GENERIC HECKE ALGEBRAS FOR MONOMIAL GROUPS

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ABSTRACT. In this paper we define a two-variable, generic Hecke algebra, $\mathcal{H}$, for each complex reflection group $G(b, 1, n)$. The algebra $\mathcal{H}$ specializes to the group algebra of $G(b, 1, n)$ and also to an endomorphism algebra of a representation of $\mathrm{GL}_n(\mathbb{F}_q)$ induced from a solvable subgroup. We construct Kazhdan-Lusztig “ R -polynomials” for $\mathcal{H}$ and show that they may be used to define a partial order on $G(b, 1, n)$. Using a generalization of Deodhar’s notion of distinguished subexpressions we give a closed formula for the R -polynomials. After passing to a one-variable quotient of the ring of scalars, we construct Kazhdan-Lusztig polynomials for $\mathcal{H}$ that reduce to the usual Kazhdan-Lusztig polynomials for the symmetric group when $b = 1$.

1. INTRODUCTION

Suppose G is a finite group of Lie type and k is a field whose characteristic is relatively prime to $|G|$. Then the irreducible k -representations of G are partitioned into Harish-Chandra series (see for example [7]). An irreducible representation is cuspidal if it is the only member in its Harish-Chandra series. Non-cuspidal representations can be constructed and their character values computed using the theory of endomorphism algebras of induced representations.

The prototypical example of a Harish-Chandra series that does not consist of just one cuspidal representation is the principal series. In this case, the endomorphism algebra of interest is the endomorphism algebra of a representation induced from the trivial representation of a Borel subgroup of G . This algebra is a specialization of a generic algebra, the Iwahori-Hecke algebra of the Weyl group W of G . The Iwahori-Hecke algebra also specializes to the group algebra of W .

Broué, Malle, and Michel (see [2], [3], [4]) have shown that if the characteristic of k is different from the defining characteristic of G , then the classical Harish-Chandra theory just outlined can be extended to describe the blocks of kG . A new feature that arises when the characteristic of k divides $|G|$ is that the endomorphism rings used to analyze non-cuspidal representations are deformations of group algebras of complex reflection groups that are not Weyl groups or even Coxeter groups.

In order to study Iwahori-Hecke algebras, Kazhdan and Lusztig [13] and Lusztig [14] have developed a powerful theory for analyzing representations of Iwahori-Hecke algebras which in turn plays a central role in describing the irreducible representations of G .

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In this paper we consider a variation of the above themes and define a two-variable generic algebra, $\mathcal{H}$, for finite general linear groups that specializes to the group algebra of the complex reflection group $G(b, 1, n)$ and also specializes to the endomorphism algebra of an induced representation of $\mathrm{GL}_n(\mathbb{F}_q)$. This last algebra is closely related to the principal block of $\mathrm{GL}_n(\mathbb{F}_q)$ when $b = |\mathrm{GL}_n(\mathbb{F}_q)|_l$ where l is the characteristic of k (see §3).

Starting from the observation that the Bruhat-Chevalley order on a finite Coxeter group is determined by the non-vanishing of the “ R -polynomials” of Kazhdan and Lusztig, we construct R -polynomials for $\mathcal{H}$ and show that the resulting relation on $G(b, 1, n)$ is a partial order. This partial order is the Bruhat order on the symmetric group when $b = 1$ but it is not the Bruhat order on hyperoctahedral groups when $b = 2$.

Passing to a single variable quotient of our two-variable ring of scalars we define Kazhdan-Lusztig polynomials for $G(b, 1, n)$. When $b = 1$ these are the usual Kazhdan-Lusztig polynomials for the symmetric group.

Our construction has a generalization to arbitrary finite Chevalley groups or $\mathbb{F}_q$ -points of a reductive algebraic group. When the underlying root system is not of type A , the groups that arise are no longer complex reflection groups. In order to retain the connection with complex reflection groups and to make the exposition of the ideas as accessible as possible, in this paper we consider only the case of the general linear group.

A specialization of the algebra $\mathcal{H}$ has been considered by Cabanes and Enguehard [5, Chapter 23]. Because our construction is generic it opens up the possibility of relating the specialized algebra studied by Cabanes and Enguehard with other algebras used to study representations of finite reductive groups. In particular, because $\mathcal{H}$ specializes to the group algebra of $G(b, 1, n)$ it should be related to a suitable specialization of an Ariki-Koike algebra. It also seems likely that $\mathcal{H}$ is a cellular algebra in the sense of Graham and Lehrer [12]. We hope to consider these questions, as well as the representation theory of $\mathcal{H}$, in future work.

For the rest of this paper, G will denote $\mathrm{GL}_n(\mathbb{F}_q)$ where q is a prime power. Suppose that a and b are relatively prime, positive integers with $ab = q - 1$. The multiplicative group of $\mathbb{F}_q$ is a cyclic group with order $q - 1$ and so it factors as the direct product of a cyclic group of order a , which we will denote by F_a , and a cyclic group of order b , which we will denote by F_b . Let H be the subgroup of G consisting of diagonal matrices. Let H_a and H_b be the subgroups of H with entries in F_a and F_b respectively. Then clearly $H \cong H_a \times H_b$.

Let U denote the subgroup of G consisting of upper triangular, unipotent matrices and define $B_a = H_a B$. Notice that $B_a = U$ when $a = 1$ and B_a is a Borel subgroup of G when $a = q - 1$.

Let e denote the centrally primitive idempotent in the group algebra $\mathbb{C}B_a$ corresponding to the trivial representation of B_a . In §2 we study the subalgebra $e\mathbb{C}Ge$. This algebra is isomorphic to the opposite algebra of the endomorphism algebra of the representation of G induced from the trivial representation of B_a . Let W denote the subgroup of G consisting of permutation matrices. It is easily seen that the subgroup WH_b of G is isomorphic to $G(b, 1, n)$ and that WH_b is in fact a complete set of (B_a, B_a) -double coset representatives. In §2 we prove some multiplication relations in $e\mathbb{C}Ge$. These relations are analogous to the braid and

quadratic relations in the Iwahori-Hecke algebra of W and also to the relations in the Bernstein-Zelevinsky presentation of the extended, affine Hecke algebra of W .

In §3 we define a generic algebra $\mathcal{H}$ using the relations from §2 as a model. This is entirely analogous to the construction of the Iwahori-Hecke algebra. We show that our generic algebra has a basis indexed by WH_b using an argument that goes back to [1, Ch. 4, Ex. 23]. We then record some of the standard properties of the Iwahori-Hecke algebra that remain true in our setup and give an application to the characters in the principal block of G in the case when H_b is a Sylow- l subgroup of G and l is the characteristic of k .

In the general case when G_0 is a finite Chevalley group with the defining characteristic equal to p and U is a Sylow- p subgroup of G_0 , Yokonuma [16] has given a presentation by generators and relations for the Hecke ring $\mathcal{H}(G_0, U)$ and defined a generic algebra for (G_0, U) . The Hecke ring $\mathcal{H}(G, P)$, for a subgroup P of G_0 containing U , is then realized as a subalgebra of the Hecke ring $\mathcal{H}(G_0, U)$ obtained by summing basis elements of $\mathcal{H}(G_0, U)$. Our results in §2 and §3 may alternatively be derived from Yokonuma's results in the same way that the Hecke ring $\mathcal{H}(G, B)$ and the Iwahori-Hecke algebra are derived from the group algebra of G_0 by analyzing the structure constants for a particular choice of basis.

In §4 we construct Kazhdan-Lusztig “ R -polynomials” and show that they can be used to define a partial order on WH_b . In the proof, we relate intervals in the partial order with intervals in the Bruhat order on W and certain subsets of H_b .

The algebra $\mathcal{H}$ is a $\mathbb{Z}[\mathbf{a}, \mathbf{v}, \mathbf{v}^{-1}]$ -algebra where $\mathbf{a}$ and $\mathbf{v}$ are indeterminates. In §5, after passing to a quotient isomorphic to $\mathbb{Z}_b[\mathbf{v}, \mathbf{v}^{-1}]$, where $\mathbb{Z}_b$ denotes the localization of $\mathbb{Z}$ at b , we construct a Kazhdan-Lusztig basis and Kazhdan-Lusztig polynomials following an argument in [15]. We compute these “Kazhdan-Lusztig” polynomials when $n = 3$ and b is arbitrary. When $b = 1$, these are the usual Kazhdan-Lusztig polynomials and so are known a priori to be the constant polynomial 1 in all cases. In contrast, when $b \neq 1$, some of our polynomials have positive degree.

Finally, in §6 we adapt Deodhar's ideas in [10] to describe the lower order ideals in the poset $(WH_b, \leq)$ and to give a closed form expression for the R -polynomials. These arguments use an analog in WH_b of a reduced expression of an element in W .

2. THE ALGEBRA $e\mathbb{C}Ge$

Recall that $G = \mathrm{GL}_n(\mathbb{F}_q)$ and that $q - 1 = ab$ where a and b are relatively prime.

In this section $\mathcal{H}_{q,a}$ will denote the “Hecke algebra” $e\mathbb{C}Ge$, where $\mathbb{C}G$ is the group algebra of G and e is the centrally primitive idempotent $|B_a|^{-1} \sum_{b \in B_a} b$ in $\mathbb{C}B_a$. Thus, $\mathcal{H}_{q,a}$ is anti-isomorphic to the endomorphism ring of the induced representation $\mathrm{ind}_{B_a}^G 1_{B_a}$.

Suppose $\{x_1, \dots, x_m\}$ is a complete set of (B_a, B_a) double coset representatives and $D_i = B_a x_i B_a$ is the double coset containing x_i . It is well known (see [6, Proposition 11.34]) that if we consider $\mathbb{C}G$ as $\mathbb{C}$ -valued functions on G and let $T_{x_i} = |B_a|^{-1} \chi_i$, where χ_i is the characteristic function of D_i , then the T_{x_i} 's are a basis of $\mathcal{H}_{q,a}$. Moreover, the multiplication in $\mathcal{H}_{q,a}$ is given by

$$T_{x_i} T_{x_j} = \sum_{k=1}^m \mu_{x_i, x_j, x_k} T_{x_k}, \text{ where } \mu_{x_i, x_j, x_k} = |B_a|^{-1} |D_i \cap x_k D_j^{-1}|.$$

For a permutation matrix w in W , define $U_w^- = \{u \in U \mid wuw^{-1} \in w_0Uw_0\}$, where w_0 is the permutation matrix with 1's on the antidiagonal. Then by the strong form of the Bruhat decomposition for G we have $G = \coprod_{w \in W} U_{w^{-1}}^- wHU$ with uniqueness of expression. Since $H \cong H_b \times H_a$, this proves the next lemma.

Lemma 2.1. *Every element in G has a unique expression as a product $u_1wt_at_2u_2$ where w is in W , u_1 is in $U_{w^{-1}}^-$, h_a is in H_a , h_b is in H_b , and u is in U . In particular, the subgroup WH_b of G is a complete set of (B_a, B_a) -double coset representatives.*

It follows that $\dim \mathcal{H}_{q,a} = |WH_b| = b^n n!$.

Fix a generator, ζ , of $\mathbb{F}_q^\times$. Then ζ^b generates F_a and ζ^a generates F_b . In order to determine the structure constants $\mu_{x,y,z}$ for x, y , and z in WH_b we need the following lemma, whose easy proof will be omitted.

Lemma 2.2. *Suppose $u = \begin{bmatrix} 1 & \zeta^c \\ 0 & 1 \end{bmatrix}$ is in $\text{GL}_2(\mathbb{F}_q)$ with $0 < c < q-1$ and $s = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Write $c = am + bn$ where $0 \leq m \leq b-1$ and $0 \leq n \leq a-1$. Then $sus = u_1sh_bh_au_2$ where*

$$u_1 = u_2 = \begin{bmatrix} 1 & \zeta^{-c} \\ 0 & 1 \end{bmatrix}, \quad h_b = \begin{bmatrix} \zeta^{am} & 0 \\ 0 & (-1)^{b-1}\zeta^{-am} \end{bmatrix}, \quad \text{and} \quad h_a = \begin{bmatrix} \zeta^{bn} & 0 \\ 0 & (-1)^b\zeta^{-bn} \end{bmatrix}.$$

Notice that $-1 = (\zeta^b)^{a/2}$ is in F_a if b is odd and $-1 = (\zeta^a)^{b/2}$ is in F_b if b is even.

For $1 \leq i \leq n-1$ define

$$s_i = \begin{matrix} & i & i+1 \\ i & & \\ i+1 & & \end{matrix} \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 0 & 1 & \\ & & 1 & 0 & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}$$

so $\mathcal{S} = \{s_1, \dots, s_{n-1}\}$ is a set of Coxeter generators for W .

For α in F_b and $1 \leq i \neq j \leq n$, define $h_i(\alpha)$ to be the diagonal matrix whose i th entry is α and whose other diagonal entries are 1 and define $h_{i,j}(\alpha) = h_i(\alpha)h_j((-1)^{b-1}\alpha^{-1})$.

Corollary 2.3. *Suppose $s = s_i$ is in $\mathcal{S}$.*

- (1) *If u is in U_s^- and the non-zero, off-diagonal entry of u is ζ^c where $c = am + bn$, then $sus = u_1sh_b^u h_a^u u_2$ where $h_b^u = h_{i,i+1}(\zeta^{am})$.*
- (2) *The relation $u_1 \sim u_2$ if and only if $h_b^{u_1} = h_b^{u_2}$ is an equivalence relation on U_s^- and each equivalence class contains a elements.*

Proof. The first statement follows immediately from the lemma.

