*

Lemma 3.11. *Assume ξ is nilpotent. Then T_ξ is nilpotent.*

Proof. We replace $G = Sp(V)$ by $G = O(V)$ and β by $\beta|_{W \times W}$ in the proof of Lemma 2.6. Moreover, when we apply $\text{Ad}(\phi(a))$ to T_ξ , we regard $\phi(a)$ as a linear map restricting to the subspace W of V so that $\phi(a) \in O(W)$. Also notice that $T_\xi \in \mathfrak{o}(W) = \{x \in \mathfrak{gl}(W) | \beta(xw, w) = 0, \forall w \in W\}$, since $\beta(T_\xi w, w) = \beta_\xi(w, w) = 0$ for all $w \in W$. Then the same argument as in the proof of Lemma 2.6 applies since $\beta|_{W \times W}$ is nondegenerate. $\square$

3.3. In this subsection assume $\mathbf{k}$ is algebraically closed. By Lemma 3.8, every form module $(V_\xi, \alpha, \beta_\xi)$ can be reduced to the form $V_\xi = V_{2m+1} \oplus W_\xi$, where V_{2m+1} has a basis $\{v_i, i = 0, \dots, m, u_i, i = 0, \dots, m-1\}$ as in Lemmas 3.5 and 3.6. We have that $(V_\xi, \alpha, \beta_\xi)$ is determined by V_{2m+1} and $(W_\xi, \alpha|_{W_\xi}, \beta_\xi|_{W_\xi \times W_\xi})$. Now we consider $(W_\xi, \alpha|_{W_\xi}, \beta_\xi|_{W_\xi \times W_\xi}) := (W, \alpha|_W, \beta_\xi|_{W \times W})$ and let $T_\xi : W \rightarrow W$ be defined as in subsection 3.2. It follows that $\beta_\xi|_{W \times W}$ is determined by T_ξ and $\beta|_{W \times W}$.

Since $T_\xi \in \mathfrak{o}(W)$ is nilpotent (Lemma 3.11), we can view W as a $k[T_\xi]$ -module. By the classification of nilpotent orbits in $\mathfrak{o}(W)$ (see [2], sections 3.5 and 3.9), W is equivalent to $W_{l_1}(m_1) \oplus \dots \oplus W_{l_s}(m_s)$ for some $m_1 \geq \dots \geq m_s$, $l_1 \geq \dots \geq l_s$ and $m_1 - l_1 \geq \dots \geq m_s - l_s$, where $[(m_i + 1)/2] \leq l_i \leq m_i$ (the same notation as in [13], Proposition 2.3).

Lemma 3.12. *Assume $m < k - l$. We have $V_{2m+1} \oplus W_l(k) \cong V_{2m+1} \oplus W_{k-m}(k)$.*

Proof. Assume $V_{2m+1} = \text{span}\{v_0, \dots, v_m, u_0, \dots, u_{m-1}\}$, where v_i, u_i are chosen as in Lemma 3.5 and Lemma 3.6. Assume $V_{2m+1} \oplus W_l(k)$ and $V_{2m+1} \oplus W_{k-m}(k)$ correspond to ξ_1 and ξ_2 , respectively. Let $T_1 = T_{\xi_1} : W_l(k) \rightarrow W_l(k)$ and $T_2 = T_{\xi_2} : W_{k-m}(k) \rightarrow W_{k-m}(k)$. There exist ρ_1, ρ_2 such that $W_l(k) = \text{span}\{\rho_1, \dots, T_1^{k-1}\rho_1, \rho_2, \dots, T_1^{k-1}\rho_2\}$, $T_1^k \rho_1 = T_1^k \rho_2 = 0$, $\alpha(T_1^i \rho_1) = \delta_{i, l-1}$, $\alpha(T_1^i \rho_2) = 0$, $\beta(T_1^i \rho_1, T_1^j \rho_1) = \beta(T_1^i \rho_2, T_1^j \rho_2) = 0$ and $\beta(T_1^i \rho_1, T_1^j \rho_2) = \delta_{i+j, k-1}$. There exist τ_1, τ_2 such that $W_{k-m}(k) = \text{span}\{\tau_1, \dots, T_2^{k-1}\tau_1, \tau_2, \dots, T_2^{k-1}\tau_2\}$, $T_2^k \tau_1 = T_2^k \tau_2 = 0$, $\alpha(T_2^i \tau_1) = \delta_{i, k-m-1}$, $\alpha(T_2^i \tau_2) = 0$, $\beta(T_2^i \tau_1, T_2^j \tau_1) = \beta(T_2^i \tau_2, T_2^j \tau_2) = 0$ and $\beta(T_2^i \tau_1, T_2^j \tau_2) = \delta_{i+j, k-1}$. Define $g : V_{2m+1} \oplus W_l(k) \rightarrow V_{2m+1} \oplus W_{k-m}(k)$ by $gv_i = v_i$, $gu_i = u_i + (T_2^{k-(m+l)+i} + T_2^i)\tau_2$, $gT_1^j \rho_2 = T_2^j \tau_2$, $gT_1^j \rho_1 = T_2^j \tau_1 + v_{k-1-j} + v_{m+l-1-j}$, where $v_i = 0$, if $i < 0$ or $i > m$. Then g is the isomorphism we want. $\square$

Lemma 3.13. *Assume $m \geq k - l_i, i = 1, 2$. We have $V_{2m+1} \oplus W_{l_1}(k) \cong V_{2m+1} \oplus W_{l_2}(k)$ if and only if $l_1 = l_2$.*

Proof. Assume $k - m \leq l_1 < l_2$. We show that $V_{2m+1} \oplus W_{l_1}(k) \not\cong V_{2m+1} \oplus W_{l_2}(k)$. Let $(V_1, \alpha, \beta_1) = V_{2m+1} \oplus W_{l_1}(k)$ and $(V_2, \alpha, \beta_2) = V_{2m+1} \oplus W_{l_2}(k)$. Let $T_1 = T_{\xi_1} : W_{l_1}(k) \rightarrow W_{l_1}(k)$ and $T_2 = T_{\xi_2} : W_{l_2}(k) \rightarrow W_{l_2}(k)$. Assume there exists $g : V_{2m+1} \oplus W_{l_1}(k) \rightarrow V_{2m+1} \oplus W_{l_2}(k)$ a linear isomorphism satisfying $\beta_2(gv, gw) = \beta_1(v, w)$ and $\alpha(gv) = \alpha(v)$. Define $\varphi : W_{l_1}(k) \rightarrow W_{l_2}(k)$ by $w_1 \mapsto (gw_1 \text{ projects to } W_{l_2}(k))$.

Then we have $\beta(\varphi(w_1), \varphi(w'_1)) = \beta(w_1, w'_1)$, $\beta_2(\varphi(w_1), \varphi(w'_1)) = \beta_1(w_1, w'_1)$ and $T_2(\varphi(w)) = \varphi(T_1(w))$ (see the proof of Lemma 3.9).

Let v_i , $i = 0, \dots, m$, and u_i , $i = 0, \dots, m-1$, be a basis of V_{2m+1} as in Lemmas 3.5 and 3.6. Choose a basis $T_i^j \rho_i, T_i^j \tau_i$, $j = 0, \dots, k-1$, $i = 1, 2$ of $W_{l_i}(k)$ such that $T_i^k \rho_i = T_i^k \tau_i = 0$, $\beta(T_i^{j_1} \rho_i, T_j^{j_2} \tau_j) = \delta_{j_1+j_2, k-1} \delta_{i,j}$, $\beta(T_i^{j_1} \rho_i, T_j^{j_2} \rho_j) = \beta(T_i^{j_1} \tau_i, T_j^{j_2} \tau_j) = 0$, $\alpha(T_i^j \rho_i) = \delta_{j, l_i-1}$ and $\alpha(T_i^j \tau_i) = 0$. We have

$$gv_i = av_i, i = 0, \dots, m, \quad gu_i = u_i/a + \sum_{l=0}^m a_{il} v_l + \sum_{l=0}^{k-1} x_{il} T_2^l \rho_2 + \sum_{l=0}^{k-1} y_{il} T_2^l \tau_2.$$

Now we can assume

$$\begin{aligned} gT_1^j \rho_1 &= \sum_{i=0}^{k-1-j} a_i T_2^{i+j} \rho_2 + \sum_{i=0}^{k-1-j} b_i T_2^{i+j} \tau_2 + \sum_{i=0}^m c_{ij} v_i + \sum_{i=0}^{m-1} d_{ij} u_i, \quad j = 0, \dots, k-1, \\ gT_1^j \tau_1 &= \sum_{i=0}^{k-1-j} e_i T_2^{i+j} \rho_2 + \sum_{i=0}^{k-1-j} f_i T_2^{i+j} \tau_2 + \sum_{i=0}^m g_{ij} v_i + \sum_{i=0}^{m-1} h_{ij} u_i, \quad j = 0, \dots, k-1. \end{aligned}$$

We have

$$\begin{aligned} \beta(gv_i, gT_1^j \rho_1) &= \beta(v_i, T_1^j \rho_1) = 0 \Rightarrow d_{ij} = 0, \quad i = 0, \dots, m-1, \quad j = 0, \dots, k-1, \\ \beta(gv_i, gT_1^j \tau_1) &= \beta(v_i, T_1^j \tau_1) = 0 \Rightarrow h_{ij} = 0, \quad i = 0, \dots, m-1, \quad j = 0, \dots, k-1, \\ \beta(gu_i, gT_1^j \rho_1) &= \beta(u_i, T_1^j \rho_1) = 0 \Rightarrow \frac{c_{ij}}{a} + \sum_{l=0}^{k-1-j} (x_{il} b_{k-1-j-l} + y_{il} a_{k-1-j-l}) = 0, \\ \beta_\xi(gu_i, gT_1^j \rho_1) &= \beta_\xi(u_i, T_1^j \rho_1) = 0 \Rightarrow \frac{c_{i+1,j}}{a} \\ &\quad + \sum_{l=0}^{k-2-j} (x_{il} b_{k-2-j-l} + y_{il} a_{k-2-j-l}) = 0. \end{aligned}$$

The last two equations imply that

$$(1) \quad c_{m,k-1} = 0, c_{ij} = c_{i+1,j-1}, i = 0, \dots, m-1, j = 0, \dots, k-1.$$

Similarly, we have

$$(2) \quad g_{m,k-1} = 0, g_{ij} = g_{i+1,j-1}, i = 0, \dots, m-1, j = 0, \dots, k-1.$$

We also have

$$\begin{aligned} \alpha(gT_1^{l_1-1} \rho_1) &= \alpha(T_1^{l_1-1} \rho_1) = 1 \Rightarrow c_{m,l_1-1}^2 + a_{l_2-l_1}^2 + \sum_{i=0}^{k+1-2l_1} a_i b_{k+1-2l_1-i} = 1, \\ \alpha(gT_1^{l_2-1} \rho_1) &= \alpha(T_1^{l_2-1} \rho_1) = 0 \Rightarrow c_{m,l_2-1}^2 + a_0^2 + \sum_{i=0}^{k+1-2l_2} a_i b_{k+1-2l_2-i} = 0, \\ \alpha(gT_1^{l_2-1} \tau_1) &= \alpha(T_1^{l_2-1} \tau_1) = 0 \Rightarrow g_{m,l_2-1}^2 + e_0^2 + \sum_{i=0}^{k+1-2l_2} a_i b_{k+1-2l_2-i} = 0. \end{aligned}$$

Since $l_2 > l_1 \geq [k+1]/2$, we have $k+1-2l_2 < 0$. Thus we get $a_0 = c_{m,l_2-1}$ and $e_0 = g_{m,l_2-1}$. Since $l_2 > k-m$, by equations (1) and (2) we get $c_{m,l_2-1} = g_{m,l_2-1} = 0$ (if $l_2 = k$) or $c_{m,l_2-1} = c_{0,l_2+m-1} = 0$, $g_{m,l_2-1} = g_{0,l_2+m-1} = 0$ (if

$l_2 < k$). Thus $a_0 = e_0 = 0$. But from $\beta(g\rho_1, gT_1^{k-1}\tau_1) = \beta(\rho_1, T_1^{k-1}\tau_1) = 1$ we have $a_0f_0 + e_0b_0 = 1$. This is a contradiction. $\square$

It follows that for any $V = (V_\xi, \alpha, \beta_\xi)$, there exist a unique $m \geq 0$ and a unique sequence of modules $W_{l_i}(k_i)$, $i = 1, \dots, s$ such that

$$V \cong V_{2m+1} \oplus W_{l_1}(k_1) \oplus \dots \oplus W_{l_s}(k_s),$$

$[(k_i + 1)/2] \leq l_i \leq k_i$, $k_1 \geq k_2 \geq \dots \geq k_s$, $l_1 \geq l_2 \geq \dots \geq l_s$ and $m \geq k_1 - l_1 \geq k_2 - l_2 \geq \dots \geq k_s - l_s$. We call this the *normal form* of the module V . Two form modules are equivalent if and only if their normal forms are the same. Hence to each nilpotent orbit we associate a pair of partitions

$$(m, k_1 - l_1, \dots, k_s - l_s)(l_1, \dots, l_s)$$

where $l_1 \geq l_2 \geq \dots \geq l_s \geq 0$ and $m \geq k_1 - l_1 \geq k_2 - l_2 \geq \dots \geq k_s - l_s \geq 0$. This defines a bijection from the set of nilpotent orbits to the set $\{(\nu, \mu) | \nu = (\nu_0, \nu_1, \dots, \nu_s), \mu = (\mu_1, \mu_2, \dots, \mu_s), |\mu| + |\nu| = n, \nu_i \leq \mu_i, i = 1, \dots, s\}$, which has cardinality $p_2(n) - p_2(n-2)$.

