EXPLICIT DOUBLING INTEGRALS FOR $\widetilde{\mathrm{Sp}}_2(\mathbb{Q}_2)$ USING “GOOD TEST VECTORS”

CHRISTIAN A. ZORN

ABSTRACT. In a previous paper (see <http://www.math.ohio-state.edu/~czorn/works.html>), we computed examples of the doubling integral for constituents of the unramified principal series of $\mathrm{Sp}_2(F)$ and $\widetilde{\mathrm{Sp}}_2(F)$ where F was a non-dyadic field. These computations relied on certain “good test vectors” and “good theta test sections” motivated by the non-vanishing of theta lifts. In this paper, we aim to prove a partial analog for $\widetilde{\mathrm{Sp}}_2(\mathbb{Q}_2)$. However, due to several complexities, we compute the doubling integral only for certain irreducible principal series representations induced from characters with ramified quadratic twists. We develop some 2-adic analogs for the machinery in the paper mentioned above; however, these tend to be more delicate and have more restrictive hypotheses than the non-dyadic case. Ultimately, this paper and the one mentioned above develop several computations intended to be used for future research into the non-vanishing of theta lifts.

1. INTRODUCTION

In a previous paper [15], we computed examples of the doubling integral for constituents of the unramified principal series of the groups $\mathrm{Sp}_2(F)$ and $\widetilde{\mathrm{Sp}}_2(F)$ where F was a non-dyadic p -adic field. This computation relied on the choice of “good test vectors” and “good theta test sections” that were motivated by the theory of local theta lifts [4]. The restriction to non-dyadic fields was due to a similar restriction on some of the intermediate computations. The purpose of this paper is to prove a partial 2-adic analogue of the previous paper for the group $\widetilde{\mathrm{Sp}}_2(\mathbb{Q}_2)$. We use the same basic method as [15]; however, there are some obstructions in the 2-adic case that we shall mention as they become relevant. Therefore, many of our 2-adic results in this paper have more restrictive assumptions.

The results of this paper lay the foundation for some results about the non-vanishing of local theta lifts from certain genuine principal series representations of $\widetilde{\mathrm{Sp}}_2(\mathbb{Q}_2)$ to the orthogonal groups $\mathrm{O}(V)$ where (V, Q) is a quadratic space satisfying a condition on its discriminant. The connection comes from a result of Rallis (in [6]), motivated by the theory of seesaw dual reductive pairs (see [3]), that for $(\pi, \mathcal{V}_\pi)$ a irreducible genuine admissible representation of $\widetilde{\mathrm{Sp}}_n(F)$ (F an arbitrary local field with characteristic 0), the local theta lift $\Theta_\psi(\pi)$ to the group $\mathrm{O}(V)$ is non-vanishing if and only if $\mathrm{Hom}_{\widetilde{\mathrm{Sp}}_n(F) \times \widetilde{\mathrm{Sp}}_n(F)}(R_{2n}(V), \mathcal{V}_\pi \otimes V_{\pi^\vee}) \neq 0$, where $R_{2n}(V)$ are the $\mathrm{O}(V)$ -coinvariants of the Weil representation $(\omega_V, S(V^{2n}))$ for $\widetilde{\mathrm{Sp}}_{2n}(F)$.

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In order to find such a homomorphism, we intertwine the $\widetilde{\mathrm{Sp}}_{2n}(F)$ -representation $R_{2n}(V)$ with a degenerate principal series representation $I_{2n}(s_0, \chi_V)$ (where s_0 and χ_V depend on (V, Q)). Furthermore, we have a process of extending a function $\varphi \in S(V^{2n})$ to a function $\varphi^{(r)} \in S(V_r^{2n})$, where V_r is a space in the Witt tower of V having dimension $\dim_F(V) + 2r$. We then intertwine all of the $S(V_r^{2n})$ with the degenerate principal series representations $I_{2n}(s_0 + r, \chi_V)$. If we abuse notation and call all such intertwining operators λ_V , we can produce a section $\Phi_{s_0+r} \in I_{2n}(s_0 + r, \chi_V)$ by $\Phi_{s_0+r} = \lambda_V(\varphi^{(r)})$. Such a section is called a “theta test section”. In the original doubling construction of [2], we would then integrate the section Φ_{s_0+r} along a section $\mathrm{Sp}_n(F) \rightarrow \widetilde{\mathrm{Sp}}_n(F)$ against a matrix coefficient $m_{v, v^\vee}$ for $(\pi, \mathcal{V}_\pi)$ to get the integral $Z(s_0 + r, \chi_V, \Phi, v, v^\vee)$. However, our construction will integrate Φ_{s_0+r} against a vector $v \in \mathcal{V}_\pi$, so we get an operator $Z(s_0 + r, \chi_V, \Phi, \cdot) \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{V}_\pi, \mathcal{V}_\pi)$. This Hom set is equivalent to $\mathcal{V}_\pi \otimes \mathcal{V}_{\pi^\vee}$ as a $\widetilde{\mathrm{Sp}}_n(F) \times \mathrm{Sp}_n(F)$ -representation. Thus the map $\varphi^{(r)} \mapsto (v \mapsto Z(s_0 + r, \chi_V, \lambda_V(\varphi^{(r)}), v))$ is an element of $\mathrm{Hom}_{\widetilde{\mathrm{Sp}}_n(F) \times \mathrm{Sp}_n(F)}(R_{2n}(V_r), \mathrm{Hom}_{\mathbb{C}}(\mathcal{V}_\pi, \mathcal{V}_\pi))$ which are equivalent to the analogous Hom sets in Rallis’s criterion.

For a judicious choice of “theta test section” $\{\Phi_{s_0+r}\}_{r=0}^\infty$ and vector $v \in \mathcal{V}_\pi$, the vector v is an eigenvector for the operators $Z(s_0 + r, \chi_V, \Phi, \cdot)$ having eigenvalues of the form $\frac{P(s_0+r)}{Q(s_0+r)}$ where $P(s), Q(s) \in \mathbb{C}[q^s, q^{-s}]$ are the product of various Euler factors. Such a section is called a “good theta test section” and the vector v is called a “good test vector”.

In [15] (where F is non-dyadic), it was shown that there existed a parahoric subgroup $I \subset \mathrm{Sp}_2(F)$ so that the space of vectors fixed by I in $\mathcal{V}_\pi$ was one-dimensional (we denote such a space as $\mathcal{V}_\pi^I$). We also demonstrated a splitting of $I \rightarrow \widetilde{\mathrm{Sp}}_2(F)$, so that a similar statement held in the metaplectic case (see Section 2 of [15] or Section 3.1 or [14]). In [15], these vectors are the “good test vectors”.

For our 2-adic analysis, we face a couple of obstructions to finding an analogous result. First, the 2-adic analogue to the splitting construction of [15] yields a splitting of a compact open subgroup (denoted $I_\alpha^*(2)$) obtained by reducing certain entries mod 4 rather than mod 2. Second, this paper will look at genuine principal series representations $(\pi, \mathcal{V}_\pi)$ induced from characters having ramified quadratic twists. Consequently, we will need to find an appropriate character $\varsigma : I_\alpha^*(2) \rightarrow \mathbb{C}^1$ so that the space of vectors having the property

$$\pi(k)f = \varsigma(k)f$$

for all $k \in I_\alpha^*(2)$ is non-empty and can be made explicit. We denote such a space $\mathcal{V}_\pi^{(I_\alpha^*(2), \varsigma)}$. For certain basic ς , we find an appropriate choice of “good theta test section” to define an operator $Z(r, \chi_V, \Phi, \cdot)$ on $\mathcal{V}_\pi^{(I_\alpha^*(2), \varsigma)}$ with $\mathrm{Rank}(Z(r, \chi_V, \Phi, \cdot)) \leq 1$. When this rank is 1, the unique space of vectors in the image of $Z(r, \chi_V, \Phi, \cdot)$ will be referred to as the “good test vectors”.

The main result has several technical conditions, so we refer the reader to Theorem 5.11 for the precise statement of the main theorem. It can be summarized as follows. For the irreducible principal series representations of $\widetilde{\mathrm{Sp}}_2(\mathbb{Q}_2)$ under consideration, “good test vectors” and “good theta test sections” exist and we can compute $Z(s_0 + r, \chi_V, \Phi, f)$ explicitly for this data. To simplify notation we set $G_n = \mathrm{Sp}_n(F)$ and $\widetilde{G}_n = \widetilde{\mathrm{Sp}}_n(F)$. In Sections 2 and 3 some results involve F being

an arbitrary p -adic field and some are stated for $F = \mathbb{Q}_2$, so we seek to explicitly state when F is general p -adic or $\mathbb{Q}_2$. In Sections 4 and 5, $F = \mathbb{Q}_2$ exclusively.

This paper is divided into several sections. In Section 2, we define all of the basic tools of this paper. We define several important subgroups of G_n and $\widetilde{G}_n$. It is worth noting that we use an atypical cover as our metaplectic group $\widetilde{G}_n$. In particular, $\widetilde{G}_n$ will be a central extension of G_n by $\mathbb{C}^1 = \{z \in \mathbb{C}^\times \mid |z| = 1\}$. This realization of the metaplectic group offers us several splittings of subgroups of G_n that are not available to the double cover. The splittings are defined in Section 8.5 of [5] and their properties are further explored in [14]. Specializing to $n = 2$, we also define a compact-open subgroup $I_\alpha^*(2) \subset G_2$ that is the analog to the various parahoric subgroups in [15]. Using a convention from [15], we let $\mathbf{I}_\alpha^*(2)$ be the image of $I_\alpha^*(2)$ in $\widetilde{G}_2$ under a splitting map that we will explicitly describe.

In Section 3, we introduce the doubling integral. We will only define the version of the doubling integral employed in Section 8 of [5]. We will also explain the connection between the non-vanishing of the local doubling integral and the non-vanishing of theta lifts for the dual reductive pair $(\widetilde{\mathrm{Sp}}_n(F), \mathrm{O}(V))$ for (V, Q) an odd-dimensional orthogonal space (for general F).

Section 4 is devoted to some character theory of $\mathbf{I}_\alpha^*(2)$ along with finding vectors in $\mathcal{V}_\pi$ that are eigenvectors for the action of $\mathbf{I}_\alpha^*(2)$. We study characters ς of $\mathbf{I}_\alpha^*(2)$ for which there may exist vectors in the space $S(V^2)$ of the Weil representation $(\omega_V, S(V^2))$ so that $S(V^2)^{(\mathbf{I}_\alpha^*(2), \varsigma)} \neq 0$. Our “good test vectors” and “good theta test sections” need to transform under $\mathbf{I}_\alpha^*(2)$ by the same character ς .

In the final section we actually compute the doubling integral for certain irreducible principal series representations of $\widetilde{G}_2$. This computation has several intermediate steps, one of which is related to the computation of local densities of quadratic forms. For this intermediate step we will rely on some computations in [12].

This paper represents an extension of the author’s thesis research (Univ. of Maryland, 2007) to the case that $F = \mathbb{Q}_2$. It was suggested by the author’s thesis advisor S. Kudla with an eye toward proving a global result regarding theta lifts from $\widetilde{\mathrm{Sp}}_2$ to various orthogonal groups.

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2. NOTATION AND BASIC PROPERTIES OF SYMPLECTIC AND METAPLECTIC GROUPS

We first introduce some notation and constructions that apply to any p -adic field that we will later specialize to $\mathbb{Q}_2$. Let F be a finite extension of $\mathbb{Q}_p$ for some prime p . F has a ring of integers $\mathcal{O}$ with maximal ideal $\mathcal{P}$ and residue field $\mathbb{F}_q$. We also fix an “unramified” additive character ψ which is trivial on $\mathcal{O}$ but non-trivial on $\mathcal{P}^{-1}$. In the case that $p = 2$, we will be even more explicit and define $\psi(x) = \exp(-2\pi i \bar{x})$, where $\bar{x}$ is the image of x under the natural maps

$\mathbb{Q}_2 \rightarrow \mathbb{Q}_2/\mathbb{Z}_2 \rightarrow \mathbb{Q}/\mathbb{Z}$. By Pontrjagin duality, all other additive characters can be realized as ψ_a where $\psi_a(x) = \psi(ax)$ for some $a \in F$. Because they arise in many of our computations, let $\xi_a : F^\times \rightarrow \mathbb{C}^\times$ be the quadratic character $\xi_a(x) = (x, a)_F$ where $(\cdot, \cdot)_F$ is the Hilbert symbol on F .

Following [15], we specify some quadratic spaces over $F = \mathbb{Q}_2$. First, we fix a convention regarding quadratic spaces. For any space V with quadratic form Q , we define the symmetric inner product $(\cdot, \cdot)_V$ by $(x, y)_V = Q(x+y) - Q(x) - Q(y)$ for all $x, y \in V$. With this definition, $Q(x) = \frac{1}{2}(x, x)_V$. Now, let $V_{0,\kappa}^+ = M_2^0(F)$ be the trace zero elements of the matrix algebra $M_2(F)$ with $Q_{0,\kappa}^+(x) = -\kappa \det(x) = \kappa x^2$ or let $V_{0,\kappa}^- = D^0$ be the trace zero elements in the quaternion algebra

$$D^- = \{a + ib + jc + ijd \mid a, b, c, d \in F, i^2 = j^2 = -1, ij = -ji\}$$

with $Q_{0,\kappa}^-(x) = -\kappa N_{D/F}(x) = \kappa x^2$. Let us also fix bases for these spaces. For $V_{0,\kappa}^+$, let $E_{i,j}$ be the elementary matrix having a 1 in the i, j -th entry and 0 in the other entries. Then $V_{0,\kappa}^+$ has a basis $\{E_{1,1} - E_{2,2}, E_{1,2}, E_{2,1}\}$. For $V_{0,\kappa}^-$ we have a basis $\{i, j, ij\}$. According to Chapter IV of [8], quadratic spaces over a p -adic field are determined (up to equivalence) by their dimension, discriminant and their Hasse invariant. Following Section IV.2 of [8], we see that $\epsilon(V_{0,\kappa}^\pm) = (\pm\kappa, -1)_{\mathbb{Q}_2} = \pm\xi_\kappa(-1)$. Finally, the discriminant is equivalent to a quadratic character $\chi_V : F^\times \rightarrow \mathbb{C}^1$ given by

$$\chi_V(x) = (x, (-1)^{\frac{m(m-1)}{2}} \det(V))_F$$

where $\det(V)$ is the square class of the determinant of a matrix whose i, j -th entry is given by $(v_i, v_j)_V$ for some basis $\{v_1, v_2, \dots, v_m\}$ of V . For $V = V_{\kappa}^\pm$, one can verify that $\chi_V = \xi_{2\kappa}$.

In the theory of quadratic forms, any quadratic space (V, Q) can be written as an orthogonal direct sum $V = V_a \oplus V_h^r$ where V_a is anisotropic and (V_h, Q_h) is a two-dimensional isotropic space called a hyperbolic plane. In particular, (V_h, Q_h) is equivalent to the quadratic space $(F^2, Q(x, y) = xy)$. For future use, we specify a basis $\{e_0, e'_0\}$ of V_h so that $(e_0, e'_0)_{V_h} = 1$. The spaces $V_a \oplus V_h^r$ share a quadratic character χ_{V_a} for all $r \geq 0$ and the set $\{V_a \oplus V_h^r\}$ is called the Witt tower containing V_a . For $V = V_a \oplus V_h^r$, we call r the Witt index of V . One can verify that $V_{0,\kappa}^+$ has Witt index 1 whereas $V_{0,\kappa}^-$ is anisotropic (hence has Witt index 0). As in [15], our constructions are most naturally done using spaces of dimension 5, so we let (V_κ^+, Q_κ^+) (resp. (V_κ^-, Q_κ^-)) be the five-dimensional space in the Witt tower containing $(V_{0,\kappa}^+, Q_{0,\kappa}^+)$ (resp. $(V_{0,\kappa}^-, Q_{0,\kappa}^-)$). Hence (V_κ^+, Q_κ^+) (resp. (V_κ^-, Q_κ^-)) has Witt index 2 (resp. 1).

2.1. Symplectic and metaplectic group. Let $G_n = \mathrm{Sp}_n(F)$ be the rank n symplectic group where

$$\mathrm{Sp}_n(F) = \left\{ g \in \mathrm{GL}_{2n}(F) \mid {}^t g \begin{pmatrix} & I_n \\ -I_n & \end{pmatrix} g = \begin{pmatrix} & I_n \\ -I_n & \end{pmatrix} \right\}$$

with I_n the $n \times n$ identity matrix. We fix a set of standard parabolic subgroups as follows. We start with a filtration $T_n \subset B_n \subset G_n$, where T_n is the diagonal maximal torus and B_n is a Borel subgroup having matrices of the form

$$\begin{pmatrix} A & \\ & {}^t A^{-1} \end{pmatrix} \begin{pmatrix} I & X \\ & I \end{pmatrix}$$

where A is an upper-triangular matrix in $\mathrm{GL}_n(F)$ and $X \in \mathrm{Sym}_n(F)$. Associated to this filtration, we get a set of simple roots $\Delta_n = \Delta(B_n, T_n)$. For each subset $\Omega \subset \Delta_n$ one can construct a parabolic P_Ω such that $B_n \subset P_\Omega$ (see Section 3.6 of [9]). Note that $B_n = P_\emptyset$. Each of these parabolics has a Levi decomposition

$$P_\Omega = M_\Omega N_\Omega$$

with

$$(2.1) \quad M_\Omega \simeq \mathrm{GL}_{r_1}(F) \times \mathrm{GL}_{r_2}(F) \times \cdots \times \mathrm{GL}_{r_k}(F) \times \mathrm{Sp}_{n'}(F)$$

where $r = \sum_{i=1}^k r_i$ and $n' = n - r$. Note that if $r_1 = n$, the corresponding P_Ω (which we also denote P_n) is called the **Siegel parabolic subgroup**. Notice that if $n = 2$, we denote $\Delta_2 = \{\alpha, \beta\}$ where β is the long root. Thus the singletons $\{\alpha\}$ and $\{\beta\}$ are the only non-empty proper subsets of Δ_2 . So to simplify notation we let

$$P_\alpha := P_{\{\alpha\}} \quad \text{and} \quad P_\beta := P_{\{\beta\}}.$$

Notice that P_α is our standard Siegel parabolic subgroup; moreover, we call P_β the standard **Klingen parabolic subgroup**.

The results of this paper also involve a more detailed analysis of compact open subgroups than was needed in [15]. We start by fixing a maximal compact subgroup, $K_n = \mathrm{Sp}_n(\mathbb{Z}_2)$. For $i \in \mathbb{Z}^+$, we have $\mathbb{Z}_2/2^i\mathbb{Z}_2 \simeq \mathbb{Z}/2^i\mathbb{Z}$ and a map

$$\mathrm{proj}_i : \mathrm{Sp}_n(\mathbb{Z}_2) \rightarrow \mathrm{Sp}_n(\mathbb{Z}/2^i\mathbb{Z})$$

that reduces the entries of $k \in \mathrm{Sp}_n(\mathbb{Z}_2) \bmod 2^i$. Then for any standard parabolic subgroup $P_\Omega(\mathbb{Z}/2^i\mathbb{Z})$, we define a **standard parahoric subgroup of level i** to be

$$I_\Omega(i) := \mathrm{proj}_i^{-1} P_\Omega(\mathbb{Z}/2^i\mathbb{Z}).$$

Notice that for each $\Omega \subset \Delta_n$, $I_\Omega(j) \subset I_\Omega(i)$ for $i \leq j$. Fortunately, the results of this paper are derived using subgroups of this type where $i = 2$.

Let us also give a brief description of $\widetilde{\mathrm{Sp}}_n(F)$ that applies to an arbitrary p -adic F . We will use a non-standard realization of this group that has been employed in Chapter 8 of [5] as well as [14] and [15]. The group $\widetilde{G}_n$ is a central extension of G_n by $\mathbb{C}^1 = \{z \in \mathbb{C}^\times \mid |z| = 1\}$. Given the exact sequence,

$$1 \rightarrow \mathbb{C}^1 \rightarrow \widetilde{G}_n \rightarrow G_n \rightarrow 1,$$

there exists a section $G_n \rightarrow \widetilde{G}_n$ such that elements $g' \in \widetilde{G}_n$ have the form

$$g' = [g, z]$$

with a product

$$[g_1, z_1] \cdot [g_2, z_2] = [g_1 g_2, z_1 z_2 c_L(g_1, g_2)]$$

where $c_L(\cdot, \cdot)$ is the Leray cocycle defined in Section 4 of [7]. This cocycle differs from the typical (Rao) cocycle (see Section 5 of [7]) used to define the metaplectic double cover $\widetilde{G}_n^{(2)}$. However, both cocycles are equivalent in $H^2(G_n, \mathbb{C}^1)$ and thus define isomorphic extensions by $\mathbb{C}^1$. We will denote elements of $\widetilde{G}_n$ as $[g, z]_L$ since we use the Leray cocycle.

Our choice of extension splits over several subgroups of G_n . Some of these splittings do not occur with the metaplectic double cover. While we will summarize the construction of these splittings, a more detailed analysis of the splittings can be found in Section 8.5 of [5] and Section 3 of [14]. While the results in [14] are stated

for $\tilde{G}_n$ over a non-dyadic p -adic field, the group structure theory can be applied to the $\tilde{G}_n$ over $\mathbb{Q}_2$ without alteration. Consider the following lemma which is valid for any p -adic F . This content of this lemma is stated, but not proved, in Section 8.5.1 of [5]. The computations that prove it are found at the beginning of Section 2 of [15].

Lemma 2.1. *Let W be a $2n$ -dimensional symplectic space such that $G_n = \mathrm{Sp}(W)$. Also let $W = X + Y$ be the complete polarization of W with $\mathrm{Stab}_{G_n}(Y) = P_n$. Then for $g \in G_n$ and $p \in P_n$,*

$$c_L(g, p) = c_L(p, g) = 1.$$

Moreover, for $g_1, g_2 \in G_n$ with $g_1 g_2 \in P_n$,

$$c_L(g_1, g_2) = 1.$$

This result gives us our first important splitting map. For our Siegel parabolic subgroup $P_n \subset G_n$, there is a splitting

$$P_n \rightarrow \tilde{G}_n \quad p \rightarrow [p, 1]_L$$

(see Section 8.5 of [5] as well).

