ADMISSIBLE DIAGRAMS IN QUANTUM NILPOTENT ALGEBRAS AND COMBINATORIC PROPERTIES OF WEYL GROUPS

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ABSTRACT. Consider a complex simple Lie algebra $\mathfrak{g}$ of rank n . Denote by Π a system of simple roots, by W the corresponding Weyl group, consider a reduced expression $w = s_{\alpha_1} \circ \cdots \circ s_{\alpha_t}$ (each $\alpha_i \in \Pi$) of some $w \in W$ and call *diagram* any subset of $\llbracket 1, \dots, t \rrbracket$. We denote by $U^+[w]$ (or $U_q^w(\mathfrak{g})$) the “quantum nilpotent” algebra as defined by Jantzen in 1996

We prove (theorem 5.3.1) that the *positive diagrams* naturally associated with the positive subexpressions (of the reduced expression of w chosen above) in the sense of R. Marsh and K. Rietsch (or equivalently the subexpressions without defect in the sense of V. Deodhar), coincide with the *admissible diagrams* constructed by G. Cauchon which describe the natural stratification of $\text{Spec}(U^+[w])$.

This theorem implies in particular (corollaries 5.3.1 and 5.3.2):

- (1) The map $\zeta : \Delta = \{j_1 < \cdots < j_s\} \mapsto u = s_{\alpha_{j_1}} \circ \cdots \circ s_{\alpha_{j_s}}$ is a bijection from the set of admissible diagrams onto the set $\{u \in W \mid u \leq w\}$.
- (2) For each admissible diagram $\Delta = \{j_1 < \cdots < j_s\}$, $s_{\alpha_{j_1}} \circ \cdots \circ s_{\alpha_{j_s}}$ is a reduced expression of $u = \zeta(\Delta)$.
- (3) The map $\zeta' : \Delta = \{j_1 < \cdots < j_s\} \mapsto u' = s_{\alpha_{j_s}} \circ \cdots \circ s_{\alpha_{j_1}}$ is a bijection from the set of admissible diagrams onto the set $\{u \in W \mid u \leq v = w^{-1}\}$.
- (4) For each admissible diagram $\Delta = \{j_1 < \cdots < j_s\}$, $s_{\alpha_{j_s}} \circ \cdots \circ s_{\alpha_{j_1}}$ is a reduced expression of $u' = \zeta'(\Delta)$.

If the Lie algebra $\mathfrak{g}$ is of type A_n and w is chosen in order that $U^+[w]$ is the algebra of quantum matrices $O_q(M_{p,m}(k))$ with $m = n - p + 1$ (see section 2.1), then, the admissible diagrams are the J-diagrams in the sense of A. Postnikov (<http://arxiv.org/abs/math/0609764>). In this particular case, the assertions 3 and 4 have also been proved (with quite different methods) by A. Postnikov and by T. Lam and L. Williams.

1. INTRODUCTION

Given a complex simple Lie algebra $\mathfrak{g}$ of rank n , denote by Π a system of simple roots, by W the corresponding Weyl group, and consider a reduced expression $w = s_{\alpha_1} \circ \cdots \circ s_{\alpha_t}$ of some $w \in W$.

Denote by $U^+[w]$ (or $U_q^w(\mathfrak{g})$) the “quantum nilpotent” algebra associated to w as in [12] and by $X_1, \dots, X_t$ the canonical generators of $U^+[w]$ associated with this chosen reduced expression of w (see section 2.1 for more details).

The natural action of the torus (namely the multiplicative group of the K_λ , where λ runs through the additive group generated by the roots system; see [22,

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Chapter 4]) on the prime spectrum $\text{Spec}(U^+[w])$ induces a finite stratification which is completely described by the *admissible diagrams* (also called Cauchon diagrams) which are some particular subsets of $\llbracket 1, \dots, t \rrbracket$ (see [4]). Those admissible diagrams are very closely connected with the quantum commutation rules satisfied by the canonical generators. For example, in the A_n type, one can choose w such that $U^+[w]$ is a quantum matrix algebra

$$O_q(M_{p,m}(k)) = k\langle X_{i,j} \rangle (1 \leq i \leq p, 1 \leq j \leq m),$$

and we know [5] that the commutation relations

$$(*) \quad X_{u,v} X_{i,j} = X_{i,j} X_{u,v} - (q - q^{-1}) X_{i,v} X_{u,j} (i < u, j < v)$$

determine the shape of admissible diagrams by the following observation:

If $\mathcal{P}$ is any completely prime ideal of $O_q(M_{p,m}(k))$, we have immediately

$$(**) \quad X_{u,v} \in \mathcal{P} \Rightarrow (X_{i,v} \in \mathcal{P} \text{ or } X_{u,j} \in \mathcal{P}).$$

This implies [5] that $\Delta \subset \llbracket 1, \dots, p \rrbracket \times \llbracket 1, \dots, m \rrbracket$ (the set of indexes) is an admissible diagram if and only if, for each i, j, u, v as above, we have:

$$(u, v) \in \Delta \Rightarrow ((i, v) \in \Delta \text{ or } (u, j) \in \Delta).$$

This means that the admissible diagrams are the unions of truncated rows and columns, namely the J-diagrams in the sense of A. Postnikov (see [20]).

So, we see in this example that admissible diagrams are quantum objects since, in the nonquantum case (when $q = 1$), the formulas (*) become $X_{u,v} X_{i,j} = X_{i,j} X_{u,v}$, so that observation (**) is no longer valid and the admissible diagrams become invisible.

On the other hand, R. Marsh and K. Rietsch [18] defined the notion of positive subexpression of the reduced expression of w considered above. These positive subexpressions are defined by particular subsets of $\llbracket 1, \dots, t \rrbracket$ that we call the *positive diagrams*. R. Marsh and K. Rietsch proved in [18] that they are in one-to-one natural correspondence with the elements of the Weyl group which are smaller or equal to w (for the Bruhat order). In this paper, we prove (theorem 5.3. 1) that *the positive diagrams coincide with the admissible diagrams*, which can be interpreted saying that *R. Marsh and K. Rietsch's positive subexpressions are quantum objects*.

In particular, this implies (corollary 5.3. 1):

- (1) The map $\zeta : \Delta = \{j_1 < \dots < j_s\} \mapsto u = s_{\alpha_{j_1}} \circ \dots \circ s_{\alpha_{j_s}}$ is a bijection from the set of admissible diagrams onto the set $\{u \in W | u \leq w\}$.
- (2) Consider an admissible diagram $\Delta = \{j_1 < \dots < j_s\}$ and some integer $i \in \llbracket 1, \dots, t \rrbracket$. Set $\Delta \cap \llbracket i+1, \dots, t \rrbracket = \{j_c, \dots, j_s\} (1 \leq c \leq s)$. Then the expression $s_{\alpha_i} \circ s_{\alpha_{j_c}} \circ \dots \circ s_{\alpha_{j_s}}$ is reduced. In particular, $s_{\alpha_{j_1}} \circ \dots \circ s_{\alpha_{j_s}}$ is a reduced expression of $\zeta(\Delta)$.
- (3) Consider some $u \in W$ with $u \leq w$. Then, the only admissible diagram Δ such that $\zeta(\Delta) = u$ is recursively defined as follows:
 - $1 \in \Delta \Leftrightarrow l(s_{\alpha_1} \circ u) = l(u) - 1 \Leftrightarrow u^{-1}(\alpha_1)$ is a negative root;
 - consider some integer $i \in \llbracket 1, \dots, t-1 \rrbracket$, assume that $\Delta \cap \llbracket 1, \dots, i \rrbracket = \{j_1, \dots, j_d\}$ and set $u_i = s_{\alpha_{j_d}} \circ \dots \circ s_{\alpha_{j_2}} \circ s_{\alpha_{j_1}} \circ u (u_i = u \text{ if } \Delta \cap \llbracket 1, \dots, i \rrbracket = \emptyset)$. Then $(i+1 \in \Delta) \Leftrightarrow (l(s_{\alpha_{i+1}} \circ u_i) = l(u_i) - 1) \Leftrightarrow (u_i^{-1}(\alpha_{i+1}) \text{ is a negative root})$.

It also implies (corollary 5.3. 2):

- (1) The map $\zeta' : \Delta = \{j_1 < \dots < j_s\} \mapsto u' = s_{\alpha_{j_s}} \circ \dots \circ s_{\alpha_{j_1}}$ is a bijection from the set of admissible diagrams onto the set $\{u \in W | u \leq v = w^{-1}\}$.
- (2) Consider an admissible diagram $\Delta = \{j_1 < \dots < j_s\}$ and some integer $i \in \llbracket 1, \dots, t \rrbracket$. Set $\Delta \cap \llbracket i+1, \dots, t \rrbracket = \{j_c, \dots, j_s\} (1 \leq c \leq s)$. Then the expression $s_{\alpha_{j_s}} \circ \dots \circ s_{\alpha_{j_c}} \circ s_{\alpha_i}$ is reduced. In particular, $s_{\alpha_{j_s}} \circ \dots \circ s_{\alpha_{j_1}}$ is a reduced expression of $\zeta'(\Delta)$.

In the particular case $U^+[w] = O_q(M_{p,m}(k))$ mentioned above, this last result gives back a theorem of A. Postnikov [20] and a theorem of T. Lam and L. Williams [17].

In the general case, let us denote by $\Delta \leftrightarrow w^\Delta$ the one-to-one correspondence constructed by R. Marsh and K. Rietsch [18] between the positive diagrams and the elements of the Weyl group which are smaller or equal to w .

Let us also denote by $\Delta \leftrightarrow \mathcal{P}_\Delta$ the one-to-one correspondence constructed by G. Cauchon [4] between the admissible diagrams and the prime ideals of $U^+[w]$ which are invariant under the torus action.

By theorem 5.3.1 of this paper, it turns out that there exists a natural one-to-one correspondence

$$w^\Delta \leftrightarrow \mathcal{P}_\Delta$$

between the elements of the Weyl group which are smaller or equal to w and the prime ideals of $U^+[w]$ which are invariant under the torus action.

Let us recall that, in the particular case $U^+[w] = O_q(M_{p,m}(k))$ mentioned above, S. Launois [14] has constructed (with quite different methods) such a one-to-one correspondence which, moreover, preserves the ordering (where the Weyl group is provided with the Bruhat order and the spectrum $\text{Spec}(O_q(M_{p,m}(k)))$ is provided with the inclusion order). This leads us to ask the following questions (unsolved at the moment):

- **Question 1.** Does this Launois correspondence coincide with ours in the particular case $U^+[w] = O_q(M_{p,m}(k))$?
- **Question 2.** Does our correspondence $w^\Delta \leftrightarrow \mathcal{P}_\Delta$ preserve the ordering in the general case (where the Weyl group is provided with the Bruhat order and the spectrum $\text{Spec}(U^+[w])$ is provided with the inclusion order)?

(A positive answer would supply the Bruhat order with a nice quantum interpretation.)

2. BACKGROUND ON WEYL GROUPS

Following J. C. Jantzen ([12]), we use the following conventions throughout this paper:

- $\mathfrak{g}$ is a complex simple Lie algebra.
- Φ is the (irreducible) root system of $\mathfrak{g}$ with respect to a fixed Cartan sub-algebra.
- Π is a fixed basis of Φ , Φ^+ denotes the subset of positive roots, and we set $n = |\Pi|$, $N = |\Phi^+|$ ($1 \leq n \leq N$).
- W is the Weyl group of Φ and $(\cdot, \cdot)$ is the unique scalar product on the real vector space V generated by Φ , such that $\|\beta\|^2 = 2$ ($\|\beta\| := \sqrt{(\beta, \beta)}$) for all short roots β in Φ .
- For any β in Φ , we set $d_\beta = \frac{\|\beta\|^2}{2}$, $\beta^\vee = \frac{\beta}{d_\beta}$, and s_β denotes the reflection with respect to β ($s_\beta(x) = x - (\beta^\vee, x)\beta$ for any x in V).

- k is a field with $\text{char}(k) \neq 2$ and, in addition, $\text{char}(k) \neq 3$ if Φ has type G_2 .
- $q \in k^* := k \setminus \{0\}$, q is not a root of unity.
- The k -algebra $U_q(\mathfrak{g})$ and its canonical generators $E_\alpha, F_\alpha, K_\alpha^{\pm 1}$ ($\alpha \in \Pi$) are defined as in [12]. We denote by U^+ (or $U_q^+(\mathfrak{g})$) the subalgebra generated by the E_α ($\alpha \in \Pi$).
- Denote by $\mathbb{Z}\Pi$ the root lattice. The algebra $U_q(\mathfrak{g})$ is $\mathbb{Z}\Pi$ -graded and, if α in Π , $\deg(E_\alpha) = \alpha$, $\deg(F_\alpha) = -\alpha$, $\deg(K_\alpha^{\pm 1}) = 0$.
- For any $\rho = \sum_{\alpha \in \Pi} m_\alpha \alpha$ ($m_\alpha \in \mathbb{Z}$) in the root lattice $\mathbb{Z}\Pi$, we set $K_\rho = \prod_{\alpha \in \Pi} K_\alpha^{m_\alpha}$. The multiplicative group $T = \{K_\rho | \rho \in \mathbb{Z}\Pi\}$ is called the Torus (of $U_q(\mathfrak{g})$). This group acts on the algebra $U_q(\mathfrak{g})$ by

$$K_\rho \cdot u = K_\rho^{-1} u K_\rho (\forall u \in U_q(\mathfrak{g})).$$

In particular, for any homogeneous element u of $U_q(\mathfrak{g})$ with degree $\deg(u) = \gamma \in \mathbb{Z}\Pi$, we have $K_\rho \cdot u = q^{-(\rho, \gamma)} u$.

2.1. Quantum algebras $U^+[w]$. Consider any $w \in W$, set $t = l(w)$ and consider a reduced expression

$$(1) \quad w = s_{\alpha_1} \circ \dots \circ s_{\alpha_t} \quad (\alpha_i \in \Pi \text{ for } 1 \leq i \leq t).$$

It is well known that

$$\beta_1 = \alpha_1, \beta_2 = s_{\alpha_1}(\alpha_2), \dots, \beta_t = s_{\alpha_1} \dots s_{\alpha_{t-1}}(\alpha_t)$$

are distinct positive roots and that the set $\{\beta_1, \dots, \beta_t\}$ does not depend on the reduced expression (1) of w . For any $\alpha \in \Pi$, define the braid automorphism T_α of the algebra $U_q(\mathfrak{g})$ as in ([12], p. 153), and set

$$X_{\beta_1} = E_{\alpha_1}, X_{\beta_2} = T_{\alpha_1}(E_{\alpha_2}), \dots, X_{\beta_t} = T_{\alpha_1} \dots T_{\alpha_{t-1}}(E_{\alpha_t}).$$

The following results are classical ([12], chapter 8):

- $X_{\beta_1}, \dots, X_{\beta_t}$ are all in U^+ . Each X_{β_i} ($1 \leq i \leq t$) is homogeneous of degree $\deg(X_{\beta_i}) = \beta_i$.
- We denote by $U^+[w]$ (or $U_q^w(\mathfrak{g})$) the subalgebra of U^+ generated by $X_{\beta_1}, \dots, X_{\beta_t}$. This algebra does not depend on the above reduced expression (1) of w (although the variables $X_{\beta_1}, \dots, X_{\beta_t}$ depend on (1)).
- The ordered monomials $X^{\underline{a}} := X_{\beta_1}^{a_1} \dots X_{\beta_t}^{a_t}$, $\underline{a} = (a_1, \dots, a_t) \in \mathbb{N}^t$, are a basis of $U^+[w]$ (as a k -vector space).
- Since the above generators X_{β_i} are homogeneous, the $\mathbb{Z}\Pi$ -graduation of $U_q(\mathfrak{g})$ induces a $\mathbb{Z}\Pi$ -graduation of $U^+[w]$ and the action of the Torus T on $U_q(\mathfrak{g})$ induces, by restriction, an action of T on $U^+[w]$.
- If $1 \leq i < j \leq t$, we have the following straightening formula due to Levendorskii and Soibelman:

$$(2) \quad X_{\beta_j} X_{\beta_i} - q^{-(\beta_i, \beta_j)} X_{\beta_i} X_{\beta_j} = P_{j,i},$$

$$(3) \quad P_{j,i} = \sum_{\underline{a}=(a_{i+1}, \dots, a_{j-1})} c_{\underline{a}} X_{\beta_{i+1}}^{a_{i+1}} \dots X_{\beta_{j-1}}^{a_{j-1}},$$

with $\underline{a} \in \mathbb{N}^{j-i-1}$, $c_{\underline{a}} \in k$, and $c_{\underline{a}} \neq 0$ for only finitely many $\underline{a}$. $P_{j,i}$ is homogeneous with degree $\beta_i + \beta_j$ so that, if $j = i + 1$, we have $P_{j,i} = 0$.

The reader will observe a little difference between those formulas and the original Levendorskii–Soibelman one ([16], prop. 5.5.2.), in which the left member of (2) is $X_{\beta_j} X_{\beta_i} - q^{(\beta_i, \beta_j)} X_{\beta_i} X_{\beta_j}$. The reason is that Levendorskii

and Soibelman use a version of the quantum group $U_q(\mathfrak{g})$ which slightly differs from ours. Under our conventions, a direct proof of formulas (2) and (3) is given in [19].

- When $w = w_0$, we have $t = N$, $\{\beta_1, \dots, \beta_t\} = \Phi^+$ and $U^+[w] = U^+$.

An example. Assume, for sake of simplicity, that $k = \mathbb{C}$ is the complex number field, that $\mathfrak{g}$ has type A_n with $n \geq 3$ and that the simple roots $\epsilon_1, \dots, \epsilon_n$ are ordered such that the Dynkin diagram is

$$\epsilon_1 \text{ --- } \epsilon_2 \text{ --- } \epsilon_3 \text{ --- } \dots \text{ --- } \epsilon_{n-1} \text{ --- } \epsilon_n.$$

Consider the following particular reduced expression of the longest element in W :

$$(4) \quad w_0 = s_{\epsilon_1} \circ (s_{\epsilon_2} \circ s_{\epsilon_1}) \circ \dots \circ (s_{\epsilon_j} \circ s_{\epsilon_{j-1}} \circ \dots \circ s_{\epsilon_1}) \circ \dots \circ (s_{\epsilon_n} \circ s_{\epsilon_{n-1}} \circ \dots \circ s_{\epsilon_1}).$$

Denote by $Y_{1,2}, Y_{1,3}, Y_{2,3}, \dots, Y_{1,j+1}, Y_{2,j+1}, \dots, Y_{j,j+1}, \dots, Y_{1,n+1}, Y_{2,n+1}, \dots, Y_{n,n+1}$ the canonical generators of U^+ with respect to this reduced decomposition and observe that:

Lemma 2.1.1. (1) $Y_{j,j+1} = E_{\alpha_j}$ for $1 \leq j \leq n$.

$$(2) \quad Y_{i,j+1} = Y_{i,j} Y_{j,j+1} - q^{-1} Y_{j,j+1} Y_{i,j} \text{ for } 1 \leq i < j \leq n.$$

Proof. (1) This results from the equality

$$s_{\epsilon_1} \circ (s_{\epsilon_2} \circ s_{\epsilon_1}) \circ \dots \circ (s_{\epsilon_j} \circ s_{\epsilon_{j-1}} \circ \dots \circ s_{\epsilon_2})(\epsilon_1) = \epsilon_j.$$

(2) Recall (see section 2.1) that

$$Y_{i,j+1} = T_{\epsilon_1} \circ (T_{\epsilon_2} \circ T_{\epsilon_1}) \circ \dots \circ (T_{\epsilon_{j-1}} \circ T_{\epsilon_{j-2}} \circ \dots \circ T_{\epsilon_1}) \circ (T_{\epsilon_j} \circ T_{\epsilon_{j-1}} \circ \dots \circ T_{\epsilon_{j+2-i}})(E_{\epsilon_{j+1-i}}) \quad (i > 1).$$

(In the case $i = 1$, this formula becomes $Y_{1,j+1} = T_{\epsilon_1} \circ (T_{\epsilon_2} \circ T_{\epsilon_1}) \circ \dots \circ (T_{\epsilon_{j-1}} \circ T_{\epsilon_{j-2}} \circ \dots \circ T_{\epsilon_1})(E_{\epsilon_j})$.)

Assume $i > 1$. Since $\{\epsilon_{j-i}, \dots, \epsilon_1\} \perp \{\epsilon_j, \dots, \epsilon_{j+2-i}\}$ and $\{\epsilon_{j-i-1}, \dots, \epsilon_1\} \perp \epsilon_{j+1-i}$, $Y_{i,j+1}$ is equal to

$$T_{\epsilon_1} \circ (T_{\epsilon_2} \circ T_{\epsilon_1}) \circ \dots \circ (T_{\epsilon_{j-1}} \circ T_{\epsilon_{j-2}} \circ \dots \circ T_{\epsilon_{j+1-i}}) \circ (T_{\epsilon_j} \circ T_{\epsilon_{j-1}} \circ \dots \circ T_{\epsilon_{j+2-i}}) \circ T_{\epsilon_{j-i}}(E_{\epsilon_{j+1-i}})$$

with, by [12],

$$T_{\epsilon_{j-i}}(E_{\epsilon_{j+1-i}}) = E_{\epsilon_{j-i}} E_{\epsilon_{j+1-i}} - q^{-1} E_{\epsilon_{j+1-i}} E_{\epsilon_{j-i}}.$$

Moreover, we have

$$\begin{aligned} & s_{\epsilon_1} \circ (s_{\epsilon_2} \circ s_{\epsilon_1}) \circ \dots \circ (s_{\epsilon_{j-1}} \circ s_{\epsilon_{j-2}} \circ \dots \circ s_{\epsilon_{j+1-i}}) \circ (s_{\epsilon_j} \circ s_{\epsilon_{j-1}} \circ \dots \circ s_{\epsilon_{j+2-i}})(\epsilon_{j+1-i}) \\ &= s_{\epsilon_1} \circ (s_{\epsilon_2} \circ s_{\epsilon_1}) \circ \dots \circ (s_{\epsilon_{j-1}} \circ s_{\epsilon_{j-2}} \circ \dots \circ s_{\epsilon_{j+1-i}})(\epsilon_{j+1-i} + \epsilon_{j+2-i} + \dots + \epsilon_j) \\ &= s_{\epsilon_1} \circ (s_{\epsilon_2} \circ s_{\epsilon_1}) \circ \dots \circ (s_{\epsilon_{j-2}} \circ s_{\epsilon_{j-3}} \circ \dots \circ s_{\epsilon_1})(\epsilon_j) = \epsilon_j. \end{aligned}$$

This implies that

$$\begin{aligned} & T_{\epsilon_1} \circ (T_{\epsilon_2} \circ T_{\epsilon_1}) \circ \dots \circ (T_{\epsilon_{j-1}} \circ T_{\epsilon_{j-2}} \circ \dots \circ T_{\epsilon_{j+1-i}}) \\ & \circ (T_{\epsilon_j} \circ T_{\epsilon_{j-1}} \circ \dots \circ T_{\epsilon_{j+2-i}})(E_{\epsilon_{j+1-i}}) = E_{\epsilon_j} = Y_{j,j+1}. \end{aligned}$$

Since $\{\epsilon_j, \epsilon_{j-1}, \dots, \epsilon_{j+2-i}\} \perp \epsilon_{j-i}$, we have

$$\begin{aligned} & T_{\epsilon_1} \circ (T_{\epsilon_2} \circ T_{\epsilon_1}) \circ \dots \circ (T_{\epsilon_{j-1}} \circ T_{\epsilon_{j-2}} \circ \dots \circ T_{\epsilon_{j+1-i}}) \circ (T_{\epsilon_j} \circ T_{\epsilon_{j-1}} \circ \dots \circ T_{\epsilon_{j+2-i}})(E_{\epsilon_{j-i}}) \\ &= T_{\epsilon_1} \circ (T_{\epsilon_2} \circ T_{\epsilon_1}) \circ \dots \circ (T_{\epsilon_{j-1}} \circ T_{\epsilon_{j-2}} \circ \dots \circ T_{\epsilon_{j+1-i}})(E_{\epsilon_{j-i}}) = Y_{i,j}. \end{aligned}$$

$$\text{So, } Y_{i,j+1} = Y_{i,j} Y_{j,j+1} - q^{-1} Y_{j,j+1} Y_{i,j}.$$

In the case $i = 1$, we get

$$\begin{aligned} Y_{1,j+1} &= T_{\epsilon_1} \circ (T_{\epsilon_2} \circ T_{\epsilon_1}) \circ \cdots \circ (T_{\epsilon_{j-1}} \circ T_{\epsilon_{j-2}} \circ \cdots \circ T_{\epsilon_1})(E_{\epsilon_j}) \\ &= T_{\epsilon_1} \circ (T_{\epsilon_2} \circ T_{\epsilon_1}) \circ \cdots \circ (T_{\epsilon_{j-2}} \circ T_{\epsilon_{j-3}} \circ \cdots \circ T_{\epsilon_1}) \circ T_{\epsilon_{j-1}}(E_{\epsilon_j}) \end{aligned}$$

with, by [12],

$$T_{\epsilon_{j-1}}(E_{\epsilon_j}) = E_{\epsilon_{j-1}}E_{\epsilon_j} - q^{-1}E_{\epsilon_j}E_{\epsilon_{j-1}}.$$

We observe that

$$T_{\epsilon_1} \circ (T_{\epsilon_2} \circ T_{\epsilon_1}) \circ \cdots \circ (T_{\epsilon_{j-2}} \circ T_{\epsilon_{j-3}} \circ \cdots \circ T_{\epsilon_1})(E_{\epsilon_j}) = E_{\epsilon_j} = Y_{j,j+1},$$

and that

$$T_{\epsilon_1} \circ (T_{\epsilon_2} \circ T_{\epsilon_1}) \circ \cdots \circ (T_{\epsilon_{j-2}} \circ T_{\epsilon_{j-3}} \circ \cdots \circ T_{\epsilon_1})(E_{\epsilon_{j-1}}) = Y_{1,j}.$$

$$\text{So, } Y_{1,j+1} = Y_{1,j}Y_{j,j+1} - q^{-1}Y_{j,j+1}Y_{1,j}.$$

□

Denote by v a square root of q and, as in [1], denote by $e_{i,j}$ the generators of U^+ constructed by H. Yamane [22]. We know that:

- $e_{j,j+1} = E_{\epsilon_j}$ for $1 \leq j \leq n$.
- $e_{i,j+1} = ve_{i,j}e_{j,j+1} - v^{-1}e_{j,j+1}e_{i,j}$ for $1 \leq i < j \leq n$.