The relation in the second statement is clearly an equivalence relation. If u is in U_s^- and the non-zero, off-diagonal entry of u is ζ^{am+bn} , then it follows from the first statement that the equivalence class containing u is the set of all elements in U_s^- whose non-zero, off-diagonal entry is $\zeta^{am+bn'}$ where $0 \leq n' \leq a-1$. $\square$

Using the natural projection $WH_b \rightarrow W$ we can lift the length function, $\ell: W \rightarrow \mathbb{N}$, to WH_b . Then $\ell(wd) = \ell(dw) = \ell(w)$ for w in W and d in H_b .

Let $\mathcal{T} = \{ws_iw^{-1} \mid 1 \leq i \leq n-1, w \in W\}$ be the set of reflections in W . For $t = ws_iw^{-1}$ in $\mathcal{T}$, define

$$(2.3.1) \quad X_t = \{wh_{i,i+1}(\alpha)w^{-1} \mid \alpha \in F_b\}.$$

Then X_t does not depend on the choice of (w, i) . Also, if b is odd, then X_t is a subgroup of H_b isomorphic to F_b ; if b is even, then X_t is a coset of a subgroup of H_b isomorphic to F_b .

We can now describe the multiplication in $\mathcal{H}_{q,a}$.

Theorem 2.4. *The multiplication in the Hecke algebra $\mathcal{H}_{q,a}$ is determined by the following relations.*

- (1) *If d and d' are in H_b , then $T_{d'}T_d = T_{d'd}$. Thus, the rule $d \mapsto T_d$ extends linearly to an algebra isomorphism between the group algebra $\mathbb{C}H_b$ and the subset of $\mathcal{H}_{q,a}$ spanned by $\{T_d \mid d \in H_b\}$.*
- (2) *If d is in H_b and w is in W , then $T_wT_d = T_{wd}$ and $T_dT_w = T_{dw}$.*
- (3) *If w is in W and s is in S , then*

$$T_wT_s = \begin{cases} T_{ws} & \text{if } \ell(ws) > \ell(w), \\ qT_{ws} + a \sum_{d \in X_s} T_{wd} & \text{if } \ell(ws) < \ell(w). \end{cases}$$

Proof. Recall that for x and y in WH_b we have

$$T_xT_y = \sum_{z \in WH_b} \mu_{x,y,z} T_z$$

where $\mu_{x,y,z} = |B_a|^{-1} |B_axB_a \cap zB_ay^{-1}B_a|$. The statements in the theorem are all proved by computing $\mu_{x,y,z}$ for appropriate choices of x , y and z . The most complicated case is showing that $T_wT_s = qT_{ws} + a \sum_{d \in X_s} T_{wd}$ when $\ell(ws) < \ell(w)$. We prove this statement and omit the other computations.

Fix w in W and $s = s_i$ in S with $\ell(ws) < \ell(w)$. Then for x in WH_b we have

$$\begin{aligned} \mu_{w,s,x} &= \frac{1}{|B_a|} |B_awB_a \cap xB_asB_a| \\ &= \frac{1}{|B_a|} |\{(u, b) \in U_s^- \times B_a \mid xusb \in B_awB_a\}| \\ &= |\{u \in U_s^- \mid xus \in B_awB_a\}|. \end{aligned}$$

Suppose $x = w_1d$ where w_1 is in W and d is in H_b .

If $\ell(w_1s) > \ell(w_1)$, then for u is in U_s^- we have $w_1dus = u'w_1sd'$ where $u' = w_1dud^{-1}w_1^{-1}$ is in U and $d' = sds$ is in H_b . Thus, xus is in B_awB_a if and only if $w_1 = ws$ and $d = 1$. Therefore, $\mu_{w,s,ws} = q$ and $\mu_{w,s,x} = 0$ otherwise.

If $\ell(w_1s) < \ell(w_1)$, then $w_1dus = w_1s(sds)$ is in B_awB_a if and only if $w_1 = ws$ and $d = 1$. But then $\ell(w_1s) = \ell(w) > \ell(ws) = \ell(w_1)$, a contradiction, so $\mu(w, s, x) = 0$ in this case. If $\ell(w_1s) < \ell(w_1)$ and u is in U_s^- with $u \neq 1$, then $w_1dus = u'w_1dh_b^u b'$ where $u' = w_1dsu_1sd^{-1}w_1^{-1}$ is in U , $b' = h_a^u u_2$ is in B_a , and $sus = u_1sh_b^u h_a^u u_2$ as in Corollary 2.3. Thus, xus is in B_awB_a if and only if $w_1 = w$ and $d = (h_b^u)^{-1}$. It follows that $\mu_{w,s,x} = 0$ unless $w_1 = w$ and d is in X_s . For $x = wd$ with d in X_s we have

$$\mu_{w,s,wd} = |\{u \in U_s^- \mid d = (h_b^u)^{-1}\}|.$$

It follows from Corollary 2.3 that for a given $h_{i,i+1}(\alpha)$, there are a elements, u , in U_s^- with $h_b^u = h_{i,i+1}(\alpha)$ and so $\mu_{w,s,wd} = a$ for d in X_s .

We have shown that if $\ell(ws) < \ell(w)$, then

$$\mu_{w,s,x} = \begin{cases} q & x = ws, \\ a & x = wd, d \in X_s, \\ 0 & \text{otherwise,} \end{cases}$$

and so $T_w T_s = qT_{ws} + a \sum_{d \in X_s} T_{wd}$. This completes the proof of the theorem. $\square$

We conclude this section by recording some properties of H_b , and the subgroups X_t that will be used later.

Proposition 2.5. *For every t in $\mathcal{T}$, the set X_t is closed under taking inverses. If $t_1, \dots, t_{r+1}$ are in $\mathcal{T}$, then*

$$X_{t_1} \cdots X_{t_r} X_{t_{r+1}} = X_{t_1} \cdots X_{t_r} X_{t_1 \cdots t_r t_{r+1} t_r \cdots t_1}.$$

Proof. The fact that $X_t = X_t^{-1}$ follows immediately from the definition.

To prove the second statement we use induction on r .

Suppose $r = 1$. If $t_1 t_2 = t_2 t_1$, then the result is clear. Suppose $t_1 t_2 \neq t_2 t_1$. There are distinct i, j , and k with $1 \leq i, j, k \leq n$ so that t_1 interchanges the i th and j th standard basis vectors of $\mathbb{F}_q^n$, t_2 interchanges the j th and k th standard basis vectors of $\mathbb{F}_q^n$, and $t_1 t_2 t_1$ interchanges the i th and k th standard basis vectors of $\mathbb{F}_q^n$. Then $h_{i,j}(\alpha) h_{j,k}(\beta) = h_{i,j}(\alpha \beta^{-1}) h_{i,k}(\beta)$ for α and β in F_b and so

$$\begin{aligned} X_{t_1} X_{t_2} &= \{ h_{i,j}(\alpha) h_{j,k}(\beta) \mid \alpha, \beta \in F_b \} \\ &= \{ h_{i,j}(\alpha \beta^{-1}) h_{i,k}(\beta) \mid \alpha, \beta \in F_b \} \\ &= \{ h_{i,j}(\alpha_1) h_{i,k}(\beta_1) \mid \alpha_1, \beta_1 \in F_b \} \\ &= X_{t_1} X_{t_1 t_2 t_1}. \end{aligned}$$

Now suppose that $r > 1$. Then by induction and the case when $r = 1$ we have

$$\begin{aligned} X_{t_1} X_{t_2} \cdots X_{t_r} X_{t_{r+1}} &= X_{t_1} X_{t_2} \cdots X_{t_r} X_{t_2 \cdots t_r t_{r+1} t_r \cdots t_2} \\ &= X_{t_2} \cdots X_{t_r} X_{t_1} X_{t_2 \cdots t_r t_{r+1} t_r \cdots t_2} \\ &= X_{t_2} \cdots X_{t_r} X_{t_1} X_{t_1 \cdots t_r t_{r+1} t_r \cdots t_1} \\ &= X_{t_1} \cdots X_{t_r} X_{t_1 \cdots t_r t_{r+1} t_r \cdots t_1}. \end{aligned}$$

This completes the proof of the proposition. $\square$

Define $X_0 = \{ h_1(\zeta^{ai}) \mid 0 \leq i \leq b-1 \}$, so X_0 is a subgroup of H_b .

Proposition 2.6. *The multiplication mapping*

$$X_0 \times X_{s_1} \times \cdots \times X_{s_{n-1}} \rightarrow H_b$$

is a bijection.

Proof. Since the domain and codomain of the mapping both have cardinality b^n , it is enough to show that the mapping is injective.

We use induction on n , the base case being when $n = 2$.

Suppose $h_1(\zeta^{ai}) h_{1,2}(\alpha_1) = h_1(\zeta^{aj}) h_{1,2}(\beta_1)$. Then comparing $(2, 2)$ -entries we see that $(-1)^{b-1} \alpha_1 = (-1)^{b-1} \beta_1$ and so $\alpha_1 = \beta_1$. It follows that $h_1(\zeta^{ai}) = h_1(\zeta^{aj})$.

Suppose $n > 2$ and

$$h_1(\zeta^{ai}) h_{1,2}(\alpha_1) \cdots h_{n-1,n}(\alpha_{n-1}) = h_1(\zeta^{aj}) h_{1,2}(\beta_1) \cdots h_{n-1,n}(\beta_{n-1}).$$

Then comparing (n, n) -entries we see that $(-1)^{b-1}\alpha_{n-1} = (-1)^{b-1}\beta_{n-1}$ and so $\alpha_{n-1} = \beta_{n-1}$. Therefore,

$$h_1(\zeta^{ai})h_{1,2}(\alpha_1) \cdots h_{n-2,n-1}(\alpha_{n-2}) = h_1(\zeta^{aj})h_{1,2}(\beta_1) \cdots h_{n-2,n-1}(\beta_{n-2}).$$

By induction we have $h_1(\zeta^{ai}) = h_1(\zeta^{aj})$ and $\alpha_i = \beta_i$ for $1 \leq i \leq n-2$. $\square$

3. THE GENERIC ALGEBRA $\mathcal{H}$

In this section we define a generic algebra that specializes to the algebra $\mathcal{H}_{q,a}$ from §2. This algebra depends only on the triple $(W, \mathcal{S}, b)$ where $(W, \mathcal{S})$ is a Coxeter group with W a symmetric group, and b is a positive integer. Such a triple determines the wreath product $(\mathbb{Z}/b\mathbb{Z}) \wr W$. In order to keep the notation to a minimum we will always consider the particular representation of this group as the subgroup WH_b of $\mathrm{GL}_n(\mathbb{F}_q)$ from §2. In addition, we will continue to use the notation already introduced for WH_b . In particular, ℓ is the length function, $\mathcal{S}$ is a fixed set of Coxeter generators of W , and for s in $\mathcal{S}$, X_s is the subset of H_b defined in (2.3.1).

Let $A = \mathbb{Z}[\mathbf{a}, \mathbf{v}]$ where $\mathbf{a}$ and $\mathbf{v}$ are indeterminates. Set $\mathbf{q} = \mathbf{v}^2$ and define $\mathcal{H}$ to be the A -algebra with generators $\{t_s \mid s \in \mathcal{S}\} \cup \{t_d \mid d \in H_b\}$ and relations

$$(3.0.1) \quad t_d t_{d'} = t_{dd'},$$

$$(3.0.2) \quad t_d t_{s_i} = t_{s_i} t_{s_i d s_i},$$

$$(3.0.3) \quad t_{s_i} t_{s_j} = t_{s_j} t_{s_i} \quad \text{if } |j - i| > 1,$$

$$(3.0.4) \quad t_{s_i} t_{s_{i+1}} t_{s_i} = t_{s_{i+1}} t_{s_i} t_{s_{i+1}},$$

$$(3.0.5) \quad t_{s_i}^2 = \mathbf{q} 1_{\mathcal{H}} + \mathbf{a} \sum_{d \in X_{s_i}} t_d t_{s_i},$$

where $1 \leq i < j \leq n-1$ and d and d' are in H_b .

It follows from relations (3.0.3) and (3.0.4) and Matsumoto's Theorem [11, theorem 1.2.2] that if w is in W , $w = s_{i_1} \cdots s_{i_p}$, and $\ell(w) = p$, then the product $t_{s_1} \cdots t_{s_p}$ depends only on w and not on $s_{i_1}, \dots, s_{i_p}$ so we may define t_w unambiguously by $t_w = t_{s_1} \cdots t_{s_p}$. We also define $t_{wd} = t_w t_d$ and $t_{dw} = t_d t_w$ for w in W and d in H_b . Using relation (3.0.2) and induction it is easy to see that t_x is unambiguously defined for every x in WH_b .

Lemma 3.1. *The following relations hold for x in WH_b , d in H_b , and s in $\mathcal{S}$:*

- (1) $t_x t_d = t_{xd}$ and $t_d t_x = t_{dx}$;
- (2) $t_s t_x = \begin{cases} t_{sx} & \text{if } \ell(sx) > \ell(x), \\ \mathbf{q} t_{sx} + \mathbf{a} \sum_{d \in X_s} t_{dx} & \text{if } \ell(sx) < \ell(x); \end{cases}$
- (3) $t_x t_s = \begin{cases} t_{xs} & \text{if } \ell(xs) > \ell(x), \\ \mathbf{q} t_{xs} + \mathbf{a} \sum_{d \in X_s} t_{xd} & \text{if } \ell(xs) < \ell(x). \end{cases}$

Proof. The first statement follows easily from relation (3.0.2) and the definitions.