Next, we define the standard parabolic subgroups of $\tilde{G}_n$. For any standard parabolic $P_\Omega \subset G_n$, we define a standard parabolic subgroup $\tilde{P}_\Omega \subset \tilde{G}_n$ as the full inverse image of P_Ω under the canonical projection $\tilde{G}_n \rightarrow G_n$. If P_Ω has Levi decomposition described in (2.1), then Section 3.1 of [14] shows that the standard parabolic subgroup $\tilde{P}_\Omega$ has a Levi decomposition

$$\tilde{P}_\Omega = \tilde{M}_\Omega \mathbf{N}_\Omega$$

where

$$(2.2) \quad \tilde{M}_\Omega \simeq \mathrm{GL}_{r_1}(F) \times \mathrm{GL}_{r_2}(F) \times \cdots \times \mathrm{GL}_{r_k} \times \tilde{G}_{n'}$$

and $\mathbf{N}_\Omega = \{[n, 1]_L \mid n \in N_\Omega\}$. If $n' = 0$, we use the convention that $\tilde{G}_0 = \mathbb{C}^1$.

For our computations, we will also need an explicit section of the Weyl group $\mathcal{W}_{G_2}$ to $\tilde{G}_2$. First note that if we set $\mathcal{W}_{\tilde{G}_n} = N_{\tilde{G}_n}(\tilde{M}_\emptyset)/\tilde{M}_\emptyset$, then $\mathcal{W}_{\tilde{G}_n} \simeq \mathcal{W}_{G_n}$. If $n = 2$, we will also fix a section $\mathcal{W}_{G_2} \rightarrow G_2$ described in Section 1 of [15] by

$$w_\alpha \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad w_\beta \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

on the generators. This is shown to be well defined in Section 2 of [15]. In order to simplify notation, we also introduce the following convention. Let $r_1, r_2, \dots, r_n$ be a collection of (not necessarily distinct) simple roots for G_2 . Then we set

$$w_{r_1 r_2 \cdots r_n} = w_{r_1} w_{r_2} \cdots w_{r_n}.$$

In order to state our results, we must also recall the notion of length for our Weyl group $\mathcal{W}_{G_2}$. Let the **length** of w (denoted $\ell(w)$) be defined as

$$\ell(w) = \min \left\{ n \in \mathbb{N} \mid w = \prod_{i=1}^n w_i, w_i \in \{w_\alpha, w_\beta\} \right\}.$$

This is also shown to be well defined in Section 2 of [15].

2.2. The principal series of $\widetilde{G}_n$. Now let us standardize some notation for induced representations of $\widetilde{G}_n$. Let $\widetilde{P}_\Omega = \widetilde{M}_\Omega \mathbf{N}_\Omega$ be the standard parabolic subgroup with $\widetilde{M}_\Omega$ as in (2.2). Also let $(\pi_i, \mathcal{V}_i)$ be admissible representations of $\mathrm{GL}_{r_i}(F)$ and let $(\sigma', \mathcal{V}_{\sigma'})$ be a genuine admissible representation of $\widetilde{G}_{n'}$. We define the representation

$$\mathrm{Ind}_{\widetilde{P}_\Omega}^{\widetilde{G}_n}(\pi_1 \otimes \pi_2 \otimes \cdots \otimes \pi_j \otimes \sigma')$$

via normalized induction (note $\delta_{\widetilde{P}} := \delta_P$) where $\widetilde{G}_n$ acts via right translation. The notion of admissibility for $\widetilde{G}_n$ and its preservation via parabolic induction is discussed in Section 4.1 of [14]. The condition that σ' be genuine ensures that this representation is also genuine. In order to simplify notation, we borrow a convention of Tadić [10]. The induced representation above will be denoted

$$\pi_1 \times \pi_2 \times \cdots \times \pi_j \rtimes \sigma'.$$

Note that if $n' = 0$ our convention will be to let

$$\sigma' = \sigma_0 : \mathbb{C}^1 \rightarrow \mathbb{C}^1 \quad \sigma_0(z) = z.$$

2.3. Splittings of compact open subgroups. In this section, $F = \mathbb{Q}_2$ and we consider splittings of certain compact open subgroups of G_n . We use the notation and definitions found in Chapter I of [4] and Chapter 8 of [5]. However, we choose to include this material because many of the computations in Section 5 rely on the following definitions and quantities.

In order to define this splitting, we must define the Schrödinger model of the Weil representation. Let $(W, \langle, \rangle)$ be a symplectic as in Lemma 2.1 so that $G_n \simeq \mathrm{Sp}(W)$ and $W = X + Y$ where $\mathrm{Stab}_{G_n}(Y) = P_n$. Let (V, Q) be an m -dimensional quadratic space over $\mathbb{Q}_2$, $V^n = V \otimes_F X$ and $S(V^n)$ the space of Schwartz functions on V^n . We have a (possibly projective) representation $(r_V, S(V^n))$ of G_n defined as follows.

Let $g \in G_n$ so that $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ blockwise, then

$$(2.3) \quad r_V(g)\varphi(x) = \int_{V^n/\ker(c)} \psi(\mathrm{tr}(\frac{1}{2}(xa, xb) + (xb, yc) + \frac{1}{2}(yc, yd)))\varphi(xa + yc) d_g y$$

where (x, y) is an $n \times n$ matrix with $(x, y)_{i,j} = (x_i, y_j)_V$, and the measure $d_g y$ is normalized to make $r_V(g)$ unitary. Now let

$$\begin{aligned} m : \mathrm{GL}_n(F) &\rightarrow P_n & m(a) &= \begin{pmatrix} a & \\ & {}_t a^{-1} \end{pmatrix}, \\ n^+ : \mathrm{Sym}_n(F) &\rightarrow P_n & n^+(b) &= \begin{pmatrix} I_n & b \\ & I_n \end{pmatrix}, \end{aligned}$$

and let w_n be the long Weyl group element. Then for $\varphi \in S(V^n)$ equations (8.5.6)–(8.5.8) of [5] give expressions for $r_V(m(a))\varphi(x)$, $r_V(n^+(b))\varphi(x)$, and $r_V(w_n)\varphi(x)$.

This (possibly projective) representation of G_n extends to a (possibly genuine) representation of the group $\widetilde{G}_n$, the so-called Schrödinger model of the Weil representation $(\omega_V, S(V^n))$. In order to define this representation, we introduce several quantities that are scattered throughout [7] and are restated more concisely in Section I.4 of [4]. Because all of these quantities are used in Section 5, we include their definitions.

First, let $m = \dim_F(V)$ and let

$$w_j = \begin{pmatrix} I_{n-j} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_j \\ 0 & 0 & I_{n-j} & 0 \\ 0 & -I_j & 0 & 0 \end{pmatrix}.$$

We have that

$$G_n = \bigcup_{0 \leq j \leq n} P_n w_j P_n$$

is a disjoint union. So for $g \in G_n$, $g = p_1 w_j^{-1} p_2$ where w_j is uniquely determined. So we define a function

$$x(g) = x(p_1 w_j^{-1} p_2) := \det(p_1 p_2|_V) \in F^\times / (F^\times)^2.$$

Next, for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, let $j(g) = \text{rank}(c)$. Following Section I.4 of [4], one can show that for $g \in P_n w_i P_n$, $j(g) = i$.

Now we let $\eta = \psi_{\frac{1}{2}}$ and we use $\gamma_F(\eta)$ to denote the Weil index of the function $x \mapsto \eta(x^2)$. Furthermore, Remark 4.4 of [4] and Definition A.6 of [7] define

$$\gamma(\eta \circ V) := \gamma_F(\det(V), \eta) \gamma_F(\eta)^m \epsilon(V),$$

$$\gamma_F(a, \eta) := \frac{\gamma_F(\eta_a)}{\gamma_F(\eta)},$$

$$\gamma(V) := \gamma_F(\eta)^{-1} \gamma_F(\det(V), \eta)^{-1} \epsilon(V),$$

where for $a \in F^\times$. These quantities are computed in Theorems A.4 and A.7 of [7].

With all these quantities, we can define the Schrödinger model of the Weil representation for $\tilde{G}_n$ as follows. For $g \in G_n$ and $z \in \mathbb{C}^1$, we have

$$(2.4) \quad \omega_V([g, z]_L) \varphi(x) = \chi_V(x(g)) (z \gamma(\eta)^{j(g)})^\bullet \gamma(\eta \circ V)^{-j(g)} r_V(g) \varphi(x)$$

where

$$\bullet = \begin{cases} 0 & \text{if } \dim_F(V) \text{ is even,} \\ 1 & \text{if } \dim_F(V) \text{ is odd.} \end{cases}$$

Notice that this representation is genuine precisely when $\dim_F(V)$ is odd.

We now summarize the arguments in Section 8.5.4 of [5] that define a splitting of compact open subgroups of G_n to $\tilde{G}_n$. Consider the Schrödinger model for the Weil representation where $(V, Q_V) = (\mathbb{Q}_2, Q_0)$ where $Q_0(x) = x^2$. In this case, we also simplify our notation by setting $r_V = r$ and $\omega_V = \omega$. From Section 8.5.4 of [5], we have

$$(2.5) \quad \omega([g_1, 1]_L) \omega([g_2, 1]_L) = c_L(g_1, g_2) \omega([g_1 g_2, 1]_L).$$

Suppose that we have a compact open subgroup $K \subset G_n$, a function $\varphi \in S(\mathbb{Q}_2)$, and a map $\lambda_n : K \rightarrow \mathbb{C}$ so that $\omega([k, 1]_L) \varphi = \lambda_n(k)^{-1} \varphi$. Then by (2.5) we see that for any $k_1, k_2 \in K$ that

$$\lambda_n(k_1)^{-1} \lambda_n(k_2)^{-1} = c_L(k_1, k_2) \lambda_n(k_1 k_2)^{-1}.$$

Then we can form a splitting $\underline{\lambda}_n : K \mapsto \tilde{G}_n$ given by $\underline{\lambda}_n(k) := [k, \lambda_n(k)]_L$. So we see that one produces splitting for a compact open subgroup $K \subset G_n$ to $\tilde{G}_n$ using an eigenfunction in $S(F^n)$ for the action of K under ω . We use the same choice of $\varphi^0 \in S(\mathbb{Q}_2^n)$ as in Lemma 8.5.11 of [5] which also gives us the following lemma.

Lemma 2.2. *Let $\varphi^0 \in S(\mathbb{Q}_2^n)$ be the characteristic function of $\mathbb{Z}_2^n \subset \mathbb{Q}_2^n$. Then there exists a function $\lambda_n : I_n(2) \rightarrow \mathbb{C}^1$ so that*

$$\omega([k, 1]_L)\varphi^0(x) = \lambda_n(k)^{-1}\varphi^0(x)$$

where $\lambda_n(k)$ is determined explicitly on a set of generators for $I_n(2)$ in Lemma 8.5.11 of [5].

As alluded to in Section 1, we would like to find a slightly larger compact open subgroup of G_n containing $I_n(2)$ to which we can extend λ_n . To motivate our choice, let us consider the operators $\omega([k, 1]_L)$ for a particular choice of $k \in I_n(2)$. In particular, let $n^-(c) := {}^t n^+(c)$ and choose $c = (c_{i,j}) \in \mathrm{Sym}_2(4\mathbb{Z}_2)$. Then we have $n^-(c) = w_n^{-1}n^+(-c)w_n$ and $w_n^{-1} = m(-I)w_n$. Thus

$$\begin{aligned} \omega([n^-(c), 1]_L)\varphi^0(x) &= \omega([n^-(c), 1]_L)\omega([w_n, 1]_L)^{-1}\chi_V(-1)^n\gamma(2, \eta)^n\widehat{\varphi^0}(x) \\ &= \xi_2(-1)^n\gamma(2, \eta)^n\omega([n^-(c)w_n, c_L(n^-(c), w_n)]_L)\widehat{\varphi^0}(x) \\ &= \xi_2(-1)^n\gamma(2, \eta)^n\omega([w_n, 1]_L[n^+(-c), c_L(n^-(c), w_n)]_L)\widehat{\varphi^0}(x). \end{aligned}$$

Since $(x, y)_V = Q_0(x+y) - Q_0(x) - Q_0(y) = 2xy$ for all $x, y \in \mathbb{Q}_2$, we have $\widehat{\varphi^0}(x) = 2^{-\frac{n}{2}} \mathrm{char}_{(2^{-1}\mathbb{Z}_2)^n}(x)$. Furthermore, if $x = (x_1, x_2, \dots, x_n)$ with $x_i \in (2^{-1}\mathbb{Z}_2)^n$, then

$$(2.6) \quad \mathrm{tr}(x, -xc) = -2 \left(\sum_{i=1}^n c_{ii}x_i^2 + \sum_{\substack{i,j=1 \\ i \neq j}}^n c_{ij}x_i x_j \right) \in 2\mathbb{Z}_2.$$

Therefore,

$$\begin{aligned} &\xi_2(-1)^n\gamma(2, \eta)^n\omega([w_n, 1]_L)\omega([n^+(-c), c_L(n^-(c), w_n)]_L)\widehat{\varphi^0}(x) \\ &= c_L(n^-(c), 1)\omega([w_n, 1]_L)\xi_2(-1)^n\gamma(2, \eta)^{-1}\widehat{\varphi^0}(x) \\ &= c_L(n^-(c), w_n)\omega([w_n, 1]_L)\omega([w_n, 1]_L)^{-1}\varphi^0(x) = c_L(n^-(c), w_n)\varphi^0(x). \end{aligned}$$

Notice that in (2.6) we have that $c_{i,j} = c_{j,i}$ so $c_{i,j}x_i x_j = c_{j,i}x_j x_i$ and

$$\sum_{\substack{i,j=1 \\ i \neq j}}^n c_{ij}x_i x_j = \sum_{\substack{i,j=1 \\ i < j}}^n 2c_{i,j}x_i x_j.$$

So if we choose $c = (c_{i,j}) \in \mathrm{Sym}_2(2\mathbb{Z}_2)$ with $c_{i,i} \in 4\mathbb{Z}_2$ for all $1 \leq i \leq n$, (2.6) would still yield $\mathrm{tr}(x, -xc) \in \mathbb{Z}_2$ and our arguments above would show that $\omega([n^-(c), 1]_L)\varphi^0(x) = c_L(n^-(c), w_n)\varphi^0(x)$.

Let us now restrict ourselves to the case that $n = 2$ (where $I_n(2) = I_\alpha(2)$) and fix the following notation. Let $m : \mathrm{GL}_2(\mathbb{Q}_2) \rightarrow M_\alpha$ be as in the previous section and let us refine the map $n^+ : \mathrm{Sym}_2(\mathbb{Q}_2)$ so that

$$n^+(x_1, x_2, x_3) = \begin{pmatrix} 1 & x_1 & x_2 \\ & 1 & x_2 & x_3 \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

Also we let $n^-(x_1, x_2, x_3) = {}^t n^+(x_1, x_2, x_3)$. We now define a larger subgroup $I_\alpha^*(2)$ containing $I_\alpha(2)$ for which φ^0 is an eigenfunction for that action of $I_\alpha^*(2)$ by ω .

Lemma 2.3. *The element $n^-(0, 2, 0)$ normalizes $I_\alpha(2)$.*

Proof. We begin by noting that $I_\alpha(2)$ is generated by elements of the form

$$m(A), \quad n^+(b_1, b_2, b_3), \quad \text{and} \quad n^-(c_1, c_2, c_3)$$

where $A \in \mathrm{GL}_2(\mathbb{Z}_2)$, $b_i \in \mathbb{Z}_2$, and $c_i \in 4\mathbb{Z}_2$ for all $i \in \{1, 2, 3\}$. If we set $n_2 := n^-(0, 2, 0)$, we need only show that $n_2 k n_2^{-1} \in I_\alpha(2)$ for the generating set. We let $\overline{N}_\alpha = {}^t N_\alpha \simeq \mathrm{Sym}_2(\mathbb{Q}_2)$ be the lower triangular Siegel unipotent subgroup. Therefore, $\overline{N}_\alpha \simeq \mathrm{Sym}_2(\mathbb{Q}_2)$ is an abelian group containing n_2 and $n^-(c_1, c_2, c_3)$ for any $c_1, c_2, c_3 \in 4\mathbb{Z}_2$. Thus for $n^-(c_1, c_2, c_3) \in I_\alpha(2)$,

$$n_2 n^-(c_1, c_2, c_3) n_2^{-1} = n^-(c_1, c_2, c_3) \in I_\alpha(2).$$

Next, consider $n^+(b_1, b_2, b_3)$. A direct computation shows that

$$n_2 n^+(b_1, b_2, b_3) n_2^{-1} = \begin{pmatrix} 1 - 2b_2 & 2b_1 & b_1 & b_2 \\ 2b_3 & 1 - 2b_2 & b_2 & b_3 \\ 4b_3 & 4b_2 & 1 + 2b_2 & 2b_3 \\ 4b_2 & 4b_1 & 2b_1 & 1 + 2b_2 \end{pmatrix} \in I_\alpha(2).$$

Lastly, we consider $m(A)$ with $A \in \mathrm{GL}_2(\mathbb{Z}_2)$. As before, a direct computation shows that

$$n_2 m(A) n_2^{-1} = \begin{pmatrix} A & 0 \\ 2(wA - {}^t A^{-1}w) & {}^t A^{-1} \end{pmatrix} \quad \text{where} \quad w = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_2).$$

If we set

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

and let $d = \det(A) \in \mathbb{Z}_2^\times$, then

$$wA - {}^t A^{-1}w = \begin{pmatrix} a_{21} + a_{21}d^{-1} & a_{22} - a_{22}d^{-1} \\ a_{11} - a_{11}d^{-1} & a_{12} + a_{12}d^{-1} \end{pmatrix}.$$

Thus each entry has the form $a(1 \pm d^{-1})$ where $d^{-1} \in \mathbb{Z}_2^\times$. But notice that $d^{-1} \in \mathbb{Z}_2^\times$ implies that $d^{-1} = 1 + \sum_{n=1}^\infty d_n 2^n$ with $d_n \in \{0, 1\}$. This tells us that $(1 \pm d^{-1}) \in 2\mathbb{Z}_2$ and therefore we get

$$2(wA - {}^t A^{-1}w) \in M_2(4\mathbb{Z}_2).$$

So ultimately, we have that $n_2 m(A) n_2^{-1} \in I_\alpha(2)$ and the lemma is proved. $\square$

So we let $I_\alpha^*(2)$ be the subgroup of G_2 generated by $I_\alpha(2)$ and $n^-(0, 2, 0)$. With this definition and Lemma 2.3, the following corollary is immediate.

Corollary 2.4. *The subgroup $I_\alpha^*(2)$ contains $I_\alpha(2)$ as a subgroup of index 2.*

Proof. It is clear that $n^-(0, 2, 0)^2 \in I_\alpha(2)$ and Lemma 2.3 shows that $n^-(0, 2, 0) \in N_{G_2}(I_\alpha(2))$. So for any $k \in I_\alpha^*(2)$ we have that

$$k = n^-(0, 2, 0)^i k'$$

with $i \in \mathbb{Z}$ and $k' \in I_\alpha(2)$. Further, since $n^-(0, 2, 0)^2 \in I_\alpha(2)$ we can choose $i \in \{0, 1\}$. $\square$

So combining the results of Lemma 2.2, the formulas for $\lambda_n(k)$ from Lemma 8.5.11 of [5], and Corollary 2.4, we can construct a splitting $\underline{\lambda}_2 : \mathbf{I}_\alpha^*(2) \rightarrow \widetilde{\mathrm{Sp}}_2(\mathbb{Q}_2)$ by $\underline{\lambda}_2(k) = [k, \lambda_2(k)]_L$ where

$$\begin{aligned} \lambda_2(m(A)) &= \chi_V(\det(A)) = \xi_2(\det(A)), \\ \lambda_2(n^+(b_1, b_2, b_3)) &= 1, \\ \lambda_2(n^-(c_1, c_2, c_3)) &= c_L(n^-(c_1, c_2, c_3), w_n)^{-1} \\ (2.7) \quad &= \gamma(\psi \circ Q(Y, Yw_n^{-1}, Yn^-(c_1, c_2, c_3)))^{-1} \end{aligned}$$

on our generating set for $\mathbf{I}_\alpha^*(2)$. As in [15], for any $k \in \mathbf{I}_\alpha^*(2)$ we set $\mathbf{k} := [k, \lambda_2(k)]_L$ and for any $K' \subseteq \mathbf{I}_\alpha^*(2)$ we let $\mathbf{K}' = \{\underline{\lambda}_2(k) \mid k \in K'\}$.

3. THE DOUBLING INTEGRAL

As in [15], our goal is to compute a version of the doubling integral of Piatetski-Shapiro and Rallis for certain “good test vectors”. So we now closely follow Chapter 8 of [5] and [15] to define our doubling integral. In this section F can be any p -adic field, since the results are uniform for all residual characteristic.

3.1. The doubling integral as an operator. First, let us define

$$\delta = \begin{pmatrix} & & I_n & \\ & I_n & & \\ -I_n & I_n & & \\ & & I_n & I_n \end{pmatrix} \in G_{2n}$$

where I_n is the $n \times n$ identity matrix. For $g_1, g_2 \in G_n$ with $g_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$, let us define an embedding of $G_n \times G_n \rightarrow G_{2n}$ given by

$$\underline{i}_0(g_1, g_2) := \begin{pmatrix} a_1 & & b_1 & \\ & a_2 & & b_2 \\ c_1 & & d_1 & \\ & c_2 & & d_2 \end{pmatrix}.$$

We further modify our embedding as follows. Let

$$g^\vee := \begin{pmatrix} I_n & \\ & -I_n \end{pmatrix} g \begin{pmatrix} I_n & \\ & -I_n \end{pmatrix}$$

and

$$\underline{i}(g_1, g_2) := \underline{i}_0(g_1, g_2^\vee).$$

The following lemma addresses the lift of this map to the metaplectic cover.

Lemma 3.1 (Lemma 8.5.4 in [5]). *The map ${}^\vee : G_n \rightarrow G_n$ has a lift to ${}^\vee : \widetilde{G}_n \rightarrow \widetilde{G}_n$ given in Lera coordinates by*

$$[g, z]_L^\vee := [g^\vee, z^{-1}]_L = [g^\vee, \bar{z}]_L.$$

So with this we get the following corollary.