3.4. In this subsection, we classify the form modules $(V_\xi, \alpha, \beta_\xi)$ over $\mathbf{F}_q$. We have $V_\xi = V_{2m+1} \oplus W_\xi$ for some m and W_ξ (Lemmas 3.5–3.8 are valid over $\mathbf{F}_q$). By the classification of $(W_\xi, \alpha|_{W_\xi}, T_\xi)$ over $\mathbf{F}_q$, we have $W_\xi \cong \bigoplus W_{l_i}^{\epsilon_i}(k_i)$ where $\epsilon_i = 0$ or δ , $m_1 \geq \dots \geq m_s$, $l_1 \geq \dots \geq l_s$, $m_1 - l_1 \geq \dots \geq m_s - l_s$ and $[(m_i + 1)/2] \leq l_i \leq m_i$ (the same notation as in [13], Proposition 3.1).

Lemma 3.14. *Assume $m \geq k - l$ and $l > m$. We have $V_{2m+1} \oplus W_l^0(k) \cong V_{2m+1} \oplus W_l^\delta(k)$.*

Proof. Let $W_l^0(k) = (W_1, \alpha, T_1)$ and $W_l^\delta(k) = (W_2, \alpha, T_2)$. Take ρ_1, ρ_2 such that $W_l^0(k) = \text{span}\{\rho_1, \dots, T_1^{k-1}\rho_1, \rho_2, \dots, T_1^{k-1}\rho_2\}$, $T_1^k\rho_1 = T_1^k\rho_2 = 0$, $\alpha(T_1^i\rho_1) = \delta_{i,l-1}$, $\alpha(T_1^i\rho_2) = 0$, $\beta(T_1^i\rho_1, T_1^j\rho_1) = \beta(T_1^i\rho_2, T_1^j\rho_2) = 0$ and $\beta(T_1^i\rho_1, T_1^j\rho_2) = \delta_{i+j,k-1}$. Take τ_1, τ_2 such that $W_l^\delta(k) = \text{span}\{\tau_1, \dots, T_2^{k-1}\tau_1, \tau_2, \dots, T_2^{k-1}\tau_2\}$, $T_2^k\tau_1 = T_2^k\tau_2 = 0$, $\alpha(T_2^i\tau_1) = \delta_{i,l-1}$, $\alpha(T_2^i\tau_2) = \delta_{i,k-l}\delta$, $\beta(T_2^i\tau_1, T_2^j\tau_1) = \beta(T_2^i\tau_2, T_2^j\tau_2) = 0$ and $\beta(T_2^i\tau_1, T_2^j\tau_2) = \delta_{i+j,k-1}$. Let v_i, u_i be a basis of V_{2m+1} as in Lemmas 3.5 and 3.6. Define $g : V_{2m+1} \oplus W_l^0(k) \rightarrow V_{2m+1} \oplus W_l^\delta(k)$ by $gv_i = v_i$, $gu_i = u_i + \sqrt{\delta}T_2^{l-m-1+i}\tau_1$, $gT_1^i\rho_1 = T_2^i\tau_1$, $gT_1^i\rho_2 = T_2^i\tau_2 + \sqrt{\delta}v_{k-l+m-i}$, where $v_i = 0$ if $i < 0$ or $i > m$. $\square$

Lemma 3.15. *Assume $m \geq k - l$ and $l \leq m$. We have $V_{2m+1} \oplus W_l^0(k) \not\cong V_{2m+1} \oplus W_l^\delta(k)$.*

Proof. Let v_i , $i = 0, \dots, m$ and u_i , $i = 0, \dots, m-1$ be a basis of V_{2m+1} as in Lemmas 3.5 and 3.6. Let $W_l^0(k) = (W_0, \alpha, T_0)$ and $W_l^\delta(k) = (W_\delta, \alpha, T_\delta)$. Choose a basis $T_\epsilon^j\rho_\epsilon, T_\epsilon^j\tau_\epsilon$, $j = 0, \dots, k-1$, $\epsilon = 0, \delta$ of $W_l^\epsilon(k)$ such that $T_\epsilon^k\rho_\epsilon = T_\epsilon^k\tau_\epsilon = 0$, $\beta(T_{\epsilon_1}^{j_1}\rho_{\epsilon_1}, T_{\epsilon_2}^{j_2}\tau_{\epsilon_2}) = \delta_{j_1+j_2,k-1}\delta_{\epsilon_1,\epsilon_2}$, $\beta(T_{\epsilon_1}^{j_1}\rho_{\epsilon_1}, T_{\epsilon_2}^{j_2}\rho_{\epsilon_2}) = \beta(T_{\epsilon_1}^{j_1}\tau_{\epsilon_1}, T_{\epsilon_2}^{j_2}\tau_{\epsilon_2}) = 0$, $\alpha(T_\epsilon^j\rho_\epsilon) = \delta_{j,l-1}$ and $\alpha(T_\epsilon^j\tau_\epsilon) = \epsilon\delta_{j,k-l}\delta_{\epsilon,\delta}$. Assume there exists $g : V_{2m+1} \oplus W_l^0(k) \rightarrow V_{2m+1} \oplus W_l^\delta(k)$ a linear isomorphism satisfying $\beta(gv, gw) = \beta(v, w)$, $\beta_\delta(gv, gw) = \beta_0(v, w)$ and $\alpha(gv) = \alpha(v)$. We have

$$gv_i = av_i, i = 0, \dots, m, \quad gu_i = u_i/a + \sum_{l=0}^m a_{il}v_l + \sum_{l=0}^{k-1} x_{il}T_\delta^l\rho_\delta + \sum_{l=0}^{k-1} y_{il}T_\delta^l\tau_\delta.$$

Now we can assume

$$\begin{aligned} gT_0^j \rho_0 &= \sum_{i=0}^{k-1-j} a_i T_\delta^{i+j} \rho_\delta + \sum_{i=0}^{k-1-j} b_i T_\delta^{i+j} \tau_\delta + \sum_{i=0}^m c_{ij} v_i + \sum_{i=0}^{m-1} d_{ij} u_i, \quad j = 0, \dots, k-1, \\ gT_0^j \tau_0 &= \sum_{i=0}^{k-1-j} e_i T_\delta^{i+j} \rho_\delta + \sum_{i=0}^{k-1-j} f_i T_\delta^{i+j} \tau_\delta + \sum_{i=0}^m g_{ij} v_i + \sum_{i=0}^{m-1} h_{ij} u_i, \quad j = 0, \dots, k-1. \end{aligned}$$

By a similar argument as in the proof of Lemma 3.13, we get that $c_{m,k-1} = 0, g_{m,k-1} = 0, c_{ij} = c_{i+1,j-1}, g_{ij} = g_{i+1,j-1}, i = 0, \dots, m-1, j = 0, \dots, k-1$. Since we have $m \geq l$, $c_{m,i} = c_{0,i+m} = 0$ and $g_{m,i} = g_{0,i+m} = 0$ when $i \geq k-l$. We get that some of the equations are $a_0^2 + a_0 b_0 + \delta b_0^2 = 1, e_0^2 + e_0 f_0 + \delta f_0^2 = 0$ and $a_0 f_0 + b_0 e_0 = 1$ (when $l = (k+1)/2$) or $a_{l-1-i}^2 + \sum_{j=0}^{k-1-2i} a_j b_{k-1-2i-j} + \delta b_{k-l-i}^2 = \delta_{i,l-1}, e_{l-1-i}^2 + \sum_{j=0}^{k-1-2i} e_j f_{k-1-2i-j} + \delta f_{k-l-i}^2 = 0, k-l \leq i \leq l-1$ and $a_0 f_0 + b_0 e_0 = 1$ (when $l > (k+1)/2$). We get $a_0 = f_0 = 1, e_i = 0, i = 0, \dots, 2l-k-2$ and $e_{2l-k-1}^2 + e_{2l-k-1} + \delta = 0$. This is a contradiction. $\square$

Recall that we have the following lemma ([13], Lemma 4.4 (iii)).

Lemma 3.16. *Assume $l_1 \geq l_2$ and $k_1 - l_1 \geq k_2 - l_2$. If $l_1 + l_2 > k_1$, we have $W_{l_1}^0(k_1) \oplus W_{l_2}^0(k_2) \cong W_{l_1}^\delta(k_1) \oplus W_{l_2}^\delta(k_2)$ and $W_{l_1}^0(k_1) \oplus W_{l_2}^\delta(k_2) \cong W_{l_1}^\delta(k_1) \oplus W_{l_2}^0(k_2)$.*

Let $G(\mathbf{F}_q), \mathfrak{g}(\mathbf{F}_q)$ be the fixed points of a Frobenius map $\mathfrak{F}_q$ relative to $\mathbf{F}_q$ on $G, \mathfrak{g}$.

Proposition 3.17. *The nilpotent orbit in $\mathfrak{g}^*$ corresponding to the pair of partitions $(\nu_0, \nu_1, \dots, \nu_s)(\mu_1, \mu_2, \dots, \mu_s)$ splits into at most 2^k $G(\mathbf{F}_q)$ -orbits in $\mathfrak{g}(\mathbf{F}_q)^*$, where $k = \#\{i \geq 1 | \nu_i < \mu_i \leq \nu_{i-1}\}$.*

Proof. Let $V = V_{2m+1} \oplus W_{l_1}(\lambda_1) \oplus \dots \oplus W_{l_s}(\lambda_s)$ be the normal form of a module corresponding to (ν, μ) over $\bar{\mathbf{F}}_q$. We show that the equivalence class of V over $\bar{\mathbf{F}}_q$ decomposes into at most 2^k equivalence classes over $\mathbf{F}_q$. It is enough to show that form modules of the form $V_{2m+1} \oplus W_{l_1}^{\epsilon_1}(\lambda_1) \oplus \dots \oplus W_{l_s}^{\epsilon_s}(\lambda_s)$, $\epsilon_i = 0$ or δ , have at most 2^k equivalence classes over $\mathbf{F}_q$. Suppose $i_1, \dots, i_k$ are such that $\beta_{i_j} < \mu_{i_j} \leq \beta_{i_{j-1}}, j = 1, \dots, k$. Using Lemma 3.14, 3.15 and 3.16, one can easily verify that a form module of the above form is isomorphic to one of the following modules: $V_1^{\epsilon_1} \oplus \dots \oplus V_k^{\epsilon_k} \oplus V_{k+1}$, where $V_1^{\epsilon_1} = V_{2m+1} \oplus W_{l_1}^0(\lambda_1) \oplus \dots \oplus W_{l_{i_1-1}}^0(\lambda_{i_1-1}) \oplus W_{l_{i_1}}^{\epsilon_1}(\lambda_{i_1})$, $V_t^{\epsilon_t} = W_{l_{i_{t-1}+1}}^0(\lambda_{i_{t-1}+1}) \oplus \dots \oplus W_{l_{i_t-1}}^0(\lambda_{i_t-1}) \oplus W_{l_{i_t}}^{\epsilon_t}(\lambda_{i_t}), t = 2, \dots, k, \epsilon_i = 0$ or $\delta, i = 1, \dots, k$, and $V_{k+1} = W_{l_{i_k+1}}^0(\lambda_{i_k+1}) \oplus \dots \oplus W_{l_s}^0(\lambda_s)$. Thus the proposition follows. $\square$

Proposition 3.18. *The number of nilpotent $G(\mathbf{F}_q)$ -orbits in $\mathfrak{g}(\mathbf{F}_q)^*$ is at most $p_2(n)$.*

Proof. We have mapped the nilpotent orbits in $\mathfrak{g}(\bar{\mathbf{F}}_q)^*$ bijectively to the set $\{(\nu, \mu) | \nu = (\nu_0, \nu_1, \dots, \nu_s), \mu = (\mu_1, \mu_2, \dots, \mu_s), |\mu| + |\nu| = n, \nu_i \leq \mu_i, i = 1, \dots, s\} := \Delta$. Let $(\nu, \mu) \in \Delta, \nu = (\nu_0, \nu_1, \dots, \nu_s), \mu = (\mu_1, \mu_2, \dots, \mu_s)$. By Proposition 3.17, the nilpotent orbit corresponding to (ν, μ) splits into at most 2^k orbits in $\mathfrak{g}(\mathbf{F}_q)^*$, where $k = \#\{i \geq 1 | \nu_i < \mu_i \leq \nu_{i-1}\}$. We associate 2^k pairs of partitions to this orbit as

follows. Suppose $r_1, r_2, \dots, r_k$ are such that $\nu_{r_i} < \mu_{r_i} \leq \nu_{r_i-1}, i = 1, \dots, k$. Let

$$\begin{aligned}\nu^0 &= (\nu_0, \dots, \nu_{r_1-1}), \mu^0 = (\mu_1, \dots, \mu_{r_1-1}), \\ \nu^{1,i} &= (\nu_{r_i}, \dots, \nu_{r_{i+1}-1}), \mu^{1,i} = (\mu_{r_i}, \dots, \mu_{r_{i+1}-1}), \\ \nu^{2,i} &= (\mu_{r_i}, \dots, \mu_{r_{i+1}-1}), \mu^{2,i} = (\nu_{r_i}, \dots, \nu_{r_{i+1}-1}), i = 1, \dots, k-1, \\ \nu^{1,k} &= (\nu_{r_k}, \dots, \nu_s), \mu^{1,k} = (\mu_{r_k}, \dots, \mu_s), \\ \nu^{2,k} &= (\mu_{r_k}, \dots, \mu_s), \mu^{2,k} = (\nu_{r_k}, \dots, \nu_s).\end{aligned}$$