Using induction on $\ell(w)$ and (3.0.5) it is easily seen that (2) holds when x is replaced by w . Then (2) follows in general by writing $x = wd$ and using (1).

Using induction on $\ell(w)$ and (3.0.5) it is easily seen that (3) holds when x is replaced by w . Then (3) follows in general by writing $x = dw$ and using (1). $\square$

Theorem 3.2. *The algebra $\mathcal{H}$ is free as an A -module with basis $\{t_x \mid x \in WH_b\}$.*

Proof. It follows from the last lemma that the span of $\{t_x \mid x \in WH_b\}$ is a two-sided ideal in $\mathcal{H}$ containing the identity element so $\{t_x \mid x \in WH_b\}$ spans $\mathcal{H}$.

To show that $\{t_x \mid x \in WH_b\}$ is linearly independent, we adapt Lusztig's presentation [15, Proposition 3.3] of the argument sketched in [1, Ch. 4, Ex. 23] to the current situation.

Let E be a free A -module with basis $\{e_x \mid x \in WH_b\}$. For s and t in $\mathcal{S}$ and d in H_b , define endomorphisms P_d, Q_d, P_s , and Q_t of E by A -linearity and

$$\begin{aligned} P_d(e_x) &= e_{dx}, & Q_d(e_x) &= e_{xd}, \\ P_s(e_x) &= \begin{cases} e_{sx} & \ell(sx) > \ell(x), \\ \mathbf{q}e_{sx} + \mathbf{a} \sum_{d' \in X_s} e_{d'x} & \ell(sx) < \ell(x), \end{cases} \\ Q_t(e_x) &= \begin{cases} e_{xt} & \ell(xt) > \ell(x), \\ \mathbf{q}e_{xt} + \mathbf{a} \sum_{d' \in X_t} e_{xd'} & \ell(xt) < \ell(x). \end{cases} \end{aligned}$$

We next show that $P_y Q_z = Q_z P_y$ for y and z in $\mathcal{S} \cup H_b$.

Clearly, $P_d Q_{d'} = Q_{d'} P_d$, $P_s Q_d = Q_d P_s$, and $P_d Q_t = Q_t P_d$. The length function ℓ is constant on H_b cosets and it follows that $P_s P_d = P_d P_s$ and $Q_d Q_t = Q_t Q_d$.

It remains to show that $P_s Q_t = Q_t P_s$ for s and t in $\mathcal{S}$.

Fix x in WH_b . As in the proof of [15, Proposition 3.3] there are six cases. Suppose first that $\ell(sxt) = \ell(x) > \ell(sx) = \ell(xt)$. Then

$$P_s Q_t(e_x) = \mathbf{q}e_{sxt} + \mathbf{a}\mathbf{q} \sum_{d' \in X_t} e_{sxd'} + \mathbf{a}^2 \sum_{d \in X_s} \sum_{d' \in X_t} e_{dxd'}$$

and

$$Q_t P_s(e_x) = \mathbf{q}e_{sxt} + \mathbf{a}\mathbf{q} \sum_{d \in X_s} e_{dxt} + \mathbf{a}^2 \sum_{d \in X_s} \sum_{d' \in X_t} e_{dxd'}.$$

Say $x = w\tilde{d}$ with w in W and $\tilde{d}$ in H_b . Then $\ell(swt) = \ell(w) > \ell(sw) = \ell(wt)$. By Deodhar's Property Z [8, Theorem 1.1] we have $wt = sw$ and so $s = wtw^{-1}$. It follows from the definition that $X_s = wX_t w^{-1}$. Moreover, it follows from the definition that $tX_t = X_t t$ and $X_t t\tilde{d} = X_t \tilde{d}$. Therefore, $w\tilde{d}tX_t = wX_t \tilde{d}t$ and so $sxX_t = X_s xt$. It follows that $P_s Q_t = Q_t P_s$.

The other five cases are all easier. We omit the details.

Now let $\hat{\mathcal{H}}$ be the subalgebra of $\text{End}_A(E)$ generated by $\{P_y \mid y \in \mathcal{S} \cup H_b\}$. Consider the evaluation map, $\epsilon: \hat{\mathcal{H}} \rightarrow E$, with $\epsilon(f) = f(e_1)$. We will show that ϵ is an isomorphism of A -modules. If d is in H_b and $w = s_{i_1} \cdots s_{i_p}$ is in W with $\ell(w) = p$, then $P_{s_1} \cdots P_{s_p} P_d(e_1) = e_{wd}$ and so ϵ is surjective. To show that ϵ is injective, suppose that f is in $\hat{\mathcal{H}}$ and $f(e_1) = 0$. If d is in H_b and $w = s_{i_1} \cdots s_{i_p}$ is in W with $\ell(w) = p$, then $f' = Q_d Q_{s_p} \cdots Q_{s_1}(e_1)$ is in $\hat{\mathcal{H}}$. By what we have shown above, $ff' = f'f$ and so

$$0 = f'f(e_1) = ff'(e_1) = f(e_{wd}).$$

Since wd is arbitrary in WH_b , it follows that $f = 0$ and so ϵ is injective. Thus, ϵ is an isomorphism of A -modules.

For x in WH_b , let f_x be the unique element in $\hat{\mathcal{H}}$ with the property that $f_x(e_1) = e_x$. Then $\{f_x \mid x \in WH_b\}$ is an A -basis of $\hat{\mathcal{H}}$, $f_s = P_s$ for s in $\mathcal{S}$, and $f_d = P_d$ for d in H_b . It is easily checked that the relations (3.0.1) to (3.0.5) are satisfied by the elements $\{f_s \mid s \in \mathcal{S}\} \cup \{f_d \mid d \in H_b\}$ in $\hat{\mathcal{H}}$. It follows that there is a

homomorphism of A -algebras, $\phi: \mathcal{H} \rightarrow \widehat{\mathcal{H}}$ with $\phi(t_y) = f_y$ for y in $\mathcal{S} \cup H_b$. Since $\{f_x \mid x \in WH_b\}$ is a basis of $\widehat{\mathcal{H}}$ it follows that $\{t_x \mid x \in WH_b\}$ is linearly independent. This completes the proof of the theorem. $\square$

Corollary 3.3. *The element t_1 is the identity in $\mathcal{H}$.*

Proof. By assumption, $t_1 t_d = t_d = t_d t_1$ for d in H_b and $t_1 t_w = t_w t_1$ for w in W . Comparing coefficients in $t_d(t_1 t_w)$ and $t_d t_w$ when both are expressed as linear combinations of $\{t_x \mid x \in WH_b\}$ shows that $t_1 t_w = t_w$ for w in W . Therefore, $t_1 t_x = t_x = t_x t_1$ for every x in WH_b . $\square$

Any function on G that is constant on (B, B) -double cosets is obviously constant on (B_a, B_a) -double cosets. Thus, if e_B is the centrally primitive idempotent in $\mathbb{C}B$ corresponding to the trivial representation of B , then $e_B \mathbb{C}G e_B \subseteq e \mathbb{C}G e = \mathcal{H}_{q,a}$. Consequently, one would expect that the Iwahori-Hecke algebra of W is a subalgebra of $\mathcal{H}$. Taking into account the relation $q - 1 = ab$ and that $\mathbf{a}$ is an indeterminate we show that this is indeed the case.

The rule $d \mapsto t_d$ defines an A -algebra isomorphism between the group algebra AWH_b and the A -span of $\{t_d \mid d \in H_b\}$. Define $e_1 = \sum_{d \in H_b} t_d$, so $t_d e_1 = e_1$ for every d in H_b and $e_1^2 = |H_b| e_1 = b^n e_1$ (note that this is not the same e_1 as in the proof above). It follows from (3.0.2) that e_1 is in the center of $\mathcal{H}$ and so $\mathcal{H}e_1$ is a two-sided ideal in $\mathcal{H}$.

Corollary 3.4. *The subalgebra $\mathcal{H}e_1$ of $\mathcal{H}$ is isomorphic to the Iwahori-Hecke algebra of W with parameters $b^n \mathbf{q}$ and $b^{n+1} \mathbf{a}$.*

Proof. For w in W define $\tilde{t}_w = t_w e_1$. It follows from Theorem 3.2 that $\{\tilde{t}_w \mid w \in W\}$ is a basis of $\mathcal{H}e_1$. Clearly, the elements $\tilde{t}_s$ for s in $\mathcal{S}$ satisfy the braid relations and

$$(\tilde{t}_s)(\tilde{t}_s) = b^n t_s^2 e_1 = b^n \mathbf{q} t_1 e_1 + b^n \mathbf{a} \sum_{d \in X_s} t_s t_d e_1 = b^n \mathbf{q} \tilde{t}_1 + b^{n+1} \mathbf{a} \tilde{t}_s. \quad \square$$

In the rest of this section we record some results that follow more or less immediately from Theorem 3.2. For many of the constructions in the rest of this paper it will be necessary to assume that $\mathbf{q}$ is invertible. Also, some formulas become simpler if we rescale the basis elements t_x of $\mathcal{H}$ by negative powers of $\mathbf{v}$. Thus, we let $A_{\mathbf{v}} = \mathbb{Z}[\mathbf{a}, \mathbf{v}, \mathbf{v}^{-1}]$ denote the localization of A at $\mathbf{v}$ and define $\mathcal{H}_{\mathbf{v}} = A_{\mathbf{v}} \otimes_A \mathcal{H}$. For x in WH_b , define $T_x = \mathbf{v}^{-\ell(x)} \otimes t_x$. Then $\{T_x \mid x \in WH_b\}$ is an $A_{\mathbf{v}}$ -basis of $\mathcal{H}_{\mathbf{v}}$ and the following relations hold for x in WH_b , d in H_b , and s in $\mathcal{S}$:

$$(3.4.1) \quad T_d T_x = T_{dx} \quad \text{and} \quad T_x T_d = T_{xd},$$

$$(3.4.2) \quad T_s T_x = \begin{cases} T_{sx} & \text{if } \ell(sx) > \ell(x), \\ T_{sx} + \mathbf{a} \mathbf{v}^{-1} \sum_{d \in X_s} T_{dx} & \text{if } \ell(sx) < \ell(x), \end{cases}$$

$$(3.4.3) \quad T_x T_s = \begin{cases} T_{xs} & \text{if } \ell(xs) > \ell(x), \\ T_{xs} + \mathbf{a} \mathbf{v}^{-1} \sum_{d \in X_s} T_{xd} & \text{if } \ell(xs) < \ell(x). \end{cases}$$

The following lemma is a crucial ingredient needed to define the R -polynomials and the partial order on WH_b .

Lemma 3.5. *The elements T_x for x in WH_b are units in $\mathcal{H}_{\mathbf{v}}$. In particular,*

$$T_s^{-1} = T_s - \mathbf{av}^{-1} \sum_{d \in X_s} T_d$$

for s in $\mathcal{S}$.

Proof. Assume for a moment that T_s is invertible for every s in $\mathcal{S}$. Then T_w is invertible for every w in W . It follows from (3.4.1) that T_d is invertible for every d in H_b and thus $T_{wd} = T_w T_d$ is invertible for every wd in WH_b .

To complete the proof, suppose s is in $\mathcal{S}$ and define $\bar{T}_s = T_s - \mathbf{av}^{-1} \sum_{d \in X_s} T_d$. Then it follows from (3.4.2) and (3.4.3) that $\bar{T}_s$ is a left and right inverse for T_s . $\square$

Define $\tau: \mathcal{H}_{\mathbf{v}} \rightarrow A_{\mathbf{v}}$ by $\tau(T_x) = \delta_{x,1}$ for x in WH_b and $A_{\mathbf{v}}$ -linearity, where $\delta_{x,y}$ is the Kronecker delta.

Proposition 3.6. *If x and y are in WH_b , then $\tau(T_x T_y) = \delta_{x,y^{-1}}$. Therefore, τ is a symmetrizing trace form on $\mathcal{H}_{\mathbf{v}}$ and $\{T_{y^{-1}} \mid y \in WH_b\}$ is the basis of $\mathcal{H}_{\mathbf{v}}$ dual to the basis $\{T_x \mid x \in WH_b\}$.*

Proof. Assume that $\tau(T_x T_y) = \delta_{x,y^{-1}}$. Then $\tau(hh') = \tau(h'h)$ for h and h' in $\mathcal{H}_{\mathbf{v}}$ and the bilinear form $(h, h') \mapsto \tau(hh')$ is non-degenerate, so τ is a symmetrizing trace form on $\mathcal{H}_{\mathbf{v}}$.

We prove the formula $\tau(T_x T_y) = \delta_{x,y^{-1}}$ using induction on $\ell(x)$.

If $\ell(x) = 0$, then $x = d$ is in H_b and using (3.4.1) we have

$$\tau(T_d T_y) = \tau(T_{dy}) = \delta_{dy,1} = \delta_{d,y^{-1}}.$$

Now suppose that $\ell(x) > 0$. Write $x = dws$ where d is in H_b , w is in W , s is in $\mathcal{S}$, and $\ell(ws) > \ell(w)$. There are two cases: $\ell(sy) > \ell(y)$ and $\ell(sy) < \ell(y)$.