Corollary 3.2. *The map $\underline{i} : G_n \times G_n \rightarrow G_{2n}$ lifts to a map $i : \widetilde{G}_n \times \widetilde{G}_n \rightarrow \widetilde{G}_{2n}$ given in Lera coordinates by*

$$i([g_1, z_1]_L, [g_2, z_2]_L) := i_0([g_1, z_1]_L, [g_2, z_2]_L^\vee) = [\underline{i}_0(g_1, g_2^\vee), z_1 z_2^{-1}]_L = [\underline{i}(g_1, g_2), z_1 \bar{z}_2]_L.$$

For a character $\chi : F^\times \rightarrow \mathbb{C}^1$, let $I_{2n}(s, \chi) := \text{Ind}_{\tilde{P}_{2n}}^{\tilde{G}_{2n}}([\chi\nu^{s-\frac{1}{2}} \circ \det] \otimes \sigma_0)$ be a genuine degenerate principal series representation of $\tilde{G}_{2n}$. Let $(\pi, \mathcal{V}_\pi)$ be a genuine admissible representation of $\tilde{G}_n$, $v \in \mathcal{V}_\pi$, and $\Phi_s \in I_{2n}(s, \chi)$. We define the local doubling integral

$$(3.1) \quad Z(s, \chi_V, \Phi, v) := \int_{\text{Sp}_n(F)} \Phi_s(\delta' i(1, g')) \pi(g') v dg$$

where δ', g' are any elements projecting to $\delta \in \text{Sp}_{2n}(F)$ and $g \in \text{Sp}_n(F)$, respectively. A direct computation using Lemma 3.1 shows that $Z(s, \chi_V, \Phi, v)$ is independent of the choice of g' projecting to g (see Section 4 of [15]).

According to Section 8.4 of [5], for sufficiently large $\text{Re}(s)$, the integral converges and $Z(s, \chi_V, \Phi, v) \in \mathcal{V}_\pi$. So $Z(s, \chi_V, \Phi, \cdot)$ is an element of $\text{Hom}_{\mathbb{C}}(\mathcal{V}_\pi, \mathcal{V}_\pi)$. Section 4 of [15] compares this construction to the original doubling integral of Piatetski, Shapiro and Rallis. Instead, we will consider the following proposition that addresses the compatibility of the map $\Phi_s \mapsto Z(s, \chi_V, \Phi, \cdot)$ with the $\tilde{G}_n \times \tilde{G}_n$ action on $I_{2n}(s, \chi)$ and $\text{Hom}_{\mathbb{C}}(\mathcal{V}_\pi, \mathcal{V}_\pi)$. Notice that $I_{2n}(s, \chi)$ is viewed as a representation of $\tilde{G}_n \times \tilde{G}_n$ as the pullback through the map $i : \tilde{G}_n \times \tilde{G}_n \rightarrow \tilde{G}_{2n}$ and $\tilde{G}_n \times \tilde{G}_n$ acts on $\text{Hom}_{\mathbb{C}}(\mathcal{V}_\pi, \mathcal{V}_\pi)$ by pre- and post-composition. Using statement (ii) of Lemma 8.4.1 of [5], we get the following proposition.

Proposition 3.3. *The doubling method outlined in this section defines a $\tilde{G}_n \times \tilde{G}_n$ -intertwining map*

$$I_{2n}(s, \chi) \rightarrow \text{Hom}_{\mathbb{C}}(\mathcal{V}_\pi, \mathcal{V}_\pi) \quad \Phi_s \mapsto [f \mapsto Z(s, \chi_V, \Phi, f)].$$

Additionally, we have the following important lemma as a consequence of Lemma 8.4.1 of [5] (similar to Corollary 8.4.2 from [5]).

Lemma 3.4. *Suppose there exists a subgroup $A \subset \tilde{G}_n$ and ς a character of A such that*

$$R(i(a, 1))\Phi_s = \varsigma(a)\Phi_s, \quad \forall a \in A.$$

Then

$$Z(s, \chi_V, \Phi, f) \in \mathcal{V}_\pi^{(A, \varsigma)}.$$

One should also note that $\text{Hom}_{\mathbb{C}}(\mathcal{V}_\pi, \mathcal{V}_\pi) \simeq \mathcal{V}_\pi \boxtimes \mathcal{V}_{\pi^\vee}$ are canonically isomorphic as $\tilde{G}_n \times \tilde{G}_n$ modules. So the doubling integral can be interpreted as a map from $I_{2n}(s, \chi) \rightarrow \mathcal{V}_\pi \boxtimes \mathcal{V}_{\pi^\vee}$. Let us recall Rallis's criterion for the non-vanishing of local theta lifts mentioned in the introduction. To establish the non-vanishing of the theta lift of $(\pi, \mathcal{V}_\pi)$ to the group $\text{O}(V)$, we need to find a non-vanishing $\tilde{G}_n \times \tilde{G}_n$ -intertwining map from $R_{2n}(V)$ to $\mathcal{V}_\pi \boxtimes \mathcal{V}_{\pi^\vee}$. Since we already have a $\tilde{G}_n \times \tilde{G}_n$ -intertwining operator from $I_{2n}(s, \chi) \rightarrow \mathcal{V}_\pi \boxtimes \mathcal{V}_{\pi^\vee}$, we now construct a $\tilde{G}_n \times \tilde{G}_n$ -intertwining map from $R_{2n}(V)$ to $I_{2n}(s_0, \chi_V)$ for certain $s_0 \in \mathbb{Z}$ that we describe below. Because, $R_{2n}(V)$ is the $\text{O}(V)$ -coinvariants of the Weil representation $(\omega_V, S(V^{2n}))$, we will actually define a $\text{O}(V)$ -invariant map from $S(V^{2n})$ to $I_{2n}(s_0, \chi_V)$ that intertwines the $\tilde{G}_n \times \tilde{G}_n$ action. So ultimately, our choice of section $\{\Phi_{s_0+r}\}_{r=0}^\infty$ will actually amount to a choice of Schwartz function.

3.2. The interpolation method. Consider the following map (described in Section III.5 of [4] and attributed to Section II of [6])

$$\lambda_V : S(V^n) \rightarrow I_n(s_0, \chi_V) \quad \varphi \mapsto \lambda_V(\varphi)([g, z]_L) := \omega_V([g, z]_L)\varphi(0)$$

where $s_0 = \frac{\dim_F(V) - 2n - 1}{2}$. As was alluded to in the previous section, this map is $O(V)$ -invariant and thus factors through $R_n(V)$. It is shown in Section II of [6] that λ_V induces an injective map,

$$R_{2n}(V) \rightarrow I_{2n}(s_0, \chi_V).$$

We now formulate a method for taking a function $\varphi \in S(V^n)$ and extending it to a Schwartz function in the Schwartz space $S(V_r^n)$ where $V_r = V \oplus V_h^r$. As mentioned in Section 2, spaces in the same Witt tower have the same quadratic character, so we can then apply the map λ_{V_r} to get a function in $I_n(s_0 + r, \chi_V)$. To do this, we need the following lemma describing a property of the Weil representation when a quadratic space $V = V_1 \oplus V_2$ (an orthogonal sum) with $m_i := \dim_F(V_i)$.

Lemma 3.5 (Lemma 8.5.9 in [5]). *The Weil representation ω_V of $\widetilde{G}_n$ on $S(V^n) \simeq S(V_1^n) \otimes S(V_2^n)$ is given by*

$$\omega_V(g') = \omega_{V_1}(g') \otimes \omega_{V_2}(g') \cdot \begin{cases} \zeta(g')^{-1} & m_1 m_2 \text{ is odd,} \\ 1 & \text{otherwise,} \end{cases}$$

where $\zeta : \widetilde{G}_n \rightarrow \mathbb{C}^\times$ is a character defined Section 8.5 of [5].

Since $V_r = V \oplus V_h^r$ and $\dim_F(V_h^r) = 2r$, we see that $\omega_{V_r} = \omega_V \otimes \omega_{V_h^r}$. Recalling our basis $\{e_0, e'_0\}$ of V_h described in Section 2, we let $L_0 \subset V_h$ be the lattice $\mathcal{O}e_0 \oplus \mathcal{O}e'_0$ and $\varphi_0 \in S(V_h)$ the characteristic function of L_0 . We then let $\varphi_0^r = \bigotimes_{k=1}^r \varphi_0 \in S(V_h^r)$. For $\varphi \in S(V^n)$, we define $\varphi^{(r)} \in S(V_r^n)$ to be the function $\varphi \otimes \bigotimes_{j=1}^n \varphi_0^r$. To simplify notation, we set $\varphi_0^{rn} := \bigotimes_{j=1}^n \varphi_0^r \in S((V_h^r)^n)$. We have the following lemma regarding φ_0^{rn} .

Lemma 3.6. *The function φ_0^{rn} is fixed by the subgroup $\widetilde{K}_n = \{[k, z]_L : k \in K_n, z \in \mathbb{C}^\times\} \subset \widetilde{G}_n$. Furthermore, for any $b \in \mathrm{Sym}_n(F)$, $a \in \mathrm{GL}_n(F)$, $k \in K_n$, and $z \in \mathbb{C}^\times$, $\omega_{V_h^r}([n^+(b)m(a)k, z]_L)\varphi_0^{rn}(0) = |\det(a)|^r$.*

Proof. The proof follows from some direct computation using (2.4) and the expressions for $r_V(g)\varphi_0^{rn}$ found in Section 8.5.1 of [5]. First, consider (2.4). Since $\dim_F(V_h^r)$ is even, we need only consider χ_V and $\gamma(\eta \circ V)$. First, we have $\det(V_h^r) = (-1)^r$ so that

$$\chi_V(x) = (x, (-1)^{\frac{2r(2r-1)}{2}}(-1)^r)_F = (x, (-1)^{2r^2})_F = 1$$

for all $x \in F^\times$. So χ_V is the trivial character. Second, after Lemma 8.5.8 of [5] it is noted that for $V = V_1 \oplus V_2$ an orthogonal direct sum that $\gamma(\eta \circ V) = \gamma(\eta \circ V_1)\gamma(\eta \circ V_2)$. For us, $\gamma(\eta \circ V_h) = \gamma_F(-1, \eta)\gamma(\eta)^2\epsilon(V_h)$. Since V_h is a 2-dimensional split quadratic space $\epsilon(V_h^r) = 1$. Also, Corollary A.5 of [7] shows that $\gamma_F(-1, \eta) = \gamma_F(\eta)^{-2}$ so $\gamma(\eta \circ V_h) = 1$ and $\gamma(\eta \circ V_h^r) = 1^r = 1$.

Substituting these quantities into (2.4), we have $\omega_{V_h^r}([g, z]_L)\varphi_0(x) = r_{V_h^r}(g)\varphi_0(x)$. In order to show that φ_0^{rn} is fixed by $\widetilde{K}_n$, we need only show that $r_{V_h^r}(k)\varphi_0^{rn} = \varphi_0^{rn}$ for all $k \in K_n$. We can verify this on a generating set for K_n . In particular, we can verify it on elements of the form $n^+(b)$ with $b \in \mathrm{Sym}_n(\mathcal{O})$, $m(a)$ with $a \in \mathrm{GL}_n(\mathcal{O})$, and w_j with $1 \leq j \leq n$ from Section 2. The statement is obvious for elements of

the form $n^+(b)$ and $m(a)$ using equations (8.5.6) and (8.5.7) of [5]. To verify that $r_{V_h^r}(w_j)\varphi_0^{rn} = \varphi_0^{rn}$, one only need notice that the Fourier transform $\widehat{\varphi}_0$ given by

$$\widehat{\varphi}_0^r(x) = \int_{V_h^r} \psi((x, y)_{V_h^r}) \varphi_0^r(y) dy$$

has the property that $\widehat{\varphi}_0^r = \varphi_0^r$ and that the operators $r_{V_h^r}(w_j)$ compute this Fourier transform on the first j -components of φ_0^r in φ_0^{rn} (see Section 8.5.1 of [5]). This shows that φ_0^{rn} is invariant under $\widehat{K}_n$.

Finally, for $b \in \text{Sym}_2(F)$, $a \in \text{GL}_n(F)$, $k \in K_n$, and $z \in \mathbb{C}^1$, we have

$$\begin{aligned} \omega_{V_h^r}([n(b)m(a)k, z]_L)\varphi_0^{rn}(0) &= \omega_{V_h^r}([n(b)m(a), 1]_L)(\omega_{V_h^r}([k, z]_L)\varphi_0^{rn})(0) \\ &= \omega_{V_h^r}([n(b)m(a), 1]_L)\varphi_0^{rn}(0) = |\det(a)|^r. \end{aligned} \quad \square$$

Let us define a function $\varrho : G_n \rightarrow \mathbb{C}$ by $\varrho(g) = |\det(a)|$ where $g = n(b)m(a)k$ with $b \in \text{Sym}_n(F)$, $a \in \text{GL}_n(F)$, and $k \in K_n$. So we see that for any $\varphi \in S(V^n)$, $[\lambda_{V_r}(\varphi^{(r)})]([g, z]_L) = \varrho(g)^r [\lambda_V(\varphi)]([g, z]_L)$. So if we define a section $\Phi_{s_0+s} \in I_n(s_0+s, \chi_V)$ by $\Phi_{s_0+s}([g, z]_L) := \varrho(g)^s [\lambda_V(\varphi)]([g, z]_L)$, we see that the $\Phi_{s_0+r}([g, z]_L) = [\lambda_{V_r}(\varphi^{(r)})]([g, z]_L)$.

For our purposes, our sections $\{\Phi_{s_0+r}\}_{r=1}^\infty$ will always be constructed by choosing a $\varphi \in S(V^n)$ and then letting $\Phi_{s_0+r} = \lambda_{V_r}(\varphi^{(r)})$. Because we have already seen how $\lambda_{V_r}(\varphi^{(r)}) = \varrho^r \lambda_V(\varphi)$, we will conflate our notation and use λ_V to denote all of the maps from $S(V_r^n)$ to $I_n(s_0+r, \chi_V)$. Let us now mention the types of functions that we will employ as $\varphi \in S(V^n)$.

Let L be a lattice (free $\mathcal{O}$ -submodule) in V . We let $L^\# = \{x \in V \mid \psi((x, y)_V) = 1, \text{ for all } y \in L\}$ be the dual lattice. We call L a **good lattice** if $\varpi L^\# \subset L \subset L^\#$. Let $\{L_i\}_{i=1}^n$ be a collection of good lattices in V and let $\varphi_{L_i} \in S(V)$ be the characteristic function of L_i . We then form the function $\varphi_{L_1 \oplus L_2 \oplus \dots \oplus L_n} = \bigotimes_{i=1}^n \varphi_{L_i} \in S(V^n)$. Sections $\{\Phi_{s_0+r}\}_{r=0}^\infty$ of the form $\Phi_{s_0+r} = \lambda_V(\varphi_{L_1 \oplus L_2 \oplus \dots \oplus L_n}^{(r)})$ are called “good theta test sections”.

4. CHARACTERS OF $\mathbf{I}_\alpha^*(2)$ AND $\mathbf{I}_\alpha(2)$ EIGENFUNCTIONS

In this section, $F = \mathbb{Q}_2$. Let $(\pi, \mathcal{V}_\pi) = \chi_1 \times \chi_2 \rtimes \sigma_0$ in the standard induced model where the χ_i have ramified quadratic twists such that $\chi_i(-1) = -1$ for $i \in \{1, 2\}$. As Lemma 3.4 suggests, we are interested in finding function $f \in \mathcal{V}_\pi$ that are eigenfunctions for the action of $\mathbf{I}_\alpha^*(2)$. The purpose of this section is to make these functions explicit.

4.1. Some characters of $\mathbf{I}_\alpha^*(2)$. Because our the inducing data for $(\pi, \mathcal{V}_\pi)$ in Theorem 5.11 as well as $I(s_0+r, \chi_V)$ contain ramified quadratic characters, we are interested in determining vectors in $\mathcal{V}_\pi$ that transform by a character under the action of $\mathbf{I}_\alpha^*(2)$. So we need to find characters $\varsigma : \mathbf{I}_\alpha^*(2) \rightarrow \mathbb{C}^1$ so that the set of vectors $f \in \mathcal{V}_\pi$ such that

$$f(g'\mathbf{k}) = \varsigma(\mathbf{k})f(g')$$

for all $g' \in \widetilde{G}_n$ and $\mathbf{k} \in \mathbf{I}_\alpha^*(2)$ is non-zero. By composition with the splitting map

$$\underline{\lambda}_2 : I_\alpha^*(2) \rightarrow \widetilde{G}_2$$

we get a character $\underline{\varsigma} : I_\alpha^*(2) \rightarrow \mathbb{C}^1$ given by

$$\underline{\varsigma}(k) = \varsigma(\underline{\lambda}_2(k)) = \varsigma(\mathbf{k}).$$

Because $I_\alpha^*(2) \simeq \mathbf{I}_\alpha^*(2)$ canonically, we will abuse notation and use ς to denote a character on either $I_\alpha^*(2)$ or $\mathbf{I}_\alpha^*(2)$.

In order to study the characters of $I_\alpha^*(2)$, we will decompose $I_\alpha^*(2)$ as a product of several subgroups whose character theory is well known. We then construct characters of $I_\alpha^*(2)$ as products of characters of the subgroups. To do this, we need to find certain compatibility conditions on the characters of the subgroups.

Let us return to the notation of Section 2.3. For any $k \in I_\alpha^*(2)$,

$$(4.1) \quad k = n^+(B)m(A)n^-(C)$$

where $B \in \mathrm{Sym}_2(\mathbb{Z}_2)$, $A \in \mathrm{GL}_2(\mathbb{Z}_2)$ and $C \in \mathrm{Sym}_2(\mathbb{Z}_2)$ of the form

$$C = \begin{pmatrix} 4c_1 & 2c_2 \\ 2c_2 & 4c_3 \end{pmatrix}.$$

So we see that $I_\alpha^*(2) = N_\alpha(\mathbb{Z}_2)M_\alpha(\mathbb{Z}_2)\overline{N}_\alpha^*$, where $\overline{N}_\alpha^* = \overline{N}_\alpha(\mathbb{Z}_2) \cap I_\alpha^*(2)$ and $\overline{N}_\alpha(\mathbb{Z}_2) = {}^tN_\alpha(\mathbb{Z}_2)$. Next we refine this notation a bit further. For $x \in F$ and $t_1, t_2 \in F^\times$, let

$$m^+(x) := m \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}, \quad m(t_1, t_2) := m \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix}, \quad \text{and} \quad m^-(x) := m \begin{pmatrix} 1 & \\ x & 1 \end{pmatrix}.$$

For any Weil representation $(\omega_V, S(V^2))$, the subgroup M_α acts by left translation twisted by a the character $\chi_V \circ \det$. As such, we will be interested in characters of $M_\alpha(\mathbb{Z}_2)$ having a similar form. Let $\chi : \mathbb{Z}_2^\times \rightarrow \mathbb{C}^1$ be a character. We will abuse notation and consider χ a character of $M_\alpha(\mathbb{Z}_2) \simeq \mathrm{GL}_2(\mathbb{Z}_2)$ by composition with the determinant map.

Remark 4.1. Notice that $M_\alpha(\mathbb{Z}_2) \simeq \mathrm{GL}_2(\mathbb{Z}_2)$, which has characters that are non-trivial on $\mathrm{SL}_2(\mathbb{Z}_2)$ (and are thus not of the form $\chi \circ \det$). For instance, take the projection $\mathrm{proj} : M_\alpha(\mathbb{Z}_2) \simeq \mathrm{GL}_2(\mathbb{Z}_2) \rightarrow \mathrm{GL}_2(\mathbb{F}_2)$. Since $\mathrm{GL}_2(\mathbb{F}_2) \simeq S_3$ (the symmetric group on 3 elements), we can compose the the projection with the sign character on S_3 to produce a character of $M_\alpha(\mathbb{Z}_2)$ for which the matrix $m^+(1)$ maps to -1 . So characters of the form $\chi \circ \det$ do not exhaust the characters of $M_\alpha(\mathbb{Z}_2)$.

Next, both $N_\alpha(\mathbb{Z}_2)$ and $\overline{N}_\alpha^*$ are isomorphic to $(\mathbb{Z}_2)^3$ (in fact, $N_\alpha(\mathbb{Z}_2) \simeq \mathrm{Sym}_2(F)$ in an obvious way, but we need not use this fact). The characters of both subgroups have the forms

$$\underline{\psi}^+(n^+(x_1, x_2, x_3)) = \psi_1^+(x_1)\psi_2^+(x_2)\psi_3^+(x_3)$$

and

$$\underline{\psi}^-(n^-(4x_1, 2x_2, 4x_3)) = \psi_1^-(x_1)\psi_2^-(x_2)\psi_3^-(x_3)$$

where $\psi_i^\pm : \mathbb{Z}_2 \rightarrow \mathbb{C}^1$ are additive characters for all $i \in \{1, 2, 3\}$. So we will now prove that a character $\varsigma : I_\alpha^*(2) \rightarrow \mathbb{C}^1$ can be written as a triple $(\underline{\psi}^+, \chi, \underline{\psi}^-)$ where

$$\varsigma(n^+(B)m(A)n^-(C)) = \underline{\psi}^+(n^+(B))\chi(m(A))\underline{\psi}^-(n^-(C)).$$

However, for an arbitrary triple $(\underline{\psi}^+, \chi, \underline{\psi}^-)$ this product of characters need not determine a well-defined character; in general, such products do not. So we will compute ς on various commutators to determine the criteria for the ς to be well defined. For this, we aim to prove the following proposition.

Proposition 4.2. *Let $\varsigma = (\underline{\psi}^+, \chi, \underline{\psi}^-)$ be a triple as above. Then ς defines a character of $I_\alpha^*(2)$ if and only if the following conditions hold:*

- a. $\psi_1^+ = \psi_2^+ = \psi_3^+$ with $\text{cond}(\psi_i^+) \leq 1$.
- b. $\psi_1^- = \psi_3^- = 1$ and $\text{cond}(\psi_2^-) \leq 1$.
- c. $\chi^2 = 1$ with $\chi|_{1+4\mathbb{Z}_2} = 1$.

Proof. We will show the necessity of conditions a-c by computing a character ς on various commutators. Since ς is a character, all commutators belong to the kernel of ς . We begin by proving the conditions on $\underline{\psi}^+$ and $\underline{\psi}^-$. In order to handle some calculations simultaneous in both cases, we let $x' = x$ if $n^\pm(x, 0, 0) = n^+(x, 0, 0)$ and $4x' = x$ if $n^\pm(x, 0, 0) = n^-(x, 0, 0)$. Therefore, we have $\underline{\psi}^\pm(n^\pm(x, 0, 0)) = \psi_1^\pm(x')$. We use this convention only when computing with both $\underline{\psi}^\pm$ simultaneously.