We associate to (ν, μ) the pairs of partitions $(\tilde{\nu}^{\epsilon_1, \dots, \epsilon_k}, \tilde{\mu}^{\epsilon_1, \dots, \epsilon_k})$,

$$\tilde{\nu}^{\epsilon_1, \dots, \epsilon_k} = (\nu^0, \nu^{\epsilon_1, 1}, \nu^{\epsilon_2, 2}, \dots, \nu^{\epsilon_k, k}), \tilde{\mu}^{\epsilon_1, \dots, \epsilon_k} = (\mu^0, \mu^{\epsilon_1, 1}, \mu^{\epsilon_2, 2}, \dots, \mu^{\epsilon_k, k}),$$

where $\epsilon_i \in \{1, 2\}, i = 1, \dots, k$. Notice that the pairs of partitions $(\tilde{\nu}^{\epsilon_1, \dots, \epsilon_k}, \tilde{\mu}^{\epsilon_1, \dots, \epsilon_k})$ are distinct and among them only $(\nu, \mu) = (\tilde{\nu}^{1, \dots, 1}, \tilde{\mu}^{1, \dots, 1})$ is in Δ . One can verify that $\{(\tilde{\nu}^{\epsilon_1, \dots, \epsilon_k}, \tilde{\mu}^{\epsilon_1, \dots, \epsilon_k}) | (\nu, \mu) \in \Delta\} = \{(\nu, \mu) | |\nu| + |\mu| = n\}$. $\square$

4. EVEN ORTHOGONAL GROUPS

Let V be a vector space of dimension $2n$ over $\mathbf{k}$ equipped with a nondefective quadratic form $\alpha : V \rightarrow \mathbf{k}$. Let $\beta : V \times V \rightarrow \mathbf{k}$ be the nondegenerate bilinear form associated to α . The even orthogonal group is defined as $G = O(2n) = \{g \in GL(V) \mid \alpha(gv) = \alpha(v), \forall v \in V\}$ and its Lie algebra is $\mathfrak{g} = \mathfrak{o}(2n) = \{x \in \mathfrak{gl}(V) \mid \beta(xv, v) = 0, \forall v \in V\}$. Let $G(\mathbf{F}_q), \mathfrak{g}(\mathbf{F}_q)$ be the fixed points of a split Frobenius map $\mathfrak{F}_q$ relative to $\mathbf{F}_q$ on $G, \mathfrak{g}$.

Proposition 4.1. *The numbers of nilpotent $G(\mathbf{F}_q)$ -orbits in $\mathfrak{g}(\mathbf{F}_q)$ and in $\mathfrak{g}(\mathbf{F}_q)^*$ are the same.*

The proposition can be proved in two ways.

First proof. We show that there exists a G -invariant nondegenerate bilinear form on $\mathfrak{g} = \mathfrak{o}(2n)$ (G. Lusztig). Then we can identify $\mathfrak{g}$ and $\mathfrak{g}^*$ via this bilinear form and the proposition follows. Consider the vector space $\bigwedge^2 V$ on which G acts in a natural way: $g(a \wedge b) = ga \wedge gb$. On $\bigwedge^2 V$ there is a G -invariant nondegenerate bilinear form

$$\langle a \wedge b, c \wedge d \rangle = \det \begin{bmatrix} \beta(a, c) & \beta(a, d) \\ \beta(b, c) & \beta(b, d) \end{bmatrix}.$$

Define a map $\phi : \bigwedge^2 V \rightarrow \mathfrak{o}(2n)$ by $a \wedge b \mapsto \phi_{a \wedge b}$ and extending by linearity where $\phi_{a \wedge b}(v) = \beta(a, v)b - \beta(b, v)a$. This map is G -equivariant since we have $\phi_{ga \wedge gb} = g\phi_{a \wedge b}g^{-1}$. One can easily verify that ϕ is a bijection. Define

$$\langle \phi_{a \wedge b}, \phi_{c \wedge d} \rangle_{\mathfrak{o}(2n)} = \langle a \wedge b, c \wedge d \rangle$$

and extend it to $\mathfrak{o}(2n)$ by linearity. This defines a G -invariant nondegenerate bilinear form on $\mathfrak{o}(2n)$.

Second proof. Let ξ be an element of $\mathfrak{g}^*$. There exists $X \in \mathfrak{gl}(V)$ such that $\xi(x) = \text{tr}(Xx)$ for any $x \in \mathfrak{g}$. We define a linear map $T_\xi : V \rightarrow V$ by

$$\beta(T_\xi v, v') = \beta(Xv, v') + \beta(v, Xv'), \text{ for all } v, v' \in V.$$

Lemma 4.2. *T_ξ is well defined.*

Proof. The same proof as in Lemma 3.1 shows that $\beta(T_\xi v, v')$ is well defined and thus T_ξ is well defined. $\square$

Lemma 4.3. *Two elements $\xi, \zeta \in \mathfrak{g}^*$ lie in the same G -orbit if and only if there exists $g \in G$ such that $gT_\xi g^{-1} = T_\zeta$.*

Proof. Assume $\xi(x) = \text{tr}(X_\xi x), \zeta(x) = \text{tr}(X_\zeta x), \forall x \in \mathfrak{g}$. Then ξ, ζ lie in the same G -orbit if and only if there exists $g \in G$ such that $\text{tr}(gX_\xi g^{-1}x) = \text{tr}(X_\zeta x), \forall x \in \mathfrak{g}$. This is equivalent to $\beta((gX_\xi g^{-1} + X_\zeta)v, w) + \beta(v, (gX_\xi g^{-1} + X_\zeta)w) = 0, \forall v, w \in V$, which is true if and only if $gT_\xi g^{-1} = T_\zeta$. $\square$

Note that $\beta(T_\xi v, v) = 0$ for any $v \in V$. Thus $T_\xi \in \mathfrak{g}$. We have, in fact, defined a bijection $\theta : \mathfrak{g}^* \rightarrow \mathfrak{g}, \xi \mapsto T_\xi$. This induces a bijection $\theta|_{\mathcal{N}'} : \mathcal{N}' \rightarrow \mathcal{N}$, where $\mathcal{N}'$ (resp. $\mathcal{N}$) is the set of all nilpotent elements (unstable vectors) in $\mathfrak{g}^*$ (resp. $\mathfrak{g}$). Moreover, $\theta|_{\mathcal{N}'}$ is G -equivariant by Lemma 4.3. The proposition follows.

5. SPRINGER CORRESPONDENCE

In this section, we assume $\mathbf{k}$ is algebraically closed. Throughout subsections 5.1–5.5, let G be a simply connected algebraic group of type B, C or D over $\mathbf{k}$ and let $\mathfrak{g}$ be the Lie algebra of G . Fix a Borel subgroup B of G with Levi decomposition $B = TU$. We denote r as the dimension of T . Let U^- be a maximal unipotent subgroup opposite to B . Let $\mathfrak{b}, \mathfrak{t}, \mathfrak{n}$ and $\mathfrak{n}^-$ be the Lie algebra of B, T, U and U^- , respectively. Let $\mathcal{B}$ be the variety of Borel subgroups of G . Let $\mathfrak{g}^*$ be the dual vector space of $\mathfrak{g}$. Let

$$\mathfrak{t}' = \{\xi \in \mathfrak{g}^* | \xi(\mathfrak{n} \oplus \mathfrak{n}^-) = 0\}, \mathfrak{n}' = \{\xi \in \mathfrak{g}^* | \xi(\mathfrak{b}) = 0\}, \mathfrak{b}' = \{\xi \in \mathfrak{g}^* | \xi(\mathfrak{n}) = 0\}.$$

An element ξ in $\mathfrak{g}^*$ is called semisimple (resp. nilpotent) if there exists $g \in G$ such that $g.\xi \in \mathfrak{t}'$ (resp. $\mathfrak{n}'$) (see [4]). The proofs in this section are entirely similar to those of [7, 8, 13]. For completeness, we include the proofs here.

5.1. Let $Z = \{(\xi, B_1, B_2) \in \mathfrak{g}^* \times \mathcal{B} \times \mathcal{B} | \xi \in \mathfrak{b}'_1 \cap \mathfrak{b}'_2\}$ and $Z' = \{(\xi, B_1, B_2) \in \mathfrak{g}^* \times \mathcal{B} \times \mathcal{B} | \xi \in \mathfrak{n}'_1 \cap \mathfrak{n}'_2\}$. Let $\mathfrak{c}$ be a nilpotent orbit in $\mathfrak{g}^*$.

Lemma 5.1. (i) *We have $\dim(\mathfrak{c} \cap \mathfrak{n}') \leq \frac{1}{2} \dim \mathfrak{c}$.*

(ii) *Given $\xi \in \mathfrak{c}$, we have $\dim\{B_1 \in \mathcal{B} | \xi \in \mathfrak{n}'_1\} \leq (\dim G - r - \dim \mathfrak{c})/2$.*

(iii) *We have $\dim Z = \dim G$ and $\dim Z' = \dim G - r$.*

Proof. We have a partition $Z = \bigcup_{\mathcal{O}} Z_{\mathcal{O}}$ according to the G -orbits $\mathcal{O}$ on $\mathcal{B} \times \mathcal{B}$ where $Z_{\mathcal{O}} = \{(\xi, B_1, B_2) \in Z | (B_1, B_2) \in \mathcal{O}\}$. Define in the same way a partition $Z' = \bigcup_{\mathcal{O}} Z'_{\mathcal{O}}$. Consider the maps from $Z_{\mathcal{O}}$ and $Z'_{\mathcal{O}}$ to $\mathcal{O}$: $(\xi, B_1, B_2) \mapsto (B_1, B_2)$. We have $\dim Z_{\mathcal{O}} = \dim(\mathfrak{b}'_1 \cap \mathfrak{b}'_2) + \dim \mathcal{O} = \dim(\mathfrak{b}_1 \cap \mathfrak{b}_2) + \dim \mathcal{O} = \dim G$ and $\dim Z'_{\mathcal{O}} = \dim(\mathfrak{n}'_1 \cap \mathfrak{n}'_2) + \dim \mathcal{O} = \dim(\mathfrak{n}_1 \cap \mathfrak{n}_2) + \dim \mathcal{O} = \dim G - r$. Thus (iii) follows.

Let $Z'(\mathfrak{c}) = \{(\xi, B_1, B_2) \in Z' | \xi \in \mathfrak{c}\} \subset Z'$. From (iii), we have $\dim Z'(\mathfrak{c}) \leq \dim G - r$. Consider the map $Z'(\mathfrak{c}) \rightarrow \mathfrak{c}, (\xi, B_1, B_2) \mapsto \xi$. We have $\dim Z'(\mathfrak{c}) = \dim \mathfrak{c} + 2 \dim\{B_1 \in \mathcal{B} | \xi \in \mathfrak{n}'_1\} \leq \dim G - r$. Thus (ii) follows.

Consider the variety $\{(\xi, B_1) \in \mathfrak{c} \times \mathcal{B} | \xi \in \mathfrak{n}'_1\}$. By projecting it to the first coordinate, and using (ii), we see that it has dimension $\leq (\dim G - r + \dim \mathfrak{c})/2$. If we project it to the second coordinate, we get $\dim(\mathfrak{c} \cap \mathfrak{n}') + \dim \mathcal{B} \leq (\dim G - r + \dim \mathfrak{c})/2$ and (i) follows. $\square$

5.2. Recall that an element ξ in $\mathfrak{g}^*$ is called regular if the connected centralizer $Z_G^0(\xi)$ in G is a maximal torus of G ([4]).

Lemma 5.2 ([4], Lemma 3.2). *There exist regular semisimple elements in $\mathfrak{g}^*$ and they form an open dense subset in $\mathfrak{g}^*$.*

Remark 5.3. Lemma 5.2 is not always true when G is not simply connected.

5.3. Let $\mathfrak{t}'_0, Y'$ be the set of semisimple regular elements in $\mathfrak{t}', \mathfrak{g}^*$, respectively. By Lemma 5.2, $\dim Y' = \dim G$. Let

$$\tilde{Y}' = \{(\xi, gT) \in Y' \times G/T \mid g^{-1}.\xi \in \mathfrak{t}'_0\}.$$

Define

$$\pi' : \tilde{Y}' \rightarrow Y' \text{ by } \pi'(\xi, gT) = \xi.$$

The Weyl group $W = NT/T$ acts (freely) on $\tilde{Y}'$ by $n : (\xi, gT) \mapsto (\xi, gn^{-1}T)$.

Lemma 5.4. $\pi' : \tilde{Y}' \rightarrow Y'$ is a principal W -bundle.