First, suppose that $\ell(sy) > \ell(y)$. Since $\ell(xs) < \ell(x)$ and $\ell(y^{-1}s) > \ell(y^{-1})$, it follows that $x \neq y^{-1}$ and so $\delta_{x,y^{-1}} = 0$. Using (3.4.1), (3.4.2), and induction we have

$$\tau(T_x T_y) = \tau(T_{dw} T_s T_y) = \tau(T_{dw} T_{sy}) = \delta_{dw,y^{-1}s}.$$

Since $x \neq y^{-1}$ it follows that $dw \neq y^{-1}s$. Therefore, $\tau(T_x T_y) = 0$.

Second, suppose that $\ell(sy) < \ell(y)$. Then using (3.4.1), (3.4.2), and induction we have

$$\begin{aligned} \tau(T_x T_y) &= \tau(T_{dw} T_s T_y) \\ &= \tau \left(T_{dw} (T_{sy} + \mathbf{av}^{-1} \sum_{d_1 \in X_s} T_{d_1 y}) \right) \\ &= \tau(T_{dw} T_{sy}) + \mathbf{av}^{-1} \sum_{d_1 \in X_s} \tau(T_{dw} T_{d_1 y}) \\ &= \delta_{dw,y^{-1}s} + \mathbf{av}^{-1} \sum_{d_1 \in X_s} \mathbf{q}^{\ell(dw)} \delta_{dw,(d_1 y)^{-1}}. \end{aligned}$$

If $dw = (d_1 y)^{-1}$ for some d_1 in X_s , then $\ell(y^{-1} d_1^{-1} s) = \ell(dws) > \ell(dw) = \ell(y^{-1} d_1^{-1})$, so $\ell(y^{-1} s) > \ell(y^{-1})$, a contradiction. Therefore, $dw \neq (d_1 y)^{-1}$ for d_1 in X_s and so

$$\tau(T_x T_y) = \delta_{dw,y^{-1}s} = \delta_{x,y^{-1}}$$

as desired. $\square$

Next, recall that a *specialization* of $A_{\mathbf{v}}$ is a ring homomorphism from $A_{\mathbf{v}}$ to a commutative ring B . Given a specialization, $\phi: A_{\mathbf{v}} \rightarrow B$, we can consider the B -algebra $\mathcal{H}_{\mathbf{v}} \otimes_{A_{\mathbf{v}}} B$.

Let $\phi_{1,0}: A_{\mathbf{v}} \rightarrow \mathbb{C}$ be a ring homomorphism with $\phi_1(\mathbf{a}) = 0$ and $\phi_1(\mathbf{v}) = \pm 1$. Then clearly the $\mathbb{C}$ -algebra $\mathcal{H}_{1,0} = \mathcal{H}_{\mathbf{v}} \otimes_{A_{\mathbf{v}}} \mathbb{C}$ is isomorphic to the group algebra $\mathbb{C}WH_b$.

Let $\phi_{q,a}: A_{\mathbf{v}} \rightarrow \mathbb{C}$ be a ring homomorphism with $\phi_{q,a}(\mathbf{a}) = a$ and $\phi_{q,a}(\mathbf{v}) = \pm\sqrt{q}$. Then clearly the $\mathbb{C}$ -algebra $\mathcal{H}_{\mathbf{v}} \otimes_{A_{\mathbf{v}}} \mathbb{C}$ is isomorphic to the algebra $\mathcal{H}_{q,a}$ from §2.

The arguments in [7, §68] involving Tit's Deformation Theorem, together with the fact that a group algebra is always isomorphic to its opposite algebra, prove the following theorem.

Theorem 3.7. *If K is a field with characteristic zero that is a splitting field for KWH_b and with the property that $\text{ind}_{B_a}^G 1_{B_a}$ is a split KG -module, then the algebras $\text{End}_{KG}(\text{ind}_{B_a}^G 1_{B_a})$ and KWH_b are isomorphic.*

Now suppose that l is a prime that divides $q-1$ and does not divide $n!$. Say $l^r | q-1$ and $l^{r+1} \nmid q-1$. Set $b = l^r$. Then H_b is a Sylow l -subgroup of G . Let $(\mathcal{O}, k, K)$ be a sufficiently large l -modular system for G and let $\mathcal{B}$ denote the principal block of $\mathcal{O}G$. We assume that q is a square in K and we consider $\mathcal{B}$ as a two-sided ideal in $\mathcal{O}G$. Let $\phi: A_{\mathbf{v}} \rightarrow K$ be a ring homomorphism with $\phi(\mathbf{a}) = a$ and $\phi(\mathbf{v}) = \pm\sqrt{q}$.

Corollary 3.8. *The $\mathcal{O}$ -module $\text{ind}_{B_a}^G 1_{B_a}$ is a biprojective $\mathcal{B} \times (\mathcal{H} \otimes_{A_{\mathbf{v}}} K)$ -bimodule and induces a Morita equivalence between the principal block, $\mathcal{B}$, of $\mathcal{O}G$ and the specialized algebra $\mathcal{H} \otimes_{A_{\mathbf{v}}} K$.*

Proof. This follows from the theorem using the argument in [5, Chapter 23]. $\square$

4. R -POLYNOMIALS AND A PARTIAL ORDER ON WH_b

In this section we follow the constructions of Kazhdan and Lusztig in [13] and define a “bar” involution of $\mathcal{H}_{\mathbf{v}}$ and R -polynomials in $A_{\mathbf{v}}$. We use the non-vanishing of the R -polynomials to define a relation on WH_b and we show that this relation is a partial order on WH_b . This partial order reduces to the Bruhat order in case $b = 1$.

Define a ring endomorphism $\bar{}: A_{\mathbf{v}} \rightarrow A_{\mathbf{v}}$ by $\mathbb{Z}$ -linearity, $\bar{\mathbf{v}} = \mathbf{v}^{-1}$, and $\bar{\mathbf{a}} = -\mathbf{a}\mathbf{v}^{-2}$. Notice that:

- $\bar{\bar{\mathbf{a}}} = -\mathbf{a}\mathbf{v}^{-2} = \mathbf{a}\mathbf{v}^2\mathbf{v}^{-2} = \mathbf{a}$ so $\bar{}$ is an involution, and
- $\overline{\mathbf{a}\mathbf{v}^{-1}} = -\mathbf{a}\mathbf{v}^{-1}$.

Next, extend $\bar{}$ to an endomorphism of $\mathcal{H}_{\mathbf{v}}$ by defining

$$\overline{\sum_{x \in WH_b} \gamma_x T_x} = \sum_{x \in WH_b} \overline{\gamma_x} \overline{T_x^{-1}} = \sum_{x \in WH_b} \overline{\gamma_x} \overline{T_x}$$

where $\overline{T_x}$ is defined to be $T_{x^{-1}}^{-1}$. Notice that $\overline{T_d} = T_d$ for d in H_b .

Proposition 4.1. *The endomorphism $\bar{}: \mathcal{H}_{\mathbf{v}} \rightarrow \mathcal{H}_{\mathbf{v}}$ is a ring isomorphism of order two.*

Proof. Clearly, $\bar{}$ has order two and so is a bijection. Since $\mathcal{H}_{\mathbf{v}}$ is generated by $\{T_d \mid d \in H_b\} \cup \{T_s \mid s \in \mathcal{S}\}$, to show it is a ring homomorphism, it is enough to

show that

$$\overline{T_d T_x} = \overline{T_d} \overline{T_x} \quad \text{and} \quad \overline{T_s T_x} = \overline{T_s} \overline{T_x}$$

for x in WH_b , d in H_b , and s in $\mathcal{S}$. These equations are easily shown to be true using (3.4.1) and (3.4.2). $\square$

For y in WH_b we may express $\overline{T_y}$ in terms of the basis $\{T_x \mid x \in WH_b\}$ of $\mathcal{H}_v$. For x and y in WH_b , define elements $R_{x,y}^*$ in A_v by

$$\overline{T_y} = \sum_{x \in WH_b} \overline{R_{x,y}^*} T_x.$$

Clearly, $R_{1,1}^* = 1$. If d is in H_b , then $\overline{T_d} = T_d$ and so $R_{x,d}^* = 0$ if $x \neq d$ and $R_{d,d}^* = 1$. It follows from Lemma 3.5 that for s in $\mathcal{S}$ we have $R_{x,s}^* = 0$ unless x is in $\{s\} \cup X_s$, $R_{s,s}^* = 1$, and $R_{d,s}^* = \mathbf{a}v^{-1}$ for d in X_s .

Proposition 4.2. *Suppose x and y are in WH_b .*

- (1) *If d is in H_b , then $R_{x,yd}^* = R_{xd^{-1},y}^*$ and $R_{x,dy}^* = R_{d^{-1}x,y}^*$.*
- (2) *If s is in $\mathcal{S}$ and $\ell(sy) < \ell(y)$, then*

$$R_{x,y}^* = \begin{cases} R_{sx,sy}^* & \text{if } \ell(sx) < \ell(x), \\ R_{sx,sy}^* + \mathbf{a}v^{-1} \sum_{d \in X_s} R_{dx,sy}^* & \text{if } \ell(sx) > \ell(x). \end{cases}$$

- (3) *If s is in $\mathcal{S}$ and $\ell(ys) < \ell(y)$, then*

$$R_{x,y}^* = \begin{cases} R_{xs,ys}^* & \text{if } \ell(xs) < \ell(x), \\ R_{xs,ys}^* + \mathbf{a}v^{-1} \sum_{d \in X_s} R_{xd,ys}^* & \text{if } \ell(xs) > \ell(x). \end{cases}$$

Proof. To prove the first statement we compute $\overline{T_{yd}}$. Using (3.4.1) and the fact that $\overline{T_d} = T_d$ we have

$$\overline{T_{yd}} = \overline{T_y} T_d = \sum_x \overline{R_{x,y}^*} T_x T_d = \sum_x \overline{R_{x,y}^*} T_{xd} = \sum_x \overline{R_{xd^{-1},y}^*} T_x.$$

On the other hand, $\overline{T_{yd}} = \sum_x \overline{R_{x,yd}^*} T_x$. Comparing coefficients of T_x we see that $R_{x,yd}^* = R_{xd^{-1},y}^*$.

A similar argument shows that $R_{x,dy}^* = R_{d^{-1}x,y}^*$.

Next, by Lemma 3.5, (3.4.1), and (3.4.2) we have

$$\overline{T_y} = \overline{T_s} \overline{T_{sy}} = \sum_{\ell(sx) < \ell(x)} \overline{R_{sx,sy}^*} T_x + \sum_{\ell(sx) > \ell(x)} \left(\overline{R_{sx,sy}^*} - \mathbf{a}v^{-1} \sum_{d \in X_s} \overline{R_{d^{-1}x,sy}^*} \right) T_x.$$

The second statement of the proposition follows by comparing coefficients of T_x and using that $\mathbf{a}v^{-1} = -\mathbf{a}v^{-1}$ and $X_s = X_s^{-1}$.

The proof of the third statement is similar to the proof of the second statement and is omitted. $\square$

Corollary 4.3. *For any x and y in WH_b , if $R_{x,y}^* \neq 0$, then $\ell(y) \geq \ell(x)$ and $R_{x,y}^*$ is a polynomial in $\mathbf{a}v^{-1}$ with non-negative integer coefficients and degree at most $\ell(y) - \ell(x)$. In particular, $R_{x,y}^* = 0$ if $\ell(x) \not\leq \ell(y)$. Moreover, $R_{y,y}^* = 1$.*

We will obtain a closed form for the polynomials $R_{x,y}^*$ in §6.

Proof. We prove the statements using induction on $\ell(y)$.

If $\ell(y) = 0$, then $y = d$ is in H_b and we have seen that $R_{x,d}^* = 0$ unless $x = d$ and that $R_{d,d}^* = 1$.

Now suppose $\ell(y) > 0$. Choose s in $\mathcal{S}$ with $\ell(sy) < \ell(y)$. Then by Proposition 4.2 and induction, $R_{y,y}^* = R_{sy,sy}^* = 1$.

Now suppose x is in WH_b and $R_{x,y}^* \neq 0$.

If $\ell(sx) < \ell(x)$, then $R_{x,y}^* = R_{sx,sy}^*$ by Proposition 4.2. Since $\ell(y) - \ell(x) = \ell(sy) - \ell(sx)$, it follows by induction that $R_{x,y}^*$ is a polynomial in $\mathbf{av}^{-1}$ with non-negative integer coefficients and degree at most $\ell(y) - \ell(x)$.

If $\ell(sx) > \ell(x)$, then $R_{x,y}^* = R_{sx,sy}^* + \mathbf{av}^{-1} \sum_{d \in X_s} R_{dx,sy}^*$ by Proposition 4.2, and so either $R_{sx,sy}^* \neq 0$ or $R_{dx,sy}^* \neq 0$ for some d in X_s . Since $\ell(sy) - \ell(sx) = \ell(y) - \ell(x) - 2$ and $\ell(sy) - \ell(dx) = \ell(y) - \ell(x) - 1$ for any d in H_b , it follows by induction that $R_{x,y}^*$ is a polynomial in $\mathbf{av}^{-1}$ with non-negative integer coefficients and degree at most $\ell(y) - \ell(x)$. $\square$

For x and y in WH_b , define

$$x \leq y \quad \text{if} \quad R_{x,y}^* \neq 0.$$

The main result in this section is that $\leq$ is a partial order on WH_b .

Notice that when $b = 1$, then $WH_b = W$. It is pointed out in [13] that $R_{x,y}^* \neq 0$ if and only if x is less than or equal to y in the Bruhat-Chevalley order and so the relation $\leq$ is the Bruhat-Chevalley order in this case.