This implies:

- $v^{i-j}e_{i,j+1} = (v^{i-j+1}e_{i,j})e_{j,j+1} - e_{j,j+1}q^{-1}(v^{i-j+1}e_{i,j})$ for $1 \leq i < j \leq n$
and, from lemma 2.1. 1, we deduce (by induction on $j - i$):

Lemma 2.1.2. $Y_{i,j+1} = v^{i-j}e_{i,j+1}$ for $1 \leq i \leq j \leq n$.

Now, consider an integer p with $1 < p < n$ and set

$$(5) \quad w = (s_{\epsilon_p} \circ s_{\epsilon_{p-1}} \circ \cdots \circ s_{\epsilon_1}) \circ (s_{\epsilon_{p+1}} \circ s_{\epsilon_p} \circ \cdots \circ s_{\epsilon_2}) \circ \cdots \circ (s_{\epsilon_n} \circ s_{\epsilon_{n-1}} \circ \cdots \circ s_{\epsilon_{n-p+1}}).$$

If $p < j < n$, we observe that

$$\{\epsilon_{j-p}, \dots, \epsilon_1\} \perp \{\epsilon_{j+1}, \dots, \epsilon_{j+1-p+1}, \dots, \epsilon_n, \dots, \epsilon_{n-p+1}\}.$$

This implies that

$$w_0 = w_1 \circ w \circ w_2$$

with

$$\begin{aligned} w_1 &= s_{\epsilon_1} \circ (s_{\epsilon_2} \circ s_{\epsilon_1}) \circ \cdots \circ (s_{\epsilon_{p-1}} \circ s_{\epsilon_{p-2}} \circ \cdots \circ s_{\epsilon_1}), \\ w_2 &= s_{\epsilon_1} \circ (s_{\epsilon_2} \circ s_{\epsilon_1}) \circ \cdots \circ (s_{\epsilon_{n-p}} \circ s_{\epsilon_{n-p-1}} \circ \cdots \circ s_{\epsilon_1}). \end{aligned}$$

If d, d_1, d_2 denote the number of simple reflections which appear respectively in the expressions of w, w_1, w_2 given above, we observe that $d + d_1 + d_2$ is equal to the length of w_0 . This implies:

Lemma 2.1.3. (1) (5) is a reduced expression of w .

(2) If $p \leq j \leq n$, then

$$T_{w_1} \circ (T_{\epsilon_p} \circ T_{\epsilon_{p-1}} \circ \cdots \circ T_{\epsilon_1}) \circ \cdots \circ (T_{\epsilon_{j-1}} \circ T_{\epsilon_{j-2}} \circ \cdots \circ T_{\epsilon_{j-p+1}})(E_{\epsilon_j}) = Y_{j,j+1}.$$

(1) If $p \leq j \leq n$ and $1 \leq i \leq p$, then

$$T_{w_1} \circ (T_{\epsilon_p} \circ T_{\epsilon_{p-1}} \circ \cdots \circ T_{\epsilon_1}) \circ \cdots \circ (T_{\epsilon_j} \circ T_{\epsilon_{j-1}} \circ \cdots \circ T_{\epsilon_{j-i+2}})(E_{\epsilon_{j-i+1}}) = Y_{i,j+1}.$$

This implies that the canonical generators $X_{i,j+1}$ of $U^+[w]$ satisfy

$$X_{i,j+1} = T_{w_1}^{-1}(Y_{i,j+1})$$

for $p \leq j \leq n$ and $1 \leq i \leq p$.

Since the Yamane generators $e_{i,j+1}$ ($p \leq j \leq n$, $1 \leq i \leq p$) satisfy the commutation relations of quantum matrices, it follows from lemma 2.1. 2 that the generators $Y_{i,j+1}$ satisfy the same property. Now, as $T_{w_1}^{-1}$ is an automorphism of $U_q(\mathfrak{g})$ and as the ordered monomials in the variables $X_{i,j+1}$ (where the order on these variables is defined by the inverse lexicographic order on the indexes $(i, j+1)$) are a basis of $U^+[w]$, we conclude that

Proposition 2.1.1. *$U^+[w]$ is the algebra $O_q(M_{p,m}(k))$ with $m = n - p + 1$ and with canonical generators the variables $X_{i,j+1}$ ($p \leq j \leq n$, $1 \leq i \leq p$).*

2.2. Diagrams. We assume that w , t , the simple roots α_i , and the positive roots β_i ($1 \leq i \leq t$), are defined as in section 2.1, and we set

$$(6) \quad v = w^{-1} = s_{\alpha_t} \circ \cdots \circ s_{\alpha_1}.$$

Definition 2.2.1. A diagram with respect to (1) is any subset Δ of $\llbracket 1, \dots, t \rrbracket$. (If there is no possible ambiguity, we omit to make precise “with respect to (1)”.)

In the following, we sometimes also omit the symbol $\circ$ in the composition of maps.

Consider a diagram Δ . For any $i \in \llbracket 1, \dots, t \rrbracket$, we set

$$s_{\alpha_i}^\Delta = \begin{cases} s_{\alpha_i} & \text{if } i \in \Delta \\ Id & \text{if } i \notin \Delta \end{cases}$$

and we denote

- $w^\Delta = s_{\alpha_1}^\Delta \cdots s_{\alpha_t}^\Delta$,
- $v^\Delta = (w^\Delta)^{-1} = s_{\alpha_t}^\Delta \cdots s_{\alpha_1}^\Delta$,
- for any $i \in \llbracket 1, \dots, t, t+1 \rrbracket$, $w_i^\Delta = s_{\alpha_i}^\Delta \cdots s_{\alpha_t}^\Delta$ ($w_1^\Delta = w^\Delta$, $w_{t+1}^\Delta = Id$),
- for any $i \in \llbracket 0, 1, \dots, t \rrbracket$, $v_i^\Delta = (w_{t-i+1}^\Delta)^{-1} = s_{\alpha_t}^\Delta \cdots s_{\alpha_{t-i+2}}^\Delta s_{\alpha_{t-i+1}}^\Delta$ ($v_0^\Delta = (w_{t+1}^\Delta)^{-1} = Id$, $v_t^\Delta = v^\Delta$).

Remark (the case of example 2.1). Assume we are in the situation of example 2.1, so that

$$\begin{aligned} w &= s_{\alpha_1} \circ s_{\alpha_2} \circ \cdots \circ s_{\alpha_t} = (s_{\epsilon_p} \circ s_{\epsilon_{p-1}} \circ \cdots \circ s_{\epsilon_1}) \\ &\quad \circ (s_{\epsilon_{p+1}} \circ s_{\epsilon_p} \circ \cdots \circ s_{\epsilon_2}) \circ \cdots \circ (s_{\epsilon_n} \circ s_{\epsilon_{n-1}} \circ \cdots \circ s_{\epsilon_{n-p+1}}). \end{aligned}$$

Set $m = n - p + 1$, so that $t = mp$, and consider a rectangular tableau consisting of $p \times m$ boxes labeled from 1 to mp as mentioned in the following figure.

1	$p + 1$		$(m - 1)p + 1$ ↓
p	$2p$		mp

Following A. Postnikov [20], let us draw this tableau with p wires going along the rows and m wires going along the columns. Label the ends of the wires from 1 to $n + 1$ as in the following figure.

We observe that, if W is identified as usual with the symmetric group $\mathcal{S}_{n+1}$ so that each s_{ϵ_i} is the transposition $(i, i + 1)$, the wiring diagram defined above corresponds to the permutation $v = w^{-1}$.

Now, consider any diagram Δ (i.e. any subset of $\llbracket 1, mp \rrbracket$), color in dark the boxes whose labels are in Δ , and replace the corresponding crossing in noncolored boxes by a noncrossing:

white box

colored box

Then, if $\Delta = \{i_1 < \dots < i_l\}$, we observe that this new wiring diagram corresponds to the permutation $s_{\epsilon_{i_l}} \circ \dots \circ s_{\epsilon_{i_1}} = v^\Delta$.

So, using *A. Postnikov's conventions* ([20], pp. 71 and 72) we can say:

- (1) Δ is a pipe dream or a wiring diagram.
- (2) $u = v^\Delta$ is the element of W corresponding to the pipe dream Δ and Δ is a pipe dream (or a wiring diagram) of u .

In particular, we have:

- (3) v is the element of W corresponding to the full pipe dream $\Delta = \llbracket 1, \dots, t \rrbracket$.
- (4) Id is the element of W corresponding to the empty pipe dream $\Delta = \emptyset$.

Now, in the general case, let us denote by $\leq$ the Bruhat order on the Weyl group W . Since (1) is a reduced expression of w , we have that (6) is a reduced expression of v . Let us recall ([13], corollary A.1.8) that an element u of W satisfies $u \leq w$ (resp. $u \leq v$) if and only if it can be written as a product of simple reflections obtained by omitting some of the s_{α_i} in the reduced expression (1) (resp. (6)). So, we have immediately:

Lemma 2.2.1. (1) *The map $\Delta \mapsto u = w^\Delta$ (resp. $u = v^\Delta$) is surjective from the set of diagrams onto the set $\{u \in W | u \leq w\}$ (resp. $\{u \in W | u \leq v\}$).*

- (2) *The map $u \mapsto u^{-1}$ is a bijection from $\{u \in W | u \leq w\}$ onto $\{u \in W | u \leq v\}$.*

Lemma 2.2.2. $\mathbf{v}^\Delta := (v_0^\Delta, v_1^\Delta, \dots, v_t^\Delta)$ is a subexpression of (6) in the sense of Marsh and Rietsch [18].

Proof. This is because $v_{t-1}^\Delta = s_{\alpha_t}^\Delta \dots s_{\alpha_{t-i+2}}^\Delta \implies v_t^\Delta = v_{t-1}^\Delta s_{\alpha_{t-i+1}}^\Delta$ with

$$s_{\alpha_{t-i+1}}^\Delta = \begin{cases} s_{\alpha_{t-i+1}} & \text{if } t-i+1 \in \Delta \\ Id & \text{if } t-i+1 \notin \Delta \end{cases}. \quad \square$$

Lemma 2.2.3. *If $\mathbf{u} = (u_0, u_1, \dots, u_t)$ is a subexpression of v , there exists a unique diagram Δ such that $\mathbf{v}^\Delta = \mathbf{u}$. So, the map $\Delta \mapsto \mathbf{v}^\Delta$ is a bijection from the set of diagrams onto the set of subexpressions of v .*

Proof. Since $\mathbf{u}$ is a subexpression of v , we have $u_0 = Id$ and

$$(\forall i \in \llbracket 1, \dots, t \rrbracket) (u_{i-1})^{-1} u_i \in \{s_{t-i+1}, Id\}.$$

Given any diagram Δ , we have $\mathbf{v}^\Delta = (v_0^\Delta, v_1^\Delta, \dots, v_t^\Delta)$. Since $v_0^\Delta = u_0 = Id$, we have $\mathbf{v}^\Delta = \mathbf{u}$ if and only if

$$(\forall i \in \llbracket 1, \dots, t \rrbracket) (u_{i-1})^{-1} u_i = (v_{i-1}^\Delta)^{-1} v_i^\Delta = s_{\alpha_{t-i+1}}^\Delta.$$

Now, there exists a unique diagram Δ which satisfies those conditions. It is defined by

$$(\forall i \in \llbracket 1, \dots, t \rrbracket) (t-i+1 \in \Delta) \Leftrightarrow (u_{i-1})^{-1} u_i = s_{t-i+1}. \quad \square$$

Definition 2.2.2. We say that a diagram Δ is positive with respect to the reduced decomposition (1) (or that Δ is positive if there is no ambiguity) if $\mathbf{v}^\Delta = (v_0^\Delta, v_1^\Delta, \dots, v_t^\Delta)$ is a positive subexpression of (6) in the sense of Marsh and Rietsch [18], namely:

$$v_{i-1}^\Delta < v_{i-1}^\Delta s_{\alpha_{t-i+1}} \quad (\forall i \in \llbracket 1, \dots, t \rrbracket).$$

Observation 2.2.1. The full diagram and the empty diagram are both positive.

Proof. • If $\Delta = \llbracket 1, \dots, t \rrbracket$ is the full diagram, we have $v_{i-1}^\Delta = s_{\alpha_t} \dots s_{\alpha_{t-i+2}}$ for any $i \in \llbracket 1, \dots, t \rrbracket$ ($v_0^\Delta = Id$). Since (6) is a reduced expression of v , $s_{\alpha_t} \dots s_{\alpha_{t-i+2}} s_{\alpha_{t-i+1}}$ is a reduced expression of $v_{i-1}^\Delta s_{\alpha_{t-i+1}}$. So, clearly, we have $v_{i-1}^\Delta < v_{i-1}^\Delta s_{\alpha_{t-i+1}}$.

• If $\Delta = \emptyset$, we have $v_{i-1}^\Delta = Id$, so that $v_{i-1}^\Delta < v_{i-1}^\Delta s_{\alpha_{t-i+1}}$. $\square$

Consider any $i \in \llbracket 1, \dots, t \rrbracket$, set $j = t - i + 1$ and $\{j_1 < \dots < j_s\} = \Delta \cap \llbracket j + 1, \dots, t \rrbracket = \Delta \cap \llbracket t - i + 2, \dots, t \rrbracket$, so that $v_{i-1}^\Delta = s_{\alpha_t} \dots s_{\alpha_{t-i+2}} = s_{\alpha_{j_s}} \dots s_{\alpha_{j_1}}$ and $v_{i-1}^\Delta s_{\alpha_{t-i+1}} = s_{\alpha_{j_s}} \dots s_{\alpha_{j_1}} s_{\alpha_j}$.

If w_1 and w_2 are in W , it follows from ([13], corollary A.1.8) that

$$\begin{aligned} (w_1 < w_2) &\Leftrightarrow (w_1 \neq w_2 \text{ and } w_1 \text{ is obtained by omitting some} \\ &\quad \text{simple reflections in a reduced expression of } w_2) \\ &\Leftrightarrow (w_1^{-1} < w_2^{-1}). \end{aligned}$$

So, with $w_1 = v_{i-1}^\Delta$ and $w_2 = v_{i-1}^\Delta s_{\alpha_{t-i+1}}$, we obtain:

$$(v_{i-1}^\Delta < v_{i-1}^\Delta s_{\alpha_{t-i+1}}) \Leftrightarrow (s_{\alpha_{j_1}} \dots s_{\alpha_{j_s}} < s_{\alpha_j} s_{\alpha_{j_1}} \dots s_{\alpha_{j_s}}).$$

Let us set $u = s_{\alpha_{j_1}} \dots s_{\alpha_{j_s}}$ and recall that $l(s_{\alpha_j} u) = l(u) \pm 1$. If $l(s_{\alpha_j} u) = l(u) + 1$, we have that $u < s_{\alpha_j} u$ ([13], A.1.6). Conversely, if $u < s_{\alpha_j} u$, we have $l(u) < l(s_{\alpha_j} u)$ ([13], A.1.6) and, consequently, $l(s_{\alpha_j} u) = l(u) + 1$. So, we have:

$$(v_{i-1}^\Delta < v_{i-1}^\Delta s_{\alpha_{t-i+1}}) \Leftrightarrow l(s_{\alpha_j} s_{\alpha_{j_1}} \dots s_{\alpha_{j_s}}) = 1 + l(s_{\alpha_{j_1}} \dots s_{\alpha_{j_s}}).$$

Since the mapping $i \mapsto j = t - i + 1$ is a bijection of $\llbracket 1, \dots, t \rrbracket$, we conclude:

Lemma 2.2.4. The diagram Δ is positive if and only if, for any $j \in \llbracket 1, \dots, t \rrbracket$,

$$\Delta \cap \llbracket j + 1, \dots, t \rrbracket = \{j_1 < \dots < j_s\} \Rightarrow l(s_{\alpha_j} s_{\alpha_{j_1}} \dots s_{\alpha_{j_s}}) = 1 + l(s_{\alpha_{j_1}} \dots s_{\alpha_{j_s}}).$$

Now, we can prove the following characterization of positive diagrams.

Proposition 2.2.1. The diagram Δ is positive if and only if, for any $j \in \llbracket 1, \dots, t \rrbracket$,

$$\Delta \cap \llbracket j + 1, \dots, t \rrbracket = \{j_1 < \dots < j_s\} \Rightarrow l(s_{\alpha_j} s_{\alpha_{j_1}} \dots s_{\alpha_{j_s}}) = 1 + s.$$

Proof. Assume that Δ is positive, choose $j, j_1, \dots, j_s$ as above, and consider any l with $1 < l < s$. We have $\Delta \cap \llbracket j_l + 1, \dots, t \rrbracket = \{j_{l+1} < \dots < j_s\}$ and, by lemma 2.2.4, $l(s_{\alpha_{j_l}} s_{\alpha_{j_{l+1}}} \dots s_{\alpha_{j_s}}) = 1 + l(s_{\alpha_{j_{l+1}}} \dots s_{\alpha_{j_s}})$. From this, we get that $l(s_{\alpha_{j_1}} \dots s_{\alpha_{j_s}}) = s$ and, again by lemma 2.2.4, $l(s_{\alpha_j} s_{\alpha_{j_1}} \dots s_{\alpha_{j_s}}) = 1 + s$.

Conversely, assume that this equality holds for any j and choose $j, j_1, \dots, j_s$ as above. We have $\Delta \cap \llbracket j_1 + 1, \dots, t \rrbracket = \{j_2 < \dots < j_s\}$ and, by assumption, $l(s_{\alpha_{j_1}} s_{\alpha_{j_2}} \dots s_{\alpha_{j_s}}) = 1 + (s - 1) = s$. So, $l(s_{\alpha_j} s_{\alpha_{j_1}} \dots s_{\alpha_{j_s}}) = 1 + l(s_{\alpha_{j_1}} \dots s_{\alpha_{j_s}})$ and Δ is positive by lemma 2.2.4. $\square$

2.3. Some properties of positive diagrams. In this section, we use the same conventions as in section 2.2.

Proposition 2.3.1.

- (1) *The map: $\Delta \mapsto u = v^\Delta$ is a bijection from the set of positive diagrams onto the set $\{u \in W \mid u \leq v\}$.*
- (2) *The map: $\Delta \mapsto u = w^\Delta$ is a bijection from the set of positive diagrams onto the set $\{u \in W \mid u \leq w\}$.*

Proof. (1) By lemma 2.2.1, $f: \Delta \mapsto u = v^\Delta$ is a map from the set of positive diagrams in the set $\{u \in W \mid u \leq v\}$. Consider any $u \in W$ with $u \leq v$. By ([18], Lemma 3.5.), there exists a unique positive subexpression $\mathbf{u}^+ = (u_0, u_1, \dots, u_t)$ of (6) with $u_t = u$. By lemma 2.2.3, there exists a unique diagram Δ which satisfies $\mathbf{v}^\Delta = (v_0^\Delta, v_1^\Delta, \dots, v_t^\Delta) = \mathbf{u}^+$. So, Δ is positive (def. 2.2.2) and $v^\Delta = v_t^\Delta = u_t = u$. This proves that f is surjective.

If Δ' is a positive diagram such that $v^{\Delta'} = u$, then $\mathbf{v}^{\Delta'} = (v_0^{\Delta'}, v_1^{\Delta'}, \dots, v_t^{\Delta'})$ is a positive subexpression of (6) with $v_t^{\Delta'} = v^{\Delta'} = u$. So, we have $\mathbf{v}^{\Delta'} = \mathbf{u}^+ \Rightarrow \mathbf{v}^{\Delta'} = \mathbf{v}^\Delta$ and, by lemma 2.2.3, $\Delta' = \Delta$. This proves that f is bijective.

- (2) Denote by g the bijection $u \mapsto u^{-1}$ from $\{u \in W \mid u \leq w\}$ onto $\{u \in W \mid u \leq v\}$ (lemma 2.2.1).

Then $g^{-1} \circ f: \Delta \mapsto u = (v^\Delta)^{-1} = w^\Delta$ is a bijection from the set of positive diagrams onto the set $\{u \in W \mid u \leq w\}$. $\square$

Consider any $p \in \llbracket 1, \dots, t \rrbracket$, $p \neq 1, t$, and set

$$(7) \quad w_1 = s_{\alpha_1} \dots s_{\alpha_p},$$

$$(8) \quad w_2 = s_{\alpha_{p+1}} \dots s_{\alpha_t}.$$

We have $w = w_1 w_2$ and, since (1) is a reduced expression of w , (7) and (8) are reduced expressions of w_1 and w_2 , respectively.

Denote by Δ any subset of $\llbracket 1, \dots, p \rrbracket$, so that Δ is a diagram with respect to (7) and with respect to (1) both.

Proposition 2.3.2. $(\Delta \text{ is positive with respect to (7)}) \Leftrightarrow (\Delta \text{ is positive with respect to (1)})$.

Proof. Assume that Δ is positive with respect to (1) and consider any $j \in \llbracket 1, \dots, p \rrbracket$. We have $\Delta \cap \llbracket j+1, \dots, p \rrbracket = \Delta \cap \llbracket j+1, \dots, t \rrbracket$ and, using the characterization of positive diagrams given in proposition 2.2.1, we obtain that Δ is positive with respect to (7).

Assume that Δ is positive with respect to (7) and consider any $j \in \llbracket 1, \dots, t \rrbracket$. If $j \leq p$, we have $\Delta \cap \llbracket j+1, \dots, t \rrbracket = \Delta \cap \llbracket j+1, \dots, p \rrbracket$, so that the characterization given in proposition 2.2.1 is satisfied. If $j > p$, we have $\Delta \cap \llbracket j+1, \dots, t \rrbracket = \emptyset$, so that the characterization given in proposition 2.2.1 is again satisfied. $\square$

Now, consider a nonempty diagram Δ (with respect to (1)) and two integers j, m in $\llbracket 1, \dots, t \rrbracket$ with $j < m$ and $m \in \Delta$. Let us recall that $(\beta_1, \dots, \beta_t)$ is the sequence of positive roots associated to the reduced expression (1) of w (section 2.1) and let us denote:

- $\overline{\Delta} = \llbracket 1, \dots, t \rrbracket \setminus \Delta$,

- $\Delta \cap \llbracket j+1, \dots, m-1 \rrbracket = \{j_1 < \dots < j_r\}$ (unless this set is empty),
- $\overline{\Delta} \cap \llbracket j+1, \dots, m-1 \rrbracket = \{l_1 < \dots < l_p\}$ (unless this set is empty),
- $(\gamma_1, \dots, \gamma_p, \gamma_{p+1})$ is the sequence of (not necessarily positive) roots defined recursively by $\gamma_{p+1} = \beta_m$ and, for $1 \leq i \leq p$, $\gamma_i = s_{\beta_{l_i}}(\gamma_{i+1})$,
- for $1 \leq i \leq p$, we set $a_i = (\beta_{l_i}^\vee, \gamma_{i+1})$.

Until the end of this section, we assume that $\overline{\Delta} \cap \llbracket j+1, \dots, m-1 \rrbracket$ is nonempty, so that p and the roots γ_i ($1 \leq i \leq p+1$) are well defined.

Lemma 2.3.1.

$$\gamma_1 = \beta_m - a_p \beta_{l_p} - \dots - a_1 \beta_{l_1}.$$

Proof. For $1 \leq i \leq p$, we have $\gamma_i = \gamma_{i+1} - a_i \beta_{l_i}$. Summing these equalities, we get $\gamma_1 = \gamma_{p+1} - a_p \beta_{l_p} - \dots - a_1 \beta_{l_1}$ and, by definition, $\gamma_{p+1} = \beta_m$. $\square$

Set

$$(9) \quad w' = s_{\alpha_1} \dots s_{\alpha_{m-1}}$$

and observe that, since (1) is a reduced expression of w , (9) is a reduced expression of w' .