Proof. We show that if $\xi \in \mathfrak{g}^*, g, g' \in G$ are such that $g^{-1}.\xi \in \mathfrak{t}'_0$ and $g'^{-1}.\xi \in \mathfrak{t}'_0$, then $g' = gn^{-1}$ for some $n \in NT$. Let $g^{-1}.\xi = \xi_1 \in \mathfrak{t}'_0, g'^{-1}.\xi = \xi_2 \in \mathfrak{t}'_0$, then we have $Z_G^0(\xi) = Z_G^0(g.\xi_1) = gZ_G^0(\xi_1)g^{-1} = gTg^{-1}$. Similarly, $Z_G^0(\xi) = Z_G^0(g'.\xi_2) = g'Tg'^{-1}$, hence $g'^{-1}g \in NT$. $\square$

Let

$$X' = \{(\xi, gB) \in \mathfrak{g}^* \times G/B \mid g^{-1}.\xi \in \mathfrak{b}'\}.$$

Define

$$\varphi' : X' \rightarrow \mathfrak{g}^* \text{ by } \varphi'(\xi, gB) = \xi.$$

The map φ' is G -equivariant with G -action on X' given by

$$g_0 : (\xi, gB) \mapsto (g_0.\xi, g_0gB).$$

Lemma 5.5. (i) X' is an irreducible variety of dimension equal to $\dim G$.

(ii) φ' is proper and $\varphi'(X') = \mathfrak{g}^*$.

(iii) $(\xi, gT) \rightarrow (\xi, gB)$ is an isomorphism $\rho : \tilde{Y}' \xrightarrow{\sim} \varphi'^{-1}(Y')$.

Proof. (i) and (ii) are easy.

(iii) We first show that ρ is a bijection. Suppose $(\xi_1, g_1T), (\xi_2, g_2T) \in \tilde{Y}'$ are such that $(\xi_1, g_1B) = (\xi_2, g_2B)$, then we have $g_1^{-1}.\xi_1 \in \mathfrak{t}'_0, g_2^{-1}.\xi_2 \in \mathfrak{t}'_0$ and $\xi_1 = \xi_2, g_2^{-1}g_1 \in B$. A similar argument as in the proof of Lemma 5.4 shows $g_2^{-1}g_1 \in NT$, hence $g_2^{-1}g_1 \in B \cap NT = T$ and it follows that $g_1T = g_2T$. Thus ρ is injective. For $(\xi, gB) \in \varphi'^{-1}(Y')$, we have $\xi \in Y', g^{-1}.\xi \in \mathfrak{b}'$, hence there exists $b \in B, \xi_0 \in \mathfrak{t}'_0$ such that $g^{-1}.\xi = b.\xi_0$. Then $\rho(\xi, gbT) = (\xi, gB)$ and it follows that ρ is surjective.

Now we show that ρ is an isomorphism of varieties. The proof is entirely similar to the Lie algebra case (see for example [3], Lemma 13.4). Let $\mathfrak{b}'_0$ be the set of regular semisimple elements in $\mathfrak{b}'$. Consider the natural projection maps

$$f : \tilde{Y}' \rightarrow G/T, \quad f' : X' \rightarrow G/B.$$

Let U^- be as in the first paragraph of this section. Then U^-B/B (resp. U^-B/T) is an open subset in G/B (resp. G/T). We have isomorphisms

$$\begin{aligned} \mathfrak{t}'_0 \times U \times U^- &\xrightarrow{\sim} f^{-1}(U^-B/T), (\xi, u, u^-) \mapsto ((u^-u).\xi, u^-uT) \\ \mathfrak{b}'_0 \times U^- &\xrightarrow{\sim} f'^{-1}(U^-B/B), (\xi, u^-) \mapsto (u^-. \xi, u^-B). \end{aligned}$$

Notice that $f^{-1}(U^-B/T)$ is the inverse image of $f'^{-1}(U^-B/B)$ under ρ . Hence under the two isomorphisms above, the map ρ corresponds to the following isomorphism

$$\mathfrak{t}'_0 \times U \times U^- \xrightarrow{\sim} \mathfrak{b}'_0 \times U^-, \quad (\xi, u, u^-) \mapsto (u\xi, u^-).$$

It follows that ρ^{-1} is a morphism on $f'^{-1}(U^-B/B)$. Since $f'^{-1}(gU^-B/B)$ with $g \in G$ cover $\varphi'^{-1}(Y')$ and ρ, f' are G -equivariant, we see that ρ^{-1} is a morphism everywhere. $\square$

By Lemma 5.4, the map $\pi' : \tilde{Y}' \rightarrow Y'$ is quasi-finite. Since π' is proper, it follows that π' is a finite covering (see [11], I 1.10). Thus $\pi'_! \bar{\mathbb{Q}}_{l\tilde{Y}'}$ is a well-defined local system on Y' and the intersection cohomology complex $IC(\mathfrak{g}^*, \pi'_! \bar{\mathbb{Q}}_{l\tilde{Y}'})$ is well defined.

Proposition 5.6. $\varphi'_! \bar{\mathbb{Q}}_{lX'}$ is canonically isomorphic to $IC(\mathfrak{g}^*, \pi'_! \bar{\mathbb{Q}}_{l\tilde{Y}'})$. Moreover, $\text{End}(\varphi'_! \bar{\mathbb{Q}}_{lX'}) = \text{End}(\pi'_! \bar{\mathbb{Q}}_{l\tilde{Y}'}) = \mathbb{Q}_l[W]$.

Proof. Using $\rho : \tilde{Y}' \xrightarrow{\sim} \varphi'^{-1}(Y')$ and $\bar{\mathbb{Q}}_{lX'}|_{\varphi'^{-1}(Y')} \cong \bar{\mathbb{Q}}_{l\tilde{Y}'}$, we have $\varphi'_! \bar{\mathbb{Q}}_{lX'}|_{Y'} = \pi'_! \bar{\mathbb{Q}}_{l\tilde{Y}'}$, by base change theorem (see for example [11]). Since φ' is proper and X' is smooth of dimension equal to $\dim Y'$, we have that the Verdier dual (see for example [3], 12.13) $\mathfrak{D}(\varphi'_! \bar{\mathbb{Q}}_{lX'}) = \varphi'_!(\mathfrak{D} \bar{\mathbb{Q}}_{lX'}) \cong \varphi'_! \bar{\mathbb{Q}}_{lX'}[2 \dim Y']$. Hence by the definition of intersection cohomology complex, it is enough to prove that

$$\forall i > 0, \dim \text{supp} \mathcal{H}^i(\varphi'_! \bar{\mathbb{Q}}_{lX'}) < \dim Y' - i.$$

For $\xi \in \mathfrak{g}^*$, the stalk $\mathcal{H}^i(\varphi'_! \bar{\mathbb{Q}}_{lX'})$ coincides with $H_c^i(\varphi'^{-1}(\xi), \bar{\mathbb{Q}}_l)$. Hence it is enough to show $\forall i > 0, \dim\{\xi \in \mathfrak{g}^* | H_c^i(\varphi'^{-1}(\xi), \bar{\mathbb{Q}}_l) \neq 0\} < \dim Y' - i$. If $H_c^i(\varphi'^{-1}(\xi), \bar{\mathbb{Q}}_l) \neq 0$, then $i \leq 2 \dim \varphi'^{-1}(\xi)$. Hence it is enough to show that

$$\forall i > 0, \dim\{\xi \in \mathfrak{g}^* | \dim \varphi'^{-1}(\xi) \geq i/2\} < \dim Y' - i.$$

Suppose this is not true for some i , then $\dim\{\xi \in \mathfrak{g}^* | \dim \varphi'^{-1}(\xi) \geq i/2\} \geq \dim Y' - i$. Let $V = \{\xi \in \mathfrak{g}^* | \dim \varphi'^{-1}(\xi) \geq i/2\}$, it is closed in $\mathfrak{g}^*$ but not equal to $\mathfrak{g}^*$. Consider the map $p : Z \rightarrow \mathfrak{g}^*$, $(\xi, B_1, B_2) \mapsto \xi$. We have $\dim p^{-1}(V) = \dim V + 2 \dim \varphi'^{-1}(\xi) \geq \dim V + i \geq \dim Y'$ (for some $\xi \in V$). Thus by Lemma 5.1 (iii), $p^{-1}(V)$ contains some $Z_{\mathcal{O}}$, $\mathcal{O} = G$ -orbit of (B, nBn^{-1}) in $\mathcal{B} \times \mathcal{B}$, where n is some element in NT . If $\xi \in \mathfrak{t}'_0$, then $(\xi, B, nBn^{-1}) \in Z_{\mathcal{O}}$, hence ξ belongs to the projection of $p^{-1}(V)$ to $\mathfrak{g}^*$ which has dimension $\dim V < \dim Y'$. But this projection is G -invariant hence contains all Y' . We get a contradiction.

Since π' is a principal W -bundle, we have $\text{End}(\pi'_! \bar{\mathbb{Q}}_{l\tilde{Y}'}) = \bar{\mathbb{Q}}_l[W]$ (see for example [3], Lemma 12.9). It follows that $\text{End}(\varphi'_! \bar{\mathbb{Q}}_{lX'}) = \bar{\mathbb{Q}}_l[W]$. $\square$

5.4. In this subsection, we introduce some sheaves on the variety of semisimple G -orbits in $\mathfrak{g}^*$ similar to [7, 8].

Lemma 5.7 ([4], Theorem 4(ii)). *A G -orbit in $\mathfrak{g}^*$ is closed if and only if it consists of semisimple elements.*

Let $\mathbf{A}$ be the set of closed G -orbits in $\mathfrak{g}^*$. By geometric invariant theory, $\mathbf{A}$ has a natural structure of affine variety and there is a well-defined morphism $\sigma : \mathfrak{g}^* \rightarrow \mathbf{A}$ such that $\sigma(\xi)$ is the G -conjugacy class of ξ_s , where $\xi = \xi_s + \xi_n$ is the Jordan decomposition of ξ (see [4] for the notion of Jordan decomposition for $\xi \in \mathfrak{g}^*$). There is a unique $\varsigma \in \mathbf{A}$ such that $\sigma^{-1}(\varsigma) = \{\xi \in \mathfrak{g}^* | \xi \text{ nilpotent}\}$.

Recall that $Z = \{(\xi, B_1, B_2) \in \mathfrak{g}^* \times \mathcal{B} \times \mathcal{B} | \xi \in \mathfrak{b}'_1 \cap \mathfrak{b}'_2\}$. Define $\tilde{\sigma} : Z \rightarrow \mathbf{A}$ by $\tilde{\sigma}(\xi, B_1, B_2) = \sigma(\xi)$. For $a \in \mathbf{A}$, let $Z^a = \tilde{\sigma}^{-1}(a)$.

Lemma 5.8. *We have $\dim Z^a \leq d_0$, where $d_0 = \dim G - r$.*

Proof. Define $m : Z^a \rightarrow \sigma^{-1}(a)$ by $(\xi, B_1, B_2) \mapsto \xi$. Let $c \subset \sigma^{-1}(a)$ be a G -orbit. Consider $m : m^{-1}(c) \rightarrow c$. We have $\dim m^{-1}(c) \leq \dim c + 2(\dim G - r - \dim c)/2 = \dim G - r$ (use Lemma 5.1(ii)). Since $\sigma^{-1}(a)$ is a union of finitely many G -orbits, it follows that $\dim Z^a \leq d_0$. $\square$

Let $\mathcal{T} = \mathcal{H}^{2d_0} \bar{\sigma}_! \bar{\mathbb{Q}}_{lZ}$. Recall that we set $Z_{\mathcal{O}} = \{(\xi, B_1, B_2) \in Z \mid (B_1, B_2) \in \mathcal{O}\}$, where $\mathcal{O}$ is an orbit of G action on $\mathcal{B} \times \mathcal{B}$. Let $\mathcal{T}^{\mathcal{O}} = \mathcal{H}^{2d_0} \sigma_!^0 \bar{\mathbb{Q}}_l$, where $\sigma^0 : Z_{\mathcal{O}} \rightarrow \mathbf{A}$ is the restriction of $\bar{\sigma}$ on $Z_{\mathcal{O}}$.

Lemma 5.9. *We have $\mathcal{T}^{\mathcal{O}} \cong \bar{\sigma}_! \bar{\mathbb{Q}}_l$, where $\bar{\sigma} : \mathfrak{t}' \rightarrow \mathbf{A}$ is the restriction of σ .*

Proof. The fiber of the natural projection $pr_{23} : Z_{\mathcal{O}} \rightarrow \mathcal{O}$ at $(B, nBn^{-1}) \in \mathcal{O}$ (where $n \in NT$) can be identified with $V = \mathfrak{b}' \cap n.\mathfrak{b}'$. Let $\mathcal{T}'^{\mathcal{O}} = \mathcal{H}^{2d_0-2\dim \mathcal{O}} \sigma_!'^0 \bar{\mathbb{Q}}_l$, where $\sigma' : V \rightarrow \mathbf{A}$ is $\xi \mapsto \sigma(\xi)$. Let $\mathcal{T}''^{\mathcal{O}} = \mathcal{H}^{2d_0+2\dim H} \sigma_!''^0 \bar{\mathbb{Q}}_l$, where $H = B \cap nBn^{-1}$ and $\sigma'' : G \times V \rightarrow \mathbf{A}$ is $(g, \xi) \mapsto \sigma(\xi)$. Consider the composition $G \times V \xrightarrow{pr_2} V \xrightarrow{\sigma'} \mathbf{A}$ (equal to σ'') and the composition $G \times V \xrightarrow{p} H \backslash (G \times V) = Z_{\mathcal{O}} \xrightarrow{\sigma^0} \mathbf{A}$ (equal to σ''), we obtain

$$\begin{aligned} \mathcal{T}''^{\mathcal{O}} &= \mathcal{H}^{2d_0+2\dim H} (\sigma_!'^0 pr_{2!} \bar{\mathbb{Q}}_l) = \mathcal{H}^{2d_0+2\dim H} (\sigma_!'^0 \bar{\mathbb{Q}}_l[-2\dim G]) = \mathcal{T}'^{\mathcal{O}}, \\ \mathcal{T}''^{\mathcal{O}} &= \mathcal{H}^{2d_0+2\dim H} (\sigma_!^0 p_! \bar{\mathbb{Q}}_l) = \mathcal{H}^{2d_0+2\dim H} (\sigma_!^0 \bar{\mathbb{Q}}_l[-2\dim H]) = \mathcal{T}^{\mathcal{O}}. \end{aligned}$$

It follows that $\mathcal{T}^{\mathcal{O}} = \mathcal{T}'^{\mathcal{O}} = \mathcal{H}^{2d_0-2\dim \mathcal{O}} (\sigma_!'^0 \bar{\mathbb{Q}}_l)$, since $\dim \mathcal{O} = \dim G - \dim H$.