We have seen that for s in $\mathcal{S}$ and x in WH_b , $R_{x,s}^* \neq 0$ if and only if x is in X_s , so $x \leq s$ if and only if x is in X_s . In particular, if 1 is the identity in WH_b , then $1 \leq s$ if and only if b is odd. It follows that in general, the restriction of $\leq$ to W is not the Bruhat-Chevalley order on W .

Suppose w_1 and w_2 are in W and d_1 and d_2 are in H_b . Then by Proposition 4.2, $R_{w_1 d_1, w_2 d_2}^* = R_{w_1 d_1 d_2^{-1}, w_2}^*$. Define

$$\Omega_{w_1, w_2} = \{d \in H_b \mid R_{w_1 d, w_2}^* \neq 0\}.$$

Then for w_1, w_2 in W and d_1, d_2 in H_b we have

$$w_1 d_1 \leq w_2 d_2 \text{ if and only if } d_1 d_2^{-1} \in \Omega_{w_1, w_2}.$$

In order to understand the relation $\leq$ we need to describe the subsets Ω_{w_1, w_2} of H_b .

Lemma 4.4. *Suppose w_1 and w_2 are in W , s is in $\mathcal{S}$, and $\ell(sw_2) < \ell(w_2)$. Set $t = w_1^{-1}sw_1$. Then*

$$\Omega_{w_1, w_2} = \begin{cases} \Omega_{sw_1, sw_2} & \text{if } \ell(sw_1) < \ell(w_1), \\ \Omega_{sw_1, sw_2} \cup \Omega_{w_1, sw_2} X_t & \text{if } \ell(sw_1) > \ell(w_1). \end{cases}$$

Proof. If d is in H_b , then

$$R_{w_1 d, w_2}^* = \begin{cases} R_{sw_1 d, sw_2}^* & \text{if } \ell(sw_1) < \ell(w_1), \\ R_{sw_1 d, sw_2}^* + \mathbf{av}^{-1} \sum_{d' \in X_s} R_{d' w_1 d, sw_2}^* & \text{if } \ell(sw_1) > \ell(w_1). \end{cases}$$

By definition $X_t = w_1^{-1}X_s w_1$ so we may rewrite the last equation as

$$R_{w_1 d, w_2}^* = \begin{cases} R_{sw_1 d, sw_2}^* & \text{if } \ell(sw_1) < \ell(w_1), \\ R_{sw_1 d, sw_2}^* + \mathbf{av}^{-1} \sum_{d_1 \in X_t} R_{w_1 d_1 d, sw_2}^* & \text{if } \ell(sw_1) > \ell(w_1). \end{cases}$$

It follows immediately that $\Omega_{w_1, w_2} = \Omega_{sw_1, sw_2}$ if $\ell(sw_1) < \ell(w_1)$.

Suppose $\ell(sw_1) > \ell(w_1)$. Since $R_{x,y}^*$ is in $\mathbb{N}[\mathbf{av}^{-1}]$, it follows that $R_{w_1 d, w_2}^* \neq 0$ if and only if $R_{sw_1 d, sw_2}^* \neq 0$ or there is a d_1 in X_t with $R_{w_1 d_1 d, sw_2}^* \neq 0$. Clearly, $d_1 d$ is in Ω_{w_1, sw_2} if and only if d is in $\Omega_{w_1, sw_2} d_1^{-1}$. Since $X_t^{-1} = X_t$ we have

$$\begin{aligned} \Omega_{w_1, w_2} &= \Omega_{sw_1, sw_2} \cup \left(\bigcup_{d_1 \in X_t} \Omega_{w_1, sw_2} d_1^{-1} \right) \\ &= \Omega_{sw_1, sw_2} \cup \Omega_{w_1, sw_2} X_t^{-1} = \Omega_{sw_1, sw_2} \cup \Omega_{w_1, sw_2} X_t. \quad \square \end{aligned}$$

In the next proposition, 1 denotes the identity in WH_b and $\leq_B$ denotes the Bruhat-Chevalley order on W .

Proposition 4.5. *Suppose w_1 and w_2 are in W . Then the following statements hold:*

- (1) $\Omega_{w_1, w_2} \neq \emptyset$ if and only if $w_1 \leq_B w_2$.
- (2) $\Omega_{w_1, w_1} = \{1\}$.
- (3) If $w_1 <_B w_2$, then $\Omega_{w_1, w_2} = X_{t_1} \cdots X_{t_r}$ where $t_1, \dots, t_r$ is any sequence of reflections in $\mathcal{T}$ with $w_2 = w_1 t_1 \cdots t_r$ and $\ell(w_1 t_1 \cdots t_i) = \ell(w_1) + i$ for $1 \leq i \leq r$.

Proof. We will prove the proposition using induction on $\ell(w_2)$.

If $\ell(w_2) = 0$, then $w_2 = 1$. We have seen that for x in WH_b , $R_{x,1}^* = 0$ if $x \neq 1$ and $R_{1,1}^* = 1$. Thus $\Omega_{w,1} = \emptyset$ for $w \neq 1$ and $\Omega_{1,1} = \{1\}$ and so (1), (2), and (3) hold in this case.

Now suppose that $\ell(w_2) > 0$ and fix s in $\mathcal{S}$ with $\ell(sw_2) < \ell(w_2)$. Then by Lemma 4.4 we have $\Omega_{w_2, w_2} = \Omega_{sw_2, sw_2}$ and so by induction, $\Omega_{w_2, w_2} = \{1\}$. This shows that (2) holds.

Suppose w_1 is in W and $w_1 \neq w_2$.

We consider first the case when $\ell(sw_1) < \ell(w_1)$.

By Lemma 4.4 we have $\Omega_{w_1, w_2} = \Omega_{sw_1, sw_2}$ and so by induction, $\Omega_{w_1, w_2} \neq \emptyset$ if and only if $sw_1 <_B sw_2$. By Deodhar's Property Z we have $sw_1 \leq_B sw_2$ if and only if $w_1 \leq_B w_2$. Therefore, $\Omega_{w_1, w_2} \neq \emptyset$ if and only if $w_1 \leq_B w_2$ and so (1) holds.

Suppose $w_1 <_B w_2$ and $t_1, \dots, t_r$ is any sequence of reflections in $\mathcal{T}$ with $w_2 = w_1 t_1 \cdots t_r$ and $\ell(w_1 t_1 \cdots t_i) = \ell(w_1) + i$ for $1 \leq i \leq r$. Define $v_0 = w_1$ and $v_i = w_1 t_1 \cdots t_i$ for $1 \leq i \leq r$. Notice that $v_r = w_2$.

There are two possibilities: either $\ell(sv_i) < \ell(v_i)$ for $1 \leq i \leq r$, or there is an i with $\ell(sv_i) > \ell(v_i)$ and $\ell(sv_{i+1}) < \ell(v_{i+1})$.

Suppose that $\ell(sv_i) < \ell(v_i)$ for $1 \leq i \leq r$. Then $sw_2 = sw_1 t_1 \cdots t_r$ and $\ell(sw_1 t_1 \cdots t_i) = \ell(sw_1) + i$ for $1 \leq i \leq r$. Using Lemma 4.4 and induction we have

$$\Omega_{w_1, w_2} = \Omega_{sw_1, sw_2} = X_{t_1} \cdots X_{t_r}.$$

Now suppose there is an i with $\ell(sv_i) > \ell(v_i)$ and $\ell(sv_{i+1}) < \ell(v_{i+1})$. Then by Deodhar's Property Z, $sv_i \leq_B v_{i+1}$. But $\ell(sv_i) = \ell(v_{i+1})$ and so $sv_i = v_{i+1}$. Hence, $sw_1 t_1 \cdots t_i = w_1 t_1 \cdots t_i t_{i+1}$ and $w_1^{-1} sw_1 = t_1 \cdots t_i t_{i+1} t_i \cdots t_1$.

Fix i so that i is maximal with $\ell(sv_i) > \ell(v_i)$. Then $1 < i < r$, $v_{i+1} = sv_i$, and for $j > i$ we have $\ell(sv_j) < \ell(v_j)$. Since $v_{i+1} = sv_i$ we have

$$sv_{i+2} = sv_i t_{i+1} t_{i+2} = v_{i+1} t_{i+1} t_{i+2} = v_1 t_{i+2}.$$

Set

$$t'_1 = w_1^{-1}sw_1, \quad \text{and} \quad t'_j = \begin{cases} t_{j-1} & \text{for } 2 \leq j \leq i+1, \\ t_j & \text{for } i+2 \leq j \leq r. \end{cases}$$

Then $sw_2 = sw_1 t'_1 \cdots t'_r$ and $\ell(sw_1 t'_1 \cdots t'_j) = \ell(sw_1) + j$ for $1 \leq j \leq r$. Using the induction hypothesis, the fact that $w_1^{-1}sw_1 = t_1 \cdots t_i t_{i+1} t_i \cdots t_1$, and Proposition 2.5 we have

$$\begin{aligned} \Omega_{w_1, w_2} &= \Omega_{sw_1, sw_2} \\ &= X_{w_1^{-1}sw_1} X_{t_1} \cdots X_{t_i} X_{t_{i+2}} \cdots X_{t_r} \\ &= X_{t_1} \cdots X_{t_i} X_{t_1 \cdots t_i t_{i+1} t_i \cdots t_1} X_{t_{i+2}} \cdots X_{t_r} \\ &= X_{t_1} \cdots X_{t_r}. \end{aligned}$$

We have shown that (3) holds in both cases and thus have completed the proof of the proposition when $\ell(sw_1) < \ell(w_1)$.

For the remainder of the proof we assume that $\ell(sw_1) > \ell(w_1)$. Then by Lemma 4.4 we have $\Omega_{w_1, w_2} = \Omega_{sw_1, sw_2} \cup \Omega_{w_1, sw_2} X_t$ where $t = w_1^{-1}sw_1$. By Deodhar's Property Z the following three conditions are equivalent:

- $sw_1 \leq_B w_2$,
- $w_1 \leq_B w_2$, and
- $w_1 \leq_B sw_2$.

If $w_1 \leq_B w_2$, then $w_1 \leq_B sw_2$ and so by induction $\Omega_{w_1, sw_2} \neq \emptyset$. Therefore, $\Omega_{w_1, w_2} \neq \emptyset$. Conversely, if $\Omega_{w_1, w_2} \neq \emptyset$, then either $\Omega_{sw_1, sw_2} \neq \emptyset$ or $\Omega_{w_1, sw_2} \neq \emptyset$. In the first case, it follows by induction that $sw_1 \leq_B sw_2$. But then $sw_1 \leq_B w_2$ and so $w_1 \leq_B w_2$. In the second case, it follows by induction that $w_1 \leq_B sw_2$ and so again $w_1 \leq_B w_2$. This proves (1).

Now suppose that $w_1 <_B w_2$ and $t_1, \dots, t_r$ is any sequence of reflections in $\mathcal{T}$ with $w_2 = w_1 t_1 \cdots t_r$ and $\ell(w_1 t_1 \cdots t_i) = \ell(w_1) + i$ for $1 \leq i \leq r$. As above, define $v_0 = w_1$ and $v_i = w_1 t_1 \cdots t_i$ for $1 \leq i \leq r$. As above we have $v_r = w_2$.

We consider the subset $\Omega_{w_1, sw_2} X_t$ of H_b . Choose i maximal with $\ell(sv_i) > \ell(v_i)$. Then we have seen that $w_1^{-1}sw_1 = t_1 \cdots t_i t_{i+1} t_i \cdots t_1$ and $sv_{i+2} = v_i t_{i+2}$. Set

$$t'_j = \begin{cases} t_j & \text{for } 1 \leq j \leq i, \\ t_{j+1} & \text{for } i+1 \leq j \leq r-1. \end{cases}$$

Then $sw_2 = w_1 t'_1 \cdots t'_{r-1}$ and $\ell(w_1 t'_1 \cdots t'_j) = \ell(w_1) + j$ for $1 \leq j \leq r-1$. Using the induction hypothesis and Proposition 2.5 we have

$$\begin{aligned} \Omega_{w_1, sw_2} X_t &= X_{t_1} \cdots X_{t_i} X_{t_{i+2}} \cdots X_{t_r} X_t \\ &= X_{t_1} \cdots X_{t_i} X_{t_1 \cdots t_i t_{i+1} t_i \cdots t_1} X_{t_{i+2}} \cdots X_{t_r} \\ &= X_{t_1} \cdots X_{t_r}. \end{aligned}$$

There are two cases, either $sw_1 \not\leq_B sw_2$ or $sw_1 \leq_B sw_2$.

Suppose that $sw_1 \not\leq_B sw_2$. Then $\Omega_{sw_1, sw_2} = \emptyset$ and so

$$\Omega_{w_1, w_2} = \Omega_{w_1, sw_2} X_t = X_{t_1} \cdots X_{t_r}.$$

Finally, suppose that $sw_1 \leq_B sw_2$. Then $\Omega_{sw_1, sw_2} \neq \emptyset$. We will show that $\Omega_{sw_1, sw_2} \subseteq \Omega_{w_1, sw_2} X_t$ and so

$$\Omega_{w_1, w_2} = \Omega_{sw_1, sw_2} \cup \Omega_{w_1, sw_2} X_t = \Omega_{w_1, sw_2} X_t = X_{t_1} \cdots X_{t_r}.$$

Choose reflections $t'_2, \dots, t'_{r-1}$ so that $sw_2 = sw_1 t'_2 \cdots t'_{r-1}$ and $\ell(sw_1 t'_2 \cdots t'_i) = \ell(sw_1) + i - 1$ for $2 \leq i \leq r - 1$. Then by induction $\Omega_{sw_1, sw_2} = X_{t'_2} \cdots X_{t'_{r-1}}$. Set $t'_1 = t$. Then $sw_2 = w_1 t'_1 \cdots t'_{r-1}$ and $\ell(w_1 t'_1 \cdots t'_i) = \ell(w_1) + i$ for $1 \leq i \leq r - 1$ and so by induction

$$\Omega_{w_1, sw_2} X_t = X_t X_{t'_2} \cdots X_{t'_{r-1}} X_t = X_t \Omega_{sw_1, sw_2} X_t.$$

The subset $X_t X_t$ of H_b is a subgroup and so $\Omega_{w_1, sw_2} X_t$ contains Ω_{sw_1, sw_2} as claimed.