Lemma 2.3.2. *For any $i \in \llbracket 1, \dots, p \rrbracket$, denote by w'_i the element of W obtained by omitting $s_{\alpha_{l_i}}, \dots, s_{\alpha_{l_p}}$ in (9). Then, we have $\gamma_i = w'_i(\alpha_m)$.*

Proof. Assume that $i = p$ and set $u_p = s_{\alpha_1} \dots s_{\alpha_{l_p}-1}$, so that:

$$\begin{aligned} w' &= u_p s_{\alpha_{l_p}} s_{\alpha_{l_p+1}} \dots s_{\alpha_{m-1}} \Rightarrow u_p^{-1} w' = s_{\alpha_{l_p}} s_{\alpha_{l_p+1}} \dots s_{\alpha_{m-1}} \\ &\Rightarrow u_p s_{\alpha_{l_p}} u_p^{-1} w' = u_p s_{\alpha_{l_p+1}} \dots s_{\alpha_{m-1}} = w'_p. \end{aligned}$$

As $u_p(\alpha_{l_p}) = \beta_{l_p}$, we conclude that $w'_p = s_{\beta_{l_p}} w'$. So, $w'_p(\alpha_m) = s_{\beta_{l_p}} w'(\alpha_m) = s_{\beta_{l_p}}(\beta_m) = \gamma_p$.

Assume $i < p$, $\gamma_{i+1} = w'_{i+1}(\alpha_m)$, and set $u_i = s_{\alpha_1} \dots s_{\alpha_{l_i}-1}$. We have $w'_{i+1} = u_i s_{\alpha_{l_i}} \sigma$ with $\sigma \in W$ and $w'_i = u_i \sigma$. From this, we deduce (as in the case $i = p$) that $u_i s_{\alpha_{l_i}} u_i^{-1} w'_{i+1} = w'_i$. So, $w'_i(\alpha_m) = s_{\beta_{l_i}} w'_{i+1}(\alpha_m) = s_{\beta_{l_i}}(\gamma_{i+1}) = \gamma_i$. $\square$

Now, we can prove:

Proposition 2.3.3. *Assume that*

$$\beta_j + \beta_m = a_p \beta_{l_p} + \dots + a_1 \beta_{l_1}.$$

Then Δ is not positive (with respect to (1)).

Proof. By lemma 2.3.1, we have $\beta_j = -\gamma_1$ and, by lemma 2.3.2, we can write $\beta_j = -w'_1(\alpha_m)$. Recall that w'_1 is obtained by omitting $s_{\alpha_{l_1}}, \dots, s_{\alpha_{l_p}}$ in the reduced expression (9) of w' and that

$$\begin{aligned} \{l_1 < \dots < l_p\} &= \overline{\Delta} \cap \llbracket j+1, \dots, m-1 \rrbracket \\ &= \llbracket j+1, \dots, m-1 \rrbracket \setminus \{j_1 < \dots < j_r\}. \end{aligned}$$

- In the case $\Delta \cap \llbracket j+1, \dots, m-1 \rrbracket = \emptyset$, we have $\{l_1, \dots, l_p\} = \llbracket j+1, m-1 \rrbracket$, so that $w'_1 = s_{\alpha_1} \dots s_{\alpha_{j-1}} s_{\alpha_j}$. So, we have $\beta_j = s_{\alpha_1} \dots s_{\alpha_{j-1}}(\alpha_j) = -s_{\alpha_1} \dots s_{\alpha_{j-1}} s_{\alpha_j}(\alpha_m)$. This implies that $\alpha_j = -s_{\alpha_j}(\alpha_m)$, so $\alpha_m = \alpha_j$. Now, we have $\Delta \cap \llbracket j+1, \dots, t \rrbracket = \{m = j'_1 < \dots < j'_l\}$ ($l \geq 1$).

Since $\alpha_j = \alpha_m = \alpha_{j'_1}$, $l(s_{\alpha_j} s_{\alpha_{j'_1}} \dots s_{\alpha_{j'_l}}) \leq l-1$, proposition 2.2.1 implies that Δ is not positive.

- Otherwise, we have

$$w'_1 = s_{\alpha_1} \dots s_{\alpha_{j-1}} s_{\alpha_j} s_{\alpha_{j_1}} \dots s_{\alpha_{j_r}} = \eta s_{\alpha_j} s_{\alpha_{j_1}} \dots s_{\alpha_{j_r}}$$

with $\eta = s_{\alpha_1} \dots s_{\alpha_{j-1}}$.

Since $\beta_j = \eta(\alpha_j)$, we have $\beta_j = -w'_1(\alpha_m) \Rightarrow \alpha_j = -s_{\alpha_j} s_{\alpha_{j_1}} \dots s_{\alpha_{j_r}}(\alpha_m) \Rightarrow s_{\alpha_j} s_{\alpha_{j_1}} \dots s_{\alpha_{j_r}}(\alpha_m) = -\alpha_j$ is a negative root. By ([10], lemma 10.2.C), this implies that $l(s_{\alpha_j} s_{\alpha_{j_1}} \dots s_{\alpha_{j_r}} s_{\alpha_m}) < r + 2$.

As $m \in \Delta$, we have $\Delta \cap \llbracket j + 1, \dots, t \rrbracket = \{j_1 < \dots < j_s\}$ with $s > r$ and $j_{r+1} = m$. So, $l(s_{\alpha_j} s_{\alpha_{j_1}} \dots s_{\alpha_{j_s}}) \leq l(s_{\alpha_j} s_{\alpha_{j_1}} \dots s_{\alpha_{j_r}} s_{\alpha_m}) + s - (r + 1) < r + 2 + s - (r + 1) = s + 1$ and, by proposition 2.2.1, Δ is not positive. $\square$

3. BACKGROUND ON THE DELETING DERIVATIONS ALGORITHM

3.1. Conventions. In this section, we use the conventions of section 2.1 and we set $R = U^+[w]$. In order to simplify the notation, we set

- $X_i = X_{\beta_i}$ for any $i \in \llbracket 1, \dots, t \rrbracket$, so that $R = k\langle X_1, \dots, X_t \rangle$. Moreover, $X_1, \dots, X_t$ are called the canonical generators (with respect to the reduced decomposition (1)) of R .
- Recall (section 2.1) that, for each ρ in the root lattice $\mathbb{Z}\Pi$, the map $h_\rho: u \mapsto K_\rho.u$ is in $\text{Aut}(R)$, the group of automorphisms of the algebra R .
- Let us set $H = \{h_\rho | \rho \in \mathbb{Z}\Pi\}$ and observe that H is an abelian subgroup of $\text{Aut}(R)$.
- Recall that R is $\mathbb{Z}\Pi$ -graded and that, for any homogeneous element u of degree γ in R , for any $\rho \in \mathbb{Z}\Pi$, we have $h_\rho(u) = q^{-(\rho, \gamma)} u$.
- For each $(i, j) \in (\llbracket 1, \dots, t \rrbracket)^2$, set $\lambda_{i,j} = q^{-(\beta_i, \beta_j)}$, $q_i = \lambda_{i,i} = q^{-\|\beta_i\|^2}$, and observe that q_i is not a root of unity.
- If $1 \leq i < j \leq t$, the Levendorskii-Soibelman formula can be written as

$$(10) \quad X_j X_i - \lambda_{j,i} X_i X_j = P_{j,i},$$

$$(11) \quad P_{j,i} = \sum_{\underline{a}=(a_{i+1}, \dots, a_{j-1})} c_{\underline{a}} X_{i+1}^{a_{i+1}} \dots X_{j-1}^{a_{j-1}}$$

with $\underline{a} \in \mathbb{N}^{j-i-1}$, $c_{\underline{a}} \in k$, and $c_{\underline{a}} \neq 0$ for only finitely many $\underline{a}$. Moreover, $P_{j,i}$ is homogeneous with degree $\beta_i + \beta_j$ so that, if $j = i + 1$, we have $P_{j,i} = 0$. In the general case, this also implies that, if $c_{\underline{a}} \neq 0$, then $a_{i+1}\beta_{i+1} + \dots + a_{j-1}\beta_{j-1} = \beta_i + \beta_j$.

- Since β_i and β_j are positive roots, $\beta_i + \beta_j$ is nonzero. So, if $c_{\underline{a}} \neq 0$, then $\underline{a}$ is nonzero.
- From this, we get that the generators $X_1, \dots, X_t$ of R satisfy the equalities and the assumption 6.1.1. of ([4], section 6.1.).
- Moreover, since the ordered monomials $X^{\underline{a}} := X_1^{a_1} \dots X_t^{a_t}$, $\underline{a} = (a_1, \dots, a_t) \in \mathbb{N}^t$, are a basis of R , it follows from ([4], propositions 6.1.1. and 6.1.2.) that R satisfies the conventions of ([4], section 3.1.). In particular, R is an iterated Ore extension of the ground field k and so, R is a noetherian domain.
- We denote by $F = \text{Fract}(R)$ the division ring of fractions of R .
- By ([4], proposition 6.1.1.), we also get that R is the k -algebra generated by the “variables” $X_1, \dots, X_t$ subject to the relations (10).

- For each $l \in \llbracket 1, \dots, t \rrbracket$, we set $h_l = h_{\beta_l} \in H$, and observe that, if $i \in \llbracket 1, \dots, t \rrbracket$, we have $h_l(X_i) = \lambda_{l,i} X_i$.
- Since each X_i is homogeneous, it is an H -eigenvector and, since $\lambda_{1,1} = q^{-(\beta_1, \beta_1)}$ is not a root of unity, the assumption 4.1.2. of [4] is satisfied. Since each $\lambda_{i,j}$ is a power of q , the assumption 4.1.1. of [4] is also satisfied. As explained in [4] (proof of lemma 4.2.2.) this implies that each prime ideal of R is completely prime.
- Recall that each automorphism $h \in H$ can be extended to a (unique) automorphism (denoted h also) of F , so that H can be seen as a subgroup of $\text{Aut}(F)$.

3.2. The algebras $R^{(m)}$. Recall ([4], section 3) that, for any $m \in \llbracket 2, \dots, t+1 \rrbracket$, there exists a family $(X_1^{(m)}, \dots, X_t^{(m)})$ of new “variables” in F , called the canonical generators (with respect to the reduced decomposition (1)) of the algebra $R^{(m)} = k\langle X_1^{(m)}, \dots, X_t^{(m)} \rangle$, and which satisfies the following properties:

- If $1 \leq i < j \leq t$, we have the following simplified Levendorskii-Soibelman formula:

$$(12) \quad X_j^{(m)} X_i^{(m)} - \lambda_{j,i} X_i^{(m)} X_j^{(m)} = P_{j,i}^{(m)} \quad \text{with}$$

$$(13) \quad \diamond \quad m \leq j \Rightarrow P_{j,i}^{(m)} = 0,$$

$$(14) \quad \diamond \quad j < m \Rightarrow P_{j,i}^{(m)} = \sum_{\underline{a}=(a_{i+1}, \dots, a_{j-1})} c_{\underline{a}} (X_{i+1}^{(m)})^{a_{i+1}} \dots (X_{j-1}^{(m)})^{a_{j-1}},$$

where the coefficients $c_{\underline{a}}$ are the same as in (11). (So, $P_{j,i}^{(m)} = 0$ in the case $i+1 = j < m$ and, in the general case, if $c_{\underline{a}} \neq 0$, we have $a_{i+1}\beta_{i+1} + \dots + a_{j-1}\beta_{j-1} = \beta_i + \beta_i$, so that $\underline{a}$ is nonzero.)

- This implies that the generators $X_1^{(m)}, \dots, X_t^{(m)}$ of $R^{(m)}$ still satisfy the equalities and the assumption 6.1.1. of ([4], section 6.1.).
- The ordered monomials $(X^{(m)})^{\underline{a}} := (X_1^{(m)})^{a_1} \dots (X_t^{(m)})^{a_t}$, $\underline{a} = (a_1, \dots, a_t) \in \mathbb{N}^t$, are a basis of $R^{(m)}$. So, $R^{(m)}$ still satisfies the conventions of ([4], section 3.1.). In particular, $R^{(m)}$ is an iterated Ore extension of the ground field k and so, $R^{(m)}$ is a noetherian domain.
- We have $\text{Fract}(R^{(m)}) = \text{Fract}(R) = F$.
- By ([4], proposition 6.1.1.), $R^{(m)}$ is the k -algebra generated by the “variables” $X_1^{(m)}, \dots, X_t^{(m)}$ subject to the relations (12).
- For each ρ in the root lattice $\mathbb{Z}\Pi$ and for each $i \in \llbracket 1, \dots, t \rrbracket$, we still have $h_{\rho}(X_i^{(m)}) = q^{-(\rho, \beta_i)} X_i^{(m)}$. This implies that $h_{\rho}(R^{(m)}) = R^{(m)}$. So H can still be seen as a subgroup of $\text{Aut}(R^{(m)})$ and each $X_i^{(m)}$ is an H -eigenvector. If l and i are in $\llbracket 1, \dots, t \rrbracket$, we have $h_l(X_i^{(m)}) = \lambda_{l,i} X_i^{(m)}$ and, as above, this implies that $R^{(m)}$ satisfies the assumptions 4.4.2 and 4.4.1 of [4]. As explained in ([4], proof of lemma 4.2.2), this implies that each prime ideal of $R^{(m)}$ is completely prime.
- If $u \in F$, and if $\gamma \in \mathbb{Z}\Pi$, we say that u is homogeneous of degree γ , if $h_{\rho}(u) = q^{-(\rho, \gamma)} u$ for all ρ in $\mathbb{Z}\Pi$. So, for example, each $X_i^{(m)}$ is homogeneous of degree β_i .
- If $u_1, \dots, u_r$ are homogeneous of same degree γ , then any linear combination of $u_1, \dots, u_r$ (with coefficients in k) is homogeneous of degree γ .

- Clearly, if u_1 is homogeneous of degree γ_1 and u_2 is homogeneous of degree γ_2 , then $u_1 u_2$ is homogeneous of degree $\gamma_1 + \gamma_2$.
- Likewise, if u is nonzero and homogeneous of degree γ , then u^{-1} is homogeneous of degree $-\gamma$.
- So, if $u_1, \dots, u_r$ are nonzero and homogeneous of degrees $\gamma_1, \dots, \gamma_r$, respectively, if $\underline{a} = (a_1, \dots, a_r) \in \mathbb{Z}^r$, then $u^{\underline{a}} := u_1^{a_1} \dots u_r^{a_r}$ is homogeneous of degree $a_1 \gamma_1 + \dots + a_r \gamma_r$.
- For each $j \in \llbracket 1, \dots, t \rrbracket$, denote by $\delta_j^{(m)}$ the left h_j -derivation of $R^{(m)}$ defined by $X_j^{(m)}$. This means that, for each $a \in R^{(m)}$, $\delta_j^{(m)}(a) := X_j^{(m)} a - h_j(a) X_j^{(m)}$.
- If $j \in \llbracket 1, \dots, t \rrbracket$, we have

$$\begin{aligned} h_j(k\langle X_1^{(m)}, \dots, X_{j-1}^{(m)} \rangle) &= k\langle X_1^{(m)}, \dots, X_{j-1}^{(m)} \rangle, \\ \delta_j^{(m)}(k\langle X_1^{(m)}, \dots, X_{j-1}^{(m)} \rangle) &\subset k\langle X_1^{(m)}, \dots, X_{j-1}^{(m)} \rangle. \end{aligned}$$

So, h_j induces an automorphism, still denoted h_j , of the algebra $k\langle X_1^{(m)}, \dots, X_{j-1}^{(m)} \rangle$. It is the only automorphism of $k\langle X_1^{(m)}, \dots, X_{j-1}^{(m)} \rangle$ which satisfies $h_j(X_i^{(m)}) = \lambda_{j,i} X_i^{(m)}$ for any i .

Likewise, $\delta_j^{(m)}$ induces a left h_j -derivation of $k\langle X_1^{(m)}, \dots, X_{j-1}^{(m)} \rangle$, still denoted $\delta_j^{(m)}$. It satisfies the following properties:

- (1) If $1 \leq i < j$, we have $\delta_j^{(m)}(X_i^{(m)}) = P_{j,i}^{(m)}$. This implies that $\delta_j^{(m)}$ is zero on $k\langle X_1^{(m)}, \dots, X_{j-1}^{(m)} \rangle$, and that $X_j^{(m)} a = h_j(a) X_j^{(m)}$ for any a in $k\langle X_1^{(m)}, \dots, X_{j-1}^{(m)} \rangle$, as soon as $j \geq m$.

- (2) $\delta_j^{(m)}$ is locally nilpotent on $k\langle X_1^{(m)}, \dots, X_{j-1}^{(m)} \rangle$.

- (3) We have

$$R^{(m)} = k[X_1^{(m)}][X_2^{(m)}; h_2, \delta_2^{(m)}] \dots [X_{m-1}^{(m)}; h_{m-1}, \delta_{m-1}^{(m)}][X_m^{(m)}; h_m] \dots [X_t^{(m)}; h_t].$$

- For each $i \in \llbracket 1, \dots, t \rrbracket$, we have $X_i^{(t+1)} = X_i$, so that $R^{(t+1)} = R$.

Until the end of this section, we assume that $m \in \llbracket 2, \dots, t \rrbracket$.

- Recall that $q_m = \lambda_{m,m}$ is not a root of unity and define quantum integers $[l]_{q_m}$ (resp. quantum factorial $[l]!_{q_m}$) for $l \in \mathbb{N}$, as in ([4], section 2).

- For each $i \in \llbracket 1, \dots, t \rrbracket$, we have

$$(1) \quad m \leq i \Rightarrow X_i^{(m)} = X_i^{(m+1)}. \quad \text{In particular, we have } X_m^{(m)} = X_m^{(m+1)}.$$

$$(2) \quad i < m \Rightarrow X_i^{(m)} = X_i^{(m+1)} + \sum_{l=1}^{+\infty} C_l^{(m+1)} (X_m^{(m+1)})^{-l} \text{ with}$$

$$C_l^{(m+1)} = \frac{(1 - q_m)^{-l}}{[l]!_{q_m}} \lambda_{m,i}^{-l} (\delta_m^{(m+1)})^l (X_i^{(m+1)}).$$

- We observe that, since $k\langle X_1^{(m+1)}, \dots, X_{m-1}^{(m+1)} \rangle$ is $\delta_m^{(m+1)}$ -stable, we have $C_l^{(m+1)} \in k\langle X_1^{(m+1)}, \dots, X_{m-1}^{(m+1)} \rangle$ and, since $\delta_m^{(m+1)}$ is locally nilpotent on $k\langle X_1^{(m+1)}, \dots, X_{m-1}^{(m+1)} \rangle$, only finitely many $C_l^{(m+1)}$ are nonzero.
- If $i < m$ and $X_m^{(m+1)} X_i^{(m+1)} - \lambda_{m,i} X_i^{(m+1)} X_m^{(m+1)} = P_{m,i}^{(m+1)} = 0$, then $\delta_m^{(m+1)}(X_i^{(m+1)}) = 0$, and $X_i^{(m)} = X_i^{(m+1)}$.

- There exists a unique homomorphism of k -algebras

$$\Theta^{(m)} : k\langle X_1^{(m+1)}, \dots, X_{m-1}^{(m+1)} \rangle \rightarrow k\langle X_1^{(m)}, \dots, X_{m-1}^{(m)} \rangle$$

which satisfies $\Theta^{(m)}(X_i^{(m+1)}) = X_i^{(m)}$ for $1 \leq i \leq m-1$.

- When $m = 2$, we set

$$(1) \quad Z_i = X_i^{(2)} \text{ for } i \in \llbracket 1, \dots, t \rrbracket.$$

$$(2) \quad \overline{R} = R^{(2)}.$$

- $\overline{R} = k\langle Z_1, \dots, Z_t \rangle$ is the k -algebra generated by the variables $Z_1, \dots, Z_t$ subject to the relations:

$$(15) \quad (1 \leq i < j \leq t) \Rightarrow Z_j Z_i = \lambda_{j,i} Z_i Z_j.$$

- For any $i \in \llbracket 1, \dots, t \rrbracket$, we have $X_i^{(2)} = \dots = X_i^{(i+1)} = Z_i$.
- $S_m = \{(X_m^{(m+1)})^l | l \in \mathbb{N}\} = \{(X_m^{(m)})^l | l \in \mathbb{N}\}$ is a multiplicative system of regular elements in $R^{(m)}$ and in $R^{(m+1)}$. It satisfies the Ore condition (on both sides) in each one of those rings and we have $R^{(m)} S_m^{-1} = R^{(m+1)} S_m^{-1}$.

3.3. Prime and H -prime spectrum of $R^{(m)}$, admissible diagrams. Consider any integer $m \in \llbracket 2, \dots, t+1 \rrbracket$. As usual, we denote by $\text{Spec}(R^{(m)})$ (resp. $H - \text{Spec}(R^{(m)})$) the set of all prime ideals (resp. H -invariant prime ideals) of $R^{(m)}$.

Recall (see section 3.2) that each prime ideal of $R^{(m)}$ is completely prime. If $m \in \llbracket 2, \dots, t \rrbracket$, we denote by

$$\phi_m : \text{Spec}(R^{(m+1)}) \rightarrow \text{Spec}(R^{(m)})$$

the canonical injection defined in ([4], section 4.3).

Moreover, if $m' \leq m+1$ and if $\mathcal{P} \in \text{Spec}(R^{(m+1)})$, then $\mathcal{P}' = \phi_{m'} \circ \dots \circ \phi_m(\mathcal{P})$ is called the canonical image of $\mathcal{P}$ in $\text{Spec}(R^{(m')})$. (If $m' = m+1$, this canonical image is $\mathcal{P}$ itself.)

Let us now recall the main properties of the maps ϕ_m .

- Denote by $\mathcal{P}^{(m+1)}$ any prime ideal in $\text{Spec}(R^{(m+1)})$ and set $\mathcal{P}^{(m)} = \phi_m(\mathcal{P}^{(m+1)})$.
- (1) If $X_m^{(m+1)} \notin \mathcal{P}^{(m+1)}$, then $\mathcal{P}^{(m+1)} \cap S_m = \emptyset$ and

$$\mathcal{P}^{(m)} = R^{(m)} \cap \mathcal{P}^{(m+1)} S_m^{-1}.$$

In this case, $X_m^{(m)} = X_m^{(m+1)} \notin \mathcal{P}^{(m)}$.

- (2) If $X_m^{(m+1)} \in \mathcal{P}^{(m+1)}$, there exists a unique algebra homomorphism $g : R^{(m)} \rightarrow R^{(m+1)} / \mathcal{P}^{(m+1)}$ which satisfies $g(X_i^{(m)}) = X_i^{(m+1)} + \mathcal{P}^{(m+1)}$ for all $i \in \llbracket 1, \dots, t \rrbracket$, and $\mathcal{P}^{(m)} = \text{Ker}(g)$. In this case, $X_m^{(m)} = X_m^{(m+1)} \in \mathcal{P}^{(m)}$.

- (3) For any $h \in H$, we have $h(\mathcal{P}^{(m)}) = \phi_m(h(\mathcal{P}^{(m+1)}))$ ([4], lemma 5.5.5.) so that, by the injectivity of ϕ_m , $\mathcal{P}^{(m+1)}$ is H -invariant if and only if $\mathcal{P}^{(m)}$ is H -invariant.

- A diagram Δ is said admissible (with respect to the reduced decomposition (1)) if there exists a prime ideal $\mathcal{P}$ of $R (= R^{(t+1)})$ whose canonical image $\mathcal{P}^{(2)}$ in $\text{Spec}(\overline{R}) (= \text{Spec}(R^{(2)}))$ satisfies:

$$\mathcal{P}^{(2)} \cap \{Z_i | 1 \leq i \leq t\} = \{Z_i | i \in \Delta\}.$$

- Consider a diagram Δ and denote by $\mathcal{P}_\Delta^{(2)} = (\{Z_i | i \in \Delta\})$ the ideal generated by the variables Z_i with $i \in \Delta$. Then, by ([4], proposition 5.5.1.), we have
 - (1) $\mathcal{P}_\Delta^{(2)} \in H - \text{Spec}(R^{(2)})$.
 - (2) Conversely, for any $\mathcal{Q} \in H - \text{Spec}(R^{(2)})$, there exists a (unique) diagram Δ such that $\mathcal{Q} = \mathcal{P}_\Delta^{(2)}$, namely $\Delta = \{i \in \llbracket 1, t \rrbracket \mid Z_i \in \mathcal{Q}\}$.
- A diagram Δ is admissible if and only if there exists $\mathcal{P}_\Delta \in \text{Spec}(R)$ ($= \text{Spec}(R^{(t+1)})$) such that

$$\mathcal{P}_\Delta^{(2)} = \phi_2 \circ \cdots \circ \phi_t(\mathcal{P}_\Delta).$$

(See [4], theorem 5.5.1. and observe that, since each ϕ_i is injective, $\mathcal{P}_\Delta$ is unique.)

- The map $\Delta \mapsto \mathcal{P}_\Delta$ is a bijection from the set of admissible diagrams onto the set $H - \text{Spec}(R)$ ($= H - \text{Spec}(R^{(t+1)})$). In fact, if Δ is admissible, then $\mathcal{P}_\Delta$ is H -invariant because $\mathcal{P}_\Delta^{(2)} = \phi_2 \circ \cdots \circ \phi_t(\mathcal{P}_\Delta)$ is. So, $\Delta \mapsto \mathcal{P}_\Delta$ is a map from the set of admissible diagrams into $H - \text{Spec}(R)$. It is injective because the map $\Delta \mapsto \mathcal{P}_\Delta^{(2)}$ is injective. If $\mathcal{P} \in H - \text{Spec}(R)$, then $\phi_2 \circ \cdots \circ \phi_t(\mathcal{P}) \in H - \text{Spec}(R^{(2)})$. So $\phi_2 \circ \cdots \circ \phi_t(\mathcal{P}) = \mathcal{P}_\Delta^{(2)}$ with Δ an admissible diagram such that $\mathcal{P} = \mathcal{P}_\Delta$.