Now the map $\sigma' : V \rightarrow \mathbf{A}$ factors as $V \xrightarrow{a'} \mathfrak{t}' \xrightarrow{\bar{\sigma}} \mathbf{A}$. Since the map a' has fibers $\mathfrak{n}' \cap n.\mathfrak{n}'$ isomorphic to affine spaces of dimension $d_0 - \dim \mathcal{O}$, we have $a_!'^0 \bar{\mathbb{Q}}_l \cong \bar{\mathbb{Q}}_l[-2(d_0 - \dim \mathcal{O})]$ (see [5], VI Lemma 2.3). Hence $\mathcal{T}^{\mathcal{O}} \cong \mathcal{H}^{2d_0-2\dim \mathcal{O}} (\bar{\sigma}_! a_!'^0 \bar{\mathbb{Q}}_l) \cong \mathcal{H}^0 \bar{\sigma}_! \bar{\mathbb{Q}}_l$. Since $\bar{\sigma}$ is a finite covering (Lemma 5.10), it follows that $\mathcal{T}^{\mathcal{O}} \cong \bar{\sigma}_! \bar{\mathbb{Q}}_l$. $\square$

Lemma 5.10. *The map $\bar{\sigma} : \mathfrak{t}' \rightarrow \mathbf{A}$ is a finite covering.*

Proof. By [4], Theorem 4 (i), the natural map $\mathbf{k}[\mathfrak{g}^*]^G \rightarrow \mathbf{k}[\mathfrak{t}']^W$ is an isomorphism of algebras. It follows that the variety $\mathbf{A}$ is isomorphic to $\mathfrak{t}'/W$. Thus the map $\bar{\sigma}$ is finite. $\square$

Denote $\mathcal{T}_{\varsigma}$ and $\mathcal{T}_{\varsigma}^{\mathcal{O}}$ the stalk of $\mathcal{T}$ and $\mathcal{T}^{\mathcal{O}}$ at ς , respectively.

Lemma 5.11. *For $w \in W$, let $\mathcal{O}_w$ be the G -orbit on $\mathcal{B} \times \mathcal{B}$ which contains $(B, n_w B n_w^{-1})$. There is a canonical isomorphism $\mathcal{T}_{\varsigma} \cong \bigoplus_{w \in W} \mathcal{T}_{\varsigma}^{\mathcal{O}_w}$.*

Proof. We have $\bar{\sigma}^{-1}(\varsigma) = Z' = \{(\xi, B_1, B_2) \in \mathfrak{g}^* \times \mathcal{B} \times \mathcal{B} \mid \xi \in \mathfrak{n}'_1 \cap \mathfrak{n}'_2\}$. We have a partition $Z' = \bigsqcup_{w \in W} Z'_{\mathcal{O}_w}$, where $Z'_{\mathcal{O}_w} = \{(\xi, B_1, B_2) \in Z' \mid (B_1, B_2) \in \mathcal{O}_w\}$. Since $\dim Z' = d_0$, we have an isomorphism

$$H_c^{2d_0}(Z', \bar{\mathbb{Q}}_l) = \bigoplus_{w \in W} H_c^{2d_0}(Z'_{\mathcal{O}_w}, \bar{\mathbb{Q}}_l),$$

which is $\mathcal{T}_{\varsigma} \cong \bigoplus_{w \in W} \mathcal{T}_{\varsigma}^{\mathcal{O}_w}$. $\square$

Recall that we have $\bar{\mathbb{Q}}_l[W] = \text{End}(\pi_!^* \bar{\mathbb{Q}}_{l\tilde{Y}'}) = \text{End}(\varphi_!^* \bar{\mathbb{Q}}_{lX'})$. In particular, $\varphi_!^* \bar{\mathbb{Q}}_{lX'}$ is naturally a W -module and $\varphi_!^* \bar{\mathbb{Q}}_{lX'} \otimes \varphi_!^* \bar{\mathbb{Q}}_{lX'}$ is naturally a W -module (with W acting on the first factor). This induces a W -module structure on $\mathcal{H}^{2d_0} \sigma_! (\varphi_!^* \bar{\mathbb{Q}}_{lX'} \otimes \varphi_!^* \bar{\mathbb{Q}}_{lX'}) = \mathcal{T}$. Hence we obtain a W -module structure on the stalk $\mathcal{T}_{\varsigma}$.

Lemma 5.12. *Let $w \in W$. Multiplication by w in the W -module structure of $\mathcal{T}_\zeta = \bigoplus_{w' \in W} \mathcal{T}_\zeta^{\mathcal{O}_{w'}}$ defines for any $w' \in W$ an isomorphism $\mathcal{T}_\zeta^{\mathcal{O}_{w'}} \xrightarrow{\sim} \mathcal{T}_\zeta^{\mathcal{O}_{ww'}}$.*

Proof. We have an isomorphism

$$f : Z'_{\mathcal{O}_{w'}} \xrightarrow{\sim} Z'_{\mathcal{O}_{ww'}}, (\xi, gBg^{-1}, gn_{w'}Bn_{w'}^{-1}g^{-1}) \mapsto (\xi, gn_w^{-1}Bn_wg^{-1}, gn_{w'}Bn_{w'}^{-1}g^{-1}).$$

This induces an isomorphism

$$H_c^{2d_0}(Z'_{\mathcal{O}_{w'}}, \bar{\mathbb{Q}}_l) \xrightarrow{\sim} H_c^{2d_0}(Z'_{\mathcal{O}_{ww'}}, \bar{\mathbb{Q}}_l),$$

which is just multiplication by w . $\square$

5.5. Let $\hat{W}$ be the set of simple modules (up to isomorphism) for the Weyl group W of G (A description of $\hat{W}$ is given for example in [10]). Given a semisimple object M of some abelian category such that M is a W -module, we write $M_\rho = \text{Hom}_{\bar{\mathbb{Q}}_l[W]}(\rho, M)$ for $\rho \in \hat{W}$. We have $M = \bigoplus_{\rho \in \hat{W}} (\rho \otimes M_\rho)$ with W acting on the ρ -factor and M_ρ is in our abelian category. In particular, we have

$$\pi'_! \bar{\mathbb{Q}}_{l\tilde{Y}'} = \bigoplus_{\rho \in \hat{W}} (\rho \otimes (\pi'_! \bar{\mathbb{Q}}_{l\tilde{Y}'})_\rho),$$

where $(\pi'_! \bar{\mathbb{Q}}_{l\tilde{Y}'})_\rho$ is an irreducible local system on Y . We have

$$\varphi'_! \bar{\mathbb{Q}}_{lX'} = \bigoplus_{\rho \in \hat{W}} (\rho \otimes (\varphi'_! \bar{\mathbb{Q}}_{lX'})_\rho),$$

where $(\varphi'_! \bar{\mathbb{Q}}_{lX'})_\rho = IC(G^*, (\pi'_! \bar{\mathbb{Q}}_{l\tilde{Y}'})_\rho)$. Moreover, for $a \in \mathbf{A}$, we have $\mathcal{T}_a = \bigoplus_{\rho \in \hat{W}} (\rho \otimes (\mathcal{T}_a)_\rho)$. Set

$$\mathfrak{g}^{*\varsigma} = \{\xi \in \mathfrak{g}^* \mid \sigma(\xi) = \varsigma\}, X'^\varsigma = \varphi'^{-1}(\mathfrak{g}^{*\varsigma}) \subset X'.$$

We have $\mathfrak{g}^{*\varsigma} = \{\xi \in \mathfrak{g}^* \mid \xi \text{ nilpotent}\}$. Let $\varphi'^\varsigma : X'^\varsigma \rightarrow \mathfrak{g}^{*\varsigma}$ be the restriction of $\varphi' : X' \rightarrow \mathfrak{g}^*$.

Lemma 5.13. *There exists a nilpotent element ξ in $\mathfrak{g}^*$ such that the set $\{B_1 \in \mathcal{B} \mid \xi \in \mathfrak{n}'_1\}$ is finite.*

Proof. Let R be the root system of G relative to T . We have a weight space decomposition $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid \text{Ad}(t)x = \alpha(t)x, \forall t \in T\}$ is one-dimensional for $\alpha \in R$ (see for example [12]). Let $\alpha_i, i = 1, \dots, r$ be a set of simple roots in R such that $\mathfrak{b} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha$ and x_α , and let $\{x_\alpha, \alpha \in R, h_{\alpha_i}\}$ be a Chevalley basis in $\mathfrak{g}$. Let x_α^* and $h_{\alpha_i}^*$ be the dual basis in $\mathfrak{g}^*$. Set $\xi = \sum_{i=1}^r x_{-\alpha_i}^*$. Then $\xi \in \mathfrak{n}'$.

We show that $\{B_1 \in \mathcal{B} \mid \xi \in \mathfrak{n}'_1\} = \{B\}$. Assume $g\xi \in \mathfrak{n}'$. We have $\xi(g^{-1}\mathfrak{b}g) = 0$. By Bruhat decomposition, we can write $g^{-1} = vn_wb$, where $v \in U \cap wUw^{-1}$ and $n_w \in NT$ is a representative for $w \in W$. Assume $w \neq 1$. There exists $1 \leq i \leq r$ such that $w^{-1}\alpha_i < 0$. Let $\alpha = -w^{-1}\alpha_i > 0$. We have $\xi(\text{Ad}(vn_w)x_\alpha) = \xi(c\text{Ad}(v)x_{-\alpha_i}) = \xi(cx_{-\alpha_i}) = c$, where c is a nonzero constant. This contradicts $\xi(g^{-1}\mathfrak{b}g) = 0$. Thus $w = 1$ and $g^{-1}\mathfrak{n}' = \mathfrak{n}'$. $\square$

Lemma 5.14. (i) X'^ς and $\mathfrak{g}^{*\varsigma}$ are irreducible varieties of dimension $d_0 = \dim G - r$.

(ii) We have $(\varphi'_! \bar{\mathbb{Q}}_{lX'})|_{\mathfrak{g}^{*\varsigma}} = \varphi'^\varsigma_! \bar{\mathbb{Q}}_{lX'^\varsigma}$. Moreover, $\varphi'^\varsigma_! \bar{\mathbb{Q}}_{lX'^\varsigma}[d_0]$ is a semisimple perverse sheaf on $\mathfrak{g}^{*\varsigma}$.

(iii) We have $(\varphi'_! \bar{\mathbb{Q}}_{lX'})_\rho|_{\mathfrak{g}^{*\varsigma}} \neq 0$ for any $\rho \in \hat{W}$.

Proof. (i) We have $X^\varsigma = \{(\xi, gB) \in \mathfrak{g}^* \times \mathcal{B} \mid g^{-1}\xi \in \mathfrak{n}\}$. By projection to the second coordinate, we see that $\dim X^\varsigma = \dim \mathfrak{n}' + \dim \mathcal{B} = \dim G - r$. The map φ'^ς is surjective and the fiber at some point ξ is finite (see Lemma 5.13). It follows that $\dim \mathfrak{g}^{*\varsigma} = \dim G - r$. This proves (i).

The first assertion of (ii) follows from the base change theorem. Since φ'^ς is proper, by a similar argument as in the proof of Proposition 5.6, to show that $\varphi'^\varsigma_! \bar{\mathbb{Q}}_{lX^\varsigma}[d_0]$ is a perverse sheaf, it suffices to show that

$$\forall i \geq 0, \dim \operatorname{supp} \mathcal{H}^i(\varphi'^\varsigma_! \bar{\mathbb{Q}}_{lX^\varsigma}) \leq \dim \mathfrak{g}^{*\varsigma} - i.$$

It is enough to show that $\forall i \geq 0, \dim\{\xi \in \mathfrak{g}^{*\varsigma} \mid \dim(\varphi'^\varsigma)^{-1}(\xi) \geq i/2\} \leq \dim \mathfrak{g}^{*\varsigma} - i$. If this is not true for some $i \geq 0$, it would follow that the variety $\{(\xi, B_1, B_2) \in \mathfrak{g}^* \times \mathcal{B} \times \mathcal{B} \mid \xi \in \mathfrak{n}'_1 \cap \mathfrak{n}'_2\}$ has dimension greater than $\dim \mathfrak{g}^{*\varsigma} = \dim G - r$, which contradicts Lemma 5.1. This proves that $\varphi'^\varsigma_! \bar{\mathbb{Q}}_{lX^\varsigma}[d_0]$ is a perverse sheaf. It is semisimple by the decomposition theorem [1]. This proves (ii).