It follows that (3) holds when $\ell(sw_1) < \ell(w_1)$. This completes the proof of the proposition. $\square$

Theorem 4.6. *The relation $\leq$ on WH_b is a partial order.*

Proof. We have seen that $R_{x,x}^* = 1$ for all x in WH_b and so the relation is reflexive.

Suppose $x = w_1 d_1$, $y = w_2 d_2$, and $z = w_3 d_3$ are in WH_b with w_1, w_2, w_3 in W and d_1, d_2, d_3 in H_b .

If $x \leq y$ and $y \leq x$, then $d_1 d_2^{-1}$ is in Ω_{w_1, w_2} , so by Proposition 4.5 we have $w_1 \leq_B w_2$. Similarly, $w_2 \leq_B w_1$ and so $w_1 = w_2$. But then $\Omega_{w_1, w_2} = \{1\}$ and so $d_1 = d_2$. Therefore, $x = y$ and so the relation is anti-symmetric.

If $x \leq y$ and $y \leq z$, then $d_1 d_2^{-1}$ is in Ω_{w_1, w_2} and $d_2 d_3^{-1}$ is in Ω_{w_2, w_3} . It follows from Proposition 4.5 that $w_1 \leq_B w_2$ and $w_2 \leq_B w_3$. Thus $w_1 \leq_B w_2 \leq_B w_3$ and so using Proposition 4.5 again we see that $\Omega_{w_1, w_3} = \Omega_{w_1, w_2} \Omega_{w_2, w_3}$. Hence, $d_1 d_3^{-1} = d_1 d_2^{-1} d_2 d_3^{-1}$ is in Ω_{w_1, w_3} . Therefore, $x \leq z$ and so the relation is transitive. $\square$

We conclude this section with some properties of the partial order and some examples.

Proposition 4.7. *Multiplication by d is a poset automorphism of $(WH_b, \leq)$ for every d in H_b .*

Proof. Clearly $x \mapsto xd$ and $x \mapsto dx$ are bijective mappings and it follows from Proposition 4.2 that $x \leq y$ if and only if either $dx \leq dy$ or $xd \leq yd$. $\square$

Define $H' = X_{s_1} \times \cdots \times X_{s_{n-1}}$ and recall that $X_0 = \{h_1(\zeta^{a_i}) \mid 0 \leq i \leq b-1\}$. By Proposition 2.6, the multiplication mapping from $X_0 \times H'$ to H_b is a bijection.

Proposition 4.8. *If b is odd, then H' is a subgroup of H_b that contains $X_{t_1} \cdots X_{t_r}$ for every $t_1, \dots, t_r$ in $\mathcal{T}$.*

Proof. Since b is odd, each X_t is a subgroup of H_b and so $X_{s_1} \cdots X_{s_{n-1}}$ is a subgroup of H_b . To prove the proposition it is enough to show that X_t is contained in $X_{s_1} \cdots X_{s_{n-1}}$ for every t in $\mathcal{T}$.

Suppose t is in $\mathcal{T}$ and t interchanges the i th and j th standard basis vector of $\mathbb{F}_q^n$ with $i < j$. Then $X_t = \{h_{i,j}(\alpha) \mid \alpha \in F_b\}$. We may assume that $j - 1 > 1$. The result follows since

$$h_{i,j}(\alpha) = h_{i,i+1}(\alpha) h_{i+1,i+2}(\alpha) h_{i+2,i+3}(\alpha) \cdots h_{j-1,j}(\alpha). \quad \square$$

Proposition 4.9. *If b is odd, then WH' is a normal subgroup of WH_b and the cosets of WH' are the connected components of the Hasse diagram of the poset $(WH_b, \leq)$.*

Proof. It is straightforward to check that WH' is a normal subgroup of WH_b and it follows from Proposition 2.6 that X_0 is a complete set of coset representatives of WH' in WH_b .

It follows from Proposition 4.7 that the posets $(WH', \leq)$ and $(WH'd_0, \leq)$ are isomorphic for every d_0 in X_0 .

Suppose w_1d_1 and w_2d_2 are in WH' and α_1 and α_2 are in F_b with $w_1d_1h_1(\alpha_1) \leq w_2d_2h_1(\alpha_2)$. Then $d_1d_2^{-1}h_1(\alpha_1\alpha_2^{-1})$ is in Ω_{w_1,w_2} . By the last proposition and Proposition 4.5 we have $\Omega_{w_1,w_2} \subseteq H'$ and so it follows from Proposition 2.6 that $h_1(\alpha_1\alpha_2^{-1}) = 1$. Therefore, $h_1(\alpha_1) = h_1(\alpha_2)$. This shows that if x and y are in WH' and d_0 and d'_0 are in X_0 , then no element of $WH'd_0$ is related to any element of $WH'd'_0$ if $d_0 \neq d'_0$.

Finally, suppose w_1d_1 and w_2d_2 are in WH' . We have seen that $\Omega_{1,w}$ is a subgroup of H' for every w in W . It follows that $d_1 \leq w_1d_1$ and $d_2 \leq w_2d_2$. Also, every simple reflection in $\mathcal{S}$ occurs in any reduced expression for the longest element w_0 , so $\Omega_{1,w_0} = H'$ and hence $d_1 \leq w_0d_1d_2$ and $d_2 \leq w_0d_1d_2$. This shows that the Hasse diagram of the poset $(WH', \leq)$ is connected. $\square$

If b is even, then it is not hard to show that the Hasse diagram of $(WH_b, \leq)$ still has b connected components, but they are somewhat more complicated to describe. This is illustrated in the next example.

Example 4.10. Suppose $n = 3$ and $b = 2$. Then the Hasse diagram of WH_b has two connected components. The connected component containing the identity in WH_2 is given in Figure 1. In this diagram we have denoted $h_i(-1)$ simply by d_i for $i = 1, 2, 3$. By Proposition 4.7, the other connected component of the Hasse diagram is obtained by multiplying on the left by d_1 .

5. THE KAZHDAN-LUSZTIG BASIS AND KAZHDAN-LUSZTIG POLYNOMIALS

Recall that a , b , and q are related by the equation $ab = q - 1$. In this section we define a ring of scalars for $\mathcal{H}$ so that the equation $ba = \mathbf{v}^2 - 1$ can be solved for $\mathbf{a}$. Using this new ring of scalars we can define a Kazhdan-Lusztig basis of $\mathcal{H}$ and Kazhdan-Lusztig polynomials for WH_b following the construction in [15, §5].

Let $\mathbb{Z}_b$ denote the localization of $\mathbb{Z}$ at b and define I to be the principal ideal in $\mathbb{Z}_b[\mathbf{a}, \mathbf{v}]_{\mathbf{v}}$ generated by $ba - (\mathbf{v}^2 - 1)$. Then $\mathbf{a}\mathbf{v}^{-1} \equiv \mathbf{v}/b - \mathbf{v}^{-1}/b \pmod{I}$. We set $\tilde{A} = \mathbb{Z}_b[\mathbf{a}, \mathbf{v}]_{\mathbf{v}}/I$. The restriction of the natural projection $\mathbb{Z}_b[\mathbf{a}, \mathbf{v}]_{\mathbf{v}} \rightarrow \tilde{A}$ to the subring $\mathbb{Z}_b[\mathbf{v}]_{\mathbf{v}}$ is a ring isomorphism. In the following we identify $\tilde{A}$ with $\mathbb{Z}_b[\mathbf{v}]_{\mathbf{v}}$.

Now define $\tilde{\mathcal{H}} = \tilde{A} \otimes_A \mathcal{H}$ and again denote $\mathbf{v}^{-\ell(x)} \otimes t_x$ by T_x . Then $\{T_x \mid x \in WH_b\}$ is an $\tilde{A}$ -basis of $\tilde{\mathcal{H}}$ and the quadratic relation (3.4.2) becomes

$$(5.0.1) \quad T_s T_x = \begin{cases} T_{sx} & \text{if } \ell(sx) > \ell(x), \\ T_{sx} + (\mathbf{v}/b - \mathbf{v}^{-1}/b) \sum_{d \in X_s} T_{dx} & \text{if } \ell(sx) < \ell(x), \end{cases}$$

for x in WH_b and s in $\mathcal{S}$. Also, $\overline{ba - (\mathbf{v}^2 - 1)} = -\mathbf{v}^{-2}(ba - (\mathbf{v}^2 - 1))$ and so the bar involution passes to the quotient $\tilde{A}$ and to $\tilde{\mathcal{H}}$. Notice that polynomials $R_{x,y}^*$ defined in §4 are now polynomials in the quantity $\mathbf{v}/b - \mathbf{v}^{-1}/b$.

Define

$$\begin{aligned} \tilde{A}_{\leq 0} &= \mathbb{Z}_b[\mathbf{v}^{-1}], \quad \tilde{A}_{< 0} = \mathbf{v}^{-1}\mathbb{Z}_b[\mathbf{v}^{-1}], \\ \tilde{\mathcal{H}}_{\leq 0} &= \bigoplus_{x \in WH_b} \tilde{A}_{\leq 0} T_x, \quad \text{and} \quad \tilde{\mathcal{H}}_{< 0} = \bigoplus_{x \in WH_b} \tilde{A}_{< 0} T_x. \end{aligned}$$

With this notation, the proof of Theorem 5.2 in [15] applies word-for-word to prove the following theorem.

FIGURE 1. Identity component of the Hasse diagram for $G(2, 1, 3)$

Theorem 5.1. *Suppose y is in WH_b . Then there is a unique element C_y in $\tilde{\mathcal{H}}$ such that (1) $\overline{C_y} = C_y$ and (2) $C_y \equiv T_y \pmod{\tilde{\mathcal{H}}_{<0}}$.*

Clearly, $\{C_y \mid y \in WH_b\}$ is an $\tilde{A}$ -basis of $\tilde{\mathcal{H}}$.

Define polynomials $P_{x,y}^*$ for x and y in WH_b by

$$C_y = \sum_{x \in WH_b} P_{x,y}^* T_x.$$

Then as in [15, §5] we have:

- $P_{x,y}^* \neq 0$ implies that $x \leq y$.
- $P_{y,y}^* = 1$.
- For $x < y$, $P_{x,y}^*$ is a polynomial in $\mathbf{v}^{-1}$ with $P_{x,y}^*(0) = 0$ and degree (in $\mathbf{v}^{-1}$) equal to $\ell(y) - \ell(x)$.
- $P_{x,y}^*$ may be computed recursively once the R -polynomials are known using the equation

$$\overline{P_{x,y}^*} - P_{x,y}^* = R_{x,y}^* + \sum_{\substack{z \\ x < z < y}} R_{x,z}^* P_{z,y}^*.$$

Corollary 5.2. For y in WH_b and d in H_b we have $C_{yd} = C_y T_d$ and $C_{dy} = T_d C_y$.

Proof. Recall first that $\overline{T_d} = T_d$ and so $\overline{C_y T_d} = C_y T_d$. Next, $T_x T_d = T_{xd}$ and $x \leq yd$ if and only if $xd^{-1} \leq y$, so we have

$$C_y T_d = \sum_{\substack{x \\ x \leq yd}} P_{x,yd}^* T_{xd} = \sum_{\substack{z \\ z \leq yd}} P_{zd^{-1},y}^* T_z = T_{yd} + \sum_{\substack{z \\ z < yd}} P_{zd^{-1},y}^* T_z.$$

Therefore, $C_x T_d$ is in $\tilde{\mathcal{H}}_{\leq 0}$ and $C_y T_d - T_{yd}$ is in $\tilde{\mathcal{H}}_{< 0}$. It follows from Theorem 5.1 that $C_{yd} = C_y T_d$.

A similar argument shows that $C_{dy} = T_d C_y$. $\square$

Corollary 5.3. For x and y in WH_b and d in H_b we have $P_{x,yd}^* = P_{xd^{-1},y}^*$ and $P_{x,dy}^* = P_{d^{-1}x,y}^*$.

For x and y in WH_b , define

$$R_{x,y} = v^{\ell(y)-\ell(x)} R_{x,y}^* \quad \text{and} \quad P_{x,y} = v^{\ell(y)-\ell(x)} P_{x,y}^*.$$

Then $\overline{T_y} = \sum_{x \in WH_b} v^{\ell(y)-\ell(x)} \overline{R_{x,y}} T_x$ and $C_y = v^{-\ell(y)} \sum_{x \in WH_b} P_{x,y} T_x$.

Proposition 5.4. For x and y in WH_b , $R_{x,y}$ is a polynomial in $\mathbf{v}^2$ with degree (in $\mathbf{v}^2$) at most $\ell(y) - \ell(x)$.

Proof. This follows immediately from the recursion formula in Proposition 4.2. $\square$

Corollary 5.5. For x and y in WH_b , $P_{x,y}$ is a polynomial in $\mathbf{v}^2$.