4. NEW RESULTS ON H -INVARIANT PRIME IDEALS

In this section, we consider an integer $m \in \llbracket 2, \dots, t+1 \rrbracket$, and we denote by $\mathcal{P}^{(m)}$ an H -invariant prime ideal of $R^{(m)}$. We set $A^{(m)} = R^{(m)}/\mathcal{P}^{(m)}$ and we observe that this algebra is a noetherian domain (since $R^{(m)}$ is noetherian and $\mathcal{P}^{(m)}$ is completely prime). Set $D_m = \text{Frac}(A^{(m)})$ its division ring of fractions. Denote by $f_m : R^{(m)} \rightarrow A^{(m)}$ the canonical homomorphism and, for each $i \in \llbracket 1, \dots, t \rrbracket$, set $x_i^{(m)} = f_m(X_i^{(m)})$ the canonical image of $X_i^{(m)}$ in $A^{(m)}$. If h is an element of the group H , we have $h(\mathcal{P}^{(m)}) = \mathcal{P}^{(m)}$ and, consequently, h induces an automorphism of the algebra $A^{(m)}$, denoted $\bar{h}$, which satisfies $\bar{h} \circ f_m = f_m \circ h$. This automorphism can be extended to an automorphism, still denoted $\bar{h}$ of the division ring D_m .

4.1. A necessary and sufficient condition for $\mathcal{P}^{(m)}$ to be in $\text{Im}(\phi_m)$. Assume that $m \leq t$ and recall (section 3.2) that $R^{(m)}$ is the k -algebra generated by the variables $X_1^{(m)}, \dots, X_t^{(m)}$ subject to the relations (12):

$$X_j^{(m)} X_i^{(m)} - \lambda_{j,i} X_i^{(m)} X_j^{(m)} = P_{j,i}^{(m)}(X_{i+1}^{(m)}, \dots, X_{j-1}^{(m)})$$

for $i < j$, and with $P_{j,i}^{(m)}(X_{i+1}^{(m)}, \dots, X_{j-1}^{(m)}) \in k\langle X_{i+1}^{(m)}, \dots, X_{j-1}^{(m)} \rangle$.

Proposition 4.1.1. $\mathcal{P}^{(m)} \in \text{Im}(\phi_m)$ if and only if one of two following conditions is satisfied:

- a) $X_m^{(m)} \notin \mathcal{P}^{(m)}$.
- b) $X_m^{(m)} \in \mathcal{P}^{(m)}$ and $\Theta^{(m)}(\delta_m^{(m+1)}(X_i^{(m+1)})) \in \mathcal{P}^{(m)}$ for $1 \leq i \leq m-1$.

Proof. Assume that $\mathcal{P}^{(m)} \in \text{Im}(\phi_m)$, so that $\mathcal{P}^{(m)} = \phi_m(\mathcal{P}^{(m+1)})$ with $\mathcal{P}^{(m+1)} \in \text{Spec}(R^{(m+1)})$, and assume that condition a) is not satisfied. This implies that $\mathcal{P}^{(m)} = \ker(g)$ where $g : R^{(m)} \rightarrow R^{(m+1)}/\mathcal{P}^{(m+1)}$ is the homomorphism which transforms each $X_i^{(m)}$ into $x_i^{(m+1)} = X_i^{(m+1)} + \mathcal{P}^{(m+1)}$. Consider $1 \leq i \leq m-1$.

Recall that $\delta_m^{(m+1)}(X_i^{(m+1)}) = P_{m,i}^{(m+1)}(X_{i+1}^{(m+1)}, \dots, X_{m-1}^{(m+1)})$ and that $\Theta^{(m)} : k\langle X_1^{(m+1)}, \dots, X_{m-1}^{(m+1)} \rangle \rightarrow k\langle X_1^{(m)}, \dots, X_{m-1}^{(m)} \rangle$ is the homomorphism which transforms each $X_l^{(m+1)}$ into $X_l^{(m)}$. Since $X_m^{(m)} \in \mathcal{P}^{(m)}$, we have $X_m^{(m+1)} \in \mathcal{P}^{(m+1)}$ ([4], proposition 4.3.1.) and so, $\delta_m^{(m+1)}(X_i^{(m+1)}) \in \mathcal{P}^{(m+1)}$. Now, we have

$$\begin{aligned} g(\Theta^{(m)}(\delta_m^{(m+1)}(X_i^{(m+1)}))) &= g(\Theta^{(m)}(P_{m,i}^{(m+1)}(X_{i+1}^{(m+1)}, \dots, X_{m-1}^{(m+1)}))) \\ &= g(P_{m,i}^{(m+1)}(X_{i+1}^{(m)}, \dots, X_{m-1}^{(m)})) = P_{m,i}^{(m+1)}(x_{i+1}^{(m+1)}, \dots, x_{m-1}^{(m+1)}) \\ &= P_{m,i}^{(m+1)}(X_{i+1}^{(m+1)}, \dots, X_{m-1}^{(m+1)}) + \mathcal{P}^{(m+1)} = 0. \end{aligned}$$

This implies that $\Theta^{(m)}(\delta_m^{(m+1)}(X_i^{(m+1)})) \in \ker(g) = \mathcal{P}^{(m)}$. If condition a) is satisfied, then $\mathcal{P}^{(m)} \in \text{Im}(\phi_m)$ by ([4], lemma 4.3.1). Assume that condition b) is satisfied. So, if $1 \leq i \leq m-1$, we have, as previously, $P_{m,i}^{(m+1)}(X_{i+1}^{(m)}, \dots, X_{m-1}^{(m)}) = \Theta^{(m)}(\delta_m^{(m+1)}(X_i^{(m+1)})) \in \mathcal{P}^{(m)}$. So, in $A^{(m)} = R^{(m)}/\mathcal{P}^{(m)}$, we have

$$P_{m,i}^{(m+1)}(x_{i+1}^{(m)}, \dots, x_{m-1}^{(m)}) = 0.$$

Since $P_{m,i}^{(m)} = 0$ (see section 3.2), we can write

$$x_m^{(m)} x_i^{(m)} - \lambda_{m,i} x_i^{(m)} x_m^{(m)} = P_{m,i}^{(m)}(x_{i+1}^{(m)}, \dots, x_{m-1}^{(m)}) = 0 = P_{m,i}^{(m+1)}(x_{i+1}^{(m)}, \dots, x_{m-1}^{(m)}).$$

If $1 \leq i \leq j-1$ with $j \neq m$, we have (see section 3.2)

$$x_j^{(m)} x_i^{(m)} - \lambda_{j,i} x_i^{(m)} x_j^{(m)} = P_{j,i}^{(m)}(x_{i+1}^{(m)}, \dots, x_{j-1}^{(m)}) = P_{j,i}^{(m+1)}(x_{i+1}^{(m)}, \dots, x_{j-1}^{(m)}).$$

So, by the universal property of algebras defined by generators and relations, there exists a (unique) homomorphism $\epsilon : R^{(m+1)} \rightarrow A^{(m)}$ which transforms each $X_l^{(m+1)}$ into $x_l^{(m)}$. This homomorphism is surjective, and its kernel $\ker(\epsilon) = \mathcal{P}^{(m+1)}$ is a prime ideal of $R^{(m+1)}$. We observe that, since $X_m^{(m)} \in \mathcal{P}^{(m)}$, we have $X_m^{(m+1)} \in \mathcal{P}^{(m+1)}$, and that ϵ induces an isomorphism

$$\bar{\epsilon} : R^{(m+1)}/\mathcal{P}^{(m+1)} \rightarrow A^{(m)} = R^{(m)}/\mathcal{P}^{(m)}$$

which transforms each $x_l^{(m+1)}$ into $x_l^{(m)}$. Recall that $f_m : R^{(m)} \rightarrow R^{(m)}/\mathcal{P}^{(m)}$ denotes the canonical homomorphism.

So, $g = (\bar{\epsilon})^{-1} \circ f_m : R^{(m)} \rightarrow R^{(m+1)}/\mathcal{P}^{(m+1)}$ is the homomorphism which transforms each $X_l^{(m)}$ into $x_l^{(m+1)}$. As $\ker(g) = \ker(f_m) = \mathcal{P}^{(m)}$, we conclude that $\mathcal{P}^{(m)} = \phi_m(\mathcal{P}^{(m+1)})$. $\square$

Corollary 4.1.1. *Assume $2 \leq m \leq t+1$ and consider some $\mathcal{Q} \in \text{Spec}(R^{(m)})$.*

Assume that $X_m^{(m)} \notin \mathcal{Q}$, ..., $X_t^{(m)} \notin \mathcal{Q}$.

Then there exists $\mathcal{P} \in \text{Spec}(R)$ such that $\mathcal{Q}$ is the canonical image of $\mathcal{P}$ in $\text{Spec}(R^{(m)})$. Moreover, if $\mathcal{P}^{(2)}$ is the canonical image of $\mathcal{P}$ in $\text{Spec}(R^{(2)})$, then

$$\mathcal{P}^{(2)} \cap \{Z_m, \dots, Z_t\} = \emptyset.$$

Proof. We prove this by decreasing induction on m .

- If $m = t+1$, we have $\mathcal{Q} \in \text{Spec}(R)$ and $\mathcal{P} = \mathcal{Q}$ satisfies the required properties.

- Assume $2 \leq m \leq t$. By proposition 4.1.1, we have $\mathcal{Q} = \phi_m(\mathcal{Q}')$ with $\mathcal{Q}' \in \text{Spec}(R^{(m+1)})$. Since $X_m^{(m)} \notin \mathcal{Q}$, we have (see section 3.3) $X_m^{(m+1)} \notin \mathcal{Q}'$ and, if we set $S_m = \{(X_m^{(m+1)})^d | d \in \mathbb{N}\}$, then

$$\mathcal{Q} = R^{(m)} \cap \mathcal{Q}' S_m^{-1}.$$

Assume $X_j^{(m+1)} \in \mathcal{Q}'$ with $j \geq m$. Since $X_j^{(m+1)} = X_j^{(m)}$ (see section 3.2), we have $X_j^{(m)} \in R^{(m)} \cap \mathcal{Q}' \subseteq R^{(m)} \cap \mathcal{Q}' S_m^{-1} = \mathcal{Q}$, which is false. So, we have:

- $X_m^{(m+1)} \notin \mathcal{Q}'$.
- $X_{m+1}^{(m+1)} \notin \mathcal{Q}', \dots, X_t^{(m+1)} \notin \mathcal{Q}'$ and, by the induction assumption, there exists $\mathcal{P} \in \text{Spec}(R)$ such that $\mathcal{Q}'$ is the canonical image of $\mathcal{P}$ in $\text{Spec}(R^{(m+1)})$. This implies that $\mathcal{Q} = \phi_m(\mathcal{Q}')$ is the canonical image of $\mathcal{P}$ in $\text{Spec}(R^{(m)})$. Moreover, again by the induction assumption, we have

$$\mathcal{P}^{(2)} \cap \{Z_{m+1}, \dots, Z_t\} = \emptyset.$$

As $X_m^{(m+1)} \notin \mathcal{Q}'$, we know ([4], proposition 5.2.1) that $Z_m \notin \mathcal{P}^{(2)}$. □

4.2. Some properties of $A^{(m)}$. For each integer $j \in \llbracket 2, \dots, m-1 \rrbracket$, we set $\mathcal{P}^{(j)} = \phi_j \circ \dots \circ \phi_{m-1}(\mathcal{P}^{(m)})$. Let us, first, construct in $A^{(m)}$, a new version of the deleting derivations algorithm.

Proposition 4.2.1. *For each integer $j \in \llbracket 2, \dots, m-1 \rrbracket$, there exists a sequence $(x_1^{(j)}, \dots, x_t^{(j)})$ in $D_m = \text{Frac}(A^{(m)})$ and a subalgebra $A^{(j)} = k\langle x_1^{(j)}, \dots, x_t^{(j)} \rangle$ of D_m which satisfy the following properties:*

- (1) *If l and i are in $\llbracket 1, \dots, t \rrbracket$, we have $\overline{h_l}(x_i^{(j)}) = \lambda_{l,i} x_i^{(j)}$, $\overline{h_l}(A^{(j)}) = A^{(j)}$ and $\overline{h_l}$ induces, by restriction, an automorphism (still denoted $\overline{h_l}$) of $A^{(j)}$.*
- (2) *Choose $l \in \llbracket 1, \dots, t \rrbracket$ and denote by $d_l^{(j)}$ the left $\overline{h_l}$ -derivation of $A^{(j)}$ associated to $x_l^{(j)}$ ($d_l^{(j)}(a) = x_l^{(j)} a - \overline{h_l}(a) x_l^{(j)}$ for all $a \in A^{(j)}$). Then $d_l^{(j)}(k\langle x_1^{(j)}, \dots, x_{l-1}^{(j)} \rangle) \subset k\langle x_1^{(j)}, \dots, x_{l-1}^{(j)} \rangle$ and $d_l^{(j)}$ is locally nilpotent on $k\langle x_1^{(j)}, \dots, x_{l-1}^{(j)} \rangle$.*
- (3) *There exists a unique homomorphism $f_j : R^{(j)} \rightarrow A^{(j)}$ which transforms each $X_i^{(j)}$ into $x_i^{(j)}$. f_j is surjective and $\ker(f_j) = \mathcal{P}^{(j)}$.*
- (4) *If $l \in \llbracket 1, \dots, t \rrbracket$, we have $d_l^{(j)} \circ f_j = f_j \circ \delta_l^{(j)}$ and $\overline{h_l} \circ f_j = f_j \circ h_l$.*
- (5) *If $l, e \in \llbracket 1, \dots, t \rrbracket$, we have $\overline{h_l} \circ \overline{h_e} = \overline{h_e} \circ \overline{h_l}$ and $\overline{h_l} \circ d_e^{(j)} = \lambda_{l,e} d_e^{(j)} \circ \overline{h_l}$.*
- (6) *If $x_j^{(j+1)} = 0$, then $x_i^{(j)} = x_i^{(j+1)}$ for each $i \in \llbracket 1, \dots, t \rrbracket$.*
- (7) *If $x_j^{(j+1)} \neq 0$, and $i \in \llbracket 1, \dots, t \rrbracket$, then*
 - (a) $j \leq i \Rightarrow x_i^{(j)} = x_i^{(j+1)}$. *In particular, we have $x_j^{(j)} = x_j^{(j+1)}$.*

(b) $i < j \Rightarrow x_i^{(j)} = x_i^{(j+1)} + \sum_{d=1}^{+\infty} c_d^{(j+1)} (x_j^{(j+1)})^{-d}$ with

$$c_d^{(j+1)} = \frac{(1 - q_j)^{-d}}{[d]!_{q_j}} \lambda_{j,i}^{-d} (d_j^{(j+1)})^d (x_i^{(j+1)}).$$

(We observe that, since $k\langle x_1^{(j+1)}, \dots, x_{j-1}^{(j+1)} \rangle$ is $d_j^{(j+1)}$ -stable, we have $c_d^{(j+1)} \in k\langle x_1^{(j+1)}, \dots, x_{j-1}^{(j+1)} \rangle$ and, since $d_j^{(j+1)}$ is locally nilpotent on $k\langle x_1^{(j+1)}, \dots, x_{j-1}^{(j+1)} \rangle$, only finitely many $c_d^{(j+1)}$ are nonzero.)

Proof. The existence of variables $x_i^{(j)}$ ($i \in \llbracket 1, \dots, t \rrbracket$, $j \in \llbracket 2, \dots, m-1 \rrbracket$) and algebras $A^{(j)} = k\langle x_1^{(j)}, \dots, x_t^{(j)} \rangle$ satisfying conditions (3), (6) and (7) is a consequence of [4, Propositions 5.4.1, 5.4.2], replacing R by $R^{(m)}$ and $\mathcal{P}$ by $\mathcal{P}^{(m)}$.

Using (3), we observe that, if $\overline{h}_l \circ f_j = f_j \circ h_l$ (as a homomorphism from $R^{(j)}$ towards $D^{(m)}$) holds for any l , then properties (1), (2), (4) and (5) are all satisfied. In fact,

1. For all l and i in $\llbracket 1, \dots, t \rrbracket$, we have $\overline{h}_l(x_i^{(j)}) = \overline{h}_l \circ f_j(X_i^{(j)}) = f_j \circ h_l(X_i^{(j)}) = \lambda_{l,i} f_j(X_i^{(j)}) = \lambda_{l,i} x_i^{(j)}$. This immediately implies that $\overline{h}_l(A^{(j)}) = A^{(j)}$ and $\overline{h}_l$ induces, by restriction, an automorphism of $A^{(j)}$.
4. If $P \in R^{(j)}$, we have $d_l^{(j)}(f_j(P)) = x_l^{(j)} f_j(P) - \overline{h}_l(f_j(P)) x_l^{(j)} = f_j(X_l^{(j)} P - h_l(P) X_l^{(j)}) = f_j \circ \delta_l^{(j)}(P)$. This proves that $d_l^{(j)} \circ f_j = f_j \circ \delta_l^{(j)}$.
2. By (4), we have $d_l^{(j)}(k\langle x_1^{(j)}, \dots, x_{l-1}^{(j)} \rangle) = d_l^{(j)}(f_j(k\langle X_1^{(j)}, \dots, X_{l-1}^{(j)} \rangle)) = f_j(\delta_l^{(j)}(k\langle X_1^{(j)}, \dots, X_{l-1}^{(j)} \rangle)) \subset f_j(k\langle X_1^{(j)}, \dots, X_{l-1}^{(j)} \rangle) = k\langle x_1^{(j)}, \dots, x_{l-1}^{(j)} \rangle$. Moreover, if $a \in k\langle x_1^{(j)}, \dots, x_{l-1}^{(j)} \rangle$, we can write $a = f_j(P)$ with $P \in k\langle X_1^{(j)}, \dots, X_{l-1}^{(j)} \rangle$. Since $\delta_l^{(j)}$ is locally nilpotent on $k\langle X_1^{(j)}, \dots, X_{l-1}^{(j)} \rangle$, we have $(\delta_l^{(j)})^r(P) = 0$ for some $r \in \mathbb{N}$, and $(d_l^{(j)})^r(a) = (d_l^{(j)})^r \circ f_j(P) = f_j \circ (\delta_l^{(j)})^r(P) = 0$. So, $d_l^{(j)}$ is locally nilpotent on $k\langle x_1^{(j)}, \dots, x_{l-1}^{(j)} \rangle$.
5. If i in $\llbracket 1, \dots, t \rrbracket$, we have $\overline{h}_l \circ \overline{h}_e(x_i^{(j)}) = \lambda_{l,i} \lambda_{e,i} x_i^{(j)} = \overline{h}_e \circ \overline{h}_l(x_i^{(j)})$ and, since $A^{(j)}$ is generated by the elements $x_i^{(j)}$, we have $\overline{h}_l \circ \overline{h}_e = \overline{h}_e \circ \overline{h}_l$. Consider two elements a, b in $A^{(j)}$ such that the maps $\overline{h}_l \circ d_e^{(j)}$ and $\lambda_{l,e} d_e^{(j)} \circ \overline{h}_l$ coincide on a and b . Then, we have $\overline{h}_l \circ d_e^{(j)}(ab) = \overline{h}_l(d_e^{(j)}(a)b + \overline{h}_e(a)d_e^{(j)}(b)) = \lambda_{l,e} d_e^{(j)}(\overline{h}_l(a))\overline{h}_l(b) + \lambda_{l,e} \overline{h}_e(\overline{h}_l(a))d_e^{(j)}(\overline{h}_l(b)) = \lambda_{l,e} d_e^{(j)} \circ \overline{h}_l(ab)$. So, we just have to prove that $\overline{h}_l \circ d_e^{(j)}$ and $\lambda_{l,e} d_e^{(j)} \circ \overline{h}_l$ coincide on each $x_i^{(j)}$. Now, $\overline{h}_l \circ d_e^{(j)}(x_i^{(j)}) = \overline{h}_l(x_e^{(j)} x_i^{(j)} - \lambda_{e,i} x_i^{(j)} x_e^{(j)}) = \lambda_{l,e} \lambda_{l,i} (x_e^{(j)} x_i^{(j)} - \lambda_{e,i} x_i^{(j)} x_e^{(j)}) = \lambda_{l,e} \lambda_{l,i} d_e^{(j)}(x_i^{(j)}) = \lambda_{l,e} d_e^{(j)}(\overline{h}_l(x_i^{(j)}))$. So we can conclude that $\overline{h}_l \circ d_e^{(j)} = \lambda_{l,e} d_e^{(j)} \circ \overline{h}_l$.

Now, it remains to verify that, for any l in $\llbracket 1, \dots, t \rrbracket$ and for any $j \in \llbracket 2, m \rrbracket$, we have $\overline{h}_l \circ f_j = f_j \circ h_l$. As $h_l(X_i^{(j)}) = \lambda_{l,i} x_i^{(j)}$, it is enough to check that $\overline{h}_l(x_i^{(j)}) = \lambda_{l,i} x_i^{(j)}$ for each i .

If $j = m$, the constructions of f_m and of the variables $x_i^{(m)}$ (see the beginning of section 4) imply straightaway the equality $\overline{h} \circ f_m = f_m \circ h$ for any h in H .

Assume that $j \in \llbracket 2, m-1 \rrbracket$ and $\overline{h}_l \circ f_{j+1} = f_{j+1} \circ h_l$ for any $l \in \llbracket 1, \dots, t \rrbracket$. So properties (1) to (7) hold for $j+1$.

Now, consider l and i in $\llbracket 1, \dots, t \rrbracket$.

If $j \leq i$, by property (4) at the rank $j + 1$, we have $\overline{h}_l(x_i^{(j)}) = \overline{h}_l(x_i^{(j+1)}) = \lambda_{l,i} x_i^{(j+1)} = \lambda_{l,i} x_i^{(j)}$.

If $i < j$, it follows from property (5) at the rank $j + 1$ that

$$\overline{h}_l \circ d_j^{(j+1)} = \lambda_{l,j} d_j^{(j+1)} \circ \overline{h}_l.$$

So, for each $d \in \mathbb{N}$, we have

$$\begin{aligned} \overline{h}_l(c_d^{(j+1)}) &= \overline{h}_l\left(\frac{(1-q_j)^{-d}}{[d]!_{q_j}} \lambda_{j,i}^{-d} (d_j^{(j+1)})^d (x_i^{(j+1)})\right) \\ &= \frac{(1-q_j)^{-d}}{[d]!_{q_j}} \lambda_{j,i}^{-d} \lambda_{l,j}^d (d_j^{(j+1)})^d \circ \overline{h}_l(x_i^{(j+1)}) = \lambda_{l,i} \lambda_{l,j}^d c_d^{(j+1)}. \end{aligned}$$

From this, $\overline{h}_l(c_d^{(j+1)}(x_j^{(j+1)})^{-d}) = \lambda_{l,i} c_d^{(j+1)}(x_j^{(j+1)})^{-d}$, which implies that $\overline{h}_l(x_i^{(j)}) = \lambda_{l,i} x_i^{(j)}$. $\square$

Recall (see section 3.3) that, since $\mathcal{P}^{(m)}$ is H -invariant, each $\mathcal{P}^{(j)}$ ($2 \leq j \leq m$) is H -invariant. In particular, $\mathcal{P}^{(2)}$ is H -invariant. Set

$$\Delta = \{i \in \llbracket 1, \dots, t \rrbracket \mid Z_i \in \mathcal{P}^{(2)}\} = \{j_1 < \dots < j_s\} \text{ (unless this set is empty)}$$

and

$$\overline{\Delta} = \llbracket 1, \dots, t \rrbracket \setminus \Delta = \{l_1 < \dots < l_e\} \text{ (unless this set is empty)}.$$

Recall that $\mathcal{P}^{(2)} = \mathcal{P}_{\Delta}^{(2)}$ (see section 3.3), $\overline{R} = R^{(2)}$ and set $\overline{A} = A^{(2)}$. For each $i \in \llbracket 1, \dots, t \rrbracket$, set $z_i = x_i^{(2)}$ and observe that $f_2 : \overline{R} \rightarrow \overline{A}$ transforms each $Z_i = X_i^{(2)}$ into z_i . So, we can also describe Δ and $\overline{\Delta}$ as follows:

$$\begin{aligned} \Delta &= \{i \in \llbracket 1, \dots, t \rrbracket \mid z_i = 0\}, \\ \overline{\Delta} &= \{i \in \llbracket 1, \dots, t \rrbracket \mid z_i \neq 0\}. \end{aligned}$$

Observation 4.2.1. Assume that $\overline{\Delta}$ is empty. Then each $x_i^{(m)}$ ($i \in \llbracket 1, \dots, t \rrbracket$) is zero.

Proof. If $\overline{\Delta}$ is empty, we have $x_j^{(j+1)} = z_j = 0$ for each $j \in \llbracket 1, \dots, m-1 \rrbracket$. By (proposition 4.2.1, (6)), this implies that, if $i \in \llbracket 1, \dots, t \rrbracket$ and $j \in \llbracket 1, \dots, m-1 \rrbracket$, we have $x_i^{(j+1)} = x_i^{(j)}$. So, each $x_i^{(m)} = x_i^{(2)} = z_i = 0$. $\square$

Until the end of section 4.2, we assume that $\overline{\Delta}$ is nonempty. So, we have:

Proposition 4.2.2. (1) $\overline{A} = k\langle z_{l_1}, \dots, z_{l_e} \rangle$.

(2) If $1 \leq i < d \leq e$, we have $z_{l_d} z_{l_i} = \lambda_{l_d, l_i} z_{l_i} z_{l_d}$.