Now we prove (iii). By Lemma 5.9, we have $\mathcal{T}_\varsigma^{\mathcal{O}_1} = H_c^0(\mathfrak{t}' \cap \sigma^{-1}(\varsigma), \bar{\mathbb{Q}}_l) \neq 0$. From Lemma 5.12, we see that the W -module structure defines an injective map $\bar{\mathbb{Q}}_l[W] \otimes \mathcal{T}_\varsigma^{\mathcal{O}_1} \rightarrow \mathcal{T}_\varsigma$. Since $\mathcal{T}_\varsigma^{\mathcal{O}_1} \neq 0$, we have $(\bar{\mathbb{Q}}_l[W] \otimes \mathcal{T}_\varsigma^{\mathcal{O}_1})_\rho \neq 0$ for any $\rho \in \hat{W}$, hence $(\mathcal{T}_\varsigma)_\rho \neq 0$. We have $\mathcal{T}_\varsigma = H_c^{2d_0}(\mathfrak{g}^{*\varsigma}, \varphi'_! \bar{\mathbb{Q}}_{lX'} \otimes \varphi'_! \bar{\mathbb{Q}}_{lX'})$, hence

$$\bigoplus_{\rho \in \hat{W}} \rho \otimes (\mathcal{T}_\varsigma)_\rho = \bigoplus_{\rho \in \hat{W}} \rho \otimes H_c^{2d_0}(\mathfrak{g}^{*\varsigma}, (\varphi'_! \bar{\mathbb{Q}}_{lX'})_\rho \otimes \varphi'_! \bar{\mathbb{Q}}_{lX'}).$$

This implies that $(\mathcal{T}_\varsigma)_\rho = H_c^{2d_0}(\mathfrak{g}^{*\varsigma}, (\varphi'_! \bar{\mathbb{Q}}_{lX'})_\rho \otimes \varphi'_! \bar{\mathbb{Q}}_{lX'})$. Thus it follows from $(\mathcal{T}_\varsigma)_\rho \neq 0$ that $(\varphi'_! \bar{\mathbb{Q}}_{lX'})_\rho|_{\mathfrak{g}^{*\varsigma}} \neq 0$ for any $\rho \in \hat{W}$. $\square$

Let $\mathfrak{A}'$ be the set of all pairs $(c', \mathcal{F}')$ where c' is a nilpotent G -orbit in $\mathfrak{g}^*$ and $\mathcal{F}'$ is an irreducible G -equivariant local system on c' (up to isomorphism).

Proposition 5.15. (i) *The restriction map*

$$\operatorname{End}_{\mathcal{D}(\mathfrak{g}^*)}(\varphi'_! \bar{\mathbb{Q}}_{lX'}) \rightarrow \operatorname{End}_{\mathcal{D}(\mathfrak{g}^{*\varsigma})}(\varphi'^\varsigma_! \bar{\mathbb{Q}}_{lX^\varsigma})$$

is an isomorphism.

(ii) *For any $\rho \in \hat{W}$, there is a unique $(c', \mathcal{F}') \in \mathfrak{A}'$ such that $(\varphi'_! \bar{\mathbb{Q}}_{lX'})_\rho|_{\mathfrak{g}^{*\varsigma}}[d_0]$ is $IC(\bar{c}', \mathcal{F}')[\dim c']$ regarded as a simple perverse sheaf on $\mathfrak{g}^{*\varsigma}$ (zero outside c'). Moreover, $\rho \mapsto (c', \mathcal{F}')$ is an injective map $\gamma: \hat{W} \rightarrow \mathfrak{A}'$.*

Proof. (i) Recall that we have

$$\varphi'_! \bar{\mathbb{Q}}_{lX'} = \bigoplus_{\rho \in \hat{W}} \rho \otimes (\varphi'_! \bar{\mathbb{Q}}_{lX'})_\rho \text{ where } (\varphi'_! \bar{\mathbb{Q}}_{lX'})_\rho[\dim \mathfrak{g}^*]$$

are simple perverse sheaves on $\mathfrak{g}^*$. Thus we have $\varphi'_! \bar{\mathbb{Q}}_{lX'}|_{\mathfrak{g}^{*\varsigma}} = \varphi'^\varsigma_! \bar{\mathbb{Q}}_{lX^\varsigma} = \bigoplus_{\rho \in \hat{W}} \rho \otimes (\varphi'_! \bar{\mathbb{Q}}_{lX'})_\rho|_{\mathfrak{g}^{*\varsigma}}$ (we use Lemma 5.14 (ii)). The restriction map $\operatorname{End}_{\mathcal{D}(\mathfrak{g}^*)}(\varphi'_! \bar{\mathbb{Q}}_{lX'}) \rightarrow \operatorname{End}_{\mathcal{D}(\mathfrak{g}^{*\varsigma})}(\varphi'^\varsigma_! \bar{\mathbb{Q}}_{lX^\varsigma})$ is factored as

$$\begin{aligned} & \bigoplus_{\rho \in \hat{W}} \operatorname{End}_{\mathcal{D}(\mathfrak{g}^*)}(\rho \otimes (\varphi'_! \bar{\mathbb{Q}}_{lX'})_\rho) \\ & \xrightarrow{b} \bigoplus_{\rho \in \hat{W}} \operatorname{End}_{\mathcal{D}(\mathfrak{g}^{*\varsigma})}(\rho \otimes (\varphi'_! \bar{\mathbb{Q}}_{lX'})_\rho|_{\mathfrak{g}^{*\varsigma}}) \xrightarrow{c} \operatorname{End}_{\mathcal{D}(\mathfrak{g}^{*\varsigma})}(\varphi'^\varsigma_! \bar{\mathbb{Q}}_{lX^\varsigma}) \end{aligned}$$

where

$$b = \bigoplus_{\rho} b_{\rho}, b_{\rho} : \text{End}(\rho) \otimes \text{End}_{\mathcal{D}(\mathfrak{g}^*)}((\varphi'_! \bar{\mathbb{Q}}_{lX'})_{\rho}) \rightarrow \text{End}(\rho) \otimes \text{End}_{\mathcal{D}(\mathfrak{g}^{*\varsigma})}((\varphi'_! \bar{\mathbb{Q}}_{lX'})_{\rho}|_{\mathfrak{g}^{*\varsigma}}).$$

By Lemma 5.14 (iii), $(\varphi'_! \bar{\mathbb{Q}}_{lX'})_{\rho}|_{\mathfrak{g}^{*\varsigma}} \neq 0$, thus

$$\text{End}_{\mathcal{D}(\mathfrak{g}^*)}((\varphi'_! \bar{\mathbb{Q}}_{lX'})_{\rho}) = \bar{\mathbb{Q}}_l \subset \text{End}_{\mathcal{D}(\mathfrak{g}^{*\varsigma})}((\varphi'_! \bar{\mathbb{Q}}_{lX'})_{\rho}|_{\mathfrak{g}^{*\varsigma}}).$$

It follows that b_{ρ} and thus b is injective. Since c is also injective, the restriction map is injective. Hence it remains to show that

$$\dim \text{End}_{\mathcal{D}(\mathfrak{g}^{*\varsigma})}(\varphi'_! \bar{\mathbb{Q}}_{lX'^{\varsigma}}) = \dim \text{End}_{\mathcal{D}(\mathfrak{g}^*)}(\varphi'_! \bar{\mathbb{Q}}_{lX'}).$$

For A, A' , two simple perverse sheaves on a variety X , we have $H_c^0(X, A \otimes A') = 0$ if and only if A is not isomorphic to $\mathfrak{D}(A')$ and $\dim H_c^0(X, A \otimes \mathfrak{D}(A)) = 1$ (see [9], section 7.4). We apply this to the semisimple perverse sheaf $\varphi'_! \bar{\mathbb{Q}}_{lX'^{\varsigma}}[d_0]$ on $\mathfrak{g}^{*\varsigma}$ and get

$$\begin{aligned} \dim \text{End}_{\mathcal{D}(\mathfrak{g}^{*\varsigma})}(\varphi'_! \bar{\mathbb{Q}}_{lX'^{\varsigma}}) &= \dim H_c^0(\mathfrak{g}^{*\varsigma}, \varphi'_! \bar{\mathbb{Q}}_{lX'^{\varsigma}}[d_0] \otimes \mathfrak{D}(\varphi'_! \bar{\mathbb{Q}}_{lX'^{\varsigma}}[d_0])) \\ &= \dim H_c^0(\mathfrak{g}^{*\varsigma}, \varphi'_! \bar{\mathbb{Q}}_{lX'^{\varsigma}}[d_0] \otimes \varphi'_! \bar{\mathbb{Q}}_{lX'^{\varsigma}}[d_0]) \\ &= \dim H_c^{2d_0}(\mathfrak{g}^{*\varsigma}, \varphi'_! \bar{\mathbb{Q}}_{lX'^{\varsigma}} \otimes \varphi'_! \bar{\mathbb{Q}}_{lX'^{\varsigma}}) \\ &= \dim H_c^{2d_0}(\mathfrak{g}^{*\varsigma}, \varphi'_! \bar{\mathbb{Q}}_{lX'} \otimes \varphi'_! \bar{\mathbb{Q}}_{lX'}) = \dim \mathcal{T}_{\varsigma} = \sum_{w \in W} \dim \mathcal{T}_{\varsigma}^{\mathcal{O}_w}. \end{aligned}$$

(The fourth equality follows from Lemma 5.14(ii) and the last one follows from Lemma 5.11.)

We have $\mathcal{T}_{\varsigma}^{\mathcal{O}_w} = H_c^0(\bar{\sigma}^{-1}(\varsigma), \bar{\mathbb{Q}}_l)$ (see Lemma 5.9), hence $\dim \mathcal{T}_{\varsigma}^{\mathcal{O}_w} = 1$ and

$$\sum_{w \in W} \dim \mathcal{T}_{\varsigma}^{\mathcal{O}_w} = |W| = \dim \text{End}_{\mathcal{D}(\mathfrak{g}^*)}(\varphi'_! \bar{\mathbb{Q}}_{lX'}).$$

Thus (i) is proved.

From the proof of (i) we see that both b and c are isomorphisms. It follows that the perverse sheaf $(\varphi'_! \bar{\mathbb{Q}}_{lX'})_{\rho}|_{\mathfrak{g}^{*\varsigma}}[d_0]$ on $\mathfrak{g}^{*\varsigma}$ is simple and that for $\rho, \rho' \in \hat{W}$, we have $(\varphi'_! \bar{\mathbb{Q}}_{lX'})_{\rho}|_{\mathfrak{g}^{*\varsigma}}[d_0] \cong (\varphi'_! \bar{\mathbb{Q}}_{lX'})_{\rho'}|_{\mathfrak{g}^{*\varsigma}}[d_0]$ if and only if $\rho = \rho'$. Since the simple perverse sheaf $(\varphi'_! \bar{\mathbb{Q}}_{lX'})_{\rho}|_{\mathfrak{g}^{*\varsigma}}[d_0]$ is G -equivariant and $\mathfrak{g}^{*\varsigma}$ consists of finitely many nilpotent G -orbits, $(\varphi'_! \bar{\mathbb{Q}}_{lX'})_{\rho}|_{\mathfrak{g}^{*\varsigma}}[d_0]$ must be as in (ii). $\square$

5.6. In this subsection let $G = SO_N(\mathbf{k})$ (resp. $Sp_{2n}(\mathbf{k})$) and $\mathfrak{g} = \mathfrak{o}_N(\mathbf{k})$ (resp. $\mathfrak{sp}_{2n}(\mathbf{k})$) be the Lie algebra of G . Let G_s be a simply connected group over $\mathbf{k}$ of the same type as G and let $\mathfrak{g}_s$ be the Lie algebra of G_s . For q a power of 2, let $G(\mathbf{F}_q)$, $\mathfrak{g}(\mathbf{F}_q)$ be the fixed points of a split Frobenius map $\mathfrak{F}_q$ relative to $\mathbf{F}_q$ on G , $\mathfrak{g}$. Let $G_s(\mathbf{F}_q)$, $\mathfrak{g}_s(\mathbf{F}_q)$ be defined as $G(\mathbf{F}_q)$, $\mathfrak{g}(\mathbf{F}_q)$. Let $\mathfrak{A}'$ be the set of all pairs $(c', \mathcal{F}')$ where c' is a nilpotent G -orbit in $\mathfrak{g}^*$ and $\mathcal{F}'$ is an irreducible G -equivariant local system on c' (up to isomorphism). Let $\mathfrak{A}'_s$ be defined for G_s as in the introduction. We show that the number of elements in $\mathfrak{A}'_s$ is equal to the number of elements in $\mathfrak{A}'$.

We first show that the number of elements in $\mathfrak{A}'$ is equal to the number of nilpotent $G(\mathbf{F}_q)$ -orbits in $\mathfrak{g}(\mathbf{F}_q)^*$ (for q large). To see this we can assume $\mathbf{k} = \mathbf{F}_2$. Pick representatives $\xi_1, \dots, \xi_M$ for the nilpotent G -orbits in $\mathfrak{g}^*$. If q is large enough, the Frobenius map $\mathfrak{F}_q$ keeps ξ_i fixed and acts trivially on $Z_G(\xi_i)/Z_G^0(\xi_i)$. Then the number of $G(\mathbf{F}_q)$ -orbits in the G -orbit of ξ_i is equal to the number of irreducible representations of $Z_G(\xi_i)/Z_G^0(\xi_i)$ hence to the number of G -equivariant irreducible

local systems on the G -orbit of ξ_i . Similarly, the number of elements in $\mathfrak{A}'_s$ is equal to the number of nilpotent $G_s(\mathbf{F}_q)$ -orbits in $\mathfrak{g}_s(\mathbf{F}_q)^*$.