Proof. This follows immediately from the last proposition and the recursion formula for $P_{x,y}^*$. $\square$

The $\tilde{A}$ -basis $\{C_y \mid y \in WH_b\}$ is a Kazhdan-Lusztig basis of $\tilde{\mathcal{H}}$ and the polynomials $P_{x,y}$ are Kazhdan-Lusztig polynomials for WH_b .

Example 5.6. We compute the Kazhdan-Lusztig R -polynomials $R_{x,y}$ and the Kazhdan-Lusztig polynomials $P_{x,y}$ when W is the symmetric group S_3 .

Since $R_{x,y} = P_{x,y} = 0$ unless $x \leq y$, $R_{x,x} = P_{x,x} = 1$, $R_{x,yd} = R_{xd^{-1},y}$ and $P_{x,yd} = P_{xd^{-1},y}$, it is enough to compute $R_{x,w}$ and $P_{x,w}$ for x in WH_b and w in W with $x < w$. Notice that $x < w$ implies $\ell(x) < \ell(w)$.

If $x < w$ and $\ell(w) - \ell(x) = 1$, then

$$R_{x,w}(\mathbf{v}) = \mathbf{v}^2/b - 1/b \quad \text{and} \quad P_{x,y}(\mathbf{v}) = 1/b.$$

If $x < w$ and $\ell(w) - \ell(x) = 2$, then

$$R_{x,w}(\mathbf{v}) = \mathbf{v}^4/b^2 - 2\mathbf{v}^2/b^2 + 1/b^2 \quad \text{and} \quad P_{x,y}(\mathbf{v}) = 1/b^2.$$

If $x < w$ and $\ell(w) - \ell(x) = 3$, then $w = s_1 s_2 s_1$ and $\ell(x) = 0$, so $x = d$ is in $\Omega_{1,s_1 s_2 s_1} = X_{s_1}^2 X_{s_2}$. There are two cases depending on whether or not d is in $s_1 X_{s_2} s_1$:

$$R_{d,s_1 s_2 s_1}(\mathbf{v}) = \begin{cases} \frac{1}{b^2} \mathbf{v}^6 - \frac{3-b}{b^2} \mathbf{v}^4 + \frac{3-b}{b^2} \mathbf{v}^2 - \frac{1}{b^2} & d \in s_1 X_{s_2} s_1, \\ \frac{1}{b^2} \mathbf{v}^6 - \frac{3}{b^2} \mathbf{v}^4 + \frac{3}{b^2} \mathbf{v}^2 - \frac{1}{b^2} & d \in X_{s_1}^2 X_{s_2} \setminus s_1 X_{s_2} s_1, \end{cases}$$

$$P_{d,s_1 s_2 s_1}(\mathbf{v}) = \begin{cases} \frac{b-1}{b^2} \mathbf{v}^2 + \frac{1}{b^2} & d \in s_1 X_{s_2} s_1, \\ \frac{-1}{b^2} \mathbf{v}^2 + \frac{1}{b^2} & d \in X_{s_1}^2 X_{s_2} \setminus s_1 X_{s_2} s_1. \end{cases}$$

Notice that in general $P_{x,y}$ does not have non-negative coefficients and that the degree of $P_{x,y}$ depends on b . In particular, if $b = 1$, so $WH_b = W$, then $d = 1$ is in $s_1 X_{s_2} s_1$ and $P_{1,s_1 s_2 s_1} = 1$; whereas if $b > 1$ and odd, then $P_{1,s_1 s_2 s_1}$ has degree 2.

6. SUBEXPRESSIONS AND A CLOSED FORMULA FOR $R_{x,y}$

In this section we adapt results of Deodhar [9] to describe the lower order ideals in WH_b and to give a closed form for the polynomials $R_{x,y}^*$.

It follows from Proposition 4.2 that for y and z in WH_b and d in H_b we have

$$y \leq zd \quad \text{if and only if} \quad yd^{-1} \leq z \quad \text{and} \\ y \leq dz \quad \text{if and only if} \quad d^{-1}y \leq z.$$

For z in WH_b , define $\mathcal{L}^z = \{x \in WH_b \mid x \leq z\}$. Then if $z = wd$ with w in W and d in H_b we have $\mathcal{L}^z = \mathcal{L}^w d^{-1}$. Thus, to analyze the lower order ideals $\mathcal{L}^z$ it is enough to consider the case when $z = w$ is in W .

Recall that a tuple of elements of $\mathcal{S}$, say $(s_1, \dots, s_p)$, is said to be *reduced* if $\ell(s_1 \cdots s_p) = p$.

Fix a reduced tuple, $\mathbf{s} = (s_1, \dots, s_p)$, of elements of $\mathcal{S}$. A *distinguished subexpression* of $\mathbf{s}$ is a $p+1$ -tuple of elements of WH_b , say $\mathbf{x} = (x_0, x_1, \dots, x_p)$, with the properties:

- (DS1) $x_0 = 1$,
- (DS2) $x_{j-1}^{-1} x_j \in \{s_j\} \cup X_{s_j}$ for $1 \leq j \leq p$, and
- (DS3) $x_j \leq x_{j-1} s_j$ for $1 \leq j \leq p$.

Let $\mathcal{D}_{\mathbf{s}}$ denote the set of distinguished subexpressions of $\mathbf{s}$ and let $\pi_{\mathbf{s}}: \mathcal{D}_{\mathbf{s}} \rightarrow WH_b$ be the projection on the last factor, $\pi_{\mathbf{s}}(\mathbf{x}) = x_p$.

In the rest of this section we frequently argue using induction on the number of elements in $\mathbf{s}$. If $\mathbf{s} = (s_1, \dots, s_p)$ with $p > 1$, define $\mathbf{s}' = (s_1, \dots, s_{p-1})$.

Theorem 6.1. *If $\mathbf{s} = (s_1, \dots, s_p)$ is a reduced tuple of elements of $\mathcal{S}$ and $w = s_1 \cdots s_p$, then*

$$\mathcal{L}^w = \{ \pi_{\mathbf{s}}(\mathbf{x}) \mid \mathbf{x} \in \mathcal{D}_{\mathbf{s}} \}.$$

Proof. We prove the result using induction on p . If $p = 0$, then the result is clear and if $p = 1$, then the result follows from Proposition 4.5.

Suppose $p > 1$. Set $s = s_p$.

Choose y in WH_b with $y \leq w$.

If $ys \leq ws$, then $0 \neq R_{y,w}^* = R_{ys,ws}^*$, so $ys \leq ws$. By induction, there is an $\mathbf{x}' = (x_0, \dots, x_{p-1})$ in $\mathcal{D}_{\mathbf{s}'}$ with $\pi_{\mathbf{s}'}(\mathbf{x}') = ys$. Define $x_p = y$ and $\mathbf{x} = (x_0, \dots, x_{p-1}, x_p)$. Then $x_{p-1}^{-1}x_p = sy^{-1}y = s$ is in $\{s_p\} \cup X_{s_p}$ and $x_p = y = yss = x_{p-1}s$. Thus, $\mathbf{x}$ is in $\mathcal{D}_{\mathbf{s}}$, $\pi_{\mathbf{s}}(\mathbf{x}) = y$, and so y is in $\pi(\mathcal{D}_{\mathbf{s}})$ in this case.

If $ys \not\leq ws$, then we must have $\ell(ys) > \ell(y)$ and $R_{y,w}^* = \mathbf{av}^{-1} \sum_{d \in X_s} R_{yd,ws}^*$. Thus, there is a d in X_s with $yd \leq ws$. By induction, there is an $\mathbf{x}' = (x_0, \dots, x_{p-1})$ in $\mathcal{D}_{\mathbf{s}'}$ with $\pi_{\mathbf{s}'}(\mathbf{x}') = yd$. Define $x_p = y$ and $\mathbf{x} = (x_0, \dots, x_{p-1}, x_p)$. Then $x_{p-1}^{-1}x_p = d^{-1}y^{-1}y = d^{-1}$ is in $\{s_p\} \cup X_{s_p}$. Say $y = vd_1$ where v is in W and d_1 is in H_b . Then $\ell(vs) > \ell(v)$ because $\ell(ys) > \ell(y)$. Now $d^{-1}d_1^{-1}d_1 \leq s$, so $d_1 \leq d_1ds$, and so $vd_1 \leq vssd_1ds = vd_1ds$. It follows that $x_p = y = vd_1 \leq vd_1ds = yds = x_{p-1}s$. Thus, $\mathbf{x}$ is in $\mathcal{D}_{\mathbf{s}}$, $\pi_{\mathbf{s}}(\mathbf{x}) = y$, and so y is in $\pi_{\mathbf{s}}(\mathcal{D}_{\mathbf{s}})$ in this case also.

We have shown that $\mathcal{L}^w \subseteq \{\pi(\mathbf{x}) \mid \mathbf{x} \in \mathcal{D}_{\mathbf{s}}\}$.

Conversely, suppose $\mathbf{x}$ is in $\mathcal{D}_{\mathbf{s}}$. Then clearly $(x_0, \dots, x_{p-1})$ is in $\mathcal{D}_{\mathbf{s}'}$. Also, $x_{p-1}^{-1}x_p$ is in $\{s\} \cup X_s$, so $x_p = x_{p-1}s$ or $x_p = x_{p-1}d$ for some d in X_s . We need to show that $x_p \leq w$.

Suppose first that $x_p = x_{p-1}s$ and $\ell(x_p s) < \ell(x_p)$. Then $R_{x_p,w}^* = R_{x_p s,ws}^* = R_{x_{p-1},ws}^*$. By induction, $x_{p-1} \leq ws$. Therefore, $R_{x_{p-1},ws}^* \neq 0$ and so $R_{x_p,w}^* \neq 0$. Hence, $x_p \leq w$ in this case.

Next, suppose that $x_p = x_{p-1}s$ and $\ell(x_p s) > \ell(x_p)$. Then

$$R_{x_p,w}^* = R_{x_p s,ws}^* + \mathbf{av}^{-1} \sum_{d \in X_s} R_{x_p d,ws}^* = R_{x_{p-1},ws}^* + \mathbf{av}^{-1} \sum_{d \in X_s} R_{x_p d,ws}^*.$$

Since $(x_0, \dots, x_{p-1})$ is in $\mathcal{D}_{\mathbf{s}'}$ it follows by induction that $x_{p-1} \leq ws$ and so $R_{x_{p-1},ws}^* \neq 0$. Therefore, it follows from Corollary 4.3 that $R_{x_p,w}^* \neq 0$ and so $x_p \leq w$ in this case.

Finally, suppose that $x_p = x_{p-1}d$ where d is in X_s . Then $x_p = x_{p-1}d \leq x_{p-1}s$. By induction $x_{p-1} \leq ws$ and so $R_{x_{p-1},ws}^* \neq 0$. By Proposition 4.2 we have

$$R_{x_{p-1}s,w}^* = \begin{cases} R_{x_{p-1},ws}^* & \text{if } \ell(x_{p-1}s) < \ell(x_{p-1}), \\ R_{x_{p-1},ws}^* + \mathbf{av}^{-1} \sum_{d \in X_s} R_{x_{p-1}sd,ws}^* & \text{if } \ell(x_{p-1}s) > \ell(x_{p-1}). \end{cases}$$

It follows from Corollary 4.3 that $R_{x_{p-1}s,w}^* \neq 0$ and so $x_{p-1}s \leq w$. Therefore, $x_p \leq w$ in this case also. $\square$

Suppose $\mathbf{s} = (s_1, \dots, s_p)$ is a reduced tuple and $\mathbf{x} = (x_1, \dots, x_p)$ is in $\mathcal{D}_{\mathbf{s}}$. Define

$$I(\mathbf{x}) = \{j \mid 1 \leq j \leq p \text{ and } x_{j-1}^{-1}x_j \in X_{s_j}\}, \quad \text{and} \quad n(\mathbf{x}) = |I(\mathbf{x})|.$$

The rest of this section is devoted to the proof of the following theorem.

Theorem 6.2. Suppose $\mathbf{s} = (s_1, \dots, s_p)$ is a reduced tuple and $w = s_1 \cdots s_p$. Then

$$R_{y,w}^* = \sum_{\mathbf{x} \in \pi_{\mathbf{s}}^{-1}(y)} (\mathbf{a}\mathbf{v}^{-1})^{n(\mathbf{x})}$$

for $y \leq w$.

We will prove the theorem using induction on p and the recursion formula from Proposition 4.2. The argument is essentially the same as that in Deodhar [9], suitably modified so as to make sense in our context.

Set $w = s_1 \cdots s_p$ and $s = s_p$. In the next lemmas, $p > 1$ and y denotes an element in WH_b with $y \leq w$. We will analyze the subsets $\pi_{\mathbf{s}}^{-1}(y)$ of $\mathcal{D}_{\mathbf{s}}$.

Recall that we have defined $\mathbf{s}' = (s_1, \dots, s_{p-1})$. It follows immediately from (DS1) to (DS3) that $\mathbf{x}' = (x_0, \dots, x_{p-1})$ is in $\mathcal{D}_{\mathbf{s}'}$. Let θ denote the projection from $\mathcal{D}_{\mathbf{s}}$ to $\mathcal{D}_{\mathbf{s}'}$ given by $\theta(\mathbf{x}) = \mathbf{x}'$.

Clearly, if $\mathbf{x}$ and $\tilde{\mathbf{x}}$ are in $\pi_{\mathbf{s}}^{-1}(y)$ with $\theta(\mathbf{x}) = \theta(\tilde{\mathbf{x}})$, then $\mathbf{x} = \tilde{\mathbf{x}}$. Thus, $\theta|_{\pi_{\mathbf{s}}^{-1}(y)}$ is injective.