(3) $z_{l_1}, \dots, z_{l_e}$ are all nonzero and the Laurent (ordered) monomials $z^{\underline{a}} = z_{l_1}^{a_1} \dots z_{l_e}^{a_e}$ ($\underline{a} = (a_1, \dots, a_e) \in \mathbb{Z}^e$) are k -linearly independent.

Proof. (1) $\overline{A} = k\langle z_1, \dots, z_t \rangle$ (see proposition 4.2. 1) and, for $i \notin \{l_1, \dots, l_e\}$, we have $i \in \Delta \Rightarrow Z_i \in \mathcal{P}^{(2)} = \ker(f_2) \Rightarrow z_i = 0$.

(2) We know (section 3.2) that $Z_{l_d} Z_{l_i} = \lambda_{l_d, l_i} Z_{l_i} Z_{l_d}$. If we transform by f_2 , we obtain the required equality.

(3) Denote by $\widehat{R} = k\langle Z_{l_1}, \dots, Z_{l_e} \rangle$ the subalgebra of $\overline{R}$ generated by $Z_{l_1}, \dots, Z_{l_e}$. By the property (1), f_2 induces (by restriction) a surjective homomorphism $\widehat{f}_2 : \widehat{R} \rightarrow \overline{A}$ and $\ker(\widehat{f}_2) = \widehat{R} \cap \mathcal{P}^{(2)}$. As each $P \in \mathcal{P}^{(2)} = \mathcal{P}_{\Delta}^{(2)} = (\{Z_{j_1}, \dots, Z_{j_s}\})$ is a linear combination of monomials in which at least one

of the variables Z_{j_i} with $1 \leq i \leq s$ appears, this intersection is reduced to zero and then, $\widehat{f_2}$ is an isomorphism which transforms each Z_{l_i} into z_{l_i} . As the ordered monomials in $Z_{l_1}, \dots, Z_{l_e}$ are linearly independent, we have the same property for the ordered monomials in $z_{l_1}, \dots, z_{l_e}$. This easily implies that $z_{l_1}, \dots, z_{l_e}$ are nonzero and that the Laurent ordered monomials in $z_{l_1}, \dots, z_{l_e}$ are also linearly independent. $\square$

4.3. Each $x_i^{(m)}$ is a Laurent polynomial in $z_{l_1}, \dots, z_{l_e}$. The conventions are the same as in section 4.2 and we still assume that $\overline{\Delta}$ is nonempty.

Let us consider some i in $\llbracket 1, \dots, t \rrbracket$.

If $u \in D_m$ and if $\gamma \in \mathbb{Z}\Pi$, we say (as in section 3.2) that u is homogeneous of degree γ if $\overline{h_\rho}(u) = q^{-(\rho, \gamma)}u$ for all ρ in $\mathbb{Z}\Pi$. Since q is not a root of unity and V is spanned by Π , the degree of a nonzero homogeneous element is uniquely defined. Moreover, we immediately have the following properties:

- If $u_1, \dots, u_r$ are homogeneous of same degree γ , then any linear combination of $u_1, \dots, u_r$ (with coefficients in k) is homogeneous of degree γ .
- If u_1 is homogeneous of degree γ_1 and u_2 is homogeneous of degree γ_2 , then $u_1 u_2$ is homogeneous of degree $\gamma_1 + \gamma_2$.
- Likewise, if u is nonzero and homogeneous of degree γ , then u^{-1} is homogeneous of degree $-\gamma$.
- So, if $u_1, \dots, u_r$ are nonzero and homogeneous of degrees $\gamma_1, \dots, \gamma_r$, respectively, if $\underline{a} = (a_1, \dots, a_r) \in \mathbb{Z}^r$, then $u^{\underline{a}} := u_1^{a_1} \dots u_r^{a_r}$ is homogeneous of degree $a_1 \gamma_1 + \dots + a_r \gamma_r$.

Lemma 4.3.1. *Consider some j in $\llbracket 2, \dots, m \rrbracket$.*

- (1) *If $j \leq i + 1$, then $x_i^{(j)} = z_i$. In particular, $x_i^{(i+1)} = z_i$.*
- (2) *If $1 \leq i < l \leq t$, then $x_l^{(j)} x_i^{(j)} - \lambda_{l,i} x_i^{(j)} x_l^{(j)} = d_l^{(j)}(x_i^{(j)}) = f_j(P_{l,i}^{(j)}) \in k\langle x_{i+1}^{(j)}, \dots, x_{l-1}^{(j)} \rangle$. Moreover, if $l \geq j$ or $l = i + 1$, then $d_l^{(j)}(x_i^{(j)}) = 0$.*
- (3) *If $U \in R^{(j)}$ is homogeneous of degree γ , then $f_j(U)$ is homogeneous of same degree γ . In particular, $x_i^{(j)}$ is homogeneous of degree β_i .*
- (4) *If $j \leq l \leq t$ and $u \in k\langle x_1^{(j)}, \dots, x_{l-1}^{(j)} \rangle$, then $z_l u = \overline{h_l}(u) z_l$.*

Proof. (1) By proposition 4.2.1, we have $x_i^{(i+1)} = x_i^{(i)} = \dots = x_i^{(2)} = z_i$.

(2) By proposition 4.2.1, we have $x_l^{(j)} x_i^{(j)} - \lambda_{l,i} x_i^{(j)} x_l^{(j)} = x_l^{(j)} x_i^{(j)} - \overline{h_l}(x_i^{(j)}) x_l^{(j)} = d_l^{(j)}(x_i^{(j)}) = d_l^{(j)} \circ f_j(X_i^{(j)}) = f_j \circ \delta_l^{(j)}(X_i^{(j)})$ and we know (see section 3.2) that $\delta_l^{(j)}(X_i^{(j)}) = P_{l,i}^{(j)} \in k\langle X_{i+1}^{(j)}, \dots, X_{l-1}^{(j)} \rangle$. Moreover, if $l \geq j$ or $l = i + 1$, we know (see section 3.2) that $P_{l,i}^{(j)} = 0$. Transforming by f_j , we get the required result.

(3) Consider ρ in $\mathbb{Z}\Pi$. By proposition 4.2.1, we have $\overline{h_\rho}(f_j(U)) = f_j(h_\rho(U)) = q^{-(\rho, \gamma)} f_j(U)$.

(4) By (1), we have $z_l = x_l^{(j)}$. So, $z_l u - \overline{h_l}(u) z_l = x_l^{(j)} u - \overline{h_l}(u) x_l^{(j)} = d_l^{(j)}(u)$. By (2), if $i < l$, we have $d_l^{(j)}(x_i^{(j)}) = 0$ (since $j \leq l$). This implies that $d_l^{(j)}$ is zero on $k\langle x_1^{(j)}, \dots, x_{l-1}^{(j)} \rangle$ and, in particular, $d_l^{(j)}(u) = 0$. $\square$

Lemma 4.3.2. *Consider some j in $\llbracket 2, \dots, m \rrbracket$.*

- (1) *If $j \leq l_1$, then $x_i^{(j)} = z_i$.*

- (2) Assume that $l_1 < j$ and denote by d the greatest integer such that $l_d < j$ ($1 \leq d \leq p$). Then $x_i^{(j)} = x_i^{(l_d+1)}$.

Proof.

2. We prove this by induction on j . If $j = l_d + 1$, there is nothing to prove. Assume $j > l_d + 1$ and set $j' = j - 1 > l_d$. By lemma 4.3.1, we have $x_{j'}^{(j'+1)} = z_{j'} = 0$ (since $j' \in \Delta$). So, by proposition 4.2.1, $x_i^{(j)} = x_i^{(j'+1)} = x_i^{(j')} = x_i^{(l_d+1)}$ (by the induction assumption).
1. The proof is the same (observing that there is nothing to prove if $j = 2$). $\square$

Now, assume that $l_1 < m$ and denote by p the greatest integer such that $l_p < m$.

Proposition 4.3.1. Consider some j in $\llbracket 2, \dots, m \rrbracket$ and assume that $i < j$.

- (1) If $i < j \leq l_1$, then $x_i^{(j)} = z_i = 0$. Assume that $l_1 < j$ and denote by d the greatest integer such that $l_d < j$ ($1 \leq d \leq p$).
- (2) (a) If $l_d \leq i < j$, then $x_i^{(j)} = z_i$.
 (b) If $i < l_d < j$, then

$$x_i^{(j)} = x_i^{(l_d)} + Q_1 z_{l_d}^{-1} + \dots + Q_K z_{l_d}^{-K}$$

with $K \geq 1$ and:

- If $d = 1$, then each $Q_l \in k$.
- If $d > 1$ and $i \geq l_{d-1}$, then each $Q_l \in k$.
- If $d > 1$ and $i < l_{d-1}$, then each $Q_l \in k\langle x_{i+1}^{(l_d)}, \dots, x_{l_{d-1}}^{(l_d)} \rangle$.

Proof. (1) By (lemma 4.3.2, (1)), we have $x_i^{(j)} = z_i$ and, since $i < l_1$, we have $i \in \Delta$, so that $z_i = 0$.

- (2) (a) By (lemma 4.3.2, (2)), we have $x_i^{(j)} = x_i^{(l_d+1)}$ and, since $l_d + 1 \leq i$, this is also equal to z_i .
 (b) We prove this result by decreasing induction on i .
 As in (2)(a), we have $x_i^{(j)} = x_i^{(l_d+1)}$ and, since $x_{l_d}^{(l_d+1)} = z_{l_d} \neq 0$, it follows from (proposition 4.2.1, (7(b))) that

$$x_i^{(j)} = x_i^{(l_d)} + P_1 z_{l_d}^{-1} + \dots + P_L z_{l_d}^{-L}$$

with $L \geq 1$ and each $P_l = \mu_l (d_{l_d}^{(l_d+1)})^l (x_i^{(l_d+1)})$ ($\mu_l \in k^*$). So, by lemma 4.3.1, each $P_l \in k\langle x_{i+1}^{(l_d+1)}, \dots, x_{l_{d-1}}^{(l_d+1)} \rangle = k\langle x_{i+1}^{(j)}, \dots, x_{l_{d-1}}^{(j)} \rangle$ (by lemma 4.3.2).

Assume that $i = l_d - 1$. Then, by lemma 4.3.1, we have $d_{l_d}^{(l_d+1)}(x_i^{(l_d+1)}) = 0$. This implies that $P_l = 0$ for each l . So, $x_i^{(j)} = x_i^{(l_d)}$ and the proof is over.

Now, assume that $i < l_d - 1$ and, for $i + 1 \leq h \leq l_d - 1$,

$$x_h^{(j)} = x_h^{(l_d)} + Q_{1,h} z_{l_d}^{-1} + \dots + Q_{M,h} z_{l_d}^{-M}$$

with $M \geq 1$, and:

- each $Q_{l,h} \in k$ if $d = 1$,
- $Q_{l,h} \in k$ if $d > 1$ and $h \geq l_{d-1}$,
- $Q_{l,h} \in k\langle x_{h+1}^{(l_d)}, \dots, x_{l_{d-1}}^{(l_d)} \rangle$ if $d > 1$ and $h < l_{d-1}$.

This implies the following results:

- If $d = 1$, then each $x_h^{(j)} \in k\langle x_h^{(l_d)}, z_{l_d}^{-1} \rangle$ ($i + 1 \leq h \leq l_d - 1$).
- If $d > 1$ and $l_{d-1} \leq h \leq l_d - 1$, then $x_h^{(j)} \in k\langle x_h^{(l_d)}, z_{l_d}^{-1} \rangle$.
- If $d > 1$ and $i + 1 \leq h < l_{d-1}$, then $x_h^{(j)} \in k\langle x_h^{(l_d)}, x_{h+1}^{(l_d)}, \dots, x_{l_{d-1}}^{(l_d)}, z_{l_d}^{-1} \rangle$.

So, it turns out that each $P_l \in k\langle x_{i+1}^{(l_d)}, \dots, x_{l_{d-1}}^{(l_d)}, z_{l_d}^{-1} \rangle$.

- If $d = 1$ and $h < l_d$, we have (by the first point of the proposition) $x_h^{(l_d)} = z_h = 0$. So, in this case, each $P_l \in k\langle z_{l_d}^{-1} \rangle$ and

$$x_i^{(j)} = x_i^{(l_d)} + Q_1 z_{l_d}^{-1} + \dots + Q_K z_{l_d}^{-K}$$

with $K \geq 1$ and each Q_l in k .

- Now, assume that $d > 1$.

If $l_{d-1} < h \leq l_d - 1$, we have (by (2)(a)) $x_h^{(l_d)} = z_h = 0$ (since $h \in \Delta$). So,

- If $i \geq l_{d-1}$, each $P_l \in k\langle z_{l_d}^{-1} \rangle$ and

$$x_i^{(j)} = x_i^{(l_d)} + Q_1 z_{l_d}^{-1} + \dots + Q_K z_{l_d}^{-K}$$

with $K \geq 1$ and each Q_l in k .

- If $i < l_{d-1}$, each $P_l \in k\langle x_{i+1}^{(l_d)}, \dots, x_{l_{d-1}}^{(l_d)}, z_{l_d}^{-1} \rangle$. As, by lemma 4.3.1, $z_{l_d} = x_{l_d}^{(l_d)}$ q -commutes with $x_{i+1}^{(l_d)}, \dots, x_{l_{d-1}}^{(l_d)}$, each P_l can be written as follows:

$$P_l = S_{0,l} + S_{1,l} z_{l_d}^{-1} + \dots + S_{E,l} z_{l_d}^{-E}$$

with $E \geq 1$ and each $S_{j,l} \in k\langle x_{i+1}^{(l_d)}, \dots, x_{l_{d-1}}^{(l_d)} \rangle$. So, we conclude that, in this case,

$$x_i^{(j)} = x_i^{(l_d)} + Q_1 z_{l_d}^{-1} + \dots + Q_K z_{l_d}^{-K}$$

with $K \geq 1$ and each Q_l in $k\langle x_{i+1}^{(l_d)}, \dots, x_{l_{d-1}}^{(l_d)} \rangle$. □

Proposition 4.3.2. *Consider some j in $\llbracket 2, \dots, m \rrbracket$, assume that $i + 1 < j$ and consider some $u \in k\langle x_{i+1}^{(j)}, \dots, x_{j-1}^{(j)} \rangle$.*

- (1) *If $j \leq l_1$, then $u \in k$.*

Assume that $l_1 < j$ and denote by d the greatest integer such that $l_d < j$ ($1 \leq d \leq p$).

- (2) (a) *If $l_d \leq i$, then $u \in k$.*
 (b) *If $i < l_d$, then*

$$u = u_1 z_{l_d}^{a_1} + \dots + u_K z_{l_d}^{a_K}$$

with $K \geq 1$, $(a_1, \dots, a_K) \in \mathbb{Z}^K$ and:

- *If $d = 1$, then each $u_l \in k$.*
- *If $d > 1$ and $i \geq l_{d-1}$, then each $u_l \in k$.*
- *If $d > 1$ and $i < l_{d-1}$, then each $u_l \in k\langle x_{i+1}^{(l_d)}, \dots, x_{l_{d-1}}^{(l_d)} \rangle$.*

Proof. (1) By (proposition 4.3.1, (1)) we have $x_h^{(j)} = 0$ for any $h < j$. So, $u \in k$.

- (2) (a) Consider some integer h such that $l_d < h < j$. By (proposition 4.3.1, (2)(a)) we have $x_h^{(j)} = z_h = 0$ (since $h \in \Delta$). So, $u \in k$.

- (b) Consider some integer h with $i < h < j$.
 If $l_d < h$, then, as in (2)(a), we have $x_h^{(j)} = 0$. If $h = l_d$, then, again by (proposition 4.3.1, (2)(a)), we have $x_h^{(j)} = z_h = z_{l_d}$.
 • Assume that $d = 1$. If $h < l_d$, then, by (proposition 4.3.1, (2)(b)), we have

$$x_h^{(j)} = x_h^{(l_d)} + Q_1 z_{l_d}^{-1} + \cdots + Q_M z_{l_d}^{-M}$$

with $M \geq 1$ and each Q_l in k . By (proposition 4.3.1, (1)), we also have $x_h^{(l_d)} = 0$. So, $x_h^{(j)} \in k\langle z_{l_d}^{-1} \rangle$ and it turns out that $u \in k\langle z_{l_d}^{\pm 1} \rangle$. This implies that

$$u = u_1 z_{l_d}^{a_1} + \cdots + u_K z_{l_d}^{a_K}$$

with $N \geq 1$, $(a_1, \dots, a_N) \in \mathbb{Z}^N$ and each $u_l \in k$.

- Assume that $d > 1$ and $i \geq l_{d-1}$. If $i < h < l_d$, we have $l_{d-1} < h < l_d$ and, by (proposition 4.3.1, (2)(b)),

$$x_h^{(j)} = x_h^{(l_d)} + Q_1 z_{l_d}^{-1} + \cdots + Q_M z_{l_d}^{-M}$$

with $M \geq 1$ and each $Q_l \in k$. By (proposition 4.3.1, (1)) we also have $x_h^{(l_d)} = z_h$ and, since $h \in \Delta$, $x_h^{(l_d)} = 0$. As above, we conclude that

$$u = u_1 z_{l_d}^{a_1} + \cdots + u_K z_{l_d}^{a_K}$$

with $K \geq 1$, $(a_1, \dots, a_K) \in \mathbb{Z}^K$ and each $u_l \in k$.

- Assume that $d > 1$ and $i < l_{d-1}$. If $i < h < l_d$, we have, always by (proposition 4.3.1, (2)(b)),

$$x_h^{(j)} = x_h^{(l_d)} + Q_1 z_{l_d}^{-1} + \cdots + Q_M z_{l_d}^{-M}$$

with $M \geq 1$ and:

- If $l_{d-1} < h$, then each $Q_l \in k$ and, as in the previous case, $x_h^{(l_d)} = 0$. So, $x_h^{(j)} \in k\langle z_{l_d}^{-1} \rangle$.
- If $h = l_{d-1}$, then each $Q_l \in k$ and, consequently, $x_h^{(j)} \in k\langle x_h^{(l_d)}, z_{l_d}^{-1} \rangle = k\langle x_{l_{d-1}}^{(l_d)}, z_{l_d}^{-1} \rangle$.
- If $h < l_{d-1}$, then each $Q_l \in \langle x_{h+1}^{(l_d)}, \dots, x_{l_{d-1}}^{(l_d)} \rangle$ and, consequently, $x_h^{(j)} \in k\langle x_h^{(l_d)}, x_{h+1}^{(l_d)}, \dots, x_{l_{d-1}}^{(l_d)}, z_{l_d}^{-1} \rangle$.

So, it turns out that $u \in k\langle x_{i+1}^{(l_d)}, \dots, x_{l_{d-1}}^{(l_d)}, z_{l_d}^{\pm 1} \rangle$. As, by lemma 4.3.1, $z_{l_d} = x_{l_d}^{(l_d)}$ q -commutes with $x_{i+1}^{(l_d)}, \dots, x_{l_{d-1}}^{(l_d)}$, we conclude that

$$u = u_1 z_{l_d}^{a_1} + \cdots + u_K z_{l_d}^{a_K}$$

with $K \geq 1$, $(a_1, \dots, a_K) \in \mathbb{Z}^K$ and each u_l in $k\langle x_{i+1}^{(l_d)}, \dots, x_{l_{d-1}}^{(l_d)} \rangle$.

□

Proposition 4.3.3. Consider some j in $\llbracket 2, \dots, m \rrbracket$ with $i < j$.

- (1) If $\{l_1, \dots, l_p\} \cap \llbracket i+1, \dots, j-1 \rrbracket = \emptyset$, then $x_i^{(j)} = z_i$.

- (2) Assume that $\{l_1, \dots, l_p\} \cap \llbracket i+1, \dots, j-1 \rrbracket \neq \emptyset$ and set $\{l_1, \dots, l_p\} \cap \llbracket i+1, \dots, j-1 \rrbracket = \{l_c < \dots < l_d\}$. Then, we have

$$x_i^{(j)} = z_i + \sum_{\underline{a}=(a_c, \dots, a_d) \in F} \eta(\underline{a}) z_{l_c}^{a_c} \dots z_{l_d}^{a_d},$$

where

- F is a finite (possibly empty) subset of $\mathbb{Z}^{d-c+1}$.
- If $\preceq$ denotes the inverse lexicographic order in $\mathbb{Z}^{d-c+1}$, then, for any $\underline{a}$ in F , we have $\underline{a} \prec 0$ (i.e. $\underline{a} = (a_c, \dots, a_r, 0, \dots, 0)$ with $a_r < 0$).
- For each $\underline{a}$ in F , we have $\eta(\underline{a}) \in k^*$.

Proof. We prove the proposition by induction on $j-i$. In order to do this, we first prove assertion (1).

So, assume that $\{l_1, \dots, l_p\} \cap \llbracket i+1, \dots, j-1 \rrbracket = \emptyset$, and observe that there are two possibilities:

- (1) (a) $j \leq l_1$. By proposition 4.3.1, we know that $x_i^{(j)} = z_i$.
- (b) $l_1 < j$ and, if d is the greatest integer such that $l_d < j$, $l_d \leq i < j$. Again, by proposition 4.3.1, we know that $x_i^{(j)} = z_i$. Assume $j-i = 1$. In this case, we have $\{l_1, \dots, l_p\} \cap \llbracket i+1, \dots, j-1 \rrbracket = \emptyset$. So, we are in the case of assertion (1), and the proof is over.

Now assume $j-i > 1$. As assertion (1) is already proved, we may assume that $\{l_1, \dots, l_p\} \cap \llbracket i+1, \dots, j-1 \rrbracket = \{l_c < \dots < l_d\}$ is nonempty. So, we have $l_1 < j$, d is the greatest integer such that $l_d < j$ and $i < l_d < j$. Now, by proposition 4.3.1, we can write

$$x_i^{(j)} = x_i^{(l_d)} + Q_1 z_{l_d}^{-1} + \dots + Q_K z_{l_d}^{-K}$$

with $K \geq 1$ and three possible cases:

- (1) $d = 1$. Then each $Q_l \in k$ and, since $x_i^{(l_d)} = z_i$ (see proposition 4.3.1, (1)), the proof is over.
- (2) $d > 1$ and $i \geq l_{d-1}$. Then each $Q_l \in k$ and, since $x_i^{(l_d)} = z_i$ (see proposition 4.3.1, (2)(a)), the proof is over.
- (3) $d > 1$ and $i < l_{d-1}$ (so that $c \leq d-1$). In this case, each $Q_l \in k\langle x_{i+1}^{(l_d)}, \dots, x_{l_{d-1}}^{(l_d)} \rangle$. Since $l_d < j$, it follows from the induction assumption that the proposition is true for each $x_h^{(l_d)}$ with $i \leq h < l_d$. This implies that

- If $i+1 \leq h \leq l_{d-1}$, we have $x_h^{(l_d)} \in k\langle z_{l_c}^{\pm 1}, \dots, z_{l_{d-1}}^{\pm 1} \rangle$ (since $\{l_1, \dots, l_p\} \cap \llbracket h+1, \dots, l_{d-1} \rrbracket \subseteq \{l_c < \dots < l_{d-1}\}$ and z_h is either 0 or equal to z_{l_r} with $c \leq r \leq d-1$). So, each $Q_l \in k\langle z_{l_c}^{\pm 1}, \dots, z_{l_{d-1}}^{\pm 1} \rangle$.
-

$$x_i^{(l_d)} = z_i + \sum_{\underline{a}=(a_c, \dots, a_{d-1}) \in F_1} \eta(\underline{a}) z_{l_c}^{a_c} \dots z_{l_{d-1}}^{a_{d-1}},$$

where

- ◊ F_1 is a finite (possibly empty) subset of $\mathbb{Z}^{d-c}$.
- ◊ For each $\underline{a}$ in F_1 , we have $\underline{a} \prec 0$ and $\eta(\underline{a}) \in k^*$.

As each Q_l is a summand of (ordered) Laurent monomials in $z_{l_c}, \dots, z_{l_{d-1}}$, it turns out that

$$x_i^{(j)} = z_i + \sum_{\underline{a}=(a_c, \dots, a_d) \in F} \eta(\underline{a}) z_{l_c}^{a_c} \dots z_{l_d}^{a_d},$$

where

- ◊ F is a finite (possibly empty) subset of $\mathbb{Z}^{d-c+1}$.
- ◊ For each $\underline{a}$ in F , we have $\underline{a} \prec 0$ and $\eta(\underline{a}) \in k^*$.

□

This proposition implies immediately

Corollary 4.3.1. *Consider some integer d with $1 < d \leq p$. If $1 \leq i < l_d$, then $x_i^{(l_d)} \in k\langle z_{l_1}^{\pm 1}, \dots, z_{l_{d-1}}^{\pm 1} \rangle$.*

Proof. By proposition 4.3.3, $x_i^{(l_d)} = z_i + Q$ with Q a (possibly 0) Laurent polynomial in $z_{l_1}, \dots, z_{l_{d-1}}$. As z_i is either 0 or z_{l_r} with $1 \leq r \leq d-1$, the proof is over. □

Corollary 4.3.2. *Consider some integer d with $1 < d \leq p$. In D_m considered as a left module over $k\langle x_1^{(l_d)}, \dots, x_{l_{d-1}}^{(l_d)} \rangle$, the ordered Laurent monomials $z_{l_d}^{a_d} \dots z_{l_p}^{a_p}$ ($a_d, \dots, a_p$ in $\mathbb{Z}$) are linearly independent.*

Proof. By corollary 4.3.1, we have $k\langle x_1^{(l_d)}, \dots, x_{l_{d-1}}^{(l_d)} \rangle \subseteq k\langle z_{l_1}^{\pm 1}, \dots, z_{l_{d-1}}^{\pm 1} \rangle$. So, the corollary results immediately from the k -linear independence of the ordered Laurent monomials in $z_{l_1}, \dots, z_{l_e}$ (see proposition 4.2.2). □

4.4. A new sufficient condition for $\mathcal{P}^{(m)}$ to be in $Im(\phi_m)$. The notations are the same as in the preceding sections, but we do not assume that $\overline{\Delta}$ is nonempty a priori.