On the other hand, the number of nilpotent $G(\mathbf{F}_q)$ -orbits in $\mathfrak{g}(\mathbf{F}_q)^*$ is equal to the number of nilpotent $G_s(\mathbf{F}_q)$ -orbits in $\mathfrak{g}_s(\mathbf{F}_q)^*$. In fact, we have a morphism $G_s \rightarrow G$ which is an isomorphism of abstract groups and an obvious bijective morphism $\mathcal{N}' \rightarrow \mathcal{N}'_s$ where $\mathcal{N}'$ (resp. $\mathcal{N}'_s$) is the set of nilpotent elements in $\mathfrak{g}^*$ (resp. $\mathfrak{g}_s^*$). Thus the nilpotent orbits in $\mathfrak{g}^*$ and $\mathfrak{g}_s^*$ are in bijection and the corresponding component groups of centralizers are isomorphic. It follows that $|\mathfrak{A}'| = |\mathfrak{A}'_s|$.

Corollary 5.16. $|\mathfrak{A}'| = |\mathfrak{A}'_s| = |\hat{W}|$.

Proof. Assume G is $SO_{2n}(\mathbf{k})$. The assertion follows from the above argument, Proposition 4.1, and Corollary 6.17 in [13]. Assume G is $Sp_{2n}(\mathbf{k})$ or $O_{2n+1}(\mathbf{k})$. It follows from Proposition 5.15(ii) that $|\mathfrak{A}'| = |\mathfrak{A}'_s|$ is greater than $|\hat{W}|$. On the other hand, it is known that $|\hat{W}| = p_2(n)$ (see [10]). Hence $|\mathfrak{A}'| = |\mathfrak{A}'_s|$ is less than $|\hat{W}|$ by Proposition 2.17, Proposition 3.18 and the above argument. $\square$

Theorem 5.17. *The map γ in Proposition 5.15(ii) is a bijection.*

Corollary 5.18. *Proposition 2.15, Corollary 2.16, Proposition 2.17, Proposition 3.17, and Proposition 3.18 hold with all “at most” removed.*

Proof. For q large enough, this follows from Corollary 5.16. Now let q be an arbitrary power of 2. Let $(c', \mathcal{F}')$ be a pair in $\mathfrak{A}'_s$. Since the Springer correspondence map γ in Proposition 5.15(ii) is bijective by Corollary 5.16, there exists $\rho \in \hat{W}$ corresponding to $(c', \mathcal{F}')$ under the map γ . It follows that the pair $(\mathfrak{F}_q^{-1}(c'), \mathfrak{F}_q^{-1}(\mathcal{F}'))$ corresponds to $\mathfrak{F}_q^{-1}(\rho) \in \hat{W}$. Since the Frobenius map $\mathfrak{F}_q$ acts trivially on W and γ is injective, it follows that c' is stable under $\mathfrak{F}_q$ and $\mathfrak{F}_q^{-1}(\mathcal{F}') \cong \mathcal{F}'$. Pick a rational point ξ in c' . The G_s -equivariant local systems on c are in one-to-one correspondence with the isomorphism classes of the irreducible representations of $Z_{G_s}(\xi)/Z_{G_s}^0(\xi)$. Since $Z_{G_s}(\xi)/Z_{G_s}^0(\xi)$ is abelian (see Propositions 6.2 and 6.7) and the Frobenius map $\mathfrak{F}_q$ acts trivially on the irreducible representations of $Z_{G_s}(\xi)/Z_{G_s}^0(\xi)$, $\mathfrak{F}_q$ acts trivially on $Z_{G_s}(\xi)/Z_{G_s}^0(\xi)$. Thus it follows that the number of nilpotent $G_s(\mathbf{F}_q)$ -orbits in $\mathfrak{g}_s(\mathbf{F}_q)^*$ is independent of q hence it is equal to $|\mathfrak{A}'_s| = |\hat{W}|$. $\square$

Remark 5.19. Let G_{ad} be an adjoint algebraic group of type B, C or D over $\mathbf{k}$ and let $\mathfrak{g}_{ad}$ be its Lie algebra. Let $\mathfrak{g}_{ad}^*$ be the dual space of $\mathfrak{g}_{ad}$. In [13], we have constructed a Springer correspondence for $\mathfrak{g}_{ad}$. One can construct a Springer correspondence for $\mathfrak{g}_{ad}^*$ using the result for $\mathfrak{g}_{ad}$ and the Deligne-Fourier transform. We expect the two Springer correspondences to coincide (up to sign representation of the Weyl group). We use the approach presented above since this construction is more suitable for computing the explicit Springer correspondence.

6. CENTRALIZERS AND COMPONENT GROUPS

6.1. In this subsection assume $G = Sp(2N)$. We study some properties of the centralizer $Z_G(\xi)$ for a nilpotent element $\xi \in \mathfrak{g}^*$ and the component group $Z_G(\xi)/Z_G^0(\xi)$. Let $V = {}^*W_{\chi(m_1)}(m_1) \oplus {}^*W_{\chi(m_2)}(m_2) \oplus \cdots \oplus {}^*W_{\chi(m_s)}(m_s)$, $m_1 \geq \cdots \geq m_s$ be a form module corresponding to $\xi \in \mathfrak{g}^*$. Let T_ξ be defined as in subsection 2.1. We have $Z_G(\xi) = Z(V) = \{g \in GL(V) | \beta(gv, gw) = \beta(v, w), \alpha_\xi(gv) = \alpha_\xi(g)w, \forall v, w \in V\}$.

Proposition 6.1. $\dim Z(V) = \sum_{i=1}^s ((4i-1)m_i - 2\chi(m_i))$.

Proof. We argue by induction on s . The case $s = 1$ can be easily verified. Let $C(V) = \{g \in GL(V) \mid gT_\xi = T_\xi g\}$. Let $V_1 = {}^*W_{\chi(m_1)}(m_1)$ and $V_2 = {}^*W_{\chi(m_2)}(m_2) \oplus \cdots \oplus {}^*W_{\chi(m_s)}(m_s)$. We consider V_1 as an element in the Grassmannian variety $Gr(V, 2m_1)$ and consider the action of $C(V)$ on $Gr(V, 2m_1)$. Then the orbit of V_1 has dimension $\dim \text{Hom}_A(V_1, V_2) = 4 \sum_{i=2}^s m_i$. Now we consider the action of $Z(V)$ on $Gr(V, 2m_1)$. The orbit $Z(V)V_1$ is open dense in $C(V)V_1$ and thus has dimension $4 \sum_{i=2}^s m_i$. We claim that

(*) the stabilizer of V_1 in $Z(V)$ is the product of $Z(V_1)$ and $Z(V_2)$.

Thus using induction hypothesis and (*) we get $\dim Z(V) = \dim Z(V_1) + \dim Z(V_2) + \dim Z(V)V_1 = 3m_1 - 2\chi(m_1) + \sum_{i=2}^s ((4i-5)m_i - 2\chi(m_i)) + 4 \sum_{i=2}^s m_i = \sum_{i=1}^s ((4i-1)m_i - 2\chi(m_i))$.

Proof of (*). Assume $g : V_1 \oplus V_2 \rightarrow V_1 \oplus V_2$ lies in the stabilizer of V_1 in $Z(V)$. Let $p_{ij}, i, j = 1, 2$ be the obvious projection composed with g . Then $p_{12} = 0$. We claim that p_{11} is nonsingular. It is enough to show that p_{11} is injective. Assume $p_{11}(v_1) = 0$ for some $v_1 \in V_1$. Then we have $\beta(gv_1, gv'_1) = \beta(p_{11}v_1, gv'_1) = 0 = \beta(v_1, v'_1)$ for any $v'_1 \in V_1$. Since $\beta|_{V_1}$ is nondegenerate, we get $v_1 = 0$. Now for any $v_2 \in V_2, v_1 \in V_1$, we have $\beta(gv_1, gv_2) = \beta(p_{11}v_1, p_{21}v_2 + p_{22}v_2) = \beta(p_{11}v_1, p_{21}v_2) = \beta(v_1, v_2) = 0$. Since $\beta|_{V_1}$ is nondegenerate and p_{11} is bijective on V_1 , we get $p_{21}(v_2) = 0$. Then (*) follows. $\square$

Let $r = \#\{1 \leq i \leq s \mid \chi(m_i) + \chi(m_{i+1}) < m_i \text{ and } \chi(m_i) > \frac{m_i-1}{2}\}$.

Proposition 6.2. The component group $Z(V)/Z^0(V)$ is $(\mathbb{Z}/2\mathbb{Z})^r$.

Proof. Assume q large enough. By the same argument as in the proof of Proposition 7.1 in [13] one shows that $Z(V)/Z^0(V)$ is an abelian group of order 2^r . We show that there is a subgroup $(\mathbb{Z}/2\mathbb{Z})^r \subset Z(V)/Z(V)^0$. Thus $Z(V)/Z(V)^0$ has to be $(\mathbb{Z}/2\mathbb{Z})^r$. Let $1 \leq i_1, \dots, i_r \leq s$ be such that $\chi(m_{i_j}) > (m_{i_j} - 1)/2$ and $\chi(m_{i_j}) + \chi(m_{i_j+1}) < m_{i_j}, j = 1, \dots, r$.

Let $V_j = {}^*W_{\chi(m_{i_{j-1}+1})}(m_{i_{j-1}+1}) \oplus \cdots \oplus {}^*W_{\chi(m_{i_j})}(m_{i_j}), j = 1, \dots, r-1$, where $i_0 = 0$, and $V_r = {}^*W_{\chi(m_{i_{r-1}+1})}(m_{i_{r-1}+1}) \oplus \cdots \oplus {}^*W_{\chi(m_s)}(m_s)$. Then $V = V_1 \oplus V_2 \oplus \cdots \oplus V_r$. We have $Z(V_i)/Z^0(V_i) = \mathbb{Z}/2\mathbb{Z}, i = 1, \dots, r$. Take $g_i \in Z(V_i)$ such that $g_i Z^0(V_i)$ generates $Z(V_i)/Z^0(V_i), i = 1, \dots, r$. Let $\tilde{g}_i = Id \oplus \cdots \oplus g_i \oplus \cdots \oplus Id, i = 1, \dots, r$. Then we have $\tilde{g}_i \in Z(V)$ and $\tilde{g}_i \notin Z^0(V)$. We also have that the images of $\tilde{g}_{i_1} \cdots \tilde{g}_{i_p}$'s, $1 \leq i_1 < \cdots < i_p \leq r, p = 1, \dots, r$, in $Z(V)/Z^0(V)$ are not equal to each other. Moreover, $\tilde{g}_i^2 \in Z^0(V)$. Thus the $\tilde{g}_i Z^0(V)$'s generate a subgroup $(\mathbb{Z}/2\mathbb{Z})^r$ in $Z(V)/Z^0(V)$. $\square$

6.2. In this subsection assume $G = O(2N+1)$. We study some properties of the centralizer $Z_G(\xi)$ for a nilpotent element $\xi \in \mathfrak{g}^*$ and the component group $Z_G(\xi)/Z_G^0(\xi)$. Let (V, α, β_ξ) be a form module corresponding to $\xi \in \mathfrak{g}^*$. Assume the corresponding pair of partitions is $(\nu_0, \dots, \nu_s)(\mu_1, \dots, \mu_s)$. Let $C(V) = \{g \in GL(V) \mid \beta(gv, gw) = \beta(v, w), \beta_\xi(gv, gw) = \beta_\xi(v, w), \forall v, w \in V\}$. We have $Z_G(\xi) = Z(V) = \{g \in C(V) \mid \alpha(gv) = \alpha(v), \forall v \in V\}$.

Lemma 6.3. $|Z(V_{2m+1})(\mathbf{F}_q)| = q^m$ and $|C(V_{2m+1})(\mathbf{F}_q)| = q^{2m+1}$.