First note that $w_\alpha n^\pm(x, 0, 0)w_\alpha^{-1}n^\pm(-x, 0, 0) = n^\pm(-x, 0, x)$. Applying ς to both sides we find that $1 = \psi_1^\pm(-x')\psi_3^\pm(x')$ for all $x' \in \mathbb{Z}_2$. Thus we have

$$(4.2) \quad \psi_1^\pm = \psi_3^\pm.$$

Next consider,

$$\begin{aligned} m(a_1, a_2)n^\pm(x_1, x_2, x_3)m(a_1^{-1}, a_2^{-1})n^\pm(-x_1, -x_2, -x_3) \\ = n^\pm((a_1^2 - 1)x_1, (a_1a_2 - 1)x_2, (a_2^2 - 1)x_3). \end{aligned}$$

Applying ς to both sides yields

$$1 = \psi_1^\pm((a_1^2 - 1)x'_1)\psi_2^\pm((a_1a_2 - 1)x'_2)\psi_3^\pm((a_2^2 - 1)x'_3) = 1.$$

Now we set $x'_i = 1$ for all $i \in \{1, 2, 3\}$. If we also set $a_2 = 1$ we get

$$(4.3) \quad \psi_1^\pm(a_1^2 - 1)\psi_2^\pm(a_1 - 1) = 1,$$

whereas if we set $a_1 = 1$ we find

$$(4.4) \quad \psi_2^\pm(a_2 - 1)\psi_3^\pm(a_2^2 - 1) = 1.$$

Now let us consider the product

$$m^+(b)n^+(0, x, 0)m^+(-b)n^+(0, -x, 0) = n^+(2bx, 0, 0)$$

which implies that

$$1 = \psi_1^+(2bx)$$

for all $b, x \in \mathbb{Z}_2$. So we see that $\psi_1^+|_{2\mathbb{Z}_2} = 1$ and $\text{cond}(\psi_1^+) \leq 1$. Now let us revisit (4.3). Since $a_1 \in \mathbb{Z}_2^\times$ we have $a_1 = 1 + 2a'_1$ for some $a'_1 \in \mathbb{Z}_2$. Furthermore,

$$(a_1^2 - 1) = ((1 + 2a'_1)^2 - 1) = 4a'_1(1 + a'_1).$$

Since $a'_1 \in \mathbb{Z}_2$, then either a'_1 or $1 + a'_1$ is an element of $2\mathbb{Z}_2$. Thus $(a_1^2 - 1) \in 8\mathbb{Z}_2$ and $\psi_1^+(a_1^2 - 1) = 1$ for all $a_1 \in \mathbb{Z}_2^\times$. Therefore, (4.3) yields

$$1 = \psi_1^+(a_1^2 - 1)\psi_2^+(a_1 - 1) = \psi_2^+(a_1 - 1)$$

for all $a_1 \in \mathbb{Z}_2^\times$; but this implies $\psi_2^+|_{2\mathbb{Z}_2} = 1$ and $\text{cond}(\psi_2^+) \leq 1$.

Now consider the product

$$m^+(1)n^+(0, 0, x)m^+(-1)n^+(0, 0, -x) = n^+(x, x, 0).$$

Again, applying ς to both sides yields

$$1 = \psi_1^+(x)\psi_2^+(x)$$

for all $x \in \mathbb{Z}_2$. Thus $(\psi_1^+)^{-1} = \psi_2^+$. However, since $\text{cond}(\psi_1^+) \leq 1$, we have

$$(\psi_1^+)^2(x) = \psi_1^+(2x) = 1$$

for all $x \in \mathbb{Z}_2$. Therefore, $\psi_2^+ = (\psi_1^+)^{-1} = \psi_1^+$ which is also equal to ψ_3^+ by (4.2). This proves the necessity of condition a.

Now we consider a similar computation for the lower triangular unipotent matrices. In particular, we have

$$m^-(b)n^-(0, 2x, 0)m^-(b)n^-(0, -2x, 0) = n^-(4bx, 0, 0)$$

which, upon application of ς , tells us that

$$1 = \psi_1^-(bx)$$

for all $b, x \in \mathbb{Z}_2$. Thus $\mathrm{cond}(\psi_1^-) = 0$ or ψ_1^- is the trivial additive character on $\mathbb{Z}_2$ and (4.2) implies that ψ_3^- is trivial as well. Combining these facts with (4.3) yields

$$1 = \psi_1^-(a_1^2 - 1)\psi_2^-(a_1 - 1) = \psi_2^-(a_1 - 1)$$

for all $a_1 \in \mathbb{Z}_2^\times = 1 + 2\mathbb{Z}_2$. Therefore, $\psi_2^-|_{2\mathbb{Z}_2} = 1$ and $\mathrm{cond}(\psi_2^-) \leq 1$. This proves the necessity of condition b.

We now consider the character χ . In particular, we have the product

$$\begin{aligned} n^+(-x, 0, 0)n^-(4y, 0, 0)n^+(x, 0, 0)n^-(4y, 0, 0) \\ = n^+\left(\frac{-4x^2y}{1+4xy}, 0, 0\right)m((1+4xy)^{-1}, 1+4xy)n^-\left(\frac{-16xy^2}{1+4xy}, 0, 0\right) \end{aligned}$$

for all $x, y' \in \mathbb{Z}_2$. When we apply ς to both sides, we see that

$$1 = \psi_1^+\left(-4\frac{x^2y}{1+4xy}\right)\chi(1+4xy)\psi_1^-\left(-4\frac{xy^2}{1+4xy}\right).$$

Since $\psi_1^\pm|_{4\mathbb{Z}_2} = 1$, we have $\chi(1+4xy) = 1$ for all $x, y \in \mathbb{Z}_2$. Thus $\chi|_{1+4\mathbb{Z}_2} = 1$. Since $(\mathbb{Z}_2^\times)^2 \subset 1 + 8\mathbb{Z}_2 \subset 1 + 4\mathbb{Z}_2$, we see $\chi^2 = 1$ (i.e., χ is quadratic) and the necessity of condition c follows.

Now we show that a $\varsigma = (\underline{\psi}^+, \chi, \underline{\psi}^-)$ satisfying conditions a–c above defines a character of $I_\alpha^*(2)$ by

$$k = n^+(B)m(A)n^-(C) \mapsto \varsigma(k) := \psi^+(n^+(B))\chi(A)\psi^-(n^-(C)).$$

This amounts to showing that the definition of the character is well defined on the various commutators of elements from any two distinct subgroups amongst the

$N_\alpha(\mathbb{Z}_2)$, $M_\alpha(\mathbb{Z}_2)$ and $\overline{N}_\alpha^*$. Consider the following with $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$\begin{aligned} [m(A^{-1}), n^-(4x_1, 2x_2, 4x_3)] &= m(A^{-1})n^-(4x_1, 2x_2, 4x_3)m(A)n^-(4x_1, 2x_2, 4x_3) \\ &= n^-(4(a^2x_1 + acx_2 + c^2x_3 - x_1), 4abx_1 + 2(ad+bc)x_2 + 4cdx_3 - 2x_2, \\ &\quad 4(b^2x_1 + bdx_2 + d^2x_3 - x_3)). \end{aligned}$$

Using condition b and setting $\psi^- = \psi_2^-$, we see that

$$\varsigma([m(A^{-1}), n^-(4x_1, 2x_2, 4x_3)]) = \psi^-((ad+bc-1)x_2).$$

Next we realize that $ad+bc-1 = 2bc + (ad-bc) - 1 = 2bc + \det(A) - 1$. Since A is invertible in $\mathrm{GL}_2(\mathbb{Z}_2)$, then $\det(A) \in 1 + 2\mathbb{Z}_2$ and we have $(ad+bc-1) \in 2\mathbb{Z}_2$ and $\psi^-((ad+bc-1)x_2) = 1$. Therefore, $\varsigma([m(A^{-1}), n^-(4x_1, 2x_2, 4x_3)]) = 1$.

Next, consider the product

$$\begin{aligned} [m(A), n^+(x_1, x_2, x_3)] &= m(A)n^+(x_1, x_2, x_3)m(A^{-1})n^+(-x_1, -x_2, -x_3) \\ &= n^+(a^2x_1 + 2abx_2 + b^2x_3 - x_1, acx_1 + (ad + bc)x_2 + bdx_3 - x_2, \\ &\quad c^2x_1 + 2cdx_2 + d^2x_3 - x_3). \end{aligned}$$

Applying ς to both sides and setting $\psi^+ = \psi_i^+$ for all $i \in \{1, 2, 3\}$ (they are all equal by condition a), and using the fact that $\text{cond}(\psi^+) \leq 1$, we have

$$\begin{aligned} \varsigma([m(A), n^+(x_1, x_2, x_3)]) \\ = \psi^+((a^2 + ac + c^2 - 1)x_1)\psi^+((ad + bc - 1)x_2)\psi^+((b^2 + bd + d^2 - 1)x_3) \end{aligned}$$

for all x_1 and x_2 in $\mathbb{Z}_2$ for ς to be well defined. First, we have $\psi^+((ad + bc - 1)x_2) = 1$ for the same reason as ψ^- above. Now let us consider the factor $\psi^+((a^2 + ac + c^2 - 1)x_1)$. Notice that since A is invertible in $\text{GL}_2(\mathbb{Z}_2)$, we must have either a or c in $\mathbb{Z}_2^\times$. But then one can verify directly that $(a^2 + ac + c^2 - 1) \in 2\mathbb{Z}_2$ as long as either a or c is a unit in $\mathbb{Z}_2^\times$. Therefore, $\psi^+((a^2 + ac + c^2 - 1)x_1) = 1$. The identical argument applies to the factor $\psi^+((b^2 + bd + d^2 - 1)x_3) = 1$. So $\varsigma([m(A), n^+(x_1, x_2, x_3)]) = 1$.

Finally, let us consider the product

$$\begin{aligned} [n^+(-B), n^-(C)] &= n^+(-B)n^-(C)n^+(B)n^-(-C) \\ &= n^+(B(I + CB)^{-1} - B)m({}^t(I + CB)^{-1})n^-((I + CB)^{-1}C - C) \end{aligned}$$

where I is the 2×2 identity matrix. Applying ς to both sides, we see that

$$\begin{aligned} \varsigma([n^+(-B), n^-(C)]) \\ = \underline{\psi}^+(n^+(B - B(I + CB)^{-1}))\chi(m({}^t(I + CB)^{-1}))\underline{\psi}^-(n^-(C - (I + CB)^{-1}C)) = 1. \end{aligned}$$

We will now show that each factor vanishes when ς satisfies conditions a–c. First we notice that

$$(4.5) \quad B - B(I + CB)^{-1} = [B(I + CB) - B](1 + CB)^{-1} = BCB(I + CB)^{-1}.$$

Because it has a factor of C , the matrix in (4.5) has entries belonging to $2\mathbb{Z}_2$ so $\underline{\psi}^+(n^+(B - B(I + CB)^{-1})) = 1$. Similarly, we have

$$(4.6) \quad C - (I + CB)^{-1}C = (I + CB)^{-1}[(I + CB)C - C] = (I + CB)^{-1}CBC.$$

Since it has two factors of C , the matrix in (4.6) has entries belonging to $4\mathbb{Z}_2$ so $\underline{\psi}^-(n^-(C - (I + CB)^{-1}C)) = 1$. Finally, one can verify by direct computation that $\overline{CB}$ and BC are matrices with entries in $2\mathbb{Z}_2$ and trace in $4\mathbb{Z}_2$. So we consider,

$${}^t(I + CB)^{-1} = (I + BC)^{-1}$$

and a direct computation shows that $\det(I + BC) = 1 - \text{tr}(BC) + \det(BC) \in 1 + 4\mathbb{Z}_2$. So condition c tells us that $\chi({}^t(1 + CB)^{-1}) = 1$.

So we have now demonstrated conditions a–c are sufficient for a triple $(\underline{\psi}^+, \chi, \underline{\psi}^-)$ to define a character of $I_\alpha^*(2)$ and so this completes the proof. $\square$

4.2. The space $\mathcal{V}_\pi^{\mathbf{I}_\alpha^*(2), \varsigma}$. Let $(\pi, \mathcal{V}_\pi)$ be as in Section 4.1. Now that we have established a splitting $\underline{\lambda}_2 : I_\alpha^*(2) \rightarrow \tilde{G}_2$ and we have determined the characters of $\mathbf{I}_\alpha^*(2) \simeq I_\alpha^*(2)$, we use them to examine the subspace $\mathcal{V}_\pi^{\mathbf{I}_\alpha^*(2), \varsigma} \subset \mathcal{V}_\pi$ of functions $f \in \mathcal{V}_\pi$ such that $f(g'\mathbf{k}) = \varsigma(\mathbf{k})f(g')$ for all $g' \in \tilde{G}_2$ and $\mathbf{k} \in \mathbf{I}_\alpha^*(2)$. We will call these the ς -eigenfunctions in $\mathcal{V}_\pi$. We first notice that because $(\pi, \mathcal{V}_\pi)$ is a

principal series representation, a function $f \in \mathcal{V}_\pi^{(\mathbf{I}_\alpha^*(2), \varsigma)}$ is completely determined by its values on representatives of the double cosets

$$\tilde{P}_\emptyset \backslash \tilde{G}_2 / \mathbf{I}_\alpha^*(2).$$

So to study the ς -eigenfunctions, we begin by decomposing $\tilde{G}_2$ into double cosets of the form $\tilde{P}_\emptyset g' \mathbf{I}_\alpha^*(2)$.

Because $\{1_{G_2}\} \times \mathbb{C}^1 \subset \tilde{P}_\emptyset$, decomposing $\tilde{G}_2$ into double cosets of the form $\tilde{P}_\emptyset g' \mathbf{I}_\alpha^*(2)$ is equivalent to decomposing G_2 into double cosets of the form $P_\emptyset g \mathbf{I}_\alpha^*(2)$. If we decompose the linear group, we can take any $g' \in \tilde{G}_2$ projecting to $g \in G_2$ as a double coset representative for the metaplectic group. So we consider the linear group.

Proposition 4.3. *The group G_2 has the following decomposition into double cosets of the form $P_\emptyset g \mathbf{I}_\alpha^*(2)$,*

$$\begin{aligned} G_2 = & \bigcup_{c_i \in \{0,2\}} P_\emptyset n^-(c_1, 0, c_3) \mathbf{I}_\alpha^*(2) \cup \bigcup_{c_1 \in \{0,2\}} P_\emptyset w_\beta n^-(c_1, 0, 0) \mathbf{I}_\alpha^*(2) \\ & \cup \bigcup_{c_1 \in \{0,2\}} P_\emptyset w_{\alpha\beta} n^-(c_1, 0, 0) \mathbf{I}_\alpha^*(2) \cup P_\emptyset w_{\beta\alpha\beta} \mathbf{I}_\alpha^*(2). \end{aligned}$$

Proof. First, the Iwasawa decomposition of G_2 (see Section 3.1 of [1]) yields that $G_2 = P_\emptyset K_2$. Next consider the map

$$K_2 \rightarrow \mathrm{Sp}_2(\mathbb{F}_2)$$

that reduces the entries of $K_2 \bmod 2\mathbb{Z}_2$. We can then employ the Bruhat decomposition of $\mathrm{Sp}_2(\mathbb{F}_2)$ to yield

$$\mathrm{Sp}_2(\mathbb{F}_2) = \bigcup_{w \in \mathcal{W}_{G_2}} P_\emptyset(\mathbb{F}_2) w P_\emptyset(\mathbb{F}_2).$$

So by taking an inverse image of this decomposition in K_2 we find

$$K_2 = \bigcup_{w \in \mathcal{W}_{G_2}} I_\emptyset(1) w I_\emptyset(1).$$

We can verify through direct computation that for $k \in I_\emptyset(1)$, $w^{-1}kw \in I_\emptyset(1)$ unless $k \in N^w(\mathbb{Z}_2) \setminus N^w(2\mathbb{Z}_2)$ where

$$N^w = \{n \in N_\emptyset \mid w^{-1}nw \in {}^t N_\emptyset\}.$$

For any $w \in \mathcal{W}_{G_2}$, $N^w(\mathbb{Z}_2) \subset N_\emptyset \subset P_\emptyset$, so we get

$$(4.7) \quad G_2 = \bigcup_{w \in \mathcal{W}_{G_2}} P_\emptyset w I_\emptyset(1).$$

Now (4.7) is not quite the decomposition we set out to achieve, but the rest of our manipulations will be quite elementary. First we realize that $I_\alpha(1)$ is generated by $I_\emptyset(1)$ and w_α . First, we fix some additional notation by setting

$$\mathcal{W}_{G_2}(\alpha) := \{w_0 = 1_{G_2}, w_\beta, w_{\alpha\beta}, w_{\beta\alpha\beta}\},$$

so that $\mathcal{W}_{G_2}(\alpha)$ is fixed a set of coset representatives for $\mathcal{W}_{G_2}/\langle w_\alpha \rangle$. With these representatives (4.7) becomes

$$G_2 = \bigcup_{w \in \mathcal{W}_{G_2}(\alpha)} P_\emptyset w I_\alpha(1).$$

Now we aim to decompose $I_\alpha(1)$ as cosets of the form $gI_\alpha^*(2)$. First, for any $k \in I_\alpha(1)$ we can write k as

$$k = n^-(C)m(A)n^+(B)$$

with $A \in \mathrm{GL}_2(\mathbb{Z}_2)$, $B \in \mathrm{Sym}_2(\mathbb{Z}_2)$ and $C \in \mathrm{Sym}_2(2\mathbb{Z}_2)$. Suppose that $c \in 2(\mathbb{Z}_2^\times)$, then $c = 2(1+t)$ with $t \in 2\mathbb{Z}_2$ and $c-2 = 2t \in 4\mathbb{Z}_2$. So for $k \in I_\alpha(1)$ with $k = n^-(C)m(A)n^+(B)$, we have

$$k = n^-(c_1, 0, c_3)n^-(C')m(A)n^+(B)$$

where $c_i \in \{0, 2\}$ for $i \in \{1, 3\}$ and $n^-(C') \in I_\alpha^*(2)$. Therefore,

$$G_2 = \bigcup_{w \in \mathcal{W}_{G_2}(\alpha)} \bigcup_{c_i \in \{0, 2\}} P_\emptyset w n^-(c_1, 0, c_3) I_\alpha^*(2).$$

Finally, we seek to eliminate redundant double cosets by determining $n^-(c_1, 0, c_3)$ with $wn^-(c_1, 0, c_3)w^{-1} \in P_\emptyset$. These final computations yield the desired conclusion. $\square$

Now let $\mathbf{w} = [w, 1]_L$ for any $w \in \mathcal{W}_{G_2}$ and we have the following corollary.

Corollary 4.4. *The group $\tilde{G}_2$ has the following decomposition into double cosets of the form $\tilde{P}_\emptyset g' I_\alpha^*(2)$,*

$$\begin{aligned} \tilde{G}_2 = & \bigcup_{c_i \in \{0, 2\}} \tilde{P}_\emptyset [n^-(c_1, 0, c_2), 1]_L \mathbf{I}_\alpha^*(2) \cup \bigcup_{c_1 \in \{0, 2\}} \tilde{P}_\emptyset \mathbf{w}_\beta [n^-(c_1, 0, 0), 1]_L \mathbf{I}_\alpha^*(2) \\ & \cup \bigcup_{c_1 \in \{0, 2\}} \tilde{P}_\emptyset \mathbf{w}_{\alpha\beta} [n^-(c_1, 0, 0), 1]_L \mathbf{I}_\alpha^*(2) \cup \tilde{P}_\emptyset \mathbf{w}_{\beta\alpha\beta} \mathbf{I}_\alpha^*(2). \end{aligned}$$

Now that we have our double coset decomposition of $\tilde{G}_2$, we ascertain which of these double cosets can support a ς -eigenfunction. If the double coset $\tilde{P}_\emptyset g_0 \mathbf{I}_\alpha^*(2)$ supports such a function, then we set $f_{g_0}^\varsigma$ to be the ς -eigenfunction with

$$\mathrm{supp}(f_{g_0}^\varsigma) = \tilde{P}_\emptyset g_0 \mathbf{I}_\alpha^*(2) \quad \text{and} \quad f_{g_0}^\varsigma(g_0) = 1.$$

As we will show in the proof of the following proposition, the cocycle plays a role in determining which double cosets are unable to support an ς -eigenfunction.

Proposition 4.5. *Let $(\pi, \mathcal{V}_\pi) = \chi_1 \times \chi_2 \rtimes \sigma_0$ in the standard induced model. Further, let $\varsigma = (\psi^+, \chi, \psi^-)$ be as in Proposition 4.2. Then characters χ can be classified into four types depending on ψ^+ , ψ^- and χ . For each type of character ς , we give an explicit description of their ς -eigenspaces.*

Suppose that $\psi^\pm = 1$ and $\chi = \chi_j \xi_2$ on $\mathbb{Z}_2^\times$ for all $j \in \{1, 2\}$. Then

$$\mathcal{V}_\pi^{(\mathbf{I}_\alpha^*(2), \varsigma)} = \mathbb{C} \cdot f_{\mathbf{w}_0}^\varsigma \oplus \mathbb{C} \cdot f_{\mathbf{w}_\beta}^\varsigma \oplus \mathbb{C} \cdot f_{\mathbf{w}_{\alpha\beta}}^\varsigma \oplus \mathbb{C} \cdot f_{\mathbf{w}_{\beta\alpha\beta}}^\varsigma.$$

Next suppose that $\psi^+ = 1$, $\psi^- \neq 1$ and $\chi = \chi_j \xi_2$ on $\mathbb{Z}_2^\times$ for all $j \in \{1, 2\}$. Then

$$\mathcal{V}_\pi^{(\mathbf{I}_\alpha^*(2), \varsigma)} = \mathbb{C} \cdot f_{\mathbf{w}_0}^\varsigma \oplus \mathbb{C} \cdot f_{\mathbf{w}_\beta}^\varsigma.$$

Further, suppose that $\psi^+ \neq 1$, $\psi^- = 1$ and $\chi = \chi_j \xi_2$ on $\mathbb{Z}_2^\times$ for all $j \in \{1, 2\}$. Then

$$\mathcal{V}_\pi^{(\mathbf{I}_\alpha^*(2), \varsigma)} = \mathbb{C} \cdot f_{\mathbf{w}_{\beta\alpha\beta}}^\varsigma.$$

For all other ς ,

$$\dim_{\mathbb{C}} \mathcal{V}_\pi^{(\mathbf{I}_\alpha^*(2), \varsigma)} = 0.$$

Proof. A double coset $\tilde{P}_\emptyset g_0 \mathbf{I}_\alpha^*(2)$ supports a ς -eigenfunction as long as the characters $\chi_1 \otimes \chi_2 \otimes \sigma_0$ and ς are compatible on $\tilde{P}_\emptyset \cap g_0 \mathbf{I}_\alpha^*(2) g_0^{-1}$. As such, we will examine this intersection for each g_0 and formulate conditions on ς so that $\chi_1 \otimes \chi_2 \otimes \sigma_0$ and ς are compatible.