Assume that $\mathcal{P}^{(m)}$ is not in $Im(\phi_m)$ so that, by proposition 4.1.1, $X_m^{(m)} \in \mathcal{P}^{(m)}$ and there is some j in $\llbracket 1, \dots, m-1 \rrbracket$ such that

$$U = \Theta^{(m)}(\delta_m^{(m+1)}(X_j^{(m+1)})) \notin \mathcal{P}^{(m)}.$$

Let us choose such a j maximal. So, we have

$$\Theta^{(m)}(\delta_m^{(m+1)}(X_i^{(m+1)})) \in \mathcal{P}^{(m)}$$

for any i in $\llbracket j+1, \dots, m-1 \rrbracket$ and we observe that, since $U \notin \mathcal{P}^{(m)}$, we have $u := f_m(U) \neq 0$ in $A^{(m)}$.

Recall that

$$\Delta = \{i \in \llbracket 1, \dots, t \rrbracket \mid Z_i \in \mathcal{P}^{(2)}\} = \{j_1 < \dots < j_s\} \text{ (unless this set is empty),}$$

$$\overline{\Delta} = \llbracket 1, \dots, t \rrbracket \setminus \Delta = \{l_1 < \dots < l_e\} \text{ (unless this set is empty).}$$

Lemma 4.4.1. *$\overline{\Delta}$ is nonempty, $l_1 < m$, $m \notin \overline{\Delta}$ and, if p denotes the greatest integer such that $l_p < m$, we have $j < l_p < m$.*

Proof. First, recall that:

- If $\overline{\Delta}$ is empty, then (observation 4.2.1) each $x_l^{(m)}$ ($l \in \llbracket 1, \dots, t \rrbracket$) is zero.
- $\delta_m^{(m+1)}(X_j^{(m+1)}) = P_{m,j}^{(m+1)}$. This is 0 if $m = j+1$ or a (finite) summand

$$\sum_{\underline{a}=(a_{j+1}, \dots, a_{m-1})} c_{\underline{a}} (X_{j+1}^{(m+1)})^{a_{j+1}} \dots (X_{m-1}^{(m+1)})^{a_{m-1}}$$

with each $\underline{a}$ in $\mathbb{N}^{m-1-j}$ if $m > j+1$ and, moreover, $\underline{a}$ is nonzero when $c_{\underline{a}} \neq 0$.

- $\Theta^{(m)}: k\langle X_1^{(m+1)}, \dots, X_{m-1}^{(m+1)} \rangle \rightarrow k\langle X_1^{(m)}, \dots, X_{m-1}^{(m)} \rangle$ is an algebra homomorphism which transforms each $X_l^{(m+1)}$ into $X_l^{(m)}$.

Assume that $m = j + 1$. In this case, we have

$$\delta_m^{(m+1)}(X_j^{(m+1)}) = 0 \Rightarrow U = \Theta^{(m)}(\delta_m^{(m+1)}(X_j^{(m+1)})) = 0 \Rightarrow u = f_m(U) = 0$$

and we obtain a contradiction.

So, we have $m > j + 1$, and

$$\begin{aligned} \delta_m^{(m+1)}(X_j^{(m+1)}) &= \sum_{\underline{a}=(a_{j+1}, \dots, a_{m-1})} c_{\underline{a}}(X_{j+1}^{(m+1)})^{a_{j+1}} \dots (X_{m-1}^{(m+1)})^{a_{m-1}} \\ &\Rightarrow U = \Theta^{(m)}(\Delta_m^{(m+1)}(X_j^{(m+1)})) \\ &= \sum_{\underline{a}=(a_{j+1}, \dots, a_{m-1})} c_{\underline{a}}(X_{j+1}^{(m)})^{a_{j+1}} \dots (X_{m-1}^{(m)})^{a_{m-1}} \\ &\Rightarrow f_m(U) = \sum_{\underline{a}=(a_{j+1}, \dots, a_{m-1})} c_{\underline{a}}(x_{j+1}^{(m)})^{a_{j+1}} \dots (x_{m-1}^{(m)})^{a_{m-1}}. \end{aligned}$$

Assume that $\overline{\Delta}$ is empty. Then, each $x_l^{(m)}$ is zero and, since $\underline{a}$ is nonzero when $c_{\underline{a}} \neq 0$, we get that $u = 0$ and we still have a contradiction. So, $\overline{\Delta}$ is nonempty.

If $l_1 \geq m$, then, for each $l \in \llbracket 1, \dots, m-1 \rrbracket$, we have (by lemma 4.3.2 and, since $l < l_1$) $x_l^{(m)} = z_l = 0$. As above, this implies that $u = 0$ and we still get a contradiction. So, $l_1 < m$.

Since $X_m^{(m)} \in \mathcal{P}^{(m)}$, we have $x_m^{(m)} = 0$. As $x_m^{(m)} = z_m$ (lemma 4.3.1), we have $Z_m \in \mathcal{P}^{(2)}$ and so, $m \in \Delta$. Moreover, if $l_p \leq j$, then, for each $l \in \llbracket j+1, \dots, m-1 \rrbracket$, we have (by proposition 4.3.1, (2)(a) and, since $l_p < l < m$) $x_l^{(m)} = z_l = 0$. As above, this implies that $u = 0$ and we still get a contradiction. So, $j < l_p$. $\square$

Now, set $V = \delta_m^{(m+1)}(X_j^{(m+1)}) = X_m^{(m+1)}X_j^{(m+1)} - q^{-(\beta_m, \beta_j)}X_j^{(m+1)}X_m^{(m+1)}$ (see section 3.2), so that $U = \Theta^{(m)}(V)$.

Lemma 4.4.2. *Consider some integer l with $j < l < m$. Then we have, in $R^{(m+1)}$,*

$$\begin{aligned} X_l^{(m+1)}V &= q^{(\beta_l, \beta_m - \beta_j)}VX_l^{(m+1)} \\ &+ q^{(\beta_l, \beta_m)}[q^{-(\beta_j, \beta_l + \beta_m)}X_j^{(m+1)}\delta_m^{(m+1)}(X_l^{(m+1)}) - \delta_m^{(m+1)}(X_l^{(m+1)})X_j^{(m+1)}] \\ &+ q^{(\beta_l, \beta_m)}\delta_m^{(m+1)} \circ \delta_l^{(m+1)}(X_j^{(m+1)}). \end{aligned}$$

Proof. We observe that, as each $X_i^{(m+1)}$ is homogeneous of degree β_i (see section 3.2), we have

$$\begin{aligned} \delta_l^{(m+1)}(V) &= \delta_l^{(m+1)}(X_m^{(m+1)})X_j^{(m+1)} + q^{-(\beta_l, \beta_m)}X_m^{(m+1)}\delta_l^{(m+1)}(X_j^{(m+1)}) \\ &- q^{-(\beta_m, \beta_j)}\delta_l^{(m+1)}(X_j^{(m+1)})X_m^{(m+1)} - q^{-[(\beta_m, \beta_j) + (\beta_l, \beta_j)]}X_j^{(m+1)}\delta_l^{(m+1)}(X_m^{(m+1)}). \end{aligned}$$

So, we can write

$$\delta_l^{(m+1)}(V) = A + B - C - D$$

with

$$\begin{aligned} A &= \delta_l^{(m+1)}(X_m^{(m+1)})X_j^{(m+1)}, \\ B &= q^{-(\beta_l, \beta_m)}X_m^{(m+1)}\delta_l^{(m+1)}(X_j^{(m+1)}), \\ C &= q^{-(\beta_m, \beta_j)}\delta_l^{(m+1)}(X_j^{(m+1)})X_m^{(m+1)}, \\ D &= q^{-[(\beta_m, \beta_j) + (\beta_l, \beta_j)]}X_j^{(m+1)}\delta_l^{(m+1)}(X_m^{(m+1)}). \end{aligned}$$

Now, we compute separately the different pieces of the right member:

- $\delta_l^{(m+1)}(X_m^{(m+1)}) = X_l^{(m+1)}X_m^{(m+1)} - q^{-(\beta_l, \beta_m)}X_m^{(m+1)}X_l^{(m+1)}$. As we also have $\delta_m^{(m+1)}(X_l^{(m+1)}) = X_m^{(m+1)}X_l^{(m+1)} - q^{-(\beta_l, \beta_m)}X_l^{(m+1)}X_m^{(m+1)}$, we can write $X_l^{(m+1)}X_m^{(m+1)} = q^{(\beta_l, \beta_m)}[X_m^{(m+1)}X_l^{(m+1)} - \delta_m^{(m+1)}(X_l^{(m+1)})]$.

So, we get:

$$\delta_l^{(m+1)}(X_m^{(m+1)}) = (q^{(\beta_l, \beta_m)} - q^{-(\beta_l, \beta_m)})X_m^{(m+1)}X_l^{(m+1)} - q^{(\beta_l, \beta_m)}\delta_m^{(m+1)}(X_l^{(m+1)}).$$

- From this, we deduce that

$$\begin{aligned} \delta_l^{(m+1)}(X_m^{(m+1)})X_j^{(m+1)} &= (q^{(\beta_l, \beta_m)} - q^{-(\beta_l, \beta_m)})X_m^{(m+1)}X_l^{(m+1)}X_j^{(m+1)} \\ &\quad - q^{(\beta_l, \beta_m)}\delta_m^{(m+1)}(X_l^{(m+1)})X_j^{(m+1)}. \end{aligned}$$

But we also have

$$\delta_l^{(m+1)}(X_j^{(m+1)}) = X_l^{(m+1)}X_j^{(m+1)} - q^{-(\beta_l, \beta_j)}X_j^{(m+1)}X_l^{(m+1)},$$

so that

$$\begin{aligned} \delta_l^{(m+1)}(X_m^{(m+1)})X_j^{(m+1)} &= (q^{(\beta_l, \beta_m)} - q^{-(\beta_l, \beta_m)})X_m^{(m+1)}[q^{-(\beta_l, \beta_j)}X_j^{(m+1)}X_l^{(m+1)} + \delta_l^{(m+1)}(X_j^{(m+1)})] \\ &\quad - q^{(\beta_l, \beta_m)}\delta_m^{(m+1)}(X_l^{(m+1)})X_j^{(m+1)}. \end{aligned}$$

So, we get:

$$\begin{aligned} A &= (q^{(\beta_l, \beta_m)} - q^{-(\beta_l, \beta_m)})q^{-(\beta_l, \beta_j)}X_m^{(m+1)}X_j^{(m+1)}X_l^{(m+1)} \\ &\quad + (q^{(\beta_l, \beta_m)} - q^{-(\beta_l, \beta_m)})X_m^{(m+1)}\delta_l^{(m+1)}(X_j^{(m+1)}) \\ &\quad - q^{(\beta_l, \beta_m)}\delta_m^{(m+1)}(X_l^{(m+1)})X_j^{(m+1)}. \end{aligned}$$

- Now, we have

$$\begin{aligned} A + B - C &= (q^{(\beta_l, \beta_m)} - q^{-(\beta_l, \beta_m)})q^{-(\beta_l, \beta_j)}X_m^{(m+1)}X_j^{(m+1)}X_l^{(m+1)} \\ &\quad + q^{(\beta_l, \beta_m)}X_m^{(m+1)}\delta_l^{(m+1)}(X_j^{(m+1)}) - q^{-(\beta_m, \beta_j)}\delta_l^{(m+1)}(X_j^{(m+1)})X_m^{(m+1)} \\ &\quad - q^{(\beta_l, \beta_m)}\delta_m^{(m+1)}(X_l^{(m+1)})X_j^{(m+1)} \\ &= (q^{(\beta_l, \beta_m)} - q^{-(\beta_l, \beta_m)})q^{-(\beta_l, \beta_j)}X_m^{(m+1)}X_j^{(m+1)}X_l^{(m+1)} \\ &\quad + q^{(\beta_l, \beta_m)}[X_m^{(m+1)}\delta_l^{(m+1)}(X_j^{(m+1)}) \\ &\quad - q^{-(\beta_m, \beta_l + \beta_j)}\delta_l^{(m+1)}(X_j^{(m+1)})X_m^{(m+1)}] \\ &\quad - q^{(\beta_l, \beta_m)}\delta_m^{(m+1)}(X_l^{(m+1)})X_j^{(m+1)}. \end{aligned}$$

Since $\delta_l^{(m+1)}(X_j^{(m+1)})$ is homogeneous of degree $\beta_l + \beta_j$, we have

$$\begin{aligned} X_m^{(m+1)}\delta_l^{(m+1)}(X_j^{(m+1)}) - q^{-(\beta_m, \beta_l + \beta_j)}\delta_l^{(m+1)}(X_j^{(m+1)})X_m^{(m+1)} \\ = \delta_m^{(m+1)} \circ \delta_l^{(m+1)}(X_j^{(m+1)}). \end{aligned}$$

So, we get:

$$\begin{aligned} &A + B - C \\ &= (q^{(\beta_l, \beta_m)} - q^{-(\beta_l, \beta_m)})q^{-(\beta_l, \beta_j)}X_m^{(m+1)}X_j^{(m+1)}X_l^{(m+1)} \\ &\quad + q^{(\beta_l, \beta_m)}\delta_m^{(m+1)} \circ \delta_l^{(m+1)}(X_j^{(m+1)}) - q^{(\beta_l, \beta_m)}\delta_m^{(m+1)}(X_l^{(m+1)})X_j^{(m+1)}. \end{aligned}$$

- Using the computation of $\delta_l^{(m+1)}(X_m^{(m+1)})$ made in the first point above, we can write:

$$\begin{aligned} D &= q^{-(\beta_m, \beta_j) + (\beta_l, \beta_j)} X_j^{(m+1)} \delta_l^{(m+1)}(X_m^{(m+1)}) \\ &= q^{-(\beta_j, \beta_l + \beta_m)} X_j^{(m+1)} [(q^{(\beta_l, \beta_m)} - q^{-(\beta_l, \beta_m)}) X_m^{(m+1)} X_l^{(m+1)} - q^{(\beta_l, \beta_m)} \delta_m^{(m+1)}(X_l^{(m+1)})]. \end{aligned}$$

So, we get:

$$\begin{aligned} D &= q^{-(\beta_j, \beta_l + \beta_m)} (q^{(\beta_l, \beta_m)} - q^{-(\beta_l, \beta_m)}) X_j^{(m+1)} X_m^{(m+1)} X_l^{(m+1)} \\ &\quad - q^{-(\beta_j, \beta_l + \beta_m)} q^{(\beta_l, \beta_m)} X_j^{(m+1)} \delta_m^{(m+1)}(X_l^{(m+1)}). \end{aligned}$$

- Now, we have

$$\begin{aligned} \delta_l^{(m+1)}(V) &= A + B - C - D \\ &= q^{-(\beta_j, \beta_l)} (q^{(\beta_l, \beta_m)} - q^{-(\beta_l, \beta_m)}) [X_m^{(m+1)} X_j^{(m+1)} - q^{-(\beta_j, \beta_m)} X_j^{(m+1)} X_m^{(m+1)}] X_l^{(m+1)} \\ &\quad + q^{(\beta_l, \beta_m)} [q^{-(\beta_j, \beta_l + \beta_m)} X_j^{(m+1)} \delta_m^{(m+1)}(X_l^{(m+1)}) - \delta_m^{(m+1)}(X_l^{(m+1)}) X_j^{(m+1)}] \\ &\quad + q^{(\beta_l, \beta_m)} \delta_m^{(m+1)} \circ \delta_l^{(m+1)}(X_j^{(m+1)}). \end{aligned}$$

So, we get:

$$\begin{aligned} \delta_l^{(m+1)}(V) &= q^{-(\beta_j, \beta_l)} (q^{(\beta_l, \beta_m)} - q^{-(\beta_l, \beta_m)}) V X_l^{(m+1)} \\ &\quad + q^{(\beta_l, \beta_m)} [q^{-(\beta_j, \beta_l + \beta_m)} X_j^{(m+1)} \delta_m^{(m+1)}(X_l^{(m+1)}) - \delta_m^{(m+1)}(X_l^{(m+1)}) X_j^{(m+1)}] \\ &\quad + q^{(\beta_l, \beta_m)} \delta_m^{(m+1)} \circ \delta_l^{(m+1)}(X_j^{(m+1)}). \end{aligned}$$

As V is homogeneous of degree $\beta_j + \beta_m$, we have also

$$\delta_l^{(m+1)}(V) = X_l^{(m+1)} V - q^{-(\beta_l, \beta_j + \beta_m)} V X_l^{(m+1)},$$

which gives the required formula. $\square$

Now, let us observe that, if $j < l < m$, we have the following results:

- $\delta_m^{(m+1)}(X_l^{(m+1)}) \in k\langle X_{l+1}^{(m+1)}, \dots, X_{m-1}^{(m+1)} \rangle$ (or is zero if $l = m - 1$) (see section 3.2).
- Similarly, $V = \delta_m^{(m+1)}(X_j^{(m+1)}) \in k\langle X_{j+1}^{(m+1)}, \dots, X_{m-1}^{(m+1)} \rangle$ (observe that $j < m - 1$ by lemma 4.4.1).
- $\delta_l^{(m+1)}(X_j^{(m+1)}) \in k\langle X_{j+1}^{(m+1)}, \dots, X_{l-1}^{(m+1)} \rangle$ (or is zero if $l = j + 1$), which implies that $\delta_m^{(m+1)} \circ \delta_l^{(m+1)}(X_j^{(m+1)}) \in k\langle X_{j+1}^{(m+1)}, \dots, X_{m-1}^{(m+1)} \rangle$.

So, V , $\delta_m^{(m+1)}(X_l^{(m+1)})$, $\delta_l^{(m+1)} \circ \delta_l^{(m+1)}(X_j^{(m+1)})$, $X_l^{(m+1)}$, $X_j^{(m+1)}$ are all in $k\langle X_1^{(m+1)}, \dots, X_{m-1}^{(m+1)} \rangle$ and, if we transform the equality of lemma 4.4.2 by the algebra homomorphism $\Theta^{(m)}$, we obtain:

Lemma 4.4.3.

$$\begin{aligned} X_l^{(m)} U &= q^{(\beta_l, \beta_m - \beta_j)} U X_l^{(m)} \\ &\quad + q^{(\beta_l, \beta_m)} [q^{-(\beta_j, \beta_l + \beta_m)} X_j^{(m)} \Theta^{(m)}(\delta_m^{(m+1)}(X_l^{(m+1)})) - \Theta^{(m)}(\delta_m^{(m+1)}(X_l^{(m+1)})) X_j^{(m)}] \\ &\quad + q^{(\beta_l, \beta_m)} \Theta^{(m)}(\delta_m^{(m+1)} \circ \delta_l^{(m+1)}(X_j^{(m+1)})). \end{aligned}$$

- It results from the choice of j that $\Theta^{(m)}(\delta_m^{(m+1)}(X_l^{(m+1)})) \in \mathcal{P}^{(m)}$. Assume that $l > j + 1$, so that (by section 3.2)

$$\delta_l^{(m+1)}(X_j^{(m+1)}) = \sum_{\underline{a}=(a_{j+1}, \dots, a_{l-1})} c_{\underline{a}}(X_{j+1}^{(m+1)})^{a_{j+1}} \dots (X_{l-1}^{(m+1)})^{a_{l-1}}$$

with each $c_{\underline{a}} \in k$.

As each $X_i^{(m+1)}$ is a h_m -eigenvector, each $\delta_m^{(m+1)}(X_{j+1}^{(m+1)})^{a_{j+1}} \dots (X_{l-1}^{(m+1)})^{a_{l-1}}$ is a linear combination of products $M_1 \delta_m^{(m+1)}(X_i^{(m+1)}) M_2$ with $j < i < l$ and M_1 (resp. M_2) an ordered monomial in $X_{j+1}^{(m+1)}, \dots, X_i^{(m+1)}$ (resp. in $X_i^{(m+1)}, \dots, X_{l-1}^{(m+1)}$). This implies that $\Theta^{(m)}(\delta_m^{(m+1)} \circ \delta_l^{(m+1)}(X_j^{(m+1)}))$ is a linear combination of products $N_1 \Theta^{(m)}(\delta_m^{(m+1)}(X_i^{(m+1)})) N_2$ with $j < i < l$, N_1 (resp. N_2) an ordered monomial in $X_{j+1}^{(m)}, \dots, X_i^{(m)}$ (resp. in $X_i^{(m)}, \dots, X_{l-1}^{(m)}$).

As $\Theta^{(m)}(\delta_m^{(m+1)}(X_i^{(m+1)})) \in \mathcal{P}^{(m)}$ for $j < i < m$, we conclude that

- $\Theta^{(m)}(\delta_m^{(m+1)} \circ \delta_l^{(m+1)}(X_j^{(m+1)})) \in \mathcal{P}^{(m)}$.

We observe that this result is also true when $l = j + 1$ since, in this case, $\delta_l^{(m+1)}(X_j^{(m+1)})$ is zero (see section 3.2). So, if we transform the equality of lemma 4.4.3 by the algebra homomorphism f_m , we obtain:

Lemma 4.4.4.

$$(16) \quad x_l^{(m)} u = q^{(\beta_l, \beta_m - \beta_j)} u x_l^{(m)}.$$

Recall (see section 3.2) that $V = \delta_m^{(m+1)}(X_j^{(m+1)})$ is a homogeneous polynomial in $X_{j+1}^{(m+1)}, \dots, X_{m-1}^{(m+1)}$ of degree $\beta_m + \beta_j$. So, since $\Theta^{(m)}$ transforms each $X_i^{(m+1)}$ ($1 \leq i < m$) into $X_i^{(m)}$, $U = \Theta^{(m)}(V)$ is a homogeneous polynomial in $X_{j+1}^{(m)}, \dots, X_{m-1}^{(m)}$ of degree $\beta_m + \beta_j$. This implies:

Lemma 4.4.5. *u is a nonzero polynomial in $x_{j+1}^{(m)}, \dots, x_{m-1}^{(m)}$. Moreover, u is homogeneous of degree $\beta_m + \beta_j$.*

Proof. $u = f_m(U)$ is nonzero by construction. Since f_m transforms each $X_i^{(m+1)}$ into $x_i^{(m)}$, it turns out that u is a polynomial in $x_{j+1}^{(m)}, \dots, x_{m-1}^{(m)}$. Moreover, $u = f_m(U)$ is homogeneous of degree $\beta_m + \beta_j$ by lemma 4.3.1. $\square$

Recall (lemma 4.4.1) that $j < l_p$ and that p is the greatest integer such that $l_p < m$. Denote by c the smallest integer such that $j < l_c$ ($1 \leq c \leq p$). As $u \in k\langle x_{j+1}^{(m)}, \dots, x_{m-1}^{(m)} \rangle$, it follows from (proposition 4.3.2, (2)(b)) that

$$u = u_1 z_{l_p}^{a_1} + \dots + u_M z_{l_p}^{a_M}$$

with $M \geq 1$, $(a_1, \dots, a_M) \in \mathbb{Z}^M$ and:

- If $p = 1$, then each $u_i \in k$.
- If $p > 1$ and $j \geq l_{p-1}$, then each $u_i \in k$.
- If $p > 1$ and $j < l_{p-1}$, then each $u_i \in k\langle x_{j+1}^{(l_p)}, \dots, x_{l_{p-1}}^{(l_p)} \rangle$, so that we can write:
- If $c = p$, then each $u_i \in k$.
- If $c < p$, then each $u_i \in k\langle x_{j+1}^{(l_p)}, \dots, x_{l_{p-1}}^{(l_p)} \rangle$.

In both cases, we may assume that $u_1, \dots, u_M$ are all nonzero, that $a_1 < \dots < a_M$, and we observe that

- Each $u_i \in k\langle x_1^{(l_p)}, \dots, x_{l_p-1}^{(l_p)} \rangle$ and is homogeneous of degree $\beta_m + \beta_j - a_i \beta_{l_p}$.

In fact, since u is homogeneous of degree $\beta_m + \beta_j$ and $z_{l_p} = x_{l_p}^{(l_p)}$ is homogeneous of degree β_{l_p} , we have, for any $\rho \in \mathbb{Z}\Pi$,

$$\begin{aligned} \overline{h}_\rho(u) &= q^{-(\rho, a_1 \beta_{l_p})} \overline{h}_\rho(u_1) z_{l_p}^{a_1} + \dots + q^{-(\rho, a_M \beta_{l_p})} \overline{h}_\rho(u_M) z_{l_p}^{a_M} \\ &= q^{-(\rho, \beta_m + \beta_j)} [u_1 z_{l_p}^{a_1} + \dots + u_M z_{l_p}^{a_M}] \end{aligned}$$

and, by corollary 4.3.2, we can identify the coefficients of $z_{l_p}^{a_i}$, so that $q^{-(\rho, a_i \beta_{l_p})} \overline{h}_\rho(u_i) = q^{-(\rho, \beta_m + \beta_j)} u_i$ and $\overline{h}_\rho(u_i) = q^{-(\rho, \beta_m + \beta_j - a_i \beta_{l_p})} u_i$.