Proof. Let $V_{2m+1} = \text{span}\{v_0, \dots, v_m, u_0, \dots, u_{m-1}\}$, where v_i, u_i are chosen as in Lemma 3.5 and Lemma 3.6. Let $g \in C(V_{2m+1})$. Then $g : V_{2m+1} \rightarrow V_{2m+1}$ satisfies $\beta(gv, gw) = \beta(v, w)$ and $\beta_\xi(gv, gw) = \beta_\xi(v, w)$ for all $v, w \in V_{2m+1}$. Since $\beta_\xi(\sum_{i=0}^m v_i \lambda^i, v) + \lambda \beta(\sum_{i=0}^m v_i \lambda^i, v) = 0$, we have

$$\beta_\xi\left(\sum_{i=0}^m gv_i \lambda^i, v\right) + \lambda \beta\left(\sum_{i=0}^m gv_i \lambda^i, v\right) = 0$$

for all $v \in V_{2m+1}$. This implies $\sum_{i=0}^m gv_i \lambda^i = a \sum_{i=0}^m v_i \lambda^i$ for some $a \in \mathbf{F}_q^*$. Namely, we have $gv_i = av_i$, $i = 0, \dots, m$. Assume $gu_i = \sum_{k=0}^m a_{ik} v_k + \sum_{k=0}^{m-1} b_{ik} u_k$. We have $\beta(gv_i, gu_j) = ab_{ji} = \beta(v_i, u_j) = \delta_{i,j}$, $\beta(gu_i, gu_j) = \sum_{k=0}^{m-1} a(a_{ik} b_{jk} + b_{ik} a_{jk}) = a(a_{ij} + a_{ji}) = \beta(u_i, u_j) = 0$, $\beta_\xi(gv_{i+1}, gu_j) = ab_{ji} = \beta_\xi(v_{i+1}, u_j) = \delta_{i,j}$ and $\beta_\xi(gu_i, gu_j) = \sum_{k=1}^m a(a_{ik} b_{j,k-1} + b_{i,k-1} a_{jk}) = a(a_{i,j+1} + a_{j,i+1}) = \beta(u_i, u_j) = 0$. Thus we get $gv_i = av_i$, $i = 0, \dots, m$, and $gu_i = u_i/a + \sum_{j=0}^m a_{ij} v_j$, $i = 0, \dots, m-1$, where $a_{ij} = a_{ji}$, $a_{i,j+1} = a_{j,i+1}$, $0 \leq i, j \leq m-1$. Hence $|C(V_{2m+1})(\mathbf{F}_q)| = q^{2m+1}$.

Now assume $g \in Z(V_{2m+1})$. Then we have additional conditions $\alpha(gv_i) = a^2 \alpha(v_i) = \alpha(v_i) = \delta_{i,m} \Rightarrow a = 1$ and $\alpha(gu_i) = \alpha(u_i/a + \sum_{j=0}^m a_{ij} v_j) = a_{im}^2 + a_{ii}/a = \alpha(u_i) = 0$, $i = 0, \dots, m-1$. Hence $|Z(V_{2m+1})(\mathbf{F}_q)| = q^m$. $\square$

Write $V = V_{2m+1} \oplus W$ as in Lemma 3.8.

Lemma 6.4. $|C(V)(\mathbf{F}_q)| = |C(V_{2m+1})(\mathbf{F}_q)| \cdot |C(W)(\mathbf{F}_q)| \cdot q^{\dim W}$.

Proof. Let $g \in C(V)$. Let $p_{11} : V_{2m+1} \rightarrow V_{2m+1}$, $p_{12} : V_{2m+1} \rightarrow W$, $p_{21} : W \rightarrow V_{2m+1}$ and $p_{22} : W \rightarrow W$ be the projections composed with g . Let v_i, u_i be a basis of V_{2m+1} as before. By the same argument as in Lemma 3.9, we have $gv_i = av_i$ for some a and $p_{22} \in C(W)$. Let $w \in W$. Assume $gu_i = \sum_{j=0}^m a_{ij} v_j + \sum_{j=0}^{m-1} b_{ij} u_j + p_{12}(u_i)$. We have $\beta(v_i, u_j) = \beta(gv_i, gu_j) = ab_{ji} = \delta_{i,j}$, thus $b_{ji} = \delta_{i,j}/a$. Now $\beta(gv_i, gw) = \beta(v_i, w) = 0$ implies $p_{21}(w) = \sum_{i=0}^m b_i^w v_i$. Thus $\beta(gu_i, gw) = \beta(p_{12}(u_i), p_{22}(w)) + b_i^w/a = 0$ and $\beta_\xi(gu_i, gw) = \beta_\xi(p_{12}(u_i), p_{22}(w)) + b_{i+1}^w/a = 0$. This gives us $b_i^w = a\beta(p_{12}(u_i), p_{22}(w))$, $i = 0, \dots, m-1$, $b_i^w = a\beta_\xi(p_{12}(u_{i-1}), p_{22}(w))$, $i = 1, \dots, m$. Thus $\beta_\xi(p_{12}(u_{i-1}), p_{22}(w)) = \beta(p_{12}(u_i), p_{22}(w))$, $i = 1, \dots, m-1$. This holds for any $w \in W$. Recall that on W , we have $\beta_\xi(p_{12}(u_{i-1}), p_{22}(w)) = \beta(T_\xi p_{12}(u_{i-1}), p_{22}(w))$. Since p_{22} is nonsingular and $\beta|_{W \times W}$ is nondegenerate, we get $p_{12}(u_i) = T_\xi^i p_{12}(u_0)$, $i = 0, \dots, m-1$, and $b_i^w = a\beta(T_\xi^i p_{12}(u_0), p_{22}(w))$.

Hence we get $gv_i = av_i$, $i = 0, \dots, m$, $gu_i = \sum_{j=0}^m a_{ij} v_j + u_i/a + T_\xi^i p_{12}(u_0)$, $i = 0, \dots, m-1$, $gw = \sum_{i=0}^m a\beta(T_\xi^i p_{12}(u_0), p_{22}(w))v_i + p_{22}(w)$, $\forall w \in W$. Now note that $p_{12}(u_0)$ can be any vector in W . It is easily verified that the lemma holds. $\square$

Proposition 6.5. $\dim Z(V) = \nu_0 + \sum_{i=1}^s \nu_i(4i+1) + \sum_{i=1}^s \mu_i(4i-1)$.

Proof. Let $V = V_{2m+1} \oplus W_{l_1}(m_1) \oplus \dots \oplus W_{l_s}(m_s) = (V, \alpha, \beta_\xi)$. Let $W = W_{l_1}(m_1) \oplus \dots \oplus W_{l_s}(m_s)$. We have $\dim C(W) = \sum_{i=1}^s (4i-1)m_i$ and $\dim C(V_{2m+1}) = 2m+1$. By Lemma 6.4, $\dim C(V) = \dim C(W) + \dim V_{2m+1} + \dim W = \sum_{i=1}^s (4i-1)m_i + 2m+1 + 2\sum_{i=1}^s m_i$. Consider V_{2m+1} as an element in the Grassmannian variety $Gr(V, 2m+1)$. Let $C(V)V_{2m+1}$ be the orbit of V_{2m+1} under the action of $C(V)$. The stabilizer of V_{2m+1} in $C(V)$ is the product of $C(V_{2m+1})$ and $C(W)$. Hence $\dim C(V)V_{2m+1} = \dim C(V) - \dim C(V_{2m+1}) - \dim C(W) = 2\sum_{i=1}^s m_i$. We have $\dim Z(V)V_{2m+1} = \dim C(V)V_{2m+1}$. Hence $\dim Z(V) =$

$$\dim Z(V_{2m+1}) + \dim Z(W) + \dim Z(V)V_{2m+1} = m + \sum_{i=1}^s ((4i+1)m_i - 2l_i) = \nu_0 + \sum_{i=1}^s \nu_i(4i+1) + \sum_{i=1}^s \mu_i(4i-1). \quad \square$$

Lemma 6.6. $|Z(V)(\mathbf{F}_q)| = 2^k q^{\dim Z(V)} + \text{lower terms}$, where $k = \#\{i \geq 1 | \nu_i < \mu_i \leq \nu_{i-1}\}$.

Proof. If $\#\{i \geq 1 | \nu_i < \mu_i \leq \nu_{i-1}\} = 0$, the assertion follows from the classification of nilpotent orbits. Assume $1 \leq t \leq s$ is the minimal integer such that $\nu_t < \mu_t \leq \nu_{t-1}$. Let $V_1 = V_{2m+1} \oplus W_1$ where $W_1 = W_{l_1}(m_1) \oplus \cdots \oplus W_{l_{t-1}}(m_{t-1})$ and $W_2 = W_{l_t}(m_t) \oplus \cdots \oplus W_{l_s}(m_s)$. We show that

$$(3) \quad |Z(V)(\mathbf{F}_q)| = |Z(V_1)(\mathbf{F}_q)| \cdot |Z(W_2)(\mathbf{F}_q)| \cdot q^{r_1},$$

where $r_1 = \dim W_2 + \dim \text{Hom}_A(W_1, W_2)$. We consider V_1 as an element in the Grassmannian variety $Gr(V, \dim V_1)$. We have

$$(4) \quad |C(V)V_1(\mathbf{F}_q)| = \frac{|C(V)(\mathbf{F}_q)|}{|C(V_1)(\mathbf{F}_q)| \cdot |C(W_2)(\mathbf{F}_q)|} \\ = \frac{|C(V_{2m+1})(\mathbf{F}_q)| \cdot |C(W_1 \oplus W_2)(\mathbf{F}_q)| \cdot q^{\dim(W_1+W_2)}}{|C(V_{2m+1})(\mathbf{F}_q)| \cdot |C(W_1)(\mathbf{F}_q)| \cdot q^{\dim(W_1)} \cdot |C(W_2)(\mathbf{F}_q)|} = q^{r_1}.$$

In fact, let p_{ij} , $i, j = 1, 2, 3$ be the projections of $g \in C(V)$. Assume g is in the stabilizer of V_1 in $C(V)$. Then we have $p_{13} = p_{23} = 0$. It follows from the same argument as in Lemma 6.4 that p_{22} is nonsingular and $gv_i = av_i$, $i = 0, \dots, m$, $gu_i = \sum_{j=0}^m a_{ij}v_j + u_i/a + T_\xi^i p_{12}(u_0)$, $i = 0, \dots, m-1$, $gw_1 = \sum_{i=0}^m a\beta(T_\xi^i p_{12}(u_0), p_{22}w_1)v_i + p_{22}(w_1)$, $\forall w_1 \in W_1$, $gw_2 = \sum_{i=0}^m a\beta(T_\xi^i p_{12}(u_0), p_{22}w_2 + p_{23}w_2)v_i + p_{22}(w_2) + p_{23}(w_2)$, $\forall w_2 \in W_2$. Now $\beta(gw_1, gw_2) = \beta(p_{22}(w_1), p_{22}(w_2) + p_{23}(w_2)) = \beta(p_{22}(w_1), p_{22}(w_2)) = 0$, for any $w_1 \in W_1$ and $w_2 \in W_2$. Since p_{22} is nonsingular and $\beta|_{W_1 \times W_1}$ is nondegenerate, we get $p_{22}(w_2) = 0$ for any $w_2 \in W_2$. Thus the stabilizer of V_1 in $C(V)$ is the product of $C(V_1)$ and $C(W_2)$ and (4) follows.

We have $C(V)(V_1 \oplus W_2) \cong C(V)(V_1) \oplus C(V)(W_2)$ implies $C(V)(V_1) \cong V_1$ and $C(V)(W_2) \cong W_2$. Thus $|C(V)(V_1)(\mathbf{F}_q)| = |Z(V)V_1(\mathbf{F}_q)| = q^{r_1}$. Since the stabilizer of V_1 in $Z(V)$ is the product of $Z(V_1)$ and $Z(W_2)$, (3) follows. Now the lemma follows by induction hypothesis since we have $\dim Z(V) = \dim Z(V_1) + \dim Z(W_2) + r_1$. $\square$

Proposition 6.7. The component group $Z(V)/Z^0(V)$ is $(\mathbb{Z}/2\mathbb{Z})^k$, where $k = \#\{i \geq 1 | \nu_i < \mu_i \leq \nu_{i-1}\}$.

Proof. Lemma 6.6 and the classification of nilpotent orbits in $\mathfrak{g}(\mathbf{F}_q)^*$ (q large) show that $Z(V)/Z^0(V)$ is an abelian group of order 2^k . It is enough to show that there exists a subgroup $(\mathbb{Z}/2\mathbb{Z})^k \subset Z(V)/Z^0(V)$. Assume $V = V_{2m+1} \oplus W_{l_1}^{\epsilon_1}(m_1) \oplus \cdots \oplus W_{l_s}^{\epsilon_s}(m_s)$. Let $i_1 < i_2 < \cdots < i_k$ be the i 's such that $\nu_i < \mu_i \leq \nu_{i-1}$. Let $V_0 = V_{2m+1} \oplus W_{l_1}^{\epsilon_1}(m_1) \oplus \cdots \oplus W_{l_{i_1-1}}^{\epsilon_{i_1-1}}(m_{i_1-1})$ and $W_j = W_{l_{i_j}}^{\epsilon_{i_j}}(m_{i_j}) \oplus \cdots \oplus W_{l_{i_{j+1}-1}}^{\epsilon_{i_{j+1}-1}}(m_{i_{j+1}-1})$, $j = 1, \dots, k$, where $i_{k+1} = s+1$. We have $Z(V_0)/Z^0(V_0) = \{1\}$ and $Z(W_j)/Z^0(W_j) = \mathbb{Z}/2\mathbb{Z}$, $j = 1, \dots, k$. Take $g_j \in Z(W_j)$ such that $g_j Z^0(W_j)$ generates $Z(W_j)/Z^0(W_j)$. Let $\tilde{g}_j = Id \oplus \cdots \oplus g_j \oplus \cdots \oplus Id$, $j = 1, \dots, k$. Then $\tilde{g}_j \in Z(V)$, $\tilde{g}_j \notin Z^0(V)$, $\tilde{g}_j^2 \in Z^0(V)$ and $\tilde{g}_{j_1}\tilde{g}_{j_2}\cdots\tilde{g}_{j_r} \notin Z^0(V)$ for any $1 \leq j_1 < j_2 < \cdots < j_r \leq s$, $r = 1, \dots, k$. Thus $\tilde{g}_j Z^0(W_j)$, $j = 1, \dots, k$ generate a subgroup $\mathbb{Z}/2\mathbb{Z}^k$. $\square$

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