First, let $g_0 = \mathbf{w}_0 = [1_{G_2}, 1]_L$. To determine $\tilde{P}_\emptyset \cap \mathbf{I}_\alpha^*(2)$, we take $\mathbf{k} = [k, \lambda_2(k)]_L \in \mathbf{I}_\alpha^*(2)$ and write $k = n^+(b_1, b_2, b_3)m(A)n^-(4c_1, 2c_2, 4c_3)$ where $A \in \mathrm{GL}_2(\mathbb{Z}_2)$, $b_i, c_j \in \mathbb{Z}_2$ for $1 \leq i, j \leq 3$. We then see that $k \in \tilde{P}_\emptyset \cap \mathbf{I}_\alpha^*(2)$ as long as $c_j = 0$ for $1 \leq j \leq 3$ and $A = m^+(b_0)m(a_1, a_2)$ for $b_0 \in \mathbb{Z}_2$ and $a_i \in \mathbb{Z}_2^\times$, where we return to the notation of Section 3. Thus $\mathbf{k}$ has the form $[n^+(b_1, b_2, b_3)m^+(b_0)m(a_1, a_2), \xi_2(a_1 a_2)]_L$ and for there to exist a ς -eigenfunction supported on this double coset, we must have that

$$\chi_1(a_1)\chi_2(a_2)\xi_2(a_1 a_2) = \chi(a_1 a_2)\psi^+(b_1 + b_2 + b_3)$$

for all $a_i \in \mathbb{Z}_2^\times$ and $b_i \in \mathbb{Z}_2$. So for our characters to be compatible we need $\chi_i|_{\mathbb{Z}_2^\times} = \chi \xi_2|_{\mathbb{Z}_2^\times}$ on $\mathbb{Z}_2^\times$ for all $i \in \{1, 2\}$ and $\psi^+ = 1$.

Next, let us consider the case that $g' = \mathbf{w}_\beta = [w_\beta, 1]_L$. As before, we consider $\mathbf{k} = [k, \lambda_2(k)]_L \in \mathbf{I}_\alpha^*(2)$ with $k = n^+(b_1, b_2, b_3)m(A)n^-(4c_1, 2c_2, 4c_3)$. A direct computation shows that

$$w_\beta n^+(b_1, b_2, b_3)m(A)n^-(4c_1, 2c_2, 4c_3)w_\beta^{-1} \in P_\emptyset$$

as long as $b_3 = 0$, $c_1 = c_2 = 0$ and $m(A) = m^+(b_0)m(a_1, a_2)$. Thus

$$\begin{aligned} \mathbf{k} &= [n^+(b_1, b_2, 0)m^+(b_0)m(a_1, a_2)n^-(0, 0, 4c_3), \xi_2(a_1 a_2)\lambda_2(n^-(0, 0, 4c_3))]_L \\ &= [n^+(b_1, b_2, 0)m^+(b_0)m(a_1, a_2), \xi_2(a_1 a_2)]_L [n^-(0, 0, 4c_3), \lambda_2(n^-(0, 0, 4c_3))]_L \end{aligned}$$

where the second equality follows from the fact that $c_L(\cdot, \cdot)$ is trivial on $P_\alpha \times G_2$. We also have that

$$w_\beta n^+(b_1, b_2, 0)m^+(b_0)m(a_1, a_2)w_\beta^{-1} = m^+(b_2)n^+(b_1, -b_0, 0)m(a_1, a_2^{-1}).$$

Note that if we set $p = n^+(b_1, b_2, 0)m^+(b_0)m(a_1, a_2)$, then $c_L(w_\beta, p) = c_L(w_\beta p, w_\beta^{-1}) = 1$ by Lemma 2.1, so we get

$$\begin{aligned} (4.8) \quad \mathbf{w}_\beta [n^+(b_1, b_2, 0)m^+(b_0)m(a_1, a_2), \xi_2(a_1 a_2)]_L \mathbf{w}_\beta^{-1} \\ = [m^+(b_2)n^+(b_1, -b_0, 0)m(a_1, a_2^{-1}), \xi_2(a_1 a_2)]_L. \end{aligned}$$

Now to analyze $\mathbf{w}_\beta \mathbf{n}^-(0, 0, 4c_3) \mathbf{w}_\beta^{-1}$ we need to compute some cocycles. In particular, by Theorem 4.1 of [7] along with (2.7), we have

$$\begin{aligned} \lambda_2(n^-(0, 0, 4c_3)) &= c_L(n^+(0, 0, 4c_3), w_2)^{-1} = \gamma(\psi \circ \frac{1}{2}Q(Y, Yw_2^{-1}, Yn^-(0, 0, 4c_3)))^{-1} \\ c_L(w_\beta, n^-(0, 0, 4c_3)) &= \gamma(\psi \circ \frac{1}{2}Q(Y, Yn^-(0, 0, 4c_3), Yw_\beta)). \end{aligned}$$

Following Section 2 of [7], we find that

$$\gamma(\psi \circ \frac{1}{2}Q(Y, Yw_2^{-1}, Yn^-(0, 0, 4c_3))) = \gamma(\psi \circ \frac{1}{2}Q(Y, Yn^-(0, 0, 4c_3), Yw_\beta)) = \gamma_{\mathbb{Q}_2}(\psi_{2c_3})$$

where the last equality also uses Theorem A.4 of [7]. So using the computation along with Lemma 2.1 we see that

$$\begin{aligned}
 (4.9) \quad \mathbf{w}_\beta[n^-(0, 0, 4c_3), \lambda_2(n^-(0, 0, 4c_3))]_L \mathbf{w}_\beta^{-1} \\
 &= [w_\beta n^-(0, 0, 4c_3), c_L(w_\beta, n^-(0, 0, 4c_3)) \lambda_2(n^-(0, 0, 4c_3))]_L \\
 &= [w_\beta n^-(0, 0, 4c_3) w_\beta^{-1}, \gamma_F(\psi_{2c_3}) \gamma_F(\psi_{-2c_3})]_L \\
 &= [n^+(0, 0, -4c_3), 1]_L.
 \end{aligned}$$

Using (4.8) and (4.9), we see that

$$\begin{aligned}
 &\mathbf{w}_\beta[n^+(b_1, b_2, 0) m^+(b_0) m(a_1, a_2) n^-(0, 0, 4c_3), \xi_2(a_1 a_2) \lambda_2(n^-(0, 0, 4c_3))]_L \\
 &= [m^+(b_2) n^+(b_1, -b_0, 0) m(a_1, a_2^{-1}) n^+(0, 0, -4c_3), \xi_2(a_1 a_2)]_L \mathbf{w}_\beta;
 \end{aligned}$$

so this coset supports a ς -eigenfunction as long as

$$\chi(a_1 a_2) \psi^+(b_1 + b_2) = \chi_1(a_1) \chi_2(a_2)^{-1} \xi_2(a_1 a_2)$$

for all $a_1 \in \mathbb{Z}_2^\times$ and $b_j \in \mathbb{Z}_2$. Since χ and ξ_2 are quadratic, we see that such a ς -eigenfunction exists as long as $\chi_i|_{\mathbb{Z}_2^\times} = \chi \xi_2|_{\mathbb{Z}_2^\times}$ for all $i \in \{1, 2\}$ and $\psi^+ = 1$.

The remaining cases of cosets that can support a ς -eigenfunction are handled similarly. We shall remark about some of the intermediate results and omit most details.

Consider the case that $g_0 = \mathbf{w}_{\alpha\beta} = [w_{\alpha\beta}, 1]_L$. Mimicking the previous argument, we must consider $k \in I_\alpha^*(2)$ with $m(A) = m^+(b_0) m(a_1, a_2)$ and $b_2 = b_3 = c_1 = 0$. We also consider some cocycle computations for $c_L(n^-(0, 2c_2, 0), w_2)^{-1}$ and $c_L(w_{\alpha\beta}, n^-(0, 2c_2, 0))$. Following Theorem 4.1 of [7], we find

$$c_L(n^-(0, 2c_2, 0), w_2) = \gamma(\psi \circ 2c_2 y_1 y_2)$$

which is the Weil index of $\psi \circ Q$ for the quadratic space $(V, Q) = (F^2, Q(y_1, y_2) = 2c_2 y_1 y_2)$. Definition A.6 of [7] tells us that for any quadratic space (V, Q) with $\dim_F(V) = m$, we have

$$\epsilon(V) = \gamma(\psi \circ Q) \gamma_{\mathbb{Q}_2}(\psi)^{-m} \gamma_{\mathbb{Q}_2}(\det(Q), \psi)^{-1}.$$

So for the case of $(V, Q) = (F^2, Q(y_1, y_2) = 2c_2 y_1 y_2)$ we have

$$\epsilon(V) = \gamma(\psi \circ 2c_2 y_1 y_2) \gamma_{\mathbb{Q}_2}(\psi)^{-2} \gamma_{\mathbb{Q}_2}(-1, \psi)^{-1}.$$

However, Corollary A.5 of [7] tells us that $\gamma_{\mathbb{Q}_2}(\psi)^{-2} \gamma_{\mathbb{Q}_2}(-1, \psi)^{-1} = 1$. Furthermore, (V, Q) is a hyperbolic plane (it has an isotropic vector and $\dim_F(V) = 2$), so $\epsilon(V) = 1$. Therefore, we see that $\gamma(\psi \circ 2c_2 y_1 y_2) = 1$, so $c_L(w_{\alpha\beta}, n^-(0, 2c_2, 0)) = 1$. Ultimately, we can show that to support a function on this coset

$$\chi(a_1 a_2) \psi^+(b_1) \psi^-(c_2) = \chi_1(a_2)^{-1} \chi_2(a_1) \xi_2(a_1 a_2)$$

for all $a_i \in \mathbb{Z}_2^\times$, $b_1, c_2 \in \mathbb{Z}_2$. So as before, we find that this double coset supports a ς -eigenfunction as long as $\chi_i|_{\mathbb{Z}_2^\times} = \chi \xi_2|_{\mathbb{Z}_2^\times}$ for $i \in \{1, 2\}$ and $\psi^+ = \psi^- = 1$.

Now we consider the case that $g_0 = \mathbf{w}_{\beta\alpha\beta} = [w_{\beta\alpha\beta}, 1]_L$, which involves $k \in I_\alpha^*(2)$ with $m(A) = m^+(b_0) m(a_1, a_2)$ and $b_1 = b_2 = b_3 = 0$. All the cocycle computations for this case are identical to the previous cases up to a change in the indices on the various c_i . We can ultimately show that to support a function on this coset

$$\chi(a_1 a_2) \psi^-(c_2) = \chi_1(a_2)^{-1} \chi_2(a_1)^{-1} \xi_2(a_1 a_2)$$

for all $a_i \in \mathbb{Z}_2^\times$ and $c_2 \in \mathbb{Z}_2$. Thus we find that $\chi_i|_{\mathbb{Z}_2^\times} = \chi \xi_2|_{\mathbb{Z}_2^\times}$ for $i \in \{1, 2\}$ and $\psi^- = 1$.

According to the statement of the proposition, we should have exhausted all of the possible double cosets that support a ς -eigenfunction. Therefore the remainder of the proof is devoted to showing that the remaining double cosets cannot support such a function. For each of the remaining double coset representatives g_0 , we will find a subset $K' \subset \mathbf{I}_\alpha^*(2)$ for which ς and $\chi_1 \otimes \chi_2 \otimes \sigma_0$ cannot be compatible on $\tilde{P}_\emptyset \cap g_0 K' g_0^{-1}$. In particular, we will see that the lack of compatibility is largely due to conflicts created by the Leray cocycle. Since the remaining arguments are nearly identical, we will only consider the case that $g_0 = [n^-(2, 0, 0), 1]_L$ in full detail and only mention how to adapt the others.

Let $g_0 = [n^-(2, 0, 0), 1]_L$ and let us consider $k \in \mathbf{I}_\alpha^*(2)$ of the form $[k, \lambda_2(k)]$ where $k = n^+(b_1, 0, 0)m(a_1, a_2)n^-(4c_1, 0, 0)$. We have

$$\begin{aligned} & n^-(2, 0, 0)n^+(b_1, 0, 0)m(a_1, a_2)n^-(4c_1, 0, 0)n^-(-2, 0, 0) \\ &= n^+\left(\frac{b_1}{1+2b_1}, 0, 0\right)m\left(\frac{a_1}{1+2b_1}, a_2\right)n^-\left(2\frac{a_1^2 - (1+2b_1)(1-2c_1)}{1+2b_1}, 0, 0\right). \end{aligned}$$

So we see that $n^-(2, 0, 0)kn^-(-2, 0, 0) \in P_\emptyset$ as long as

$$(4.10) \quad a_1^2 = (1+2b_1)(1-2c_1),$$

so let us enforce this extra condition on k . So in $\tilde{G}_2$ we get

$$\begin{aligned} & [n^-(2, 0, 0), 1]_L[k, \lambda_2(k)]_L[n^-(2, 0, 0), 1]_L \\ &= [n^+\left(\frac{b_1}{1+2b_1}, 0, 0\right)m\left(\frac{a_1}{1+2b_1}, a_2\right), c_L(n^-(2, 0, 0), k)\lambda_2(k)]_L. \end{aligned}$$

We examine the product $c_L(n^-(2, 0, 0), k)\lambda_2(k)$. First we find

$$(4.11) \quad c_L(n^-(2, 0, 0), k) = \gamma(\psi \circ \frac{1}{2}Q(Y, Yk^{-1}, Yn^-(2, 0, 0))) = \gamma_{\mathbb{Q}_2}(\psi_{2c_1(1+2b_1)}).$$

It is worth noting that to obtain (4.11), we use the facts that $\gamma_{\mathbb{Q}_2}(\psi_{a^2b}) = \gamma_{\mathbb{Q}_2}(\psi_b)$ for any $a, b \in \mathbb{Q}_2$ in conjunction with $a_1^2 = (1+2b_1)(1-2c_1)$. Also, as shown previously, we have

$$\lambda_2(k) = \xi_2(a_1a_2)c_L(n^-(4c_1, 0, 0), w_2)^{-1} = \xi_2(a_1a_2)\gamma_{\mathbb{Q}_2}(\psi_{-2c_1}).$$

So we must have

$$\chi_1\left(\frac{a_1}{1+2b_1}\right)\chi_2(a_2)\gamma_{\mathbb{Q}_2}(\psi_{-2c_1})\gamma_{\mathbb{Q}_2}(\psi_{2c_1(1+2b_1)})\xi_2(a_1a_2) = \psi^+(b_1)\chi(a_1a_2)$$

for the double coset to support a function. If we set $a_1 = a_2 = 1$ and use the fact that $a_1^2 = (1+2b_1)(1-2c_1)$, this equation becomes

$$(4.12) \quad \chi_1(1-2c_1)\gamma_{\mathbb{Q}_2}(\psi_{-2c_1})\gamma_{\mathbb{Q}_2}(\psi_{2c_1(1+2b_1)}) = \psi^+(b_1).$$

Now let us simplify the product of Weil indices using Theorem A.4 and Corollary A.5 of [7] as follows:

$$\begin{aligned} & \gamma_{\mathbb{Q}_2}(\psi_{-2c_1})\gamma_{\mathbb{Q}_2}(\psi_{2c_1(1+2b_1)}) = \gamma_{\mathbb{Q}_2}(\psi_{-2c_1})\gamma_{\mathbb{Q}_2}(2c_1(1+2b_2), \psi)\gamma_{\mathbb{Q}_2}(\psi) \\ &= \gamma_{\mathbb{Q}_2}(\psi_{-2c_1})(2c_1, 1+2b_1)_{\mathbb{Q}_2}\gamma_{\mathbb{Q}_2}(2c_1, \psi)\gamma_{\mathbb{Q}_2}(1+2b_1, \psi)\gamma_{\mathbb{Q}_2}(\psi) \\ &= (2c_1, 1-2c_1)_{\mathbb{Q}_2}\gamma_{\mathbb{Q}_2}(\psi_{-2c_1})\gamma_{\mathbb{Q}_2}(\psi_{2c_1})\gamma_{\mathbb{Q}_2}(1-2c_1, \psi) \\ &= \gamma_{\mathbb{Q}_2}(1-2c_1, \psi). \end{aligned}$$

So (4.12) becomes $\chi_1(1 - 2c_1)\gamma_{\mathbb{Q}_2}(1 - 2c_1, \psi) = \psi^+(b_1)$. If we set $b_1 = -1$, then (4.10) forces $c_1 = 1$ and we find

$$\chi_1(-1)\gamma_{\mathbb{Q}_2}(-1, \psi) = \psi^+(-1).$$

By squaring both sides we find $\gamma_{\mathbb{Q}_2}(-1, \psi)^2 = \psi^+(-2)$. However, $\text{cond}(\psi^+) \leq 1$ so $\psi^+(-2) = 1$ whereas Corollary A.5 of [7] tells us that

$$\gamma_{\mathbb{Q}_2}(-1, \psi)^2 = (-1, -1)_{\mathbb{Q}_2} = -1.$$

Therefore we have arrived at a condition that cannot be satisfied. As such this double coset does not support a ς -eigenfunction.

The remaining cases are handled by a nearly identical argument. For $g_0 = [n^+(0, 0, 2), 1]_L$, we consider $\mathbf{k} \in \mathbf{I}_\alpha^*(2)$ with $k = n^+(0, 0, b_3)m(a_1, a_2)n^-(0, 0, 4c_3)$. For

$$g_0 \in \{\mathbf{w}_\beta[n^-(2, 0, 0), 1]_L, \mathbf{w}_{\alpha\beta}[n^-(2, 0, 0), 1]_L\}$$

we use exactly the same $\mathbf{k}$ in the case we worked out above. The argument is nearly identical except that some of the indices on the various characters change and the various a_i might be inverted. These adjustments to the a_i are a consequence of the coset representative having a Weyl group element as a factor. Finally, if $g_0 = [n^-(2, 0, 2), 1]_L$, we choose $\mathbf{k} \in \mathbf{I}_\alpha^*(2)$ with

$$k = n^+(b_1, 0, b_3)m(a_1, a_2)n^-(4c_1, 0, 4c_3).$$

We follow the original argument up to equation (4.17) which becomes

$$\chi_1(1 - 2c_1)\chi_2(1 - 2c_3)\gamma_{\mathbb{Q}_2}(1 - 2c_1, \psi)\gamma_{\mathbb{Q}_2}(1 - 2c_3, \psi) = \psi^+(b_1 + b_3).$$

In this case we simply set $b_1 = b_3 = -1$ which forces $c_1 = c_3 = 1$ and our contradiction follows without squaring both sides. $\square$

5. THE MAIN COMPUTATION

In this section, we compute the doubling integral for “good test vectors” and “good theta test sections”. The representation $(\pi, \mathcal{V}_\pi)$ will be an irreducible principal series representation $\chi_1 \otimes \chi_2 \rtimes \sigma_0$, where we will place certain compatibility restrictions on the characters χ_i . Let (V_κ^+, Q_κ^+) and (V_κ^-, Q_κ^-) be the five-dimensional quadratic spaces described in Section 2 and let $\kappa \in \mathbb{Z}_2^\times$. When stating results that treat the two spaces uniformly, we will use the notation $(V_\kappa^\pm, Q_\kappa^\pm)$ to denote that a statement holds for either space. Recall that for $(V_\kappa^\pm, Q_\kappa^\pm)$, $\chi_V = \xi_{2\kappa}$. Because our doubling integral will eventually involve integrating the characters $\chi_i\chi_V$ along $\mathbb{Q}_2^\times$, we require that $\chi_i\chi_V$ be unramified. So $\chi_i = \xi_{2a}|\cdot|^{t_i}$ where $t_i \in \mathbb{C}$ and $a \equiv \kappa \pmod{4}$. We also choose a character ς of $\mathbf{I}_\alpha^*(2)$ so that our “good test vector” is an element of $\mathcal{V}_\pi^{\mathbf{I}_\alpha^*(2), \varsigma}$. In this case, we choose $\varsigma = (\psi^+, \chi, \psi^-)$ so that $\psi^+ = \psi^- = 1$ and $\chi = \xi_a$. Since $a \equiv \kappa \pmod{4}$ and $\kappa \in \mathbb{Z}_2^\times$ we get that $\chi(5) = (a, 5)_{\mathbb{Q}_2} = 1$ which was required by Proposition 4.2.

We will break our computation into several steps. First, consider integral in (3.1) as it applies to the case $n = 2$ and to sections $\{\Phi_{s_0+r}\}_{r=0}^\infty$ of the form $\Phi_{s_0+r} = \lambda_V(\varphi^{(r)})$ with $\varphi \in S(V^4)$ for an arbitrary quadratic space (V, Q) . We further specify that $\varphi = \varphi_1 \otimes \varphi_2$ with $\varphi_i \in S(V^2)$ and rewrite (3.1) as a multiple integral over various subgroups of G_n and V_r^2 . The next step will involve computing an integral over K_2 and we will restrict our quadratic spaces to being one of the spaces $(V_\kappa^\pm, Q_\kappa^\pm)$. As is alluded to in the main computations of [15], these integrals will vanish if the choice of “good test vector” and “good theta test section” are not

compatible. If these integrals do not vanish, the result is another function in the space of $S(V^4)$.

The third step takes the resulting function and integrates it against an additive character over $N_\emptyset$ (the unipotent radical of the standard Borel subgroup) and V_r^2 . Such an integral is reminiscent of the formula for local densities of quadratic forms (see [11] and [12]). These local densities are difficult to compute, especially in the case that $F = \mathbb{Q}_2$, which is one of the reasons that we have such restrictive assumptions on our representations and quadratic spaces. Our integrals will actually be a limit of local densities computed in [12]. The results in [12] give the expression of 2-adic local densities as rational functions in $\mathbb{C}[2^r, 2^{-r}]$. As we will see, our integrals will be rational functions in $\mathbb{C}[2^r, 2^{-r}]$ comprised of local Euler factors. The final step in the computation integrates some characters of $\mathbb{Q}_2^\times$ over the torus $M_\emptyset$.

5.1. Expanding the integral. Let us start by expanding (3.1) into a multiple integral over the various pieces that we will evaluate individually. For this step, we let (V, Q) be an arbitrary odd-dimensional quadratic space and $\varphi = \varphi_1 \otimes \varphi_2$ with $\varphi_i \in S(V^2)$. Using the notation from Section 3.2, we have that $\varphi^{(r)} = \varphi_1^{(r)} \otimes \varphi_2^{(r)}$. We prove the following lemma.

Lemma 5.1. *Suppose that $\Phi_{s_0+r} = \lambda_V(\varphi_1^{(r)} \otimes \overline{\varphi_2^{(r)}})$, then*

$$(5.1) \quad \Phi_{s_0+r}(\delta' i(g'_0, g')) = \gamma(V)^2 \int_{V_r^2} \omega_{V_r}(g'_0) \varphi_1^{(r)}(x) \cdot \overline{\omega_{V_r}(g') \varphi_2^{(r)}(-x)} dx$$

where δ' is any element of $\widetilde{G}_2$ projecting on to δ and $\gamma(V)$ is defined in Sections 3 and 2, respectively.