By (proposition 4.3.3, (1)) we have $x_{l_p}^{(m)} = z_{l_p} = x_{l_p}^{(l_p)}$. So, using lemma 4.4.4 with $l = l_p$, we obtain:

$$z_{l_p} u = q^{(\beta_{l_p}, \beta_m - \beta_j)} u z_{l_p}$$

and, by (lemma 4.3.1, (4)),

$$z_{l_p} u = \overline{h_{l_p}}(u_1) z_{l_p}^{a_1+1} + \dots + \overline{h_{l_p}}(u_M) z_{l_p}^{a_M+1}.$$

This implies that

$$\begin{aligned} & q^{(\beta_{l_p}, \beta_m - \beta_j)} [u_1 z_{l_p}^{a_1} + \dots + u_M z_{l_p}^{a_M}] z_{l_p} \\ &= [q^{-(\beta_{l_p}, \beta_m + \beta_j - a_1 \beta_{l_p})} u_1 z_{l_p}^{a_1} + \dots + q^{-(\beta_{l_p}, \beta_m + \beta_j - a_M \beta_{l_p})} u_M z_{l_p}^{a_M}] z_{l_p}. \end{aligned}$$

As above, if $1 \leq i \leq M$, we can identify the coefficients of $z_{l_p}^{a_i+1}$, so that, since $u_i \neq 0$ and q is not a root of unity, we have $-(\beta_{l_p}, \beta_m + \beta_j - a_i \beta_{l_p}) = (\beta_{l_p}, \beta_m - \beta_j)$.

So, $a_i \|\beta_{l_p}\|^2 = 2(\beta_{l_p}, \beta_m)$ and then $a_i = (\beta_{l_p}^\vee, \beta_m)$.

This implies that $M = 1$ and we conclude:

Lemma 4.4.6.

$$u = u_p z_{l_p}^{a_p}$$

with:

- If $c = p$, then $u_p \in k^*$.
- If $c < p$, then $u_p \in k\langle x_{j+1}^{(l_p)}, \dots, x_{l_p-1}^{(l_p)} \rangle \setminus \{0\}$.
- $a_p = (\beta_{l_p}^\vee, \beta_m)$.
- u_p is homogeneous of degree $\beta_m + \beta_j - a_p \beta_{l_p} = \gamma_p + \beta_j$ with $\gamma_p = s_{\beta_{l_p}}(\beta_m)$.

Now, we extend this result as follows:

Lemma 4.4 1. Consider some integer d with $c \leq d \leq p$. Then

$$u = u_d z_{l_d}^{a_d} \dots z_{l_p}^{a_p}$$

with:

- If $c = d$, then $u_d \in k^*$.
- If $c < d$, then $u_d \in k\langle x_{j+1}^{(l_d)}, \dots, x_{l_d-1}^{(l_d)} \rangle \setminus \{0\}$.
- Denote by $(\gamma_d, \dots, \gamma_p, \gamma_{p+1})$ the sequence of (not necessarily positive) roots recursively defined by $\gamma_{p+1} = \beta_m$ and, for $d \leq i \leq p$, $\gamma_i = s_{\beta_{l_i}}(\gamma_{i+1})$. Then, each $a_i = (\beta_{l_i}^\vee, \gamma_{i+1})$.
- u_d is homogeneous of degree $\gamma_d + \beta_j$.

Proof. We proceed by decreasing induction on d .

If $d = p$, everything has been done in lemma 4.4.6.

Assume that $c \leq d < p$ and that the lemma is true at the rank $d + 1$, namely:

$$u = u_{d+1} z_{l_{d+1}}^{a_{d+1}} \cdots z_{l_p}^{a_p}$$

with:

- $u_{d+1} \in k\langle x_{j+1}^{(l_{d+1})}, \dots, x_{l_d}^{(l_{d+1})} \rangle \setminus \{0\}$.
- If $(\gamma_{d+1}, \dots, \gamma_p, \gamma_{p+1})$ is the sequence of (not necessarily positive) roots recursively defined by $\gamma_{p+1} = \beta_m$ and, for $d+1 \leq i \leq p$, $\gamma_i = s_{\beta_{l_i}}(\gamma_{i+1})$, then, each $a_i = (\beta_{l_i}^\vee, \gamma_{i+1})$.
- u_{d+1} is homogeneous of degree $\gamma_{d+1} + \beta_j$.

By (proposition 4.3.2, (2)(b)), we have:

$$u_{d+1} = v_1 z_{l_d}^{b_1} + \cdots + v_M z_{l_d}^{b_M}$$

with $M \geq 1$, $(b_1, \dots, b_M) \in \mathbb{Z}^M$ and:

- If $d = c$, then each $v_i \in k$.
- If $d > c$, then each $v_i \in k\langle x_{j+1}^{(l_d)}, \dots, x_{l_{d-1}}^{(l_d)} \rangle$.

As above, we may assume that $v_1, \dots, v_M$ are all nonzero, that $b_1 < \cdots < b_M$, and observe that

- Each v_i is homogeneous of degree $\gamma_{d+1} + \beta_j - b_i \beta_{l_d}$.

By (proposition 4.3.3, (2)), we have

$$x_{l_d}^{(m)} = z_{l_d} + \sum_{\underline{s}=(s_{d+1}, \dots, s_p) \in F} \eta(\underline{s}) z_{l_{d+1}}^{s_{d+1}} \cdots z_{l_p}^{s_p},$$

where:

- F is a finite (possibly empty) subset of $\mathbb{Z}^{p-d}$.
- If $\preceq$ denotes the inverse lexicographic order in $\mathbb{Z}^{p-d}$, then, for any $\underline{s}$ in F , we have $\underline{s} \prec 0$.
- For each $\underline{s}$ in F , we have $\eta(\underline{s}) \in k^*$.

By (lemma 4.3.1, (4)), we have

$$z_{l_d} u_{d+1} = \overline{h_{l_d}}(v_1) z_{l_d}^{b_1+1} + \cdots + \overline{h_{l_d}}(v_M) z_{l_d}^{b_M+1}.$$

For any $\underline{e} = (e_{d+1}, \dots, e_p) \in \mathbb{Z}^{p-d}$, let us denote $z^{\underline{e}} := z_{l_{d+1}}^{e_{d+1}} \cdots z_{l_p}^{e_p}$.

Again by (lemma 4.3.1, (4)), there exists $(\lambda_{\underline{e},1}, \dots, \lambda_{\underline{e},M}) \in (k^*)^M$ such that

$$z^{\underline{e}} u_{d+1} = \lambda_{\underline{e},1} v_1 z_{l_d}^{b_1} z^{\underline{e}} + \cdots + \lambda_{\underline{e},M} v_M z_{l_d}^{b_M} z^{\underline{e}}.$$

Now, if we set $\underline{a} = (a_{d+1}, \dots, a_p)$, we have

$$x_{l_d}^{(m)} u = (1) + (2)$$

with

$$\begin{aligned} (1) &= z_{l_d} u_{d+1} z^{\underline{a}} = \overline{h_{l_d}}(v_1) z_{l_d}^{b_1+1} z^{\underline{a}} + \cdots + \overline{h_{l_d}}(v_M) z_{l_d}^{b_M+1} z^{\underline{a}} \\ &= q^{-(\beta_{l_d}, \gamma_{d+1} + \beta_j - b_1 \beta_{l_d})} v_1 z_{l_d}^{b_1+1} z^{\underline{a}} + \cdots + q^{-(\beta_{l_d}, \gamma_{d+1} + \beta_j - b_M \beta_{l_d})} v_M z_{l_d}^{b_M+1} z^{\underline{a}} \end{aligned}$$

and

$$(2) = \left(\sum_{\underline{s} \in F} \eta(\underline{s}) z^{\underline{s}} \right) u_{d+1} z^{\underline{a}} = \sum_{\underline{s} \in F} \eta(\underline{s}) (\lambda_{\underline{s},1} v_1 z_{l_d}^{b_1} z^{\underline{s}} + \cdots + \lambda_{\underline{s},M} v_M z_{l_d}^{b_M} z^{\underline{s}}) z^{\underline{a}}.$$

Thus, we also have

$$ux_{l_d}^{(m)} = (1') + (2')$$

with

$$\begin{aligned} (1') &= u_{d+1} z^{\underline{a}} z_{l_d} = v_1 z_{l_d}^{b_1} z^{\underline{a}} z_{l_d} + \cdots + v_M z_{l_d}^{b_M} z^{\underline{a}} z_{l_d} \\ &= q^{-(a_{d+1}\beta_{l_{d+1}} + \cdots + a_p\beta_{l_p}, \beta_{l_d})} (v_1 z_{l_d}^{b_1+1} + \cdots + v_M z_{l_d}^{b_M+1}) z^{\underline{a}} \end{aligned}$$

and

$$(2') = u_{d+1} z^{\underline{a}} \left[\sum_{\underline{s} \in F} \eta(\underline{s}) z^{\underline{s}} \right] = \sum_{\underline{s} \in F} \eta(\underline{s}) (v_1 z_{l_d}^{b_1} z^{\underline{a}} z^{\underline{s}} + \cdots + v_M z_{l_d}^{b_M} z^{\underline{a}} z^{\underline{s}}).$$

Using lemma 4.4.4 and with $l = l_d$, we obtain:

$$(1) + (2) = q^{(\beta_{l_d}, \beta_m - \beta_j)} [(1') + (2')].$$

We observe that (1) and (1') are left linear combinations of monomials of type $z_{l_d}^h z^{\underline{a}}$ ($h \in \mathbb{Z}$) with coefficients in $A = k\langle x_{j+1}^{(l_d)}, \dots, x_{l_{d-1}}^{(l_d)} \rangle$, while (2) and (2') are left linear combinations of monomials of type $z_{l_d}^h z^{\underline{e}}$ ($h \in \mathbb{Z}, \underline{e} = \underline{a} + \underline{s}, s \in F$) with coefficients in A . Recall that, for any $\underline{s} \in F$, we have $\underline{s} \prec 0 \Rightarrow \underline{e} = \underline{a} + \underline{s} \prec \underline{a}$. So, by corollary 4.3.2, we have $(1) = q^{(\beta_{l_d}, \beta_m - \beta_j)} (1')$, which implies that, if $1 \leq i \leq M$, then

$$q^{-(\beta_{l_d}, \gamma_{d+1} + \beta_j - b_i \beta_{l_d})} = q^{(\beta_{l_d}, \beta_m - \beta_j)} q^{-(a_{d+1}\beta_{l_{d+1}} + \cdots + a_p\beta_{l_p}, \beta_{l_d})}.$$

Recall that, for $d+1 \leq h \leq p$, $\gamma_h = s_{\beta_{l_h}}(\gamma_{h+1}) = \gamma_{h+1} - (\beta_{l_h}^\vee, \gamma_{h+1})\beta_{l_h} = \gamma_{h+1} - a_h\beta_{l_h}$. If we add up all these equalities, we obtain $\gamma_{d+1} = \gamma_{p+1} - (a_{d+1}\beta_{l_{d+1}} + \cdots + a_p\beta_{l_p})$. As $\gamma_{p+1} = \beta_m$, we get the formula

$$-(\beta_{l_d}, \gamma_{d+1} + \beta_j - b_i \beta_{l_d}) = (\beta_{l_d}, \beta_m - \beta_j) - (\beta_m - \gamma_{d+1}, \beta_{l_d})$$

for any $i \in \llbracket 1, \dots, M \rrbracket$.

This implies that $b_i \|\beta_{l_d}\|^2 - (\beta_{l_d}, \gamma_{d+1}) = (\beta_{l_d}, \gamma_{d+1})$. So, as the integers b_i are distinct, we necessarily have $M = 1$, $b_1 = (\beta_{l_d}^\vee, \gamma_{d+1})$ and v_1 is homogeneous of degree $\gamma_{d+1} - b_1\beta_{l_d} + \beta_j = \gamma_d + \beta_j$ if we set $\gamma_d = \gamma_{d+1} - b_1\beta_{l_d} = s_{\beta_{l_d}}(\gamma_{d+1})$. $\square$

Now, in the case $d = c$, lemma 4.4.7 becomes:

$$u = u_c z_{l_c}^{a_c} \cdots z_{l_p}^{a_p}$$

with:

- $u_c \in k^*$.
- If $(\gamma_c, \dots, \gamma_p, \gamma_{p+1})$ is the sequence of (not necessarily positive) roots recursively defined by $\gamma_{p+1} = \beta_m$ and, for $c \leq i \leq p$, $\gamma_i = s_{\beta_{l_i}}(\gamma_{i+1})$, then each $a_i = (\beta_{l_i}^\vee, \gamma_{i+1})$.

In this case, u_c is homogeneous of degree 0 and, since each $z_{l_d} = x_{l_d}^{(l_d)}$ is homogeneous of degree β_{l_d} (see lemma 4.3.1), we obtain that u is also homogeneous of degree $a_c\beta_{l_c} + \cdots + a_p\beta_{l_p}$. So, as the degree of u is uniquely defined, we have

$$\beta_m + \beta_j = a_c\beta_{l_c} + \cdots + a_p\beta_{l_p}$$

and, by proposition 2.3.3, we conclude:

Proposition 4.4.1. *If $\mathcal{P}^{(m)}$ is not in $\text{Im}(\phi_m)$, then*

$$\Delta = \{i \in \llbracket 1, \dots, t \rrbracket \mid Z_i \in \mathcal{P}^{(2)}\}$$

is not a positive diagram with respect to (1) in the sense of definition 2.2.2.

5. CONNECTIONS BETWEEN ADMISSIBLE AND POSITIVE DIAGRAMS

Assume that w and the algebra $R = U^+[w]$ are defined as in section 2. The admissible diagrams (with respect to the reduced expression (1)) are defined as in section 3.3.

We know that the longest element w_0 has a reduced expression of the form

$$(17) \quad w_0 = s_{\alpha_1} \dots s_{\alpha_N} (\alpha_i \in \Pi \text{ for } 1 \leq i \leq N = |\Phi^+|)$$

such that $s_{\alpha_1} \dots s_{\alpha_t}$ is the reduced expression (1) of w .

In this section, we shall use the following conventions:

- Denote by $R_0 = U^+[w_0]$ the algebra associated to the reduced expression (17) as explained in section 2.1 and set $F_0 = \text{Fract}(R_0)$ its division ring of fractions.
- As in section 3.1, we set $X_i = X_{\beta_i}$ for $1 \leq i \leq N$ and we observe that $X_1, \dots, X_t$ are also the generators of $R = U^+[w]$ defined in section 2.1, so that $R = k\langle X_1, \dots, X_t \rangle$ is a subalgebra of $R_0 = k\langle X_1, \dots, X_N \rangle$.
- For any $\rho \in \mathbb{Z}\Pi$ we still denote by h_ρ the automorphism of R_0 defined as in section 3.1.
- Each h_ρ can uniquely be extended to an automorphism, still denoted h_ρ , of F_0 .
- We still denote by H the set of all the automorphisms h_ρ of F_0 ($\rho \in \mathbb{Z}\Pi$). It is a subgroup of $\text{Aut}(F_0)$.
- For any m in $\llbracket 1, \dots, N+1 \rrbracket$, we define the algebra $R_0^{(m)}$ and its canonical generators $X_1^{(m)}, \dots, X_N^{(m)}$ as in section 3.2.

5.1. A property of admissible diagrams. Consider Δ a diagram with respect to (1) (i.e. a subset of $\llbracket 1, \dots, t \rrbracket$). It is also a diagram with respect to (17) but it is not quite clear whether the properties “ Δ is admissible with respect to (1)” and “ Δ is admissible with respect to (17)” are equivalent. In this section, we clarify this point.

- Lemma 5.1.1.** (1) *There exists a unique algebra isomorphism $g : R = k\langle X_1, \dots, X_t \rangle \rightarrow k\langle X_1^{(t+1)}, \dots, X_t^{(t+1)} \rangle \subset R_0^{(t+1)}$ which transforms each X_i ($1 \leq i \leq t$) into $X_i^{(t+1)}$.*
- (2) *For any $\rho \in \mathbb{Z}\Pi$ and $B \in R$, we have $g \circ h_\rho(B) = h_\rho \circ g(B)$.*
- (3) *The map $\mathcal{P} \mapsto g(\mathcal{P})$ is a bijection from the set of H -invariant prime ideals of R onto the set of H -invariant prime ideals of $k\langle X_1^{(t+1)}, \dots, X_t^{(t+1)} \rangle$.*

Proof. (1) We know (see section 3.1) that R is the k -algebra generated by the variables $X_1, \dots, X_t$ submitted to the Levendorskii-Soibelman relations (10) and (11). As $X_1^{(t+1)}, \dots, X_t^{(t+1)}$ satisfy the “same” relations (12) and (14) (see section 3.2), there exists a unique homomorphism $g: R = k\langle X_1, \dots, X_t \rangle \rightarrow k\langle X_1^{(t+1)}, \dots, X_t^{(t+1)} \rangle \subset R_0^{(t+1)}$ which transforms each X_i ($1 \leq i \leq t$) into $X_i^{(t+1)}$. As g transforms the base of the ordered monomials $X_1, \dots, X_t$ into the base of the ordered monomials in $X_1^{(t+1)}, \dots, X_t^{(t+1)}$, it is an isomorphism.

- (2) For $1 \leq i \leq t$, we have $g \circ h_\rho(X_i) = q^{-(\rho, \beta_i)} g(X_i) = q^{-(\rho, \beta_i)} X_i^{(t+1)} = h_\rho(X_i^{(t+1)}) = h_\rho \circ g(X_i)$. This implies that the homomorphisms $g \circ h_\rho$ and $h_\rho \circ g$ coincide on R .
- (3) Results thoroughly from (1) and (2). □

Until the end of this section, we identify the canonical generators $X_1, \dots, X_t$ of R with $X_1^{(t+1)}, \dots, X_t^{(t+1)}$, respectively, so that R is identified with the subalgebra $k\langle X_1^{(t+1)}, \dots, X_t^{(t+1)} \rangle$ of $R_0^{(t+1)}$, $H\text{-Spec}(R)$ is identified with the set of H -invariant prime ideals of $k\langle X_1^{(t+1)}, \dots, X_t^{(t+1)} \rangle$ and the division ring $F = \text{Fract}(R)$ is identified with the subalgebra of F_0 defined as follows: $F = \{as^{-1} | a \in R, s \in R \setminus \{0\}\}$. (R and F are both H -invariant.)

Lemma 5.1.2. *Consider an integer m with $2 \leq m \leq t+1$.*

If $X_1^{(m)}, \dots, X_N^{(m)}$ are the canonical generators of $R_0^{(m)}$, then $X_1^{(m)}, \dots, X_t^{(m)}$ are the canonical generators of $R^{(m)}$.

Proof. We proceed by decreasing induction on m .

- If $m = t+1$, $X_1^{(t+1)}, \dots, X_t^{(t+1)}$ are the canonical generators of R by the previous identification. So, they also are the canonical generators of $R^{(t+1)}$ by the definition of this algebra.
- Assume that $2 \leq m \leq t$ and that $X_1^{(m+1)}, \dots, X_t^{(m+1)}$ are the canonical generators of $R^{(m+1)}$. Denote (temporarily) by $X_1^{(m)}, \dots, X_t^{(m)}$ the canonical generators of $R^{(m)}$. Recall (see section 3.2) that they are defined as follows:
- For each $i \in \llbracket 1, \dots, t \rrbracket$, we have
 - (1) $m \leq i \Rightarrow X_i^{(m)} = X_i^{(m+1)}$.
 - (2) $i < m \Rightarrow X_i^{(m)} = X_i^{(m+1)} + \sum_{l=1}^{+\infty} C_l^{(m+1)} (X_m^{(m+1)})^{-l}$ with

$$C_l^{(m+1)} = \frac{(1 - q_m)^{-l}}{[l]_{q_m}!} \lambda_{m,i}^{-l} (\delta_m^{(m+1)})^l (X_i^{(m+1)}).$$

Moreover, if h_m is the (unique) automorphism of $k\langle X_1^{(m+1)}, \dots, X_{m-1}^{(m+1)} \rangle$ which satisfies $h_m(X_i^{(m+1)}) = \lambda_{m,i} X_i^{(m+1)}$ ($\lambda_{m,i} = q^{-(\beta_m, \beta_i)}$), we have $\delta_m^{(m+1)}(a) = X_m^{(m+1)} a - h_m(a) X_m^{(m+1)}$ for any $a \in k\langle X_1^{(m+1)}, \dots, X_{m-1}^{(m+1)} \rangle$. As $X_1^{(m)}, \dots, X_t^{(m)}$ are defined by the same formulas, the proof is over. $\square$

Recall (see sections 3.1 and 3.2) that each prime ideal of $R_0^{(t+1)}$ (resp. R) is completely prime. So, if $\mathcal{Q}$ is any prime ideal of $R_0^{(t+1)}$, then $\mathcal{P} = \mathcal{Q} \cap R$ is a prime ideal of R . Moreover, since $R = k\langle X_1^{(t+1)}, \dots, X_t^{(t+1)} \rangle$ is H -invariant, we have: ($\mathcal{Q}$ is H -invariant) $\Rightarrow$ ($\mathcal{P}$ is H -invariant).

Lemma 5.1.3. *If $\mathcal{P}$ is any H -invariant prime ideal of R , there exists a H -invariant prime ideal $\mathcal{Q}$ of $R_0^{(t+1)}$, such that*

- $\mathcal{P} = \mathcal{Q} \cap R$.
- $\mathcal{Q} \cap \{X_{t+1}^{(t+1)}, \dots, X_N^{(t+1)}\} = \emptyset$.

Proof. Observe that $R_0^{(t+1)} = R[X_{t+1}^{(t+1)}; \sigma_{t+1}] \cdots [X_N^{(t+1)}; \sigma_N]$, where each σ_i is defined by the action of some element of H . So, lemma 5.1.3 actually follows easily from $N - t$ iterations of ([15], corollary 2.4). $\square$

Lemma 5.1.4. *Consider $\mathcal{P}_0 \in \text{Spec}(R_0)$ and $\mathcal{P} \in \text{Spec}(R)$.*

Assume that $\mathcal{P} = \mathcal{Q} \cap R$ with $\mathcal{Q} = \mathcal{P}_0^{(t+1)} =$ the canonical image of $\mathcal{P}_0$ in $\text{Spec}(R_0^{(t+1)})$.

For any $m \in \llbracket 2, \dots, t+1 \rrbracket$, we denote by $\mathcal{P}^{(m)}$ (resp. $\mathcal{P}_0^{(m)}$) the canonical image of $\mathcal{P}$ (resp. $\mathcal{P}_0$) in $\text{Spec}(R^{(m)})$ (resp. $\text{Spec}(R_0^{(m)})$) and we recall that, by lemma 5.1.2, $R^{(m)}$ is the subalgebra of $R_0^{(m)}$ generated by its t first canonical generators $X_1^{(m)}, \dots, X_t^{(m)}$. Then

$$\mathcal{P}^{(m)} = \mathcal{P}_0^{(m)} \cap R^{(m)}.$$

Proof. We prove this by decreasing induction on m .

- If $m = t+1$, we have $R^{(m)} = R$, $\mathcal{P}^{(m)} = \mathcal{P}$ and $\mathcal{P}_0^{(m)} = \mathcal{Q}$. So, we have the required equality by assumption.
- Assume $m \leq t$ and $\mathcal{P}^{(m+1)} = \mathcal{P}_0^{(m+1)} \cap R^{(m+1)}$.
 - Assume that $X_m^{(m+1)} \notin \mathcal{P}_0^{(m+1)}$, so that $X_m^{(m+1)} \notin \mathcal{P}^{(m+1)}$.

Set $S_m = \{(X_m^{(m+1)})^d \mid d \in \mathbb{N}\}$ and recall (sections 3.2 and 3.3) that

$$\begin{aligned} * \quad R^{(m)} S_m^{-1} &= R^{(m+1)} S_m^{-1}. \\ * \quad \mathcal{P}^{(m)} &= \mathcal{P}^{(m+1)} S_m^{-1} \cap R^{(m)}. \\ * \quad \mathcal{P}_0^{(m)} &= \mathcal{P}_0^{(m+1)} S_m^{-1} \cap R_0^{(m)}. \end{aligned}$$

As $R^{(m)} \subseteq R_0^{(m)}$ and $\mathcal{P}^{(m+1)} \subseteq \mathcal{P}_0^{(m+1)}$, we have

$$\mathcal{P}^{(m)} = \mathcal{P}^{(m+1)} S_m^{-1} \cap R^{(m)} \subseteq \mathcal{P}_0^{(m+1)} S_m^{-1} \cap R_0^{(m)} = \mathcal{P}_0^{(m)}.$$

This implies that

$$\mathcal{P}^{(m)} \subseteq \mathcal{P}_0^{(m)} \cap R^{(m)}.$$

Now, consider some $u \in \mathcal{P}_0^{(m)} \cap R^{(m)}$.

We have $\mathcal{P}_0^{(m)} \subseteq \mathcal{P}_0^{(m+1)} S_m^{-1} \Rightarrow u (X_m^{(m+1)})^{d_1} \in \mathcal{P}_0^{(m+1)}$ with $d_1 \in \mathbb{N}$.

We have $R^{(m)} \subseteq R^{(m+1)} S_m^{-1} \Rightarrow u (X_m^{(m+1)})^{d_2} \in R^{(m+1)}$ with $d_2 \in \mathbb{N}$.