Suppose $\mathbf{x} = (x_0, \dots, x_p)$ is in $\pi_{\mathbf{s}}^{-1}(y)$ and $p > 1$. Then $x_{p-1}^{-1}y$ is in $\{s\} \cup X_s$ and so:

- p is not in $I(\mathbf{x})$ if and only if $x_{p-1} = ys$ and
- p is in $I(\mathbf{x})$ if and only if $x_{p-1} = yd$ for some d in X_s .

Set

$$\pi_{\mathbf{s}}^{-1}(y)_p^* = \{ \mathbf{x} \in \mathcal{D}_{\mathbf{s}} \mid \pi_{\mathbf{s}}(\mathbf{x}) = y \text{ and } p \notin I(\mathbf{x}) \}$$

and

$$\pi_{\mathbf{s}}^{-1}(y)_p = \{ \mathbf{x} \in \mathcal{D}_{\mathbf{s}} \mid \pi_{\mathbf{s}}(\mathbf{x}) = y \text{ and } p \in I(\mathbf{x}) \},$$

so $\pi_{\mathbf{s}}^{-1}(y)$ is the disjoint union of $\pi_{\mathbf{s}}^{-1}(y)_p$ and $\pi_{\mathbf{s}}^{-1}(y)_p^*$.

Lemma 6.3. If $\ell(ys) < \ell(y)$, then

- (1) $ys \leq ws$,
- (2) $\pi_{\mathbf{s}}^{-1}(y) = \pi_{\mathbf{s}}^{-1}(y)_p^*$, and
- (3) $\theta|_{\pi_{\mathbf{s}}^{-1}(y)}: \pi_{\mathbf{s}}^{-1}(y) \rightarrow \pi_{\mathbf{s}'}^{-1}(ys)$ is a bijection.

Proof. By Proposition 4.2 we have $R_{y,w}^* = R_{ys,ws}^*$ and so $ys \leq ws$.

Say $\mathbf{x} = (x_0, \dots, x_p)$ is in $\pi_{\mathbf{s}}^{-1}(y)$. Then $x_p = y$, $x_{p-1}^{-1}y$ is in $\{s\} \cup X_s$, and $y \leq x_{p-1}s$. Just suppose that $x_{p-1}^{-1}y = d$ is in X_s . Then $y \leq yd^{-1}s$ and so $\ell(y) \leq \ell(ys)$, a contradiction. Therefore, p is not in $I(\mathbf{x})$. This shows that $\pi_{\mathbf{s}}^{-1}(y) = \pi_{\mathbf{s}}^{-1}(y)_p^*$.

To show that $\theta(\pi_{\mathbf{s}}^{-1}(y)) = \pi_{\mathbf{s}'}^{-1}(ys)$, suppose $\mathbf{x}' = (x_0, \dots, x_{p-1})$ is in $\mathcal{D}_{\mathbf{s}'}$ and $x_{p-1} = ys$. Set $x_p = y$ and $\mathbf{x} = (x_0, \dots, x_p)$. Then $x_{p-1}^{-1}x_p = sy^{-1}y = s$ and $x_p = y = yss = x_{p-1}s$, so $\mathbf{x}$ is in $\pi_{\mathbf{s}}^{-1}(y)$ and $\theta(\mathbf{x}) = \mathbf{x}'$. $\square$

Lemma 6.4. Suppose $y \leq w$ and $y = vd$ with v in W and d in X_s . If $\ell(ys) > \ell(y)$, then $y \leq yd_1s$ for all d_1 in X_s .

Proof. Since $\ell(ys) > \ell(y)$ it follows that $\ell(vs) > \ell(v)$. If d_1 is in X_s , then $d_1^{-1} \leq s$ and so $1 \leq ds$. Thus, $d \leq (s)(sdd_1s)$ and so $y = vd \leq (vs)(sdd_1s) = yd_1s$. $\square$

Lemma 6.5. If $\ell(ys) > \ell(y)$ and $ys \not\leq ws$, then

- (1) $\pi_{\mathbf{s}}^{-1}(y) = \pi_{\mathbf{s}}^{-1}(y)_p$ and
- (2) $\theta|_{\pi_{\mathbf{s}}^{-1}(y)}: \pi_{\mathbf{s}}^{-1}(y) \rightarrow \bigcup_{d \in X_s} \pi_{\mathbf{s}'}^{-1}(yd)$ is a bijection.

Proof. Say $\mathbf{x} = (x_0, \dots, x_p)$. Then $x_p = y$ and $x_{p-1}^{-1}y$ is in $\{s\} \cup X_s$. Just suppose that $x_{p-1}^{-1}y = s$, so $y = x_{p-1}s$. Now $\mathbf{x}' = (x_0, \dots, x_{p-1})$ is in $\mathcal{D}_{s'}$ and so by Theorem 6.1, $x_{p-1} \leq ws$. But $x_{p-1} = ys$ and by assumption $ys \not\leq ws$, a contradiction. Therefore, p is not in $I(\mathbf{x})$. This shows that $\pi_s^{-1}(y) = \pi_s^{-1}(y)_p$.

To show that $\theta(\pi_s^{-1}(y)) = \bigcup_{d \in X_s} \pi_{s'}^{-1}(yd)$, suppose d is in X_s , $\mathbf{x}' = (x_0, \dots, x_{p-1})$ is in $\mathcal{D}_{s'}$, and $x_{p-1} = yd$. Set $x_p = y$ and $\mathbf{x} = (x_0, \dots, x_p)$. Then $x_{p-1}^{-1}x_p = d^{-1}y^{-1}y = d^{-1}$ is in X_s . It follows from the last lemma that $y \leq yds$ and so $\mathbf{x}$ is in $\pi_s^{-1}(y)$ and $\theta(\mathbf{x}) = \mathbf{x}'$. $\square$

Lemma 6.6. *Suppose $\ell(ys) \leq \ell(y)$ and $ys \leq ws$, then*

$$\theta|_{\pi_s^{-1}(y)_p^*} : \pi_s^{-1}(y)_p^* \rightarrow \pi_{s'}^{-1}(ys) \quad \text{and} \quad \theta|_{\pi_s^{-1}(y)_p} : \pi_s^{-1}(y)_p \rightarrow \bigcup_{d \in X_s} \pi_{s'}^{-1}(yd)$$

are bijections.

Proof. If $\mathbf{x} = (x_0, \dots, x_p)$ is in $\pi_s^{-1}(y)_p^*$, then $x_p = y$ and $p \notin I(\mathbf{x})$, so $x_{p-1} = ys$. Thus $\theta(\mathbf{x})$ is in $\pi_{s'}^{-1}(ys)$.

To show that $\theta(\pi_s^{-1}(y)_p^*) = \pi_{s'}^{-1}(ys)$, suppose $\mathbf{x}' = (x_0, \dots, x_{p-1})$ is in $\pi_{s'}^{-1}(ys)$. Set $x_p = y$ and $\mathbf{x} = (x_0, \dots, x_p)$. Then as above, $x_{p-1}^{-1}x_p = sy^{-1}y = s$ and $x_p = y = yss = x_{p-1}s$, so $\mathbf{x}$ is in $\pi_s^{-1}(y)_p^*$ and $\theta(\mathbf{x}) = \mathbf{x}'$.

If $\mathbf{x} = (x_0, \dots, x_p)$ is in $\pi_s^{-1}(y)_p$, then $x_p = y$ and $p \in I(\mathbf{x})$, so $x_{p-1} = yd_1$ for some d_1 in X_s . Thus $\theta(\mathbf{x})$ is in $\bigcup_{d \in X_s} \pi_{s'}^{-1}(yd)$.

To show that $\theta(\pi_s^{-1}(y)_p) = \bigcup_{d \in X_s} \pi_{s'}^{-1}(yd)$, suppose d is in X_s and $\mathbf{x}' = (x_0, \dots, x_{p-1})$ is in $\pi_{s'}^{-1}(yd)$. Set $x_p = y$ and $\mathbf{x} = (x_0, \dots, x_p)$. Then as above, $x_{p-1}^{-1}x_p = d^{-1}$ and it follows from Lemma 6.4 that $y \leq yds$, so $\mathbf{x}$ is in $\pi_s^{-1}(y)_p$. $\square$

Proof of Theorem 6.2. We prove the theorem using induction on p . If $p = 1$, then the result follows from (3.4.3).

Suppose $p > 1$, set $w = s_1 \cdots s_p$ and $s = s_p$, and suppose y is in WH_b with $y \leq w$. There are three cases:

- (1) $\ell(ys) < \ell(y)$,
- (2) $\ell(ys) > \ell(y)$ and $ys \not\leq ws$, and
- (3) $\ell(ys) > \ell(y)$ and $ys \leq ws$.

Suppose first that $\ell(ys) < \ell(y)$. Then $ys \leq ws$ and $n(\theta(\mathbf{x})) = n(\mathbf{x})$ for $\mathbf{x}$ in $\pi_s^{-1}(y)$. Using Proposition 4.2, induction, and Lemma 6.3 we have

$$\begin{aligned} R_{y,w}^* &= R_{ys,ws}^* \\ &= \sum_{\mathbf{x}' \in \pi_{s'}^{-1}(ys)} (\mathbf{a}\mathbf{v}^{-1})^{n(\mathbf{x}')} \\ &= \sum_{\mathbf{x} \in \pi_s^{-1}(y)} (\mathbf{a}\mathbf{v}^{-1})^{n(\theta(\mathbf{x}))} \\ &= \sum_{\mathbf{x} \in \pi_s^{-1}(y)} (\mathbf{a}\mathbf{v}^{-1})^{n(\mathbf{x})}. \end{aligned}$$

Second, suppose that $\ell(ys) > \ell(y)$ and $ys \not\leq ws$. Then $n(\mathbf{x}) = n(\theta(\mathbf{x})) + 1$ for $\mathbf{x}$ in $\pi_s^{-1}(y)$. Using Proposition 4.2, induction, and Lemma 6.5 we have

$$\begin{aligned}
 R_{y,w}^* &= \mathbf{a}\mathbf{v}^{-1} \sum_{d \in X_s} R_{yd,ws}^* \\
 &= \mathbf{a}\mathbf{v}^{-1} \sum_{d \in X_s} \sum_{\mathbf{x}' \in \pi_{s'}^{-1}(yd)} (\mathbf{a}\mathbf{v}^{-1})^{n(\mathbf{x}')} \\
 &= \sum_{d \in X_s} \sum_{\mathbf{x}' \in \pi_{s'}^{-1}(yd)} (\mathbf{a}\mathbf{v}^{-1})^{n(\mathbf{x}') + 1} \\
 &= \sum_{\mathbf{x} \in \pi_s^{-1}(y)} (\mathbf{a}\mathbf{v}^{-1})^{n(\theta(\mathbf{x})) + 1} \\
 &= \sum_{\mathbf{x} \in \pi_s^{-1}(y)} (\mathbf{a}\mathbf{v}^{-1})^{n(\mathbf{x})}.
 \end{aligned}$$

Finally, suppose that $\ell(ys) > \ell(y)$ and $ys \leq ws$. Then $n(\mathbf{x}) = n(\theta(\mathbf{x}))$ for $\mathbf{x}$ in $\pi_s^{-1}(y)_p^*$ and $n(\mathbf{x}) = n(\theta(\mathbf{x})) + 1$ for $\mathbf{x}$ in $\pi_s^{-1}(y)_p$. Using Proposition 4.2, induction, and Lemma 6.6 we have

$$\begin{aligned}
 R_{y,w}^* &= R_{ys,ws}^* + \mathbf{a}\mathbf{v}^{-1} \sum_{d \in X_s} R_{yd,ws}^* \\
 &= \sum_{\mathbf{x}' \in \pi_{s'}^{-1}(ys)} (\mathbf{a}\mathbf{v}^{-1})^{n(\mathbf{x}')} + \mathbf{a}\mathbf{v}^{-1} \sum_{d \in X_s} \sum_{\mathbf{x}' \in \pi_{s'}^{-1}(yd)} (\mathbf{a}\mathbf{v}^{-1})^{n(\mathbf{x}')} \\
 &= \sum_{\mathbf{x}' \in \pi_{s'}^{-1}(ys)} (\mathbf{a}\mathbf{v}^{-1})^{n(\mathbf{x}')} + \sum_{d \in X_s} \sum_{\mathbf{x}' \in \pi_{s'}^{-1}(yd)} (\mathbf{a}\mathbf{v}^{-1})^{n(\mathbf{x}') + 1} \\
 &= \sum_{\mathbf{x} \in \pi_s^{-1}(ys)_p^*} (\mathbf{a}\mathbf{v}^{-1})^{n(\theta(\mathbf{x}))} + \sum_{\mathbf{x} \in \pi_s^{-1}(y)_p} (\mathbf{a}\mathbf{v}^{-1})^{n(\theta(\mathbf{x})) + 1} \\
 &= \sum_{\mathbf{x} \in \pi_s^{-1}(ys)_p^*} (\mathbf{a}\mathbf{v}^{-1})^{n(\mathbf{x})} + \sum_{\mathbf{x} \in \pi_s^{-1}(y)_p} (\mathbf{a}\mathbf{v}^{-1})^{n(\mathbf{x})} \\
 &= \sum_{\mathbf{x} \in \pi_s^{-1}(y)} (\mathbf{a}\mathbf{v}^{-1})^{n(\mathbf{x})}.
 \end{aligned}$$

This completes the proof of the theorem. $\square$

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