Proof. The proof nearly follows mimicking the proof of Lemma 8.4.3 of [5]. Following the steps of that proof yields (5.1) with $\gamma(V)^2$ replaced with $\gamma(V_r)^2$. However, a direct computation using the definition of $\gamma(V_r)$ and $\gamma(V)$ in (8.5.21) of [5] shows that $\gamma(V_r)\gamma(V)^{-1} = \epsilon(V)\epsilon(V_r)$. But $\epsilon(V), \epsilon(V_r) \in \{\pm 1\}$, so $\gamma(V_r)^2 = \gamma(V)^2$. $\square$

So we combine Lemma 5.1 with (3.1) and the fact that $G_2 = P_\emptyset K_2$ to get

(5.2)

$$\begin{aligned} & Z(s_0 + r, \chi_V, \Phi, f)(g'_0) \\ &= \gamma(V)^2 \int_{P_\emptyset} \int_{K_2} \int_{V_r^2} \omega_{V_r}(g'_0) \varphi_1^{(r)}(x) \overline{\omega_V(\mathbf{p}k') \varphi_2^{(r)}(x)} f(\mathbf{p}k') dk dx dp \\ &= \gamma(V)^2 \int_{(\mathbb{Q}_2^\times)^2} \int_{\mathbb{Q}_2} \int_{\mathrm{Sym}_2(\mathbb{Q}_2)} \int_{V_r^2} \int_{K_2} \omega_{V_r}(g'_0) \varphi_1^{(r)}(x) \overline{\omega_{V_r}(\mathbf{n}_2(b)\mathbf{m} \begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix} k') \varphi_2^{(r)}(x)} \\ & \quad \times |a_1|^{-4} |a_2|^{-3} f(\mathbf{n}_2(b)\mathbf{m} \begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix} k') dk dx db dc d^\times a \end{aligned}$$

where the factor $|a_1|^{-4}|a_2|^{-3}$ comes from the transformations $c \mapsto ca_2^{-1}$ as well as the relationship $dp = \delta_P^{-1} dn dm$, where dp is the right Haar measure on $P_\emptyset = M_\emptyset N_\emptyset$ and dm (resp. dn) is the Haar measure on $M_\emptyset$ (resp $N_\emptyset$). We also interchanged the dk and dx integrals, which we can do because K_2 is a compact set and the function $\omega_{V_r}(g'_0) \varphi_1^{(r)} \in S(V_r^2)$ is compactly supported on V_r^2 .

Next, we use the fact that $f \in \mathcal{V}_\pi$ along with (2.4) to arrive at (5.3)

$$\begin{aligned}
& Z(s_0 + r, \chi_V, \Phi, f)(g'_0) \\
&= \gamma(V)^2 \int_{(\mathbb{Q}_2^\times)^2} \chi_V(a_1 a_2) \chi_1(a_1) \chi_2(a_2) |a_1 a_2|^{-2} |a_1 a_2|^{\frac{5+2s_0+2r}{2}} \\
&\quad \times \int_{\mathbb{Q}_2} \int_{\text{Sym}_2(\mathbb{Q}_2)} \int_{V_r^2} \omega_{V_r}(g'_0) \varphi_1^{(r)}(x) \left(\int_{K_2} f(k') [\omega_{V_r}(k') \varphi_2^{(r)}](-x \begin{pmatrix} a_1 & c \\ 0 & a_2 \end{pmatrix}) dk \right) \\
&\quad \times \psi\left(-\frac{1}{2} \text{tr}(b[Q(x)])\right) dx db dc d^\times a \\
&= \gamma(V)^2 \int_{(\mathbb{Q}_2^\times)^2} \chi_V \chi_1(a_1) \chi_V \chi_2(a_2) |a_1 a_2|^{r+\frac{1}{2}+s_0} \int_{\mathbb{Q}_2} \int_{\text{Sym}_2(\mathbb{Q}_2)} \int_{V_r^2} \omega_{V_r}(g'_0) \varphi_1^{(r)}(x) \\
&\quad \times I(f, \varphi_2^{(r)})(-x \begin{pmatrix} a_1 & c \\ 0 & a_2 \end{pmatrix}) \psi\left(-\frac{1}{2} \text{tr}(b[Q(x)])\right) dx db dc d^\times a
\end{aligned}$$

where

$$I(f, \varphi^{(r)})(x) := \int_{K_2} f(k') \overline{\omega_{V_r}(k') \varphi^{(r)}(x)} dk$$

and $[Q(x)] \in \text{Sym}_2(\mathbb{Q}_2)$ with $[Q(x)]_{i,j} = (x_i, x_j)_V$. We also have the following lemma regarding $I(f, \varphi^{(r)})(x)$.

Lemma 5.2. *For $\varphi \in S(V^2)$, $I(f, \varphi^r)(x) = I(f, \varphi)^{(r)}(x)$.*

Proof. For F non-dyadic, this result is Lemma 7.2 of [15]. Here we use essentially the same proof. The proof boils down to the factorization $\omega_{V_r}(g) \varphi^{(r)} = \omega_V(g) \varphi \otimes \omega_{V_h^r}(g) \varphi_0^2$. However, Lemma 3.6 shows that φ_0^2 is fixed by the action of $\tilde{K}_2$ under $\omega_{V_h^r}$. With that we can follow the proof of Lemma 7.2 of [15] and obtain our result. $\square$

5.2. Evaluating $I(f, \varphi)(x)$. At this juncture, evaluating the necessary expressions will require an explicit quadratic space and an explicit choice of $\varphi_i \in S(V^2)$. From this point forward, our quadratic spaces (V, Q) will be of the form $(V_\kappa^\pm, Q_\kappa^\pm)$ and we will continue to use the notation (V, Q) only as a convenience to simplify our formula. Also, since $\dim(V_\kappa^\pm) = 5$, we have that $s_0 = 0$. Let us define the following lattices in $V_\kappa^\pm$:

$$L_{\kappa,1}^+ = \mathbb{Z}_2(E_{1,1} - E_{2,2}) \oplus \mathbb{Z}_2 E_{1,2} \oplus \mathbb{Z}_2 E_{2,1} \oplus \mathbb{Z}_2 e_0 \oplus \mathbb{Z}_2 e'_0$$

so that

$$(L_{\kappa,1}^+)^{\#} = 2^{-1} \mathbb{Z}_2(E_{1,1} - E_{2,2}) \oplus \mathbb{Z}_2 E_{1,2} \oplus \mathbb{Z}_2 E_{2,1} \oplus \mathbb{Z}_2 e_0 \oplus \mathbb{Z}_2 e'_0$$

and

$$L_{\kappa,1}^- = \mathbb{Z}_2 i \oplus \mathbb{Z}_2 j \oplus \mathbb{Z}_2 ij \oplus \mathbb{Z}_2 e_0 \oplus \mathbb{Z}_2 e'_0$$

so that

$$(L_{\kappa,1}^-)^{\#} = 2^{-1} \mathbb{Z}_2 i \oplus 2^{-1} \mathbb{Z}_2 j \oplus 2^{-1} \mathbb{Z}_2 ij \oplus \mathbb{Z}_2 e_0 \oplus \mathbb{Z}_2 e'_0.$$

Additionally, we let $\varphi_{L_{\kappa,1}^+ \oplus L_{\kappa,1}^+}$ (resp. $\varphi_{L_{\kappa,1}^- \oplus L_{\kappa,1}^-}$) be the characteristic function of $L_{\kappa,1}^+ \oplus L_{\kappa,1}^+ \subset (V_{\kappa,1}^+)^2$ (resp. $L_{\kappa,1}^- \oplus L_{\kappa,1}^- \subset (V_{\kappa,1}^-)^2$). Because these are the

only lattices of interest and the roles of their characteristic functions are identical in $I(f, \varphi)(x)$, we will conflate notation and let L_1 stand for the various $L_{\kappa,1}^\pm$ in subsequent computations.

Now set $\Phi_r^\pm = \lambda_V(\varphi_{L_1 \oplus L_1}^{(r)} \otimes \overline{\varphi_{L_1 \oplus L_1}^{(r)}})$, which we compute using (2.4) with $\mathbf{k} = [k, \lambda_2(k)]_L$ and $k \in I_\alpha^*(2)$ with $k = n^+(B)m(A)n^-(C) \in I_\alpha^*(2)$. We see that

$$\begin{aligned} \Phi_r^\pm(g'i(\mathbf{k}, 1)) &= [\omega_V(g')\omega_V(i(\mathbf{k}, 1))\varphi_{L_1 \oplus L_1}^{(r)} \otimes \overline{\varphi_{L_1 \oplus L_1}^{(r)}}](0) \\ &= [\omega_V(g')(\omega_V(\mathbf{k})\varphi_{L_1 \oplus L_1}^{(r)} \otimes \overline{\varphi_{L_1 \oplus L_1}^{(r)}})](0) \\ &= \chi_V \xi_2(\det(A))[\omega_V(g')\varphi_{L_1 \oplus L_1}^{(r)} \otimes \overline{\varphi_{L_1 \oplus L_1}^{(r)}}](0) \\ &= \chi_V \xi_2(\det(A))\Phi_r^\pm(g'). \end{aligned}$$

Because χ_V was chosen so that $\chi_V \xi_2|_{\mathbb{Z}_2^\times} = \chi$, where χ was used to define ς , $Z(r, \chi_V, \Phi^\pm, \cdot)$ will be an operator on the finite dimensional space $\mathcal{V}_\pi^{\mathbf{I}_\alpha^*(2), \varsigma}$. Since Proposition 4.5 gives a basis for $\mathcal{V}_\pi^{\mathbf{I}_\alpha^*(2), \varsigma}$ of functions of the form $f_\mathbf{w}^\varsigma$, we compute $I(f_\mathbf{w}^\varsigma, \varphi_{L_1 \oplus L_1}^{(r)})$.

Proposition 5.3. *For $f_\mathbf{w}^\varsigma$ and $\varphi_{L_1 \oplus L_1}$, we have*

$$(5.4) \quad I(f_\mathbf{w}^\varsigma, \varphi_{L_1 \oplus L_1}) = \mathrm{vol}(\mathbf{I}_\alpha^*(2)) 2^{\ell(w)+j(w)} \chi_V(\mathbf{x}(w)) \left(\frac{\gamma(V)}{[L_1^\# : L_1]^{\frac{1}{2}}} \right)^{j(w)} \varphi_{L_1 \oplus L_1}.$$

Proof. Since the integral is independent of the choice of $k' \in \tilde{G}_2$ over $k \in K$, we have

$$\begin{aligned} I(f_\mathbf{w}^\varsigma, \varphi_{L_1 \oplus L_1})(x) &= \int_{K_2} f_\mathbf{w}^\varsigma(k') \overline{\omega_V(k') \varphi_{L_1 \oplus L_1}(x)} dk \\ &= \sum_{w \in \mathcal{W}_{G_2}(\alpha)} \int_{\mathbf{N}^w \mathbf{w} \mathbf{I}_\alpha^*(2)} f_\mathbf{w}^\varsigma(\mathbf{k}) \overline{\omega_V(\mathbf{k}) \varphi_{L_1 \oplus L_1}(x)} dk \end{aligned}$$

where $\mathbf{N}^w$ is a set of representatives for the cosets $\mathbf{I}_\alpha^*(2)/(\mathbf{I}_\alpha^*(2) \cap \mathbf{w}^{-1}\mathbf{I}_\alpha^*(2)\mathbf{w})$. One can verify that the elements of $\mathbf{N}^w$ can be taken to be elements of $\mathbf{N}_\emptyset$ (which we do). Notice that both f and φ transforms on the right by ς for $\mathbf{I}_\alpha^*(2)$. Also, notice that our $f_\mathbf{w}^\varsigma$ is supported on only the double coset $P_\emptyset \mathbf{w} \mathbf{I}_\alpha^*(2)$, so

$$(5.5) \quad I(f_\mathbf{w}^\varsigma, \varphi_{L_1 \oplus L_1})(x) = \mathrm{vol}(\mathbf{I}_\alpha^*(2)) f_\mathbf{w}^\varsigma(\mathbf{w}) \sum_{\mathbf{n} \in \mathbf{N}^w} \omega_V(\mathbf{n}\mathbf{w}) \varphi_{L_1 \oplus L_1}(x)$$

which we compute in the same way as in Section 7.2 of [15]. We will compute this for $\mathbf{w} = \mathbf{w}_{\beta\alpha\beta}$. The remaining cases are similar and simpler. Let us write $\varphi_{L_1 \oplus L_1}(x)$ as $\varphi_{L_1} \otimes \varphi_{L_1}$ where $\varphi_{L_1} \in S(V)$ is the characteristic function of $L_1 \subset V$. Then (2.4) gives us

$$\omega_V(\mathbf{w}_{\beta\alpha\beta})(\varphi_{L_1} \otimes \varphi_{L_1}) = \chi_V(\mathbf{x}(w_{\beta\alpha\beta})) \gamma(V)^2 (\widehat{\varphi_{L_1}} \otimes \widehat{\varphi_{L_1}}).$$

Since $\widehat{\varphi_{L_1}} = [L_1^\# : L_1]^{-\frac{1}{2}} \varphi_{L_1^\#}(x)$ with $\varphi_{L_1^\#}$ the characteristic function of $L_1^\# \subset V$, we see that

$$\omega_V(\mathbf{w}_{\beta\alpha\beta})(\varphi_{L_1 \oplus L_1}) = \chi_V(-1) \gamma(V)^2 [L_1^\# : L_1]^{-1} \varphi_{L_1^\# \oplus L_1^\#}$$

where $\varphi_{L_1^\# \oplus L_1^\#}$ is the characteristic function of $L_1^\# \oplus L_1^\# \subset V^2$.

A simple computation shows that $\mathbf{N}^{w_{\alpha\beta\alpha}} = \{\mathbf{n}^+(b_1, b_2, b_3) \mid b_1, b_3 \in \mathbb{Z}_2/4\mathbb{Z}_2, b_2 \in \mathbb{Z}_2/2\mathbb{Z}_2\}$. Since $N^{w_{\alpha\beta\alpha}} \subset N_\alpha$, which is abelian, we can the action of each subgroup corresponding to a b_i individually. For $x \in V^2$ let $x = (x_1, x_2)$ with $x_i \in V$, and (2.4) gives us the following equalities:

$$\begin{aligned}\omega_V(\mathbf{n}^+(b_1, 0, 0))(\varphi_{L_1^\# \oplus L_1^\#})(x) &= \psi(b_1 Q(x_1))(\varphi_{L_1^\# \oplus L_1^\#})(x), \\ \omega_V(\mathbf{n}^+(0, b_2, 0))(\varphi_{L_1^\# \oplus L_1^\#})(x) &= \psi(b_2(x_1, x_2)_V)(\varphi_{L_1^\# \oplus L_1^\#})(x), \\ \omega_V(\mathbf{n}^+(0, 0, b_3))(\varphi_{L_1^\# \oplus L_1^\#})(x) &= \psi(b_3 Q(x_2))(\varphi_{L_1^\# \oplus L_1^\#})(x).\end{aligned}$$

In (5.5) we have to sum over the various $\mathbf{n}^+(b_1, b_2, b_3)$, so we get

$$\begin{aligned}\sum_{b_1 \in \mathbb{Z}_2/4\mathbb{Z}_2} \omega_V(\mathbf{n}^+(b_1, 0, 0))(\varphi_{L_1^\# \oplus L_1^\#})(x) &= 4 \operatorname{char}_{\mathbb{Z}_2}(Q(x_1) + 2Q(x_1))(\varphi_{L_1^\# \oplus L_1^\#})(x), \\ \sum_{b_2 \in \mathbb{Z}_2/2\mathbb{Z}_2} \omega_V(\mathbf{n}^+(0, b_2, 0))(\varphi_{L_1^\# \oplus L_1^\#})(x) &= 2 \operatorname{char}_{\mathbb{Z}_2}((x_1, x_2)_V)(\varphi_{L_1^\# \oplus L_1^\#})(x), \\ \sum_{b_3 \in \mathbb{Z}_2/4\mathbb{Z}_2} \omega_V(\mathbf{n}^+(0, 0, b_3))(\varphi_{L_1^\# \oplus L_1^\#})(x) &= 4 \operatorname{char}_{\mathbb{Z}_2}(Q(x_2) + 2Q(x_2))(\varphi_{L_1^\# \oplus L_1^\#})(x).\end{aligned}$$

Combining these yields

$$\sum_{\mathbf{n} \in \mathbf{N}^w} \omega_V(\mathbf{n})(\varphi_{L_1^\# \oplus L_1^\#})(x) = 2^5 \mu(x)(\varphi_{L_1^\# \oplus L_1^\#})(x)$$

where

$$\mu(x) = \operatorname{char}_{\mathbb{Z}_2}(Q(x_1) + 2Q(x_1)) \operatorname{char}_{\mathbb{Z}_2}(Q(x_2) + 2Q(x_2)) \operatorname{char}_{\mathbb{Z}_2}((x_1, x_2)_V).$$

Analyzing μ , we see that $\mu(x) = 1$ if and only if $x_i \in L_1$ for both $i \in \{1, 2\}$. Thus

$$\mu(x) \varphi_{L_1^\# \oplus L_1^\#}(x) = \varphi_{L_1 \oplus L_1}(x).$$

Also, $\ell(w_{\beta\alpha\beta}) = 3$ and $j(w_{\beta\alpha\beta}) = 2$. So, our result is proved. $\square$

Having evaluated $I(f_{\mathbf{w}}^s, \varphi_{L_1 \oplus L_1}^{(r)})(x)$, we substitute this into (5.3) to get

$$\begin{aligned}(5.6) \quad Z(r, \chi_V, \Phi^\pm, f_{\mathbf{w}}^s)(g'_0) &= \gamma(V)^2 \operatorname{vol}(\mathbf{I}_\alpha^*(2)) 2^{\ell(w)+j(w)} \chi_V(x(w)) \left(\frac{\gamma(V)}{[L_1^\# : L_1]^{\frac{1}{2}}} \right)^{j(w)} \\ &\quad \times \int_{(\mathbb{Q}_2^\times)^2} \chi_V \chi_1(a_1) \chi_V \chi_2(a_2) |a_1 a_2|^{r+\frac{1}{2}} \\ &\quad \times \int_{\mathbb{Q}_2} \Lambda(r, \begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix}; \omega_{V_r}(g'_0) \varphi_{L_1 \oplus L_1}, \varphi_{L_1 \oplus L_1}) dc d^\times a,\end{aligned}$$

where as a function of $r \geq 0$, $g \in \operatorname{GL}_2(\mathbb{Q}_2)$, and $\varphi_1, \varphi_2 \in S(V^2)$

$$\Lambda(r, g; \varphi_1, \varphi_2) = \int_{\operatorname{Sym}_2(\mathbb{Q}_2)} \int_{V_r^2} \varphi_1^{(r)}(x) \varphi_2^{(r)}(xg) \psi(-\frac{1}{2} \operatorname{tr}(b[Q(x)])) dx db.$$

Our next goal will be to evaluate $\Lambda(r, g; \omega_V(\mathbf{w}) \varphi_{L_1 \oplus L_1}, \varphi_{L_1 \oplus L_1})$ for upper triangular $g \in \operatorname{GL}_2(\mathbb{Q}_2)$.

5.3. Evaluating $\Lambda(r, g; \varphi_1, \varphi_2)$. For a lattice $L \subset V$, $\Lambda(r, 1_{\mathrm{GL}_2}; \varphi_{L \oplus L}, \varphi_{L \oplus L})$ bears a resemblance to the local density quantities computed in [11] and [12]. The goal for this section will be to reduce our expression $\Lambda(r, \begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix}; \omega_{V_r}(g'_0) \varphi_{L_1 \oplus L_1}, \varphi_{L_1 \oplus L_1})$ to something that can be evaluated using [12]. Since $Z(r, \chi_V, \Phi^\pm, \cdot)$ operates on the space $\mathcal{V}_\pi^{(\mathbf{I}_\alpha^*(2), \varsigma)}$ and such functions are completely determined by their values $f(\mathbf{w})$ for $w \in \mathcal{W}_{G_2}(\alpha)$, it suffices for us restrict g'_0 to being an element $\mathbf{w}$ with $w \in \mathcal{W}_{G_2}(\alpha)$. With this restriction, we have the following proposition.

Proposition 5.4. *For $w \in \mathcal{W}_{G_2}(\alpha)$, we have*

$$\begin{aligned} & \Lambda(r, \begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix}; \omega_V(\mathbf{w}) \varphi_{L_1 \oplus L_1}, \varphi_{L_1 \oplus L_1}) \\ &= \chi_V(x(w)) \left(\frac{\gamma(V)}{[L_1^\# : L_1]^{1/2}} \right)^{j(w)} \Lambda(r, \begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix}; \varphi_{L_1 \oplus L_1}, \varphi_{L_1 \oplus L_1}). \end{aligned}$$

Proof. For the proof of this proposition, we require the following lemma.

Lemma 5.5. *Let $L \subset V$ be an arbitrary lattice and let $y_1, y_2 \in L_1^\#$ be such that $\mathrm{ord}_{\mathbb{Q}_2}(Q(y_i)) < 0$ and $2y_i \in L_1$, then*

$$\begin{aligned} (5.7) \quad & \Lambda(r, \begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix}; \varphi_{L \oplus L}, \varphi_{y_1^* + L_1, y_2^* + L_1}) \\ &= \int_{\mathrm{Sym}_2(\mathbb{Q}_2)} \int_{V_r^2} \varphi_{L \oplus L}^{(r)}(x) \varphi_{(y_1^* + L_1) \oplus (y_2^* + L_1)}^{(r)}(x \begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix}) \psi(-\frac{1}{2} \mathrm{tr}(b[Q(x)])) dx db = 0 \end{aligned}$$

for $y_i^* \in \{y_i, 0\}$ with y_1^* and y_2^* not both zero.