Since $m \leq t$, $X_m^{(m+1)}$ is in $R^{(m+1)}$ and so, with $d = \max(d_1, d_2)$, we obtain that

$$u (X_m^{(m+1)})^d \in \mathcal{P}_0^{(m+1)} \cap R^{(m+1)} = \mathcal{P}^{(m+1)} \Rightarrow u \in \mathcal{P}^{(m+1)} S_m^{-1} \cap R^{(m)} = \mathcal{P}^{(m)}.$$

This implies that

$$\mathcal{P}^{(m)} = \mathcal{P}_0^{(m)} \cap R^{(m)}.$$

- Assume that $X_m^{(m+1)} \in \mathcal{P}_0^{(m+1)}$, so that, since $m \leq t$,

$$X_m^{(m+1)} \in \mathcal{P}_0^{(m+1)} \cap R^{(m+1)} = \mathcal{P}^{(m+1)}.$$

Denote by

$$g : R^{(m)} \rightarrow \frac{R^{(m+1)}}{\mathcal{P}^{(m+1)}}$$

the only algebra homomorphism which transforms each $X_i^{(m)}$ ($1 \leq i \leq t$) into $g(X_i^{(m)}) = X_i^{(m+1)} + \mathcal{P}^{(m+1)}$ (the canonical image of $X_i^{(m+1)}$ in $\frac{R^{(m+1)}}{\mathcal{P}^{(m+1)}}$). We recall (section 3.3) that

$$\mathcal{P}^{(m)} = \ker(g).$$

Denote by

$$g_0 : R_0^{(m)} \rightarrow \frac{R_0^{(m+1)}}{\mathcal{P}_0^{(m+1)}}$$

the only algebra homomorphism which transforms each $X_i^{(m)}$ ($1 \leq i \leq N$) into $g_0(X_i^{(m)}) = X_i^{(m+1)} + \mathcal{P}_0^{(m+1)}$ (the canonical image of $X_i^{(m+1)}$ in $\frac{R_0^{(m+1)}}{\mathcal{P}_0^{(m+1)}}$), so that

$$\mathcal{P}_0^{(m)} = \ker(g_0).$$

The canonical homomorphism $R_0^{(m+1)} \rightarrow \frac{R_0^{(m+1)}}{\mathcal{P}_0^{(m+1)}}$ restricted to $R^{(m+1)}$ has its kernel equal to $\mathcal{P}_0^{(m+1)} \cap R^{(m+1)} = \mathcal{P}^{(m+1)}$. So, it induces an injective homomorphism

$$\epsilon : \frac{R^{(m+1)}}{\mathcal{P}^{(m+1)}} \rightarrow \frac{R_0^{(m+1)}}{\mathcal{P}_0^{(m+1)}}$$

which transforms each $X_i^{(m+1)} + \mathcal{P}^{(m+1)}$ ($1 \leq i \leq t$) into $X_i^{(m+1)} + \mathcal{P}_0^{(m+1)}$. So,

$$\epsilon \circ g : R^{(m)} \rightarrow \frac{R_0^{(m+1)}}{\mathcal{P}_0^{(m+1)}}$$

is an algebra homomorphism which transforms each $X_i^{(m)}$ ($1 \leq i \leq t$) into $X_i^{(m+1)} + \mathcal{P}_0^{(m+1)} = g_0(X_i^{(m)})$. This implies that $\epsilon \circ g$ is the restriction of g_0 to $R^{(m)}$ and so, $\ker(\epsilon \circ g) = \mathcal{P}_0^{(m)} \cap R^{(m)}$. Since ϵ is injective, we also have $\ker(\epsilon \circ g) = \ker(g) = \mathcal{P}^{(m)}$ and the proof is over. $\square$

Proposition 5.1.1. *Consider a diagram Δ with respect to (1) (i.e. a subset of $\llbracket 1, \dots, t \rrbracket$), so that Δ is also a diagram with respect to (17)). Then Δ is admissible with respect to (1) if and only if Δ is admissible with respect to (17).*

Proof. • Assume that Δ is admissible with respect to (17). This means (see section 3.3) that there exists some $\mathcal{P}_0$ in $\text{Spec}(R_0)$ such that its canonical image $\mathcal{P}_0^{(2)}$ in $\text{Spec}(R_0^{(2)})$ satisfies:

$$\mathcal{P}_0^{(2)} \cap \{Z_1, \dots, Z_N\} = \{Z_i | i \in \Delta\}.$$

Denote by $\mathcal{Q} = \mathcal{P}_0^{(t+1)}$ the canonical image of $\mathcal{P}_0$ in $\text{Spec}(R_0^{(t+1)})$ and set $\mathcal{P} = \mathcal{Q} \cap R$. We know that $\mathcal{P} \in \text{Spec}(R)$ and, by lemma 5.1.4, its canonical image $\mathcal{P}^{(2)}$ in $\text{Spec}(R^{(2)})$ satisfies

$$\mathcal{P}^{(2)} = \mathcal{P}_0^{(2)} \cap R^{(2)}.$$

So,

$$\mathcal{P}^{(2)} \cap \{Z_1, \dots, Z_t\} = \mathcal{P}_0^{(2)} \cap R^{(2)} \cap \{Z_1, \dots, Z_t\} = \mathcal{P}_0^{(2)} \cap \{Z_1, \dots, Z_t\}.$$

Since Δ is a subset of $\llbracket 1, \dots, t \rrbracket$, this implies that

$$\mathcal{P}^{(2)} \cap \{Z_1, \dots, Z_t\} = \{Z_i | i \in \Delta\}.$$

So Δ is admissible with respect to (1).

- Assume that Δ is admissible with respect to (1). This implies (see section 3.3) that there exists some $\mathcal{P}$ in $H - \text{Spec}(R)$ whose canonical image $\mathcal{P}^{(2)}$ in $\text{Spec}(R^{(2)})$ is the ideal of $R^{(2)}$ generated by $\{Z_i | i \in \Delta\}$ ($\mathcal{P}^{(2)} = 0$ if $\Delta = \emptyset$).

Since $\mathcal{P}$ is H -invariant, it follows from lemma 5.1.3 that there exists a (H -invariant) prime ideal $\mathcal{Q}$ of $R_0^{(t+1)}$, such that

- $\mathcal{P} = \mathcal{Q} \cap R$.
- $\mathcal{Q} \cap \{X_{t+1}^{(t+1)}, \dots, X_N^{(t+1)}\} = \emptyset$.

Now, by corollary 4.1.1, there exists $\mathcal{P}_0$ in $\text{Spec}(R_0)$ whose canonical image $\mathcal{P}_0^{(t+1)}$ in $\text{Spec}(R_0^{(t+1)})$ is equal to $\mathcal{Q}$ and whose canonical image $\mathcal{P}_0^{(2)}$ in $\text{Spec}(R_0^{(2)})$ does not meet the set $\{Z_{t+1}, \dots, Z_N\}$.

As above, by lemma 5.1.4, we have

$$\mathcal{P}^{(2)} = \mathcal{P}_0^{(2)} \cap R^{(2)}.$$

Since $\mathcal{P}_0^{(2)}$ does not meet the set $\{Z_{t+1}, \dots, Z_N\}$, we have

$$\mathcal{P}_0^{(2)} \cap \{Z_1, \dots, Z_N\} = \mathcal{P}_0^{(2)} \cap \{Z_1, \dots, Z_t\}.$$

So, since $\{Z_1, \dots, Z_t\} \subseteq R^{(2)}$, we can write

$$\begin{aligned} \mathcal{P}_0^{(2)} \cap \{Z_1, \dots, Z_N\} &= \mathcal{P}_0^{(2)} \cap R^{(2)} \cap \{Z_1, \dots, Z_t\} \\ &= \mathcal{P}^{(2)} \cap \{Z_1, \dots, Z_N\} = \{Z_i | i \in \Delta\}. \end{aligned}$$

This proves that Δ is admissible with respect to (17). $\square$

5.2. A partial result.

Lemma 5.2.1. *For any diagram Δ (with respect to (1)) we have:*

$$\Delta \text{ positive} \Rightarrow \Delta \text{ admissible}.$$

Proof. As in section 3.3, we denote by $\mathcal{P}_\Delta^{(2)}$ the ideal of $\overline{R} = R^{(2)}$ generated by the canonical generators Z_i with $i \in \Delta$ ($\mathcal{P}_\Delta^{(2)} = 0$ if $\Delta = \emptyset$). Recall (see section 3.3) that $\mathcal{P}_\Delta^{(2)} \in H - \text{Spec}(R^{(2)})$.

Consider an integer m with $2 \leq m \leq t+1$. We first prove, by induction on m , that there exists a prime ideal $\mathcal{P}^{(m)} \in \text{Spec}(R^{(m)})$ such that $\mathcal{P}_\Delta^{(2)}$ is the canonical image of $\mathcal{P}^{(m)}$ in $\text{Spec}(R^{(2)})$.

- If $m = 2$, we just have to choose $\mathcal{P}^{(2)} = \mathcal{P}_\Delta^{(2)}$.
- Assume that, for some m with $2 \leq m \leq t$, there exists a prime ideal $\mathcal{P}^{(m)} \in \text{Spec}(R^{(m)})$ such that $\mathcal{P}_\Delta^{(2)}$ is the canonical image of $\mathcal{P}^{(m)}$ in $\text{Spec}(R^{(2)})$. Since Δ is positive, we deduce from proposition 4.4.1 that $\mathcal{P}^{(m)} = \phi_m(\mathcal{P}^{(m+1)})$ with $\mathcal{P}^{(m+1)}$ in $\text{Spec}(R^{(m+1)})$.

So, $\mathcal{P}_\Delta^{(2)}$ is the canonical image of $\mathcal{P}^{(m+1)}$ in $\text{Spec}(R^{(2)})$, and our affirmation is proved.

In particular, there exists $\mathcal{P} \in \text{Spec}(R^{(t+1)}) = \text{Spec}(R)$ such that $\mathcal{P}_\Delta^{(2)}$ is the canonical image of $\mathcal{P}$ in $\text{Spec}(R^{(2)})$. This means (see section 3.3) that Δ is admissible. $\square$

Proposition 5.2.1. *Assume that $w = w_0$. Then, the positive diagrams (with respect to (1)) coincide with the admissible diagrams (with respect to (1)).*

Proof. By lemma 5.2.1, the set of positive diagrams is contained in the set of admissible diagrams. By proposition 2.3.1, the number of positive diagrams is equal to the number of u in the Weyl group W such that $u \leq w$ (where $\leq$ denotes

the Bruhat order). Since we assume that $w = w_0$, we get that the number of positive diagrams is equal to the cardinal $|W|$ of W . Since the set of admissible diagrams is in one-to-one correspondence with $H - \text{Spec}(R) = H - \text{Spec}(U^+)$, the number of admissible diagrams does not depend on the reduced decomposition of w . In [19], Antoine Mériaux gives a precise description of admissible diagrams, for each type of the simple complex Lie algebra $\mathfrak{g}$, and for a particular reduced decomposition of w_0 so that he can compute their number and check that it is precisely equal to $|W|$. This implies that the positive diagrams coincide with the admissible diagrams. $\square$

Let us observe that, in this proof, we use a result of Antoine Mériaux to prove that the number of admissible diagrams (which is also the cardinal of $H - \text{Spec}(U^+)$) is equal to $|W|$. This equality can also be obtained using results of M. Gorelik, N. Andruskiewitsch and F. Dumas ([9] and [2]) or T. J. Hodges and T. Levasseur and M. Toro [11], but this should require some (minor) restricted assumptions on the choice of the ground field k ($\text{char}(k) = 0$) or on the parameter q (q transcendental).

5.3. The general case. We can now prove, in the general case (i.e. w is no longer assumed to be the longest element of W):

Theorem 5.3.1. *The positive diagrams (with respect to (1)) coincide with the admissible diagrams (with respect to (1)).*

Proof. Consider a diagram Δ with respect to (1) (i.e. a subset of $\llbracket 1, \dots, t \rrbracket$), so that Δ is also a diagram with respect to (17).

- Assume that Δ is positive (with respect to (1)). By proposition 2.3.2, Δ is positive with respect to (17) and, by proposition 5.2.1, Δ is admissible with respect to (17). Now, by proposition 5.1.1, Δ is admissible with respect to (1).
- Assume that Δ is admissible (with respect to (1)). By proposition 5.1.1, Δ is admissible with respect to (17) and, by proposition 5.2.1, Δ is positive with respect to (17). Now, by proposition 2.3.2, Δ is positive with respect to (1). $\square$

Corollary 5.3.1. *(All the diagrams are with respect to (1).)*

- (1) *The map $\zeta: \Delta = \{j_1, \dots, j_s\} \mapsto u = w^\Delta = s_{\alpha_{j_1}} \circ \dots \circ s_{\alpha_{j_s}}$ is a bijection from the set of admissible diagrams onto the set $\{u \in W \mid u \leq w\}$ ($\zeta(\emptyset) = \text{Id}$).*
- (2) *Consider an admissible diagram $\Delta = \{j_1, \dots, j_s\}$ and some integer $i \in \llbracket 1, \dots, t \rrbracket$. Set $\Delta \cap \llbracket i+1, \dots, t \rrbracket = \{j_c, \dots, j_s\}$ ($1 \leq c \leq s$). Then the expression $s_{\alpha_i} \circ s_{\alpha_{j_c}} \circ \dots \circ s_{\alpha_{j_s}}$ is reduced. In particular, $s_{\alpha_{j_1}} \circ \dots \circ s_{\alpha_{j_s}}$ is a reduced expression of $\zeta(\Delta)$.*
- (3) *Consider some $u \in W$ with $u \leq w$. Then, the only admissible diagram Δ such that $\zeta(\Delta) = u$ is recursively defined as follows:*
 - $1 \in \Delta \Leftrightarrow l(s_{\alpha_1} \circ u) = l(u) - 1 \Leftrightarrow u^{-1}(\alpha_1)$ is a negative root.
 - Consider some integer $i \in \llbracket 1, \dots, t-1 \rrbracket$ and assume that $\Delta \cap \llbracket 1, \dots, i \rrbracket = \{j_1, \dots, j_d\}$.
Set $u_i = s_{\alpha_{j_d}} \circ \dots \circ s_{\alpha_{j_2}} \circ s_{\alpha_{j_1}} \circ u$ ($u_i = u$ if $\Delta \cap \llbracket 1, \dots, i \rrbracket = \emptyset$). Then $i+1 \in \Delta \Leftrightarrow l(s_{\alpha_{i+1}} \circ u_i) = l(u_i) - 1 \Leftrightarrow u_i^{-1}(\alpha_{i+1})$ is a negative root.

Proof. As the admissible diagrams coincide with the positive diagrams, (1) is proposition 2.3.1 (assertion (2)) and (2) is proposition 2.2.2. Now, let us prove (3).

• Assume that $1 \in \Delta$, so that $\Delta = \{j_1, \dots, j_s\}$ with $j_1 = 1$. As $u = \zeta(\Delta) = s_{\alpha_{j_1}} \circ \dots \circ s_{\alpha_{j_s}}$, we have $l(s_{\alpha_1} \circ u) = l(s_{\alpha_{j_2}} \circ \dots \circ s_{\alpha_{j_s}}) = s - 1 = l(u) - 1$.

Conversely, assume that $l(s_{\alpha_1} \circ u) = l(u) - 1$. If $1 \notin \Delta$, then $\Delta \cap \llbracket 2, \dots, t \rrbracket = \Delta$ and, by (2), $l(s_{\alpha_1} \circ u) = l(u) + 1$. So, we obtain a contradiction.

Now, we have classically: $l(s_{\alpha_1} \circ u) = l(u) - 1 \Leftrightarrow l(u^{-1} \circ s_{\alpha_1}) = l(u^{-1}) - 1 \Leftrightarrow u^{-1}(\alpha_1)$ is a negative root.

• Assume that $i + 1 \in \Delta$, so that $\Delta \cap \llbracket i + 1, \dots, t \rrbracket = \{j_{d+1}, \dots, j_s\}$ with $j_{d+1} = i + 1$. As $u_i = s_{\alpha_{j_d}} \circ \dots \circ s_{\alpha_{j_2}} \circ s_{\alpha_{j_1}} \circ u = s_{\alpha_{j_{d+1}}} \circ \dots \circ s_{\alpha_{j_s}}$, we have $l(s_{\alpha_{i+1}} \circ u) = l(s_{\alpha_{j_{d+2}}} \circ \dots \circ s_{\alpha_{j_s}}) = s - (d + 1) = l(u_i) - 1$.

Conversely, assume that $l(s_{\alpha_{i+1}} \circ u_i) = l(u_i) - 1$. If $i + 1 \notin \Delta$, then $\Delta \cap \llbracket i + 2, \dots, t \rrbracket = \{j_{d+1}, \dots, j_s\}$ and, by (2), $l(s_{\alpha_{i+1}} \circ u_i) = l(u_i) + 1$. So, we still have a contradiction and we conclude as above. $\square$

From corollary 5.3.1 (assertions (1) and (2)) and lemma 2.2.1 (assertion (2)), we deduce:

Corollary 5.3.2. (1) *The map $\zeta' : \Delta = \{j_1, \dots, j_s\} \mapsto u' = v^\Delta = s_{\alpha_{j_s}} \circ \dots \circ s_{\alpha_{j_1}}$ is a bijection from the set of admissible diagrams onto the set $\{u \in W \mid u \leq v = w^{-1}\}$ ($\zeta'(\Delta) = (\zeta(\Delta))^{-1}, \zeta'(\emptyset) = Id$).*
 (2) *Consider an admissible diagram $\Delta = \{j_1, \dots, j_s\}$ and some integer $i \in \llbracket 1, \dots, t \rrbracket$. Set $\Delta \cap \llbracket i + 1, \dots, t \rrbracket = \{j_c, \dots, j_s\}$ ($1 \leq c \leq s$). Then the expression $s_{\alpha_{j_s}} \circ \dots \circ s_{\alpha_{j_c}} \circ s_{\alpha_i}$ is reduced. In particular, $s_{\alpha_{j_s}} \circ \dots \circ s_{\alpha_{j_1}}$ is a reduced expression of $\zeta'(\Delta)$.*

If w and its reduced decomposition are chosen as in the example of section 2.1, we know (proposition 2.1.1) that $U^+[w]$ is the algebra of quantum matrices $O_q(M_{p,m}(k))$ with $m = n - p + 1$ and with the same canonical generators (as $U^+[w]$). So, by [5], the admissible diagrams are the J-diagrams. In this case, corollary 5.3.2 has also been proved (with quite different methods) by A. Postnikov ([20], theorem 19.1) and by T. Lam and L. Williams ([17], theorem 5.3).

In the general case, using the bijection between the set of admissible diagrams and $H - \text{Spec}(U^+[w])$ recalled in section 3.3, the two bijections of proposition 2.3.1 and the coincidence of admissible and positive diagrams (theorem 5.3.1), we get

Corollary 5.3.3. (1) *There exists a natural bijection from the set $\{u \in W \mid u \leq v = w^{-1}\}$ onto $H - \text{Spec}(U^+[w])$. It is defined by*

$$v^\Delta \mapsto \mathcal{P}_\Delta$$

where Δ runs through the set of admissible diagrams.

(2) *There exists a natural bijection from the set $\{u \in W \mid u \leq w\}$ onto $H - \text{Spec}(U^+[w])$. It is defined by*

$$w^\Delta \mapsto \mathcal{P}_\Delta$$

where Δ runs through the set of admissible diagrams.

When $U^+[w]$ is the quantum matrix algebra $O_q(M_{p,m}(k))$ as above, S. Launois [14] has constructed (with different methods) a bijection from $\{u \in W \mid u \leq v\}$ onto $H - \text{Spec}(O_q(M_{p,m}(k)))$ which, moreover, preserves the ordering (where the Weyl group is provided with the Bruhat order and $H - \text{Spec}(O_q(M_{p,m}(k)))$ is provided with the inclusion). So, it seems natural to ask the following questions:

Question 5.3.1. Assume w and its reduced decomposition chosen as in the example of section 2.1.

Does the S. Launois bijection from $\{u \in W | u \leq v\}$ onto $H - \text{Spec}(O_q(M_{p,m}(k))) = H - \text{Spec}(U^+[w])$ coincide with the first bijection of corollary 5.3.3? *If not, is there a simple relation between those two bijections?*

Question 5.3.2. In the general case, are the two bijections of corollary 5.3.3 isomorphisms of ordered sets, where the set $\{u \in W | u \leq v\}$, (resp. $\{u \in W | u \leq w\}$) is provided with the Bruhat order and $H - \text{Spec}(U^+[w])$ is provided with the inclusion?

Under stricter assumptions than in the present paper (namely assuming that k has characteristic zero and that q is transcendental over $\mathbb{Q}$), M. Yakimov [22] constructs an order isomorphism between the set $\{u \in W | u \leq w\}$ equipped with the Bruhat order and $H - \text{Spec}(U^+[w])$ equipped with inclusion. This raises a new question:

Question 5.3.3. Is this Yakimov bijection the same as the one in the corollary 5.3.3 (2)?

REFERENCES

- [1] J. Alev and F. Dumas, *Sur le corps des fractions de certaines algèbres quantiques*, J. Algebra **170** (1994), 229-265. MR1302839 (96c:16033)
- [2] N. Andruskiewitsch and F. Dumas, *On the automorphisms of $U_q^+(\mathfrak{g})$* “Quantum Groups” (B. Enriquez, ed.), IRMA Lectures in Mathematical and Theoretical Physics **12** European Math. Society (2008), 107-133. MR2432991 (2009i:17019)
- [3] K. A. Brown and K. R. Goodearl, *Lectures on algebraic quantum groups*, Advanced Courses in Mathematics, CRM Barcelona, Birkhäuser Verlag, Basel, 2002. MR1898492 (2003f:16067)
- [4] G. Cauchon, *Effacement des dérivations et spectres premiers des algèbres quantiques*, J. Algebra **260** (2003), 476-518. MR1967309 (2004g:16044)
- [5] G. Cauchon, *Spectre premier de $O_q(M_n(k))$: image canonique et séparation normale*. J. Algebra **260** (2003), 419-569. MR1967310 (2004g:16045)
- [6] V. Deodhar, *A combinatorial setting for questions in Kazhdan-Lusztig theory*, Geometriae Dedicata **36** (1990), 95-119. MR1065215 (91h:20075)
- [7] K. R. Goodearl and E. S. Letzter, *Prime factor algebras of the coordinate ring of quantum matrices*, Proc. Amer. Math. Soc. **121** (1994), 1017-1025. MR1211579 (94j:16066)
- [8] K. R. Goodearl and R. B. Warfield Jr., *An introduction to noncommutative noetherian rings*, London Math. Soc. Student texts **16**, Cambridge University Press, Cambridge (1989). MR1020298 (91c:16001)
- [9] M. Gorelik, *The prime and primitive spectra of a quantum Bruhat cell translate*, J. Algebra **227** (2000), 211-253. MR1754232 (2001d:17012)
- [10] J. E. Humphreys, *Introduction to Lie Algebras and Representations Theory*, Graduate Texts in Mathematics **9**, Springer-Verlag, New York (1980). MR499562 (81b:17007)
- [11] T. J. Hodges and T. Levasseur and M. Toro, *Algebraic structure of multiparameter quantum groups*, Advances in Maths. **126** (1997), 52-92. MR1440253 (98e:17022)
- [12] J. C. Jantzen, *Lectures on quantum groups*, Graduate Studies in Mathematics **6**, Amer. Math. Soc., Providence (1996). MR1359532 (96m:17029)
- [13] A. Joseph, *Quantum Groups and Their Primitive Ideals*, Ergebnisse der Mathematik **(3) 29**, Springer-Verlag, Berlin (1995). MR1315966 (96d:17015)
- [14] S. Launois, *Combinatorics of H -primes in quantum matrices*, J. Algebra **309** (2007), 139-167. MR2301235 (2007k:05221)
- [15] S. Launois, T. H. Lenagan and L. Rigal, *Quantum unique factorisation domains*, J. London Math. Soc. **(2) 74** (2006), 321-340. MR2269632 (2007h:16047)
- [16] S. Z. Levendorskii and Ya. S. Soibelman, *Algebras of functions on a compact quantum group, Schubert cells and quantum tori*, Comm. Math. Phys. **139** (1991), 141-170. MR1116413 (92h:58020)

- [17] T. Lam and L. Williams, *Total positivity for cominuscule grassmannians*, New York Journal of Mathematics **14** (2008), 53-99. MR2383586 (2008m:05307)
- [18] R. Marsh and K. Rietsch, *Parametrisations of flag varieties*, Representation Theory, **8** (2004), 212-242. MR2058727 (2005c:14061)
- [19] A. Mériaux, *Les diagrammes de Cauchon pour $U_q^+(\mathfrak{g})$* , J. Algebra **323** (2010), 1060-1097.
- [20] A. Postnikov, *Total positivity, Grassmannians and Networks*, Preprint, 2006: <http://arxiv.org/abs/math/0609764>.
- [21] M. Yakimov, *Invariant prime ideals in quantizations of nilpotent Lie algebras*, Preprint, 2009: <http://arxiv.org/abs/0905.0852>
- [22] H. Yamane, *A Poincaré-Birkhoff-Witt theorem for quantized universal enveloping algebra of type A_N* , Publ. Res. Inst. Math. Sci. **25** (1989), 503-520. MR1018513 (91a:17016)

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