Proof. This proof is several cases of direct computation where we attempt to find an element $b' \in \mathrm{Sym}_2(\mathbb{Q}_2)$ so that the translation $b \mapsto b + b'$ has the effect of multiplying our integral by a constant different than 1. Since $x \in V_r^2$, we have $x = (x_1, x_2)$ with $x_i \in V_r$. Let us suppose that $y_1^* = y_1$ and $y_2^* \in \{y_2, 0\}$. Then the integral in (5.7) becomes,

$$\int_{\mathrm{Sym}_2(\mathbb{Q}_2)} \int_{L^{(r)} \oplus L^{(r)}} \varphi_{(y_1 + L_1) \oplus (y_2^* + L_1)}^{(r)}(x \begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix}) \psi(-\frac{1}{2} \mathrm{tr}[Q(x)]b) dx db.$$

One of the requirements for the integrand to be non-vanishing is that

$$a_1 x_1 \in y_1 + L_1.$$

Thus $a_1 x_1 = y_1 + x_0$ for some $x_0 \in L_1$. We also have

$$Q(y_1 + x_0) = (y_1, x_0)_V + Q(y_1) + Q(x_0).$$

Since $y_1 \in L_0^\#$, $(y_1, x_0) \in \mathbb{Z}_2$. Also, $Q(x_0) \in \mathbb{Z}_2$ for $x_0 \in L_1$, so

$$\mathrm{ord}_{\mathbb{Q}_2}(Q(a_1 x_1)) = \mathrm{ord}_{\mathbb{Q}_2}(Q(y_1)) < 0$$

and since $Q(a_1 x_1) = a_1^2 Q(x_1)$, we have

$$\mathrm{ord}_{\mathbb{Q}_2}(Q(x_1)) = \mathrm{ord}_{\mathbb{Q}_2}(Q(y_1)) - 2\mathrm{ord}_{\mathbb{Q}_2}(a_1) < -2\mathrm{ord}_{\mathbb{Q}_2}(a_1)$$

for all $x_1 \in a_1^{-1}y_1 + a_1^{-1}L_1$. Because we have chosen our ψ to be trivial on $\mathbb{Z}_2$ but non-trivial on $2^{-1}\mathbb{Z}_2$, we choose a $u \in \mathbb{Z}_2^\times$ so that $\psi(-uQ(y_1)) \neq 1$ and translate b by

$$b \mapsto b + \begin{pmatrix} ua_1^2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Following this translation through equation (5.7), yields

$$-\frac{1}{2}\text{tr}\left(\begin{pmatrix} ua_1^2 & 0 \\ 0 & 0 \end{pmatrix} [Q(x)]\right) = -\frac{1}{2}ua_1^2(x_1, x_1)_V = -uQ(a_1x_1) = -uQ(y_1).$$

So we have

$$\begin{aligned} \Lambda(r, \begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix}; \varphi_{L \oplus L}, \varphi_{(y_1+L_1) \oplus (y_2^*+L_1)}) \\ = \psi(-uQ(y_1)) \Lambda(r, \begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix}; \varphi_{L \oplus L}, \varphi_{(y_1+L_1) \oplus (y_2^*+L_1)}). \end{aligned}$$

So we see that $\Lambda(r, \begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix}; \varphi_{L \oplus L}, \varphi_{(y_1+L_1) \oplus (y_2^*+L_1)})$ vanishes in this case.

Now we want to consider the case that $y_1^* = 0$. We will use a similar translation argument to show that the integral vanishes. However, our choice of translation will actually depend upon the relative divisibilities of a_1 , a_2 and c by 2. In particular, we have three cases.

First, let us suppose that $\text{ord}_{\mathbb{Q}_2}(a_1) \leq \text{ord}_{\mathbb{Q}_2}(c)$. Consequently, $ca_1^{-1} \in \mathbb{Z}_2$ and

$$\begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix} = \begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix} \begin{pmatrix} 1 & ca_1^{-1} \\ & 1 \end{pmatrix}.$$

Moreover, the function $\varphi_{L_1 \oplus (y_2+L_1)}^{(r)}(x)$ is invariant under right translation by

$$\begin{pmatrix} 1 & ca_1^{-1} \\ & 1 \end{pmatrix}.$$

So our integral from (5.7) becomes

$$(5.8) \quad \int_{\text{Sym}_2(\mathbb{Q}_2)} \int_{L^{(r)} \oplus L^{(r)}} \varphi_{L_1 \oplus (y_2+L_1)}^{(r)}(x \begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix}) \psi(-\frac{1}{2}\text{tr}(b[Q(x)])) dx db.$$

Now we argue exactly as in the first case except that we exchange a_1, x_1 and y_1 with a_2, x_2 and y_2 , respectively.

Second, let us suppose that $\text{ord}_{\mathbb{Q}_2}(a_2) \leq \text{ord}_{\mathbb{Q}_2}(c)$. As a consequence, $ca_2^{-1} \in \mathbb{Z}_2$ and

$$\begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix} = \begin{pmatrix} 1 & ca_2^{-1} \\ & 1 \end{pmatrix} \begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix}.$$

However, as a function of $g \in \text{GL}_2(\mathbb{Q}_2)$, $\Lambda(r, g; \varphi_{L \oplus L}, \varphi_{L_1 \oplus (y_2+L_1)})$ is right $\text{GL}_2(\mathbb{Z}_2)$ invariant. So we translate x and b as

$$x \mapsto x \begin{pmatrix} 1 & -ca_2^{-1} \\ & 1 \end{pmatrix} \quad \text{and} \quad b \mapsto \begin{pmatrix} 1 & ca_2^{-1} \\ & 1 \end{pmatrix} b \begin{pmatrix} 1 & \\ & ca_2^{-1} \end{pmatrix}$$

both of which preserve their respective measures dx and db . As in the previous case, integral in (5.7) becomes

$$(5.9) \quad \int_{\text{Sym}_2(\mathbb{Q}_2)} \int_{L^{(r)} \oplus L^{(r)}} \varphi_{L_1 \oplus (y_2+L_1)}^{(r)}(x \begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix}) \psi(-\frac{1}{2}\text{tr}(b[Q(x)])) dx db$$

and as we have already seen that this integral vanishes.

Finally, let us suppose that $\mathrm{ord}_{\mathbb{Q}_2}(c) < \mathrm{ord}_{\mathbb{Q}_2}(a_i)$ for $i \in \{1, 2\}$. Then $a_i c^{-1} \in 2\mathbb{Z}_2$ for $i \in \{1, 2\}$. Because $2y_2 \in L_1$, we have that $\varphi_{L_1 \oplus (y_2 + L_1)}^{(r)}(x)$ is invariant under right translation by

$$\begin{pmatrix} 1 & \\ -a_1 c^{-1} & 1 \end{pmatrix}.$$

We also have the matrix decomposition

$$\begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & a_2 c^{-1} \end{pmatrix} \begin{pmatrix} a_1 a_2 c^{-1} & \\ & c \end{pmatrix} \begin{pmatrix} 1 & \\ a_1 c^{-1} & 1 \end{pmatrix}.$$

So by using the invariance of $\varphi_{L_1 \oplus y_2 + L_1}^{(r)}(x)$ and by translating x and b as

$$x \mapsto x \begin{pmatrix} a_2 c^{-1} & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad b \mapsto \begin{pmatrix} a_2 c^{-1} & -1 \\ 1 & 0 \end{pmatrix} b \begin{pmatrix} a_2 c^{-1} & -1 \\ 1 & 0 \end{pmatrix},$$

our integral in (5.7) becomes

$$(5.10) \quad \int_{\mathrm{Sym}_2(\mathbb{Q}_2)} \int_{L^{(r)} \oplus L^{(r)}} \varphi_{L_1 + (y_2 \oplus L_1)}^{(r)}(x \begin{pmatrix} a_1 a_2 c^{-1} & \\ & c \end{pmatrix}) \psi(-\frac{1}{2} \mathrm{tr}(b[Q(x)])) dx db.$$

As before, the translations preserve the measures dx and db and we prove the vanishing integral in nearly the same way as all the others. The only difference for this case is that c satisfies the role of one of our a_i used in the previous cases. $\square$

We also get the following similar lemma.

Lemma 5.6. *Let L, L_1, y_i^* be as in Lemma 5.5. Then as long as the y_i^* are not both zero, we have*

$$\Lambda(r, \begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix}; \varphi_{(y_1^* + L_1) \oplus (y_2^* + L_1)}, \varphi_{L \oplus L}) = 0.$$

Proof. This follows from Lemma 5.5 after we translate x by

$$x \mapsto x \begin{pmatrix} a_1^{-1} & -c a_1^{-1} a_2^{-1} \\ & a_2^{-1} \end{pmatrix}.$$

By translating x in this way and keeping track of the various scalars coming from the translations of the measures we see that

$$\begin{aligned} & \Lambda(r, \begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix}; \varphi_{(y_1^* + L_1) \oplus (y_2^* + L_1)}, \varphi_{L \oplus L}) \\ &= |a_1 a_2|^{-(2r+1)} \Lambda(r, \begin{pmatrix} a_1^{-1} & -c a_1^{-1} a_2^{-1} \\ & a_2^{-1} \end{pmatrix}; \varphi_{L \oplus L}, \varphi_{(y_1^* + L_1) \oplus (y_2^* + L_1)}). \end{aligned}$$

The term on the right-hand side vanishes by Lemma 5.5. $\square$

To complete the proof of Proposition 5.4, let us reconsider

$$\Lambda(r, \begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix}; \omega_V(\mathbf{w}) \varphi_{L_1 \oplus L_1}, \varphi_{L_1 \oplus L_1}).$$

A direct computation of the Weil representation on $S(V^2)$ shows us that

$$\omega_V(\mathbf{w}) \varphi_{L_1 \oplus L_1}(x) = \chi_V(x(w)) \left(\frac{\gamma(V)}{[L_1^\# : L_1]^{1/2}} \right)^{j(w)} \cdot \begin{cases} \varphi_{L_1 \oplus L_1}(x) & \text{if } \mathbf{w} = \mathbf{w}_0, \\ \varphi_{L_1 \oplus L_1^\#}(x) & \text{if } \mathbf{w} = \mathbf{w}_\beta, \\ \varphi_{L_1^\# \oplus L_1}(x) & \text{if } \mathbf{w} = \mathbf{w}_{\alpha\beta}, \\ \varphi_{L_1^\# \oplus L_1^\#}(x) & \text{if } \mathbf{w} = \mathbf{w}_{\beta\alpha\beta}. \end{cases}$$

Finally, we notice that for all $y \in L_1^\# \setminus L_1$ that $Q(y) \notin \mathbb{Z}_2$, so

$$\begin{aligned} \Lambda(r, \begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix}, \varphi_{L_1^\# \oplus L_1^\#}, \varphi_{L_1 \oplus L_1}) \\ = \sum_{y_i \in L_1^\# / L_1} \Lambda(r, \begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix}, \varphi_{(y_1+L_1) \oplus (y_2+L_1)}, \varphi_{L_1 \oplus L_1}) \end{aligned}$$

so that Lemma 5.6 applies. The cases where we replace $\varphi_{L_1^\# \oplus L_1^\#}$ with $\varphi_{L_1 \oplus L_1^\#}$ or $\varphi_{L_1^\# \oplus L_1}$ follow similarly. $\square$

Substituting the results of Proposition 5.4 into (5.6) and setting $g'_0 = \mathbf{w}'$, we get

$$\begin{aligned} (5.11) \quad Z(r, \chi_V, \Phi^\pm, f_{\mathbf{w}}^s)(\mathbf{w}') \\ = \gamma(V)^2 \text{vol}(\mathbf{I}_\alpha^*(2)) 2^{\ell(w)+j(w)} \chi_V(\mathbf{x}(w)\mathbf{x}(w')) \left(\frac{\gamma(V)}{[L_1^\# : L_1]^{\frac{1}{2}}} \right)^{j(w)+j(w')} \\ \times \int_{(\mathbb{Q}_2^\times)^2} \chi_V \chi_1(a_1) \chi_V \chi_2(a_2) |a_1 a_2|^{r+\frac{1}{2}} \\ \times \int_{\mathbb{Q}_2} \Lambda(r, \begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix}; \varphi_{L_1 \oplus L_1}, \varphi_{L_1 \oplus L_1}) dc d^\times a. \end{aligned}$$

By emulating the example after Lemma 7.4 of [15], we get the following lemma.

Lemma 5.7. *For $a_i \in \mathbb{Q}_2^\times$ and $c \in \mathbb{Q}_2$, we get*

$$\begin{aligned} (5.12) \quad \int_{\mathbb{Q}_2} \Lambda(r, \begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix}; \varphi_{L_1 \oplus L_1}, \varphi_{L_1 \oplus L_1}) dc \\ = |a_1|^{-(2r+1)\iota(a_1)} |a_2|^{-(2r+1)\iota(a_2)} \frac{1 - 2^{-(2r+2)}}{1 - 2^{-(2r+1)}} W_0(r, L_1, L_1) \end{aligned}$$

where

$$\iota(a_i) = \begin{cases} 0 & \text{if } a_i \in \mathbb{Z}_2, \\ 1 & \text{if } a_i \in \mathbb{Q}_2 \setminus \mathbb{Z}_2, \end{cases}$$

and

$$W_0(r, L_1, L_1) := \int_{\text{Sym}_2(\mathbb{Q}_2)} \int_{V_r^2} \varphi_{L_1 \oplus L_1}(x) \psi\left(-\frac{1}{2} \text{tr}(b[Q(x)])\right) dx db.$$

Substituting (5.12) into (5.11) yields,

$$\begin{aligned} (5.13) \quad Z(r, \chi_V, \Phi^\pm, f_{\mathbf{w}}^s)(\mathbf{w}') \\ = \gamma(V)^2 \text{vol}(\mathbf{I}_\alpha^*(2)) 2^{\ell(w)+j(w)} \chi_V(\mathbf{x}(w)\mathbf{x}(w')) \left(\frac{\gamma(V)}{[L_1^\# : L_1]^{\frac{1}{2}}} \right)^{j(w)+j(w')} \\ \times \frac{1 - 2^{-2r-2}}{1 - 2^{-2r-1}} W_0(r, L_1, L_1) \int_{(\mathbb{Q}_2^\times)^2} \chi_V \chi_1(a_1) \chi_V \chi_2(a_2) |a_1 a_2|^{r+\frac{1}{2}} d^\times a. \end{aligned}$$

We refer to the $W_0(r, L_1, L_1)$ as Siegel unipotent functions because the integral over $\text{Sym}_2(\mathbb{Q}_2)$ is really the integral over a unipotent subgroup of the Siegel parabolic for $\widehat{G}_2$. In order to compute $W_0(r, L_1, L_1)$ we rely on some results from [12].

5.4. Relating $W_0(r, L_1, L_1)$ to local densities of quadratic forms. Next we evaluate the integrals $W_0(r, L_1, L_1)$. As in the non-dyadic case, this quantity is computed as the limit of some local densities of quadratic forms. As we will see, [12] has formulas for closely related integrals. In order to follow [12], set $X = 2^{-r-1}$ and $T \in \mathrm{Sym}_n(\mathbb{Q}_2)$. Let (V^S, Q^S) be a quadratic space so that $\dim_{\mathbb{Q}_2}(V^S) = 5$ and let $S \in \mathrm{Sym}_5(\mathbb{Q}_2)$ so that $Q^S(x) = {}^t x S x$. Let $V_r^S = V^S \oplus V_h^r$ with Q_r^S defined in the usual way. For a lattice $L \subset V^S$, let $L^{(r)} = L \oplus (\mathbb{Z}_2 e_0 \oplus \mathbb{Z}_2 e'_0)^r$. Then [12] defines the following integral:

$$W_0(X, T, L) := \int_{\mathrm{Sym}_n(\mathbb{Q}_2)} \int_{\bigoplus_{i=1}^n L^{(r)}} \psi(\mathrm{tr} b[Q_r^S(x)]) \psi(-\mathrm{tr} T b) dx db.$$

Additionally, [12] has some explicit formulas in the case that $n \in \{1, 2\}$ for $F = \mathbb{Q}_2$. It is also noted in [12] that $W_0(X, T, L)$ is independent of the choice of ψ , so we need not worry that our expression for $W_0(r, L_1, L_1)$ has an additional factor of $-\frac{1}{2}$ in the additive character. Notice that when $n = 2$,

$$W_0(r, L_1, L_1) = 2^{-1 \pm \frac{1}{2}} \lim_{T \rightarrow 0} W_0(X, T, L_1)$$

as long as this limit makes sense. Notice that the factor $2^{-1 \pm \frac{1}{2}}$ comes from a comparison of Haar measures that we will discuss in this section. Ultimately, we will show that such a limit makes sense for $V = V_\kappa^\pm$ and that $W_0(r, L_1, L_1)$ has a reasonable formula for our choices of L_1 .

It is worth mentioning that unlike the non-dyadic case, T need not be diagonalizable, so the formulas for $W_0(X, T, L_1)$ break up into several cases. Further, our formulas will be slightly different than Yang's because of a shift in dimension. The formulas in [12] are actually stated for $(V, Q) = (M_2^0(\mathbb{Q}_2), -\kappa \det)$ or $(V, Q) = (D^0, -\kappa x^2)$. However, the net effect of this difference in dimension is that the formulas in [12] use $X = 2^{-r}$. Now we prove the following lemma.

Lemma 5.8. *For $L_1 \in \{L_{\kappa,1}^\pm\}$, we have*

$$(5.14) \quad W_0(r, L_1, L_1) = \begin{cases} \frac{1}{\sqrt{2}} \frac{1-2^{-(2r+4)}}{1-2^{-(2r+1)}} = \frac{1}{\sqrt{2}} \frac{\zeta(2r+1)}{\zeta(2r+4)} & \text{if } L_1 = L_{\kappa,1}^+, \\ \frac{1}{2\sqrt{2}} \frac{(1-2^{-r})(1+2^{-r})}{1-2^{-(2r+1)}} = \frac{1}{2\sqrt{2}} \frac{\zeta(2r+1)}{\zeta(2r)} & \text{if } L_1 = L_{\kappa,1}^-. \end{cases}$$

Proof. The proof involves demonstrating that all the formulas for $W_0(X, T, L_1)$ approach the same limit as $T \rightarrow 0$. As in (1.16) of [12], T has the form

$$T = 2^{a-1} \begin{pmatrix} b & 1 \\ 1 & b \end{pmatrix}, \text{ or } 2^a \mathrm{diag}(\alpha_1, 2^t \alpha_2)$$

for $a, t \geq 0$, $\alpha_i \in \mathbb{Z}_2^\times$ and $b \in \{0, 2\}$. Theorems 5.2–5.4 of [12] compute $W_0(X, T, L_1)$ for the various T above. The essential point is that for any of these cases we have

$$\begin{aligned} \frac{W_0(X, T, L_{\kappa,1}^+)}{1-2^{-2}X^2} &= \sum_{0 \leq i \leq \frac{a-1}{2}} (2X^2)^i + O((2^{1/2}X)^a), \\ W_0(X, T, L_{\kappa,1}^-) &= 1 - \sum_{1 \leq i \leq \frac{a-1}{2}} (2X^2)^i + O((2^{1/2}X)^a). \end{aligned}$$

So regardless of T , $W_0(W, T, L_1)$ is a reasonable geometric series plus an error term that is dominated by $(2^{1/2}X)^a$. However, we notice that in our case $2^{\frac{1}{2}}X = 2^{\frac{1}{2}}2^{-r-1} = 2^{-r-\frac{1}{2}}$. So for $r \geq 0$, $\lim_{a \rightarrow \infty} (2^{1/2}X)^a = 0$.

We now let $T \rightarrow 0$ which is exactly the condition that $a \rightarrow \infty$, so

$$\begin{aligned} W_0(r, L_{\kappa,1}^+, L_{\kappa,1}^+) &= \frac{1}{\sqrt{2}} \lim_{T \rightarrow 0} W_0(X, T, L_{\kappa,1}^+) = \frac{1 - 2^{-2}X^2}{\sqrt{2}} \lim_{T \rightarrow 0} \frac{W_0(X, T, L_{\kappa,1}^+)}{1 - 2^{-2}X^2} \\ &= \frac{1 - 2^{-2}X^2}{\sqrt{2}} \lim_{a \rightarrow \infty} \left[\sum_{0 \leq i \leq \frac{a-1}{2}} (2X^2)^i + O((2^{1/2}X)^a) \right] = \frac{1 - 2^{-2}X^2}{\sqrt{2}} \sum_{i=0}^{\infty} (2X^2)^i \end{aligned}$$

and

$$\begin{aligned} W_0(r, L_{\kappa,1}^-, L_{\kappa,1}^-) &= \frac{1}{2\sqrt{2}} \lim_{T \rightarrow 0} W_0(X, T, L_{\kappa,1}^-) \\ &= \frac{1}{2\sqrt{2}} \lim_{a \rightarrow \infty} \left[1 - \sum_{1 \leq i \leq \frac{a-1}{2}} (2X^2)^i + O((2^{1/2}X)^a) \right] = \frac{1}{2\sqrt{2}} \left[1 - \sum_{i=1}^{\infty} (2X^2)^i \right]. \end{aligned}$$

The lemma now follows from evaluating the geometric series. Recall that for our purposes $X = 2^{-r-1}$ so $2X^2 = 2^{-2r-1}$ and the geometric series converge for $r \geq 0$. One should also note that the extra factors $\frac{1}{\sqrt{2}}$ and $\frac{1}{2\sqrt{2}}$ come from a slight difference in how we normalize our Haar measure dx versus how they are normalized in [12]. The measures on V_r in [12] are normalized so that $\text{vol}(L_1^{(r)}) = 1$, whereas we define ours so that

$$\int_{V_r} |\varphi_{L_1}^{(r)}|^2 = \int_{V_r} |\widehat{\varphi_{L_1}^{(r)}}|^2. \quad \square$$

5.5. The integral along the torus. The final integral in (5.13) to compute is

$$\int_{(\mathbb{Q}_2^\times)^2} \chi_V \chi_1(a_1) \chi_V \chi_2(a_2) |a_1 a_2|^{r+\frac{1}{2}} d^\times a.$$

However, this is just a Mellin transform and is quite standard to compute (see Section 7.4 of [15]). This final integral also demonstrates the necessity of the condition that $\chi_V \chi_i$ is unramified. If $\chi_V \chi_i$ were ramified for either i , then we would be integrating a ramified multiplicative character over $\mathbb{Q}_2^\times$ and the integral would vanish. However, since $\chi_V \chi_i$ is unramified for $i \in \{1, 2\}$, we get

$$\begin{aligned} \mathcal{Z}(r, \chi_V, \Phi^\pm, \chi_1, \chi_2) &:= \int_{(\mathbb{Q}_2^\times)^2} \chi_V \chi_1(a_1) \chi_V \chi_2(a_2) |a_1 a_2|^{r+\frac{1}{2}+s_0} d^\times a \\ (5.15) \quad &= \frac{\prod_{i=1}^2 L(r + \frac{1}{2}, \chi_V \chi_i) L(r + \frac{1}{2}, \chi_V \chi_i^{-1})}{\zeta(2r+2) \zeta(2r+1)}. \end{aligned}$$

Substituting the quantity $\mathcal{Z}(r, \chi_V, \Phi^\pm, \chi_1, \chi_2)$ in (5.13), we get

$$\begin{aligned} (5.16) \quad Z(r, \chi_V, \Phi^\pm, f_{\mathbf{w}}^s)(\mathbf{w}') &= \gamma(V)^2 \text{vol}(\mathbf{I}_\alpha^*(2)) \frac{1 - 2^{-2r-2}}{1 - 2^{-2r-1}} W_0(r, L_1, L_1) \mathcal{Z}(r, \chi_V, \Phi^\pm, \chi_1, \chi_2) \\ &\quad \times 2^{\ell(w)+j(w)} \chi_V(\mathbf{x}(w)\mathbf{x}(w')) \left(\frac{\gamma(V)}{[L_1^\# : L_1]^{\frac{1}{2}}} \right)^{j(w)+j(w')}. \end{aligned}$$

Notice that only the factors on the second line depend on the $w, w' \in \mathcal{W}_{G_2}(\alpha)$. So finally, we have a formula that computes the values of $Z(r, \chi_V, \Phi^\pm, f_{\mathbf{w}}^s)(\mathbf{w}')$. Since

the support of the various $f_{\mathbf{w}}^\varsigma$ are disjoint, we have that

$$Z(r, \chi_V, \Phi^\pm, f_{\mathbf{w}}^\varsigma) = \sum_{w' \in \mathcal{W}_{G_2}(\alpha)} c_{w, w'}^{\pm, \kappa} f_{\mathbf{w}'}^\varsigma \quad \text{where} \quad Z(r, \chi_V, \Phi^\pm, f_{\mathbf{w}}^\varsigma)(\mathbf{w}') = c_{w, w'}^{\pm, \kappa}.$$

So we get the following proposition.

Proposition 5.9. *As an operator on $\mathcal{V}_\pi^{(\mathbf{I}_\alpha^*(2), \varsigma)}$, $\mathrm{rank}(Z(r, \chi_V, \Phi^\pm, \cdot)) \leq 1$. Furthermore, if $\mathcal{Z}(r, \chi_V, \Phi^\pm, \chi_1, \chi_2)W_0(r, L_1, L_1) \neq 0$, then this rank is one with image generated by the function*

$$f_{\kappa, 0}^\pm = f_{\mathbf{w}_0}^\varsigma + \frac{\gamma(V)}{[L_1^\# : L_1]^{1/2}} [f_{\mathbf{w}_\beta}^\varsigma + \chi_V(-1)f_{\mathbf{w}_{\alpha\beta}}^\varsigma] + \chi_V(-1) \frac{\gamma(V)^2}{[L_1^\# : L_1]} f_{\mathbf{w}_{\alpha\beta}}^\varsigma.$$

Proof. Let $w_1, w_2 \in \mathcal{W}_{G_2}(\alpha)$ and consider $Z(r, \chi_V, \Phi^\pm, f_{\mathbf{w}_i}^\varsigma)$ for both $i \in \{1, 2\}$. First we notice that

$$c_{w_1, w'}^{\pm, \kappa} = 2^{\ell(w_1)+j(w_1)-\ell(w_2)-j(w_2)} \chi_V(x(w_1)x(w_2)) \left(\frac{\gamma(V)}{[L_1^\# : L_1]^{\frac{1}{2}}} \right)^{j(w_1)-j(w_2)} c_{w_2, w'}^{\pm, \kappa}$$

for all $w' \in \mathcal{W}_{G_2}(\alpha)$. Notice that the only factor on the right-hand side of the equation that depends on w' is $c_{w_2, w'}^{\pm, \kappa}$. So we see that

$$\begin{aligned} Z(r, \chi_V, \Phi^\pm, f_{\mathbf{w}_1}^\varsigma) &= 2^{\ell(w_1)+j(w_1)-\ell(w_2)-j(w_2)} \chi_V(x(w_1)x(w_2)) \\ &\quad \times \left(\frac{\gamma(V)}{[L_1^\# : L_1]^{\frac{1}{2}}} \right)^{j(w_1)-j(w_2)} Z(r, \chi_V, \Phi^\pm, f_{\mathbf{w}_2}^\varsigma) \end{aligned}$$

for any $w_1, w_2 \in \mathcal{W}_{G_2}(\alpha)$. But the set $\{f_{\mathbf{w}}^\varsigma\}_{w \in \mathcal{W}_{G_2}(\alpha)}$ is a basis for $\mathcal{V}_\pi^{(\mathbf{I}_\alpha^*(2), \varsigma)}$, so the image of $Z(r, \chi_V, \Phi^\pm, \cdot)$ is at most one-dimensional.

Now if $\mathcal{Z}(r, \chi_V, \Phi^\pm, \chi_1, \chi_2)W_0(r, L_1, L_1) \neq 0$, then the image of $Z(r, \chi_V, \Phi^\pm, \cdot)$ is generated by

$$\begin{aligned} f_{\kappa, 0}^\pm &= \gamma(V)^{-2} \mathrm{vol}(\mathbf{I}_\alpha^*(2))^{-1} \mathcal{Z}(r, \chi_V, \Phi^\pm, \chi_1, \chi_2)^{-1} W_0(r, L_1, L_1)^{-1} Z(r, \chi_V, \Phi^\pm, f_{\mathbf{w}_0}^\varsigma) \\ &= \sum_{w' \in \mathcal{W}_{G_2}(\alpha)} \chi_V(x(w')) \left(\frac{\gamma(V)}{[L_1^\# : L_1]^{\frac{1}{2}}} \right)^{j(w')} f_{\mathbf{w}'}^\varsigma \end{aligned}$$

where the second equality comes from using (5.16) along with $x(w_0) = 1$, $\ell(w_0) = 0$, and $j(w_0) = 0$. The result follows from evaluating this last expression. $\square$

Since $f_{\kappa, 0}^\pm \in \mathcal{V}_\pi^{(\mathbf{I}_\alpha^*(2), \varsigma)}$ and it generates the image of $Z(r, \chi_V, \Phi^\pm, \cdot)$, then it is an eigenvector for the operator $Z(r, \chi_V, \Phi^\pm, \cdot)$. This makes the $f_{\kappa, 0}^\pm$ the precise analog to the “good test vectors” in [15] for F non-dyadic. Thus for our 2-adic calculations the $f_{\kappa, 0}^\pm$ will be called the “good test vectors”.

Let us now consider $\chi_V(-1)$, $\gamma(V)$ and $[L_1^\# : L_1]$ for our various quadratic spaces $V \in \{V_\kappa^\pm\}$.

Lemma 5.10. *For $\kappa \in \mathbb{Z}_2^\times$, let $\vartheta(\kappa) \in \{0, 1\}$ corresponding to $(\kappa - 1)/2 \pmod 2$. Then*

$$\begin{aligned}\chi_V(-1) &= (-1)^{\vartheta(\kappa)}, \\ \gamma(V) &= -(-i)^{\vartheta(\kappa)}\epsilon(V), \\ [L_1^\# : L_1]^{1/2} &= \begin{cases} \sqrt{2} & \text{if } V = V_\kappa^+, \\ 2\sqrt{2} & \text{if } V = V_\kappa^-. \end{cases}\end{aligned}$$

Proof. The first equation is simply a direct computation of $\chi_V(-1) = \xi_{2\kappa}(-1) = (-1, 2\kappa)_{\mathbb{Q}_2}$ which can be found in many places including Section III.1.2 of [8]. Recall from Section 2 that

$$\gamma(V) = \chi_V(-1)\gamma_{\mathbb{Q}_2}(\eta)\gamma(\eta \circ V)^{-1}.$$

As a consequence of Definition A.6 of [7], we have

$$(5.17) \quad \epsilon(V) = \gamma(\eta \circ V)\gamma_{\mathbb{Q}_2}(\eta)^{-5}\gamma_{\mathbb{Q}_2}(\det(V), \eta)^{-1}.$$

Substituting this into the expression for $\gamma(V)$ yields

$$(5.18) \quad \gamma(V) = \chi_V(-1)\gamma_{\mathbb{Q}_2}(\eta)^{-4}\gamma_{\mathbb{Q}_2}(2\kappa, \eta)^{-1}\epsilon(V).$$

However, we notice two important facts. First from Section A.3 of [7] we have

$$\gamma_{\mathbb{Q}_2}(\eta) = \gamma_{\mathbb{Q}_2}(\psi_{\frac{1}{2}}) = \gamma_{\mathbb{Q}_2}(\frac{1}{2}, \psi)\gamma_{\mathbb{Q}_2}(\psi) = \gamma(2, \psi)\gamma_{\mathbb{Q}_2}(\psi) = \gamma_{\mathbb{Q}_2}(\psi).$$

With this we can show

$$\gamma_{\mathbb{Q}_2}(2\kappa, \eta) = \frac{\gamma_{\mathbb{Q}_2}(\eta_{2\kappa})}{\gamma_{\mathbb{Q}_2}(\eta)} = \frac{\gamma_{\mathbb{Q}_2}(\psi_\kappa)}{\gamma_{\mathbb{Q}_2}(\psi)} = \gamma_{\mathbb{Q}_2}(\kappa, \psi).$$

Second, we use Proposition A.12 of [7] to show

$$\begin{aligned}\gamma_{\mathbb{Q}_2}(\psi) &= 2^{-1/2}(1 + \psi(\frac{1}{4})) = 2^{-1/2}(1 - i) = \mathbf{e}(-\frac{1}{4}), \\ \gamma_{\mathbb{Q}_2}(\kappa, \psi) &= \psi(-\frac{1}{4})^{\vartheta(\kappa)} = \mathbf{e}(\frac{1}{2})^{\vartheta(\kappa)}\end{aligned}$$

where $\mathbf{e}(x) = \exp(\pi ix)$. Notice $\mathbf{e}(-\frac{1}{4})^{-4} = \mathbf{e}(-1)^{-1} = -1$, so (5.18) becomes

$$(5.19) \quad \gamma(V) = -(-1)^{\vartheta(\kappa)}(-1)\mathbf{e}(\frac{1}{2})^{\vartheta(\kappa)}\epsilon(V)$$

which simplifies to the second equation in the statement of the lemma.

Finally, we compute $[L_1^\# : L_1]$ directly. In particular, if $V = V_\kappa^+$, then

$$L_1 = \mathbb{Z}_2(E_{1,1} - E_{2,2}) \oplus \mathbb{Z}_2 E_{1,2} \oplus \mathbb{Z}_2 E_{2,1} \oplus \mathbb{Z}_2 e_0 \oplus \mathbb{Z}_2 e'_0$$

and

$$L_1^\# = 2^{-1}\mathbb{Z}_2(E_{1,1} - E_{2,2}) \oplus \mathbb{Z}_2 E_{1,2} \oplus \mathbb{Z}_2 E_{2,1} \oplus \mathbb{Z}_2 e_0 \oplus \mathbb{Z}_2 e'_0.$$

On the other hand, if $V = V_\kappa^-$, then

$$L_1 = \mathbb{Z}_2 i \oplus \mathbb{Z}_2 j \oplus \mathbb{Z}_2 ij \oplus \mathbb{Z}_2 e_0 \oplus \mathbb{Z}_2 e'_0$$

and

$$L_1^\# = 2^{-1}\mathbb{Z}_2 i \oplus 2^{-1}\mathbb{Z}_2 j \oplus 2^{-1}\mathbb{Z}_2 ij \oplus \mathbb{Z}_2 e_0 \oplus \mathbb{Z}_2 e'_0.$$

The equality regarding $[L_1^\# : L_1]$ follows directly from the description of the lattices above. $\square$

With all of these explicit computations, we can finally prove our main theorem.

Theorem 5.11. *Let $(\pi, \mathcal{V}_\pi) = \chi_1 \times \chi_2 \rtimes \sigma_0$ be an irreducible principal series representation of $\widetilde{\mathrm{Sp}}_2(\mathbb{Q}_2)$ in the standard induced model. Let (V, Q) be one of the spaces $(V_\kappa^\pm, Q_\kappa^\pm)$ where $\kappa \in \mathbb{Z}_2^\times$. As mentioned above, restrict our choices of χ_i so that $\chi_i = \xi_{2a}$ on $\mathbb{Z}_2^\times$ with $\kappa \equiv a \pmod{4}$.*

For $V = V_\kappa^+$ and

$$\Phi_r^+ = \lambda_V(\varphi_{L_1^+ \oplus L_1^+}^{(r)} \otimes \overline{\varphi_{L_1^+ \oplus L_1^+}^{(r)}}),$$

the $\mathrm{rank}(Z(r, \chi_V, \Phi^+, \cdot)) = 1$ and its image is generated by the eigenvector $f_{\kappa,0}^+$ having eigenvalue

$$(5.20) \quad \gamma(V)^2 \mathrm{vol}(\mathbf{I}_\alpha^*(2)) \mathcal{Z}(r, \chi_V, \Phi^\pm, \chi_1, \chi_2) W_0(r, L_1, L_1) \Upsilon_\kappa^+ \\ = \frac{\gamma(V)^2 \mathrm{vol}(\mathbf{I}_\alpha^*(2)) \Upsilon_\kappa^+}{\sqrt{2}} \frac{\prod_{i=1}^2 L(r + \frac{1}{2}, \chi_V \chi_i) L(r + \frac{1}{2}, \chi_V \chi_i^{-1})}{\zeta(2r+2)\zeta(2r+4)}$$

with $\Upsilon_\kappa^+ = 15 - 12\vartheta(\kappa)$.

Similarly, for $V = V_\kappa^-$ and

$$\Phi_r^- = \lambda_V(\varphi_{L_1^- \oplus L_1^-}^{(r)} \otimes \overline{\varphi_{L_1^- \oplus L_1^-}^{(r)}}),$$

the $\mathrm{rank}(Z(r, \chi_V, \Phi^-, \cdot)) \leq 1$ with equality for $r \geq 1$. For $r \geq 1$, the image of $Z(r, \chi_V, \Phi^-, \cdot)$ is generated by the eigenvector $f_{\kappa,0}^-$ having eigenvalue

$$(5.21) \quad \gamma(V)^2 \mathrm{vol}(\mathbf{I}_\alpha^*(2)) \mathcal{Z}(r, \chi_V, \Phi^\pm, \chi_1, \chi_2) W_0(r, L_1, L_1) \Upsilon_\kappa^- \\ = \frac{\gamma(V)^2 \mathrm{vol}(\mathbf{I}_\alpha^*(2)) \Upsilon_\kappa^-}{2\sqrt{2}} \frac{\prod_{i=1}^2 L(r + \frac{1}{2}, \chi_V \chi_i) L(r + \frac{1}{2}, \chi_V \chi_i^{-1})}{\zeta(2r+2)\zeta(2r)}$$

with $\Upsilon_\kappa^- = 3 - 3\vartheta(\kappa)$. In particular, if $\kappa \equiv 3 \pmod{4}$, then $\Upsilon_\kappa^- = 0$ and the operator $Z(r, \chi_V, \Phi^-, \cdot)$ is nilpotent.

Proof. First note that by (5.14) and (5.15), we have

$$\mathcal{Z}(r, \chi_V, \Phi^\pm, \chi_1, \chi_2) W_0(r, L_1, L_1) = \begin{cases} \frac{1}{2^{\frac{1}{2}}} \frac{\prod_{i=1}^2 L(r + \frac{1}{2}, \chi_V \chi_i) L(r + \frac{1}{2}, \chi_V \chi_i^{-1})}{\zeta(2r+4)\zeta(2r+2)} & \text{if } V = V_\kappa^+, \\ \frac{1}{2^{\frac{3}{2}}} \frac{\prod_{i=1}^2 L(r + \frac{1}{2}, \chi_V \chi_i) L(r + \frac{1}{2}, \chi_V \chi_i^{-1})}{\zeta(2r+2)\zeta(2r)} & \text{if } V = V_\kappa^-. \end{cases}$$

Thus $\mathcal{Z}(r, \chi_V, \Phi^\pm, \chi_1, \chi_2) W_0(r, L_1, L_1) = 0$ if and only if $V = V_\kappa^-$ and $r = 0$. In all other cases, $\mathrm{rank}(Z(r, \chi_V, \Phi^\pm, \cdot)) = 1$ and the image of the operator is generated by $f_{\kappa,0}^\pm$. To compute $f_{\kappa,0}^\pm$ explicitly, we first use the results of Section IV.2 of [8] to show that $\epsilon(V_\kappa^\pm) = \mp 1$ (notice the reversal of sign). Then by applying the explicit formulas of Lemma 5.10 to Proposition 5.9 $f_{\kappa,0}^\pm$ we see that

$$f_{\kappa,0}^\pm = \begin{cases} f_{\mathbf{w}_0}^\pm + 2^{-\frac{1}{2}} f_{\mathbf{w}_\beta}^\pm + 2^{-\frac{1}{2}} f_{\mathbf{w}_{\alpha\beta}}^\pm + 2^{-1} f_{\mathbf{w}_{\beta\alpha\beta}}^\pm & \text{if } 4 | (\kappa - 1) \text{ and } V = V_\kappa^+, \\ f_{\mathbf{w}_0}^\pm + 2^{-\frac{1}{2}} i f_{\mathbf{w}_\beta}^\pm - 2^{-\frac{1}{2}} i f_{\mathbf{w}_{\alpha\beta}}^\pm + 2^{-1} f_{\mathbf{w}_{\beta\alpha\beta}}^\pm & \text{if } 4 | (\kappa - 3) \text{ and } V = V_\kappa^+, \\ f_{\mathbf{w}_0}^\pm - 2^{-\frac{3}{2}} f_{\mathbf{w}_\beta}^\pm - 2^{-\frac{3}{2}} f_{\mathbf{w}_{\alpha\beta}}^\pm + 2^{-3} f_{\mathbf{w}_{\beta\alpha\beta}}^\pm & \text{if } 4 | (\kappa - 1) \text{ and } V = V_\kappa^-, \\ f_{\mathbf{w}_0}^\pm - 2^{-\frac{3}{2}} i f_{\mathbf{w}_\beta}^\pm + 2^{-\frac{3}{2}} i f_{\mathbf{w}_{\alpha\beta}}^\pm + 2^{-3} f_{\mathbf{w}_{\beta\alpha\beta}}^\pm & \text{if } 4 | (\kappa - 3) \text{ and } V = V_\kappa^-. \end{cases}$$

In particular, $f_{\kappa,0}^\pm(\mathbf{w}_0) = 1$, so we can compute its eigenvalue for $Z(r, \chi_V, \Phi^\pm, \cdot)$ by evaluating $Z(r, \chi_V, \Phi^\pm, f_{\kappa,0}^\pm)(\mathbf{w}_0)$. In particular, we have

$$Z(r, \chi_V, \Phi^\pm, f_{\kappa,0}^\pm)(\mathbf{w}_0) = (c_{w_0, w_0}^{\pm, \kappa} + c_{w_\beta, w_0}^{\pm, \kappa} + c_{w_{\alpha\beta}, w_0}^{\pm, \kappa} + c_{w_{\beta\alpha\beta}, w_0}^{\pm, \kappa}).$$

Using (5.16) with the substitutions $\mathbf{x}(w_0) = 1$ and $j(w_0) = 0$, we see that

$$Z(r, \chi_V, \Phi^\pm, f_{\kappa,0}^\pm)(\mathbf{w}_0) = \gamma(V)^2 \mathrm{vol}(\mathbf{I}_\alpha^*(2)) \mathcal{Z}(r, \chi_V, \Phi^\pm, \chi_1, \chi_2) W_0(r, L_1^\pm, L_1^\pm) \Upsilon_\kappa^\pm$$

where

$$\Upsilon_{\kappa}^{\pm} = (1 + 4 \frac{\gamma(V)^2}{[L_1^{\#} : L_1]})(1 + 8 \frac{\gamma(V)^2}{[L_1^{\#} : L_1]}).$$

Using Lemma 5.10 to compute $\Upsilon_{\kappa}^{\pm}$ and get

$$\Upsilon_{\kappa}^{\pm} = \begin{cases} 15 & \text{if } \kappa \equiv 1 \pmod{4} \text{ and } V = V_{\kappa}^{+}, \\ 3 & \text{if } \kappa \equiv 3 \pmod{4} \text{ and } V = V_{\kappa}^{+}, \\ 3 & \text{if } \kappa \equiv 1 \pmod{4} \text{ and } V = V_{\kappa}^{-}, \\ 0 & \text{if } \kappa \equiv 3 \pmod{4} \text{ and } V = V_{\kappa}^{-}. \end{cases}$$

This completes the proof. $\square$

Remark 5.12. The numerator contains the Euler product

$$L(r + \frac{1}{2}, \chi_V \chi_1) L(r + \frac{1}{2}, \chi_V \chi_2) L(r + \frac{1}{2}, \chi_V \chi_1^{-1}) L(r + \frac{1}{2}, \chi_V \chi_2^{-1}).$$

This is essentially the same Euler product that one gets when computing the doubling integral for non-dyadic F and π a spherical representation (see Introduction of [15]).

Remark 5.13. Also, we notice that $Z(r, \chi_V, \Phi^{-}, \cdot)$ vanished when $r = 0$ for all quadratic spaces. This is not surprising when one considers the expectation coming from the theta dichotomy conjectures. For a pair of spaces $\{V_{\kappa}^{\pm}\}$, one expects that a genuine irreducible admissible representation $(\pi, \mathcal{V}_{\pi})$ of G_2 should only have a non-vanishing lift to $O(V_{\kappa}^{\pm})$ for one of the two spaces $(V_{\kappa}^{\pm}, Q_{\kappa}^{\pm})$. Since $Z(r, \chi_V, \Phi^{\pm}, \cdot)$ detects non-vanishing theta lifts to these groups when $r = 0$, the vanishing of $Z(r, \chi_V, \Phi^{-}, \cdot)$ is not unreasonable.

Thus we have “good test vectors” and “good theta test sections” for some genuine principal series representations of $\widetilde{\mathrm{Sp}}_2(\mathbb{Q}_2)$ and $\widetilde{\mathrm{Sp}}_2(F)$ with F non-dyadic. In future work, we would like to use these computations to derive the non-vanishing of local theta lifts from $\widetilde{\mathrm{Sp}}_2(F)$ to various $O(V)$ at the non-archimedean places as well as pursue some global non-vanishing results for global theta lifts.

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DEPARTMENT OF MATHEMATICS, THE OHIO STATE UNIVERSITY, 231 W. 18TH AVE., COLUMBUS, OHIO 43210

E-mail address: `czorn@math.ohio-state.edu`